The Gribov problem in Noncommutative Quantum Field Theory

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Outline

- The Gribov problem in standard gauge theory

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- Noncommutative QED on the Moyal plane

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- Perspectives

General setting for first part:

- M smooth manifold (space-time, rigorously, should be compact)
- G finite dimensional Lie group
- $\pi : P \to M$ principal G-bundle

Definition Aut(P)

An *automorphism* of P is a *diffeomorphism* $\varphi : P \to P$ which is G-equivariant, that is $\varphi(p \cdot g) = \varphi(p) \cdot g$ for all $p \in P$ and $g \in G$.

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Definition $\mathcal{G}(P)$

The gauge group of P is $\mathcal{G}(P) :=$ker(H). Its elements are called *gauge transformations* or also vertical automorphisms.

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$$S[A] = \frac{1}{4} Tr \int F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \int A_\mu^a M^{\mu\nu} A_\nu^a$$

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This amounts to choose a surface $\Sigma_f \subset \mathcal{A}$ which possibly intersects the gauge orbits only once: a section for the principal bundle

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The classical action is invariant $\longrightarrow$ insensitive to the gauge fixing.

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To perform the change of variable $[d\alpha] \to [df(A)]$:
insert the Jacobian

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and integrate over [df] with the delta function:

$$[d\mu(\mathcal{A})]\ \mathrm{Det}\Delta\ \ \delta(\mathrm{f}(\mathrm{A}) - \mathrm{h}(\mathrm{x})) = [\mathrm{d}\mu(\mathcal{B})]$$

.

The gauge fixing is not enough to remove unphysical degrees of freedom if the theory is non-Abelian.

- Roughly: consider the gauge orbit

$$A_\mu^g = gA_\mu g^{-1} + \partial_\mu g g^{-1} \simeq A_\mu + D_\mu \alpha$$

with $D_\mu = \partial_\mu + iA_\mu$

- the gauge fixing condition $\partial^\mu A_\mu^g = 0$ yields

$$\partial_\mu D^\mu \alpha = 0$$

which may have nontrivial solutions.

- Notice that $-(\partial_\mu D^\mu)\delta^{(4)}(x-y)\delta^{ab}$ is exactly the FP determinant for this choice of gauge fixing

Partial Summary

- The Gribov problem manifests itself for non Abelian gauge theories.
- It amounts to existence of zero modes of the FP operator, which generate unphysical field configurations.
- These configurations are improperly taken into account in the functional integral .

Back to global approach

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for $G = U(N)$ $\Pi_5 = Z$, $N \ge 3$; $\Pi_5 = Z_2$, $N = 2$; $\Pi_5 = 0$, $N = 1$

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Summary: Non-Abelian gauge theories do not admit global sections

This amounts to the FP operator Δ having non trivial zero modes (the determinant of the Jacobian changes its sign when the surface of gauge fixing meets the gauge orbits more than once).

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$$G^{GZ}(p) \simeq \frac{p^2}{p^4 + a^4}$$

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- and replace with an algebra of functions on $\mathbb{R}^n$, with noncommutative star product

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Notice:

$$x^i \star f = \frac{i}{2} \theta^{ij} \partial_j f$$

The action for a scalar field theory

$$S[\phi] = \int \partial_\mu \phi \star \partial^\mu \phi + \lambda \phi^{\star 4}$$

the integral is a trace

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and extended to tempered distributions by duality.
Θ is block diagonal, antisymmetric with θ_i real.

$$\Theta = \begin{pmatrix} 0 & -\theta_1 & \\ \theta_1 & 0 & \\ & & .. \end{pmatrix}$$

with these defs R^{2n}_θ is unital and involutive. It contains $\mathcal{S}$ and polynomials. Constants are in the center.

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  d, i_{∂_μ} defined algebraically. Forms are constructed by duality.

The differential calculus

- We consider the minimal derivation based differential calculus over the Moyal algebra. [Dubois-Violette, Masson, Wallet]
  $\partial_\mu = -i\theta^{-1}_{\mu\nu}[x^\nu, \cdot]_\star$ generate the Lie algebra of derivations (inner, not a left module)
  d, i_{∂_μ} defined algebraically. Forms are constructed by duality.

