Time function: Classical and Quantum

Some ideas presented by:

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Time is money, but Space is a very long Time

Time function: Classical and Quantum

The problem of Time in Quantum Mechanics

Is Time a quantum observable (self-adjoint operator)? →

- *Time of arrival;*
- *Time of occurrence;*
- *Tunneling Time.*

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$$[\mathbf{H}\,;\mathbf{T}] = -\imath\,\mathbb{I}$$

Pauli's theorem

- *Counterexamples;*
- *Maximally symmetric Time operators;*
- *Time POVMs.*

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- Counterexamples;
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Time in Q.M. is dynamical.

Self-adjoint operators are not enough.

Our proposal:

A Time function to define simultaneity.

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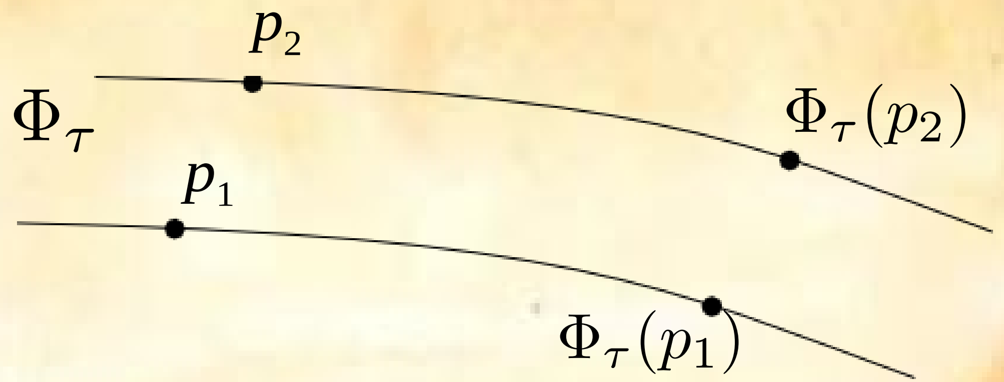
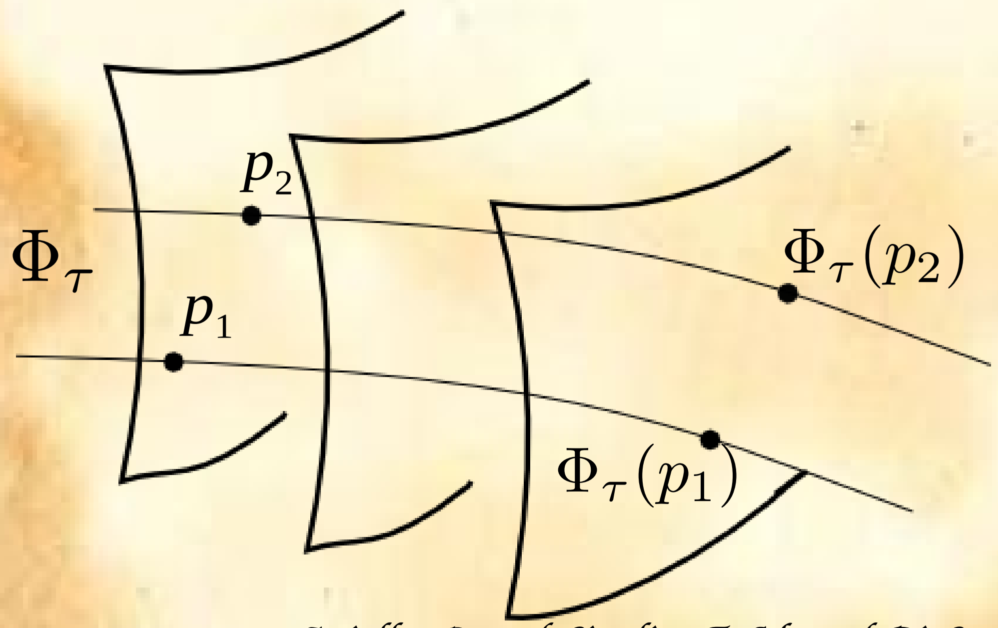
Geometric aspects of simultaneity

Twofold role of Time in physical phenomena

Causality: *the dynamical evolution is perceived as an ordered sequence of states of the system*



