MATERIAL GROUPOIDS AND ALGEBROIDS

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The idea

- The notion of *material groupoid* was first introduced in our article: M.E. and M. de L. (1998), Geometrical Theory of Uniform Cosserat Media, *Journal of Geometry and Physics* **26**, 127-180.
- Since it is one of the most intuitive ideas that one can think of in relation to the properties of material bodies, one may ask the following two questions:

1. Why did it take that long to be identified?
2. Why is it not in more widespread use?

A poetic answer

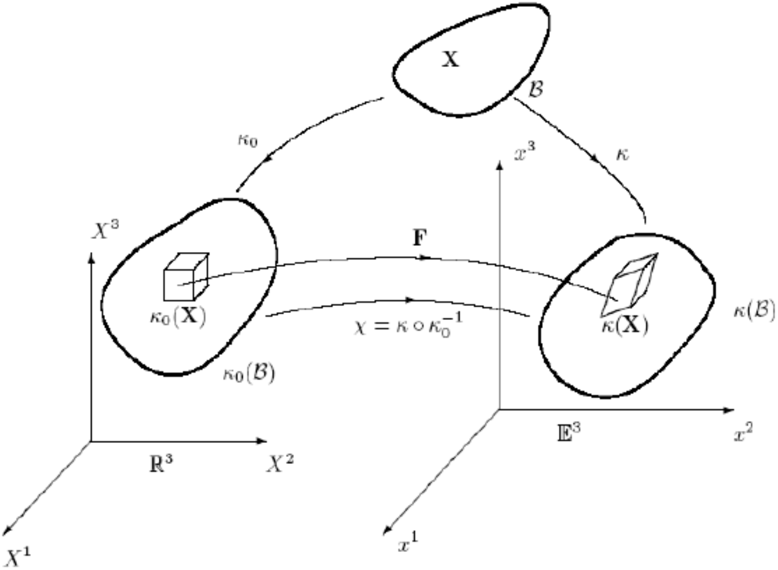
"... as beautiful as ... the chance encounter of a sewing machine and an umbrella on an operating table."

Chants de Maldoror, Comte de Lautréamont (1846-1870)

This presentation

- Basic engineering level
- For more details ...

Recall ... continuum kinematics



Naive picture

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- **Moreover, we need to establish a means of evaluating whether or not the comparison has been successful. If it has, we draw and arrow from X to Y. Otherwise, no arrow is drawn.**

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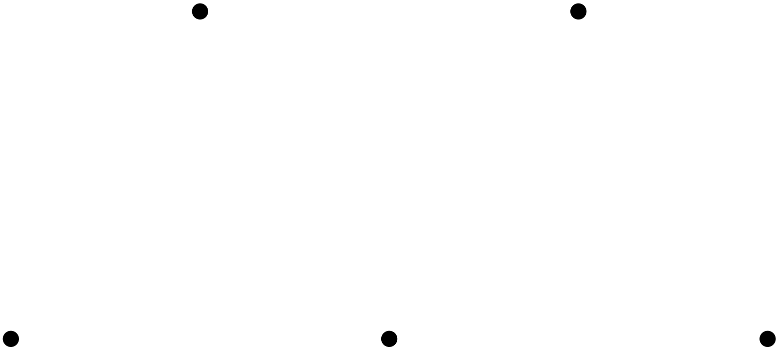
- Moreover, we need to establish a means of evaluating whether or not the comparison has been successful. If it has, we draw and arrow from X to Y. Otherwise, no arrow is drawn.
- **Since 'having the same property' is surely an *equivalence relation*, every point $X \in \mathcal{B}$ should be assigned at least one loop-shaped arrow. Moreover, for every arrow drawn from X to Y there should also be an arrow drawn from Y to X. Finally, if there is an arrow from X to Y and another arrow from Y to Z, there should also be an arrow from X to Z. The set $\mathcal{Z}$ of all these arrows constitutes the *material groupoid associated with the chosen material property*. The body $\mathcal{B}$ is the *base* of the material groupoid.**

