SOME REMARKS ON DIRAC-KÄHLER OPERATOR

F.Di Cosmo, G.Marmo, J.M.Peréz-Pardo, A.Zampini

XXV International Fall Workshop on Geometry and Physics
Madrid
2 September 2016

Introduction
Kahler Formalism and Dirac-Kahler equation
Lie Groups
Homogeneous Spaces
Conclusions

Contents

- Introduction
- Kahler Formalism and Dirac-Kahler Equation
- Lie Groups
- Homogeneous Spaces
- Conclusions

Contents

- Introduction
- Kahler Formalism and Dirac-Kahler Equation
- Lie Groups
- Homogeneous Spaces
- Conclusions

Contents

- Introduction
- Kahler Formalism and Dirac-Kahler Equation
- Lie Groups
- Homogeneous Spaces
- Conclusions

Contents

- Introduction
- Kahler Formalism and Dirac-Kahler Equation
- Lie Groups
- Homogeneous Spaces
- Conclusions

Contents

- Introduction
- Kahler Formalism and Dirac-Kahler Equation
- Lie Groups
- Homogeneous Spaces
- Conclusions

Introduction

- Dirac Equation was written in 1928 by Dirac. On $\mathbb{R}^4$ with the Minkowsky metric tensor η, it assumes the following form

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

where $\{\gamma^\mu\}$ are matrices satisfyng the rules of the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\}_+ = 2\eta^{\mu\nu}$$

The group of even invertible elements of a Clifford Algebra is called Spin Group.

- He introduced this equation as a square-root of the Klein-Gordon equation, in the attempt of finding a relativistic description of quantum particles.
- A generalization of this equation to Riemannian manifold, has lead to the development of the so-called Spin Geometry.

Introduction

- Dirac Equation was written in 1928 by Dirac. On $\mathbb{R}^4$ with the Minkowsky metric tensor η, it assumes the following form

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi = 0$$

  where $\{\gamma^{\mu}\}$ are matrices satisfyng the rules of the Clifford algebra

$$\{\gamma^{\mu}, \gamma^{\nu}\}_{+} = 2\eta^{\mu\nu}$$

  The group of even invertible elements of a Clifford Algebra is called Spin Group.
- He introduced this equation as a square-root of the Klein-Gordon equation, in the attempt of finding a relativistic description of quantum particles.
- A generalization of this equation to Riemannian manifold, has lead to the development of the so-called Spin Geometry.

Introduction

- Dirac Equation was written in 1928 by Dirac. On $\mathbb{R}^4$ with the Minkowsky metric tensor η, it assumes the following form

  $$(i\gamma^\mu \partial_\mu - m)\,\psi = 0$$

  where $\{\gamma^\mu\}$ are matrices satisfyng the rules of the Clifford algebra

  $$\{\gamma^\mu, \gamma^\nu\}_+ = 2\eta^{\mu\nu}$$

  The group of even invertible elements of a Clifford Algebra is called Spin Group.
- He introduced this equation as a square-root of the Klein-Gordon equation, in the attempt of finding a relativistic description of quantum particles.
- A generalization of this equation to Riemannian manifold, has lead to the development of the so-called Spin Geometry.

Introduction

- In 1962 Kahler introduced a different formulation of Dirac equation in which spinor fields are described by means of inhomogeneous differential forms
- During the 80's Kahler formulation was deeply investigated in the attempt of unifying gravitational theory and Quantum Field Theory. Therefore the main efforts were made to generalize this construction to Riemannian manifold.
- The aim of our work is to exploit the tensorial behaviour of differential forms and to introduce in Kahler formulation unfolding and reduction techniques. Therefore we have considered a different generalization of Kahler formulation, in the direction of Lie Groups and Homogeneous spaces.
- The main motivation of this preliminary work is the fact that some manifolds with boundaries can be treated as orbifolds $\rightarrow$ Hints for the analysis of relationships between boundary conditions and topology

