Minimal hypersurfaces in manifolds with density

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Abstract

We provide uniqueness and non-existence results of ϕ-minimal graphs in warped product Riemannian manifolds with density. Applications to PDE's are provided. This work is based on [11].

1. Introduction

Recently, the study of ϕ-minimal submanifolds, and in particular ϕ-minimal hypersurfaces, has attracted the attention of many researchs (see, for instance [4], [8]). This kind of submanifolds appears as critical points of a functional given by the area functional with a factor depending on the density function. Recall that a *manifold with density* is a Riemannian manifold $(\overline{M}, \overline{g})$ endowed with a positive smooth function $e^{-\phi}$, called density, which is used to measure geometric objects in $\overline{M}$, as area and volume. The prototypical example of a manifold with density is the Gauss space, which appeared in Probability and Statistics. This manifold is the Euclidean space, $\mathbb{R}^n$, with the Gaussian probability density $e^{c-|x|^2}$, where c is a constant chosen to normalize. As we will see, the manifolds with density are a good candidate to generalize the classical notion of minimal submanifolds (they are given by critical points of the area functional). Some concepts related to minimality have its counterparts in ϕ-minimality, as ϕ-stability of hypersurfaces (for instance, [6]), of intrinsic interest.

Let $(\overline{M}^{n+1}, \overline{g})$ be a Riemannian manifold, and consider $\phi : \overline{M} \to \mathbb{R}$ a smooth function, for which the density of $\overline{M}$ is $e^{-\phi}$. In this setting, consider $x : S^n \to \overline{M}$ an immersed hypersurface with induced metric $g = x^*\overline{g}$, volume element dS and second fundamental form $A : TS^2 \to T^\perp S$. Recall that the mean curvature of S is $H = \frac{1}{n}\mathrm{trace}_g A$. We can consider on S the weighted area

$$A(\phi, x) := \int_S dS_\phi = \int_S e^{-\phi(x)}\, dS\,.$$

Then, S is said to be *ϕ-minimal* if it is a critical point of $A(\phi, \cdot)$ among all normal immersions of S into $\overline{M}$ (in the non-compact case, among that that fixes the boundary). Observe that the minimal hypersurfaces are ϕ-minimal when ϕ is the constant function. On the other hand, it is defined the ϕ-mean curvature of S, H_ϕ, by

$$H_\phi = H + \frac{1}{n}\overline{g}(\overline{\nabla}\phi, N)\,.$$

Then, the ϕ-minimal hypersurfaces must satisfy $H_\phi = 0$, i.e.,

$$nH = -\overline{g}(\overline{\nabla}\phi, N)\,. \tag{1}$$

As example, the n-sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is ϕ-minimal for $\phi(z) = -\frac{n}{2}|z|^2$ on $\mathbb{R}^{n+1}$. In a more general setting, in $\mathbb{R}^n$ with a spherical density satifying some convexity assumption, the spheres, among other classical hypersurfaces, have been characterized (see, for instance, [2] and [3]).

The study of minimal hypersurfaces is classical, and it has been deep and fruitful. In 1914, S. Bernstein [1] proved that

The only entire solutions to the minimal surface equation in $\mathbb{R}^3$ are the affine functions.

An amusing research topic, which has made many advances on geometric analysis, has been the possible extension of the Bernstein result to higher dimension. A very notable contribution in this area was made by J. Moser [9] in 1961, who obtain the following general result,

The only entire solutions u to the minimal surface equation in $\mathbb{R}^{n+1}$ such that $|Du| \le C$, for some $C \in \mathbb{R}^+$, are the affine functions

$$u(x_1, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n + c\,,$$

$a_i, c \in \mathbb{R}$, $1 \le i \le n$, *with* $\sum_{i=1}^n a_i^2 \le C^2$.

In 1968, J. Simons [12] proved a result that yield a proof of the extension of the Bernstein theorem for $n \le 7$. Moreover, there is a counterexample $u \in C^\infty(\mathbb{R}^n)$ for each $n \ge 8$ (with ubounded $|Du|$). The main aim of this work is to investigate new uniqueness results for some kind of hypersurfaces that, in some sense, generalize the minimal hypersurfaces. So, we are interested in study uniqueness of ϕ-minimal hypersurfaces. They will live in a different class of Riemannian manifolds than $\mathbb{R}^n$, and such that the structure of the hypersurface is controlled. This control on its topology is given by requering that the hypersurface is a graph on a manifold (M, g). Then, the natural choice of ambient space is the product manifold of M and an open interval of the real line, I. However, we consider on it a more general metric than the Riemannian product one.