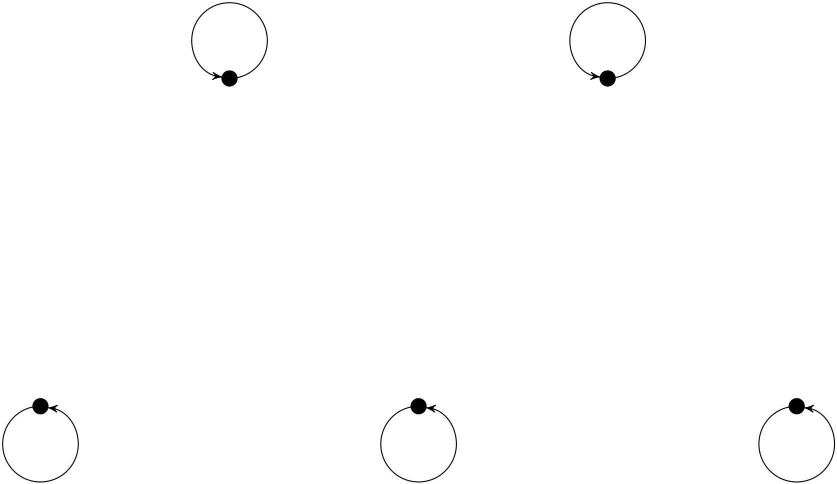
Intuitive picture

A collection $(\mathcal{Z})$ of arrows joining some points of a set $(\mathcal{M})$.



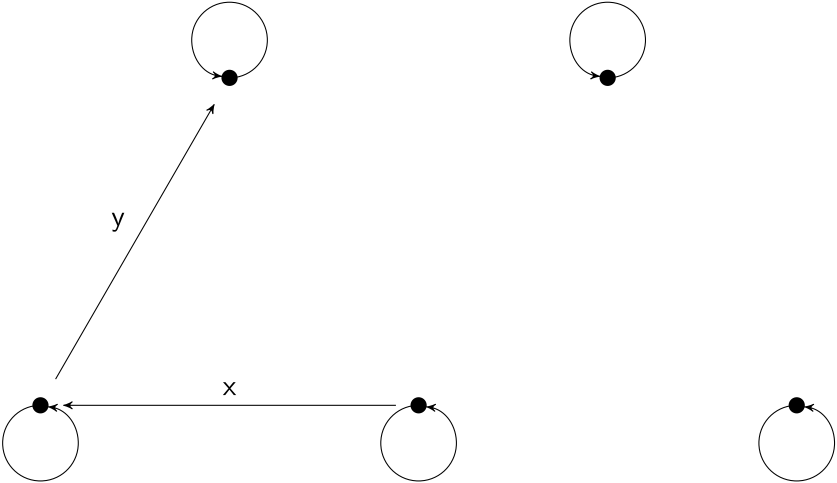
Intuitive picture

The identities must be included.



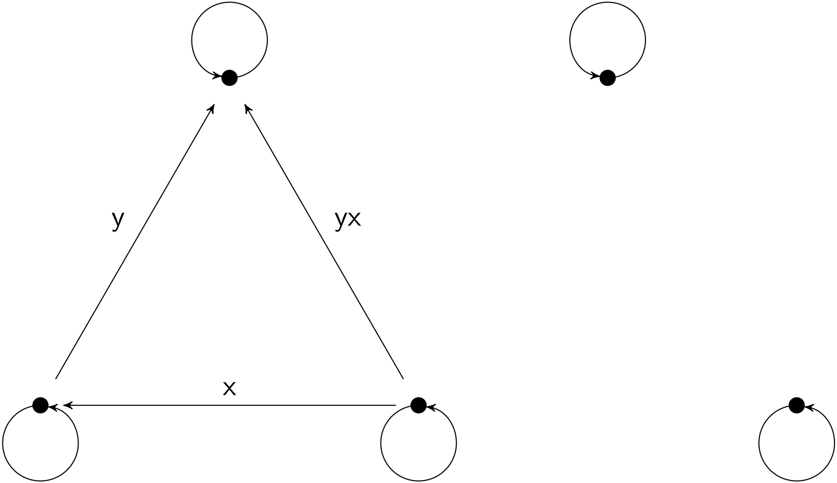
Intuitive picture

Some points of $\mathcal{M}$ may be connected.



