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Graded geometry in physics – above and beyond

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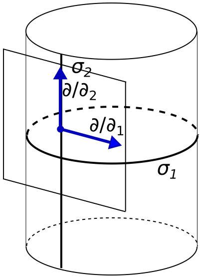
Graded geometry in physics and mechanics

- Graded geometry and generalized geometry:
  Q-manifolds/bundles,
  (twisted) Poisson manifolds, Dirac structures

- Application to sigma models:
  gauging via equivariant Q-cohomology; supersymmetry
  (j/w Thomas Strobl – Lyon, Alexei Kotov – Curitiba,
  Jean-Philippe Michel – Louvain)

- Application to mechanics:
  port-Hamiltonian systems and Dirac structures
  (in progress, j/w Aziz Hamdouni – La Rochelle)

Graded manifolds – example



Consider functions on $T[1]\Sigma$.

$\sigma^1, \ldots, \sigma^d$ – coordinates on Σ:
$deg(\sigma^\mu) = 0$, $\sigma^{\mu_1}\sigma^{\mu_2} = \sigma^{\mu_2}\sigma^{\mu_1}$.
$deg(h(\sigma^1, \ldots, \sigma^d)) = 0$.

$\theta^1, \ldots, \theta^d$ – fiber linear coordinates:
$deg(\theta^\mu) := 1$, $\theta^{\mu_1}\theta^{\mu_2} = -\theta^{\mu_2}\theta^{\mu_1}$

Arbitrary homogeneous function on $T[1]\Sigma$ of $deg = p$:
$f = f_{\mu_1\ldots\mu_p}(\sigma^1, \ldots, \sigma^d)\theta^{\mu_1}\ldots\theta^{\mu_p}$.

Graded commutative product: $f \cdot g = (-1)^{deg(f)deg(g)} g \cdot f$

$f \leftrightarrow \omega = f_{\mu_1\ldots\mu_p} \mathrm{d}\sigma^{\mu_1} \wedge \cdots \wedge \mathrm{d}\sigma^{\mu_p} \in \Omega(\Sigma)$

$\rightarrow$ <u>"Definition" of a graded manifold</u>
– manifold with a ($\mathbb{Z}$-)grading defined on the sheaf of functions.

Graded manifolds/Q-manifolds (DG-manifolds)

D. Roytenberg: "...graded manifolds are just manifolds with a few bells and whistles..."

$T[1]\Sigma$, $deg(\sigma^\mu) = 0$, $deg(\theta^\mu) = 1$, $f_{\mu_1 \ldots \mu_p}(\sigma^1, \ldots, \sigma^d)\theta^{\mu_1} \ldots \theta^{\mu_p}$

Consider a vector field $Q = \theta^\mu \frac{\partial}{\partial \sigma^\mu}$

$$\left.\begin{array}{l} \deg Q = 1 \\ Q(f \cdot g) = (Qf) \cdot g + (-1)^{1 \cdot deg(f)} f \cdot (Qg) \\ [Q, Q] \equiv 2Q^2 = 0 \end{array}\right\} \leftarrow \mathrm{d}_{\text{de Rham}}$$

$\rightarrow$ Definition of a Q-structure: a vector field on a graded manifold, which is of degree 1 and squaring to zero.

Remark. Gradings can be encoded in the Euler vector field $\epsilon = deg(q^\alpha) q^\alpha \frac{\partial}{\partial q^\alpha}$ (can be a "definition").

Remark. (Ask J. Grabowski for details ;)) Gradings can be encoded in the homogeneity structure $h \colon \mathbb{R}_+ \times \mathcal{M} \rightarrow \mathcal{M}$ such that $(q^1, \ldots, q^N) \mapsto h_t(q^1, \ldots, q^N) \equiv (t^{deg(q^1)} q^1, \ldots, t^{deg(q^N)} q^N)$.

Poisson manifold $\rightarrow (T^*[1]M, Q_\pi)$

Consider a Poisson manifold M,
$\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\rightarrow C^\infty(M)$.

A Poisson bracket can be written as $\{f,g\}=\pi(\mathrm{d}f,\mathrm{d}g)$, where $\pi\in\Gamma(\Lambda^2 TM)$ is a bivector field. $\pi^{ij}(x)=\{x^i,x^j\}$.

