b-Symplectic manifolds: going to infinity and coming back

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XXV International Fall Workshop on Geometry and Physics
Madrid

Outline

The restricted 3-body problem (joint with Delshams)

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem on elliptic orbits with eccentricity e.

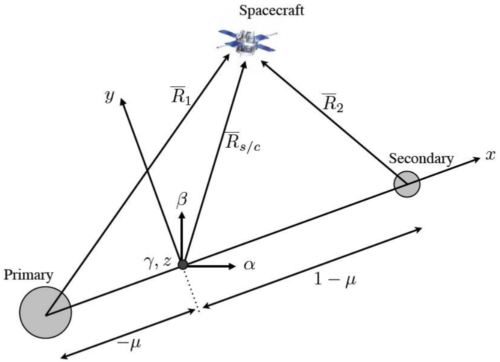


Figure: Circular 3-body problem

Planar restricted 3-body problem

- By Newton's law of universal gravitation

$$\frac{d^2q}{dt^2} = (1-\mu)\frac{q_1 - q}{|q_1 - q|^3} + \mu\frac{q_2 - q}{|q_2 - q|^3} \tag{1}$$

with $q_1 = q_1(t)$ the position of the planet with mass $1-\mu$ at time t and $q_2 = q_2(t)$ the position of the one with mass μ.

- Introducing the momentum $p = dq/dt$ and the time-dependent self-potential of the small body $U(q,t) = \frac{1-\mu}{|q-q_1|} + \frac{\mu}{|q-q_2|}$ Equation (1) can be rewritten as a Hamiltonian system with Hamiltonian $H(q,p,t) = p^2/2 - U(q,t), \quad (q,p) \in \mathbf{R}^2 \times \mathbf{R}^2,$.
- The primaries move around their center of mass on ellipses and it is useful to introduce polar coordinates for $q = (X,Y) \in \mathbf{R}^2 \setminus \{0\}$, $X = r\cos\alpha, Y = r\sin\alpha, \qquad (r,\alpha) \in \mathbf{R}^+ \times \mathbb{T}$
- The momenta $p = (P_X, P_Y)$ are transformed in such a way that the total change of coordinates $(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$ is canonical, i.e. the symplectic structure remains the same.

Planar restricted 3-body problem

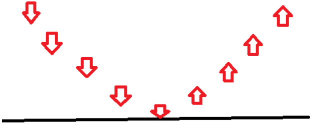
- To study the behaviour at $r = \infty$, we introduce **McGehee coordinates** (x, α, y, G), where
$$r = \frac{2}{x^2}, \quad x \in \mathbf{R}^+.$$
- This transformation is non-canonical i.e. the symplectic structure changes: for $x > 0$ it is given by
$$-\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$$
which extends to a b^3-symplectic structure on $\mathbf{R} \times \mathbb{T} \times \mathbb{R}^2$.
- The Poisson bracket is
$$\{f, g\} = -\frac{x^3}{4}\left(\frac{\partial f}{\partial g}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) + \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial G} - \frac{\partial f}{\partial G}\frac{\partial g}{\partial \alpha}$$
- The integrable 2-body problem for $\mu = 0$ is integrable with respect to the singular ω.

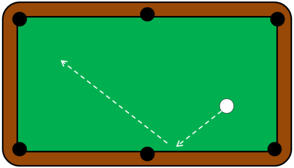
Model for these systems

$$\omega = \mathbf{\frac{1}{x_1^n} dx_1 \wedge dy_1 + \sum_{i \geq 2} dx_i \wedge dy_i}$$

Close to $x_1 = 0$, the systems behave like,

and not like,





The 3-body problem

- Consider the system of three bodies with masses m_1, m_2, m_3 and positions $\mathbf{q_1} = (q_1, q_2, q_3), \mathbf{q_2} = (q_4, q_5, q_6), \mathbf{q_3} = (q_7, q_8, q_9) \in \mathbb{R}^3$.
- Define the 9×9 matrix $M := \text{diag}(m_1, m_1, m_1, m_2, m_2, m_2, m_3, m_3, m_3)$.
- Assume central coordinates $(m_1\mathbf{q_1} + m_2\mathbf{q_2} + m_3\mathbf{q_3} = 0)$.
- Introduce the following "McGehee"-coordinates:

$$r := \sqrt{q^T M q}, \qquad s := \frac{q}{r}, \qquad z := p\sqrt{r}. \tag{2}$$

- $r = 0$ corresponds to triple collisions. Essentially, these are spherical coordinates since s lies on the unit-sphere in $\mathbb{R}^9$ with respect to the metric given by M.