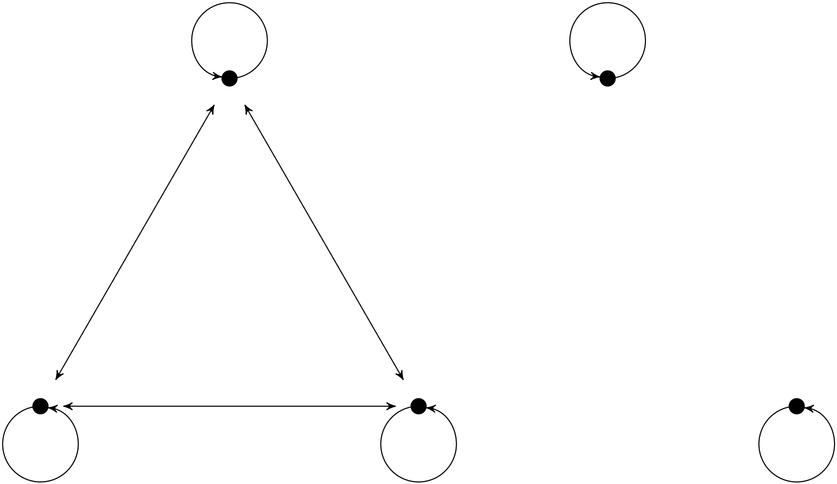
Intuitive picture

Compositions must be included.



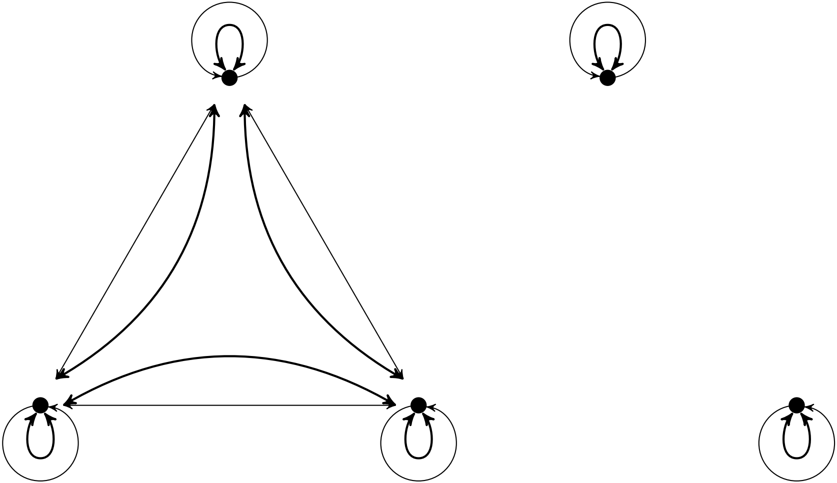
Intuitive picture

Inverses must be included.



Intuitive picture

Self connections form local groups and imply further connections.



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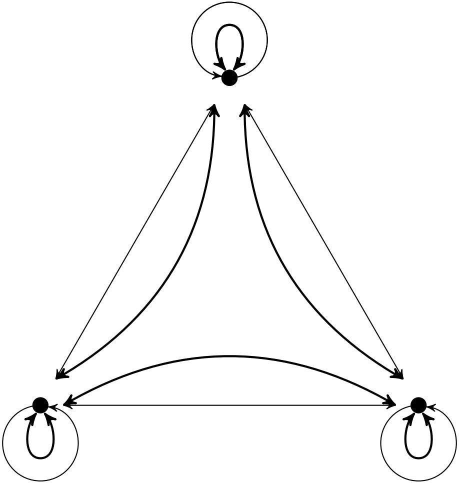
- **If every pair of points is joined by an arrow, the body is *uniform* with respect to the chosen property. In the mathematical terminology, we speak of a *transitive* groupoid. At the other extreme, if no two different points are connected by an arrow (so that the only surviving arrows are the loops), we have a *totally intransitive groupoid*.**

Depicting a transitive groupoid



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- **The question of *homogeneity* of a uniform body can be formulated in terms of the *material algebroid* associated with the material groupoid.**

Formal definition of a groupoid

Just like a group, a groupoid is a set $\mathcal{Z}$ with two operations, also called *inverse* and *product*. There is, however, an essential difference in respect of the product operation, namely, the product in a groupoid is not defined for all pairs of elements of $\mathcal{Z}$. In other words, given two elements $x, y \in \mathcal{Z}$, the product xy may or may not exist. The operations must satisfy

1. Associativity

$$\exists\, xy, yz \Leftrightarrow \exists\, x(yz) \Leftrightarrow \exists\, (xy)z \Rightarrow (xy)z = x(yz). \tag{1}$$