Consider $T^*[1]M$ (coords. $x^i(0), p_i(1)$), with a degree 1 vector field

$$Q_\pi=\left\{\frac{1}{2}\pi^{ij}p_ip_j,\cdot\right\}_{T^*M}=\pi^{ij}(x)p_j\frac{\partial}{\partial x^i}-\frac{1}{2}\frac{\partial\pi^{jk}}{\partial x^i}p_jp_k\frac{\partial}{\partial p_i}$$

Jacobi identity:

$$\begin{gathered}\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0\\ \Leftrightarrow [\pi,\pi]_{SN}=0 \quad \Leftrightarrow Q_\pi^2=0\end{gathered}$$

Remark. For $H\in\Omega^3_{cl}(M)$, one can twist the picture to $[\pi,\pi]_{SN}=< H,\pi\otimes\pi\otimes\pi >\Leftrightarrow Q^2_{\pi,H}=0$

Courant algebroids, Dirac structures

Let us construct on $E = TM \oplus T^*M$ a twisted exact Courant algebroid structure, governed by a closed 3-form H on M.
The symmetric pairing: $< v \oplus \eta, v' \oplus \eta' >= \eta(v') + \eta'(v)$,
the anchor: $\rho(v \oplus \eta) = v$
the H-twisted bracket (Dorfman):

$$[v \oplus \eta, v' \oplus \eta'] = [v, v']_{\mathrm{Lie}} \oplus (\mathcal{L}_v \eta' - \iota_{v'} \mathrm{d}\eta + \iota_v \iota_{v'} H). \qquad (1)$$

A Dirac structure $\mathcal{D}$ is a maximally isotropic (Lagrangian) subbundle of an exact Courant algebroid E closed with respect to the bracket (1).

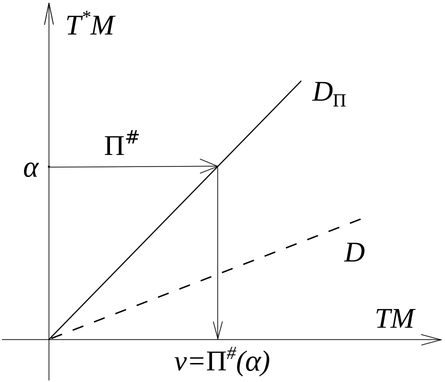
Trivial example: $\mathcal{D} = TM$ for $H = 0$.

Dirac structures: Poisson example.

Example. $\mathcal{D} = \mathit{graph}(\Pi^\sharp)$

Isotropy $\Leftrightarrow$
π^{ij} antisymmetric.

Involutivity $\Leftrightarrow$
Π (twisted) Poisson.



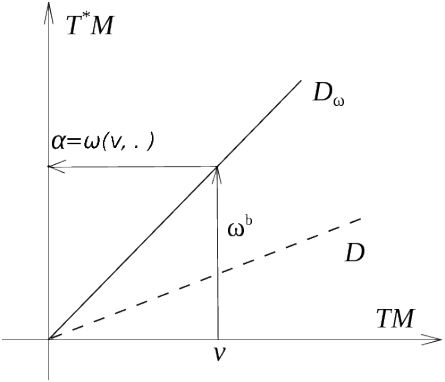
$\mathcal{D}_\Pi = \{(\Pi^\sharp\alpha, \alpha)\}$

Dirac structures: symplectic example.

Example. $\mathcal{D} = graph(\omega)$

Isotropy $\Leftrightarrow$
ω_{ij} antisymmetric.

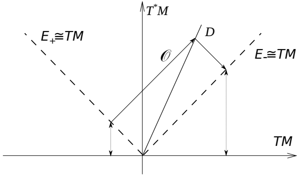
Involutivity $\Leftrightarrow$
ω closed.



$$\mathcal{D}_\omega = \{(v, \iota_v \omega)\}$$

Dirac structures: general

Choose a metric on $M \Rightarrow TM \oplus T^*M \cong TM \oplus TM$,
Introduce the eigenvalue subbundles $E_\pm = \{v \oplus \pm v\}$
of the involution $(v, \alpha) \mapsto (\alpha, v)$. Clearly, $E_+ \cong E_- \cong TM$.



