Symmetries and Conservation Laws in Multisymplectic Second-order Field Theories: the Gravitational Field

Jordi Gaset, Narciso Román-Roy (Dept. Mathematics, UPC)

SUMMARY: *Generalizing previous results for first-order field theories [3, 4], conserved quantities and different kinds of symmetries are studied for second order Lagrangian field theories. As an application we analyze the case of the Hilbert-Einstein formulation of the gravitational field.* (Work in progress).

Jet bundles. Second-order Lagrangian field theories

(See [9]). Let $E \xrightarrow{\pi} M$ be a fiber bundle, $\dim E = m+n$, over an orientable m-dimensional manifold M, whose volume form is $\eta \in \Omega^m(M)$, and let $J^k\pi$ be the *kth-order jet bundle*. The *kth prolongation* of a local section of π, $\phi \in \Gamma(\pi)$, to $J^k\pi$ is denoted by $j^k\phi \in \Gamma(\bar{\pi}^k)$. Points in $J^k\pi$ are denoted by $j^k_x\phi$, $x \in M$ and $\phi \in \Gamma(\pi)$ being a representative of the equivalence class. If $r \leqslant k$, natural projections are:

$$\begin{array}{cccccc} \pi^k_r\colon J^k\pi \longrightarrow J^r\pi & & \pi^k\colon J^k\pi \longrightarrow E & & \bar{\pi}^k\colon J^k\pi \longrightarrow M \\ j^k_x\phi \longmapsto j^r_x\phi & & j^k_x\phi \longmapsto \phi(x) & & j^k_x\phi \longmapsto x \end{array}$$

Observe that $\pi^s_r \circ \pi^k_s = \pi^k_r$, $\pi^k_0 = \pi^k$, $\pi^k_k = \mathrm{Id}_{J^k\pi}$, and $\bar{\pi}^k = \pi \circ \pi^k$. A section $\psi \in \Gamma(\bar{\pi}^k)$ is *holonomic* if $j^k(\pi^k \circ \psi) = \psi$; that is, ψ is the kth prolongation of a section $\phi = \pi^k \circ \psi \in \Gamma(\pi)$.

If (x^i, u^α), $1 \leqslant i \leqslant m$, $1 \leqslant \alpha \leqslant n$, are local coordinates in E adapted to the bundle structure, such that $\eta = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$; then local coordinates in $J^k\pi$ are denoted (x^i, u^α_I), with $0 \leqslant |I| \leqslant k$ (I is a multi-index; that is, an element of $\mathbb{Z}^m$ where every component is positive, the ith position of the multi-index is denoted $I(i)$, and $|I| = \displaystyle\sum_{i=1}^m I(i)$ is the length of the multi-index. An expression as $|I| = k$ means that the expression is taken for every multi-index of length k. The element $1_i \in \mathbb{Z}^m$ is defined as $1_i(j) = \delta^j_i$, $n(ij)$ is a combinatorial factor which $n(ij) = 1$ for $i = j$, and $n(ij) = 2$ for $i \neq j$). The *coordinate total derivatives* are [6, 9]: $D_i = \dfrac{\partial}{\partial x^i} + \displaystyle\sum_{|I|=0}^{k} u^\alpha_{I+1_i}\dfrac{\partial}{\partial u^\alpha_I}$. For every function f, $D_i f := \mathrm{L}_{D_i} f$.

Definition 1. [4]. *An* m-multivector field *in $J^k\pi$ is a skew-symmetric contravariant tensor of order m in $J^k\pi$. The set of m-multivector fields in $J^k\pi$ is denoted $\mathfrak{X}^m(J^k\pi)$.*
A multivector field $\mathbf{X} \in \mathfrak{X}^m(J^k\pi)$ is said to be locally decomposable*, for every $p \in J^k\pi$, there is an open neighbourhood $U_p \subset J^k\pi$ and $X_1, \ldots, X_m \in \mathfrak{X}(U_p)$ such that $\mathbf{X}|_{U_p} = X_1 \wedge \ldots \wedge X_m$.*
Locally decomposable m-multivector fields $\mathbf{X} \in \mathfrak{X}^m(J^k\pi)$ are locally associated with m-dimensional distributions $D \subset \mathrm{T}J^k\pi$. Then, $\mathbf{X}$ is integrable *if its associated distribution is integrable. In particular, $\mathbf{X}$ is* holonomic *if it is integrable and its integral sections are holonomic sections of $\bar{\pi}^k$.*

