On Singular Lagrangians

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Overview

Singular metrics

The theory of singular (pseudo)riemannian metrics is wellknown, according to [2, 7] and the references therein. A singular lagrangians analogous to the (pseudo)riemannian one is intended to be given below.
First let us recall briefly some facts from (pseudo)riemannian metrics.
A *singular (pseudo)riemannian metric* is a bilinear map g (or a two-covariant tensor) in the fibers of TM, where $\pi_{TM} : TM \to M$ is the tangent bundle of a smooth manifold M. In order to involve a regularity condition (on foliations) we suppose that the nullity N_g of g has a constant rank, say $r > 1$. In [2, Definition 3.1.1] it is defined a natural Koszul derivative of g and according to [2, Lemma 3.1.2], a Koszul derivative of g exists iff (if and only if) $\mathcal{L}_U g = 0$ for every section $U \in \Gamma(N_g)$, where $\mathcal{L}$ denotes the Lie derivative; if it is the case, the distribution $N_g \subset TM$ is integrable ([2, Lemma 3.1.4, a)]), giving rise to a (pseudo)riemannian foliation and a transverse and regular (pseudo)riemannian $\bar{g} : N_{\mathcal{F}} \to \boldsymbol{R}$, where $\mathcal{F}$ is the regular foliation having as tangent bundle $\tau\mathcal{F} = N_g$ and $N_{\mathcal{F}} = \mathcal{TM}/\tau\mathcal{F}$ is its normal bundle.

Singular Lagrangians

Recall that if $\pi_E : E \to M$ is a vector bundle, then $\pi_{VE} : VE = \ker \tau\pi_E \to E$ is its vertical bundle, where $\tau\pi_E : TE \to TM$ denotes the differential map of π_E and ker denotes the kernel of the epimorpsm of vector bundles. We use local coordinates adapted to the vector bundle structure: (x^i) on M, (x^i, y^a) on E, (x^i, X^i) on TM, (x^i, y^a, Y^a) on VE, (x^i, y^a, X^i, Y^a) on TE, such that the vector bundle projections have local natural forms.

A lagrangian is a smooth map $L : TM \to \boldsymbol{R}$, or, by extension, in general, $L : E \to \boldsymbol{R}$. Notice that we can take $L : TM_* \to \boldsymbol{R}$, where $TM_* \subset TM$ (the slashed bundle) is obtained removing the image of the null section; but, for sake of simplicity we use $L : TM \to \boldsymbol{R}$.

Notice that every two-covariant tensor (called, by extension a *metric tensor*) gives rise to a quadratic lagrangian.

Preliminaries on foliations

Let us consider an $(n+m)$-dimensional manifold M, connected and orientable. A codimension n foliation $\mathcal{F}$ on M is defined by a foliated cocycle $\{U_i, \varphi_i, f_{i,j}\}$.

Every fibre of φ_i is called a *plaque* of the foliation. The manifold M is decomposed into submanifolds, called *leafs* of $\mathcal{F}$. If $U \subset M$ is an open subset, then there is an *induced foliation* $\mathcal{F}_U$.

We denote by $T\mathcal{F}$ the tangent bundle to $\mathcal{F}$ and by $\Gamma(T\mathcal{F})$ the module of its global sections, i.e. the vector fields on M tangent to $\mathcal{F}$. The *normal bundle* of $\mathcal{F}$ is $N\mathcal{F} = TM/T\mathcal{F}$. A vector field on M is *transverse* if it locally projects to the transversal manifold.

A system of local coordinates adapted to $\mathcal{F}$ are coordinates $(x^u, x^{\bar{u}})$, $u = 1, \ldots, m$, $\bar{u} = 1, \ldots, n$ on an open subset U, where $\mathcal{F}_U$ is trivial and defined by the equations $dx^{\bar{u}} = 0$, $\bar{u} = 1, \ldots, n$.

