Worldline time reversal and Massless Supermultiplets

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Based on Worldline CPT and Massless Supermultiplets
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Motivation

Particles vs Fields: which is more useful/fundamental?

- Particles: Dirac, Tomonaga, Feynman (appendix to QED paper)
- Fields: Heisenberg & Pauli, Schwinger

Feynman's diagrams originated from his particle viewpoint. Dyson re-interpreted them as visual aids to Schwinger's field theory calculations. Fields triumphed, until the emergence of ....

String Theory

This was a return to Feynman's viewpoint but with strings instead of particles. A bonus: Worldsheet physics $\rightarrow$ Spacetime physics

This led to a *particle revival* (still ongoing). And a question.

➽ Is there a Worldline physics $\rightarrow$ Spacetime physics bonus?

Unitary irreps of 4D Poincaré

$m^2 > 0$: Classified by m^2 and spin s

Spin from Pauli-Lubanski pseudo-vector $L^m = \frac{1}{2}\epsilon^{mnpq} P_n J_{pq}$

$$L^2 = s(s+1)m^2 \qquad (2s \in \mathbb{Z})$$

$m = 0$: Classified by helicity h

$$L^m = h\, P^m \qquad (2h \in \mathbb{Z})$$

[N.B. Ignore "continuous spin" irreps]

$m^2 < 0$: Tachyons: excluded by susy

CPT

CPT theorem: any **local** relativistic QFT on 4D Minkowski spacetime is CPT invariant $\Rightarrow$ free-field multiplets are CPT self-conjugate

- $m^2 > 0$. Unitary irrep of spin s has helicity content

$$h_s = (-s, -s+1, \cdots, s-1, s)$$

CPT self-conjugate since $CPT : h \to -h$

- $m = 0$. Helicity content for spin $s \neq 0$ is $h_s = (-s, s)$

CPT self-conjugate but **reducible**.

Can we not get the irreducible $s \neq 0$ rep. by quantization of some massless particle action?

Massless Particle

Reparametrization invariant phase-space action for massless particle is

$$S = \int dt \left\{ \dot{x}^m p_m - \frac{1}{2} e\, p^2 \right\}$$

where $e(t)$ is Lagrange multiplier for mass-shell constraint $p^2 = 0$.
Poincaré Noether charges are

$$P_m = p_m \,, \qquad J_{mn} = x_m p_n - x_n p_m$$

The particle has zero spin because $L^m = 0$.

Phase Space Dimension

Phase space has dimension $2 \times 4 = 8$ but physical phase space has dimension $8 - 2 = 6$ because of the mass-shell constraint and the reparametrization invariance.

Worldline CPT

Action is invariant under worldline time reversal, which we can combine with "spacetime reversal" (PT) to get transformations

$$t \to -t\,; \quad e(t) \to e(-t) \quad \left.\begin{array}{c} x^m(t) \to -x^m(-t) \\ p_m(t) \to p_m(-t) \end{array}\right\} \qquad (\dagger)$$

Realized in QT by anti-unitary operator $K = B \circ K_0$
$K_0 = c.c.$ and B is unitary.

➽ $K\Psi$ is charge conjugate of wavefunction Ψ
[**more on this later**]

Moral: The transformation (†) becomes CPT in quantum theory.
Call it "Worldline CPT".

Kramers degeneracy

$K^2 = \pm 1$

$$\left.\begin{array}{lll} \text{K involutive}: & \Rightarrow & K^2 = \epsilon\,, \quad |\epsilon|^2 = 1 \\ K = B \circ K_0 : & \Rightarrow & K^2 = BB^* = B(B^T)^{-1} \end{array}\right\} \Rightarrow B = \epsilon\, B^T \;\; \Rightarrow \;\; \epsilon^2 = 1$$

- $K^2 = 1$. In this case we can impose $K\Psi = \Psi$, a reality condition.

- $K^2 = -1$. In this case Ψ and $K\Psi$ are linearly independent.

If $\left[K, \widehat{\mathrm{Ham}}\right] = 0$ then Ψ and $K\Psi$ are also degenerate.

