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Translating Solitons, Semi-Riemannian Manifolds, and Lie Groups

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Summary

Introduction

This talk is based on

M.-A. Lawn, —, *Translating solitons from semi-Riemannian foliations*, http://arxiv.org/abs/1607.04571

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A hypersurface $F : M \to \mathbb{R}^{n+1}$ (standard metric) is called *Translating Soliton* if its mean curvature vector $\mathbf{H}$ satisfies

$$\mathbf{H} = v^{\perp}, \tag{1}$$

for some constant (unit) vector $v \in \mathbb{R}^{n+1}$. Usually, $v = (0, \ldots, 0, 1)^t$. It is possible to construct a family

$$F_t : M \to \mathbb{R}^{n+1} \; \forall t \in \mathbb{R}, \; x \in M, \quad F_0(x) = F(x)$$

that is a solution to the *mean curvature flow*

$$\frac{d}{dt} F_t = \vec{H}_t. \tag{2}$$

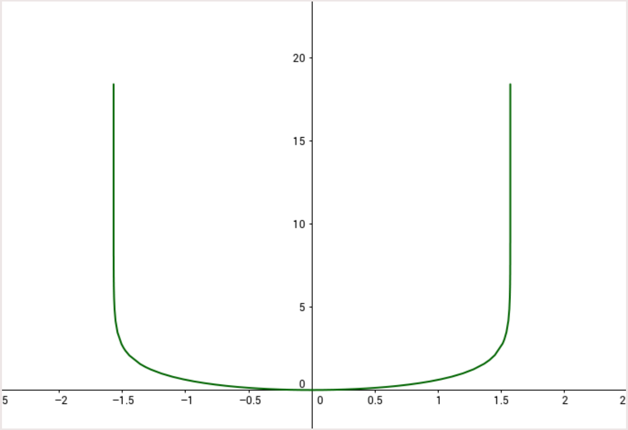
Famous Examples

- For a good list of known examples in Euclidean Space:
  - F. Martín, A. Savas-Halilaj, K. Smoczyk, *On the topology of translating solitons of the mean curvature flow*, Cal. Var. Partial Diff. Eq. 54 (2015), no. 3, 2853-2882.
- Rotationally invariant translating solitons in Minkowski Space:
  - G. Li, D. Tian, C. Wu, *Translating Solitons of Mean Curvature Flow of Noncompact Submanifolds*, Math. Phys. Anal. Geom **14**(2011), 83–99
- Examples with infinite genus:
  - X. H. Nguyen, *Doubly periodic self-translating surfaces for the mean curvature flow*, Geom. Dedicata 174 (2015), 177–185.

(La Muerte)

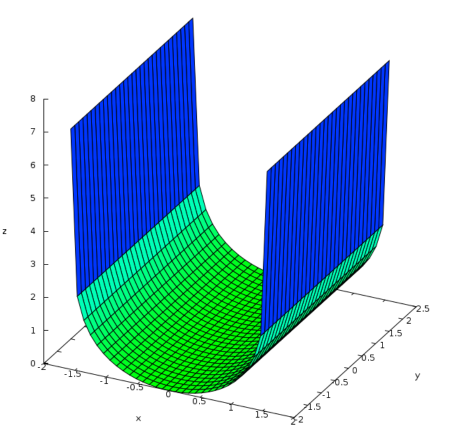
The Grim Reaper

In $\mathbb{R}^2$, the graph of the function $y = -\log(\cos(x))$, $x \in (-\pi/2, \pi/2)$.



The Grim Reaper Cylinder in $\mathbb{R}^n$

$F : (-\pi/2, \pi/2) \times \mathbb{R}^{n-2} \to \mathbb{R}^n,\ F(x, y) = (-\log(\cos(x)), y).$



The Translating Paraboloid

S. J. Altschuler, L. F. Wu, Calc. Var. Partial Diff. Eq, **2**(1994), Issue 1, pp 101-111.

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The translating soliton is

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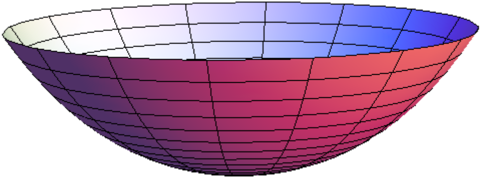
$$F : \mathbb{R}^{n-1} \to \mathbb{R}^n,\ F(x) = (x, u(x)),$$
$$u(x) = f(|x|), \forall x \in \mathbb{R}^{n-1},$$

where f is the solution to

$$f''(s) = \left(1 + f'(s)^2\right)\left(1 - \frac{n-1}{s} f'(s)\right), \quad f'(0) = 0,\ f(0) = a \in \mathbb{R}. \quad (3)$$

Unique $C^\infty[0, +\infty)$, strictly increasing solution.