Causality *and* ***Simultaneity*** *are "transversal to each other"*



Simultaneity: *mutual relation between states of different dynamical trajectories*

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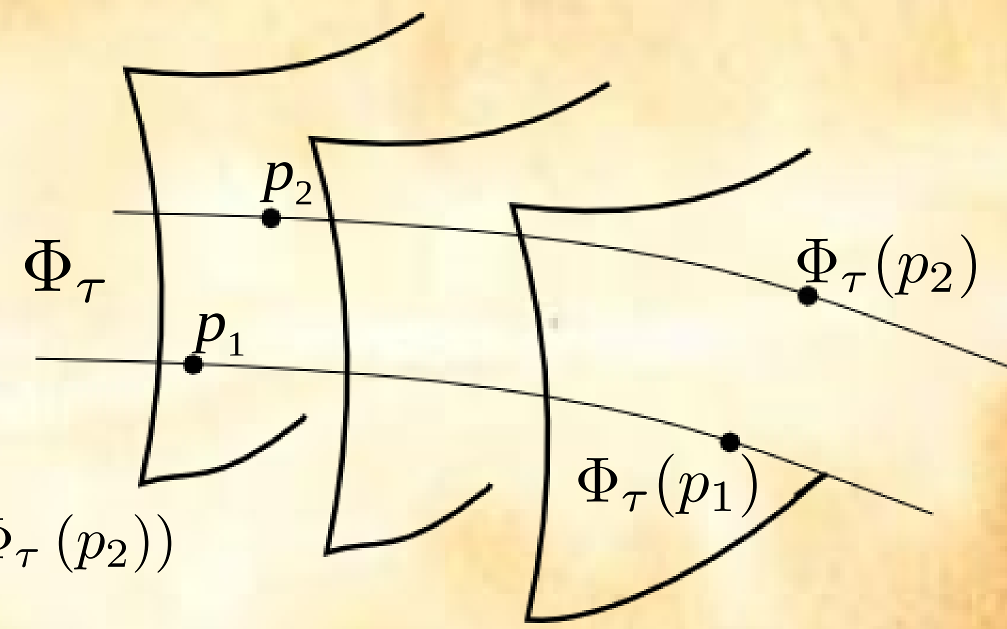
Geometric aspects of simultaneity

How to define simultaneity

Time function:

$$T(p) \neq T(\Phi_\tau(p)) \quad \forall \tau \neq 0$$

A state can never be simultaneous to its dynamical evolution



$$T(p_1) = T(p_2) \implies T(\Phi_\tau(p_1)) = T(\Phi_\tau(p_2))$$

Simultaneity between states is preserved under the dynamical evolution

The level sets define the simultaneous states

- *The time functions is a dynamical object*
- *It can be defined both for classical and quantum systems without the need of quantization*
- *In the quantum setting, it is* ***not*** *associated to any self-adjoint operator*

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Free point particle:

Dynamical trajectories:

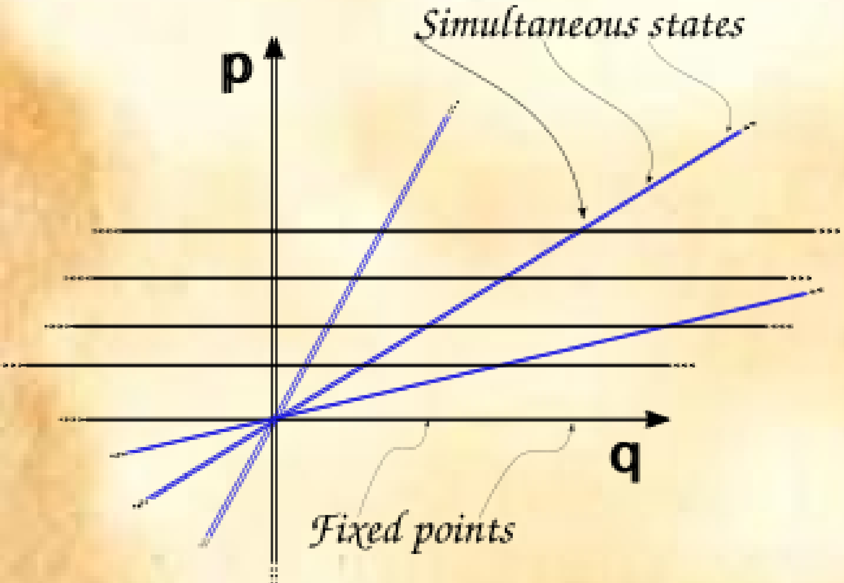
$$\Phi_\tau(\vec{q};\vec{p}) = \begin{cases} p_j(\tau) = p_j \\ q_j(\tau) = p_j\tau + q_j \end{cases}$$