This is Kramers degeneracy: A doubling of the states in time-reversal invariant theories with $K^2 = -1$

Twistor action [Penrose, Shirafuji]

Solve $p^2 = 0$ in terms of *commuting* Majorana spinor U:

$$p_m = \bar{U}\Gamma_m U \qquad (\Rightarrow \;\; p^2 \equiv 0)$$

$$\Rightarrow \quad \dot{x}^m p_m = \dot{\bar{U}} W + \frac{d}{dt}(\dots)\,, \qquad \boxed{W = \not{x} U} \;\; (\text{incidence relation})$$

Spinor pair (U, W) is a twistor: spinor of conformal group $SU(2,2)$

- **Problem**: U and W are not independent because $\bar{U}\gamma_5 W \equiv 0$
- **Resolution**: impose identity as spin-shell constraint with Lagrange multiplier $\varsigma(t)$, to get twistor action

$$S_0 = \int dt \left\{ \dot{\bar{U}} W - \varsigma U \gamma_5 W \right\}$$

Spin-shell constraint & $U(1)$ gauge invariance

- Poincaré Noether charges are now

$$\left.\begin{array}{r} P_m = p_m(U)\,, \\ J_{mn} = -\frac{1}{2}\bar{U}\Gamma_{mn}W \end{array}\right\} \quad \Rightarrow \quad \boxed{h = \frac{1}{2}\bar{U}\gamma_5 W}$$

Spin-shell constraint $h = 0 \quad \Rightarrow$ particle has zero helicity.

- $\varsigma(t)$ is gauge potential for $\Rightarrow$ chiral $U(1)$ gauge invariance:

$$\begin{array}{rcl} U & \to & e^{\vartheta\gamma_5}U \\ W & \to & e^{-\vartheta\gamma_5}W \\ \varsigma & \to & \varsigma + \dot{\vartheta} \end{array} \qquad (\gamma_5^2 = -1)$$

$\Rightarrow$ Physical phase space dimension $= 2 \times 4 - 2 = 6$, as required

WCS term and non-zero helicity

To get helicity $h = \frac{n}{2}$ just change spin-shell constraint to

$$\frac{1}{2}\left(\bar{U}\gamma_5 W - n\right) = 0$$

Equivalent to adding to action the Worldline Chern Simons term

$$S_{WCS} = n\int dt\, \varsigma(t)$$

In Euclidean path integral, with $t \sim t + T$, we may make large $U(1)$ gauge transformation with parameter $\vartheta(t) = 2\pi t/T$:

$$\delta S_{WCS} = 2\pi\, n \qquad \Rightarrow \quad \delta\left[e^{iS_{WCS}}\right] = 0 \quad \text{iff} \quad \boxed{n \in \mathbb{Z}}$$

$\Rightarrow$ helicity is quantized in units of $\frac{1}{2}$.

Broken CPT

Spin-zero twistor action: $S_0 = \int dt \left\{ \dot{\bar{U}} W - \varsigma U \gamma_5 W \right\}$

Still invariant under worldline CPT. Transformations are now

$$t \to -t\,; \qquad \boxed{\varsigma(t) \to -\varsigma(-t)\,,}$$
$$U(t) \to U(-t)\,, \qquad W(t) \to -W(-t)$$

This implies $h(t) \to -h(-t)$ but $h(t) = h$, constant of motion:

$\Rightarrow \;\; CPT : h \to -h$

But CPT : $\boxed{S_{WCS} \to -S_{WCS}}$ $\;\;\Rightarrow$ non-zero helicity breaks CPT

What about CPT theorem?

No contradiction: wave equation for $\Psi(U)$ is non-local in spacetime.

Worldline time reversal for fermions

Action for "real" anticommuting variable ψ:

$S = \int dt\, i\psi\dot{\psi}$ [factor of i because $(\psi\psi')^* = -(\psi\psi')$]

- **Problem**: $\psi(t) \to \sigma\psi(-t) \;\Rightarrow\; S \to -\sigma^2 S = -S$ for $\sigma = \pm 1$
- **Resolution**: include a factor of i: $\psi(t) \to i\sigma\psi(-t)$

But did we not say that ψ is "real"? No, we said it was ""real"".

➻ "Reality" for anticommuting variables is a formal rule: $\psi^* = \psi$, but $\psi^* = -\psi$ is a physically equivalent rule.