Translating Paraboloid

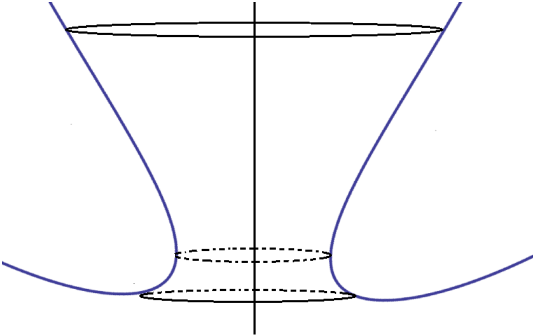


Note

$u(x) = u(Ax)$, $\forall x \in \mathbb{R}^{n-1}$, $\forall A \in SO(n-1)$.

The Translating Catenoid

It can be seen as the desingularization of two translating paraboloids connected by a small neck of some radius.



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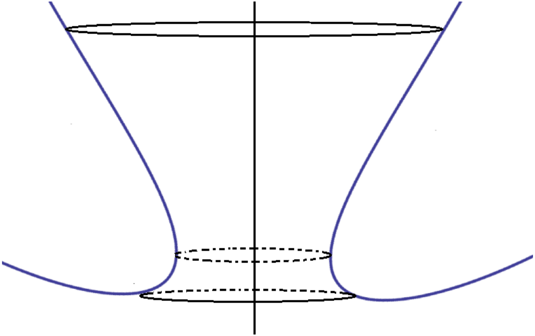

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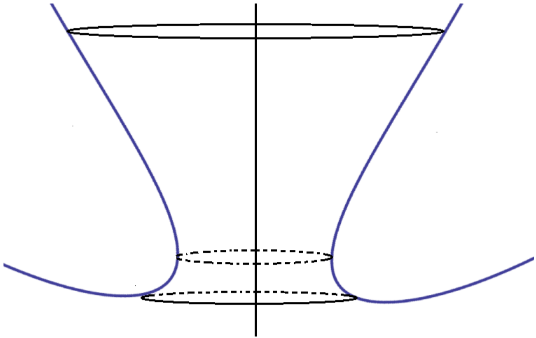

The Translating Paraboloid and the Translating Catenoid
are **rotationally invariant**
$\equiv$ **invariant by the action of** $SO(n-1)$ **by isometries**

To-do List

- A definition of Translating Solitons in Semi-Riemannian products $\mathbf{M} \times \mathbb{R}$.

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- Lie Groups in action (by isometries).
- Examples
- Future Works.

Summary

Defining (vertical) Translating Solitons

(M, g) a connected semi-Riemann manifold, $\dim M = n \geq 2$, index $0 \leq \alpha \leq n-1$.
$\varepsilon = \pm 1$, the semi-Riemannian product $\bar{M} = M \times \mathbb{R}$, $\langle , \rangle = g + \varepsilon \mathrm{d}t^2$.
$\partial_t \in \mathfrak{X}(\bar{M})$ is Killing, unit, spacelike when $\varepsilon = +1$ and timelike when $\varepsilon = -1$.
Let $F : \Gamma \to \bar{M}$ be a submanifold with mean curvature vector $\vec{H}$. $\partial_t^{\perp}$ = the normal component of ∂_t along F.

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Definition 1

With the previous notation, we will call F a (vertical) translating soliton of the mean curvature flow, or simply, a translating soliton, if $\vec{H} = \partial_t^\perp$.

Summary

Graphical Translating Solitons

Given $u \in C^2(M)$, we construct its graph map $F : M \to M \times \mathbb{R}$, $F(x) = (x, u(x))$. If ν is the upward normal vector along F with $\varepsilon' = \mathrm{sign}(\langle \nu, \nu \rangle) = \pm 1$.

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Proposition 1

Assume that $F : (M, \gamma = F^* \langle , \rangle) \to (\bar{M}, \langle , \rangle = g + \varepsilon\, dt^2)$ is a semi-Riemannian hypersurface. Then, F is a (vertical) translating soliton if, and only if, function u satisfies

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{\varepsilon'(\varepsilon + |\nabla u|_g^2)}} \right) = \frac{1}{\sqrt{\varepsilon'(\varepsilon + |\nabla u|_g^2)}}. \tag{4}$$

Summary

Submersions

Consider $\pi : (M, g_M) \to (B, g_B)$ a smooth map among semi-Riemannian mfds such that:

- π is surjective,
- $\mathcal{V}_x = \ker(d\pi_x) =$, $\dim \mathcal{V} =$constant $\forall x \in M$.
- for each $x \in M$, $d\pi_x : \mathcal{V}_x^\perp \to T_{\pi(x)}B$ is a linear isometry.

With all this, we say that π is a *semi-Riemannian submersion*.

For each $p \in B$, $\pi^{-1}\{p\} \subset M$ is a submanifold called *a fiber*.