For a second-order classical field theory, a *second-order Lagrangian density* is a $\bar{\pi}^2$-semibasic m-form $\mathcal{L} \in \Omega^m(J^2\pi)$; then $\mathcal{L} = L\,(\bar{\pi}^2)^*\eta$, where $L \in \mathrm{C}^\infty(J^2\pi)$ is the *Lagrangian function.* The Lagrangian phase bundle is $J^3\pi$ and natural coordinates adapted to the fibration are $(x^i, u^\alpha, u^\alpha_i, u^\alpha_I, u^\alpha_J)$; $1 \leq i \leq m$, $1 \leq \alpha \leq n$, and I, J are multiindices with $|I| = 2$, $|J| = 3$. The Poincaré-Cartan m-form $\Theta_\mathcal{L} \in \Omega^m(J^3\pi)$ can be unambiguously constructed using the canonical structures of $J^k\pi$ [1, 5, 6] and it is given by

$$\begin{aligned}\Theta_\mathcal{L} = &\left(\frac{\partial L}{\partial u^\alpha_i} - \sum_{j=1}^m \frac{1}{n(ij)}\frac{d}{dx^j}\frac{\partial L}{\partial u^\alpha_{1_i+1_j}}\right)(\mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i - u^\alpha_i \mathrm{d}^m x) + \frac{1}{n(ij)}\frac{\partial L}{\partial u^\alpha_{1_i+1_j}}(\mathrm{d}u^\alpha_i \wedge \mathrm{d}^{m-1}x_j \\ &- u^\alpha_{1_i+1_j}\mathrm{d}^m x) + L\mathrm{d}^m x \equiv L^i_\alpha \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i + L^{ij}_\alpha \mathrm{d}u^\alpha_i \wedge \mathrm{d}^{m-1}x_j + (L - L^i_\alpha u^\alpha_i - L^{ij}_\alpha u^\alpha_{1_i+1_j})\mathrm{d}^m x\,,\end{aligned}$$

where $L^i_\alpha = \dfrac{\partial L}{\partial u^\alpha_i} - \displaystyle\sum_{j=1}^m D_j L^{ij}_\alpha$, $L^{ij}_\alpha = \dfrac{1}{n(ij)}\dfrac{\partial L}{\partial u^\alpha_{1_i+1_j}}$. The Poincaré-Cartan $(m+1)$-form is $\Omega_\mathcal{L} := -\mathrm{d}\Theta_\mathcal{L}$.

A *second-order Lagrangian system* is specified as $(J^3\pi, \Omega_\mathcal{L})$. The solutions to the Lagrangian variational problem posed by $\mathcal{L}$ are holonomic sections $j^3\phi\colon M \to J^3\pi$ which are the integral submanifolds of a locally decomposable holonomic multivector field $\mathcal{X}_\mathcal{L} \in \mathfrak{X}^m(J^3\pi)$ satisfying the equation

$$\boxed{i(\mathcal{X}_\mathcal{L})\Omega_\mathcal{L} = 0\,. \qquad (1)}$$

Let $\ker^m \Omega_\mathcal{L} := \{\mathcal{X} \in \mathfrak{X}^m(\mathcal{M}) \mid i(\mathcal{X})\Omega_\mathcal{L} = 0\}$. If $\omega = (\bar{\pi}^3)^*\eta$, let $\ker^m_\omega \Omega_\mathcal{L}$ be the set of m-multivector fields satisfying the equation (1) and the $\bar{\pi}^3$-transversality condition $i(\mathcal{X})\omega \neq 0$, but being not necessarily locally decomposable. Then, we denote by $\ker^m_{\omega(I)} \Omega_\mathcal{L}$ the set of integrable m-multivector fields satisfying both conditions. Oviously we have that $\ker^m_{\omega(I)} \Omega_\mathcal{L} \subset \ker^m_\omega \Omega_\mathcal{L} \subset \ker^m \Omega_\mathcal{L}$.