Classical Hamiltonian Mechanics:

- *The space of States is a symplectic manifold* $(\mathcal{P};\omega)$
- *The observables are smooth functions:*
- *The dynamical evolution* Φ_τ *is the flow of a vector field* Γ *associated to a Hamiltonian function* H *:*

$$i_\Gamma \omega = dH$$

The Time function is not defined on the fixed points of the dynamics (particles at rest).



Time function:

$$T(\vec{q};\vec{p}) = \frac{\vec{p}\cdot\vec{q}}{p^2} \implies T\circ\Phi_\tau(\vec{q};\vec{p}) = \tau + \frac{\vec{p}\cdot\vec{q}}{p^2}$$

In general, the Time function will be defined on an open dense subset $\mathcal{P}_*$ *of the space of states, which is invariant w.r.t. the dynamical evolution*

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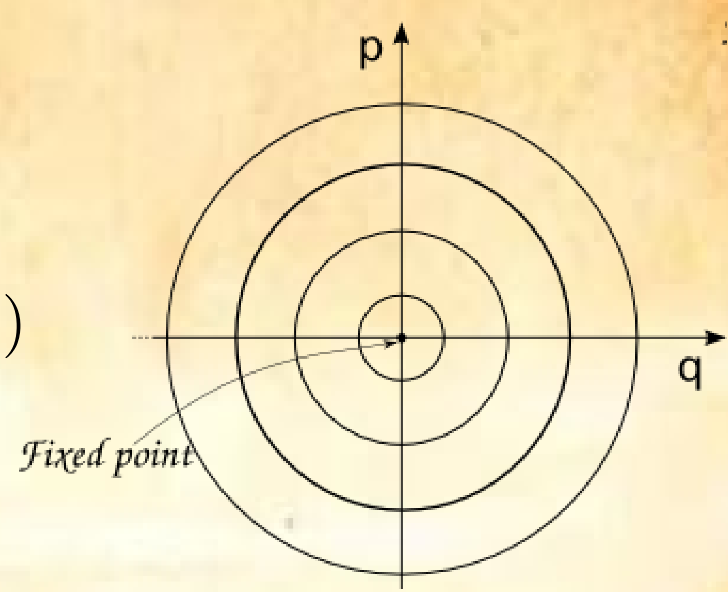
1-D harmonic oscillator:

Dynamical trajectories:

$$\Phi_\tau\,(q\,;p) = \begin{cases} p(\tau) = -m\nu q\sin(\nu\tau) - p\cos(\nu\tau) \\[2ex] q(\tau) = q\cos(\nu\tau) - \frac{p}{m\nu}\sin(\nu\tau) \end{cases}$$



Because of periodic orbits the global transversality between causality and simultaneity is lost

The simultaneity relation becomes periodic

Some well-known examples of periodic simultaneity relations:

Mechanical clock

Ancient Maya calendar

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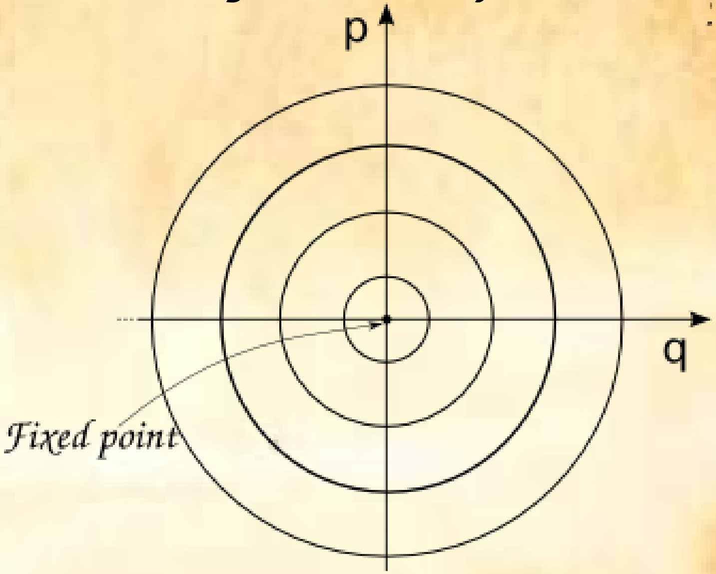
1-D harmonic oscillator