Given $u \in C^2(B)$,

- $F : B \to B \times \mathbb{R}$, $F(p) = (p, u(p))$, and
- $\tilde{F} : M \to M \times \mathbb{R}$, $\tilde{F}(x) = (x, u \circ \pi(x))$.

In othe words, the following diagram is conmutative:

$$\begin{array}{ccc} (M, g_M) & \xrightarrow{\widetilde{F}} & (M \times \mathbb{R}, g_M + \varepsilon\, dt^2) \\ \pi \big\downarrow & & \big\downarrow \pi\times 1 \\ (B, g_B) & \xrightarrow{F} & (B \times \mathbb{R}, g_B + \varepsilon\, dt^2) \end{array}$$

$$F(p) = (p, u(p)), \quad \tilde{F}(x) = (x, u(\pi(x))).$$

Remark: $\pi \times 1 : M \times \mathbb{R} \to B \times \mathbb{R}$ is another semi-Riemannian submersion.

$$\begin{array}{ccc} (M, g_M) & \xrightarrow{\widetilde{F}} & (M \times \mathbb{R}, g_M + \varepsilon\, dt^2) \\ \Big\downarrow \pi & & \Big\downarrow \pi\times 1 \\ (B, g_B) & \xrightarrow{F} & (B \times \mathbb{R}, g_B + \varepsilon\, dt^2) \end{array}$$

$$F(p) = (p, u(p)), \quad \tilde{F}(x) = (x, u(\pi(x))).$$

When the mean curvature of the fibers $\pi^{-1}\{p\} \subset M$ vanish, the submersion is called harmonic.

Theorem 2

Let $\pi : (M, g_M) \to (B, g_B)$ be a harmonic semi-Riemannian submersion. Let $u \in C^2(B)$, $F : B \to B \times \mathbb{R}$, and F and $\tilde{F}$ as before. Then $\widetilde{F}$ is a graphical translating soliton if and only if F is a graphical translating soliton.

Case $\dim B = 1$

Take $\pi : (M, g_M) \to (I, \widetilde{\varepsilon} ds^2)$, where I is an open interval.
Each $s \in I$, the fiber $\pi^{-1}(s)$ is a hypersurface in M. Assume each fiber has constant mean curvature (CMC). For each $s \in I$, $h(s)$ =the mean curvature of $\pi^{-1}(s)$. We say that h *represents the mean curvature of the fibers.*

Theorem 3

Let I be an open interval and $\pi : (M, g_M) \to (I, \widetilde{\varepsilon} ds^2)$ be a semi-Riemannian submersion with CMC fibers, and function h representing the mean curvature of the fibers. Given $u \in C^2(M)$ which is constant along the fibers of π, let $\widetilde{F} : M \to M \times \mathbb{R}$, $\widetilde{F}(x) = (x, u(x))$, be its graph map, and $f : I \to \mathbb{R}$ be the map such that $u = f \circ \pi$. Then $\widetilde{F}$ is a translating soliton if and only if function f is a solution to

$$f''(s) = \big(\widetilde{\varepsilon} + \varepsilon f'(s)^2\big)\big(1 - f'(s)h(s)\big). \tag{5}$$

Summary

Σ a Lie group acting on the manifold M by isometries and s.t. the projection to the space of orbits $\pi : M \to M/\Sigma$ is a well-defined smooth map, where M/Σ is diffeomorphic to an open interval.

Corollary 4

(M, g) a connected semi-Riemannian manifold. Σ a Lie group acting by isometries on M and $\pi : (M, g_M) \to (I, \widetilde{\varepsilon} ds^2)$ a semi-Riemannian submersion s.t. the fibers of π are orbits of the action, with function $h : I \to \mathbb{R}$ representing the mean curvature of the fibers. Given $u \in C^2(M)$, its graph map $F : M \to M \times \mathbb{R}$, $F(x) = (x, u(x))$ for any $x \in M$. Then, F is a Σ-invariant translating soliton if, and only if, there exists a solution $f \in C^2(I, \mathbb{R})$ to (5) *such that $u = f \circ \pi$.*

An Algorithm

(M, g) a semi-Riemannian mfd. Σ a Lie subgroup of $\text{Iso}(M, g)$, I open interval. Choose $\varepsilon \in \{\pm 1\}$. The metric in $M \times \mathbb{R}$ is $\langle , \rangle = g + \varepsilon\, dt^2$.

