INVARIANT SOLUTIONS TO THE HETEROTIC EQUATIONS OF MOTION ON COMPACT QUOTIENTS OF LIE GROUPS

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Objectives

1. Provide new explicit smooth six-dimensional non-Kähler solutions to the Strominger system and the heterotic equations of motion.
2. Our solutions are based on the invariant Hermitian geometry on six-dimensional solvmanifolds studied by the authors in previous works.

Strominger system and equations of motion

[Str86] analyzed heterotic superstring background with space-time supersymmetry. The model requires a 10-dimensional manifold of the form:

$$\mathcal{M} = \underbrace{M^{3,1}}_{\text{Lorenztian}} \times \underbrace{M^6}_{\text{Hermitian mfd generalizing Calabi-Yau mfd}}, \quad g = e^{2D(x)}\begin{pmatrix} g_{\mu,\nu}(x) & 0 \\ 0 & g_{m,n}(y)\end{pmatrix}$$

- Equations of motion (EM) are derived from the Lagrangian:
  $$S = \int e^{-2\phi}\left[s^g + 4(\nabla^g\phi)^2 - \tfrac{1}{2}|H|^2 - \tfrac{\alpha'}{4}\left(Tr|F^A|^2 - Tr|R|^2\right)\right]\sqrt{-g}\, d^{10}x,$$
  where ϕ is a scalar field called *dilaton*. Very restrictive.
- Strominger system (SS): less restrictive than the equations of motion. If the dilaton is constant, we look for $(M^6, J, F, \Psi, \nabla, A)$ satisfying:
  - Gravitino eq.: $Hol(\nabla^+) \subseteq SU(3)$, being ∇^+ the Bismut connection associated to the Hermitian structure (J, F). Equiv: Ψ is a holomorphic (3,0)-form i.e. $\bar\partial\Psi = 0$ trivializing the holomorphic canonical bundle.
  - Dilatino eq.: (M, J, F) is balanced, i.e. $dF^2 = 0$.
  - Gaugino eq.: A is Donaldson-Uhlenbeck-Yau instanton, i.e. $\Omega^A \in \mathfrak{su}(3)$.
  - The Green-Schwarz anomaly cancellation condition:
    $$dT = 2\pi^2\alpha'(p_1(\nabla) - p_1(A)) = \frac{\alpha'}{4}(\mathrm{tr}(\Omega\wedge\Omega) - \mathrm{tr}(\Omega^A\wedge\Omega^A)), \qquad (1)$$
    - $\alpha' \in \mathbb{R}\setminus\{0\}$ (better in physics $\alpha' > 0$).
    - T is the torsion 3-form associated to ∇^+. It is identified to $T = JdF$.
    - Ω is the curvature form of a metric connection ∇ on TM.
- Relation between (EM) and (SS) [Iva10]:
  A solution to (SS) satisfies (EM) $\iff \nabla$ in (1) is an instanton.