Dynamical trajectories:



Periodic Time function *(with period τ_T)*

- $T\left(\Phi_\tau\left(p\right)\right) = T\left(\Phi_{\tau + k\tau_T}\left(p\right)\right)\ \forall k \in \mathbb{Z}$

- $T\left(p_1\right) = T\left(p_2\right) \implies T\left(\Phi_\tau\left(p_1\right)\right) = T\left(\Phi_\tau\left(p_2\right)\right)$

The periodic Time function takes values in the circle, the non-trivial topology of the circle allows to handle the periodicity of the orbits

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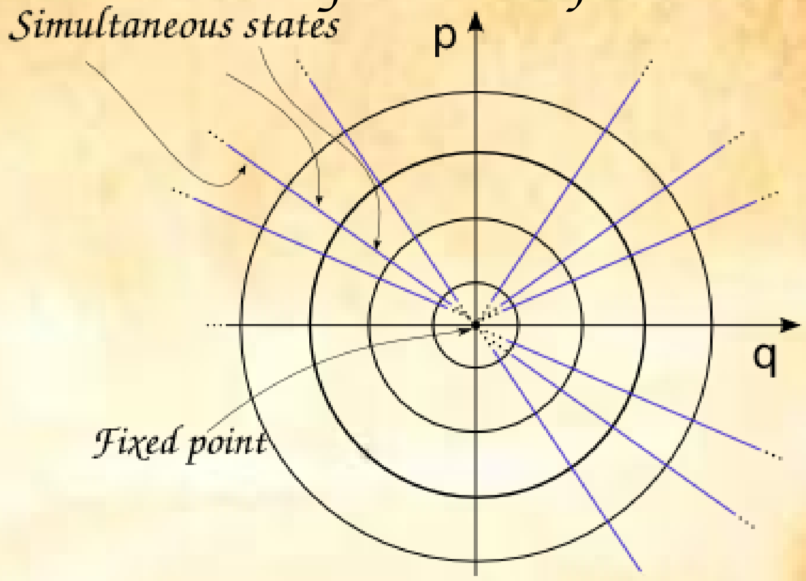
1-D harmonic oscillator



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The periodic Time function takes values in the circle, the non-trivial topology of the circle allows to handle the periodicity of the orbits

The reduced space of states is diffeomorphic to the cilinder:

$$\Psi : \mathcal{P}_* \cong \mathbb{R}^2 - \{(0,0)\} \to S^1 \times \mathbb{R}^+$$

The periodic Time function is the just the projection on the circle: $T = pr_{S^1} \circ \Psi$

The simultaneous states are points on a radial line

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Geometric formulation of Quantum Mechanics

The carrier space is a Hilbert space

$$\mathcal{H} \cong \mathbb{C}^n$$

The observables are self-adjoint linear operators on the Hilbert space:

$$\mathcal{B}(\mathcal{H}) \ni \mathbf{A} \, : \; \mathbf{A}^\dagger = \mathbf{A}$$

The pure states p_ψ *are rays of the Hilbert space*

$$|\psi\rangle \sim |\phi\rangle \longleftrightarrow |\psi\rangle = r e^{\imath\theta} |\phi\rangle \,, \quad r \in \mathbb{R}^+ \,, \; e^{\imath\theta} \in U(1)$$