Consistency with QM: $\mathrm{Ham}(t) \to \mathrm{Ham}(-t) \quad \Rightarrow [K, \widehat{\mathrm{Ham}}] = 0$

Spinning particle [Brink, Howe, Di Vecchia; Deser Zumino]

A spin $\frac{1}{2}$ particle in Minkowski$_D$ has classical description via "spinning particle" action, with local worldline supersymmetry. In light-cone gauge

$$S = S_{\rm bosonic} + \int dt\ \frac{i}{2}\boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}}$$

where $\boldsymbol{\lambda}$ is anticommuting $(D-2)$-vector. "Worldline CPT" includes

$$\boldsymbol{\lambda}(t) \to i\eta\,\boldsymbol{\lambda}(-t) \quad \Rightarrow K\hat{\boldsymbol{\lambda}}K^{-1} = \eta\,\hat{\boldsymbol{\lambda}} \qquad (\eta = \pm 1)$$

For $D = 2 + 2n$, one can show that $\boxed{K^2 = \eta^n (-1)^{\frac{1}{2}n(n-1)}}$

Kramers degeneracy and Majorana spinors [Henty, Howe, PKT]

If $K^2 = 1$ is not possible (e.g. $D = 6$) then $K\Psi = \Psi$ is not consistent.
$\Rightarrow$ non-existence of Majorana spinors

4D Susy: massless supermultiplets

Susy algebra

Supersymmetric field theories have $\mathcal{N}$ spinor charges. In Weyl spinor notation they are $\{Q^i_\alpha; \alpha = 1, 2; i = 1, \ldots, \mathcal{N}\}$, and complex conjugates $(Q^i_\alpha)^* \equiv \bar{Q}_{\dot\alpha\, i}$. Algebra is

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = 0\,, \qquad \left\{Q^i_\alpha, \bar{Q}_{\dot\alpha\, j}\right\} = \delta^i_j P_{\alpha\dot\alpha}\,,$$

where $2x2$ Hermitian matrix P is equivalent to 4-momentum.

Massless super-Poincaré irreps

Choose frame $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$ Q^i_1 are $\mathcal{N}$ Clifford algebra creation operators, which we use to build supermultiplet from state $|h\rangle$ annihilated by $\bar{Q}_{1\, i}$. Get $2^{\mathcal{N}}$ states with maximum helicity $h + \frac{\mathcal{N}}{2}$.

Massless Supermultiplet Peculiarities

- $\mathcal{N} = 4$. Starting from $|-1\rangle$ we get the CPT self-conjugate Maxwell multiplet with helicities $h = (-1; -\frac{1}{2} \times 4; \cdots, 1)$
- $\mathcal{N} = 2$. Starting from $|-\frac{1}{2}\rangle$ we get (?) $h = (-\frac{1}{2}, 0, 0, \frac{1}{2})$. In fact, the only CPT self-conjugate $\mathcal{N} = 2$ massless supermultiplet is the **hypermultiplet**

$$h = (-\frac{1}{2}, 0, 0, \tfrac{1}{2}) \times 2 \qquad \textbf{Why } \times \mathbf{2} \text{ ?}$$

- $\mathcal{N} = 1$. Starting from $|s\rangle$ we get the superhelicity s supermultiplet $h = (s, s + \frac{1}{2})$. This is not CPT self-conjugate.

  What happened to the CPT theorem?

4D massless superparticle [Brink, Schwarz]

$\mathcal{N}$-extended superparticle action is

$$S = \int dt \left\{ \left(\dot{x}^m + i \sum_{i=1}^{\mathcal{N}} \bar{\theta}_i \gamma^m \dot{\theta}_i \right) p_m - \frac{1}{2} e\, p^2 \right\},$$

where θ_i are $\mathcal{N}$ anticommuting Majorana spinors (spinor indices supressed)

- Super-Poincaré invariant (with $\mathcal{N}$ spinor charges)
- But also has fermionic gauge invariance [Siegel], which complicates quantization.

Light-cone gauge quantization:

State space is irreducible $\mathcal{N}$-extended supermultiplet

Supertwistors [Ferber]

Supertwistor = spinor of ($\mathcal{N}$-extended) superconformal group

Equivalently: complex $(4|n)$-plet of $SU(2,2|N)$:

$$(U, W; \xi^i) \qquad (i = 1, \ldots, \mathcal{N})$$

For superparticle, $\boxed{\xi_i = \bar{U}(\mathbb{I} + i\gamma_5)\theta_i}$ (anticommuting scalars)

➽ Superparticle action in supertwistor variables is

$$S = \int dt \left\{ \dot{\bar{U}} W + i \sum_{i=1}^{\mathcal{N}} \bar{\xi}_i \dot{\xi}_i - 2\varsigma H \right\} \qquad H = h - \frac{1}{2} \sum_{i=1}^{\mathcal{N}} \bar{\xi}_i \xi_i \,.$$

H is "classical superhelicity". Constraint is $H = 0$.