Introduction

- In 1962 Kahler introduced a different formulation of Dirac equation in which spinor fields are described by means of inhomogeneous differential forms
- During the 80's Kahler formulation was deeply investigated in the attempt of unifying gravitational theory and Quantum Field Theory. Therefore the main efforts were made to generalize this construction to Riemannian manifold.
- The aim of our work is to exploit the tensorial behaviour of differential forms and to introduce in Kahler formulation unfolding and reduction techniques. Therefore we have considered a different generalization of Kahler formulation, in the direction of Lie Groups and Homogeneous spaces.
- The main motivation of this preliminary work is the fact that some manifolds with boundaries can be treated as orbifolds $\rightarrow$ Hints for the analysis of relationships between boundary conditions and topology

Introduction

- In 1962 Kahler introduced a different formulation of Dirac equation in which spinor fields are described by means of inhomogeneous differential forms
- During the 80's Kahler formulation was deeply investigated in the attempt of unifying gravitational theory and Quantum Field Theory. Therefore the main efforts were made to generalize this construction to Riemannian manifold.
- The aim of our work is to exploit the tensorial behaviour of differential forms and to introduce in Kahler formulation unfolding and reduction techniques. Therefore we have considered a different generalization of Kahler formulation, in the direction of Lie Groups and Homogeneous spaces.
- The main motivation of this preliminary work is the fact that some manifolds with boundaries can be treated as orbifolds $\rightarrow$ Hints for the analysis of relationships between boundary conditions and topology

Introduction

- In 1962 Kahler introduced a different formulation of Dirac equation in which spinor fields are described by means of inhomogeneous differential forms
- During the 80's Kahler formulation was deeply investigated in the attempt of unifying gravitational theory and Quantum Field Theory. Therefore the main efforts were made to generalize this construction to Riemannian manifold.
- The aim of our work is to exploit the tensorial behaviour of differential forms and to introduce in Kahler formulation unfolding and reduction techniques. Therefore we have considered a different generalization of Kahler formulation, in the direction of Lie Groups and Homogeneous spaces.
- The main motivation of this preliminary work is the fact that some manifolds with boundaries can be treated as orbifolds $\rightarrow$ Hints for the analysis of relationships between boundary conditions and topology

Kahler-Atiyah Algebra

In 1962 Kahler proposed an inner calculus over the exterior algebra of a manifold M, equipped with a metric tensor g.
He provided a representation of the Clifford algebra $Cl(TM, g)$ on the exterior algebra $\Lambda(M)$ by defining the so called $\vee$-product

$$\phi \vee \omega = \sum_s \frac{(-1)^{s(s-1)/2}}{s!} g^{a_1 b_1} \cdots g^{a_s b_s} (\mu^s \{ i_{e_{a_1}} \cdots i_{e_{a_s}} \phi \}) \wedge \{ i_{e_{b_1}} \cdots i_{e_{b_s}} \omega \}$$

where $\phi \in \Lambda^k(M)$ and $\omega \in \Lambda^p(M)$.
This representation, however, is not irreducible, being defined on a 2^m-dimensional module.

Dirac-Kahler equation

By means of the previous Clifford product, Kahler wrote a Dirac-type equation on the exterior algebra $\Lambda(M)$.
By analogy with the usual Dirac operator, Dirac-Kahler operator is written

$$\mathcal{D} = idx^a \vee \nabla_{\partial_a} \tag{1}$$

It is possible to show that this operator coincides with the following one:

$$\mathcal{D} = (d + \delta) \tag{2}$$

Therefore Dirac-Kahler equation is

$$(d + \delta - m)\psi = 0$$

(2) is clearly a square-root of the Laplace-de Rham operator $\Delta = (d + \delta)^2$ but it does not need to be represented in terms of matrix-valued differential operators.