The 3-body problem

- The standard symplectic form $\sum_{i=1}^{9} dq_i \wedge dp_i$ becomes in the new coordinates $(r, s_1, \ldots, s_8, z_1, \ldots, z_9)$.

$$\sum_{i=1}^{8}\left(\frac{s_i}{\sqrt{r}}dr \wedge dz_i + \sqrt{r}ds_i \wedge dz_i - \frac{z_i}{2\sqrt{r}}ds_i \wedge dr\right) +$$
$$+\frac{1}{\sqrt{\overline{m}_9 r\mu}}\left(\mu dr \wedge dz_9 - r\sum_{i=1}^{8}\overline{m}_i s_i ds_i \wedge dz_9 + \frac{z_9}{2}\sum_{i=1}^{8}\overline{m}_i s_i ds_i \wedge dr\right),$$

with $\mu := 1 - \sum_{i=1}^{8} s_i^2 \overline{m}_i$.

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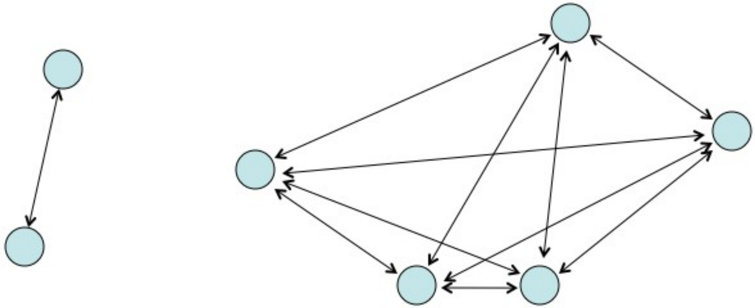
$$\bigwedge_{i=1}^{9} dq_i \wedge dp_i = \sqrt{\frac{\mu r^7}{\overline{m}_9}} ds_1 \wedge dz_1 \wedge ds_2 \wedge dz_2 \wedge \ldots \wedge ds_8 \wedge dz_8 \wedge dr \wedge dz_9,$$

It is a $\frac{7}{2}$-folded symplectic structure. (In the n-body problem m-folded symplectic for a certain m).

Other examples

- Kustaanheimo-Stiefel regularization for n-body problem (useful for binary collisions) $\rightsquigarrow$ folded-type symplectic structures with hyperbolic singularities



- two fixed-center problem via Appell's transformation $\rightsquigarrow$ combination of folded-type and b^m-symplectic structures $\rightsquigarrow$ Dirac structures.

b-Symplectic/b-Poisson structures

Definition

Let (M^{2n}, Π) be an (oriented) Poisson manifold such that the map

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

is transverse to the zero section, then $Z = \{p \in M | (\Pi(p))^n = 0\}$ is a hypersurface called *the critical hypersurface* and we say that Π is a **Poisson b-structure** on (M, Z).

Symplectic foliation of a b-Poisson manifold

The symplectic foliation has dense symplectic leaves and codimension 2 symplectic leaves whose union is Z.

Darboux normal forms

Theorem (Guillemin-M.-Pires)

For all $p \in Z$, there exists a Darboux coordinate system $x_1, y_1, \dots, x_n, y_n$ centered at p such that Z is defined by $x_1 = 0$ and

$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

Darboux for b^n-symplectic structures

$$\Pi = x_1^n \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

or dually

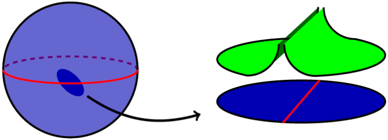
$$\omega = \frac{1}{x_1^n} dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$$

Dimension 2

Radko classified b-Poisson structures on compact oriented surfaces:

- Geometrical invariants: The topology of S and the curves γ_i where Π vanishes.
- Dynamical invariants: The periods of the "modular vector field" along γ_i.
- Measure: The regularized Liouville volume of S, $V_h^{\varepsilon}(\Pi) = \int_{|h|>\varepsilon} \omega_\Pi$ for h a function vanishing linearly on the curves $\gamma_1, \ldots, \gamma_n$ and ω_Π the "dual "form to the Poisson structure.

