A geometric Hamilton–Jacobi theory and Nambu–Poisson structures

Cristina Sardón
ICMAT-CSIC

XXV International Fall Workshop, in Geometry and Physics, Madrid 2016

Contents

What is a Hamilton Jacobi equation?

The Hamilton–Jacobi equation (HJE) is a differential equation in partial derivatives used in Classical Mechanics to obtain equations of motion.

$$H + \frac{\partial S}{\partial t} = 0 \tag{1}$$

where S is called a **Hamilton's principal function** and it is a function of $S(q_1, \ldots, q_n, t)$ of the n coordinates that define the configuration of the system and time. Here H is the Hamiltonian of the system

$$H\left(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_1}, t\right) \tag{2}$$

where the conjugate momenta correspond with the first derivatives of the funcion S with respect to the coordinates.

Where does the HJE derive from?

Their derivation comes from a generating type 2 function, such that

$$(q_1, \dots, q_n, p_1, \dots, p_n) \to (Q_1, \dots, Q_n, P_1, \dots, P_n) \tag{3}$$

the initial variables are turned into others in which a new Hamiltonian is retrieved as

$$K(Q, P, t) = H(q, p, t) + \frac{\partial F_2(q, P, t)}{\partial t}, \quad p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P} \tag{4}$$

and the new Hamilton equations take the form

$$\dot{Q} = \frac{\partial K}{\partial P}, \quad \dot{P} = -\frac{\partial K}{\partial Q} \tag{5}$$

We choose a F_2 such that $K = 0$ so the Hamilton equations become trivial

$$\dot{P} = 0, \quad \dot{Q} = 0 \tag{6}$$

This means that the new coordinates are generalized constants of motion $P = \alpha$ and $Q = \beta$.

Where does the HJE derive from?

Setting $F_2(q, \alpha, t) = S(q, t) + A$, where A is an arbitrary constant, then the HJE appears explicitly,

$$p = \frac{\partial F_2}{\partial q} = \frac{\partial S}{\partial q} \to H(q, p, t) + \frac{\partial F_2}{\partial t} = 0 \tag{7}$$

and then,

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{8}$$

Hamilton's principal action and the Hamiltonian function are related to action

$$\frac{dS}{dt} = \sum_i \frac{\partial S}{\partial q^i}\dot{q}^i + \frac{\partial S}{\partial t} = \sum_{i=1} p_i \dot{q}_i - H = L \to \tag{9}$$

$$S = \int L dt \tag{10}$$

If the Hamiltonian does not depend on time, then $W = S + Et$

$$H\left(q, \frac{\partial S}{\partial q}, t\right) = E \tag{11}$$

Example: The eikonal equation

The eikonal equation is a nonlinear PDE encountered in problems of wave propagation. It provides a link between physical (wave) optics and geometric (ray) optics. The Hamiltonian is

$$H(t, q, p) = -\sqrt{n^2 - p_1^2 - p_2^2} \tag{12}$$

The HJE for this Hamiltonian is

$$-\sqrt{n^2 - \left(\frac{\partial \Phi}{\partial q^1}\right)^2 - \left(\frac{\partial \Phi}{\partial q^2}\right)^2} + \frac{\partial \Phi}{\partial t} = 0 \tag{13}$$

which equals

$$\left(\frac{\partial \Phi}{\partial q^1}\right)^2 + \left(\frac{\partial \Phi}{\partial q^2}\right)^2 + \left(\frac{\partial \Phi}{\partial t}\right)^2 = n^2 \tag{14}$$

that is,

$$|\nabla \phi|^2 = n^2 \tag{15}$$

Geometric Lagrangian Mechanics

Consider the triple $(Q, TQ, \tau_Q : TQ \to Q)$. A Lagrangian is a function $L : TQ \to \mathbb{R}$, where $L = L(q^i, \dot{q}^i)$ with (q^i) being coordinates on the manifold Q and $(\dot{q}^i)$ are the corresponding velocities. We introduce the *Poincaré–Cartan 1-form*

$$\theta_L = S^*(dL) = \frac{\partial L}{\partial \dot{q}^i} dq^i,$$

where $S = \frac{\partial}{\partial \dot{q}^i} \otimes dq^i$ and *Poincaré–Cartan two-form* is defined as $\omega_L = -d\theta_L$ The total energy of the system corresponds with

$$E_L = \Delta(L) - L \in C^\infty(TQ),$$

with $\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$. We say that $L(q, \dot{q}^i)$ is *regular* if the Hessian matrix

$$(W_{ij}) = \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right) \tag{16}$$

is invertible. From here, we recover classical expressions

$$\omega_L = dq^i \wedge dp_i, \quad \text{such that} \quad p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad E_L = \dot{q}^i p_i - L.$$

