Moduli spaces for jets of linear connections

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1. Introduction

A classical approach to local classifications of geometric structures consists on considering "normal forms", that is, finding suitable coordinate charts where the expression of geometric objects is specially simple.

In this work we adopt another approach: we will study the structure of the quotient $J^r_x\mathcal{C}/\mathrm{Diff}_x$, where $J^r_x\mathcal{C}$ denotes the smooth manifold of $r-$jets of linear connections at a point and Diff_x denotes the group of germs of diffeomorphisms leaving x fixed.

By using the *normal tensors* associated to linear connections (see [1], [4] or [5] for several examples where they have been used to study different classification problems), we give (Theorem 2) an isomorphism of ringed spaces between the quotient $J^r_x\mathcal{C}/\mathrm{Diff}_x$ and the orbit space of a linear representation of the general linear group Gl_n. This result is obtained as a corollary of a previous orbit-reduction-type statement (Theorem 1).

Finally (Theorem 3), we apply the aforementioned results to prove the absence of non-trivial (scalar) differential invariants associated to linear connections.

2. Moduli spaces of jets of linear connections

Let $\mathcal{C}\to X$ be the bundle of linear connections over X, and let us denote by $J^r\mathcal{C}\to X$ the fiber bundle of $r-$jets of linear connections on X.

For a certain point $x_0\in X$, let Diff_{x_0} be the group of local diffeomorphisms of X leaving x_0 fixed, and let $\mathrm{Diff}^r_{x_0}$ be the Lie group of $r-$jets at x_0 of those local diffeomorphisms.

The group Diff_{x_0} acts on $J^r_{x_0}\mathcal{C}$ like this: for $\tau\in\mathrm{Diff}_{x_0}$ and $j^r_{x_0}\nabla\in J^r_{x_0}\mathcal{C}$, $\tau\cdot(j^r_{x_0}\nabla)$ is the $r-$ jet at x_0 of the linear connection $\tau\cdot\nabla$, defined as:

$$(\tau\cdot\nabla)_D\bar{D} := \tau_*^{-1}\left(\nabla_{\tau_*D}(\tau_*\bar{D})\right) .$$

If we denote by $H^r_{x_0}$ the subgroup of Diff_{x_0} made up of those diffeomorphisms whose $r-$jet at x_0 coincides with that of the identity, then the action of Diff_{x_0} factors through an action of $\mathrm{Diff}^{r+2}_{x_0}$, since $H^{r+2}_{x_0}$ acts trivially on $J^r_{x_0}\mathcal{C}$.

We will say that two $r-$jets $j^r_{x_0}\nabla$, $j^r_{x_0}\bar{\nabla}\in J^r_{x_0}\mathcal{C}$ are **equivalent** if there exists a local diffeomorphism $\tau\in\mathrm{Diff}_{x_0}$ such that $j^r_{x_0}\bar{\nabla}=j^r_{x_0}(\tau^*\nabla)$.

We call **moduli space** of $r-$jets of linear connections the quotient (ringed) space

$$\mathfrak{C}^r_n := J^r_{x_0}\mathcal{C}/\mathrm{Diff}_{x_0} = J^r_{x_0}\mathcal{C}/\mathrm{Diff}^{r+2}_{x_0} .$$

(Trivial) example. *If $n=1$, any linear connection is locally isomorphic to the standard, flat connection on $\mathbb{R}$. That is why all moduli spaces $\mathfrak{C}^r_1$ reduce to a single point, for any $r\in\mathbb{N}\cup\{0\}$.*

3. Description via normal tensors

Definition. *Let $m\geq 0$ be a fixed integer and let $x\in X$. The space C_m of* **normal tensors** *of order m at x is the vector space of $(1,m+2)-$tensors T at x having the following symmetries:*

- *symmetry in the last m covariant indices:*

$$T^l_{ijk_1\ldots k_m}=T^l_{ijk_{\sigma(1)}\ldots k_{\sigma(m)}} \quad , \quad \forall\sigma\in S_m ;$$

- *the symmetrization over the $m+2$ covariant indices is zero:*

$$\sum_{\sigma\in S_{m+2}} T^l_{\sigma(i)\sigma(j)\sigma(k_1)\ldots\sigma(k_m)}=0 .$$

Every germ of linear connection ∇ around x produces a sequence of normal tensors Γ^m at x in the following way:

Let $\overline{\nabla}$ be the germ of linear connection around x that corresponds, via the exponential map $\exp_\nabla:\mathrm{T}_xX\to X$, to the canonical flat connection of T_xX.