2. Inversion

$$\forall\, x \in \mathcal{Z}\, \exists\, x^{-1},\ \ x^{-1}x,\ \ xx^{-1}\ \ \in \mathcal{Z}. \tag{2}$$

3. Units or identities

$$\exists\, xy\ \Rightarrow\ xyy^{-1} = x,\ \ x^{-1}xy = y. \tag{3}$$

Alternative (more benign) definition of a groupoid

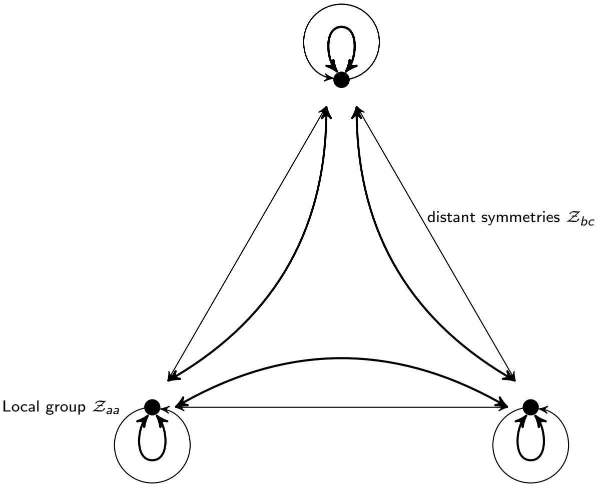
A *groupoid* consists of

1. A *total set* $\mathcal{Z}$ and a base set $\mathcal{M}$
2. Two (*projection*) surjective maps
$$\alpha : \mathcal{Z} \to \mathcal{M} \quad \text{and} \quad \beta : \mathcal{Z} \to \mathcal{M} \tag{4}$$
called, respectively, the *source* and the *target* maps
3. A binary associative operation zy (*composition*) defined only for those ordered pairs $(y, z) \in \mathcal{Z} \times \mathcal{Z}$ such that
$$\alpha(z) = \beta(y). \tag{5}$$
Moreover,
$$\alpha(zy) = \alpha(y), \qquad \beta(zy) = \beta(z). \tag{6}$$
4. An *identity* id_m at each point $m \in \mathcal{M}$, satisfying $z\, id_m = z$ whenever $\alpha(z) = m$, and $id_m\, z = z$ whenever $\beta(z) = m$
5. For each $z \in \mathcal{Z}$ there exists a (unique) *inverse* z^{-1} such that $zz^{-1} = id_{\beta(z)}$ and $z^{-1}z = id_{\alpha(z)}$

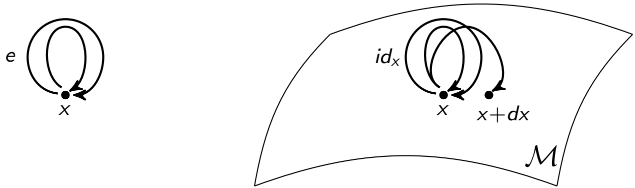
Transitive groupoids

1. A groupoid is said to be *transitive* if for each pair of points $a, b \in \mathcal{M}$ there exists at least one element of the total set with a and b as the source and target points, respectively.
2. Denoting $\mathcal{Z}_{ab} = \{z \in \mathcal{Z} \mid \beta(z) = b, \; \alpha(z) = a\}$, we may say that a groupoid is transitive if, and only if, $\mathcal{Z}_{ab} \neq \emptyset$ for all $(a, b) \in \mathcal{M} \times \mathcal{M}$. In a transitive groupoid all the local groups $\mathcal{Z}_{bb}$ are mutually conjugate. In this case, we can consider any of the local groups as the *typical group* of the transitive groupoid.