Geometric Euler-Lagrange equations

The Euler–Lagrange equations can be written in the symplectic way as

$$\iota_{\xi_L}\omega_L = dE_L \tag{17}$$

whose solution ξ_L is called the *Euler–Lagrange vector field*. If we write the Euler–Lagrange vector field,

$$\xi_L = \dot{q}^i \frac{\partial}{\partial q^i} + \xi_i(q_i, \dot{q}_i)\frac{\partial}{\partial \dot{q}^i} \tag{18}$$

its integral curves $(q^i(t), \dot{q}^i(t))$ are lifts of their projections $q(t)$ on Q and are solutions of the system of differential equations

$$\frac{dq^i(t)}{dt} = \dot{q}^i, \quad \frac{d\dot{q}^i(t)}{dt} = \xi^i,$$

The curves $q(t)$ in Q are called the solutions of ξ_L that correspond with the solutions of the Euler–Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i}, \quad 1 \leqslant i \leqslant n = \dim Q \tag{19}$$

Geometric Hamiltonian Mechanics

The passing from the Lagrangian to the Hamiltonian setting $(Q, T^*Q, \pi_Q : T^*Q \to Q)$ is introduced by a Legendre transformation, as the fibered mapping $FL : TQ \to T^*Q$ such that $\pi_Q \circ FL = \tau_Q$. The Hamiltonian is retrieved through $H = E_L \circ FL^{-1}$.

If ω_Q is the canonical sympletic form on T^*Q where (q^i, p_i) are the canonical coordinates T^*Q. then $\omega_Q = dq^i \wedge dp_i$ and therefore

$$\iota_{X_H}\omega_Q = dH \tag{20}$$

and it provides the Hamiltonian vector field X_H on T^*Q such that its integral curves $(q^i(t), p_i(t))$ satisfy the Hamilton equations

$$\begin{cases} \dot{q}^i = \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial H}{\partial q^i} \end{cases} \tag{21}$$

for all $i = 1, \ldots, n$. The Legendre transformation maps solutions ξ_L to solutions of X_H since $(FL)^*\omega_Q = \omega_L$ and that ξ_l and X_H are FL-related.

A geometric HJE

If the principal function is separable in time dependence, then we can make the Ansatz $S = W(q^1, \ldots, q^n) - Et$ where E is the total energy of the system.

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = E. \tag{22}$$

If we find a solution W, then any solution of the Hamilton equations is retrieved by $p_i = \frac{\partial W}{\partial q^i}$. Geometrically, this can be interpreted through a diagram

$$\begin{tikzcd} T^*Q \arrow[r, "X_H"] \arrow[d, "\pi"] & TT^*Q \arrow[d, "T\pi"] \\ Q \arrow[r, "X_H^{dW}"] \arrow[u, bend left, "dW"] & TQ \end{tikzcd}$$

This implies

$$X_H^{dW} = T\pi \circ X_H \circ dW \tag{23}$$

Nambu–Poisson structures

Nambu–Poisson structures (NP) arose to deal with Hamiltonian systems equipped with two or more Hamiltonian functions.

$$[A, B, C] = \frac{\partial(A, B, C)}{\partial(x, y, z)}$$

with the canonical variables satisfying $[x, y, z] = 1$, that could be interpreted as a bracket defined by the canonical volume form in $\mathbb{R}^3$. This bracket attracted a lot of scientific attention at that time.

An extension to manifolds has been developed by L. Takhtajan. Here, the geometric structure is provided by a contravariant tensor field Λ of order n.