Dirac-Kahler equation

By means of the previous Clifford product, Kahler wrote a Dirac-type equation on the exterior algebra $\Lambda(M)$.
By analogy with the usual Dirac operator, Dirac-Kahler operator is written

$$\mathcal{D} = idx^a \vee \nabla_{\partial_a} \tag{1}$$

It is possible to show that this operator coincides with the following one:

$$\mathcal{D} = (d + \delta) \tag{2}$$

Therefore Dirac-Kahler equation is

$$(d + \delta - m)\psi = 0$$

(2) is clearly a square-root of the Laplace-de Rham operator $\Delta = (d+\delta)^2$ but it does not need to be represented in terms of matrix-valued differential operators.

DIRAC-KAHLER EQUATION

By means of the previous Clifford product, Kahler wrote a Dirac-type equation on the exterior algebra $\Lambda(M)$.
By analogy with the usual Dirac operator, Dirac-Kahler operator is written

$$\mathcal{D} = idx^a \vee \nabla_{\partial_a} \tag{1}$$

It is possible to show that this operator coincides with the following one:

$$\mathcal{D} = (d + \delta) \tag{2}$$

Therefore Dirac-Kahler equation is

$$(d + \delta - m)\psi = 0$$

(2) is clearly a square-root of the Laplace-de Rham operator $\Delta = (d + \delta)^2$ but it does not need to be represented in terms of matrix-valued differential operators.

Projectors

- Irreducible representations are defined by means of a family of minimal orthogonal projectors $\{P_j\}$, providing a decomposition of the identity.
- The range of each projector is an ideal of this Clifford algebra and it is a module on which Clifford action is irreducible: these elements are called algebraic spinors.
- It is not always possible to decompose the action of Dirac-Kahler operator on the space of algebraic spinors.
- A compatibility condition has to be satisfied:

$$P_j \vee \nabla P_j = 0 \tag{3}$$

Projectors

- Irreducible representations are defined by means of a family of minimal orthogonal projectors $\{P_j\}$, providing a decomposition of the identity.
- The range of each projector is an ideal of this Clifford algebra and it is a module on which Clifford action is irreducible: these elements are called algebraic spinors.
- It is not always possible to decompose the action of Dirac-Kahler operator on the space of algebraic spinors.
- A compatibility condition has to be satisfied:

$$P_j \vee \nabla P_j = 0 \tag{3}$$

Projectors

- Irreducible representations are defined by means of a family of minimal orthogonal projectors $\{P_j\}$, providing a decomposition of the identity.
- The range of each projector is an ideal of this Clifford algebra and it is a module on which Clifford action is irreducible: these elements are called algebraic spinors.
- It is not always possible to decompose the action of Dirac-Kahler operator on the space of algebraic spinors.
- A compatibility condition has to be satisfied:

$$P_j \vee \nabla P_j = 0 \tag{3}$$

Projectors

- Irreducible representations are defined by means of a family of minimal orthogonal projectors $\{P_j\}$, providing a decomposition of the identity.
- The range of each projector is an ideal of this Clifford algebra and it is a module on which Clifford action is irreducible: these elements are called algebraic spinors.
- It is not always possible to decompose the action of Dirac-Kahler operator on the space of algebraic spinors.
- A compatibility condition has to be satisfied:

$$P_j \vee \nabla P_j = 0 \tag{3}$$

The Kahler-Atyiah Algebra on $SU(2)$

Let us consider a concrete example: $SU(2)$ as Lie group with the metric tensor

$$g = \delta_{ab}\theta^a \otimes \theta^b$$

where $\{\theta^a\}$ is a basis of left-invariant differential forms. The $\vee$-product among these differential forms is easily computed

$$\theta^a \vee \theta^b = \theta^a \wedge \theta^b + \delta^{ab}$$

The Kahler-Atyiah Algebra on $SU(2)$

Let us consider a concrete example: $SU(2)$ as Lie group with the metric tensor

$$g = \delta_{ab}\theta^a \otimes \theta^b$$

where $\{\theta^a\}$ is a basis of left-invariant differential forms. The $\vee$-product among these differential forms is easily computed

$$\theta^a \vee \theta^b = \theta^a \wedge \theta^b + \delta^{ab}$$

Irreducible representations

A family of projectors which provides a decomposition of this action into irreducible subspaces is given by

$$P^{\pm}_{\pm} = \frac{1}{4}\left(1 \pm i\theta^1 \wedge \theta^2\right) \vee \left(1 \pm i\theta^1 \wedge \theta^2\right)$$

