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On a problem of Singer about plane curves: curvature and distance from a point

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Motivation: Singer's problem [4]

Can a plane curve be determined if its curvature is given in terms of its position?

$$\kappa = \kappa(x,y), \qquad \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{3/2}} = \kappa(x(t), y(t))$$

- Euler elastic curves [5]: $\kappa(x,y) = \mu\, y$, $\mu \neq 0$.
- Plane curves with curvature depending on distance to a line [1]: $\kappa = \kappa(y)$.
- Bernoulli lemniscate [4]: $\kappa(x,y) = \sqrt{x^2+y^2} \Leftrightarrow \kappa(r) = r$.

Main Result: [2]

Prescribe $\kappa = \kappa(r)$ such that $r\kappa(r)$ continuous. The problem of determining a curve $\alpha(s) = r(s)\, e^{i\theta(s)}$ (s arc length) with curvature $\kappa(r)$ is solvable by three quadratures:

$$\int r\kappa(r)dr = G(r)+c,\ c \text{ constant};\ s = s(r) = \int \frac{r\,dr}{\sqrt{r^2-(G(r)+c)^2}} \dashrightarrow r = r(s);\ \theta(s) = \int \frac{G(r(s))+c}{r(s)^2}ds$$

- Election of constant c: $\int r\kappa(r)dr \ni \mathcal{K}(r)$ *radial primitive curvature*
- Such α is uniquely determined, up to rotations, by $\mathcal{K}(r)$

New family I: $\kappa(r) = 2\lambda + \mu/r$ ($\lambda = 1, \mu \neq 0$)

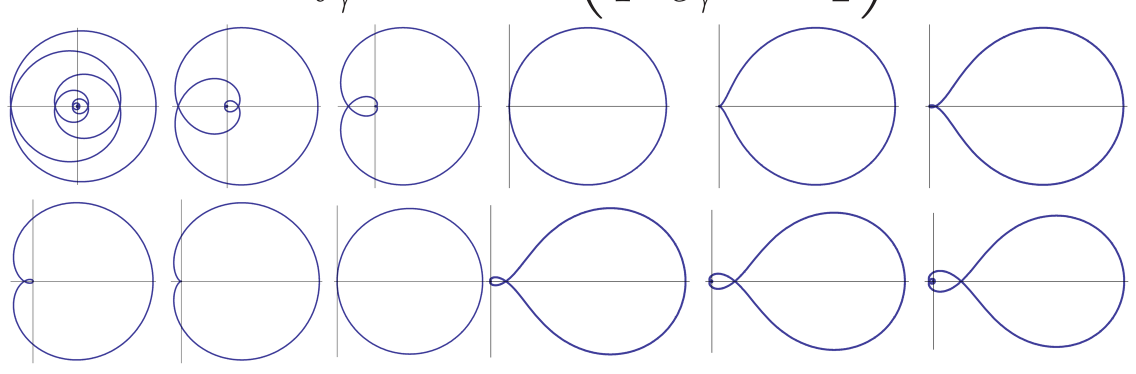
$\mathcal{K}(r) = r^2 + \mu r,\ \mu < 1 \qquad r(s) = \cos s - \mu,\ \theta(s) = s + \mu \int \frac{ds}{\cos s - \mu} \qquad \kappa(s) = \frac{2\cos s - \mu}{\cos s - \mu}$

- $\mu = \cos\gamma$, $0 < \gamma < \pi$

$(s_\gamma \equiv \sin\gamma,\ c_\gamma \equiv \cos\gamma,\ t_\gamma \equiv \tan\gamma)$

$r_\gamma(s) = \cos s - c_\gamma$

$\theta_\gamma(s) = s + \frac{2}{t_\gamma}\,\mathrm{arctanh}\left(\frac{s_\gamma}{1-c_\gamma}\tan\frac{s}{2}\right)$