Euler equations for a rotator

This generalization was thought in terms of two Hamiltonians and three dimensional phase space. The potential usefulness of this formalism was rooted in the Euler equations of a rotator.

$$\dot{x} = \frac{\partial(H,G)}{\partial(y,z)}, \quad \dot{y} = \frac{\partial(H,G)}{\partial(x,z)}, \quad \dot{z} = \frac{\partial(H,G)}{\partial(x,y)} \tag{24}$$

The evolution of a quantity F is given by

$$\frac{dF}{dt} = \nabla F(\nabla H \times \nabla G) = \frac{\partial(F.G.H)}{\partial(x,y,z)} \tag{25}$$

and the Hamiltonians are

$$G = \frac{1}{2}\left(\frac{L_x^2}{I_x} + \frac{L_y^2}{I_y} + \frac{L_z^2}{I_z}\right), \quad H = \frac{1}{2}(L_x^2 + L_y^2 + L_z^2) \tag{26}$$

An almost NP structure

Let us consider an **almost NP**, i.e., the pair (E, Λ) where E is a differentiable manifold of dimension m equipped with a $(n, 0)$-skew symmetric contravariant tensor Λ ($m \geqslant n$). The tensor Λ defines the vector bundle morphism $\sharp : \Lambda^{n-1}(T^*E) \to TE$ by $< \sharp(\alpha), \beta >= \Lambda(\alpha, \beta)$, where $\alpha \in \Lambda^{n-1}(T^*E)$ and $\beta \in T^*E$.

The bracket induced by Λ on $C^\infty(E)$ is defined as

$$\{f_1, \dots, f_n\} = \Lambda(df_1, \dots, df_n), \qquad f_1, \dots, f_n \in C^\infty(E) \tag{27}$$

This bracket has the following properties

1. $\{f_1, \dots, f_n\} = (-1)^{\epsilon(\sigma)}\{f_{\sigma(1)}, \dots, f_{\sigma(n)}\}$, with $\sigma \in \mathrm{Symm}(n)$ and $\epsilon(\sigma)$ is the parity of the permutation;
2. $\{f_1 g_1, \dots, f_n\} = f_1\{g_1, \dots, f_n\} + g_1\{f_1, \dots, f_n\}$,

which are the skew-symmetry and Leibnitz rule, correspondingly.

Nambu–Poisson tensors

Consider now an **almost Nambu–Poisson manifold** (E, Λ) with $m \geq n \geq 3$. To have dynamics, we are provided with $C^\infty(E)$ hamiltonian functions $f_1, \ldots, f_{n-1} : E \to \mathbb{R}$ whose corresponding vector field is the Hamiltonian vector field

$$X_{f_1,\ldots,f_{n-1}} = \sharp(df_1 \wedge \cdots \wedge df_{n-1}).$$

When all these vector fields are derivations of the algebra $C^\infty(E) \times \cdots \times C^\infty(E)$, that is, the following identity, known as *fundamental identity* introduced by Takhtajan, holds

$$X_{f_1,\ldots,f_{n-1}}\{g_1, \ldots, g_n\} = \sum_{i=1}^{n} \{g_1, \ldots, X_{f_1,\ldots,f_{n-1}} g_i, \ldots, g_n\} \qquad (28)$$

for all functions $f_1, \ldots, f_{n-1}, g_1, \ldots, g_n \in C^\infty(E)$ on E, then, (E, Λ) is called a **Nambu–Poisson manifold** and Λ is a **Nambu–Poisson tensor**.

Key theorem

Theorem

Let (E, Λ) be a generalized m-dimensional almost Poisson manifold of order $n \geqslant 3$.

- *If Λ is an almost NP tensor, $\mathcal{D}$ is not involutive.*
- *If Λ is a NP tensor, then the distribution $\mathcal{D}$ is completely integrable and defines a foliation on E st. when Λ is restricted to leaves of the foliation, there exist induced NP structures in each leaf. The leaves are of two kinds, for a point $x \in E$, if $\Lambda(x) \neq 0$, then the leave passing through x has dimension n and the induced NP structure derives from a volume form.*

$$\Lambda = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}. \tag{29}$$

Associated with this tensor, there exists a volume form which can be locally written as

$$\Omega = dx^1 \wedge \cdots \wedge dx^n. \tag{30}$$

If $\Lambda = 0$, then the leaf reduces to a point x and the induced Nambu–Poisson structure is trivial.