Quantum supertwistors [Shirafuji]

In quantum theory, *with standard operator ordering*,

$$\sum_{i=1}^{\mathcal{N}} \bar{\xi}_i \xi_i \to \frac{1}{2} \sum_{i=1}^{\mathcal{N}} \left[\xi_i^\dagger, \xi_i\right] = \sum_{i=1}^{\mathcal{N}} \hat{\nu}_i - \frac{\mathcal{N}}{2}$$

The $\hat{\nu}_i$ are fermi number operators: $\nu_i = 0, 1$. Quantum constraint $\hat{H}\Psi = 0$ gives helicities:

$$h = \frac{1}{2} \sum_{i=1}^{\mathcal{N}} \nu_i - \frac{\mathcal{N}}{4}$$

- $\mathcal{N} = 4$: $h = (-1, -\frac{1}{2}\times 4, 0\times 6, \frac{1}{2}\times 4, 1)$. Maxwell supermultiplet.

CPT self-conjugate because $CPT : H \to -H$.

Kramers degeneracy and the hypermultiplet

Twistor action has $\mathcal{N}$ fermi oscillators. For even $n \equiv \mathcal{N}$

$$K^2 = (-1)^{\mathcal{N}/2} = \begin{cases} 1 & \mathcal{N} = 0, 4, 8, \ldots \\ -1 & \mathcal{N} = 2, 6, \ldots \end{cases}$$

- $\mathcal{N} = 4$: $K^2 = 1 \quad \Rightarrow \; 2^{\mathcal{N}} = 2^4 = 8 + 8 \text{ states}$.
  Maxwell supermultiplet
- $\mathcal{N} = 2$: $K^2 = -1 \quad \Rightarrow \; 2 \times 2^{\mathcal{N}} = 2^3 = 4 + 4 \text{ states}$
  Hypermultiplet

Doubling due to Kramers degeneracy

$\mathcal{N} = 6 \quad \Rightarrow \; K^2 = -1$ for gravitino supermuliplet

✓ Scalars in complex **20** of $SU(6)$.

$\mathcal{N} = 1$ and the CPT anomaly

For standard operator ordering: $h = \frac{\nu}{2} - \frac{1}{4}$. This gives $h = (-\frac{1}{4}, \frac{1}{4})$, but then $2h \notin \mathbb{Z}$. To get $2h \in \mathbb{Z}$ we must add WCS term, e.g.

$$S_{WCS} = \left(n + \tfrac{1}{2}\right) \int dt\, s \qquad (n \in \mathbb{Z})$$

Now we get superhelicity n: $h = \left(n, n + \frac{1}{2}\right)$, but

- WCS coefficient $\neq 0$ for any n $\Rightarrow$ CPT anomaly

Analogy with 3D Parity anomaly [Redlich]

Integrating out single fermi oscillator gives WCS term with coefficient $\pm\frac{1}{2}$. This violates **large** $U(1)$ gauge invariance. Cancel gauge anomaly by adding "classical" WCS term $\Rightarrow$ CPT anomaly.

Canceling the CPT anomaly

- To avoid global $U(1)$ anomaly of $\mathcal{N} = 1$ superparticle action S_0, we added WCS term

$$S[Z;s] \equiv S_0[U, W; \xi; s] \pm \frac{1}{2} \int dt\, s$$

- Now cancel CPT anomaly by analogy with 3D Parity anomaly [Hagen] Take action

$$S[Z, \tilde{Z}; s, \tilde{s}] = \left(S_0 + \tilde{S}_0\right) + \frac{1}{2} S_{WCS}[s] - \frac{1}{2} S_{WCS}[\tilde{s}]$$

Quantzation now gives (for $n = 0$) reducible but CPT conjugate Wess-Zumino supermultiplet $h = \left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)$.

Conclusion: all peculiarities of massless supermultiplets can be understood from a worldline perspective!