Hamiltonian structures of Liénard equation revisited: From non standard to contact Hamiltonian

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1 The Hamiltonian formulation of a Liénard equation and Cheillini's integrability condition

Consider a class of second order differential equations (ODE) in which the damping term is proportional to the velocity $\dot{x}$, i.e.,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1.1}$$

This is equivalent to the standard system,

$$\dot{x} = y, \qquad \dot{y} = -f(x)y - g(x). \tag{1.2}$$

To deduce a Lagrangian for such a planar system we use the Jacobi Last Multiplier (JLM), M, whose relationship with the Lagrangian, $L = L(t, x, \dot{x})$, for any second-order equation of the form

$$\ddot{x} = F(t, x, \dot{x}) \tag{1.3}$$

is

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}. \tag{1.4}$$

Note that the JLM, $M = M(t, x, \dot{x})$ satisfies, by definition, the following equation

$$\frac{d}{dt}(\log M) + \frac{\partial F}{\partial \dot{x}} = 0, \tag{1.5}$$

which in the present case is

$$\frac{d}{dt}(\log M) - f(x) = 0. \tag{1.6}$$

Assuming that its formal solution is related to a new variable u defined *via*

$$M(t, x) = \exp\left(\int f(x)dt\right) := u^{1/\alpha}$$

we find that

$$\dot{u} = \alpha u f(x). \tag{1.7}$$

Let us now set

$$\dot{x} = u + W(x),$$

where the particular form of $W(x)$ is to be determined. On taking the time derivative of the last equation and using (1.7) we have

$$\ddot{x} = (\alpha f(x) + W'(x))\dot{x} - \alpha f(x) W. \tag{1.8}$$

Comparing (1.1) and (1.8) we see that

$$W'(x) = -(\alpha + 1) f(x) \quad \text{and} \;\; \alpha W(x) = \frac{g(x)}{f(x)}.$$

The consistency of these expressions leads to the integrability condition:

$$\frac{d}{dx}\left(\frac{g}{f}\right) = -\alpha(\alpha + 1) f(x), \tag{1.9}$$

which serves to determine the parameter α for given f and g. Comparison of $\frac{d}{dx}(\frac{g}{f}) = sf$ and (1.9) shows that the constant $s = -\alpha(1 + \alpha)$. Furthermore the system (1.2) is equivalent to the system

$$\dot{x} = u + \frac{1}{\alpha}\frac{g}{f}, \quad \dot{u} = \alpha u f$$

subject of course to the condition (1.9).

2 Chiellini integrability and Bi-Hamiltonian structure of polynomial class of Liénard equation

From the above discussion it is clear that the system (1.2) is equivalent to the following system

$$\dot{u} = \alpha u f(x), \quad \dot{x} = u + \frac{1}{\alpha}\left(\frac{g}{f}\right), \tag{2.1}$$

subject to Cheillini's condition for integrability (1.9). Assuming that this system admits a Hamiltonian structure we may recast (2.1) as

$$\dot{x} = u + \frac{1}{\alpha}\frac{g}{f} = -J\frac{\partial H}{\partial u}, \qquad \dot{u} = \alpha u f(x) = J\frac{\partial H}{\partial x}, \tag{2.2}$$

where H is the Hamiltonian of the system and J is symplectic, then up on equating the mixed derivative of H w.r.t. x and u we get the following linear partial differential equation determining the symplectic J:

$$J_x(u + \frac{1}{\alpha}\frac{g}{f}) + \alpha f(x) u J_u = -J f(x). \tag{2.3}$$

We assume that the function $f(x)$ is of the form

$$f(x) = a\ x^{\mu} \tag{2.4}$$

then condition (1.9) which then determines the function $g(x)$ leads to

$$g(x) = x^{\mu} + a^2\frac{\alpha(1-\alpha)}{\mu+1}x^{2\mu+1}, \tag{2.5}$$

where we have set the constant of integration to be a^{-1}, without loss of generality. Consequently the system (2.1) now appears as

$$\dot{u} = \alpha a u x^{\mu}, \tag{2.6}$$

$$\dot{x} = (u + \frac{1}{\alpha a}) + \frac{1-\alpha}{\mu+1}a x^{\mu+1}. \tag{2.7}$$

