Completeness of certain inextensible accelerated observers in General Relativity

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Abstract

We introduce the notions of uniformly accelerated, unchanged direction and circular motion in General Relativity in the realm of the Lorentzian Geometry. We analyse the completeness of the inextensible trajectories of observers which obey one of these motions, when the ambient spacetime has certain symmetries.

1. Introduction

In a non-Relativistic setting, a particle may be detected as accelerated by using an *accelerometer*. An accelerometer may be intuitively thought as a sphere in whose center there is a small round object which is supported on elastic radii of the sphere surface. If a free falling observer carries such a accelerometer, then it will notice that the small round object remains just at the center. Whereas it will be displaced if the observer obeys an accelerate motion. This argument suggests that a uniformly accelerated motion may be detected from a constant displacement of the small round object. Analogously,if the ball moves along a radius, the observer may though that its motion obeys a rectilinear trajectory. At the same way, it would be natural that if an observer checks that the small ball describes a plane uniform rotation, then it believes that it obeys a uniform circular motion. This idea has the advantage that may be used independently if the spacetime where the observer lies is relativistic or not. Now, our aim is to provide rigour to some assertions like "the accelerometer marks a constant value" or "the proper acceleration does not change its direction".

2. The Fermi-Walker connection

Let $(M, \langle\,,\,\rangle)$ be an $n(\geq 2)-\text{dimensional}$ spacetime, endowed with a fixed time orientation. We consider an observer in M as a (smooth) timelike unit curve $\gamma : I \longrightarrow M$, I an open interval of $\mathbb{R}$, such that $\gamma\,'(t)$ is future pointing for any proper time t of γ [4]. At each event $\gamma(t)$ the tangent space $T_{\gamma(t)}M$ splits as

$$T_{\gamma(t)}M = T_t \oplus R_t,$$

where $T_t = \mathrm{Span}\{\gamma\,'(t)\}$ and $R_t = T_t^{\perp}$. Endowed with the restriction of $\langle\,,\,\rangle$, R_t is a spacelike hyperplane of $T_{\gamma(t)}M$. It is interpreted as the instantaneous physical space observed by γ at t. Clearly, the observer γ is able to compare spatial directions at t. In order to compare $v_1 \in R_{t_1}$ with $v_2 \in R_{t_2}$, $t_1 < t_2$ and $|v_1| = |v_2|$, the observer γ could use, as a first attempt, the parallel transport along γ defined by the Levi-Civita covariant derivative along γ,

$$P^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_2)}M.$$

Unfortunately, this linear isometry satisfies $P^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}$ if γ is free falling (i.e., γ is a geodesic) but this property does not remain true for any general observer.

Now, for each $Y \in \mathfrak{X}(\gamma)$ put Y_t^T, Y_t^R the orthogonal projections of Y_t on T_t and R_t, respectively, i.e., $Y_t^T = -\langle Y_t, \gamma\,'(t)\rangle\,\gamma\,'(t)$ and $Y_t^R = Y_t - Y_t^T$. In this way, we define $Y^T, Y^R \in \mathfrak{X}(\gamma)$. We have [5],

Proposition 1 *There exists a unique connection $\widehat{\nabla}$ along γ such that*

$$\widehat{\nabla}_X Y = \left(\nabla_X Y^T\right)^T + \left(\nabla_X Y^R\right)^R,$$

for any $X \in \mathfrak{X}(I)$ and $Y \in \mathfrak{X}(\gamma)$.

This connection $\widehat{\nabla}$ is called the *Fermi-Walker connection of* γ. It shows the suggestive property that if $Y \in \mathfrak{X}(\gamma)$ satisfies $Y = Y^R$ (i.e., Y_t may be observed by γ at any t) then $\left(\widehat{\nabla}_X Y\right)_t \in R_t$ for any t, where ∇ is the usual Levi-Civita connection.