Each range is a two dimensional Clifford module on which the differential forms $\{\theta^a\}$ have the following representation

$$\theta^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \theta^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \theta^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These projectors, however, are not compatible with the action of the corresponding Dirac-Kahler operator.
The Levi-Civita connection indeed is

$$\nabla_{X_a}\theta^b = \frac{1}{2}\epsilon^b_{ca}\theta^c$$

Irreducible representations

A family of projectors which provides a decomposition of this action into irreducible subspaces is given by

$$P^{\pm}_{\pm} = \frac{1}{4}\left(1 \pm i\theta^1 \wedge \theta^2\right) \vee \left(1 \pm i\theta^1 \wedge \theta^2\right)$$

Each range is a two dimensional Clifford module on which the differential forms $\{\theta^a\}$ have the following representation

$$\theta^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \theta^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \theta^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These projectors, however, are not compatible with the action of the corresponding Dirac-Kahler operator.
The Levi-Civita connection indeed is

$$\nabla_{X_a}\theta^b = \frac{1}{2}\epsilon^{b}_{ca}\theta^c$$

Irreducible representations

A family of projectors which provides a decomposition of this action into irreducible subspaces is given by

$$P_{\pm}^{\pm} = \frac{1}{4}\left(1 \pm i\theta^1 \wedge \theta^2\right) \vee \left(1 \pm i\theta^1 \wedge \theta^2\right)$$

Each range is a two dimensional Clifford module on which the differential forms $\{\theta^a\}$ have the following representation

$$\theta^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \theta^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \theta^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These projectors, however, are not compatible with the action of the corresponding Dirac-Kahler operator.
The Levi-Civita connection indeed is

$$\nabla_{X_a}\theta^b = \frac{1}{2}\epsilon^b_{ca}\theta^c$$

Dirac-Kahler operator as a Matrix-Valued differential operator

A family of projectors which allows to decompose the action of Dirac-Kahler operator is

$$P_{\pm} = \frac{1}{2}\left(1 \pm i\theta^1 \wedge \theta^2 \wedge \theta^3\right)$$

The corresponding matrix-valued differential operator is

$$\mathcal{D} = \begin{pmatrix} 0 & L_{X_1} & L_{X_2} & L_{X_3} \\ L_{X_1} & -i & iL_{X_3} & -iL_{X_2} \\ L_{X_2} & -iL_{X_3} & -i & iL_{X_1} \\ L_{X_3} & iL_{X_2} & -iL_{X_1} & -i \end{pmatrix}$$

Dirac-Kahler operator as a Matrix-Valued differential operator

A family of projectors which allows to decompose the action of Dirac-Kahler operator is

$$P_{\pm} = \frac{1}{2}\left(1 \pm i\theta^1 \wedge \theta^2 \wedge \theta^3\right)$$

The corresponding matrix-valued differential operator is

$$\mathcal{D} = \begin{pmatrix} 0 & L_{X_1} & L_{X_2} & L_{X_3} \\ L_{X_1} & -i & iL_{X_3} & -iL_{X_2} \\ L_{X_2} & -iL_{X_3} & -i & iL_{X_1} \\ L_{X_3} & iL_{X_2} & -iL_{X_1} & -i \end{pmatrix}$$