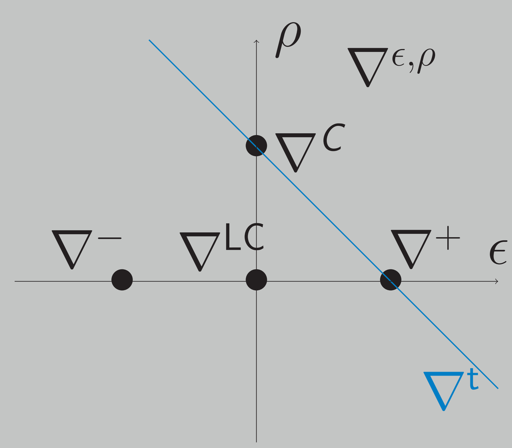
Antecedents

- There are several proposals for connections ∇ in (1):
  - Levi-Civita: ∇^{LC}. Torsion-free.
  - Bismut-Strominger: $\nabla^+ = \nabla^{LC} + \frac{1}{2}T$, where $T = JdF$.
  - Chern: $\nabla^C = \nabla^{LC} + \frac{1}{2}C$, where $C(\cdot,\cdot,\cdot) = dF(J\cdot,\cdot,\cdot)$.
  - Hull: $\nabla^- = \nabla^{LC} - \frac{1}{2}T$.
  - Hermitian connections (Gauduchon): $\nabla^t = \nabla^{LC} + \frac{1-t}{4}T + \frac{1+t}{4}C$, $t \in \mathbb{R}$. $(\nabla^+ = \nabla^{t=-1}, \nabla^c = \nabla^{t=1})$.
- Solutions to SS & EM:
  - [FIUV09] First explicit solutions based on nilmanifolds (compact quotients of nilpotent Lie groups) with $\alpha' > 0$, constant dilaton and non-flat instanton A.
    - (SS) for $\nabla = \nabla^{LC}, \nabla^+$.
    - (EM) for $\nabla = \nabla^+$ in a nilmanifold with underlying Lie algebra $\mathfrak{h}_3$.
  - [UV15] Explicit solutions for (SS) based on the nilpotent Lie algebra $\mathfrak{h}_{19}^-$ with $\alpha' > 0$, constant dilaton, non-flat instanton and $\nabla = \nabla^C$ in (1).
  - [FIUV14] non-invariant explicit solutions with non-constant dilaton for (SS) on nilmanifolds which are $\mathbb{T}^2$-bundles over $\mathbb{T}^4$.
  - [AG14],[FY15] explicit solutions for (SS) on compact quotient of $SL(2,\mathbb{C})$ with respect to $\nabla^{t<0}$ (flat) and $\nabla^{t<-1}$ (non-flat).

A new family of metric connections

In [OUV16] we define a new family of metric connections:

$\nabla^{\varepsilon,\rho} = \nabla^{LC} + \varepsilon T + \rho C$, $(\varepsilon,\rho) \in \mathbb{R}^2$.
This family includes all the previous connections:
$\nabla^{LC} = \nabla^{0,0}$, $\quad \nabla^- = \nabla^{-1/2,0}$,
$\nabla^+ = \nabla^{1/2,0}$, $\quad \nabla^C = \nabla^{0,1/2}$,
$\nabla^t = \nabla^{\varepsilon,1/2-\varepsilon}$.



Result: $\nabla^{\varepsilon,\rho}$ is Hermitian $\iff \varepsilon + \rho = \frac{1}{2}$.

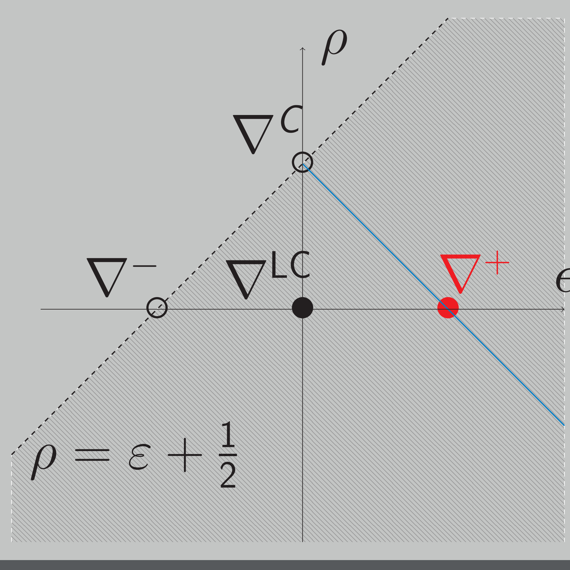
Results: Our strategy

We choose $(M^6, J, F, \Psi, \nabla, A)$, where:
- $M^6 = G\backslash\Gamma$ compact quotient of Lie group G $\Longrightarrow \mathfrak{g}$ associated Lie algebra.
- (J, F, Ψ) is an invariant Hermitian balanced SU(3)-structure on M^6.
- $\nabla = \nabla^{\varepsilon,\rho}$.
- A is an SU(3)-instanton with respect to the previous structure.

Results: nilpotent case

$\mathfrak{h}_3$ is the Lie algebra underlying the product of the 5-dimensional Heisenberg Lie group with $\mathbb{R}$. Consider:
- $J: d\omega^1 = d\omega^2 = 0$, $d\omega^3 = \omega^{1\bar1} - \omega^{2\bar2}$.
- $F_t = \frac{i}{2}(\omega^{1\bar1} + \omega^{2\bar2} + t^2\omega^{3\bar3})$.
- SU(3)-instantons: ∇^+ and A_λ defined by: $(\sigma^{A_\lambda})^1_2 = -(\sigma^{A_\lambda})^3_4 = \lambda(e^5 + e^6)$.

Results (with $\alpha' > 0$):
- (SS) using A_λ with respect to $\nabla^{\varepsilon,\rho}$ if and only if $\rho < \varepsilon + \frac{1}{2}$.
- (EM) if $\nabla = \nabla^+$.