(Almost) Dirac structure – a graph of an orthogonal operator $\mathcal{O} \in \Gamma(\mathrm{End}(TM))$:
$(v, \alpha) = ((\mathrm{id} - \mathcal{O})w, g((\mathrm{id} + \mathcal{O})w, \cdot))$
Dirac structure – a graph of an orthogonal operator $\mathcal{O} \in \Gamma(\mathrm{End}(TM))$ subject to the (twisted Jacobi-type) integrability condition

$$g\left(\mathcal{O}^{-1}\nabla_{(\mathrm{id}-\mathcal{O})\xi_1}(\mathcal{O})\xi_2, \xi_3\right) + cycl(1,2,3) = \frac{1}{2}H((\mathrm{id}-\mathcal{O})\xi_1, (\mathrm{id}-\mathcal{O})\xi_2, (\mathrm{id}-\mathcal{O})\xi_3).$$

Remark. If the operator $(\mathrm{id} + \mathcal{O})$ is invertible, one recovers D_Π with $\Pi = \dfrac{\mathrm{id} - \mathcal{O}}{\mathrm{id} + \mathcal{O}}$ (Cayley transform), integrability $\Leftrightarrow [\Pi, \Pi]_{SN} = \langle H, \Pi^{\otimes 3}\rangle$.

Remark (!) Any $\mathcal{D}[1]$ can be equipped with a Q-structure $\Leftrightarrow$ Lie algebroid structure on TM inherited from the symplectic realization of the Courant algebroid.

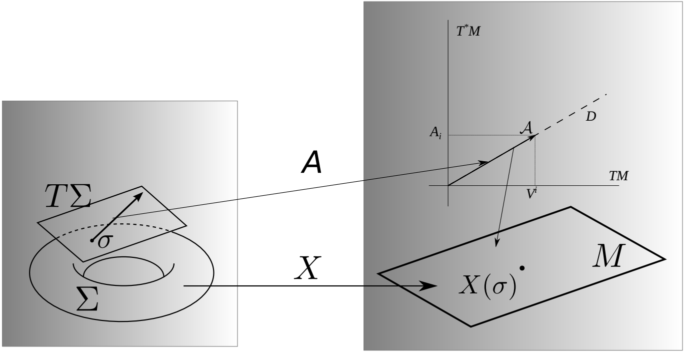
Q-morphisms

Given two Q-manifolds $(\mathcal{M}_1, Q_1)$, $(\mathcal{M}_2, Q_2)$, a degree preserving map $f : \mathcal{M}_1 \to \mathcal{M}_2$, is a <u>$Q$-morphism</u> iff $Q_1 f^* - f^* Q_2 = 0$.

Proposition Given a degree preserving map between Q-manifolds $(\mathcal{M}_1, Q_1)$ and $(\mathcal{M}, Q)$, there exists a Q-morphism between the Q-manifolds $(\mathcal{M}_1, Q_1)$ and $(\mathcal{M}_2, Q_2) = (T[1]\mathcal{M}, \mathrm{d}_{DR} + \mathcal{L}_Q)$

$$\begin{tikzcd}
 & T[1]\mathcal{M} \arrow[d] \\
\mathcal{M}_1 \arrow[ur, "f"] \arrow[r, "a"] & \mathcal{M}
\end{tikzcd}$$

Application 1: Sigma models, gauging



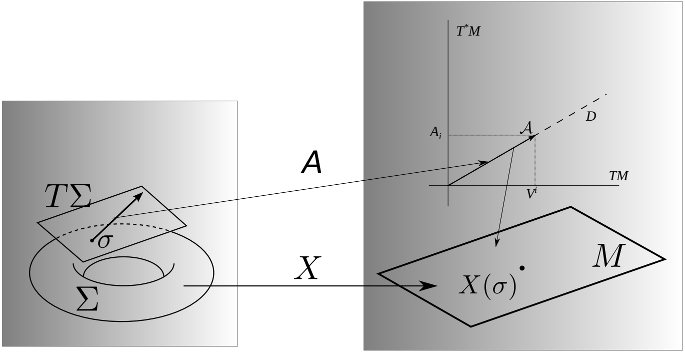
World-sheet (space-time) $\mapsto$ Target