In general if we assume

$$\dot{x} = u + \frac{1}{\alpha}\frac{g}{f} = -J\frac{\partial H}{\partial u}, \qquad \dot{u} = \alpha u f(x) = J\frac{\partial H}{\partial x}, \tag{2.8}$$

where H is the Hamiltonian of the system and J is symplectic, then up on equating the mixed derivative of H w.r.t. x and u we get the following linear partial differential equation determining the symplectic

J:

$$J_x(u + \frac{1}{\alpha}\frac{g}{f}) + \alpha f(x) u J_u = J f(x). \tag{2.9}$$

For the chosen form of $f(x)$ and $g(x)$ given in (2.5) this becomes

$$J_x \left[(u + \frac{1}{\alpha a} + \frac{1-\alpha}{1+\mu} a x^{\mu+1} \right] + (a\alpha u x^{\mu}) J_u = J a x^{\mu}. \tag{2.10}$$

The Lagrange system for (2.9) is in general

$$\frac{dx}{u + \alpha^{-1} g/f} = \frac{du}{\alpha u f} = \frac{dJ}{-Jf}. \tag{2.11}$$

Its characteristics are easily found to be:

$$c_1 = J u^{1/\alpha} \tag{2.12}$$

$$c_2 = u^{(\alpha+1)/\alpha} \left[\frac{g}{f} + \frac{\alpha(\alpha+1)}{2\alpha+1} u \right], \tag{2.13}$$

where J is the component of a symplectic matrix. The general solution of (2.11) is therefore of the form $c_1 = F(c_2)$ where F is an arbitrary function. Assuming $F(c_2) = c_2$ we have

$$J = u \left(\frac{g}{f} + \frac{\alpha(\alpha+1)}{2\alpha+1} u \right). \tag{2.14}$$

It remains to calculate the Hamiltonian H which using the last expression for J in (2.8) is given by

$$H = \ln \left[|u|^{-1/\alpha} \left| \frac{g}{f} + \frac{\alpha(\alpha+1)}{2\alpha+1} u \right|^{-1/(\alpha+1)} \right]. \tag{2.15}$$

Thus we claim:

Proposition 2.1 *Let the Liénard equation satisfy Chiellini's integrability condition then the planar system* $\dot{u} = \alpha u f(x), \dot{x} = u + \frac{1}{\alpha}\frac{g}{f}$ *admits a bihamiltonian structure, with symplectic structures given by*

$$J_1 = k_1 u^{1/\alpha}, \qquad J_2 = k_2 u^{1/(1-\alpha)}, \tag{2.16}$$

where k_1 *and* k_2 *are arbitrary constants.*

3 A Lagrangian and Hamiltonians of Liénard equation

Now from (1.4) and (2.1) we have

$$\frac{\partial^2 L}{\partial \dot{x}^2} = \left(\dot{x} - \frac{1}{\alpha}\frac{g}{f}\right)^{1/\alpha},$$

so that

$$L(x, \dot{x}, t) = \frac{\left(\dot{x} - \frac{1}{\alpha}\frac{g}{f}\right)^{1/\alpha+2}}{(1/\alpha+1)(1/\alpha+2)} + h_1(x,t)\dot{x} + h_2(x,t).$$

Here $h_1(x,t)$ and $h_2(x,t)$ are arbitrary functions of integration. Inserting this Lagrangian into the Euler-Lagrange equation and using (1.1) we find that

$$h_{1t} - h_{2x} = 0.$$

Therefore choosing $h_1(x,t) = G_x$ and $h_2(x,t) = G_t$ it follows that

$$L = \frac{\left(\dot{x} - \frac{1}{\alpha}\frac{g}{f}\right)^{1/\alpha+2}}{(1/\alpha+1)(1/\alpha+2)} + \frac{dG}{dt}. \tag{3.1}$$

We can drop the total derivative term without loss of generality. The conjugate momentum is then defined in the usual manner by

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{\left(\dot{x} - \frac{1}{\alpha}\frac{g}{f}\right)^{1/\alpha+1}}{(1/\alpha+1)},$$

and therefore

$$\dot{x} = \frac{1}{\alpha}\frac{g}{f} + \left((1/\alpha+1)p\right)^{1/(1/\alpha+1)}.$$

