Mini-Course on Quantum Field Theory [1]

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Abstract:
The goal of the series of lectures is to provide a sucint introduction to the foundations of Quantum Field Theory to an audience of theoretical physicists and mathematicians with a background in Geometry and Physics. We review the principles of Quantum Field Theory in the canonical and covariant formalisms. The problem of ultraviolet divergences and its renormalization is analyzed in the canonical formalism. As an application we review the roots of Casimir effect. For simplicity we focus on the scalar field theory but the generalization for fermionic is straightforward. However, the quantization of gauge fields require extra techniques which are beyond the scope of these lectures. Finally a short introduction to functional integrals and perturbation theory in the Euclidean formalism is included in the last section.

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Contents

1 Introduction

Quantum Field Theory (QFT) is the current paradigm of Fundamental Physics. It arises from the convolution of Quantum Physics and Relativity, the two major theoretical revolutions of the XX century physics. The search for a theory of quantum fields started right after the discovery of Quantum Mechanics, but the ultraviolet problem postponed the formulation of a consistent theory till the end of the World War II. The main problem was solved in perturbation theory by the renormalization of vacuum energy, masses, charges, fundamental fields and other couplings. One further step in the formulation of the theory beyond perturbation theory was achieved by Wilson's renormalization group approach a quarter of a century later.

It is usually considered that there were *only* two conceptual revolutions in XX century physics: the theory of relativity and the quantum theory. It is not quite true, the formulation of QFT required also a radical deep conceptual change in the relations between theory and observations that might be considered as a third major revolution of physics. The need of renormalization of ultraviolet (UV) divergences required a *dramatic* solution (*a la Planck*): the parameters which appear in the Lagrangian do not coincide with those associated with observations. Moreover the parameters of the Lagrangian of interacting field theories become divergent when the UV cutoff is removed, while the physical parameters remain finite in that limit.

The first attempts to quantize field theories were initiated one year after the discovery of quantum mechanics by Heisenberg, Born, and Jordan for free fields. One year later in 1927 Dirac introduced quantum electrodynamics (QED), the first quantum theory of interacting quantum fields. The quantization of relativistic field theories was initiated by Jordan and Pauli one year later.

However, the development of the theory was suddenly stopped by the appearance of ultraviolet divergences. The situation was so desperate that Heisenberg noted in 1938 that the revolutions of special relativity and of quantum mechanics were associated with fundamental dimensional parameters: the speed of light, c, and Planck's constant, h. These delineated the domain of classical physics. He proposed that the next revolution be associated with the introduction of a fundamental unit of length, which would delineate the domain in which the concept of fields and local interactions would be applicable.

Dirac was even more pessimistic. He wrote in last paragraph of the forth edition of his book on the Principles of Quantum Mechanics: *Parece ser que hemos seguido hasta donde es posible el desarrollo lógico de las ideas de la mecánica cuántica tal y como se conocen hoy en día. Teniendo en cuenta que las dificultades son de carácter muy profundo y únicamente pueden ser superadas por un cambio drástico de los fundamentos de la teoría, probablemente tan drástico como el paso de la teoría de las órbitas de Bohr a la mecánica cuántica actual.*

The resolution of the renormalization problem required two decades to be solved by Bethe, Feynman, Schwinger and Tomonaga

After the resolution of this problem QED became a powerful predictive theory for atomic physics and provided results which matched the experimental values with the major accuracy ever found in physics. The QED prediction for the magnetic dipole moment of the muon

$$(g_\mu - 2)_{\text{theor.}} \;=\; 233\;183\;478\;\;(308)\times 10^{-11}, \tag{1.1}$$

fits impressively well with the experimentally measured value

$$(g_\mu - 2)_{\text{exp.}} \;=\; 233\;184\;600\;(1680)\times 10^{-11}. \tag{1.2}$$

However, in the late fifties is was remarked that the renormalization of the theory has another UV catastrophe due to the appearance of a singular pole in the effective electric charge of the electron. The phenomenon know as Landau pole motivated that the majority of particle physicists consider that quantum field theory was not a suitable theory for weak and strong interactions of the newly discovered *elementary* particles.

However, with the discovery of the theory of gauge fields and the formulation of the Standard Model of particle physics the perspective radically changed and today quantum field theory is the basic framework of Fundamental Physics.

• *What are the essential features of quantum field theory which make it so special?*

First, it provides a framework where the theory of relativity and the quantum theory become consistently integrated out.

Sometimes field theory is identified with particle theory. This is not absolutely correct. Field theory is a framework which goes beyond particle physics. In fact there are field theories where there is no particle interpretation of any of the states of the theory.