Associated to the Fermi-Walker connection on γ, there exist a parallel transport

$$\widehat{P}^{\gamma}_{t_1,t_2} : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_1)}M,$$

which is a lineal isometry and satisfies $\widehat{P}^{\gamma}_{t_1,t_2}(R_{t_1}) = R_{t_2}$. Therefore, given $v_1 \in R_{t_1}$ and $v_2 \in R_{t_2}$, with $t_1 < t_2$ and $|v_1| = |v_2|$, the observer γ may consider $\widehat{P}^{\gamma}_{t_1,t_2}(v_1)$ instead v_1, with the advantage to wonder if $\widehat{P}^{\gamma}_{t_1,t_2}(v_1)$ is equal to v_2 or not.

The acceleration $\dfrac{D\gamma\,'}{dt}$ satisfies $\dfrac{D\gamma\,'}{dt}(t) \in R_t$, for any t. Therefore, it may be observed by γ whereas the velocity $\gamma\,'$ is not observable by γ.

3. Uniformly accelerated motion

Definition 1 *An observer $\gamma : I \longrightarrow M$ is said to obey a **uniformly accelerated (UA) motion** if*

$$\widehat{P}^{\gamma}_{t_1,t_2}\left(\frac{D\gamma\,'}{dt}(t_1)\right) = \frac{D\gamma\,'}{dt}(t_2),$$

for any $t_1, t_2 \in I$ with $t_1 < t_2$. Equivalently, if the equation

$$\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right) = 0,$$

holds everywhere on I, i.e., $\dfrac{D\gamma\,'}{dt}$ is Fermi-Walker parallel along γ [1].

Clearly, if γ is free falling, then it is a UA observer.

The UA observers may be characterized as follows.

Proposition 2 *For any observer $\gamma : I \longrightarrow M$, the following assertions are equivalent:*

(a) *γ is a UA observer.*

(b) *γ is a solution of third-order differential equation*

$$\frac{D^2\gamma\,'}{dt^2} = \left\langle \frac{D\gamma\,'}{dt}, \frac{D\gamma\,'}{dt} \right\rangle \gamma\,'.$$

(c) *γ is a Lorentzian circle or it is free falling.*

(d) *γ has constant curvature and the remaining curvatures equal to zero.*

(e) *γ, viewed as an isometric immersion from $(I, -dt^2)$ to M, is totally umbilical with parallel mean curvature vector.*

4. Unchanged direction motion

Definition 2 *An observer $\gamma : I \longrightarrow M$ is said to obey an **unchanged direction (UD) motion** if*

$$\widehat{P}^{\gamma}_{t_1,t_2}\left(\frac{D\gamma\,'}{dt}(t_1)\right) = \lambda(t_1,t_2)\,\frac{D\gamma\,'}{dt}(t_2),$$

for a certain proportional factor λ and for any $t_1, t_2 \in I$ with $t_1 < t_2$ [2].

Clearly, if an observer γ is a free falling then it obeys a UD motion. More generally, a UA observer satisfies (1) with $\lambda = 0$. Thus, it obeys a UD motion. Of course, the family of UD observers is much bigger than the one of the UA observers.

Definition 3 *An observer $\gamma : I \longrightarrow M$ is said to obey a **piecewise unchanged direction motion** if γ satisfies the equation*

$$\left|\frac{D\gamma\,'}{dt}\right|^2 \frac{D^2\gamma\,'}{dt^2} = \frac{1}{2}\frac{d}{dt}\left|\frac{D\gamma\,'}{dt}\right|^2 \frac{D\gamma\,'}{dt} + \left|\frac{D\gamma\,'}{dt}\right|^4 \gamma\,'.$$

Some geometric characterizations of this kind of observers are shown in the following proposition.

Proposition 3 *For any observer $\gamma : I \longrightarrow M$ with nowhere zero acceleration the following assertions are equivalent:*

(a) *γ is a piecewise UD observer.*

(b) *The development of γ is a piecewise planar curve in the tangent space of every point.*

(c) *γ has all the curvatures equal to zero except (possibly) the first one.*

(d) *γ, viewed as an isometric immersion from $(I, -dt^2)$ to M, is (totally umbilical) with parallel normalized mean curvature vector whenever it is defined.*

The following result shows a sort of first integral, which will be useful tool in order to study the completeness of the inextensible trajectories.

