Bi-Jacobi fields and the biharmonic flow

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1. Introduction

Biharmonic curves on a Riemannian manifold M are the critical points of the bienergy functional

$$J(\gamma) = \frac{1}{2}\int_0^T \left\langle \frac{D^2\gamma}{dt^2}, \frac{D^2\gamma}{dt^2} \right\rangle dt$$

and are seen as a natural generalization of geodesics ([3, 6]). The theory of Jacobi fields and conjugate points along geodesics can be extended to biharmonic curves. Two characterizations of Jacobi fields along geodesics can be used: the null vectors of the second variation of the energy functional and the variational vector fields associated with a one-parameter family of geodesics. The generalized Jacobi fields along biharmonic curves are called bi-Jacobi fields (see [2] for the theory of bi-Jacobi fields in general and [4] for bi-Jacobi fields along geodesics).

In this work we extend some properties of Jacobi fields along geodesics to bi-Jacobi fields. We relate bi-Jacobi fields to the biharmonic flow on the third order tangent bundle T^3M. We consider a connection map on T^3M and the corresponding nonlinear connection defined by its kernel ([1]). We describe the connection map and the decomposition of TT^3M into vertical and horizontal subbundles in terms of bi-Jacobi fields.

2. Bi-Jacobi fields

A bi-Jacobi field along a biharmonic curve γ is a vector field that is obtained as the variational vector field of a variation of γ through biharmonic curves. It is well known that W is a bi-Jacobi field along γ if and only if it satisfies the following fourth order differential equation

$$\begin{aligned} &\frac{D^4W}{dt^4} + (\nabla^2_V R)(W,V)V + (\nabla_W R)(\frac{DV}{dt},V)V + R(R(Y,V)V,V)V + R(W,\frac{D^2V}{dt^2})V \\ &+2\left[(\nabla_V R)(\frac{DW}{dt},V)V + (\nabla_V R)(W,\frac{DV}{dt})V + R(\frac{D^2W}{dt^2},V)V\right] \\ &+3\left[(\nabla_V R)(W,V)\frac{DV}{dt} + R(W,V)\frac{D^2V}{dt^2} + R(W,\frac{DV}{dt})\frac{DV}{dt}\right] + 4R(\frac{DW}{dt},V)\frac{DV}{dt} = 0, \end{aligned}$$

where V is the velocity vector field $d\gamma/dt$.

Proposition 1 *The set $\mathcal{J}_\gamma$ of all bi-Jacobi fields along a biharmonic curve γ is a vector space isomorphic to $(T_{\gamma(t_0)}M)^4$ under the mapping $W \mapsto (W(t_0), (DW/dt)(t_0), (D^2W/dt^2)(t_0), (D^3W/dt^3)(t_0))$. On $\mathcal{J}_\gamma$ we have the symplectic structure defined by*

$$\begin{aligned} \omega(W,Z) \;=\; &\frac{1}{2}\left(< \frac{D^3W}{dt^3}, Z > - < \frac{D^3Z}{dt^3}, W > + < \frac{DW}{dt}, \frac{D^2Z}{dt^2} > - < \frac{DZ}{dt}, \frac{D^2W}{dt^2} >\right)_{|t=t_0} \\ &+ \left(< R(\frac{DW}{dt},V)V, Z > - < R(\frac{DZ}{dt},V)V, W > + < R(W,V)\frac{DV}{dt}, Z > - < R(Z,V)\frac{DV}{dt}, W >\right)_{|t=t_0}. \end{aligned}$$

If the invariant along γ, $< D^2V/dt^2, V > -\frac{1}{2} < DV/dt, DV/dt >$, is different from zero, then $\mathcal{J}_\gamma$ splits into the ω-nondegenerate subspace of tangential bi-Jacobi fields and the ω-orthogonal complement.

3. A connection map on T^3M

Let (T^3M, π_3, M) be the third order tangent bundle of M and J the almost tangent structure on T^3M. A connection map on T^3M is a morphism of vector bundles

$$K = (K_1, K_2, K_3) : (TT^3M, \tau_{T^3M}, T^3M) \to (TM^{(3)}, \tau^{(3)}, M),$$

which satisfies

$$K_3 \circ J = K_2,\; K_2 \circ J = K_1,\; K_1 \circ J = \pi_{3*},$$

where $TM^{(3)} = TM \oplus TM \oplus TM$.

Let $u \in T^3M$and γ_u be the unique biharmonic curve with initial conditions $\gamma_u^3(0) = u$, where γ_u^3 is the lift to T^3M of γ_u. For each $\xi \in T_uT^3M$, consider an adapted curve w to ξ on T^3M and the variation of γ_u given by $\alpha(s,t) = \gamma_{w(s)}(t)$. The map $K_u : T_uT^3M \to (T_{\pi_3(u)}M)^3, \xi \mapsto K_u(\xi) = (K_{1,u}(\xi); K_{2,u}(\xi); K_{3,u}(\xi))$, defined by

$$K_{1,u}(\xi) = \nabla_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial s}_{|s=t=0},\; K_{2,u}(\xi) = \frac{1}{2!}\nabla^2_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial s}_{|s=t=0},\; K_{3,u}(\xi) = \frac{1}{3!}\nabla^3_{\frac{\partial\alpha}{\partial t}} \frac{\partial\alpha}{\partial s}_{|s=t=0},$$

is a connection map on T^3M.

The kernel of the connection map K, $N_1 = \ker K$, defines a nonlinear connection on T^3M and gives the decomposition $T_uT^3M = N_1(u) \oplus V_1(u)$, where $V_1 = \ker \pi_{3*}$. From this connection, taking $J(N_1(u)) = N_2(u)$, $J(N_2(u)) = N_3(u)$, $JV_1 = V_2$ and $JV_2 = V_3$, we obtain the decomposition

$$T_uT^3M = N_1(u) \oplus N_2(u) \oplus N_3(u) \oplus V_3(u).$$

Using the linear isomorphism $j_u : T_uT^3M \to (T_{\pi_3(u)}M)^4; \xi \mapsto j_u(\xi) = (\pi_{3*|u}(\xi), K_u(\xi))$ we can identify ξ with

$$j_u(\xi) = (W(0), \frac{DW}{dt}(0), \frac{1}{2!}\frac{D^2W}{dt^2}(0), \frac{1}{3!}\frac{D^3W}{dt^3}(0)),$$

where W is the bi-Jacobi field associated with the variation α.

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