Using the standard Legendre transformation the Hamiltonian is found to be

$$H = p\dot{x} - L = \frac{1}{\alpha}p\frac{g}{f} + \frac{\alpha}{2\alpha+1}\left(\frac{\alpha+1}{\alpha}p\right)^{\frac{2\alpha+1}{\alpha+1}}. \tag{3.2}$$

The corresponding canonical equations are:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{1}{\alpha}\frac{g}{f} + \left(\frac{\alpha+1}{\alpha}p\right)^{\frac{\alpha}{\alpha+1}}, \tag{3.3}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = (\alpha+1)fp. \tag{3.4}$$

In terms of the scaled variable $\tilde{p} := (\alpha+1)/\alpha p$ the Hamiltonian (3.2) has the following appearance,

$$H(x,\tilde{p};\alpha) = \frac{1}{\alpha+1}\tilde{p}\frac{g}{f} + \frac{\alpha}{2\alpha+1}\tilde{p}^{\frac{2\alpha+1}{\alpha+1}}. \tag{3.5}$$

Note that α is a parameter and H changes if α changes. The canonical Poisson bracket accordingly becomes $\{x,\tilde{p}\} = (\alpha+1)/\alpha$. From Chiellini condition the function $g = K(x)f$ and hence the Hamiltonian assumes the form

$$H(x,\tilde{p};\alpha) = \frac{\tilde{p}}{\alpha+1}K(x) + \frac{\alpha}{2\alpha+1}\tilde{p}^{\frac{2\alpha+1}{\alpha+1}}. \tag{3.6}$$

If $K(x)$ is a linear function of x then the first factor reduces to the well known Berry-Keating Hamiltonian. Thus the Liénard Hamiltonian is a deformation of the Berry-Keating Hamiltonian.

4 Constant of motion of damped oscillator from Non standard Hamiltonian

Recall the Hamiltonian of the Liénard equation in terms of $(\dot{x}, x)$,

$$H = \frac{(\dot{x} - \frac{1}{\alpha}\frac{g}{f})^{1+1/\alpha}}{2+1/\alpha}(\dot{x} + \frac{1}{1+\alpha}\frac{g}{f}). \tag{4.1}$$

Proposition 4.1 *The Hamiltonian H is a first integral of the Liénard equation.*

Proof : By direct calculation. □

Let us rewrite Eqn. (4.1)

$$H = \frac{(\dot{x} - \frac{1}{\alpha}\frac{g}{f})^{1/\alpha}}{2+1/\alpha}\big(\dot{x}^2 - \frac{1}{\alpha(1+\alpha)}\frac{g}{f}\dot{x} - \frac{1}{\alpha(1+\alpha)}(\frac{g}{f})^2\big). \tag{4.2}$$

Let us focus on to damped oscillator, where $f(x) = \gamma$ and $g(x) = x$.

Proposition 4.2 *The equation (4.2) reduces to constant of motion*

$$I = e^{\gamma t}(\dot{x}^2 + \gamma x\dot{x} + x^2)$$

of damped harmonic oscillator when for $f(x) = \gamma$ and $g(x) = x$.

Proof : It is clear from the Chiellini integrability condition $\frac{1}{\gamma^2} = -\alpha(\alpha+1)$. Substituting $f(x), g(x)$ we obtain

$$(\dot{x}^2 - \frac{1}{\alpha(1+\alpha)}\frac{g}{f}\dot{x}\frac{1}{\alpha(1+\alpha)}(\frac{g}{f})^2) = (\dot{x}^2 + \gamma x\dot{x} + x^2).$$

Our next step is to show that $\frac{(\dot{x}-\frac{1}{\alpha}\frac{g}{f})^{1/\alpha}}{2+1/\alpha} = e^{\gamma t}$. If we take a log of both sides and differentiating w.r.t. time t, we obtain

$$\ddot{x} - \frac{1}{\alpha}\frac{d}{dx}\big(\frac{g}{f}\big)\dot{x} = \alpha\gamma\big(\dot{x} - \frac{1}{\alpha}\frac{g}{f}\big), \qquad \text{where} \quad f(x) = \gamma, g(x) = x.$$

Once using Chiellini condition one can show that the above equation satisfies damped oscillator equation. □

Let us now focus on to Liénard equation.