Results: semisimple case

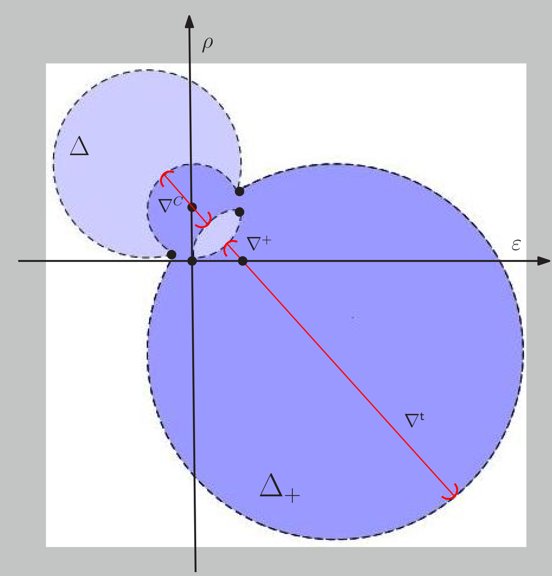
$\mathfrak{sl}(2,\mathbb{C})$ is the Lie algebra of $SL(2,\mathbb{C}) = \{M \in GL(2,\mathbb{C}),\, |\det M = 1\}$.
- $J: d\omega^1 = \omega^{23}$, $\quad d\omega^2 = -\omega^{13}$, $\quad d\omega^3 = \omega^{12}$, $\quad F = \frac{i}{2}(\omega^{1\bar1} + \omega^{2\bar2} + \omega^{3\bar3})$.
- $\nabla^{\varepsilon,\rho}$ is an SU(3)-instanton $\iff \nabla^{\varepsilon,\rho} = \nabla^C$ (flat) or $\nabla^{\varepsilon,\rho} = \nabla^+$ (non-flat).
- Consider:
  $\beta(\varepsilon,\rho) = 1 + 4\varepsilon + 4\varepsilon^2 + 32\varepsilon^3 - 12\rho - 24\varepsilon\rho - 32\varepsilon^2\rho + 36\rho^2 + 32\varepsilon\rho^2 - 32\rho^3$.

Results (with $\alpha' > 0$):
- If $\beta(\varepsilon,\rho) \neq 0$, consider $A = \nabla^c$ and sign (α') = sign $(\beta(\varepsilon,\rho))$. Then: (SS) for ∇^{LC} and $\nabla^{t<0}$ and (EM) for ∇^+.
- If $\beta(\varepsilon,\rho) \neq 0$, consider $A = \nabla^c$ and sign (α') = sign $(\beta(\varepsilon,\rho) - 8)$. Then, (SS) for $\nabla^{t<-1}$.

Results: solvable case

$\mathfrak{g}_7$ is a solvable Lie algebra admitting lattices and complex structures with holomorphically trivial canonical bundle:
- $J: \; d\omega^1 = i\,\omega^1\wedge(\omega^3 + \omega^{\bar3})$, $\; d\omega^2 = -i\,\omega^2\wedge(\omega^3 + \omega^{\bar3})$, $\; d\omega^3 = \omega^{1\bar1} - \omega^{2\bar2}$.
- $2F_u = i(\omega^{1\bar1} + \omega^{2\bar2} + \omega^{3\bar3}) + u\,\omega^{1\bar2} - \bar u\,\omega^{2\bar1}$, $u \in \mathbb{C}$, $|u| < 1$.
- SU(3)-instantons: ∇^+ (only if $u = 0$) and $A_{\lambda,\mu}$ defined by the following connection 1-forms: $(\sigma^{A_{\lambda,\mu}})^1_2 = -(\sigma^{A_{\lambda,\mu}})^3_4 = \lambda e^5 + \mu e^6$.

Results (with $\alpha' > 0$):
- $u = 0$:
  - (SS) using $A_{\lambda,\mu}$ with respect to $\nabla^{\varepsilon,\rho}$ if and only if $\rho < \varepsilon + \frac{1}{2}$.
  - (EM) if $\nabla = \nabla^+$.
- $u \neq 0$: (SS) using $A_{\lambda,\mu}$ with respect to $\nabla^{\varepsilon,\rho}$ if and only if $(\varepsilon,\rho) \in \Delta_+$. In particular for ∇^c.



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