Other classification schemes: For b^n-symplectic structures (not necessarily oriented) $\rightsquigarrow$ Scott, M.-Planas.



Higher dimensions: Some compact examples.

- The product of (R, π_R) a Radko compact surface with a compact symplectic manifold (S, ω) is a b-Poisson manifold.
- corank 1 Poisson manifold (N, π) and X Poisson vector field $\Rightarrow$ $(S^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi)$ is a b-Poisson manifold if,
  1. f vanishes linearly.
  2. X is transverse to the symplectic leaves of N.

  We then have as many copies of N as zeroes of f.

Poisson Geometry of the critical hypersurface

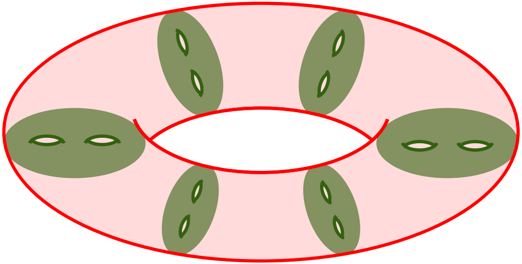
This last example is semilocally the *canonical* picture of a b-Poisson structure .

1. The critical hypersurface Z has an induced regular Poisson structure of corank 1.
2. There exists a Poisson vector field transverse to the symplectic foliation induced on Z.

The singular hypersurface

Theorem (Guillemin-M.-Pires)

If $\mathcal{L}$ contains a compact leaf L, then Z is the mapping torus of the symplectomorphism $\phi : L \to L$ determined by the flow of a Poisson vector field v transverse to the symplectic foliation.

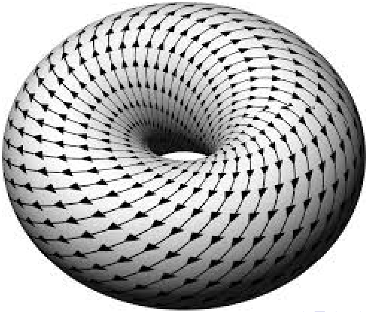


This description also works for b^n-symplectic structures.

A dual approach...

- b-Poisson structures can be seen as symplectic structures modeled over a Lie algebroid (the b-tangent bundle).
- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$



- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- A **b-symplectic form** is a closed, nondegenerate, b-form of degree 2.
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.

What else?

b-integrable systems

Definition

b-integrable system A set of b-functions[a] $f_1, \ldots, f_n$ on (M^{2n}, ω) such that

- $f_1, \ldots, f_n$ Poisson commute.
- $df_1 \wedge \cdots \wedge df_n \neq 0$ as a section of $\Lambda^n({}^bT^*(M))$ on a dense subset of M and on a dense subset of Z

[a] $c \log|x| + g$

Example

The symplectic form $\frac{1}{h} dh \wedge d\theta$ defined on the interior of the upper hemisphere H_+ of S^2 extends to a b-symplectic form ω on the double of H_+ which is S^2. The triple $(S^2, \omega, log|h|)$ is a b-integrable system.

Example

If $(f_1, \ldots, f_n)$ is an integrable system on M, then $(\log|h|, f_1, \ldots, f_n)$ on $H_+ \times M$ extends to a b-integrable on $S^2 \times M$.

Action-angle coordinates for b-integrable systems

The compact regular level sets of a b-integrable system are (Liouville) tori.

Theorem (Kiesenhofer-M.-Scott)

Around a Liouville torus there exist coordinates $(p_1, \dots, p_n, \theta_1, \dots, \theta_n) : U \to B^n \times \mathbf{T}^n$ *such that*

$$\omega|_U = \frac{c}{p_1} dp_1 \wedge d\theta_1 + \sum_{i=2}^{n} dp_i \wedge d\theta_i, \tag{3}$$

and the level sets of the coordinates $p_1, \dots, p_n$ *correspond to the Liouville tori of the system.*

Reformulation of the result

Integrable systems semilocally $\leftrightsquigarrow$ twisted cotangent lift[a] of a $\mathbb{T}^n$ action by translations on itself to $(T^*\mathbb{T}^n)$.

