PMCTs

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Goals

- understanding compactness in Poisson Geometry.
- examples.
- their structure (... classification?).
- fundamental properties.

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Ongoing joint work with: David Martinez Torres, Rui Loja Fernandes:

- Poisson Manifolds of Compact Types (PMCT 1), arXiv:1510.07108
- Regular Poisson manifolds of compact types (PMCT 2):1603.00064.

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Breakthrough in the strong compact case: David Martinez Torres:

- A Poisson manifold of strong compact type, arXiv:1312.7267.

$$\{\text{Lie Theory}\} \hookrightarrow \{\text{Poisson Geometry}\}$$

$$(\mathfrak{g}, [\cdot,\cdot]) \mapsto (\mathfrak{g}^*, \pi_{\mathrm{lin}}).$$

$$\left(\text{structure constants } c_{ij}^k \mapsto \text{bivector } \sum c_{ij}^k x_k \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right)$$

Note

Roughly speaking:

$$\textit{Lie Theory} = \textit{linear Poisson Geometry}.$$

Aim: add "compactness" to the story.

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Lie algebra $\mathfrak{g}$ is of compact type

if there exists an integrating

Lie group G which is compact

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If the canonical integration of G (the unique simply connected one) is compact: $\mathfrak{g}$- of strong compact type.

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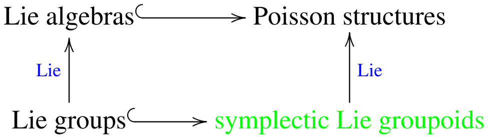

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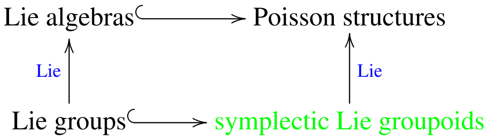

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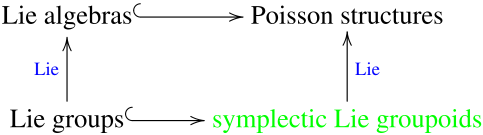

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- compact: if $\mathcal{G}$ is compact.
- s-proper: if the source map $s : \mathcal{G} \to M$ is proper.
- proper: if $(s, t) : \mathcal{G} \to M \times M$ is proper.

Definition

Consider one of the compactness types

$$\mathcal{C} \in \{\text{compact}, \text{s-proper}, \text{proper}\}.$$

Say (M, π) is of $\mathcal{C}$-type: if it comes from a symplectic Lie groupoid

$$(\mathcal{G}, \Omega) \rightrightarrows M$$

which has property $\mathcal{C}$.

Add the adjective "*strong*": if $\mathcal{G}$ is the canonical integration.

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Note: makes sense for general Dirac structures!

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$$\begin{array}{ccccc} \text{strong compact} & \Longrightarrow & \text{strong s-proper} & \Longrightarrow & \text{strong proper} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{compact} & \Longrightarrow & \text{s-proper} & \Longrightarrow & \text{proper} \end{array}$$

Example (Symplectic manifolds)

For symplectic (S, ω):

- (s-)compact type $\iff$ S is compact.
- proper type: always.
- strong: if also $\pi_1(S)$ is finite.

(integrating groupoids: $S \times S \rightrightarrows M$, or the fundamental groupoid).

Example (Lie algebras)

For $(\mathfrak{g}^*, \pi_{\text{lin}})$:

- compact type: never.
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Example (the Cartan Dirac structure ≡ the non-linear analogue of $\mathfrak{g}^*$)

- It lives on G-simply connected compact Lie group.
- Constructed using $\langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g}$ Ad-invariant.
- No longer Poisson, but not far from it ... i.e. it is Dirac:
- i.e. comes with a *pre*-symplectic foliation: the conjugacy classes $\mathcal{C}_g \subset G$ endowed with (GHJW '97)

$$\omega_g(\widehat{u}, \widehat{v}) := \left\langle \frac{\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}}{2} u, v \right\rangle_{\mathfrak{g}}$$

Fact: G with this Dirac structure is *always of strong compact type*! (integrating groupoid: just $G \ltimes G$ coming from conjugation).

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Example (The zero Poisson structure)

$(M, \pi \equiv 0)$-not of any strong $\mathcal{C}$-type. The canonical integration:

$$(T^*M, \Omega_{\text{can}}) \rightrightarrows M.$$

Other possible integrations: quotients T^*M/Λ modulo lattices

$$\Lambda \subset T^*M$$

which are Lagrangian (so that Ω_{can} descends). This is part of:

$$\left\{ \begin{array}{c} \text{integral affine} \\ \text{structures } \Lambda \text{ on } M \end{array} \right\} \stackrel{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{symplectic torus bundles over } M \end{array} \right\}$$

So: proper integrations of $(M, 0) \longleftrightarrow$ integral affine structures on M.