The exterior algebra on S^2

HOPF BUNDLE

$$\begin{tikzcd} SU(2) \arrow[d, "\pi"'] & U(1) \arrow[l, "R"'] \\ S^2 & \end{tikzcd}$$

It is possible to consider the exterior algebra $\Lambda(S^2)$ as a subalgebra $\Lambda_0(S^3)$ of the exterior algebra $\Lambda(S^3)$. These satisfy the following conditions:

- $i_{X_3}\alpha = 0$
- $L_{X_3}\alpha = 0$

Differential forms $\alpha \in \Lambda_0(S^3)$ can be written as

$$\alpha = f + \alpha_+\theta^+ + \alpha_-\theta^- + i\omega\theta^- \wedge \theta^+$$

where $\theta^\pm = \theta^1 \mp i\theta^2$ and the coefficients satisfy the eigenvalue equation

$$L_{X_3} f = L_{X_3}\omega = 0$$
$$L_{X_3}\alpha_\pm = \mp i\alpha_\pm$$

The exterior algebra on S^2

HOPF BUNDLE

$$\begin{tikzcd} SU(2) & U(1) \arrow[l, "R"'] \\ S^2 \arrow[from=1-1, "\pi"'] \end{tikzcd}$$

It is possible to consider the exterior algebra $\Lambda(S^2)$ as a subalgebra $\Lambda_0(S^3)$ of the exterior algebra $\Lambda(S^3)$. These satisfy the following conditions:

- $i_{X_3}\alpha = 0$
- $L_{X_3}\alpha = 0$

Differential forms $\alpha \in \Lambda_0(S^3)$ can be written as

$$\alpha = f + \alpha_+\theta^+ + \alpha_-\theta^- + i\omega\theta^- \wedge \theta^+$$

where $\theta^\pm = \theta^1 \mp i\theta^2$ and the coefficients satisfy the eigenvalue equation

$$L_{X_3} f = L_{X_3}\omega = 0$$
$$L_{X_3}\alpha_\pm = \mp i\alpha_\pm$$

The exterior algebra on S^2

HOPF BUNDLE

$$\begin{tikzcd} SU(2) \arrow[d, "\pi"'] & U(1) \arrow[l, "R"'] \\ S^2 & \end{tikzcd}$$

It is possible to consider the exterior algebra $\Lambda(S^2)$ as a subalgebra $\Lambda_0(S^3)$ of the exterior algebra $\Lambda(S^3)$. These satisfy the following conditions:

- $i_{X_3}\alpha = 0$
- $L_{X_3}\alpha = 0$

Differential forms $\alpha \in \Lambda_0(S^3)$ can be written as

$$\alpha = f + \alpha_+\theta^+ + \alpha_-\theta^- + i\omega\theta^- \wedge \theta^+$$

where $\theta^\pm = \theta^1 \mp i\theta^2$ and the coefficients satisfy the eigenvalue equation

$$L_{X_3} f = L_{X_3}\omega = 0$$
$$L_{X_3}\alpha_\pm = \mp i\alpha_\pm$$

Quadratic Form

- If one considers the Killing-Cartan metric on the sphere S^3 with the associated metric volume $\theta^1 \wedge \theta^2 \wedge \theta^3$, the corresponding Dirac-Kähler operator does not give a meaningful operator when its domain is restricted to $\Lambda_0(S^3)$
- This problem can be solved by introducing on S^3 a degenerate quadratic form
$$g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2$$
with a suitable Hodge dual operator $*_{S^2} : \Lambda_0(S^3) \to \Lambda_0(S^3)$:
$$\begin{aligned} *_{S^2}(f) &= if\theta^- \wedge \theta^+ \\ *_{S^2}(\alpha_-\theta^-) &= -i\alpha - \theta^- \\ *_{S^2}(\alpha_+\theta^+) &= i\alpha_+\theta^+ \\ *_{S^2}(i\omega\theta^- \wedge \theta^+) &= \omega \end{aligned}$$
- It is possible to define a hermitean product in $\Lambda_0(S^3)$ as follows
$$(\alpha, \beta) = \int_S^3 \theta^3 \wedge \bar{\alpha} \wedge *_{S^2}\beta \tag{4}$$