The space of pure states is the complex projective space $CP(n-1)$ *which is a Kahler manifold:*

$$\mathcal{H} - \{\mathbf{0}\}$$
$$\downarrow \quad \mathbb{R}^+$$
$$S^{2n-1}$$
$$\downarrow \quad U(1)$$
$$CP(n-1)$$

$$g(X\,;Y) = \omega(J(X)\,;Y) \qquad \forall X, Y \in \mathfrak{X}(CP(n-1))$$

Observables are represented by expectation value functions:

$$e_{\mathbf{A}}(p_\psi) := \frac{\langle\psi|\mathbf{A}|\psi\rangle}{\langle\psi|\psi\rangle}$$

$$e_{a\mathbf{A}+b\mathbf{B}} = a e_{\mathbf{A}} + b e_{\mathbf{B}}$$

$$\{e_{\mathbf{A}}\,; e_{\mathbf{B}}\} = e_{\imath[\mathbf{A}\,;\mathbf{B}]}$$

Unitary evolutions are generated by a Hamiltonian operator:

$$i_\Gamma \omega = d e_{\mathbf{H}}$$

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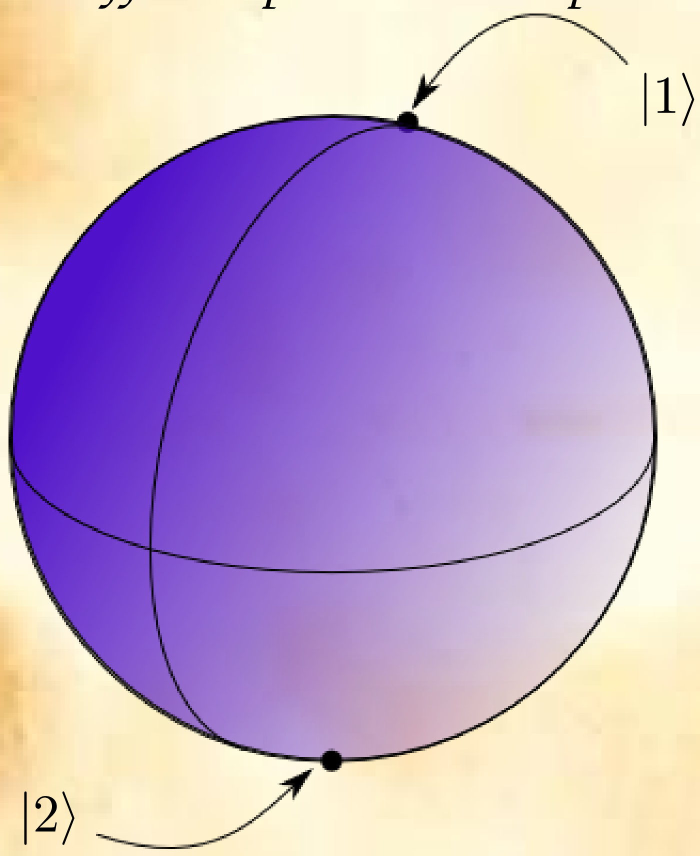
The Qubit case

The complex projective space $CP(1)$
is diffeomorphic to a 2-D sphere:



Hamiltonian operator $\mathbf{H} = \nu_1 |1\rangle\langle 1| + \nu_2 |2\rangle\langle 2|$

Eigenvalues $\nu_1 \, , \, \nu_2$ *Eigenvectors* $|1\rangle \, , |2\rangle$

Eigenprojectors $\mathbf{E}_1 = |1\rangle\langle 1| \, , \, \mathbf{E}_2 = |2\rangle\langle 2| \, , \, \mathbf{E}_1 + \mathbf{E}_2 = \mathbb{I}$

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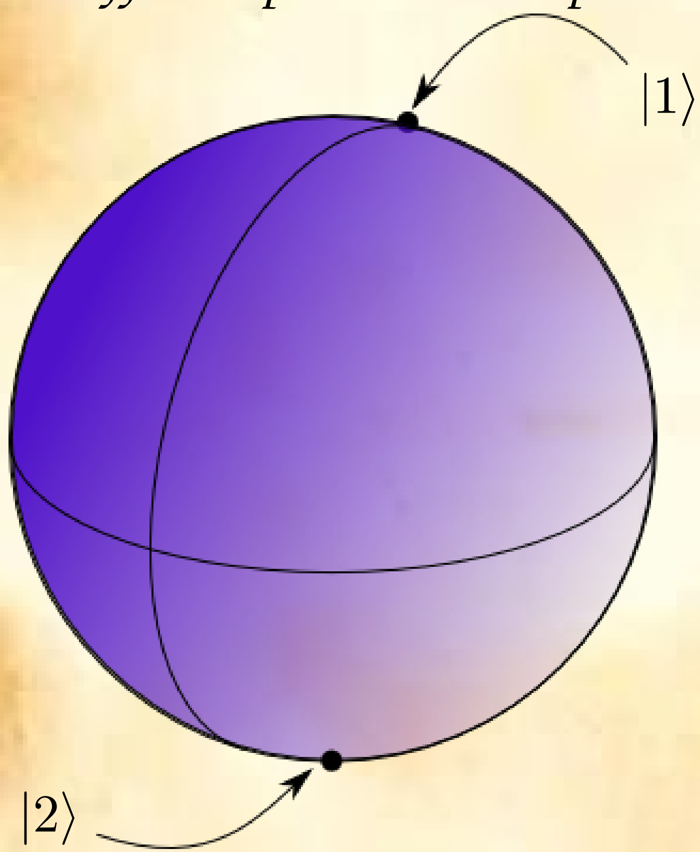