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Consider:

$$p : M \to B$$

symplectic fibration with connected fibers, everything compact.

Then the (symplectic) fibers of p make M into Poisson; this is:

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Example (Reduction)

Consider: compact Lie group G, a G-Hamiltonian space

$$\mu : (Q, \omega) \to \mathfrak{g}^*$$

and the reduced Poisson manifold

$$M = Q/G \quad \text{endowed with} \quad \pi_{\text{red}}.$$

Assume action free and proper (so M is smooth), μ-connected fibers.

Then (M, π_{red}):

- is always of proper type.
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Similalrly for $M = Q/S^1$ for an S^1-quasi-Hamiltonian space

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Warning

Compactness types are, in general (and most often!), not inherited by Poisson submanifolds.

- example: the sphere

$$\mathbb{S}_{\mathfrak{g}^*} \subset \mathfrak{g}^*$$

where $\mathfrak{g}$ is a Lie algebra of strong compact type. Then $\mathbb{S}_{\mathfrak{g}^*}$ is (almost) never of proper type.

+ example: if (M, π) is of proper type, then it comes with an "orbit type" stratification, with strata being (regular) Poisson submanifolds. They are always of proper type!

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The structure of PMCTs- main points:

0. Desingularization.
1. Orbifolds.
2. Integral Affine Geometry.
3. Symplectic gerbes.

Note

- *after step 0: we restrict to the regular case, concentrate on the space of symplectic leaves*

$$B := M/\mathcal{F}_\pi \quad (\mathcal{F}_\pi - \textit{the symplectic foliation})$$

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Desingularization of (M, π): desingularize via blow-ups, but do everything at once, intrinsically:

$$\hat{M} := \{(x, \mathfrak{t}) : x \in M, \mathfrak{t} \subset \mathfrak{g}_x \text{ maximal abelian}\},$$

where $\mathfrak{g}_x(\pi) = \mathrm{Ker}\pi^\sharp_x$, the isotropy Lie algebra at $x \in M$. Properties:

- $\hat{M}$ is smooth.
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Example

In the linear case $M = \mathfrak{g}^*$,

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It follows that the Grothendieck simultaneous resolution, the Weyl integration formula, etc are 100% Poisson geometric!

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Orbifold structure: consider the leaf space

$$B := M/\mathcal{F}_\pi$$

of the symplectic foliation $\mathcal{F}_\pi$ (very pathological in general!).

Theorem

If (M, π) is a of proper type then B is an orbifold (and any proper integration gives rise to a canonical orbifold atlass/structure on B).

If the symplectic leaves are simply-connected, then B is smooth.

Explanation: if $\mathcal{G}$ integrates (M, π), form

$$1 \to \mathcal{T} \to \mathcal{G} \to \mathcal{B} \to 1$$

where $\mathcal{T}$ is made of the connected components of the isotropies $\mathcal{G}_x$. The quotient $\mathcal{B}$ is an orbifold atlass since it has finite isotropy groups.

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If (M, π) is a of proper type then B is an orbifold (and any proper integration gives rise to a canonical orbifold atlass/structure on B).

If the symplectic leaves are simply-connected, then B is smooth.

Explanation: if $\mathcal{G}$ integrates (M, π), form

$$1 \to \mathcal{T} \to \mathcal{G} \to \mathcal{B} \to 1$$

where $\mathcal{T}$ is made of the connected components of the isotropies $\mathcal{G}_x$.
The quotient $\mathcal{B}$ is an orbifold atlass since it has finite isotropy groups.

Orbifold structure: consider the leaf space

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Integral affine geometry: it arises from the variation of the symplectic forms (see blackboard).

Different description: in the short exact sequence

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the restriction of Ω from $\mathcal{G}$ makes $\mathcal{T}$ into a pre-symplectic torus bundle over M. Fact: this comes from a symplectic torus bundle over B- i.e. from an IAS on B!

Theorem

For any regular Poisson manifold (M, π) of $\mathcal{C}$-type its leaf space $B = M/\mathcal{F}_\pi$ is an integral affine orbifold: any integration having property $\mathcal{C}$ gives rise to an integral orbifold structure on B. Moreover, the underlying classical orbifold is good.

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Classical S^1-gerbes over B: higher versions of principal S^1-bundles.

