Surfaces in Riemannian and Lorentzian 3-manifolds admitting a Killing vector field

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Graphs in Killing submersions

Some applications

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Basic properties of Killing submersions

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A Killing submersion is a Riemannian submersion $\pi : \mathbb{E} \to M$, where $\mathbb{E}$ and M are orientable and connected, such that the fibers of π are the integral curves of a Killing vector field ξ with no zeroes.

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Classification criterion

Two Killing submersions $\pi_1 : \mathbb{E}_1 \to M$ and $\pi_2 : \mathbb{E}_2 \to M$ are isomorphic if there exists an isometry $T : \mathbb{E}_1 \to \mathbb{E}_2$ such that $\pi_2 \circ T = \pi_1$.

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3. The bundle curvature τ.

$$\begin{aligned} \xi \text{ Killing} \quad &\Leftrightarrow \quad \langle \overline{\nabla}_X \xi, Y \rangle + \langle \overline{\nabla}_Y \xi, X \rangle = 0 \text{ for all } X, Y \in \mathfrak{X}(M) \\ &\Leftrightarrow \quad \omega(X, Y) = \langle \overline{\nabla}_X \xi, Y \rangle \text{ is skew-symmetric} \end{aligned}$$

We define $\tau(p) = \frac{1}{\|\xi\|}\omega(e_1, e_2)$, where $\{e_1, e_2\}$ positive orthonormal basis of $\ker(\mathrm{d}\pi)^\perp$ (horizontal distribution).

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- It is constant along the fibers $\rightsquigarrow$ $\tau \in C^\infty(M)$.
- It does not depend on the choice of ξ.
- Horizontal distribution integrable $\Leftrightarrow \tau \equiv 0$.

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Simply-connected homogeneous Riemannian 3-manifolds N

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  - Product spaces $\mathbb{H}^2(\kappa) \times \mathbb{R}$ and $\mathbb{S}^2(\kappa) \times \mathbb{R}$,
  - Heisenberg group Nil_3,
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Warped products $M \times_f \mathbb{R}$, endowed with the metric

$$\pi_M^*(\mathrm{d}s_M^2) \pm (f \circ \pi_M)\, \pi_{\mathbb{R}}^*(\mathrm{d}t^2),$$

where f is a function defined on M (1-dimensional fibers).

Existence and uniqueness

Theorem

Let M be a simply-connected surface and $\tau, \mu \in C^\infty(M)$, $\mu > 0$. Then there exists a Killing submersion $\pi : \mathbb{E} \to M$ such that

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  - $\mathbb{E}(M, \tau, \mu) = \big(\mathbb{D} \times \mathbb{R}, \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2) + \mu^2(\mathrm{d}z + \eta(y\mathrm{d}x - x\mathrm{d}y)\big)$
  - $\mathbb{L}(M, \tau, \mu) = \big(\mathbb{D} \times \mathbb{R}, \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2) - \mu^2(\mathrm{d}z - \eta(y\mathrm{d}x - x\mathrm{d}y)\big)$

$$\eta(x, y) = \int_0^1 \frac{2s\,\tau(xs,ys)\lambda(xs,ys)^2}{\mu(xs,ys)}\mathrm{d}s.$$

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The non-simply-connected case

An isometry $f : \mathbb{E}_1 \to \mathbb{E}_2$ is called a Killing isometry if it preserves the vertical direction. In particular,

$$\pi_2 \circ f = h \circ \pi_1.$$

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Lifting isometries

Assume $M_1, M_2, \mathbb{E}_1, \mathbb{E}_2$ are simply-connected, $h : M_1 \to M_2$ is an isometry and $p_1 \in \mathbb{E}_1$, $p_2 \in \mathbb{E}_2$ are such that $h(\pi_1(p_1)) = h(\pi_2(p_2))$.

- If $\tau_2 \circ h = \tau_1$ and $\mu_2 \circ h = \mu_1$, then there exists a unique orientation-preserving Killing isometry $f : \mathbb{E}_1 \to \mathbb{E}_2$ such that $f(p_1) = p_2$.

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Theorem

Let $\pi : \mathbb{E} \to M$ be a Killing submersion, and let $\rho : \widetilde{M} \to M$ and $\sigma : \widetilde{\mathbb{E}} \to \mathbb{E}$ be the universal Riemannian covering maps of M and $\mathbb{E}$, respectively.

- *There is a Killing submersion $\widetilde{\pi} : \widetilde{\mathbb{E}} \to \widetilde{M}$.*
- *There exists a group G of Killing isometries acting properly discontinuously on $\widetilde{\mathbb{E}}$ such that $\mathbb{E} = \widetilde{\mathbb{E}}/G$.*

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The mean curvature equation

Let $\pi : \mathbb{E} \to M$ be a Killing submersion

- A Killing graph over $\Omega \subset M$ is a smooth section $F : \Omega \to \pi^{-1}(\Omega) \subset \mathbb{E}$.
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$$F_u : \Omega \to \mathbb{E}, \qquad F_u(p) = \phi_{u(p)}(F_0(p)),$$

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- The vector field Z satisfies $\mathrm{div}(JZ) = \frac{-2\epsilon\tau}{\mu}$.

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Calabi-type correspondence

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$$\operatorname{div}\left(\frac{\mu\, Gu}{\sqrt{\mu^{-2} + \|Gu\|^2}}\right) = 2H\mu \iff \operatorname{div}\left(\frac{\mu\, Gu}{\sqrt{\mu^{-2} + \|Gu\|^2}} - JZ'\right) = 0$$

$$\iff \frac{\mu\, Gu}{\sqrt{\mu^{-2} + \|Gu\|^2}} - JZ' = -J\nabla v$$

We set $G'v = \nabla v - Z'$ and manipulate to obtain $\|G'v\|^2 < \mu^2$ and

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Theorem

There is a one-to-one conformal correspondence between

(a) Graphs in $\mathbb{E}(M, \tau, \mu)$ with mean curvature H.

(b) Spacelike graphs in $\mathbb{L}(M, H, \mu^{-1})$ with mean curvature τ.

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Application 1: Complete spacelike surfaces (Lorentz.)

Let $\pi : \mathbb{E} \to M$ be a Riemmannian Killing submersion. Assume that M is not compact, and Σ_u satisfies $H \geq c$.Given a regular domain $\Omega \subset\subset M$,

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- In particular, a spacetime satisfying (b) is not distinguishable.
- Example: Lorentzian Heisenberg group $\mathrm{Nil}_3^1(\tau) = \mathbb{L}(\mathbb{R}^2, \tau, 1)$.

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Consider $\pi : \mathbb{E} \to M$, M compact, bundle curvature τ and Kiling length μ.

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Existence uses (Meeks-Simon-Yau, 1982) and (Gerhardt, 1985).

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If Σ is stable then the angle function $\nu = \langle N, \xi \rangle$ satisfies either $\nu \equiv 0$ or $\nu > 0$ (ν lies in the kernel of the stability operator).

Theorem

If Σ a compact stable orientable surface immersed in $\mathbb{E}$ with constant mean curvature, then one of the following holds:

- M is compact and Σ is an entire minimal graph.
- the fibers of π are compact and Σ is everywhere tangent to the Killing direction.