Proposition 4.3 *The non standard Hamiltonian for the Liénard equation*

$$H_L = \frac{(\dot{x} - \frac{1}{\alpha}\frac{g}{f})^{1/\alpha}}{2+\frac{1}{\alpha}}\big(\dot{x}^2 - \frac{1}{\alpha(1+\alpha)}\frac{g}{f}\dot{x} - \frac{1}{\alpha(1+\alpha)}(\frac{g}{f})^2\big)$$

can be re-written as

$$H_L = e^{\int f(x)dt}\big(\dot{x}^2 - \frac{1}{\alpha(1+\alpha)}\frac{g}{f}\dot{x} - \frac{1}{\alpha(1+\alpha)}(\frac{g}{f})^2\big). \tag{4.3}$$

One can easily check that $\frac{dH_L}{dt} = 0$.

5 Non standard Hamiltonian to contact Hamiltonian

We start with the damped harmonic oscillator. Let us consider $h = (\dot{x}^2 + \gamma x\dot{x} + x^2)$. Let $y = \dot{x}$ and $s(x,y) = xy$, then h becomes contact Hamiltonian $h_c = y^2 + x^2 + \gamma s$, and the equation of motions read

$$\dot{x} = y, \qquad \dot{y} = -x - \gamma y, \qquad \dot{s} = y^2 - x^2 - \gamma s.$$

We can generalize the Hamiltonian equation to a contact manifold

$$\dot{x^i} = \frac{\partial h_c}{\partial y_i}, \qquad \dot{y_i} = -\frac{\partial h_c}{\partial x^i} - \frac{\partial h_c}{\partial s}y_i \qquad \dot{s} = y_i\frac{\partial h_c}{\partial y_i} - h_c. \tag{5.1}$$

From the first two equations it is clear that this dynamics induces a standard Hamiltonian dynamics over the physical phase space whenever the generating function h_c does not depend on s. Futhermore this set of equations is very similar to the Nosé-Hoover thermostat and Density Dynamics equations.

The contact Hamiltonian vector field X_{h_c} takes the form

$$\frac{\partial h_c}{\partial y}\frac{\partial}{\partial x} - (\frac{\partial h_c}{\partial x} + \frac{\partial h_c}{\partial s}y)\frac{\partial}{\partial y} + (y\frac{\partial h_c}{\partial y} - h_c)\frac{\partial}{\partial s}, \tag{5.2}$$

which satisfies

$$dh_c = d\eta(X_{h_c}, \cdot) - L_{X_{h_c}}\eta(\cdot). \tag{5.3}$$

Here η is contact one form that satisfies the condition $\eta \wedge (d\eta)^n \neq 0$ for a contact manifold T. There is another fundamental object called the Reeb vector field ξ which satisfies $\eta(\xi) = 1$ and $d\eta(\xi) = 0$. In terms of contact coordinates (x^i, y_i, s) 1-form η and the Reeb vector field can be expressed as

$$\eta = ds - y_i dx^i, \qquad \xi = \frac{\partial}{\partial s}.$$

Given two contact vector fields X_f and X_g on a differentiable manifold M and a symplectic form ω one obtains the following Lie bracket called the Jacobi bracket from

$$\{f, g\} = i_{[X_f, X_g]}\omega,$$

where its expression in local coordinates is given by

$$\{f, g\} = \sum_{k=1}^{n} (\frac{\partial f}{\partial x^k}\frac{\partial g}{\partial y_k} - \frac{\partial f}{\partial y_k}\frac{\partial g}{\partial x^k})$$

$$+(f - \sum_{k=1}^{n} y_k\frac{\partial f}{\partial y_k})\frac{\partial g}{\partial s} - (g - \sum_{k=1}^{n} y_k\frac{\partial g}{\partial y_k})\frac{\partial f}{\partial s}.$$

So we map non standard Hamiltonian to contact or cosymplectic mechanics, given by M. de Leon et al. , Bravetti et al.

