REMARKS ON DIRAC-KÄHLER OPERATOR

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Introduction

The purpose of this work is to investigate the role of topology in quantum theory. In particular our analysis is centered on Dirac-Kähler operator, which acts on differential forms, defining spinor fields in this formulation. In this work we will exploit the tensorial behaviour of differential forms in order to write Dirac-Kähler operator on spaces with a non trivial topology. In particular we will focus on Homogeneous Spaces. Differently from previous works [1], we are not interested in applications to general relativity. Our aim is to introduce unfolding and reduction methods [2] in Kahler scheme. We hope that this analysis would provide hints for treating manifolds with boundaries, which can be seen as quotients of manifolds with respect to the action of discrete groups.

Projectors

In this approach to Dirac equation, spinor fields are differential forms. The $\vee$-product provides representations of Clifford algebras, which are not irreducible. However it is possible to decompose the action of (7) by introducing a family of minimal orthonormal operators projectors $\{P_j\}$, providing a decomposition of the identity.
The ranges of these projectors are vector spaces on which $\vee$-product provides an irreducible representation of the corresponding Clifford algebra. If the projectors also stisfy the compatibility condition [4]

$$P_j \vee \nabla P_j = 0$$

Dirac-Kähler operator can be decomposed on the same subspaces resulting in a matrix-valued differential operator.

Kähler Formulation

In 1962 Kähler [3] showed that, given a manifold M with a metric tensor g, the exterior algebra $\Lambda(M)$ can be equipped with a new product, called $\vee$-product. It provides a representation of the Clifford Algebra generated by the tangent space T_xM (tangent space to M at x) with the metric $g(x)$. The expression of the $\vee$-product among two differential forms ϕ and ω is given by the formula:

$$\phi \vee \omega = \sum_s \frac{(-1)^{\binom{s}{2}}}{s!} g^{a_1 b_1} \cdots g^{a_s b_s} (\gamma^s \{i_{e_{a_1}} \cdots i_{e_{a_s}} \phi\}) \wedge \{i_{e_{b_1}} \cdots i_{e_{b_s}} \omega\} \tag{5}$$

where i_a is the interior product with respect to the vector field ∂_a and $\gamma(\phi) = (-1)^k \phi$ if $\phi \in \Lambda^k(M)$. Kähler also proposed a Dirac-like operator, suitable for this description, locally written as:

$$\mathcal{D}_K = dx^a \vee \nabla_{\partial_a} \tag{6}$$

where ∇_{∂_a} is the covariant derivative associated with the Levi-Civita connection of the manifold (M, g). It is possible to show that this operator is equal to

$$\mathcal{D}_K = d + \delta \tag{7}$$

where d is the exterior derivative and δ is the codifferential. Analogously to Dirac operator in spin geometry this is a square-root of Laplace-De Rham operator $\Delta = d\delta + \delta d$, but we do not need to use matrix-valued differential operators.

Dirac-Kähler operator on S^2

Let us show the construction by means of an example, the sphere S^2. This manifold can be considered as the quotient $SU(2)/U(1)$. Since T^*S^2 is not parallelizzable, we can consider the exterior algebra $\Lambda(S^2)$ as a subalgebra of the exterior algebra $\Lambda(S^3)$ ($S^3 \simeq SU(2)$). Indeed differential forms on S^3 which are pull-back of differential forms on S^2 satisfy the following condition:

$$i_{X_3}\alpha = 0$$
$$L_{X_3}\alpha = 0$$

Therefore differential forms $\alpha \in \Lambda(S^2)$ can be written in terms of the $SU(2)$-left invariant differential forms θ^j as $\alpha = f + \alpha_+\theta^+ + \alpha_-\theta^- + i\omega\theta^- \wedge \theta^+$ where $\theta^{\pm} = \theta^1 \mp i\theta^2$ and the coefficients satisfy the eigenvalue equation

$$L_{X_3} f = L_{X_3}\omega = 0 \tag{1}$$
$$L_{X_3}\alpha_{\pm} = \mp i\alpha_{\pm} \tag{2}$$

A Dirac-Kahler operator on S^3 which can be reduced to this subalgebra, can be defined by introducing a degenerate symmetric tensor

$$g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2$$

with a suitable Hodge dual operator on $\Lambda(S^2)$. This operator $*_{S^2} : \Lambda(S^2) \to \Lambda(S^2)$ acts according to the following rules

$$*_{S^2}(f) = if\theta^- \wedge \theta^+$$
$$*_{S^2}(\alpha_-\theta^-) = -i\alpha - \theta^-$$
$$*_{S^2}(\alpha_+\theta^+) = i\alpha_+\theta^+$$
$$*_{S^2}(i\omega\theta^- \wedge \theta^+) = \omega$$

It is possible to define a hermitean product in $\Lambda(S^2)$ as follows

$$(\alpha, \beta) = \int_{S^3} \theta^3 \wedge \bar{\alpha} \wedge *_{S^2}\beta$$

The operator $(-1)^{n(k-1)-1} *_{S^2} d *_{S^2}$ is the adjoint of the exterior derivative with respect to this hermitean product. Therefore one can define as Dirac-Kähler operator on the sphere S^2 the operator

$$D_K = d + *_{S^2} d *_{S^2} \tag{3}$$

which acts on the space $\Lambda(S^2)$. Its action can be reduced to two 4-dimensional modules by means of the projectors

$$P_{\pm} = \frac{1}{2}\left(1 \pm \theta^- \wedge \theta^+\right)$$

On each of these modules we can express (3) as a four-dimensional matrix-valued differential operator. In this case we get the following expression

$$D_K = \begin{pmatrix} 0 & 2(\phi_1 L_+ + (L_+\bar{\phi}_1)) & 2(\phi_2 L_+ + (L_+\bar{\phi}_2)) & 2(\phi_3 L_+ + (L_+\bar{\phi}_3)) \\ \phi_1 L_- & 0 & 0 & 0 \\ \phi_2 L_- & 0 & 0 & 0 \\ \phi_3 L_- & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

where the symbol $L_{\pm}$ denotes the Lie derivative with respect to the vector field $X_1 \pm iX_2$ and $\{\phi_j\}$ are a basis of functions satisfying (2).

Conclusions and Further Research

In this work we have shown that tensorial character of Kahler approach to spinors allows to consider dynamical systems formulated as Dirac-like equations on carrier spaces topologically non trivial, such as the Hopf bundle in the example. A key ingredient, however, has been the use of a degenerate quadratic form in order to reduce Dirac-Kahler operator to the subalgebra in analysis. Our future efforts will be directed towards manifolds with boundaries, treated as orbifolds, that are quotient of manifolds with respect to the action of groups, which can be discrete.

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