Hamiltonian operator $\mathbf{H} = \nu_1 \,|1\rangle\langle 1| + \nu_2 \,|2\rangle\langle 2|$

Eigenvalues $\nu_1\,,\ \nu_2$ Eigenvectors $|1\rangle\,, |2\rangle$

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Expectation value functions

$$e_1 := e_{\mathbf{E}_1}\,,\ e_2 := e_{\mathbf{E}_2}\,,\ e_{\mathbf{H}} = \nu_1 e_1 + \nu_2 e_2$$

These are constants of the motion in involution

$$\{e_1\,;e_2\} = 0\,,\ \{e_{\mathbf{H}}\,;e_1\} = 0\,,\ \{e_{\mathbf{H}}\,;e_2\} = 0$$

They are not linearly independent:

$$e_1 + e_2 = 1 \implies e_2 = 1 - e_1$$

The same is true for their associated Hamiltonian vector fields:

$$[X_1\,;X_2] = 0\,,[X_1\,;\Gamma] = 0\,,[X_2\,;\Gamma] = 0\,,\ X_1 + X_2 = 0$$

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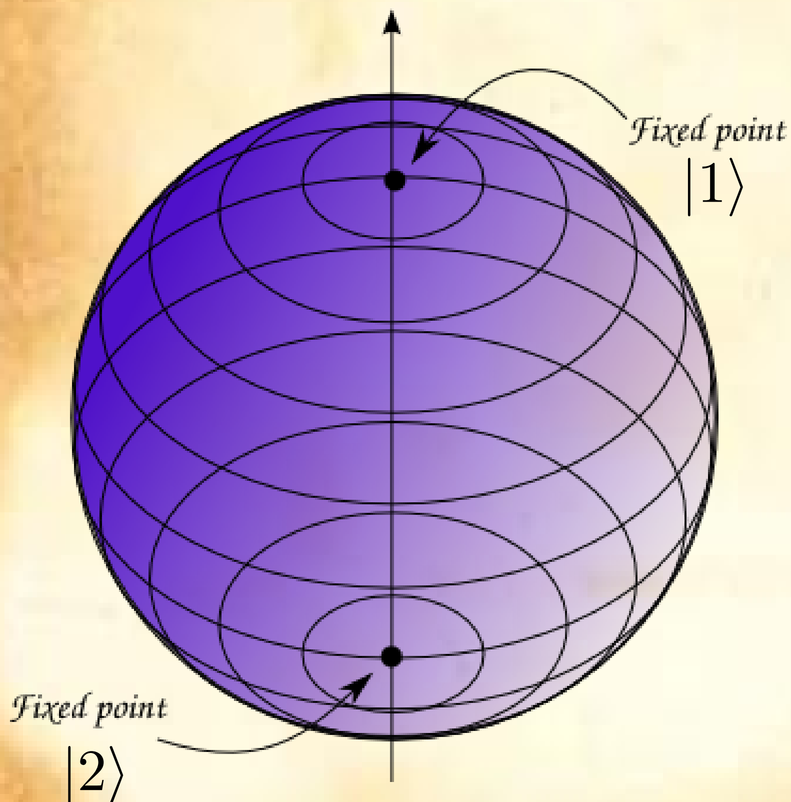

The dynamical vector field reads:

$$\Gamma = \nu_1 X_1 + \nu_2 X_2 = (\nu_1 - \nu_2)\, X_1$$

The dynamical trajectories are circles with center on the z-axis.
The North and South poles are fixed points (the eigenvectors of the Hamiltonian operator).