Symmetries and conservation laws

Definition 2. *1. A* symmetry *of $(J^k\pi, \Omega_\mathcal{L})$ is a diffeomorphism $\Phi\colon J^k\pi \to J^k\pi$ s.t. $\Phi_*(\ker^m \Omega_\mathcal{L}) \subset \ker^m \Omega_\mathcal{L}$. If $\Phi = j^k\varphi$ for a difeormorphism $\varphi\colon E \to E$, the symmetry is* natural.
2. An infinitesimal symmetry *of $(J^k\pi, \Omega_\mathcal{L})$ is a vector field $Y \in \mathfrak{X}(\mathcal{M})$ whose local flows are local symmetries or, what is equivalent, such that $[Y, \ker^m \Omega_\mathcal{L}] \subset \ker^m \Omega_\mathcal{L}$. If $Y = j^k Z$ for some $Z \in \mathfrak{X}(M)$, then the infinitesimal symmetry is* natural.

Theorem 1. *1. Let $\Phi \in \mathrm{Diff}(\mathcal{M})$ be a symmetry of $(J^k\pi, \Omega_\mathcal{L})$ such that $\Phi \in \mathrm{Diff}(J^k\pi)$ restricts to a diffeormorphism $\varphi\colon M \to M$. Then, for every $\mathcal{X} \in \ker^m_{\omega(I)} \Omega_\mathcal{L}$, Φ transforms integral submanifolds of $\mathcal{X}$ into integral submanifolds of $\Phi_*\mathcal{X}$, and hence $\Phi_*\mathcal{X} \in \ker^m_{\omega(I)} \Omega_\mathcal{L}$.*
2. Let $Y \in \mathfrak{X}(J^k\pi)$ be an infinitesimal symmetry of $(J^k\pi, \Omega_\mathcal{L})$ and F_t a local flow of Y. If $Y \in \mathfrak{X}(J^k\pi)$ is $\bar{\pi}^3$-projectable then, for every $\mathcal{X} \in \ker^m_{\omega(I)} \Omega_\mathcal{L}$, F_t transforms integral submanifolds of $\mathcal{X}$ into integral submanifolds of $F_{t*}\mathcal{X}$, and hence $F_{t*}\mathcal{X} \in \ker^m_{\omega(I)} \Omega_\mathcal{L}$.

Definition 3. *1. A* Cartan (Noether) symmetry *of $(J^k\pi, \Omega_\mathcal{L})$ is a diffeomorphism $\Phi\colon J^k\pi \to J^k\pi$ such that, $\Phi^*\Omega_\mathcal{L} = \Omega_\mathcal{L}$. If, in addition, $\Phi^*\Theta_\mathcal{L} = \Theta_\mathcal{L}$, then Φ is said to be an* exact Cartan symmetry*. If $\Phi = j^k\varphi$ for a diffeormorphism $\varphi\colon E \to E$, the Cartan symmetry is* natural.
2. An infinitesimal Cartan (Noether) symmetry *of $(J^k\pi, \Omega_\mathcal{L})$ is a vector field $Y \in \mathfrak{X}(J^k\pi)$ satisfying that $\mathrm{L}(Y)\Omega_\mathcal{L} = 0$. If, in addition, $\mathrm{L}(Y)\Theta_\mathcal{L} = 0$, then Φ is said to be an* infinitesimal exact Cartan symmetry*. If $Y = j^k Z$ for some $Z \in \mathfrak{X}(E)$, then the infinitesimal Cartan symmetry is* natural.

Canonical lifts of diffeomorphisms and vector fields preserve the canonical structures of $J^k\pi$, but not $\Omega_\mathcal{L}$.