Lagrangian submanifolds

Let (E, Λ) be a Nambu–Poisson manifold with $m > n \geq 3$, we say that a submanifold $N \subset E$ is j-Lagrangian $\forall x \in N, 1 \leqslant j \leqslant n-1$ if

$$\sharp \mathrm{Ann}^j(T_x N) = \sharp(\Lambda^{n-1}(T_x^* E)) \cap T_x N, \tag{31}$$

where the annihilator is defined as

$$\mathrm{Ann}^j(T_x N) = \{\alpha \in \Lambda^{n-1}(T_x^* E)| \quad \iota_{v_1 \wedge \cdots \wedge v_j} \alpha = 0, \forall v_1, \ldots, v_j \in T_x N\}. \tag{32}$$

The following inclusions are clearly fulfilled

$$\mathrm{Ann}^1(T_x N) \subseteq \mathrm{Ann}^2(T_x N) \subseteq \cdots \subseteq \mathrm{Ann}^{n-1}(T_x N). \tag{33}$$

Volume NP structures

We consider a **volume manifold** as a pair (E, Ω), where Ω is a volume form on the differentiable n dimensional manifold E. There is an associated $(n, 0)$-skew symmetric tensor Λ_Ω defined as

$$\Lambda_\Omega(df_1 \dots df_n) = \{f_1, \dots, f_n\} \tag{34}$$

where the bracket is defined by

$$\{f_1, \dots, f_n\}\Omega = df_1 \wedge \cdots \wedge df_n. \tag{35}$$

A particular example is $E \simeq \mathbb{R}^n$ with canonical coordinates $\{x^i, i = 1, \dots, n\}$. Here, the canonical volume form is written as $\Omega_{\mathbb{R}^n} = dx^1 \wedge \cdots \wedge dx^n$ and

$$\{f_1, \dots, f_n\} = \begin{vmatrix} \frac{\partial f_1}{\partial x^1} & \dots & \frac{\partial f_1}{\partial x^n} \\ \frac{\partial f_2}{\partial x^1} & \dots & \frac{\partial f_2}{\partial x^n} \\ \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x^1} & \dots & \frac{\partial f_n}{\partial x^n} \end{vmatrix} \tag{36}$$

HJE on volume NP

Theorem

Given a volume Nambu–Poisson structure (E, Ω) of dimension n, every submanifold of codimension 1 is $(n-1)$-Lagrangian.

Given a Nambu–Poisson structure (E, Λ), consider the map $\sharp : \Lambda^{n-1}(E) \to \mathfrak{X}(E)$ induced by Λ. Let us choose a set of functions $f_1, \dots, f_{n-1}$ in $C^\infty(E)$ and define the pairing

$$< \sharp(df_1 \wedge \cdots \wedge df_{n-1}), df_n >= \Lambda(df_1, \dots, df_{n-1}, df_n),$$

where $df_1, \dots, df_{n-1}, df_n \in \Omega^1(E)$ are one-forms in E. The characteristic distribution in this case is $\mathcal{D}_x = \sharp\Lambda^{n-1}(T_x^* E)$ and the associated Hamiltonian vector field is defined by

$$X_{f_1,\dots,f_{n-1}} = \sharp(df_1 \wedge \cdots \wedge df_{n-1}).$$

HJE on volume NP

In particular, we are interested in scenarios with a volume Nambu–Poisson structure (E, Ω) with *dim* $E = n$, whose dynamics is interpreted in terms of $(n-1)$-Hamiltonian functions $H_1, \ldots, H_{n-1} \in C^\infty(E)$, in which the Hamilton–Jacobi theory is applicable.

