Isoparametric submanifolds of complex space forms

Miguel Domínguez Vázquez

Instituto de Ciencias Matemáticas (ICMAT-CSIC)

Supported by projects EM2014/009, and MTM2013-41335-P with FEDER funds (Spain).

XXV IFWGP - Madrid 2016

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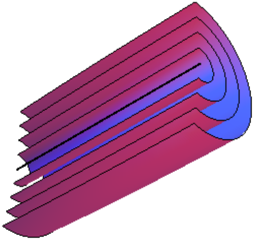
Definition

M is an isoparametric hypersurface if it and its nearby equidistant hypersurfaces have constant mean curvature



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M **homogeneous hypersurface** $\Rightarrow$ M **isoparametric hypersurface**

The classification in space forms

Theorem [Cartan, Segre (1938)]

Let M be a hypersurface in a real space form $\bar{M} \in \{\mathbb{R}^n, \mathbb{R}H^n, \mathbb{S}^n\}$. Then:

- M is isoparametric $\Leftrightarrow$ M has constant principal curvatures
- If $\bar{M} \in \{\mathbb{R}^n, \mathbb{R}H^n\}$, M is isoparametric $\Leftrightarrow$ M is homogeneous

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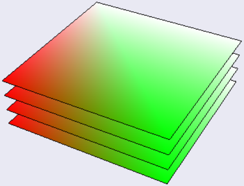
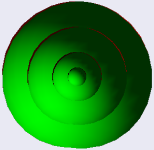
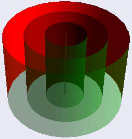
Classification in the Euclidean space $\mathbb{R}^n$ [Segre]



Parallel hyperplanes $\mathbb{R}^{n-1}$



Concentric spheres $\mathbb{S}^{n-1}$



Generalized cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k-1}$

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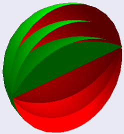
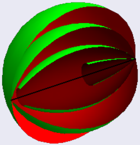
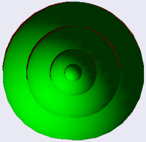
Classification in the real hyperbolic space $\mathbb{R}H^n$ [Cartan]



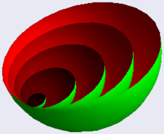
Tot. geod. $\mathbb{R}H^{n-1}$ and equidistant hypersurfaces



Tubes around a totally geodesic $\mathbb{R}H^k$



Geodesic spheres



Horospheres

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Classification in spheres $\mathbb{S}^n$

- The number of principal curvatures is $g \in \{1, 2, 3, 4, 6\}$ [Münzner]
- Homogeneous hypersurfaces are classified [Hsiang, Lawson]
- Hypersurfaces with $g \in \{1, 2, 3\}$ are homogeneous [Cartan]
- There are inhomogeneous examples with $g = 4$ [Ferus, Karcher, Münzner]
- All isoparametric hypersurfaces are homogeneous or of FKM-type [Takagi; Ozeki, Takeuchi; Stolz; Cecil, Chi, Jensen; Immervoll; Chi; Abresch; Dorfmeister, Neher; Miyaoka]

Isoparametric submanifolds

M submanifold in a Riemannian manifold $\bar{M}$

Definition [Heintze, Liu, Olmos (2006)]

M is isoparametric if

- M has flat normal bundle
- Nearby parallel submanifolds to M have CMC in radial directions
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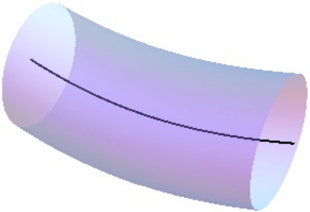

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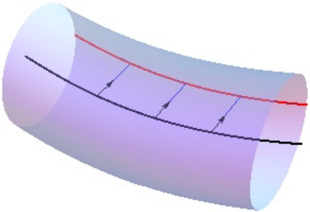

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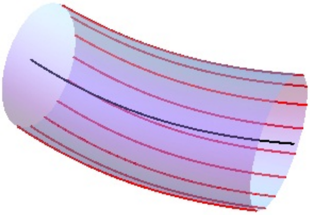

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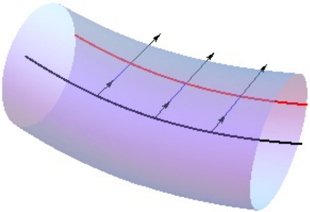

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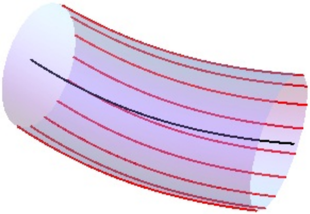