In the case of the Liénard equation things are not so simple. Define

$$s = -\frac{1}{\alpha(1+\alpha)}\frac{g}{f}\dot{x}. \tag{5.4}$$

Then we obtain

$$\begin{aligned}\dot{s} &= -\frac{1}{\alpha(1+\alpha)}\frac{d}{dx}(\frac{g}{f})\dot{x}^2 - \frac{1}{\alpha(1+\alpha)}\frac{g}{f}\big(- f(x)\dot{x} - g(x)) \\ &= f(x)\big(\dot{x}^2 + \frac{1}{\alpha(1+\alpha)}\frac{g}{f}\dot{x} + \frac{1}{\alpha(1+\alpha)}(\frac{g}{f})^2\big).\end{aligned}$$

Now once again we write the Hamiltonian H_L as $H_L \equiv h_c = y^2 - \frac{1}{\alpha(1+\alpha)}(\frac{g}{f})^2 + s$, then a straight forward computation shows

$$\dot{s} = f(x)\big(y\frac{\partial h}{\partial y} - h_c\big). \tag{5.5}$$

Unfortunately we can not recast the Liénard equation in contact mechanics form, unless $f(x) = constant$.

6 More on conformal Hamiltonian formalism

If we drop the $\dot{x}$ term from the earlier defined Lagrangian for the Liénard equation we obtain

$$L = \frac{1}{2}e^{\int^t f(x(s))ds}\big(\dot{x}^2 + \frac{1}{\alpha(\alpha+1)}(\frac{g}{f})^2\big), \tag{6.1}$$

where we introduced a normalization factor.

Let us start with the Lagrangian flow correspond to the following Euler-Lagrange equation

$$\frac{d}{dt}\big(e^{\int^t f(x(s))ds}\frac{\partial \mathcal{L}}{\partial y}\big) = e^{\int^t f(x(s))ds}\frac{\partial \mathcal{L}}{\partial x}. \tag{6.2}$$

This can be simplified to

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial y}) = \frac{\partial \mathcal{L}}{\partial x} - f(x)y, \tag{6.3}$$

this becomes the Liénard equation for $\mathcal{L} = \frac{1}{2}(y^2 + \frac{1}{\alpha(\alpha+1)}(\frac{g}{f})^2)$. Using the Legendre transformation associated to L,

$$FL_{\mathcal{L}} : TM \to T^*M, \qquad (x, y) \mapsto (x, \frac{\partial \mathcal{L}}{\partial y}(x, y)),$$

we rewrite the above equation in (conformal) Hamiltonian form

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} - f(x)p, \tag{6.4}$$

where we have tacitly used the consequence of the Legendre transform $\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial \mathcal{H}}{\partial x}$.

The vector field X_H^λ on M is conformal with real parameter λ if $i_{X_H^\lambda}\omega = dH - \lambda\theta$, where $\omega = dx \wedge dp = -d\theta$ and $H \in C^\infty(M)$. This condition is equivalent to $L_{X_H^\lambda}\omega = -\lambda\omega$.

Note that the hypothesis of exact symplectic manifold does not restrain the generality, since a symplectic manifold admits a vector field X_H^λ with $L_{X_H^\lambda}\omega = -\lambda\omega$ if and only if it is exact. If in addition, $H^1(M) = 0$, then all conformal vector fields on M are given by

$$\{X_H + \lambda Z\} \,|\, H \in C^\infty(M)\}, \qquad i_Z\omega = -\theta.$$

First step to generalize the definition of a conformal Hamiltonian vector on an exact symplectic manifold is to introduce $Z = fp\frac{\partial}{\partial p}$ such that

$$L_{X_H^f}\omega = -d(f\theta), \qquad \text{where} \quad H^1(M) = 0.$$

If $f(x)$ is an arbitrary function of x, then $H^1(M) \neq 0$, then we can not have a conformal vector field on exact symplectic manifold. Locally we can define $Z_f = f(x)p\frac{\partial}{\partial p}$ and the conformal Hamiltonian $X_H^f = X_H + Z_f$, which is a sum of Hamiltonian vector field and generalized Liouville vector field yields the vector field of the Liénard equation.

7 Deformation of Lagrangians

We will show that the Lagrangian derived here is a *deformation* of a more elementary Lagrangian L. Consider a differential function $\phi : \mathbb{R} \to \mathbb{R}$, then for a Lagrangian L, the deformation of Lagrangian function is $\phi(L)$.