- Vector bundles are replaced by right modules over the algebra $\mathbb{R}^{2n}_\theta$, with Hermitian structure h

$$h(m_1 \star f_1, m_2 \star f_2) = f_1^\dagger \star h(m_1, m_2) \star f_2, \quad f_i \in \mathbb{R}^{2n}_\theta, m_i \in \mathcal{H}$$

The connection is defined as

$$\nabla : Der(\mathbb{R}^{2n}_\theta) \times \mathcal{H} \to \mathcal{H}, \quad \nabla_\mu (m \star f) = \nabla_\mu m \star f + m \star \partial_\mu f$$

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U(1) vector bundle is replaced by the right module (one generator)

$$\mathcal{H} = \mathbb{C} \otimes \mathbb{R}_\theta^{2n}$$

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$(h(f_1, f_2) = f_1^\dagger \star f_2)$.
- The gauge connection is defined by its action on the basis

$$\nabla_\mu(\mathbf{1}) \equiv -iA(\partial_\mu) = -iA_\mu$$

so that $\quad \nabla_\mu f = \nabla_\mu(\mathbf{1} \star f) = \partial_\mu f - iA_\mu \star f$

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Properties of the gauge connection

- $(\nabla^A_\mu)^\gamma(\phi) := \gamma(\nabla^A_\mu(\gamma^{-1}\phi)) = U \star \nabla^A_\mu U^{-1} \star \phi$
- $A^U_\mu = U \star A_\mu \star U^{-1} + iU \star \partial_\mu U^{-1}$
- $F_{\mu\nu} = i([\nabla^A_\mu, \nabla^A_\nu] - \nabla^A_{[X_\mu, X_\nu]}) = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$
- $F^U_{\mu\nu} = U \star F_{\mu\nu} \star U^{-1}$

Noncommutative QED on R^{2n}_θ

The natural QED action

$$S = \int d^{2n}x \;\; F_{\mu\nu} \star F^{\mu\nu}$$

is gauge and Poincaré invariant but yields new pathologies w.r.t. the commutative case (UV/IR mixing)

Gribov copies in NC QED

Asymptotycally:

$$(f \star g)(x) = f(x) \exp\left\{ \frac{i}{2} \theta^{\rho\sigma} \overleftarrow{\partial}_\rho \overrightarrow{\partial}_\sigma \right\} g(x)$$

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Under the $U(1)$ gauge transformation in NCQED the gauge field A transforms as

$$A \to A'_\mu[\alpha] = U \star A_\mu \star U^\dagger + i\, U \star \partial_\mu U^\dagger, \qquad U \equiv \exp_\star(i\alpha)$$

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Fourier transforming we get a homogeneous Fredholm equation of second kind

$$\hat{\alpha}(k) = \int d^d q \; Q(q,k) \; \hat{\alpha}(q)$$

with the kernel Q given by

$$Q(q,k) = -\frac{2i\, k^\mu \hat{A}_\mu(k-q)}{k^2} \sin\left(\frac{1}{2}\,\theta^{\rho\sigma} q_\rho k_\sigma\right)$$

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The existence of Gribov copies has been recast into an eigenvalue equation for the operator Q. [properties]

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Unfortunately, this connection is gauge invariant: a fixed point of the gauge group $\rightarrow$ no new copies.

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$$\left(-4k^2 \Box - 8\, \varepsilon^{\mu\nu} \varepsilon^{\eta\lambda} k_\mu k_\eta \partial_\nu \partial_\lambda - \frac{4k^2}{Q\theta} + \theta^2 k^4 \right) \hat{\alpha}(k) = 0$$

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infinite number of solutions in terms of special functions.

$$\hat{\alpha}_{nm}(r,\phi) = (C_1 \cos n\phi + C_2 \sin n\phi) r^{\sqrt{3n^2+1}-1} \exp(-\frac{r^2\theta}{4}) L_m^{\sqrt{3n^2+1}}(\frac{\theta r^2}{2})$$

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can be extended to 4d case.

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• We have explicitly exhibited potentials for which the equation has an infinite number of solutions

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- Scalar case $\phi \to U \star \phi \star U^{-1}$
  - Compute $S[\phi_U] - S[\phi]$ and study the equation of "copies" and the correction to the propagator.
  - Compare with the translation invariant scalar model Gurau, Magnen, Rivasseau, Tanasa '09

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