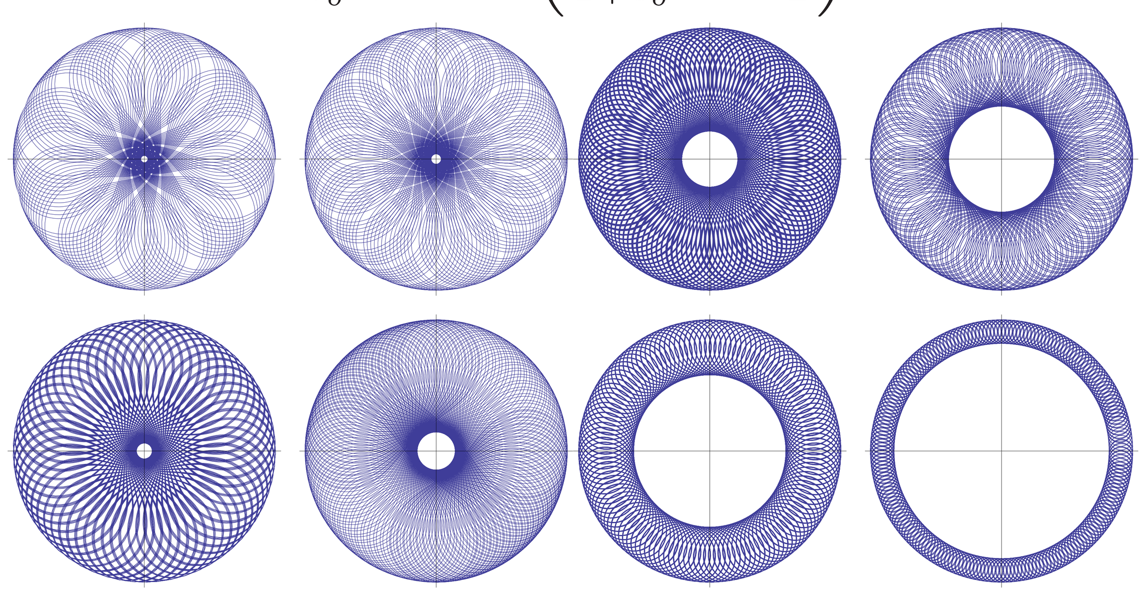


- $\mu = -\cosh\delta$, $\delta > 0$

$(s_\delta \equiv \sinh\delta,\ c_\delta \equiv \cosh\delta,\ t_\delta \equiv \tanh\delta)$

$r_\delta(s) = \cos s + c_\delta$

$\theta_\delta(s) = s - \frac{2}{t_\delta}\arctan\left(\frac{s_\delta}{1+c_\delta}\tan\frac{s}{2}\right)$



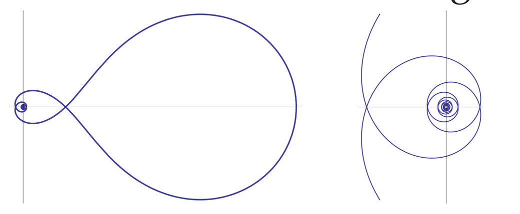
- $\mu = -1$

Inverse Norwich spiral

$r(s) = \cos s + 1$

$\theta(s) = s - \tan\frac{s}{2}$

The curve; detail near origin:



New family II: $\kappa(r) = \lambda/\sqrt{r^2+\mu}$ ($\lambda > 0, \mu \pm 1$)

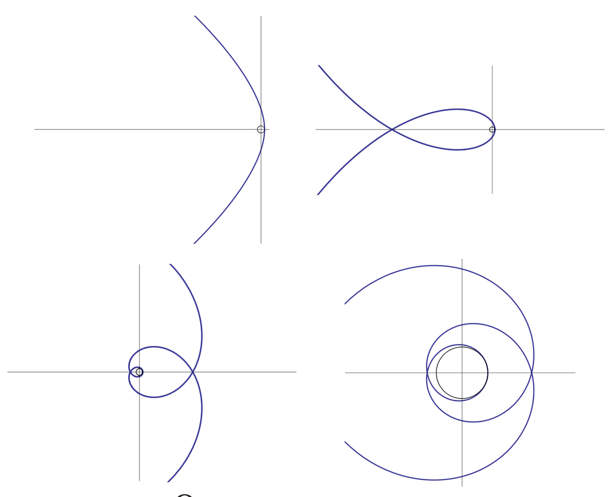
■ $\kappa(r) = \lambda/\sqrt{r^2+1}$, $\lambda < 1$: $\lambda = \sin\beta$, $\beta \in (0,\pi/2)$