A particular example of a foliation is a *fibered manifold*, called a *simple foliation*. In particular, a *locally trivial fibration*.

There are elementary examples of simple foliations that come from no trivial fibrations and the spaces of leaves are not Hausdorff separated.

Tangent bundle geometry

Let us briefly recall now some constructions from the tangent space geometry [3, 8]. Any vector field $X \in \mathcal{X}(M)$ can be lifted to a *vertical lift* $X^v \in \Gamma(VTM) \subset \mathcal{X}(TM)$ and to a *complete lift* $X^c \in \mathcal{X}(TM)$. Using local coordinates, if $X = X^i\left(x^j\right)\frac{\partial}{\partial x^i}$, then

$$X^v = X^i \frac{\partial}{\partial y^i}, \; X^c = X^i \frac{\partial}{\partial x^i} + y^j \frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial y^i}.$$

We use some simple formulas related to these two lifts:

$$(fX)^v = fX^v, \; (X+Y)^v = X^v + Y^v, \; [X^v, Y^v] = 0, \; [X^v, Y^c] = [X.Y]^v, \quad (1)$$

$$(fX)^c = fX^c + df(\Delta)X^v, \; (X+Y)^c = X^c + Y^c, \; [X^c, Y^c] = [X, Y]^c, \quad (2)$$

where $X, Y \in \mathcal{X}(M)$, $f \in \mathcal{F}(M)$, $[\cdot, \cdot]$ denotes the Lie bracket and $df(\Delta) = \frac{\partial f}{\partial x^i} y^i$ denotes the evaluation of $df = \frac{\partial f}{\partial x^i} dx^i \in \mathcal{X}^*(M)$ (the differential of f, lifted to $\mathcal{F}(TM)$) and $\Delta = y^i \frac{\partial}{\partial y^i} \in \Gamma(VTM) \subset \mathcal{X}(TM)$ (the Liouville vector field).

The extended distribution

Let us suppose now that a regular distribution $\mathcal{D} \subset TM$ is given, having dimension k.

Proposition

The $\mathcal{F}(TM)$-linear spams of the vertical lifts and respectively vertical and complete lifts of vector fields from $\mathcal{D}$ give rise to two regular distributions $\mathcal{D}^v \subset \Gamma(VTM)$ and $\mathcal{D}^{cv} \subset TTM$ of dimensions k and $2k$ respectively. Also we have:

1. $\pi_{TTM}(\mathcal{D}^v) = \bar{0}_{TM}$ *(the image of the null section $M \to TM$) and* $\pi_{TTM}(\mathcal{D}^c) = \mathcal{D}$.
2. *The distribution $\mathcal{D}^v$ is always integrable, but $\mathcal{D}^{cv}$ is integrable iff $\mathcal{D}$ is integrable.*

Thus if a (regular) foliation $\mathcal{F}$ is given on M, then, by Proposition 1, the vertical and complete lifts from $\tau\mathcal{F}$ give rise together to a foliation $\mathcal{F}^{cv}$ on the manifold TM. Notice that $(\tau\mathcal{F})^{v}$ gives rise to a foliation with leaves in the fibers of $\pi_{TM} : TM \to M$, thus it projects to the points of M.
We use now adapted coordinates to the foliation $\mathcal{F}$. Consider $(x^{u}, x^{\bar{u}})$ coordinates on M, where $(x^{\bar{u}})$ are transverse coordinates. Then $(x^{u}, x^{\bar{u}}, y^{u}, y^{\bar{u}})$ are coordinates on TM, where $(x^{\bar{u}}, y^{\bar{u}})$ are transverse coordinates, and $(x^{u}, x^{\bar{u}}, y^{\bar{u}})$ are coordinates on $N_{\mathcal{F}}$. A foliation $\mathcal{F}_{N_{\mathcal{F}}} = \Pi_{N_{\mathcal{F}}}(\mathcal{F}^{cv})$ is induced on $N\mathcal{F}$ by the canonical projection $\Pi_{N_{\mathcal{F}}} : TM \to N_{\mathcal{F}} = \mathcal{TM}/\tau\mathcal{F}$.