Simple examples

$$B \in \Omega^2(M) \qquad \text{or} \qquad H \in \Omega^3(M)$$

$$S = \int_\Sigma X^* B \qquad\qquad S = \int_{\Sigma_3} X^* H$$

Application 1: Sigma models, gauging



World-sheet (space-time) $\mapsto$ Target

Dirac sigma model (A.Kotov, P.Schaller, T.Strobl – 2005)
Functional on vector bundle morphisms from $T\Sigma$ to $\mathcal{D}$.

$$S^0_{DSM} = \int_\Sigma A_i \wedge \mathrm{d}X^i - \frac{1}{2} A_i \wedge V^i + \int_{\Sigma_3} H.$$

Gauge transformations of the DSM

Theorem (V.S., T.Strobl) Any smooth map from Σ to the space $\Gamma(\mathcal{D})$ of sections of the Dirac structure $\mathcal{D} \subset TM \oplus T^*M$ defines an infinitesimal gauge transformation of the (metric independent part of the) Dirac sigma model governed by $\mathcal{D}$ if and only if for any point $\sigma \in \Sigma$ the section $v \oplus \eta \in \Gamma(\mathcal{D})$ satisfies

$$\mathrm{d}\eta - \iota_v H = 0,$$

where d is the de Rham differential on M.

Remark. H non-degenerate – 2-plectic geometry.

DSM from gauging, extension of the algebra

Consider an algebra $\tilde{\mathcal{G}}$ of degree -1 vector fields on $T[1]D[1]$ (+ some technical assumptions) and a subalgebra $\widetilde{\mathcal{GT}} \subset \tilde{\mathcal{G}}$ defined by $\mathrm{d}\eta - \iota_v H = 0$, extended by symmetric 2-tensors

Theorem (V.S., T.Strobl). Let H be a closed 3-form on M and D a Dirac structure on $(TM \oplus T^*M)_H$ such that the pullback of H to a dense set of orbits of D is non-zero. Then the $\widetilde{\mathcal{GT}}$-equivariantly closed extension of H is unique and yields the (metric-independent part of) the Dirac sigma model on $\Sigma = \partial\Sigma^3$.

V.S., T.S., “Dirac Sigma Models from Gauging”, Journal of High Energy Physics, 11(2013)110.
V.S. “Graded geometry in gauge theories and beyond”, Journal of Geometry and Physics, Volume 87, 2015.

Tool to prove: equivariant cohomology for Q-manifolds

Let $(\mathcal{M}, Q)$ be a Q-manifold, and let $\mathcal{G}$ be a subalgebra of degree -1 vector fields ε on $\mathcal{M}$ closed w.r.t. the Q-derived bracket: $[\varepsilon, \varepsilon']_Q = [\varepsilon, [Q, \varepsilon']]$.

Definition. Call a differential form (superfunction) ω on $\mathcal{M}$ $\underline{\mathcal{G}\text{-horizontal}}$ if $\varepsilon\omega = 0$, for any $\varepsilon \in \mathcal{G}$.

Definition. Call a differential form (superfunction) ω on $\mathcal{M}$ $\underline{\mathcal{G}\text{-equivariant}}$ if $(ad_Q\varepsilon)\omega := [Q, \varepsilon]\omega = 0$, for any $\varepsilon \in \mathcal{G}$.

Definition. Call a differential form (superfunction) ω on $\mathcal{M}$ $\underline{\mathcal{G}\text{-basic}}$ if it is $\mathcal{G}$-horizontal and $\mathcal{G}$-equivariant.

Remark. For Q-closed superfunctions $\mathcal{G}$-horizontal $\Leftrightarrow$ $\mathcal{G}$-basic

Key idea to apply to gauge theories:
Replace "gauge invariant" by "equivariantly Q-closed".