$(s_\beta \equiv \sin\beta,\ c_\beta \equiv \cos\beta,\ t_\beta \equiv \tan\beta)$

$\mathcal{K}(r) = \sin\beta\sqrt{r^2+1}$

$r_\beta(t)^2 = \frac{\cosh^2(c_\beta t)}{c_\beta^2} - 1 \qquad \theta_\beta(t) = s_\beta t + \arctan\left(\frac{\tanh(c_\beta t)}{t_\beta}\right)$

$ds = \sqrt{r_\beta^2+1}dt,\ t = \sinh(c_\beta t)/c_\beta^2 \quad \kappa_\beta(s) = \frac{\sin\beta\cos\beta}{\sqrt{1+\cos^4\beta\, s^2}},\ \beta \in (0,\pi/2)$



■ $\kappa(r) = \lambda/\sqrt{r^2-1}$

- $\lambda = 1$

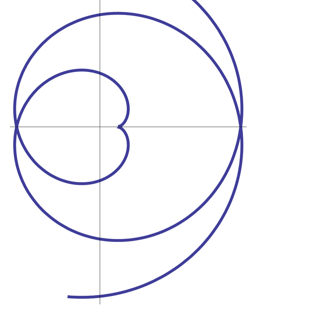
$\mathcal{K}(r) = \sqrt{r^2-1}$

Anti-clothoid

$r(s) = \sqrt{1+2s}$

$\theta(s) = \sqrt{2s} - \arctan\sqrt{2s}$

$\kappa(s) = \frac{1}{\sqrt{2s}},\ s > 0$



- $0 < \lambda < 1$: $\lambda = \sin\alpha$, $\alpha \in (0,\pi/2)$

$(s_\alpha \equiv \sin\alpha,\ c_\alpha \equiv \cos\alpha,\ t_\alpha \equiv \tan\alpha)$

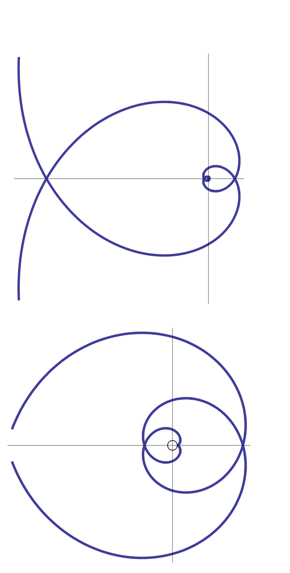
$\mathcal{K}(r) = \sin\alpha\sqrt{r^2-1}$