Introduction
Kahler Formalism and Dirac-Kahler equation
Lie Groups
Homogeneous Spaces
Conclusions

Quadratic Form

- If one considers the Killing-Cartan metric on the sphere S^3 with the associated metric volume $\theta^1 \wedge \theta^2 \wedge \theta^3$, the corresponding Dirac-Kähler operator does not give a meaningful operator when its domain is restricted to $\Lambda_0(S^3)$
- This problem can be solved by introducing on S^3 a degenerate quadratic form
$$g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2$$
with a suitable Hodge dual operator $*_{S^2} : \Lambda_0(S^3) \to \Lambda_0(S^3)$:
$$\begin{aligned} *_{S^2}(f) &= if\theta^- \wedge \theta^+ \\ *_{S^2}(\alpha_-\theta^-) &= -i\alpha - \theta^- \\ *_{S^2}(\alpha_+\theta^+) &= i\alpha_+\theta^+ \\ *_{S^2}(i\omega\theta^- \wedge \theta^+) &= \omega \end{aligned}$$
- It is possible to define a hermitean product in $\Lambda_0(S^3)$ as follows
$$(\alpha, \beta) = \int_S^3 \theta^3 \wedge \bar{\alpha} \wedge *_{S^2}\beta \qquad (4)$$

Quadratic Form

- If one considers the Killing-Cartan metric on the sphere S^3 with the associated metric volume $\theta^1 \wedge \theta^2 \wedge \theta^3$, the corresponding Dirac-Kähler operator does not give a meaningful operator when its domain is restricted to $\Lambda_0(S^3)$
- This problem can be solved by introducing on S^3 a degenerate quadratic form

$$g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2$$

with a suitable Hodge dual operator $*_{S^2} : \Lambda_0(S^3) \to \Lambda_0(S^3)$:

$$\begin{aligned} *_{S^2}(f) &= if\theta^- \wedge \theta^+ \\ *_{S^2}(\alpha_-\theta^-) &= -i\alpha - \theta^- \\ *_{S^2}(\alpha_+\theta^+) &= i\alpha_+\theta^+ \\ *_{S^2}(i\omega\theta^- \wedge \theta^+) &= \omega \end{aligned}$$

- It is possible to define a hermitean product in $\Lambda_0(S^3)$ as follows

$$(\alpha, \beta) = \int_S^3 \theta^3 \wedge \bar{\alpha} \wedge *_{S^2}\beta \tag{4}$$

Dirac-Kahler operator

- A Dirac-Kahler operator which is internal to the subalgebra $\Lambda_0(S^3)$ is given by

$$\mathcal{D} = d + \delta^{S^2} \tag{5}$$

where $\delta^{S^2} = (-1)^{n(k-1)-1} *_{S^2} d *_{S^2}$ is the adjoint of the exterior derivative with respect to the Hermitian product (4).

- Its action can be reduced from the eight-dimensional module $\Lambda_0(S^3)$, to a 4-dimensional module by means of the projector

$$P = \frac{1}{2}\left(1 \pm \theta^- \wedge \theta^+\right)$$

whose range is given by differential forms of the kind

$$\psi = f_0(1 - \theta^- \wedge \theta^+) + \sum_{j=-1}^{1} (f_j \bar{\phi}_j)\theta^-$$

- On this module (5) has the following matrix form

$$\mathcal{D} = \begin{pmatrix} 0 & 2(\phi_1 L_+ + (L_+\bar{\phi}_1)) & 2(\phi_2 L_+ + (L_+\bar{\phi}_2)) & 2(\phi_3 L_+ + (L_+\bar{\phi}_3)) \\ \phi_1 L_- & 0 & 0 & 0 \\ \phi_2 L_- & 0 & 0 & 0 \\ \phi_3 L_- & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

Dirac-Kahler operator

- A Dirac-Kahler operator which is internal to the subalgebra $\Lambda_0(S^3)$ is given by

$$\mathcal{D} = d + \delta^{S^2} \tag{5}$$

where $\delta^{S^2} = (-1)^{n(k-1)-1} *_{S^2} d *_{S^2}$ is the adjoint of the exterior derivative with respect to the Hermitian product (4).