Consider a positive smooth function f on I and an n-dimensional Riemannian manifold (M, g). With this ingredients, the product manifold $I \times M$ can be endowed with the Riemannian metric

$$\overline{g} = \pi_I^*(dt^2) + f(\pi_I)^2\pi_M^*(g)\,, \tag{2}$$

where π_I and π_M denote the projections onto I and M, respectively. This Riemannian manifold is a *warped product*, with base (I, dt^2), *fiber* (M, g) and *warping function* f. According to [10], let us denote this manifold by $I \times_f M$. Observe that these manifolds have adquired relevance, from the fact that in this case it was showed how a convex function affects the topology and curvature of a Riemannian manifold. Furthermore, it is also used some functions to build manifolds of negative sectional curvature. This class of manifolds play the role of our ambient spaces. Moreover, the fiber is assumed to be parabolic. Recall that a (noncompact) complete Riemannian manifold (M, g) is parabolic if it admits no nonconstant positive superharmonic function (for instance, see [7], and [5] for a wide treatment).

2. Main results

A key fact in our results is to assure parabolicity of a entire graph on the fiber of a warped product when some conditions are fulfilled. Hence, the method to prove our results is to study some distinguished functions on a ϕ-minimal graph in order to conclude that it is the constant graph. Among our results, we have,

Theorem 1 *Let $I \times_f M$ be a warped product Riemannian manifold (which is not a Riemannian product) endowed with a density function $e^{-\phi}$, which satisfies $||\overline{\nabla}\phi||^2 \le n^2 f'(t)^2/f(t)^2$. Assume that M is parabolic and f is bounded satisfying $(\log f)''(t) \ge \overline{\sigma}(\log f)'(t)^2$, for some negative constant $\overline{\sigma}$.*

Then, the only complete ϕ-minimal graphs on M whose normal vector field satisfies $\overline{g}(N, \partial_t)$ is bounded away from zero are the hypersurfaces $t = t_0$, $t_0 \in I$, such that $\partial_t\phi(t_0, p) = nf'(t_0)/f(t_0)$.

An interesting case appears when the warping function is monotonic. Then, we obtain,

Theorem 2 *Let $I \times_f M$ be a warped product Riemannian manifold endowed with a density function $e^{-\phi}$, which satisfies $2|f'(t)/f(t)| \ge ||\overline{\nabla}\phi||$. Assume that M is parabolic and f is non-decreasing (resp. non-increasing).*

Then, the only complete ϕ-minimal graphs on M which are bounded from above (resp. below) and whose normal vector field satisfies $\overline{g}(N, \partial_t)$ is bounded away from zero are the hypersurfaces $t = t_0$, $t_0 \in I$, such that $\partial_t\phi(t_0, p) = nf'(t_0)/f(t_0)$.

On the other hand, we are able to consider the 3-dimensional hyperbolic space, where uniqueness results are achieved,

Theorem 3 *Let $\mathbb{H}^3(-k)$ be the hyperbolic space endowed with a density $e^{-\phi}$ such that $||\nabla\phi||^2 \le k$. Consider S to be a complete ϕ-minimal surface. If there exists a conformal vector field $K \in \mathfrak{X}(\mathbb{H}^3)$ for which $g(K, N) \ge \epsilon|K|$, $\epsilon > 0$, where $N \in \mathfrak{X}(S)^\perp$, then S must be a level hypersurface of K.*

Moreover, as applications of our results, we solve new interesting PDE's, like

Theorem 4 *Let (M, g) be a parabolic Riemannian manifold. On $I \subseteq \mathbb{R}$, consider a non-decreasing function f and a non-increasing function ϕ (resp. a non-increasing function f and a non-decreasing function ϕ).*

The only entire solutions $u \in C^\infty(M)$ to

$$\mathrm{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 + |Du|^2}}\right) - \frac{f'(u)}{\sqrt{f(u)^2 + |Du|^2}}\left\{n - \frac{|Du|^2}{f(u)^2}\right\} = -\frac{f(u)}{\sqrt{f(u)^2 + |Du|^2}}\phi'(u)\,,$$

which satisfy $|Du| \le Cf(u)$, for some $C \in \mathbb{R}^+$, and are bounded from above (resp. from below) are the constants.

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