Homothetic solitons for the inverse mean curvature flow

Ana M. Lerma

(joint work with Ildefonso Castro)

Universidad de Jaén

Abstract

We study solutions to the inverse mean curvature flow (IMCF) which evolve by homotheties of a given submanifold with arbitrary dimension and codimension.

Introduction

Homothetic soliton for the IMCF

$\phi : M^n \to \mathbb{R}^m$ isometric immersion

H nonvanishing mean curvature vector of M

$\perp$ the projection onto the normal bundle of M

$$-\frac{H}{|H|^2} = a\phi^\perp, \quad a \neq 0 \text{ constant}$$

- a *velocity constant* of ϕ.
- If $a < 0$, M is called *shrinker* for the IMCF.
- If $a > 0$, M is called *expander* for the IMCF.
- If ϕ is a homothetic soliton $\overset{\forall \rho > 0}{\Longrightarrow}$ $\rho\phi$ is a homothetic soliton with the same constant velocity.
- Spherical minimal submanifolds are expanders for the IMCF ($a = 1/n$).

Examples of expanders for the IMCF:

1. The standard sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.
2. The cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$ ($a = 1/k$).
3. The Clifford immersion
$$\mathbb{S}^{n_1}(\sqrt{n_1}) \times \mathbb{S}^{n_2}(\sqrt{n_2}) \hookrightarrow \mathbb{R}^{n+2}, \quad n_1 + n_2 = n.$$
4. The product of n circles
$$\mathbb{S}^1 \times \overset{n)}{\cdots} \times \mathbb{S}^1 \hookrightarrow \mathbb{R}^{2n}.$$
5. The Hopf immersion $\mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{C}^n \equiv \mathbb{R}^{2n}$:
$$(e^{it}, (x_1, \ldots, x_n)) \mapsto \sqrt{n}\, e^{it}\, (x_1, \ldots, x_n).$$

Lagrangian submanifolds

$(\mathbb{R}^{2n} \equiv \mathbb{C}^n, \langle \,,\, \rangle, J)$ complex Euclidean space

$\omega(\cdot,\cdot) = \langle J\cdot, \cdot \rangle$ Kaehler two-form

$\phi : M^n \to \mathbb{C}^n$ **Lagrangian** if $\phi^*\omega = 0$

$\Updownarrow$

$J : TM \cong T^\perp M$

Solitons for the IMCF

Any homothetic soliton ϕ for the IMCF with constant velocity a satisfies that:

- $\langle H, \phi \rangle = -1/a$.
- $\triangle |\phi|^2 = 2(n - 1/a)$.

Remark: The first condition characterizes the homothetic solitons for the IMCF of codimension one, including plane curves.

$\Rightarrow$

Theorem

Let $\phi : M^n \to \mathbb{R}^m$ be a homothetic soliton for the IMCF with constant velocity a. If M is closed, then $a = 1/n$ and ϕ must be a spherical minimal immersion.

Closed expanders for the IMCF

- $M^n \subset \mathbb{R}^{n+1}$ closed homothetic soliton hypersurface for the IMCF $\Rightarrow$ $M = \mathbb{S}^n$, up to dilations.

Rigidity results for the Clifford torus

The Clifford torus is the only closed homothetic soliton for the IMCF...

... which is an embedded torus in $\mathbb{R}^4$.

... with constant or non negative Gauss curvature in $\mathbb{R}^4$ (besides the two-sphere $\mathbb{S}^2$).

... which satisfies the pinching condition $\frac{|\sigma|^2}{|H|^2} \leq \frac{3n-4}{n(2n-3)}$ when codimension and dimension are equal to n.

Theorem

Let $\phi : M^n \to \mathbb{R}^{2n} \equiv \mathbb{C}^n$ be a closed **Lagrangian** homothetic soliton for the inverse mean curvature flow. Then, up to dilations, $M = \mathbb{S}^1 \times N^{n-1}$, where N is a closed $(n-1)$-manifold, and ϕ is given by
$$\phi(e^{it}, x) = e^{it}\psi(x),$$
where $\psi : N^{n-1} \to \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ is a minimal Legendrian immersion.