[a] We replace the Liouville form by $c \log |p_1| d\theta_1 + \sum_{i=2}^{n} p_i d\theta_i$.

Intermezzo on twisted b-cotangent lifts

Consider $G := S^1 \times \mathbf{R}^+ \times S^1$ acting on $M := S^1 \times \mathbf{R}^2$:
$(\varphi, a, \alpha) \cdot (\theta, x_1, x_2) := (\theta + \varphi, aR_\alpha(x_1, x_2))$, with R_α rotation.
Its twisted b-cotangent lift gives focus-focus singularities on b-symplectic manifolds.
The logarithmic Liouville one-form is $\lambda := \log|p| d\theta + y_1 dx_1 + y_2 dx_2$ and the moment map is $\mu := (f_1, f_2, f_3)$ with

$$f_1 = \langle \lambda, X_1^{\#} \rangle = \log|p|,$$
$$f_2 = \langle \lambda, X_2^{\#} \rangle = x_1 y_1 + x_2 y_2,$$
$$f_3 = \langle \lambda, X_3^{\#} \rangle = x_1 y_2 - y_1 x_2.$$

Proof

1. Topology of the foliation. In a neighbourhood of a compact connected fiber the b-integrable system F is diffeomorphic to the b-integrable system on $W := \mathbf{T}^n \times B^n$ given by the projections $p_1, \ldots, p_{n-1}$ and $\log |p_n|$.

2. Uniformization of periods: We want to define integrals whose (b-)Hamiltonian vector fields induce a $\mathbf{T}^n$ action. Start with $\mathbf{R}^n$-action:

$$\begin{array}{rcrcl} \Phi & : & \mathbf{R}^n \times (\mathbf{T}^n \times B^n) & \to & \mathbf{T}^n \times B^n \\ & & ((t_1, \ldots, t_n), m) & \mapsto & \Phi^{(1)}_{t_1} \circ \cdots \circ \Phi^{(n)}_{t_n}(m). \end{array}$$

Uniformize to get a $\mathbf{T}^n$ action with fundamental vector fields Y_i.

3. The vector fields Y_i are Poisson vector fields (check $\mathcal{L}_{Y_i}\mathcal{L}_{Y_i}\omega = 0$).

4. The vector fields Y_i are Hamiltonian with primitives $\sigma_1, \ldots, \sigma_n \in {}^b C^\infty(W)$. In this step the properties of b-cohomology are essential.Use this action to drag a local normal form (Darboux-Carathéodory) in a whole neighbourhood.

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A picture...

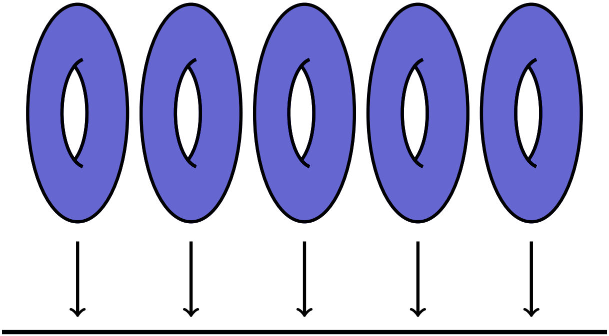


Figure: Fibration by Liouville tori

Applications to KAM theory (surviving torus under perturbations) on b-symplectic manifolds (Kiesenhofer-M.-Scott).