Theorem 4 *Let $a : I \longrightarrow \mathbb{R}$ be a smooth function and $v, w \in T_pM$ such that $|v|^2 = -1$, $|w|^2 = 1$ and $\langle v, w\rangle = 0$. The 4-velocity of the unique UD observer γ satisfying the initial conditions*

$$\gamma(0) = p, \quad \gamma\,'(0) = v, \quad \frac{D\gamma\,'}{dt}(0) = a(0)\,w,$$

is given by

$$\gamma\,'(t) = \cosh\big(V(t)\big)\,L(t) + \sinh\big(V(t)\big)\,M(t),$$

where

$$V(t) = \int_0^t a(s)ds,$$

and L, M are two Levi-Civita parallel vector fields along γ with $L(0) = v$ and $M(0) = a(0)\,w$.

5. Uniformly circular motion

First we expose the notion of *planar motion*, to make precise when an observer intuitively considers that it is moving along a plane.

Definition 4 *An observer $\gamma : I \longrightarrow M$ obeys a **planar motion** if for some $t_0 \in I$, there exists an observable plane $\Pi_{t_0} \subset \gamma'(t_0)^{\perp} \subset T_{\gamma(t_0)}M$, such that*

$$\widehat{P}^{\gamma}_{t,t_0}\left(\frac{D\gamma\,'}{dt}(t)\right) \in \Pi_{t_0}$$

for any $t \in I$ [3].

A uniform circular motion is a particular case of planar motion.

Definition 5 *An observer $\gamma : I \longrightarrow M$, following a planar motion, is said to obey an **uniform circular (UC) motion** if*

$$\left|\frac{D\gamma\,'}{dt}\right|^2 = a^2 \qquad \text{and} \qquad \left|\frac{\widehat{D}}{dt}\left(\frac{D\gamma\,'}{dt}\right)\right|^2 = a^2w^2,$$

where $a, w > 0$ and $a < w$ [3].

Here, a is de modulus of the acceleration, and w corresponds with the angular velocity which the observer perceives.

The UC observers may be characterized as follows.

Proposition 5 *The three following assertions are equivalent*

(a) *A curve γ in M is a uniformly circular observer with centripetal acceleration a and frequency w.*

(b) *A curve γ in M is a Lorentzian helix with curvature a and torsion w.*

(c) *A curve γ in M is a solution of*

$$\frac{D}{dt}\left[\frac{D^2\gamma'}{dt^2} + \langle\gamma', \frac{D\gamma'}{dt}\rangle\frac{D\gamma'}{dt} + \left(w^2 - \left|\frac{D\gamma'}{dt}\right|^2\right)\gamma'\right] = 0.$$

An analogous result of Theorem 4 is also obtained for UC observers.

6. Completeness of the inextensible trajectories

The main completeness result, valid for UA, UD and UC observers is summarized in the following theorem.

Theorem 6 *Let M be a spacetime which admits a timelike conformal and closed vector field K. If $\inf_M \sqrt{-\langle K, K\rangle} > 0$, then, each UA, UD or UC observer $\gamma : I \longrightarrow M$ such that $\gamma(I)$ lies in a compact subset of M can be extended.*

Corolary 7 *In a compact spacetime admitting a timelike conformal and closed vector field, every UA, UD and UC observer is complete.*

By means of more analytical tools, it is also proved the completeness of these accelerated observers in certain physically relevant pp-wave spacetimes.

A (four dimensional) **Plane Wave** is a spacetime $(\mathbb{R}^4, g)$ which admits a coordinate system (u, v, x, y) such that the Lorentzian metric may be expressed as follows,

$$g = H(u, x, y)\,du^2 + 2dudv + dx^2 + dy^2,$$

where $H(u, x, y)$ is a quadratic function in the coordinates x and y with coefficients depending on u, that is,

$$H(u, x, y) = A(u)x^2 + B(u)y^2 + C(u)xy + D(u)x + E(u)y + F(u).$$

The coordinates are known as a Brinkmann coordinate system of $(\mathbb{R}^4, g)$.

Theorem 8 *Every UA, UD or UC inextensible trajectory in a **Plane Wave spacetime** admitting a global Brinkmann chart is complete.*

References

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