Definition 4. *1. A* Lagrangian symmetry *of $(J^k\pi, \Omega_\mathcal{L})$ is a diffeomorphism $\Phi\colon J^k\pi \to J^k\pi$ such that Φ leaves the canonical geometric structures of $J^k\pi$ invariant and $\Phi^*\mathcal{L} = \mathcal{L}$ (Φ leaves $\mathcal{L}$ invariant).*
A natural Lagrangian symmetry *of $(J^k\pi, \Omega_\mathcal{L})$ is a diffeomorphism $\Phi\colon J^k\pi \to J^k\pi$ such that $\Phi = j^k\varphi$, for some diffeomorphism $\varphi\colon E \to E$, and Φ leaves $\mathcal{L}$ invariant.*
2. An infinitesimal Lagrangian symmetry *of $(J^k\pi, \Omega_\mathcal{L})$ is a vector field $Y \in \mathfrak{X}(J^1E)$ such that the canonical geometric structures of $J^k\pi$ are invariant under Y and $\mathrm{L}(Y)\mathcal{L} = 0$ (Y leaves $\mathcal{L}$ invariant).*
An infinitesimal natural Lagrangian symmetry *of $(J^k\pi, \Omega_\mathcal{L})$ is a vector field $Y \in \mathfrak{X}(J^k\pi)$ such that $Y = j^k Z$, for some $Z \in \mathfrak{X}(E)$, and $\mathrm{L}(j^k Z)\mathcal{L} = 0$.*

Proposition 1. *If $\Phi\colon J^k\pi \to J^k\pi$ is a Lagrangian symmetry of $(J^k\pi, \Omega_\mathcal{L})$, then $\Phi^*\Theta_\mathcal{L} = \Theta_\mathcal{L}$, and hence it is an exact Cartan symmetry.*
If $Y \in \mathfrak{X}(J^k\pi)$ is an infinitesimal Lagrangian symmetry of $(J^k\pi, \Omega_\mathcal{L})$, then $\mathrm{L}(Y)\Theta_\mathcal{L} = 0$, and hence it is an infinitesimal exact Cartan symmetry.

Definition 5. *A* conservation law *or a* conserved quantity *of $(J^k\pi, \Omega_\mathcal{L})$ is a form $\xi \in \Omega^{m-1}(\mathcal{M})$ such that $\mathrm{L}(\mathcal{X})\xi := (-1)^{m+1} i(\mathcal{X})\mathrm{d}\xi = 0$, for every $\mathcal{X} \in \ker^m_\omega \Omega_\mathcal{L}$.*

Theorem 2. *A form $\xi \in \Omega^{m-1}(\mathcal{M})$ is a conserved quantity of $(J^k\pi, \Omega_\mathcal{L})$ if, and only if, $\mathrm{L}(\mathcal{Z})\xi = 0$, for every $\mathcal{Z} \in \ker^m \Omega$.*

Proposition 2. *If $\xi \in \Omega^{m-1}(\mathcal{M})$ is a conserved quantity of $(J^k\pi, \Omega_\mathcal{L})$ and $\mathcal{X} \in \ker^m_{\omega(I)} \Omega_\mathcal{L}$, then ξ is closed on the integral submanifolds of $\mathcal{X}$: if $j_S\colon S \hookrightarrow \mathcal{M}$ is an integral submanifold, then $\mathrm{d}j^*_S\xi = 0$.*

Theorem 3. (Noether)*: If $Y \in \mathfrak{X}(J^k\pi)$ is an infinitesimal Cartan symmetry of $(J^k\pi, \Omega_\mathcal{L})$, with $i(Y)\Omega_\mathcal{L} = \mathrm{d}\xi_Y$. Then, for every $\mathcal{X} \in \ker^m_\omega \Omega_\mathcal{L}$ (and hence for every $\mathcal{X} \in \ker^m_{\omega(I)} \Omega_\mathcal{L}$), we have that $\mathrm{L}(\mathcal{X})\xi_Y = 0$; that is, any Hamiltonian $(m-1)$-form ξ_Y associated with Y is a conserved quantity. (It is usually called a* Noether current*). As a particular case, if $\mathrm{L}(Y)\Theta_\mathcal{L} = 0$ then $\xi_Y = i(Y)\Theta_\mathcal{L}$.*

Definition 6. *If $\Omega_\mathcal{L}$ is a premultisymplectic form (i.e., it is 1-degenerate) then the vector fields $Y \in \ker \Omega_\mathcal{L}$ are the* (infinitesimal) gauge symmetries *of $(J^k\pi, \Omega_\mathcal{L})$.*