Theorem 7.1 *(Cariñena-Nũnez) Let L is a regular Lagrangian for a SODE vector field Γ_L such that $i(\Gamma_L)d\Theta_L = 0$, where Θ_L is the Poincaré-Cartan form defined as*

$$\Theta_L = \frac{\partial L}{\partial v^i}\vartheta^i + Ldt = \frac{\partial L}{\partial v^i}dq^i - E_L dt,$$

where $\vartheta^i = dq^i - v^i dt$ are the n (local) one forms which generate the contact distribution and $E_L = \Delta(L) - L$ with $\Delta = v^i \partial_{v^i}$ being the Liouville vector field. The equations of motion for $\phi(L)$, $i(\Gamma_{(\phi(L)}d\Theta_{\phi(L)} = 0$ yields

$$\Gamma_{\phi(L)}\Big(\frac{\partial L}{\partial v^i}\Big) - \frac{\partial L}{\partial q^i} + \frac{\phi''}{\phi'}\frac{dL}{dt}\frac{\partial L}{\partial v^i} = 0. \tag{7.1}$$

It is worth to note that the Poincaré-Cartan form for deformed Lagrangian is

$$\Theta_{\phi(L)} = \phi''(L)dL \wedge S^{\vee}(dL) + \phi'(L)d\Theta_L, \tag{7.2}$$

where $S^\vee$ is the dual endomorphism on differential forms of the vertical endomorphism S of $\mathbb{R} \times TQ$. Here S is a $(1,1)$ tensor field and for every π-vertical vector field S satisfies $S(\Gamma) = 0$, $S(Y) = 0$ and $S([\Gamma, Y]) = -Y$.

8 Application to Liénard system

The Lagrangian L derived earlier using JLM with Chiellini condition is a power of more elementary Lagrangian $\mathcal{L}$, thus the deformed Lagrangian of $\mathcal{L}$ is given by

$$\phi(\mathcal{L}) = \frac{\left(\dot{x} - \frac{1}{\alpha}\frac{g}{f}\right)^{1/\alpha+2}}{(1/\alpha+1)(1/\alpha+2)} = \frac{1}{(1/\alpha+1)(1/\alpha+2)}\mathcal{L}^{1/\alpha+2}, \quad (8.1)$$

where the elementary Lagrangian is given by

$$\mathcal{L} = \left(\dot{x} - \frac{1}{\alpha}\frac{g}{f}\right). \quad (8.2)$$

It is clear that $\frac{\phi''}{\phi'} = \frac{1+\alpha}{\alpha}\frac{1}{\mathcal{L}}$.

Proposition 8.1 *The Hergoltz equation obtained from (7.1)*

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v^i}\right) - \frac{\partial \mathcal{L}}{\partial q^i} + \frac{\phi''}{\phi'}\frac{dL}{dt}\frac{\partial \mathcal{L}}{\partial v^i} = 0 \quad (8.3)$$

yields the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$.

Proof : By straight forward calculation we obtain

$$\frac{\phi''}{\phi'} = \frac{1+\alpha}{\alpha}\frac{1}{\mathcal{L}}, \qquad \frac{d\mathcal{L}}{dt} = \ddot{x} + (\alpha+1)f(x)\dot{x}, \qquad \frac{\partial \mathcal{L}}{\partial x} = (\alpha+1)f(x).$$

Substituting in (8.3) we obtain the Liénard equation. □

9 Chiellini integrability condition and metriplectic structure

In this section we show that using Chiellini integrability condition the Liénard equation can be reformulated in terms of the complex Hamiltonian theory. To this end we define

$$V_x = \frac{g}{f}, \qquad V_{xx} \equiv \frac{d}{dx}(\frac{g}{f}) = \mu f. \tag{9.1}$$

Therefore in terms of the new function V a Liénard type ODE has the form therefore be written as

$$\ddot{x} + \frac{1}{\mu}V_{xx}\dot{x} + (\frac{1}{2\mu}V_x^2)_x = 0. \tag{9.2}$$

As a first-order system of ODEs it may be recast as

$$\begin{aligned} \dot{x} &= y \ , \\ \dot{y} &= -\frac{1}{\mu}V_{xx}y - (\frac{1}{2\mu}V_x^2)_x \ . \end{aligned} \tag{9.3}$$