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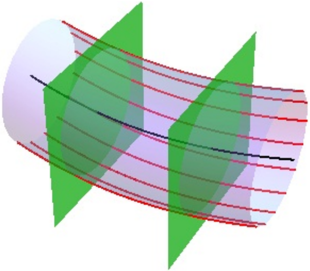

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- M extends to a complete isoparametric submanifold, which is a leaf of a global isoparametric foliation [Terng, *J. Differential Geom.* (1985)]
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- If $M \subset \mathbb{S}^n$ is an (irreducible, full) isoparametric submanifold of $\operatorname{codim} M \geq 2$, then M is an orbit of the isotropy representation of a symmetric space G/K [Thorbergsson, *Ann. of Math.* (1991)]

Complex space forms

Complex space forms

Simply connected, complete Kähler manifolds of constant holomorphic sectional curvature c:

- complex Euclidean spaces $\mathbb{C}^n$, if $c = 0$
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- The Hopf map $\pi\colon \mathrm{AdS}^{2n+1} \longrightarrow \mathbb{C}H^n$ is a semi-Riemannian submersion with $\mathbb{S}^1$-fibers

Isoparametric submanifolds in $\mathbb{C}P^n$

Idea of the approach [DV., *Trans. Amer. Math. Soc.* (2016)]

- An isoparametric submanifold in $\mathbb{C}P^n$ extends to a global isoparametric foliation on $\mathbb{C}P^n$

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- Every irreducible isoparametric foliation on $\mathbb{C}P^n$ is homogeneous if and only if $n+1$ is a prime number

The classification

Theorem [Díaz-Ramos, DV., Sanmartín-López, arXiv:1509.02498]

A connected hypersurface M in the complex hyperbolic space $\mathbb{C}H^n$ is isoparametric if and only if it is an open part of:

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4. A tube around a ruled homogeneous minimal submanifold $W_{\mathfrak{w}}$, for some proper real subspace $\mathfrak{w} \subset \mathfrak{g}_\alpha$

$\mathbb{C}H^n$ as a Lie group

- $G = SU(1,n)$ isometry group of $\mathbb{C}H^n$
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$$\mathfrak{g}_\alpha \equiv \mathbb{C}^{n-1} \quad , \mathfrak{g}_{2\alpha} \equiv \mathbb{R}$$

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- $\phi\colon AN \times \mathbb{C}H^n \to \mathbb{C}H^n$ free and transitive action

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- $\mathfrak{k}$ maximal compact in $\mathfrak{g}$
- $\mathfrak{a} \cong \mathbb{R}$ abelian
- $\mathfrak{n}$ nilpotent

- $\mathfrak{a} \oplus \mathfrak{n} \rightsquigarrow AN$ Lie subgroup of G
- $\phi\colon AN \times \mathbb{C}H^n \to \mathbb{C}H^n$ free and transitive action

The complex hyperbolic space as a Lie group

$$(\mathbb{C}H^n, \langle \cdot, \cdot \rangle) \equiv (AN, \phi_o^* \langle \cdot, \cdot \rangle)$$

- Solvable
- Left invariant metric

The examples of the classification

Examples

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- New examples [Díaz-Ramos, DV., *Math. Z.* (2012)]

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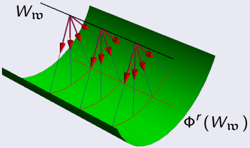
- $W_{\mathfrak{w}} = S_{\mathfrak{w}} \cdot o$ is a ruled homogeneous submanifold, $o \in \mathbb{C}H^n$
- The tubes around $W_{\mathfrak{w}}$ are isoparametric, but generically **inhomogeneous**



Sketch of the proof

- M isoparametric hypersurface in $\mathbb{C}H^n$, ξ unit normal to M
- $SX = -\bar{\nabla}_X \xi$ the shape operator of M
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$$\begin{array}{ccc} \mathrm{AdS}^{2n+1} & \qquad & \widetilde{M} = \pi^{-1}(M) \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{C}H^n & & M \end{array}$$

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With respect to the basis $\{X_1^L, \ldots, X_{2n-1}^L, V\}$, $\widetilde{S}$ is given by

$$\begin{pmatrix} \lambda_1 & & 0 & -\frac{b_1}{2} \\ & \ddots & & \vdots \\ 0 & & \lambda_{2n-1} & -\frac{b_{2n-1}}{2} \\ \frac{b_1}{2} & \ldots & \frac{b_{2n-1}}{2} & 0 \end{pmatrix}$$