KAM for b-symplectic manifolds

Theorem (Kiesenhofer-M.-Scott)

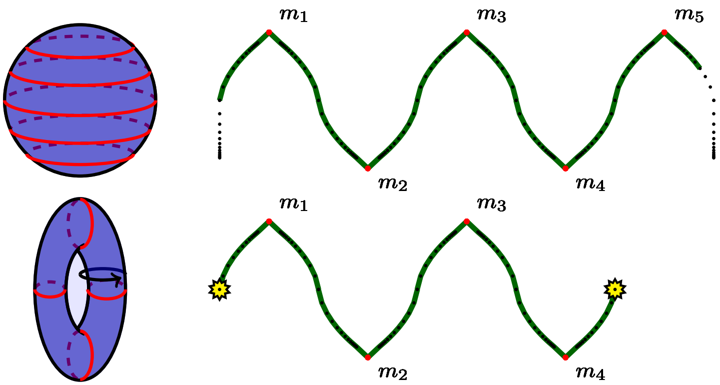
Consider $\mathbf{T}^n \times B_r^n$ with the standard b-symplectic structure and consider the b-function $H = k\log|y_1| + h(y)$ with h analytic. If the frequency map has a Diophantine value and is non-degenerate, then a Liouville torus on Z persists under sufficiently small perturbations of H. More precisely, if $|\epsilon|$ is sufficiently small, then the perturbed system

$$H_\epsilon = H + \epsilon P$$

(with $P(\varphi, y) = \log|y_1| + f_1(\tilde{\varphi}, y) + y_1 f_2(\varphi, y) + f_3(\varphi_1, y_1)$) admits an invariant torus $\mathcal{T}$.
Moreover, there exists a diffeomorphism $\mathbf{T}^n \to \mathcal{T}$ close to the identity taking the flow γ^t of the perturbed system on $\mathcal{T}$ to the linear flow on $\mathbf{T}^n$ with frequency vector $(\frac{k+\epsilon k'}{c}, \tilde{\omega})$.

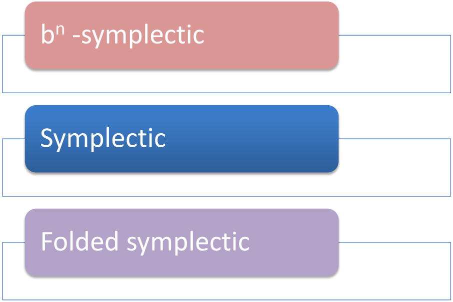
Global classification

Circle actions on b-surfaces:



Further results on b-manifolds

- Delzant theorem and convexity for $\mathbb{T}^k$-actions (Guillemin-M.-Pires-Scott).
- Symplectic topological aspects (Frejlich-Martínez-M).
- Quantization of b-symplectic manifolds (Guillemin- M.-Weitsman).
- What about b^n-symplectic manifolds? Guillemin-M.-Weitsman



Desingularizing b^m-symplectic structures

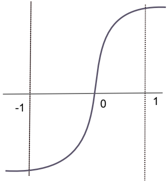
Theorem (Guillemin-M.-Weitsman)

Given a b^m-symplectic structure ω on a compact manifold (M^{2n}, Z):

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If $m = 2k+1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

$$\omega = \frac{dx}{x^{2k}} \wedge \left(\sum_{i=0}^{2k-1} \alpha_i x^i\right) + \beta \tag{4}$$

- Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be an odd smooth function satisfying $f'(x) > 0$ for all $x \in [-1, 1]$,



and satisfying

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1 \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1 \end{cases}$$

outside $[-1, 1]$.

- Scaling:

$$f_\epsilon(x) := \frac{1}{\epsilon^{2k-1}} f\left(\frac{x}{\epsilon}\right). \tag{5}$$

Outside the interval $[-\epsilon, \epsilon]$,

$$f_\epsilon(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x < -\epsilon \\ \frac{-1}{(2k-1)x^{2k-1}} + \frac{2}{\epsilon^{2k-1}} & \text{for} \quad x > \epsilon \end{cases}$$

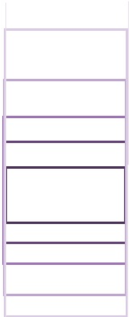
- Replace $\frac{dx}{x^{2k}}$ by df_ϵ to obtain

$$\omega_\epsilon = df_\epsilon \wedge \left(\sum_{i=0}^{2k-1} \alpha_i x^i\right) + \beta$$

which is symplectic.

Applications of deblogging

- Convexity for $\mathbb{T}^k$-actions.



- Delzant theorem and Delzant-type theorem for semitoric systems (bolytopes).
- Applications to KAM.
- Periodic orbits of problems in celestial mechanics and applications to stability (joint with Roisin Braddell, Amadeu Delshams, Cédric Oms and Arnau Planas) .