Corollaries:

1. M^2 closed Lagrangian homothetic soliton for the IMCF $\Rightarrow$ M is the Clifford torus, up to dilations.
2. M^3 closed orientable Lagrangian homothetic soliton for the IMCF with sectional curvature K verifying that $K \geq 0$ or $K \leq 0$ $\Rightarrow$ M is, up to dilations, the Hopf immersion of $\mathbb{S}^1 \times \mathbb{S}^2$ in $\mathbb{C}^3$ or the immersion of $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbb{R}^6$.
3. M^n closed Lagrangian homothetic soliton for the IMCF with sectional curvature K constant $\Rightarrow$ M is, up to dilations, the immersion of $\mathbb{S}^1 \times \overset{n)}{\cdots} \times \mathbb{S}^1$ in $\mathbb{R}^{2n}$.
4. $\phi : M^n \to \mathbb{C}^n$ closed Lagrangian homothetic soliton for the IMCF with $3n^2|\sigma|^2 \leq (11n-6)|H|^2$ $\Rightarrow$ either $n^2|\sigma|^2 = (3n-2)|H|^2$ and ϕ is given, up to dilations, by the Hopf immersion of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ in $\mathbb{C}^n$ or $n = 3$, $|\sigma|^2 = |H|^2$ and ϕ is given, up to dilations, by $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ in $\mathbb{C}^3$.
   - Closed Lagrangian homothetic solitons for the IMCF in $\mathbb{C}^n$ satisfy $n^2|\sigma|^2 \geq (3n-2)|H|^2$.
5. $\phi : M^n \to \mathbb{C}^n$ closed Lagrangian homothetic soliton for the IMCF and invariant under the action of the special orthogonal group given by
$$SO(n) \times \mathbb{C}^n \longrightarrow \mathbb{C}^n / (A, (z_1, \ldots, z_n)) \mapsto (z_1, \ldots, z_n)A$$
$\Rightarrow$ ϕ is given, up to dilations, by the Hopf immersion of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ in $\mathbb{C}^n$.

Examples of Lagrangian homothetic solitons for the IMCF

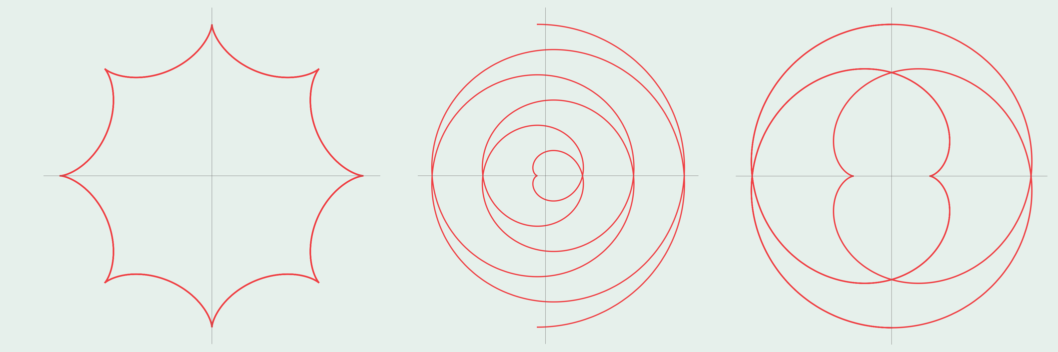
Inverse curve shortening flow

$\alpha = \alpha(t)$ regular plane curve

κ the curvature of α

$$a\kappa \langle \alpha', J\alpha \rangle = |\alpha'|$$

- The only compact regular solutions are circles centered at the origin (proved by Andrews).
- The classification of all homothetic soliton curves was given by Drugan, Lee and Wheeler.



Examples of homothetic soliton curves.

The Lagrangian homothetic solitons that are product of n plane curves are given by:

1. $\mathbb{S}^1 \times \overset{n)}{\cdots} \times \mathbb{S}^1 \hookrightarrow \mathbb{C}^n$.
2. $\mathbb{S}^1 \times \overset{k)}{\cdots} \times \mathbb{S}^1 \times \mathbb{R}^{n-k} \hookrightarrow \mathbb{C}^n$.
3. $\mathcal{C} \times \mathbb{R}^{n-1} \hookrightarrow \mathbb{C}^n$, where $\mathcal{C}$ is a homothetic soliton for the inverse curve shortening flow.

Theorem

Let $\phi : M^n \to \mathbb{C}^n$ be a Lagrangian pseudoumbilical homothetic soliton for the IMCF with constant velocity a. Then ϕ is locally congruent to
$$I \times N \longrightarrow \mathbb{C}^n, \quad (t, x) \mapsto \alpha(t)\psi(x),$$
where $\psi : N \longrightarrow \mathbb{S}^{2n-1} \subset \mathbb{C}^n$ is a minimal Legendrian immersion of a Riemannian $(n-1)$-manifold N and $\alpha : I \to \mathbb{C}^*$ is a regular plane curve such that α^n is a homothetic soliton for the inverse curve shortening flow with constant velocity na.

References

I. Castro and A.M. Lerma. *Homothetic solitons for the inverse mean curvature flow.* To appear in Results Math.

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