Basic lagrangians

We say that a lagrangian $L : TM \to \mathbb{R}$ is *basic* according to the foliation $\mathcal{F}$ on M, if L is a basic function or the foliation $\mathcal{F}^{cv}$. Using local coordinates, it has the form $L = L(x^{\bar{u}}, y^{\bar{u}})$. For such an L, the hessian is obviously singular, but we can consider the *basic hessian* H_L as a bilinear form in the fibers of $VN_{\mathcal{F}}$. We say that L is *transversally regular* if its basic hessian is nondegenerate. If it is the case, the normal bundle $N_{\mathcal{F}^{cv}}$ has a Whitney decomposition $N_{\mathcal{F}^{cv}} = VN_{\mathcal{F}} \oplus HN_{\mathcal{F}}$ and an isomorphism of $VN_{\mathcal{F}}$ and $HN_{\mathcal{F}}$, that allows to extend the nondegenerate basic hessian H_L (on $VN_{\mathcal{F}}$) to a nondegenerate basic bilinear form H'_L (on $HN_{\mathcal{F}}$), giving together a nondegenerate basic bilinear form H''_L on $N_{\mathcal{F}^{cv}}$. Using the general arguments in [4, 3.2], one can infere the following result:

Proposition

The transverse metric H''_L lifts to a singular metric H'''_L in the fibers of $\pi_{TTM} : TTM \to TM$ that projects to H''_L on $N_{\mathcal{F}^{cv}} = TTM/_T\mathcal{F}^{cv}$.

Consider a lagrangian L and denote by H_L its hessian. We have:

The Main Result

Theorem

Let $L : TM \to \mathbb{R}$ be a lagrangian. The following conditions A, B and C are equivalent:

- A *The following conditions $A1 - A3$ hold:*
  - A1 *– the nulity bundle $\mathcal{N}_{H_L} \subset VTN$ has a constant rank $r > 0$ and there is a distribution $\mathcal{D} \subset TM$ such that $\mathcal{N}_{H_L} = \mathcal{D}^v$;*
  - A2 *– there is a singular metric H_L''' in the fibers of $TTM \to TM$, that restricts to the hessian H_L on VTM, $\mathcal{N}_{H_L'''} = \mathcal{D}^{cv}$ and $X(L) = 0$, $(\forall) X \in \Gamma(\mathcal{D}^{cv})$;*
  - A3 *– $\mathcal{L}_U H_L''' = 0$, $(\forall) U \in \Gamma\left(\mathcal{N}_{H_L'''}\right)$;*
- B *The following conditions $B1 - B3$ hold:*
  - B1 *$=$A1;*
  - B2 *– the distribution $\mathcal{D}^{cv}$ is integrable giving a foliation $\mathcal{F}^{cv}$;*
  - B3 *– L is a basic function according to the foliation $\mathcal{F}^{cv}$.*
- C *The lagrangian L is transversally regular according to a regular foliation $\mathcal{F}$ on M.*

Further developments

A systematic study of projectable lagrangians and hamiltonians was performed in separated papers [5, 6, 7], but not a monograph like D. N. Kupeli's [2], which is dedicated to the study of singular scalar products. Most of physics and/or mathematics settings on lagrangians and hamiltonians can be translated into a foliated laguage, usind an appropiate dictionary. Singular lagrangians, as considered here, can be used for, where a serie of papers of O. C. Stoica (as, for example [7]) are usefull.
In order to handle the singular cases, one can use *algebroids*, or *Lie algebroids*. Using these settings, the Kupeli approach, as well as our approach, can be extended. Since these general cases would substantially increase the exposition, we prefered to reduce it to the above result.

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Thank you very much!