$r_\alpha(t)^2 = \frac{\sinh^2(c_\alpha t)}{c_\alpha^2} + 1$

$\theta_\alpha(t) = s_\alpha t - \arctan(t_\alpha\tanh(c_\alpha t))$

$ds = \sqrt{r_\alpha^2-1}dt,\ t = \cosh(c_\alpha t)/c_\alpha^2$

$\kappa_\alpha(s) = \frac{\sin\alpha\cos\alpha}{\sqrt{\cos^4\alpha\, s^2 - 1}}$



- $\lambda > 1$: $\lambda = \cosh\tau$, $\tau > 0$

$(s_\tau \equiv \sinh\tau,\ c_\tau \equiv \cosh\tau,\ t_\tau \equiv \tanh\tau)$

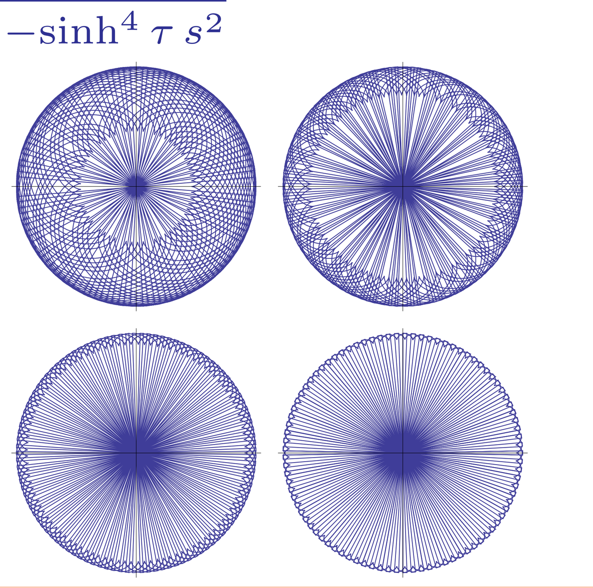
$\mathcal{K}(r) = \cosh\tau\sqrt{r^2-1}$

$r_\tau(t)^2 = \frac{\sin^2(s_\tau t)}{s_\tau^2} + 1$

$\theta_\tau(t) = c_\tau t - \arctan\left(\frac{\tan(s_\tau t)}{t_\tau}\right)$

$ds = \sqrt{r_\tau^2-1}dt,\ t = -\cos(s_\tau t)/s_\tau^2$

$\kappa_\tau(s) = \frac{\sinh\tau\cosh\tau}{\sqrt{1-\sinh^4\tau\, s^2}}$



$\kappa(r) = 1/r$: Sturm spiral [3]

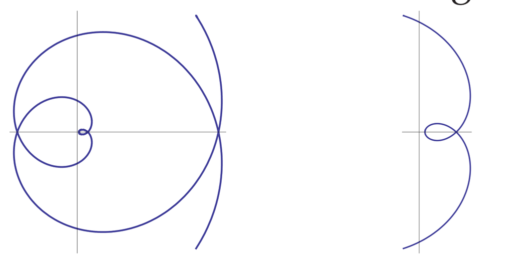
$r(t) = \rho^2(t^2+1)$

$\rho > 0$

$\theta(t) = t - 2\arctan t$

$(ds = r\,dt)$

The curve; detail near origin:



Example: Circles

$\kappa(r) \equiv \kappa_0 > 0$

$\int \kappa_0 r\,dr = \kappa_0 r^2/2 + c$

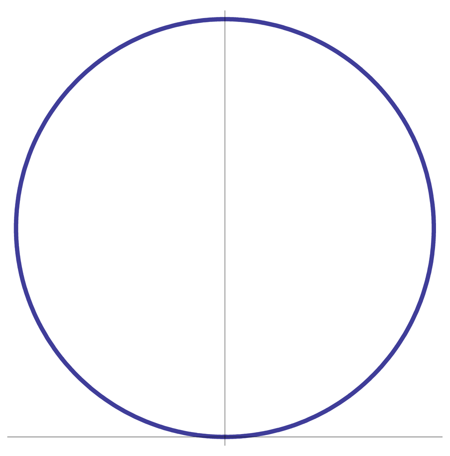
$s = \int \frac{r\,dr}{\sqrt{r^2-(\kappa_0 r^2/2+c)^2}}$

▷ $c = 0$:

$\mathcal{K}(r) = \kappa_0 r^2/2$

$s = (2/\kappa_0)\arcsin(\kappa_0 r/2)$

$r(s) = (2/\kappa_0)\sin(\kappa_0 s/2),\ \theta(s) = \kappa_0 s/2$



$\kappa(r) = \lambda\, r^{n-1}$ ($\lambda > 0, n \neq -1, 0$)

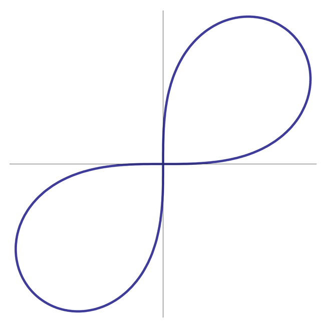
$\mathcal{K}(r) = \frac{\lambda}{n+1}r^{n+1} \qquad d\theta = \frac{\frac{\lambda}{n+1}r^{n-1}}{\sqrt{1-\left(\frac{\lambda}{n+1}\right)^2 r^{2n}}}dr$