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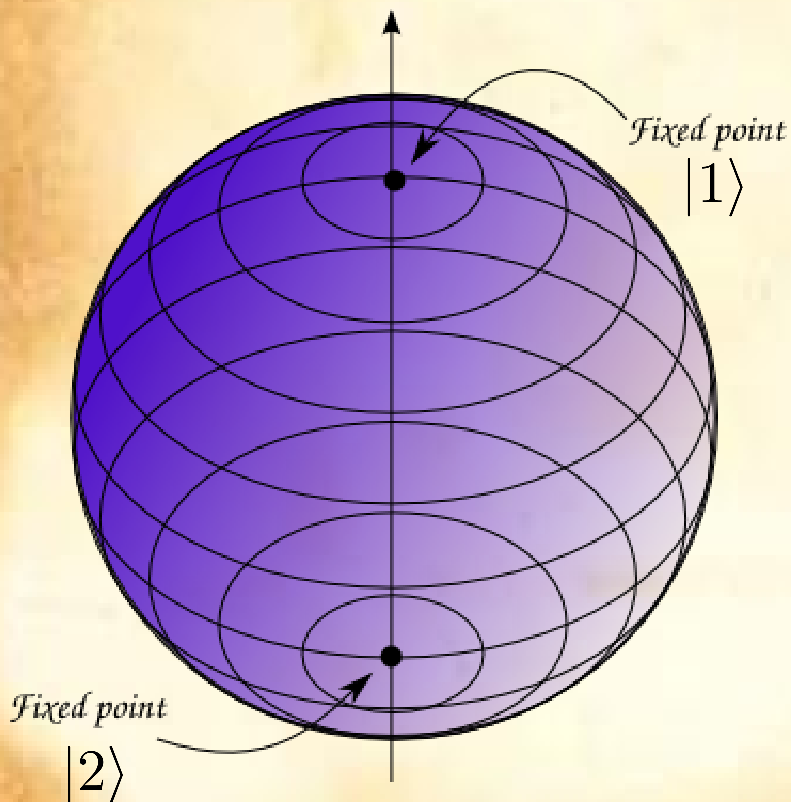

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The dynamical trajectories are circles with center on the z-axis.
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The reduced space of states is diffeomorphic to an open finite cilinder:

$$\Psi : \mathcal{P}_* \to S^1 \times I$$

Normalized vector in the Hilbert space:

$$|\psi\rangle = \frac{1}{\sqrt{(r_1)^2 + (r_2)^2}} \left(r_1\, \mathrm{e}^{\imath\theta_1} |1\rangle + r_2\, \mathrm{e}^{\imath\theta_2} |2\rangle \right) = \frac{r_2 \mathrm{e}^{\imath\theta_2}}{\sqrt{(r_1)^2 + (r_2)^2}} \left(\frac{r_1}{(r_1)^2 + (r_2)^2}\, \mathrm{e}^{\imath(\theta_1 - \theta_2)} |1\rangle + |2\rangle \right)$$

The diffeomorphism reads: $\Psi(p_\psi) = \left(\mathrm{e}^{\imath(\theta_1 - \theta_2)} ; \frac{r_1}{(r_1)^2 + (r_2)^2} \right)$