Depicting a transitive groupoid

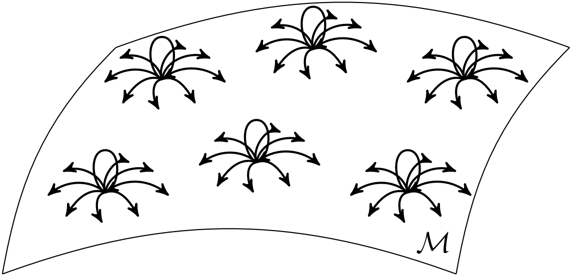


The Lie algebroid of a Lie groupoid



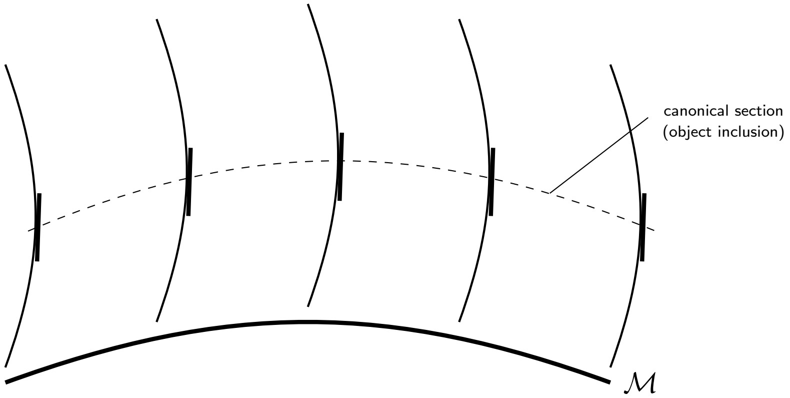
Thinking in a Lie algebra (left) and in a Lie algebroid (right)

A spider colony



The α-bundle $\mathcal{Z}_{\mathcal{M}}$ as a spider colony

The Lie algebroid as a vector bundle



The algebroid vector bundle (indicated with thicker lines) $A\mathcal{Z} \to \mathcal{M}$ as a particular sub-bundle of the tangent bundle $T\mathcal{Z}_{\mathcal{M}}$, selecting out of each fibre ('spider') of $\mathcal{Z}_{\mathcal{M}}$ the tangent space (to the fibre) at the identity.

Groupoids in Continuum Mechanics

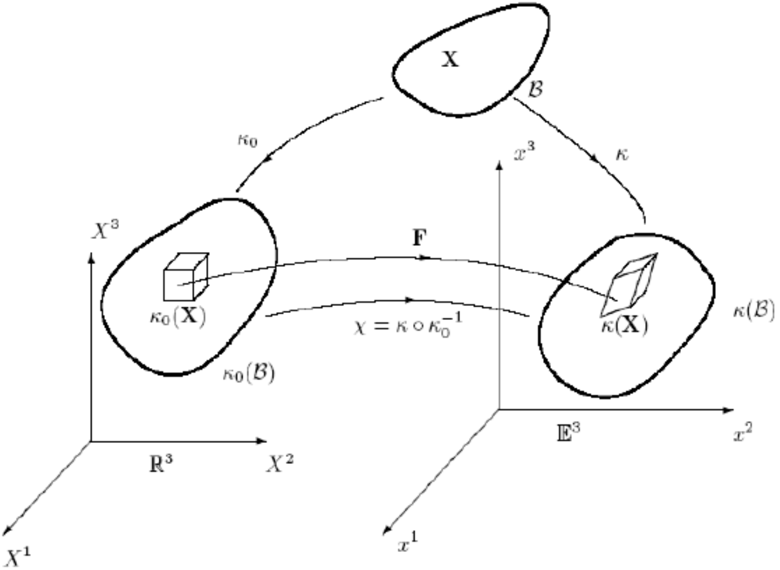
The notion of material groupoid arises quite naturally in Continuum Mechanics in the context of the constitutive theory as a result of the possible answers to the following question:

In a material body $\mathcal{B}$, given two material points, X_1 and X_2, and two instants of time, t_1 and t_2, what basis of comparison can be established between the corresponding material responses?

There are two meaningfully different answers to this question, giving rise to

1. The material groupoid
2. The material-type groupoid

Recall ... continuum kinematics



The material groupoid (I)

Consider a constitutive function $\psi = \psi(\mathbf{F})$ pertaining to a certain physical property of a *simple* material. As the general linear group $GL(3; \mathbb{R})$ acts to the right on this function, we obtain the orbit

$$\mathcal{O}_\psi = \{\phi \mid \phi(\mathbf{F}) = \psi(\mathbf{FP})\},$$

with $\mathbf{F}, \mathbf{P} \in GL(3; \mathbb{R})$.

Clearly, all the constitutive functions in this orbit *represent the same material response*. They differ only by the choice of local reference configuration. From the 'chemical' point of view, it would be impossible to distinguish between them!

The material groupoid (II)

Let $\psi = \psi(\mathbf{F}; X)$ with $X \in \mathcal{B}$ be (for now) a time-independent constitutive response for a body $\mathcal{B}$.