Let us also consider the difference tensor between ∇ and $\overline{\nabla}$:

$$\mathbb{T}(\omega,D_1,D_2):=\omega(\nabla_{D_1}D_2-\overline{\nabla}_{D_1}D_2) .$$

If $(x_1,\ldots,x_n)$ is a normal system of coordinates for ∇ around x, and Γ^k_{ij} are the Christoffel symbols of ∇ in those coordinates, then:

$$\mathbb{T}:=\sum_{i,j,k}\Gamma^k_{ij}\frac{\partial}{\partial x_k}\otimes \mathrm{d}x^i\otimes\mathrm{d}x^j .$$

Proposition. *For each $m\geq 0$, the $m-$th* **normal tensor** *of the connection ∇ at the point x is:*

$$\Gamma^m_x:=\overline{\nabla}^m_x\mathbb{T} .$$

If $(x_1,\ldots,x_n)$ is a normal chart for ∇ around x, then:

$$\Gamma^m_x=\sum_{i,j,k,a_1,\ldots,a_m}\Gamma^k_{ij;a_1\ldots a_m}(0)\frac{\partial}{\partial x_k}\otimes\mathrm{d}x^i\otimes\mathrm{d}x^j\otimes\mathrm{d}x^{a_1}\otimes\ldots\otimes\mathrm{d}x^{a_m} ,$$

where

$$\Gamma^k_{ij;a_1\ldots a_m}:=\frac{\partial\Gamma^k_{ij}}{\partial x_{a_1}\ldots\partial x_{a_m}} .$$

The tensor Γ^m_x belongs to C_m.

Reduction theorem

If ∇ is a germ of linear connection around x, let us denote $(\Gamma^0_x,\ldots,\Gamma^m_x,\ldots)$ the sequence of its normal tensors at the point x.

Theorem 1. *For each $r\in\mathbb{N}\cup\{0\}$, the map*

$$\begin{aligned} J^r_x\mathcal{C} &\xrightarrow{\pi_r} C_0\times\ldots\times C_r\\ j^r_x\nabla &\longmapsto (\Gamma^0_x,\ldots,\Gamma^r_x)\end{aligned}$$

is a surjective regular submersion, whose fibers are the orbits of H^1_x. Therefore, π_r induces an isomorphism of smooth manifolds:

$$(J^r_x\mathcal{C})\,/\,H^1_x \;=\; C_0\times\ldots\times C_r .$$

As a corollary of Theorem 1, we also get:

Theorem 2. *The moduli space of jets of linear connections is isomorphic, as a ringed space, to the orbit space of a linear representation of Gl_n:*

$$\mathfrak{C}^r_n\simeq(C_0\times\ldots\times C_r)/\mathrm{Gl}_n .$$

Non-existence of differential invariants

We define a (scalar) **differential invariant** of order $\leq r$ of linear connections to be a global differentiable function on some $\mathfrak{C}^r_n$.

Theorem 2, in combination with invariant theory for the general linear group ([2]) and a result by Luna ([3]), allows us to make this assertion:

Theorem 3. *The only differential invariants associated to linear connections are constant functions.*

[1] Epstein, D.B.A.: *Natural tensors on Riemannian manifolds*, J. Diff. Geom. **10** (1975) 631–645.
[2] Kolář, I., Michor, P.W. & Slovák, J.: *Natural operations in differential geometry*, Springer-Verlag, Berlin (1993)
[3] Luna, D. *Fonctions différentiables invariantes sous l'opération d'un groupe réductif*, Ann. Inst. Fourier, **26** 1 (1976) 33–49.
[4] Stredder, P.: *Natural differential operators on riemannian manifolds and representations of the orthogonal and special orthogonal groups*, J. Diff. Geometry **10**, (1975) 647–660.
[5] Thomas, T. Y.: *The differential invariants of generalized spaces, Chelsea Publishing Company*, New York (1991). (First edition: Cambridge University Press, 1934.)