1. Assume $\Phi : M \to \Sigma \times I$ diffeom. s.t. its restriction $\pi : M \to I$ satifies $|\nabla \pi|^2 \neq 0$.
2. By a change of variable, recompute Φ and π s.t. $|\nabla \pi|^2 = \widetilde{\varepsilon} (= \pm 1)$.
3. For each $s \in I$, compute the mean curvature $h(s)$ of the fiber $\pi^{-1}\{s\} \subset M$. Note $\pi^{-1}\{s\} \cong \Sigma$.
4. Solve $f''(s) = \left(\widetilde{\varepsilon} + \varepsilon f'(s)^2\right)\left(1 - f'(s)h(s)\right)$, for some initial values in an interval $J \subset I$.
5. The translating soliton is

$$F : \Sigma \times J \to M \times \mathbb{R}, \quad F(\sigma, s) = \left(\Phi^{-1}(\sigma, s), f(s)\right).$$

$$\left(\bar{F} : \Phi^{-1}(\Sigma \times J) \to M \times \mathbb{R}, \quad \bar{F}(x) = (x, f(\pi(x)). \right)$$

An algorithm

$$f''(s) = \left(\widetilde{\varepsilon} + \varepsilon f'(s)^2\right)\left(1 - f'(s)h(s)\right), \quad h : I \to \mathbb{R}. \tag{6}$$

Further problems

- Sometimes, the solutions to (6) can be extended to the whole I. For example, this is the case when $\varepsilon\widetilde{\varepsilon} = -1$ and $|f'(s_o)| < 1$, $s_o \in I$. Hint: $f(s) = s + b$ and $f(s) = -s + b$ are globally defined solutions.
- Some solutions to (6) do admit finite-time blow-ups. This is the case for some bounds on h, and for some initial values to (6).
- If $\Omega \subset M$ s.t. $\Omega \cong \Sigma \times I$, can we extend it to the whole M?

Summary

Desingularizing two graphical translating solitons

Corollary 5

(M, g) a connected semi-Riemannian manifold. Σ a Lie group acting by isometries on M and $\pi : M \to I$ a submersion such that the fibers of π are orbits of the action. h represents the mean curvature of the fibers. Then, for each $y_o \in \mathbb{R}$ and each $s_o \in I$ such that $h(s_o) \neq 0$, there exist a real number $\rho > 0$ and a translating soliton $F : (y_0 - \rho, y_o + \rho) \times \Sigma \to M \times \mathbb{R}$ such that it is the union of two graphical translating solitons.

Examples

1. We recover the Translating Paraboloid and the Translating Catenoid in $\mathbb{R}^n$.

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2. $SO(n-1)$-invariant spacelike *Translating Paraboloid* in $\mathbb{L}^n$.
   New: Spacelike *Translating Catenoid* in $\mathbb{L}^n$.

New examples

3. **A generalization of the Grim Reaper Cylinder in $\mathbb{R}^n$.**
If $F : M \to M \times \mathbb{R}$ is a translating soliton, given another manifold P, then

$$\bar{F} : P \times M \to P \times M \times \mathbb{R}, \quad \bar{F}(p, x) = (p, F(x)),$$

is another translating soliton.

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4. $SO(n-1)$-invariant translating solitons in De Sitter Space $d\mathbb{S}^n$.
   1. With one end $F : d\mathbb{S}^n \to d\mathbb{S}^n \times \mathbb{R}$, or two ends $F : \mathbb{S}^{n-1} \times \mathbb{R} \to d\mathbb{S}^n \times \mathbb{R}$.

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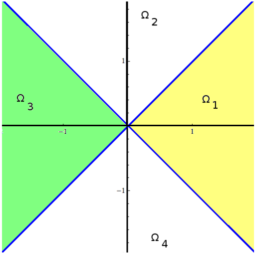
   2. Admitting blow-ups.
7. In anti-de Sitter 3-space $\mathbb{H}^3_1$, a $\mathbb{S}^1 \times \mathbb{S}^1$-invariant translating soliton. Hint: $\mathbb{S}^1$-Hopf fibration $\pi : \mathbb{H}^3_1 \to \mathbb{H}^2$.

8. In Minkowski 3-Space $\mathbb{L}^3$. Take the Boost Group

$$\Sigma = \left\{ A_\theta = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) \\ \sinh(\theta) & \cosh(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\},$$

which acts on $\mathbb{L}^2$ by isometries. We split the plane in four regions Ω_i, $i = 1, 2, 3, 4$, whose boundaries are made of two light-like geodesics crossing at the origin.



Note $\mathbb{L}^3 = \mathbb{L}^2 \times \mathbb{R}$.

On each region, we construct a boost invariant, globally defined, translating soliton, approaching the origin, $F_i : \Omega_i \to \mathbb{L}^3$.

The union of two adjacent, three or four or them, and the corresponding light-like geodesics, provide a C^∞ translating soliton in $\mathbb{L}^3$.

Summary

Future Works

1. When is the PDE of the graphical translating solitons parabolic?
2. Are these (graphical) translating solitons analytical?
3. Is there any tangency principle?
4. When are these examples complete?
5. What is the behaviour at infinity in Hadamard manifolds?

Thank you very much for your attention!