Here, we assume that the n-dimensional manifold E fibers over a manifold N of dimension $n-1$, say $\pi : E \to N$ is a fibration. Given a section γ of π, that is, $\gamma : N \longrightarrow E$ is such that $\pi \circ \gamma = Id_N$, then $\gamma(N)$ is a submanifold of E with codimension 1. The vector field $X^\gamma_{H_1,\ldots H_{n-1}}$ is then defined as

$$X^\gamma_{H_1,\ldots H_{n-1}} = T\pi \circ X_{H_1,\ldots H_{n-1}} \circ \gamma \qquad (37)$$

Theorem

The vector fields $X_{H_1,\ldots,H_{n-1}}$ and $X^\gamma_{H_1,\ldots,H_{n-1}}$ are γ-related if and only if the following equation is satisfied

$$d(H_1 \circ \gamma) \wedge \cdots \wedge d(H_{n-1} \circ \gamma) = 0. \qquad (38)$$

Geometric diagram for NP

The following diagram summarizes the above construction

$$
\begin{tikzcd}
(E,\Omega) \arrow[rr, "X_{H_1,\dots H_{n-1}}"] \arrow[d, "\pi"] & & TE \arrow[d, "T\pi"] \\
N \arrow[u, bend left, "\gamma"] \arrow[rr, "X^{\gamma}_{H_1,\dots H_{n-1}}"] & & TN
\end{tikzcd}
$$

N-coupled Riccati equations

The first-order Riccati equations appear as a system of equations containing n copies of same Ricatti equation. This system is

$$\dot{x}^{i_k} = a_0(t) + a_1(t)x^{i_k} + a_2(t){x^{i_k}}^2, \qquad i_k = 1, \dots, n \tag{39}$$

defined on $\mathcal{O} = \{(x_1, \dots, x_n)|(x_1 - x_2)\dots(x_{n-1} - x_n) \neq 0 \subset \mathbb{R}^n\}$.
The associated t-dependent vector field with this system is

$$X_t = \sum_{i_k=1}^{n} \left(a_0(t) + a_1(t)x^{i_k} + a_2(t){x^{i_k}}^2\right) \frac{\partial}{\partial x^{i_k}}.$$

A method to obtain $n-1$ presymplectic forms from which to derive Hamiltonians, is the permutation of indices. By fixing one of the coordinates, let us say l, with $l = 1, \dots, n-1$

$$\omega^{[l]} = \sum_{k<l}^{l-1} \frac{dx^k \wedge dx^l}{(x^k - x^l)^2} + \sum_{k>l}^{n} \frac{dx^l \wedge dx^k}{(x^l - x^k)^2}, \tag{40}$$

for a fixed l.

N-coupled Riccati equations

Due to its own construction, these are closed forms $d\omega^{[l]} = 0$. Equivalently, we can derive Hamiltonian functions associated with each $\omega^{[l]}$

$$
\begin{aligned}
h^{[l]} = a_0(t) &\left(\sum_{k>l}^{n} \frac{1}{x^l - x^k} + \sum_{k<l}^{l-1} \frac{1}{x^k - x^l} \right) \\
&+ \frac{a_1(t)}{2} \left(\sum_{k>l}^{n} \frac{x^l + x^k}{x^l - x^k} + \sum_{k<l}^{l-1} \frac{x^l + x^k}{x^k - x^l} \right) \\
&+ a_2(t) \left(\sum_{k>l}^{n} \frac{x^l x^k}{x^l - x^k} + \sum_{k<l}^{l-1} \frac{x^l x^k}{x^k - x^l} \right)
\end{aligned}
$$

for every fixed $l = 1, \ldots, n-1$.

N-coupled Riccati equations

According to the Nambu–Poisson theory, equations (39) must be retrived through the computation

$$\dot{x}^{i_k}=\{h^{[1]},\dots,h^{[n-1]},x^{i_k}\},\quad i_k=1,\dots,n. \tag{41}$$

and the n-dimensional bracket (36) takes the form

$$\dot{x}^{i_k}=\{h^{[1]},h^{[2]},\dots,h^{[n-1]},x^{i_k}\}=\begin{vmatrix} \frac{\partial h^{[1]}}{\partial x^1} & \frac{\partial h^{[1]}}{\partial x^2} & \dots & \frac{\partial h^{[1]}}{\partial x^n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial h^{[k]}}{\partial x^1} & \frac{\partial h^{[k]}}{\partial x^2} & \dots & \frac{\partial h^{[k]}}{\partial x^n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial h^{[n-1]}}{\partial x^1} & \frac{\partial h^{[n-1]}}{\partial x^2} & \dots & \frac{\partial h^{[n-1]}}{\partial x^n} \\ \frac{\partial x^{i_k}}{\partial x^1} & \frac{\partial x^{i_k}}{\partial x^2} & \dots & \frac{\partial x^{i_k}}{\partial x^n} \end{vmatrix}$$