Lemma 9.1 *The Lienard equation transforms under*

$$y = p - V_x. \tag{9.4}$$

to new set of first order ODE

$$\begin{aligned} \dot{x} &= p - V_x \ , \\ \dot{p} &= -(\frac{1}{\mu} - 1)V_{xx}p - (\frac{1}{2\mu}V_x^2)_x \ . \end{aligned} \tag{9.5}$$

Proof: It is clear that

$$\dot{p} = V_{xx}(p - V_x) - \frac{1}{\mu}V_{xx}(p - V_x) - \frac{1}{2\mu}(V_x^2)_x$$

$$= -(\frac{1}{\mu} - 1)V_{xx}p - (\frac{1}{2\mu}V_x^2)_x. \qquad \square$$

Our aim is next to rewrite the system (9.3) in a metriplectic and complex form. Let S be a real valued function on a m-dimensional manifold M. If M is compact smooth Riemannian manifold, the gradient vector field associated with the metric $g = \sum g_{ij}dx^i \otimes dx^j$ is given by

$$\mathrm{grad}(S) = G\big(\frac{\partial S}{\partial x_1}, \cdots, \frac{\partial S}{\partial x_m}\big),$$

where $G = (g_{ij})$ and $(x_1, \cdots, x_m)$ is a local coordinate.

P.J. Morrison introduced a natural geometrical formulation of dynamical systems that exhibit both conservative and nonconservative characteristics. A metriplectic system consists of a smooth manifold M , two smooth vector bundle maps $J^\sharp, G^\sharp : T^*M \to TM$ covering the identity, and two functions $H, S \in C^\infty(M)$, the Hamiltonian or total energy and the entropy of the system, such that it yields Poisson bracket and positive semidefinite symmetric bracket

$$J(df, dh) = \{f, h\}, \qquad G(df, dh) = (f, g),$$

respectively. Moreover, the additional requirements that H remains a conserved quantity and S continues to be dissipated. These requirements can be met if the following conditions on H and S are satisfied $\{S, F\} = 0$ and $(H, F) = 0$ for all $F \in C^\infty(M)$, i.e, $JdS = GdH = 0$. It shows that S is a Casimir function for the Poisson tensor J and dH is a null vector for the symmetric tensor G.

In this paper we work with a slightly weaker condition of metriplectic condition, i.e. $JdS + GdH = 0$.

Proposition 9.1 *The Liénard equation of motion take the following form*

$$\dot{X} = J\nabla H_1 - G\nabla S, \tag{9.6}$$

where $X = \begin{pmatrix} x \\ p \end{pmatrix}$. *Here* J *is the standard symplectic matrix and the second term represents gradient flow, where* G *is defined by*

$$G = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \alpha \end{pmatrix},$$

where α *is a parameter. The* H_1 *and* S *are given by*

$$H_1 = \frac{1}{2}p^2 + \frac{1}{2\mu}V_x^2 + \Big((\frac{1}{\mu} - 1)V_x - \alpha x\Big)p, \tag{9.7}$$

$$S = \frac{1}{2}p^2 + \alpha\big(\frac{1}{\mu}V - \frac{\alpha}{2}x^2\big). \tag{9.8}$$

Proof: It is easy to see that

$$H_{1x} = \Big[\frac{1}{\mu}V_x V_{xx} + \big((\frac{1}{\mu} - 1)V_{xx} - \alpha\big)p\Big], \tag{9.9}$$

$$H_{1p} = p + \big((\frac{1}{\mu} - 1)V_x - \alpha x\big), \tag{9.10}$$

$$S_x = \alpha[\frac{1}{\mu}V_x - \alpha x], \tag{9.11}$$

and $S_p = p$. Using all these expressions we obtain our result. $\square$

Corollary 9.1 *If* $\mu = 2$ *and* $p = V_x$ *then the Liénard equation satisfies weaker metriplectic condition, i.e.,* $JdS + GdH = 0$.

10 Complex Hamiltonian formulation and Liénard equation

Suppose S be the symplectic foliation of M. We denote by N the distribution defined as the g-orthogonal complement to S. Thus at every point m a decomposition into direct sum of sub-bundles, i.e. $T_mM = T_mS \oplus N_x$. If the Poisson bivector Π is parallel with respect to the Levi-Civita connection ∇, i.e. $\nabla\Pi = 0$. There is a classical result of Lichnerowicz that the distribution N is integrable. Hence together with the symplectic structure and the restriction of the metric g to the symplectic leaves defines a Kähler structure.