Q-morphisms for sigma models

$$\begin{tikzcd}
 & & (\mathcal{M}_1 \times \tilde{\mathcal{M}}) \circlearrowleft \hat{Q} \arrow[dll, bend left, "pr_1"] \arrow[d] \\
Q_1 \circlearrowright \mathcal{M}_1 \arrow[urr, bend left, "\hat{f}"] \arrow[rr, "\hat{a}"] & & \mathcal{M}_1 \times \mathcal{M}_2
\end{tikzcd}$$

where $\tilde{\mathcal{M}} = T[1]\mathcal{M}_2$,
$\hat{Q} = Q_1 + \tilde{Q} = Q_1 + \mathrm{d} + \mathcal{L}_{Q_2}$ – Q-structure on $\mathcal{M}_1 \times \tilde{\mathcal{M}}$

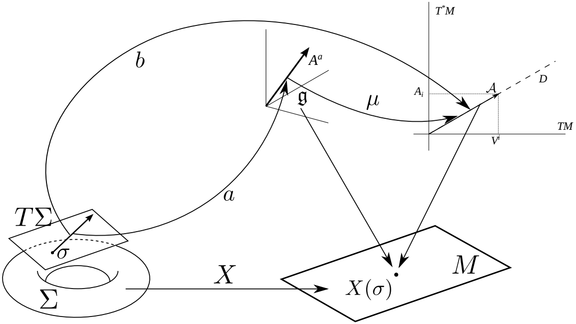
Gauge transformations: $\delta_\varepsilon \hat{f}^*(\cdot) = \hat{f}^*(V_\varepsilon \cdot) = \hat{f}^*([\hat{Q}, \hat{\varepsilon}]\cdot)$,
where $\hat{\varepsilon}$ – degree -1 vector field on $\mathcal{M}_1 \times \tilde{\mathcal{M}}$, vertical w.r.t. pr_1.

Gauge invariance of $S = \int_{\tilde{\Sigma}} \hat{f}^*(\bullet) \Leftrightarrow [\hat{Q}, \hat{\varepsilon}]\bullet = 0$
$\Leftrightarrow \bullet$ is equivariantly $\tilde{Q}$-closed (on $\tilde{\mathcal{M}}$!)

Extension of A. Kotov, T. Strobl, 2007.

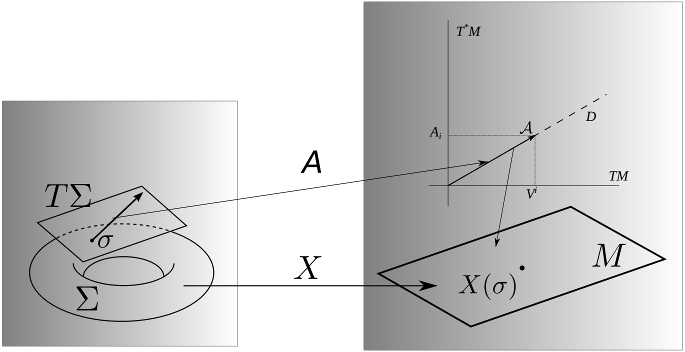
Generality of the DSM in dim = 2.

Theorem (A.Kotov, V.S., T.Strobl), Gauging of a Wess-Zumino term $\int_{\Sigma^3} H$ is not obstructed iff (under weak technical assumptions) the result is a Dirac Sigma model.



A.K., V.S., T.S, "2d Gauge theories and generalized geometry", Journal of High Energy Physics, 08(2014)021.

Application 1.5: Supersymmetrization



Space–time $\mapsto$ Target

Graded Poisson sigma model (M. Ertl, W. Kummer, T. Strobl $\sim$2000, to describe supergravity). Functional on vector bundle morphisms from $T\Sigma$ to T^*M, M – super Poisson

$$S^0_{sPSM} = \int_\Sigma \mathrm{d}X^i A_i + \frac{1}{2}\pi^{ji} A_i A_j$$

Alexandrov–Kontsevich–Schwarz–Zaboronsky procedure

Theorem (AKSZ).
Consider two graded manifolds X (source) and Y (target).
Let the *source* X be equipped with a Q-structure D and a D-invariant homogeneous (degree $-(n+1)$) non-degenerate measure μ;
the *target* Y equipped with a Q-structure Q, compatible with the symplectic structure ω, such that $deg(\omega) = n$ and the parity is opposite to the parity of μ.
Then the space of graded maps Y^X can be equipped with a QP-structure. Moreover if ω is exact one can define a functional on Y^X satisfying the classical master equation.