Several descriptions varying from "down to earth" to "more intrinsic":

1. in terms of transition functions/S^1-valued Cech cocycles on B.
2. in terms of central extensions of Lie groupoids

$$1 \to S^1 \to \mathcal{G} \to \mathcal{B} \to 1$$

where $\mathcal{B}$ models the smooth structure on B.

Here 1. $\Longleftrightarrow$ 2.: via the Dixmier-Douady class of an extension.

Several variations, e.g. the obvious ones:

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$$1 \to \mathcal{T} \to (\mathcal{G}, \Omega) \to \mathcal{B} \to 1$$

(hence Ω induces the given symplectic form on $\mathcal{T}$).

Symplectic $\mathcal{T}$-gerbes over B: symplectic Morita equivalence classes of such extensions.They form a group w.r.t. "the fusion product".

Theorem

Extensions as above are classified by a cohomology class

$$c_2(\mathcal{G}, \Omega) \in H^2(B, \mathcal{T}_{Lagr}),$$

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Symplectic gerbes-back to PMCTs: for a proper integration $(\mathcal{G}, \Omega)$ of (M, π) we get the IAS on the leaf space B, i.e. a symplectic torus bundle, and a class

$$c_2(\mathcal{G}, \Omega) \in H^2(B, \mathcal{T}_{\mathrm{Lagr}}).$$

Theorem

$c_2(\mathcal{G}, \Omega) = 0$ iff $\mathcal{G}$ arise from (free) $\mathcal{T}$-Hamiltonian reduction or, equivalently, from a proper isotropic realization of (M, π).

Here: (M, π) regular of s-proper type with 1-connected leaves; then B is smooth, we have a locally trivial fibration

$$p : M \to B,$$

and think of M as a family $\{(S_b, \omega_b)\}_{b \in B}$ of symplectic manifolds.

Fix $b_0 \in B$ with fiber (S_0, ω_0); realize $\tilde{B}$ using paths starting at b_0.

1. The Gauss-Manin connection allows us to look at

$$\gamma^*([\omega_{\gamma(1)}]) - [\omega_0] \in H^2(S_0).$$

2. The IAS on B induced by $\mathcal{G}$ gives rise to the developing map

$$\mathrm{dev} : \tilde{B} \to \mathbb{R}^q.$$

3. $s_{\mathcal{G}}^{-1}(b_0)$ =a principal $\mathbb{T}^q$-bundle over S_0; Chern classes denoted

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Theorem

One has the linear variation formula:

$$\gamma^*([\omega_{\gamma(1)}]) - [\omega_0] = dev^1(\gamma)c_1 + \ldots + dev^q(\gamma)c_q.$$

Same assumptions as before. Consider the following measures on B:

1. μ_{Aff}-corresponding to the IAS on B.
2. μ_{DH}- the push-forward of the Liouville measure on $\mathcal{G}$.

Also consider the leafwise symplectic volumes,

$$\mathrm{vol} : B \to \mathbb{R}$$

and $\iota : B \to \mathbb{Z}$ counting the components of the isotropy groups of $\mathcal{G}$.

Theorem

$$\mu_{DH} = (\iota \cdot vol)^2 \mu_{Aff}$$

and $(\iota \cdot vol)^2$ is polynomial w.r.t. the integral affine structure on B.

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2. μ_{DH}- the push-forward of the Liouville measure on $\mathcal{G}$.

Also consider the leafwise symplectic volumes,

$$\mathrm{vol} : B \to \mathbb{R}$$

and $\iota : B \to \mathbb{Z}$ counting the components of the isotropy groups of $\mathcal{G}$.

Theorem

$$\mu_{DH} = (\iota \cdot vol)^2 \mu_{Aff}$$

and $(\iota \cdot vol)^2$ *is polynomial w.r.t. the integral affine structure on B.*

With the same assumptions as before:

Theorem

If (M, π) is a regular Poisson manifold, with proper integration $(\mathcal{G}, \Omega)$, then for any $f \in C_c^\infty(M)$:

$$\int_M f(x)\, \mathrm{d}\mu_M^{\mathit{Aff}}(x) = \int_B \left(\iota(b) \int_{S_b} f(y)\, \mathrm{d}\mu_{S_b}(y) \right) \mathrm{d}\mu_{\mathit{Aff}}(b),$$

where μ_{S_b} is the Liouville measure of the symplectic leaf S_b.

With the same assumptions as before:

Theorem

If (M, π) is a regular Poisson manifold, with proper integration $(\mathcal{G}, \Omega)$, then for any $f \in C_c^\infty(M)$:

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where μ_{S_b} is the Liouville measure of the symplectic leaf S_b.