such that if we compute the determinant, we obtain

$$\dot{x}^{i_k}=(-1)^{i_k+n}\sum_{\sigma_{i_1,\dots,i_n}}(-1)^{\left(\frac{n(n-1)}{2}+i_1+\dots+i_n\right)}\frac{\partial h^{[1]}}{\partial x^{i_1}}\cdots\frac{\partial h^{[n-1]}}{\partial x^{i_n}}, \tag{42}$$

when $i_1,\dots,i_{n-1}=1,\dots,n\neq i_k$ and a particular i_k that takes any value $1,\dots,n$.

N-coupled Riccati equations

We need to conformally transform the canonical volume form $\Omega = dx^1 \wedge \cdots \wedge dx^n$ associated with the former problem (39) into another volume form $\bar{\Omega}$ corresponding with (42).

There exists a change of coordinates $\overline{x}^{\hat{j}} = f_{\hat{j}}(x^j)x^j$ for all $\hat{j} = 1, \ldots, n$ through which we derive a compatible $\overline{\Omega}$ compatible with (42) that maps (42) into (39). It takes the following form

$$\overline{\Omega} = \prod_{\hat{j}=1,\ldots,n} f_{\hat{j}} dx^1 \wedge \cdots \wedge dx^n \tag{43}$$

with

$$f_{\hat{j}} = \delta_{\hat{j}}^{[l]} \left(\sum_{l>k}^{l-1} \frac{1}{(x^k - x^l)^2} - \sum_{k>l}^{n} \frac{1}{(x^l - x^k)^2} \right) + \bar{\delta}_{\hat{j}}^{[l]} \left(\sum_{k>l}^{n} \frac{1}{(x^l - x^k)^2} - \sum_{k<l}^{l-1} \frac{1}{(x^k - x^l)^2} \right)$$

Therefore, the canonical n-dimensional Nambu–Poisson bracket takes the expression

$$\{x^1, \ldots, x^n\} = \frac{1}{\prod_{\hat{j}=1,\ldots,n} f_{\hat{j}}}. \tag{44}$$

HJE for ND Riccati equations

If we want to apply the Hamilton–Jacobi theory to this example, we have the diagram

$$\begin{tikzcd}
E = \mathcal{O} \arrow[rr, "X_{h^1\dots h^{n-1}}"] \arrow[d, "\pi"] & & TE \arrow[d, "T\pi"] \\
N \arrow[rr, "X^{\gamma}_{h^1\dots h^{n-1}}"] \arrow[u, bend left, "\gamma"] & & TN
\end{tikzcd}$$

where $\mathcal{O} = \{(x_1, \dots, x_n) | (x_1 - x_2) \dots (x_{n-1} - x_n) \neq 0 \subset \mathbb{R}^n\}$ and $N = \{(x_1, \dots, x_n) | (x_1 - x_2) \dots (x_{n-2} - x_{n-1}) \neq 0 \subset \mathbb{R}^{n-1}\}$.

The vector field $X_{h^1\dots h^{n-1}}$ can be obtained by performing the calculation

$$X_{h^1\dots h^{n-1}} = \sharp(dh^1 \wedge \dots \wedge dh^{n-1}) \tag{45}$$

HJE for ND Riccati equations

In this way, it takes the expression

$$X_{h^1\dots h^{n-1}} = \sharp\left(\sum_{\sigma_{i_1\dots i_n}} (-1)^{\frac{n(n-1)}{2}+1_1+\dots+i_n} \frac{\partial h^{[1]}}{\partial x^{i_1}} \dots \frac{\partial h^{[n-1]}}{\partial x^{i_n}} \frac{dx^{i_1}\wedge\dots\wedge dx^{i_n}}{(\prod f^{i_1\dots i_n})_{i_1,\dots,i_n}}\right)$$