Multigraded AKSZ

Multigraded = several gradings on the sheaf of functions
$\Leftrightarrow$ commuting homogeneity structures
$\Leftrightarrow$ commuting Euler vector fields. (Ask J. Grabowski again ;))

Theorem.
Consider two multigraded manifolds X (source) and Y (target). Let the *source* X be equipped with a Q-structure D and a D-invariant homogeneous (gh-degree $-(n+1)$) non-degenerate measure μ;
the *target* Y equipped with a Q-structure Q, compatible with the symplectic structure ω, such that $gh(\omega) = n$ and the total parity is opposite to the parity of μ.
Then the space of multigraded maps Y^X can be equipped with a QP-structure. Moreover if ω is exact one can define a functional on Y^X satisfying the classical master equation.

Remark. The condition on μ is now rather restrictive.

Details: V.S., arXiv:1608.07457.

Multigraded Q-bundles

Theorem (M. Bojowald, A. Kotov and T. Strobl, 2005). Solutions of the field equations of the PSM = morphisms of Lie algebroids; gauge transformations = Lie algebroid homotopies.

Remark. Equivalently: Q-morphisms / Q-homotopies $\Rightarrow$ multigraded generalization.

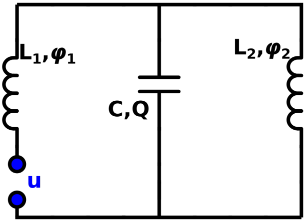
Prop. The theory resulting from the source supersymmetrization of the PSM is on-shell equivalent to the original (non-supersymmetric) one.

Prop. The source supersymmetrized Chern–Simons theory can be reformulated as the target-supersymmetrized theory with an extended algebra.

Details: V.S., arXiv:1608.07457.

Application 2. Port-Hamiltonian systems.

Example: Electric circuit (L_1, L_2, C) with a controlled port u



$$\begin{cases} \dot{Q} = \varphi_1/L_1 - \varphi_2/L_2 \\ \dot{\varphi}_1 = -Q/C + u \\ \dot{\varphi}_2 = Q/C. \end{cases}$$

$$H = \frac{\varphi_1^2}{2L_1} + \frac{\varphi_2^2}{2L_2} + \frac{Q^2}{2C}$$

Port: input u, output $e = \varphi_1/L_1$.

Port-Hamiltonian system: $\dot{\mathbf{x}} = J(\mathbf{x})\frac{\partial H}{\partial \mathbf{x}} + g(\mathbf{x})\mathbf{f}$ with

$$\mathbf{x} = \begin{pmatrix} Q \\ \varphi_1 \\ \varphi_2 \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{matrix} f = u \\ e = \varphi_1/L_1 \end{matrix}$$

$$f_s := -\dot{\mathbf{x}}, \quad e_s := \begin{pmatrix} p_Q \\ p_{\varphi_1} \\ p_{\varphi_2} \end{pmatrix} \equiv \begin{pmatrix} Q/C \\ \varphi_1/L_1 \\ \varphi_2/L_2 \end{pmatrix}.$$

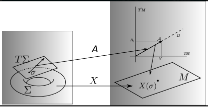
$$\dot{H} \equiv -e_s^T f_s = u\varphi_1/L_1 \Leftrightarrow$$
$$e_s^T f_s + ef = 0$$
$$\Rightarrow \text{Almost Dirac}$$

Port-hamiltonian $\rightarrow$ Dirac $\rightarrow \ldots$

A. van der Schaft, ‘Port-Hamiltonian systems...’ ICM Madrid, ’06.

Geometry of port-Hamiltonian systems:
J.I. Neimark, N.A. Fufaev, H. Yoshimura, J.E. Marsden,
T.S. Ratiu, B.M. Maschke ... **Talk by F. Gay-Balmaz tomorrow.**

My motivation:

Classical (ODE) classical mechanics	Poisson symplectic (almost) Dirac	Q-struct.	
Modern (PDE) classical mechanics	multi-symplectic Stokes–Dirac ?	???	
Gauge theories Sigma models	Dirac in 2d	Q-struct.	

Thank you for your attention!