- $X_1, \ldots, X_{2n-1}$ eigenvectors of S
- $b_i = \langle J\xi, X_i \rangle$
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M isoparametric $\Leftrightarrow$ $\widetilde{M}$ isoparametric

Sketch of the proof

Possibilities for the shape operator of $\widetilde{M}$:

Type I

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2n} \end{pmatrix}$$

Type II

$$\begin{pmatrix} \lambda_1 & 0 & & \\ \varepsilon & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{2n-1} \end{pmatrix}$$

Type III

$$\begin{pmatrix} \lambda_1 & 0 & 1 & & \\ 0 & \lambda_1 & 0 & & \\ 0 & 1 & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2} \end{pmatrix}$$

Type IV

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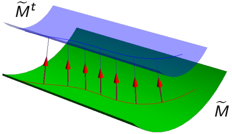

Sketch of the proof

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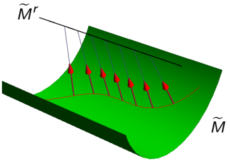

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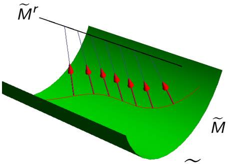

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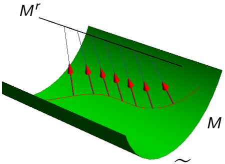

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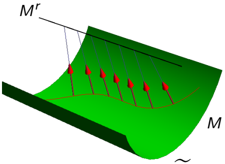

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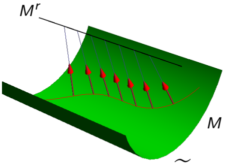

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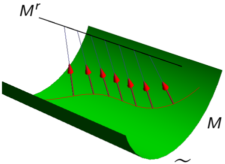

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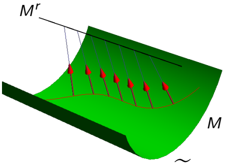

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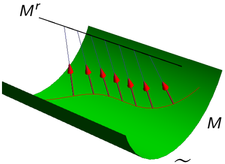

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- The horocycles determined by $(J\xi)^\top$ and x are contained in M^r, for each ξ normal to M^r

Isoparametric submanifolds in complex space forms

Isoparametric submanifolds in $\mathbb{C}H^2$

Theorem [Díaz-Ramos, DV., Vidal-Castiñeira, arXiv:1604.01237]

An isoparametric submanifold of $\mathbb{C}H^2$ is an open part of an orbit of maximal dimension of a polar action on $\mathbb{C}H^2$.

Definition [Dadok, *TAMS* (1985) & Palais, Terng, *TAMS* (1987)]

A polar action is an action $H \times \bar{M} \to \bar{M}$ of a Lie group H of isometries of a Riemannian manifold $\bar{M}$ such that there exists a totally geodesic submanifold Σ of $\bar{M}$ that intersects all the orbits and always orthogonally.

Polar actions on $\mathbb{C}H^2$ have been classified [Berndt, Díaz-Ramos, *Ann. Global Anal. Geom.* (2013)]. There are five different examples of cohomogeneity 1, and four of cohomogeneity 2.

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- If codim $M = 1$, M is an isoparametric hypersurface, and hence an orbit of a polar action of cohomogeneity 1 (by the previous classification)
- If codim $M = 2$, M is a Lagrangian flat surface with parallel mean curvature in $\mathbb{C}H^2$. Then, M is an orbit of maximal dimension of a polar action of cohomogeneity 2 [Díaz-Ramos, DV., Vidal-Castiñeira, *J. Geom. Anal.* (2016)]

Isoparametric submanifolds in complex space forms

1. Introduction
   1. Isoparametric hypersurfaces
   2. Isoparametric submanifolds
   3. Complex space forms
2. The classification problem in complex projective spaces
3. The classification problem in complex hyperbolic spaces
   1. The codimension one case
   2. The classification in the plane
4. Open problems

Some open problems

- Conclude the classification of isoparametric hypersurfaces in $\mathbb{C}P^{15}$
- Construct inhomogeneous examples of isoparametric submanifolds of codim ≥ 2 in $\mathbb{C}H^n$
- Extend the nonclassical construction of isoparametric hypersurfaces in $\mathbb{C}H^n$ to other noncompact symmetric spaces (the rank one case was done in [Díaz-Ramos, DV., *Adv. Math.* (2013)], and a partial extension to higher rank in [DV., *Int. Math. Res. Not.* (2015)])