$$\sum_{\sigma_{i_1\dots i_n}} (-1)^{\frac{n(n-1)}{2}+1_1+\dots+i_n} \frac{\partial h^{[1]}}{\partial x^{i_1}} \dots \frac{\partial h^{[n-1]}}{\partial x^{i_n}} \frac{\delta_1^{i_1}\dots\delta_{n-1}^{i_n}}{(\prod f^{i_1\dots i_n})_{i_1,\dots,i_n}} \frac{\partial}{\partial x^{i_k}}$$

where $i_1,\dots,i_n \neq i_k = 1,\dots,n$. On the other hand, the vector field

$$\mathrm{T}_\gamma\left(X^\gamma_{h^1\dots h^{n-1}}\right) = \sharp\left(\sum_{\sigma_{i_1\dots i_n}} (-1)^{\frac{n(n-1)}{2}+1_1+\dots+i_n} \frac{\partial h^{[1]}}{\partial x^{i_1}} \dots \frac{\partial h^{[n-1]}}{\partial x^{i_n}} \frac{dx^{i_1}\wedge\dots\wedge dx^{i_n}}{(\prod f^{i_1\dots i_n})_{i_1,\dots,i_n}}\right)$$

$$\sum_{\sigma_{i_1\dots i_n}} (-1)^{\frac{n(n-1)}{2}+1_1+\dots+i_n} \frac{\partial h^{[1]}}{\partial x^{i_1}} \dots \frac{\partial h^{[n-1]}}{\partial x^{i_n}} \frac{\delta_1^{i_1}\dots\delta_{n-1}^{i_n}}{(\prod f^{i_1\dots i_n})_{i_1,\dots,i_n}} \left(\frac{\partial}{\partial x^{i_k}} + \frac{\partial\gamma^n}{\partial x^{i_k}}\frac{\partial}{\partial x^n}\right)$$

where $i_1,\dots,i_n \neq i_k$ and $i_k = 1,\dots,n-1$ whilst $i_1,\dots,i_n = 1,\dots,n$.

HJE for ND Riccati equations

And we have chosen γ in such a way as $\gamma(x^1,\dots,x^{n-1},\gamma^n(x^1,\dots,x^{n-1}))$.
So, the **Hamilton–Jacobi equation** for this case reads

$$\sum_{i_k}^{n-1}\sum_{\sigma_{i_1\dots i_n}}^{1\leq i_j\leq n-1}(-1)^{\frac{n(n-1)}{2}+i_1+\dots+i_n}\frac{\partial h^{[1]}}{\partial x^{i_1}}\dots\frac{\partial h^{[n-1]}}{\partial x^{i_n}}\frac{\delta_1^{i_1}\dots\delta_{n-1}^{i_n}}{\left(\prod f^{i_1\dots i_n}\right)_{i_1,\dots,i_n}}\gamma^n_{i_k}$$
$$+\sum_{\sigma_{i_1\dots i_n}}^{1\leq i_j\leq n}(-1)^{\frac{n(n-1)}{2}+i_1+\dots+i_n}\frac{\partial h^{[1]}}{\partial x^{i_1}}\dots\frac{\partial h^{[n-1]}}{\partial x^{i_n}}\frac{\delta_1^{i_1}\dots\delta_{n-1}^{i_n}}{\left(\prod f^{i_1\dots i_n}\right)_{i_1,\dots,i_n}}=0,$$

where $\gamma^n_{i_k}$ means $\frac{\partial\gamma^n}{\partial x_{i_k}}$.

R. Ibañez, M. de León, J.C. Marrero, D. Martín de Diego, *Dynamics of generalized Poisson and Nambu–Poisson brackets,* J. Math. Phys. **38**, 2332 (1997).

M. de León, C. Sardón, *A geometric Hamilton–Jacobi theory for a Nambu–Poisson structure* https://arxiv.org/abs/1604.08904

Y. Nambu, *Generalized Hamiltonian dynamics, Phys. Rev. D* **7**, 2405-2412 (1973).

L. Takhtajan, *On foundation of the generalized Nambu Mechanics,* Comm. Math. Phys. **160**, 295–315 (1994).

Thanks for your attention!