- Its action can be reduced from the eight-dimensional module $\Lambda_0(S^3)$, to a 4-dimensional module by means of the projector

$$P = \frac{1}{2}\left(1 \pm \theta^- \wedge \theta^+\right)$$

whose range is given by differential forms of the kind

$$\psi = f_0(1 - \theta^- \wedge \theta^+) + \sum_{j=-1}^{1} (f_j \bar{\phi}_j)\theta^-$$

- On this module (5) has the following matrix form

$$\mathcal{D} = \begin{pmatrix} 0 & 2(\phi_1 L_+ + (L_+\bar{\phi}_1)) & 2(\phi_2 L_+ + (L_+\bar{\phi}_2)) & 2(\phi_3 L_+ + (L_+\bar{\phi}_3)) \\ \phi_1 L_- & 0 & 0 & 0 \\ \phi_2 L_- & 0 & 0 & 0 \\ \phi_3 L_- & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

Dirac-Kahler operator

- A Dirac-Kahler operator which is internal to the subalgebra $\Lambda_0(S^3)$ is given by
$$\mathcal{D} = d + \delta^{S^2} \tag{5}$$
where $\delta^{S^2} = (-1)^{n(k-1)-1} *_{S^2} d *_{S^2}$ is the adjoint of the exterior derivative with respect to the Hermitian product (4).
- Its action can be reduced from the eight-dimensional module $\Lambda_0(S^3)$, to a 4-dimensional module by means of the projector
$$P = \frac{1}{2}\left(1 \pm \theta^- \wedge \theta^+\right)$$
whose range is given by differential forms of the kind
$$\psi = f_0(1 - \theta^- \wedge \theta^+) + \sum_{j=-1}^{1} (f_j \bar{\phi}_j)\theta^-$$
- On this module (5) has the following matrix form
$$\mathcal{D} = \begin{pmatrix} 0 & 2(\phi_1 L_+ + (L_+\bar{\phi}_1)) & 2(\phi_2 L_+ + (L_+\bar{\phi}_2)) & 2(\phi_3 L_+ + (L_+\bar{\phi}_3)) \\ \phi_1 L_- & 0 & 0 & 0 \\ \phi_2 L_- & 0 & 0 & 0 \\ \phi_3 L_- & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

Adding Monopole

- An interesting generalization of this result is obtained by replacing the space of coefficients $\mathcal{F}_0(S^3)$ with the space of equivariant functions $\mathcal{F}_n(S^3)$, which satisfy the following condition

$$f(sg) = \rho(g)^{-1} f(s) \tag{7}$$

  These functions, therefore, must satisfy the condition

$$L_{X_3} f_j = inf_j$$

- The Dirac-Kähler operator (5) cannot be reduced to this subspace. This problem can be solved by replacing the exterior derivative with an exterior covariant derivative and the resulting Dirac-Kähler operator is

$$\mathcal{D}_{U(1)} = (d - iA) + *_{S^2}(d - iA)*_{S^2}$$

- This operator could be used to define a Dirac-type equation for $U(1)$-charged differential forms, coupled to the magnetic field generated by a monopole.