We declare $X_1, X_2 \in \mathcal{B}$ to be *materially isomorphic*[1] if $\psi(\mathbf{F}; X_1)$ and $\psi(\mathbf{F}; X_2)$ belong to the same constitutive orbit, as defined above. In other words, the two points are materially isomorphic if there is a linear map $\mathbf{P}_{12}$ (called a *material isomorphism*) such that, for all $\mathbf{F}$,

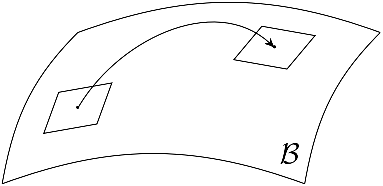
$$\psi(\mathbf{F}; X_2) = \psi(\mathbf{F}\mathbf{P}_{12}; X_1).$$

Note that a material automorphism is the same as a material symmetry in the usual sense.

If we use an arrow between two points for every material isomorphism, we obtain our first *material groupoid*. The material groupoid is transitive if, and only if, the body is *materially uniform*.

[1]Noll, 1967.

Pictorial representation



The arrow (an element of the material groupoid) represents a material isomorphism, that is, a non-singular linear map between the tangent spaces of two body points that satisfies the constitutive condition described above.

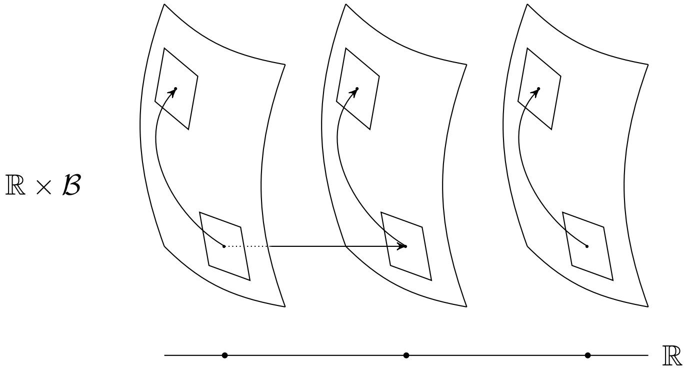
The temporal dimension

A body is said to undergo a process of *material evolution* if its constitutive descriptor is of the form

$$\psi = \psi(\mathbf{F}; X, t - t_0).$$

Accordingly, we can introduce the *body-time manifold* $\mathcal{M} = \mathbb{R} \times \mathcal{B}$ and build upon it the *body-time material groupoid* just as before. Fixing attention on a particular point $X \in \mathcal{B}$ and letting time t run, we ask: What is the physical meaning of the successive responses remaining always in the same constitutive orbit?

Pictorial representation in the body-time manifold



A few arrows representing elements of a body-time material groupoid

Remodeling

The temporal counterpart of uniformity is a special kind of material evolution, whereby a material point remains materially isomorphic to a reference material point with the passage of time. We call this special type of material evolution *remodeling*, with or without growth. Any other kind of evolution we call *aging*.

If we consider a *transitive* body-time material groupoid, its physical meaning is a body that is initially materially uniform and that evolves by pure remodeling (with no aging). In particular, it remains always materially uniform. Classical plastic (or anelastic) evolution belongs to this material class and so does the model of tissue growth pioneered by Rodriguez et al. (1994)

Remark

Whether or not one agrees with the naming of this kind of evolution as ‘remodeling’ is not at issue. What is important is to recognize that the mathematics itself forces us to single out this particular type as worthy of special consideration.

The question that arises now is the following: Given an arbitrary process of material evolution, is it possible to *canonically* resolve it into a part attributable to remodeling and a part attributable to aging? Clearly, the answer to this question has a bearing on the construction of physical models that account for physically and chemically based cause-effect relations.

This remark leads us to the introduction of a second kind of material groupoid.

Functionally graded materials (FGM)

By their very definition, FGMs are not uniform bodies. Their usual process of fabrication, however, entails a preservation of the symmetry type (isotropy, orthotropy) from point to point in the body. Using this feature as a hint, we will declare two points of a material body to be *materially isosymmetric* if their constitutive responses have conjugate symmetry groups. Clearly, material isomorphism implies material isosymmetry, but not vice versa.

The material-type groupoid

If we assign an arrow to every possible conjugation between the symmetry groups at different pairs of points in the body $\mathcal{B}$, we obtain the *material-type groupoid* of the body.