Application to the Hilbert-Einstein action

For this system, M is a 4-dimensional manifold representing space-time and E is the manifold of Lorentzian metrics on M. Local coordinates in E are denoted $(x^\mu, g_{\alpha\beta})$, with $0 \leq \alpha \leq \beta \leq 3$, and $\dim E = 14$. The induced coordinates in $J^3\pi$ are $(x^\mu,\ g_{\alpha\beta},\ g_{\alpha\beta,\mu},\ g_{\alpha\beta,\mu\nu},\ g_{\alpha\beta,\mu\nu\rho})$. The *Hilbert-Einstein Lagrangian* is

$$L_{EH} = \varrho g^{\alpha\beta} R_{\alpha\beta} = \varrho R\,,$$

where $\varrho \equiv \sqrt{|det(g_{\alpha\beta})|}$, R is the *scalar curvature*, $R_{\alpha\beta}$ are the components of the *Ricci tensor*, and $g^{\alpha\beta}$ is the inverse matrix of g. The Poincaré-Cartan 3-form $\Theta_{\mathcal{L}_{EH}}$ associated with the Lagrangian density $\mathcal{L}_{EH} = L_{EH}\,(\bar{\pi}^3)^*\eta = L_{EH}\,\mathrm{d}^4x$ is

$$\begin{aligned}\Theta_{\mathcal{L}_{EH}} &= -H\mathrm{d}^4x + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu}\mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^{m-1}x_\mu + \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu\nu}\mathrm{d}g_{\alpha\beta,\mu} \wedge \mathrm{d}^{m-1}x_\nu\,;\\ L^{\alpha\beta,\mu} &= \frac{\partial L}{\partial g_{\alpha\beta,\mu}} - \sum_{\nu=0}^{3}\frac{1}{n(\mu\nu)}D_\nu\frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}} = \frac{n(\alpha\beta)\varrho}{2}\Big(\Gamma^\alpha_{\nu\sigma}(g^{\beta\sigma}g^{\mu\nu} - g^{\beta\mu}g^{\sigma\nu}) + \Gamma^\beta_{\nu\sigma}(g^{\alpha\sigma}g^{\mu\nu} - g^{\alpha\mu}g^{\sigma\nu})\Big)\\ L^{\alpha\beta,\mu\nu} &= \frac{1}{n(\mu\nu)}\frac{\partial L}{\partial g_{\alpha\beta,\mu\nu}} = \frac{n(\alpha\beta)}{2}\varrho(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - 2g^{\alpha\beta}g^{\mu\nu})\,;\\ H &= \sum_{\alpha\leq\beta} L^{\alpha\beta,\mu}g_{\alpha\beta,\mu} + \sum_{\alpha\leq\beta} L^{\alpha\beta,I}g_{\alpha\beta,I} - L,\end{aligned}$$

where $\Gamma^\rho_{\mu\nu}$ are the *Christoffel symbols of the Levi-Civita connection* of g. As it is usual, $\Omega_{\mathcal{L}_{EH}} = -\mathrm{d}\Theta_{\mathcal{L}_{EH}}$.

As $\Omega_{\mathcal{L}_{EH}}$ is π^3_1-projectable [2, 6, 7, 8] the π^3_1-vertical vector fields in $J^3\pi$ are gauge symmetries.

Definition 7. *1. Let $F\colon M \to M$ be a diffeomorphism. The* canonical lift of F to the bundle of metrics E *is the diffeomorphism $\mathcal{F}\colon E \to E$ defined as follows: for every $(x, g_x) \in E$, then $\mathcal{F}(x, g_x) := (F(x), (F^{-1})^*(g_x))$. (Thus $\pi \circ \mathcal{F} = F \circ \pi$).*
The canonical lift of $\mathcal{F}$ to the jet bundle $J^k\pi$ *is the diffeomorphism $j^k\mathcal{F}\colon J^k\pi \to J^k\pi$ defined as follows: for every $j^k_x\phi \in J^k\pi$, then $\mathcal{F}(j^k_x\phi) := j^k(\mathcal{F} \circ \phi \circ F^{-1})(x)$.*
2. Let $Z \in \mathfrak{X}(M)$. The canonical lift of Z to the bundle of metrics E *is the vector field $Y \in \mathfrak{X}(E)$ whose associated local one-parameter groups of diffeomorphisms $\mathcal{F}_t$ are the canonical lifts to the bundle of metrics E of the local one-parameter groups of diffeomorphisms F_t of Z.*
The canonical lift of $Y \in \mathfrak{X}(E)$ to the jet bundle $J^k\pi$ *is the vector field $Y^k \equiv j^kY \in \mathfrak{X}(J^k\pi)$ whose associated local one-parameter groups of diffeomorphisms are the canonical lifts $j^1\mathcal{F}_t$ of the local one-parameter groups of diffeomorphisms $\mathcal{F}_t$ of Y.*