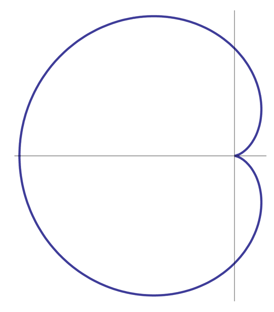
Sinusoidal spirals $r^n = \frac{n+1}{\lambda}\sin(n\theta)$

- $n = 2$:

Bernoulli lemniscate

$r^2 = \frac{3}{\lambda}\sin 2\theta$





- $n = 1/2$:

Cardioid

$r = \frac{9}{4\lambda^2}\sin^2\frac{\theta}{2}$

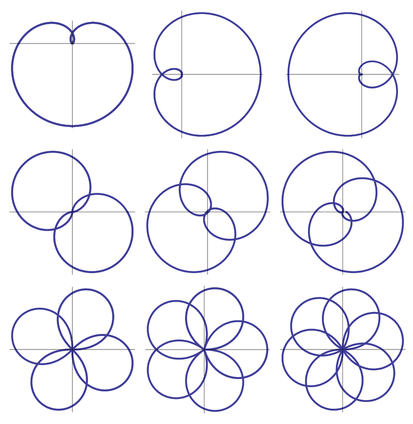
- $n \in \mathbb{Q}$:

Algebraic curves

$n = 1/3, 1/4, 1/6$

$n = 2/3, 2/5, 2/7$

$n = 4/3, 5/4, 6/5$



$\kappa(r) = \lambda/r^3 + 3\mu r$ ($\lambda \in \mathbb{R}, \mu > 0$)

$\mathcal{K}(r) = \mu r^3 - \lambda/r \quad d\theta = \frac{\mu r^4 - \lambda}{r\sqrt{r^4-(\mu r^4-\lambda)^2}}dr$

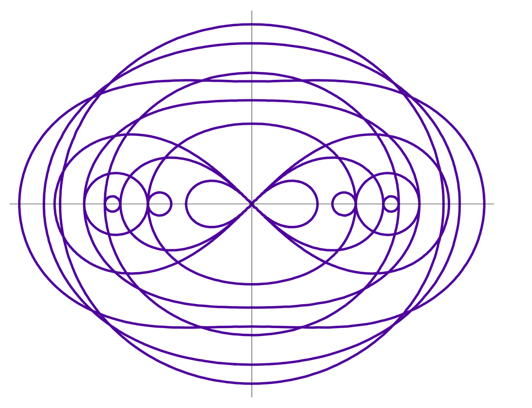
- $1 + 4\lambda\mu > 0$:

$\exists\, a, b \neq 0$ /

$\mu = \frac{1}{2b^2},\ \lambda = \frac{a^4-b^4}{2b^2}$

Cassini ovals

$r^4 - 2a^2r^2\cos 2\theta + a^4 = b^4$



$a \in \{1,2,3\}, b \in \{1,2,3,4\}$

References

[1] I. Castro and I. Castro-Infantes. *Plane curves with curvature depending on distance to a line.* Diff. Geom. Appl. **44** (2016), 77–97.

[2] I. Castro, I. Castro-Infantes and J. Castro-Infantes. *New plane curves with curvature depending on distance from the origin.* Preprint 2016 (submitted).

[3] I.M. Mladenov, M.T. Hadzhilazova, P.A. Djondjorov and V.M. Vassilev. *On the generalized Sturmian spirals.* C. R. Acad. Sci. Bulg. **64** (2011), 633–640.

[4] D. Singer. *Curves whose curvature depends on distance from the origin.* Amer. Math. Monthly **106** (1999), 835–841.

[5] D. Singer. *Curvature and variational modeling in Physics and Biophysics.* AIP Proc. **1002** (2008), 3–32.