If the material-type groupoid is transitive, we obtain a body like an FGM, which we call a *unisymmetric body*. Recall that a transitive groupoid is characterized by a *typical group*. Clearly, the typical group of a transitive material groupoid (read: uniform body) is the typical material symmetry group. But what is the typical group of our new transitive material-type groupoid? The somewhat disappointing answer is: the *normalizer* of the typical symmetry group within the general linear group. This is a much larger entity (that includes, for example, all dilatations).

The time dimension

What is the counterpart of unisymmetry in the time domain? If we fix a point and let time go on, isosymmetry implies that the material, though changing its material properties (such as elastic moduli), preserves its symmetry type. So, rubber may turn into gold! These are very common processes of aging, such as the case of osteoporosis, where there is a deterioration of the mineral quality and quantity of trabecular bone.

What is the interpretation of other kinds of aging processes? They would represent processes whereby the material symmetry type changes (e.g., from isotropy to transverse isotropy). These necessarily abrupt changes give rise to phenomena of *morphogenesis* and pattern formation.

The solid case

We mentioned the fact that the typical group of the material-type groupoid is the normalizer of the symmetry group within the general linear group. In the solid case, however, it is possible to define a reduced groupoid based on the normalizer *within the orthogonal group*. Moreover, for certain solid classes (including isotropy, transverse isotropy and orthotropy), an important result can be shown to hold true, namely: In a unisymmetric solid evolving without morphogenesis, a canonical separation can be effected between aging and remodeling components.

The solid material-type groupoid (I)

Let X and Y be two isosymmetric solid points in some reference configuration, and let A be a conjugation between the respective symmetry groups $\mathcal{G}_X$ and $\mathcal{G}_Y$, namely, $\mathcal{G}_Y = A\,\mathcal{G}_X\,A^{-1}$.

Let $K_X : T_X\mathcal{B} \to \mathbb{R}^3$ and $K_Y : T_Y\mathcal{B} \to \mathbb{R}^3$ be respective natural states, whose symmetry groups we denote by $\bar{\mathcal{G}}_X$ and $\bar{\mathcal{G}}_Y$.

$$\begin{tikzcd}
T_X\mathcal{B} \arrow[r, "A"] \arrow[d, "K_X"'] & T_Y\mathcal{B} \arrow[d, "K_Y"] \\
\mathbb{R}^3 \arrow[r, "\bar{A}"] & \mathbb{R}^3
\end{tikzcd}$$

A conjugation between $\bar{\mathcal{G}}_X$ and $\bar{\mathcal{G}}_Y$ is clearly given by

$$\bar{A} = K_Y\,A\,K_X^{-1}$$

The solid material-type groupoid (II)

<u>*Lemma*</u>*: Conjugate orthogonal subgroups of the general linear group are orthogonally conjugate.*[2] In fact, the orthogonal part of the polar decomposition of a conjugation in the general linear group between two subgroups of the orthogonal group is also a conjugation between these subgroups.

Since $\bar{\mathcal{G}}_X$ and $\bar{\mathcal{G}}_Y$ are orthogonal subgroups, the lemma applies. Let $\bar{Q}$ be an orthogonal conjugation. Then

$$Q = K_Y^{-1}\,\bar{Q}\,K_X$$

is the corresponding conjugation between $\mathcal{G}_X$ and $\mathcal{G}_Y$.

We define the *solid material-type groupoid* by means of all the conjugations attainable in this way.

[2] Coleman B and Noll W (1964).

The solid material-type groupoid (III)

What has been gained? The degrees of freedom in the choice of 'arrows' in the groupoids introduced so far as follows:

1. Material groupoid: Symmetry group $\mathcal{G}$ of source or target
2. Material-type groupoid: Normalizer $\mathcal{N}(\mathcal{G})$ of $\mathcal{G}$ within the general linear group
3. Solid material-type groupoid: Normalizer $\bar{\mathcal{N}}(\bar{\mathcal{G}})$ of $\bar{\mathcal{G}}$ *within the orthogonal group*

Corollary: If $\bar{\mathcal{N}}(\bar{\mathcal{G}}) = \bar{\mathcal{G}}$, the solid material-type groupoid (of a FGM) coincides with the material groupoid (of a uniform body). As a consequence, for solid classes with this property, we can talk about inhomogeneities (dislocations) and canonically split remodeling from aging!