Adding Monopole

- An interesting generalization of this result is obtained by replacing the space of coefficients $\mathcal{F}_0(S^3)$ with the space of equivariant functions $\mathcal{F}_n(S^3)$, which satisfy the following condition

$$f(sg) = \rho(g)^{-1} f(s) \tag{7}$$

  These functions, therefore, must satisfy the condition

$$L_{X_3} f_j = in f_j$$

- The Dirac-Kähler operator (5) cannot be reduced to this subspace. This problem can be solved by replacing the exterior derivative with an exterior covariant derivative and the resulting Dirac-Kähler operator is

$$\mathcal{D}_{U(1)} = (d - iA) + *_{S^2}(d - iA)*_{S^2}$$

- This operator could be used to define a Dirac-type equation for $U(1)$-charged differential forms, coupled to the magnetic field generated by a monopole.

Adding Monopole

- An interesting generalization of this result is obtained by replacing the space of coefficients $\mathcal{F}_0(S^3)$ with the space of equivariant functions $\mathcal{F}_n(S^3)$, which satisfy the following condition

$$f(sg) = \rho(g)^{-1} f(s) \tag{7}$$

These functions, therefore, must satisfy the condition

$$L_{X_3} f_j = inf_j$$

- The Dirac-Kähler operator (5) cannot be reduced to this subspace. This problem can be solved by replacing the exterior derivative with an exterior covariant derivative and the resulting Dirac-Kähler operator is

$$\mathcal{D}_{U(1)} = (d - iA) + *_{S^2}(d - iA)*_{S^2}$$

- This operator could be used to define a Dirac-type equation for $U(1)$-charged differential forms, coupled to the magnetic field generated by a monopole.

Conclusions

Introduction
Kahler Formalism and Dirac-Kahler equation
Lie Groups
Homogeneous Spaces
Conclusions

- We have shown how Kahler formulation can be easily applied to manifolds which are Lie groups.
- Exterior algebras of Homogeneous spaces can be treated as subalgebras of the exterior algebra of a suitable Lie group.
- Reduction of Dirac-Kahler operator to this subalgebra can be achieved by considering degenerate quadratic forms and a suitable Hermitian product on the space of differential forms.
- Future work will be devoted to extend such a construction to orbifold. We hope to obtain some hints for a deeper understanding of the relationship between topology and boundary conditions.

Conclusions

- We have shown how Kahler formulation can be easily applied to manifolds which are Lie groups.
- Exterior algebras of Homogeneous spaces can be treated as subalgebras of the exterior algebra of a suitable Lie group.
- Reduction of Dirac-Kahler operator to this subalgebra can be achieved by considering degenerate quadratic forms and a suitable Hermitian product on the space of differential forms.
- Future work will be devoted to extend such a construction to orbifold. We hope to obtain some hints for a deeper understanding of the relationship between topology and boundary conditions.

Introduction

Kahler Formalism and Dirac-Kahler equation

Lie Groups

Homogeneous Spaces

Conclusions

Conclusions

- We have shown how Kahler formulation can be easily applied to manifolds which are Lie groups.
- Exterior algebras of Homogeneous spaces can be treated as subalgebras of the exterior algebra of a suitable Lie group.
- Reduction of Dirac-Kahler operator to this subalgebra can be achieved by considering degenerate quadratic forms and a suitable Hermitian product on the space of differential forms.
- Future work will be devoted to extend such a construction to orbifold. We hope to obtain some hints for a deeper understanding of the relationship between topology and boundary conditions.

Conclusions

- We have shown how Kahler formulation can be easily applied to manifolds which are Lie groups.
- Exterior algebras of Homogeneous spaces can be treated as subalgebras of the exterior algebra of a suitable Lie group.
- Reduction of Dirac-Kahler operator to this subalgebra can be achieved by considering degenerate quadratic forms and a suitable Hermitian product on the space of differential forms.
- Future work will be devoted to extend such a construction to orbifold. We hope to obtain some hints for a deeper understanding of the relationship between topology and boundary conditions.

Introduction

Kahler Formalism and Dirac-Kahler equation

Lie Groups

Homogeneous Spaces

Conclusions

THANK YOU FOR THE ATTENTION