It is also possible to express the Hamiltonian of an equation of the Liénard type within the framework of the complex Hamiltonian theory. In this section we adopt more straight forward approach, and once again we demostrate the role of Chiellini integrability condition.

Proposition 10.1 *The equations of motion take the complex form, given by*

$$\frac{d}{dt}\begin{pmatrix} p \\ x \end{pmatrix} = \begin{pmatrix} \{H_1, p\} \\ \{H_1, x\} \end{pmatrix} + J\begin{pmatrix} \{H_2, p\} \\ \{H_2, x\} \end{pmatrix}, \tag{10.1}$$

where J is an almost complex structure defined by

$$J = \begin{pmatrix} 0 & -\alpha \\ \frac{1}{\alpha} & 0 \end{pmatrix},$$

where α is a parameter. The Hamiltonians are given by

$$H_1 = \frac{1}{2}p^2 + \frac{1}{2\mu}V_x^2 + \Big((\frac{1}{\mu} - 1)V_x - \alpha x\Big)p, \tag{10.2}$$

$$H_2 = \frac{1}{2}p^2 + \alpha\big(\frac{1}{\mu}V - \frac{\alpha}{2}x^2\big). \tag{10.3}$$

A complex structure allows one to endow a real vector space $\mathcal{V}$ with the structure of a complex vector space. In other words, given any real vector space $\mathcal{V}$ we may define its complexification by $\mathcal{V}^{\mathbb{C}} = \mathcal{V} \otimes_{\mathbb{R}} \mathbb{C}$ and J is guaranteed to have eigenvalues which satisfy $a^2 = -1$, namely $a = \pm i$. Thus we may write

$$\mathcal{V}^{\mathbb{C}} = \mathcal{V}^+ \oplus \mathcal{V}^-$$

where $\mathcal{V}^+$ and $\mathcal{V}^-$ are the eigenspaces of $+i$ and $-i$ respectively. Given such a matrix J we can define the equation of motion in terms of the complex coordinates

$$\dot{z} = \{H_{\mathbb{C}}, z\}, \tag{10.4}$$

generated by the complex Hamiltonian function $H_{\mathbb{C}} = H_1 + iH_2$, where H_1 and H_2 are as given in (10.2) and (10.3) respectively.

Lemma 10.1 *The Liénard equation can be recast as*

$$\dot{z} = \{H_1 + iH_2, z\}, \tag{10.5}$$

where $z = \frac{1}{\sqrt{2\alpha}}(\alpha x - ip)$. *The conjugate* $z^* = \frac{1}{\sqrt{2\alpha}}(\alpha x + ip)$ *and* z *satisfy*

$$\{z^*, z\} = -i. \tag{10.6}$$

Proof: Equating real and imaginary part we obtain

$$\alpha\dot{x} = \alpha\{H_1, x\} + \{H_2, p\}, \qquad -\dot{p} = \alpha\{H_2, x\} - \{H_1, p\}.$$

By normalizing these set of we obtain that these equations are equivalent to $\dot{p}$ and $\dot{x}$ equations. Thus we find our desired result. Moreover it is easy to check directly that $\{z^*, z\} = -i$. $\square$

By changing coordinates $(x, p) \to (z, z^*)$ one rewrite the Hamiltonian equation in complex form in terms of the complex Poisson bracket

$$\{K, L\} = -i\Big(\frac{\partial K}{\partial z^*}\frac{\partial L}{\partial z} - \frac{\partial K}{\partial z}\frac{\partial L}{\partial z^*}\Big). \qquad (10.7)$$

Proposition 10.2 *Suppose the Chiellini integrability condition is satisfied for the Liénard equation of motion. Then Liénard equation can be expressed in complex Hamiltonian form*

$$\dot{z} = \{H_{\mathbb{C}}, z\} = -i\frac{\partial H_{\mathbb{C}}}{\partial z^*}, \qquad (10.8)$$

with complex coordinates and complex Hamiltonian function $H_{\mathbb{C}} = H_1 + iS$*, where* H_1 *and* S *are as given in (10.2) and (10.3) respectively.*

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