In coordinates, if $Z = u^\mu(x)\dfrac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$, the canonical lift of Z to the bundle of metrics $Y \in \mathfrak{X}(E)$ is

$$Y = u^\mu\frac{\partial}{\partial x^\mu} - \sum_{\alpha\leq\beta}\left(\frac{\partial u^\mu}{\partial x^\alpha}g_{\mu\beta} + \frac{\partial u^\mu}{\partial x^\beta}g_{\mu\alpha}\right)\frac{\partial}{\partial g_{\alpha\beta}}\,, \quad \text{and then :}$$

$$\begin{aligned}Y^1 &= u^\mu\frac{\partial}{\partial x^\mu} + \sum_{\alpha\leq\beta} Y_{\alpha\beta}\frac{\partial}{\partial g_{\alpha\beta}} + \sum_{\alpha\leq\beta} Y_{\alpha\beta\mu}\frac{\partial}{\partial g_{\alpha\beta,\mu}} = u^\mu\frac{\partial}{\partial x^\mu} - \sum_{\alpha\leq\beta}\left(\frac{\partial u^\mu}{\partial x^\alpha}g_{\mu\beta} + \frac{\partial u^\mu}{\partial x^\beta}g_{\mu\alpha}\right)\frac{\partial}{\partial g_{\alpha\beta}}\\ &- \sum_{\alpha\leq\beta}\left(\frac{\partial^2 u^\nu}{\partial x^\alpha\partial x^\mu}g_{\nu\beta} + \frac{\partial^2 u^\nu}{\partial x^\beta\partial x^\mu}g_{\alpha\nu} + \frac{\partial u^\nu}{\partial x^\alpha}g_{\nu\beta,\mu} + \frac{\partial u^\nu}{\partial x^\beta}g_{\alpha\nu,\mu} + \frac{\partial u^\nu}{\partial x^\mu}g_{\alpha\beta,\nu}\right)\frac{\partial}{\partial g_{\alpha\beta,\mu}}\,.\end{aligned}$$

For every $Z \in \mathfrak{X}(M)$, we have $\mathrm{L}(Y^3)\mathcal{L}_{EH} = 0$, because $\mathcal{L}_{EH}$ is invariant under diffeomorphisms. As Y^3 is a canonical lift, it is an infinitesimal Lagrangian symmetry. Thus, Y^3 it is an exact infinitesimal Cartan symmetry, its associated conserved quantity is $\xi_Y = i(Y^3)\Theta_{\mathcal{L}_{EH}}$ and, as $\Theta_{\mathcal{L}_{EH}}$ is π^3_1-basic,

$$\boxed{\begin{aligned}\xi_Y &= i(Y^3)\Theta_{\mathcal{L}_{EH}} = i(Y^1)\Theta_{\mathcal{L}_{EH}} = \Big(Y_{\alpha\beta}L^{\alpha\beta,\mu} + Y_{\alpha\beta\nu}L^{\alpha\beta,\nu\mu} - u^\mu H\Big)\,\mathrm{d}^3x_\mu\\ &+ \Big(u^\nu L^{\alpha\beta,\mu} - u^\mu L^{\alpha\beta,\nu}\Big)\,\mathrm{d}g_{\alpha\beta} \wedge \mathrm{d}^2x_{\nu\mu} + \Big(u^\nu L^{\alpha\beta,\lambda\mu} - u^\mu L^{\alpha\beta,\lambda\nu}\Big)\,\mathrm{d}g_{\alpha\beta,\lambda} \wedge \mathrm{d}^2x_{\nu\mu}\,,\end{aligned}}$$

where $\mathrm{d}^2x_{\mu\nu} = i\left(\dfrac{\partial}{\partial x^\nu}\right) i\left(\dfrac{\partial}{\partial x^\mu}\right)\mathrm{d}^4x$.

The vector fields of the form Y^3 are the only natural infinitesimal Lagrangian symmetries [6, 8].

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