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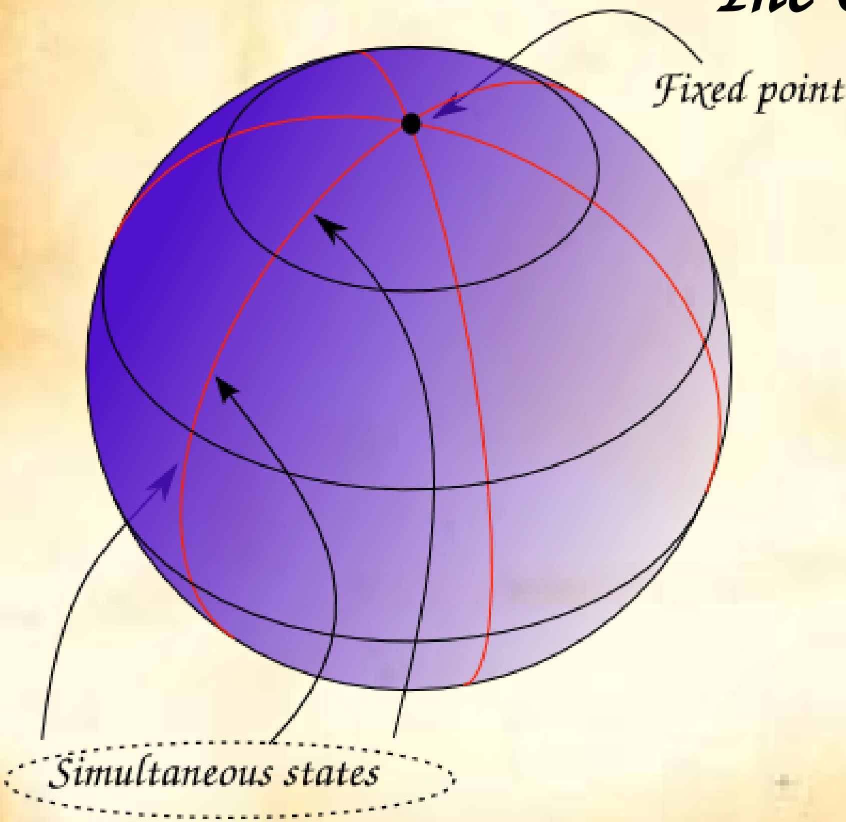

The qubit case is mathematically equivalent to the 1-D harmonic oscillator!!!

The Time function is:

$$T = pr_{S^1} \circ \Psi \qquad T\left(p_\psi\right) = \mathrm{e}^{\imath(\theta_1 - \theta_2)}$$

$$T \circ \Phi_\tau \left(p_\psi\right) = \mathrm{e}^{\imath(\theta_1 - \theta_2)}\, \mathrm{e}^{-\imath(\nu_1 - \nu_2)\tau}$$

The sets of simultaneous states are the meridians on the 2-D sphere.

The periodic Time function is **not** the expectation value function of some self-adjoint linear operator.

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Higher-dimensional generalization

Hamiltonian operator $\mathbf{H} = \sum_{j=1}^{n} \nu_j \, \mathbf{E}_j$

Expectation value functions $e_j := e_{\mathbf{E}_j} \, , \; \{e_j \, ; e_k\} = 0 = \{e_j \, ; e_{\mathbf{H}}\}$

There are (n-1) linearly independent constants of the motion.

These constants of the motion are functionally independent on the reduced space of states:

$$\mathcal{P}_* := \{p_\psi \in CP(n-1) : \; \langle j|\psi\rangle \neq 0 \;\; \forall j = 1, ..., n\}$$

The reduced space of states is diffeomorphic to a product: $\Psi : \mathcal{P}_* \to \left(S^1\right)^{(n-1)} \times I^{(n-1)}$

$$\Psi\left(p_\psi\right) = \left(\mathrm{e}^{\imath(\theta_1 - \theta_n)} \, ; \cdots \, ; \mathrm{e}^{\imath(\theta_{n-1} - \theta_n)} \, ; \frac{r_1}{N^2} \, ; \cdots \, ; \frac{r_{n-1}}{N^2}\right)$$

There is a family of periodic Time functions:

$$T_j \circ \Phi_\tau \left(p_\psi\right) = \mathrm{e}^{\imath(\theta_j - \theta_n)} \; \mathrm{e}^{\imath(\nu_n - \nu_j)\tau}$$

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Conclusions:

- The simultaneity aspect of Time in Q.M. is captured by the Time function;
- The Time function is intimately connected to the dynamics of the system;
- In general, the Time function of a quantum system is not associated to an operator;
- The Time function is well-defined for finite-level quantum systems, whereas the Time operator is not;
- The definition of a Time function is well-suited for both Classical and Quantum systems without the need to invoke a quantization procedure;
- In principle, the Time function can be defined for dissipative systems.

Thank You for Your attention

Time in Quantum Mechanics

P. Aniello and F. M. Ciaglia and F. Di Cosmo and G. Marmo and J. M. Pérez-Pardo. Time, classical and quantum. Annals of Physics, 373, 2016.

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"Without music to decorate it,
Time is just a bunch of boring production deadlines or dates by which bills must be paid."
Frank Zappa

Simultaneity and reference frames

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