But the most successful field theories admit a particle interpretation. That means that there are states which can be correctly interpreted as local particle states and in those cases field theory provides a causal framework for particle interactions where action at a distance is replaced by local field interactions. Although this is also achieved by classical field theory, the difference with the classical theory resides on the fact that in the quantum theory the interaction between the particles can be interpreted as a creation and destruction of messenger particles process. The association of forces and interactions with particle exchange is one of the most interesting features of QFT.

The particles which appear in field theory are very special: they are all identical. This means that the electrons in the earth is the same as the electrons in alpha Centauri because they are excitations of the same electron field in QED.

Another essential characteristic of relativistic field theories is that when the field theory admits particle states they are accompanied by antiparticle states, i.e. the theory requires the existence of antiparticles. This interesting property is also a source of the ultraviolet problems of the theory.

• *What are the mathematical tools of quantum field theory?*

As G. Gamow remarked the first two revolutions had at their disposal the required mathematical tools: *In their efforts to solve the riddles of Nature, physicists often looked for the help of pure mathematics, and in many cases obtained it. When Einstein wanted to interpret gravity as the curvature of four-dimensional, continuum space-time, he found waiting for him Riemann's theory of curved multidimensional space. When Heisenberg looked for some unusual kind of mathematics to describe the motion of electrons inside of an atom, noncommutative algebra was ready for him.* However, the revolution of QFT was lacking an appropriate mathematical tool. The theory of distributions to deal with singularities in a rigorous way was formulated by L. Schwartz in the late forties.

The fact that the quantum fields involve distributions is behind the existence of UV divergences which in the quantum field theory require renormalization.

The goal of the series of lectures is to summarize the foundations of QFT in the Fall Workshop of Geometry and Physics. The focus is to provide some insights of the theory of quantum fields to an audience with a solid mathematical background. However, due to the tight schedule the level of mathematical rigor will be softened. I will follow a path between the standards level fixed by von Neumann y Dirac in their approaches to quantum mechanics.

2 Quantum Mechanics and Relativity

2.1 Quantum Mechanics

A quantum theory is defined by a space of states which are projective rays of vectors $|\psi\rangle$ of a Hilbert space $\mathcal{H}$. The physical observables are Hermitian operators in this Hilbert space. In any quantum system, there is an special observable, the Hamiltonian $H(t)$, which governs the time evolution of the quantum states by the first order differential equation

$$i\partial_t|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

The symmetries of a quantum system are unitary operators U which commute with the Hamiltonian of the system. In the particular case that the Hamiltonian $H(t)$ is time independent the unitary group defined by

$$U(t) = e^{itH}$$

is as symmetry group, i.e. $[U(t), H] = 0$, and defines the dynamics of the quantum system,

$$|\psi(t)\rangle = U(t)|\psi(t)\rangle.$$

Some interesting cases of quantum systems are those who arise from the canonical quantization of classical mechanical systems. The archetype of those systems is the harmonic oscillator. Let us analyse this case in some detail because it will be useful to understand its generalization to field theory.

Classical Harmonic Oscillator

The Lagrangian of an harmonic oscillator is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2. \tag{2.1}$$

The Euler-Lagrange equations give rise to the classical Newton's equation of motion

$$\ddot{x} = -\omega^2 x. \tag{2.2}$$

The solution of (2.2) in terms of the initial Cauchy conditions $(x(0), \dot{x}(0))$ is

$$x(t) = x(0)\cos\omega t + \frac{\dot{x}(0)}{\omega}\sin\omega t \tag{2.3}$$

$$= \frac{1}{2}\left[x(0) + i\frac{\dot{x}(0)}{\omega}\right] e^{-i\omega t} + \frac{1}{2}\left[x(0) - i\frac{\dot{x}(0)}{\omega}\right] e^{i\omega t}. \tag{2.4}$$

Upon Legendre transformation

$$p = m\dot{x}, \tag{2.5}$$

and the Poisson bracket structure,

$$\{x, x\} = \{p, p\} = 0; \quad \{x, p\} = 1 \tag{2.6}$$

one obtains the Hamiltonian of the harmonic oscillator

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2, \tag{2.7}$$

and the corresponding Hamilton equations of motion

$$\dot{x} = \{x, H\} = \frac{p}{m}, \qquad \dot{p} = \{p, H\} = -m\omega^2 x \tag{2.8}$$

are equivalent to Newton's equations (2.2).

In field theory is very convenient to use the coherent variables

$$a = \sqrt{\frac{m\omega}{2}}x + \frac{i}{\sqrt{2m\omega}}p; \qquad a^* = \sqrt{\frac{m\omega}{2}}x - \frac{i}{\sqrt{2m\omega}}p, \tag{2.9}$$

in terms of which time evolution becomes

$$a(t) = a(0)\, e^{-i\omega t}, \tag{2.10}$$

which gives (2.4),

$$x(t) = \frac{1}{\sqrt{2m\omega}}\,(a(t)\; + a^*(t)\,) = \frac{1}{\sqrt{2m\omega}}\left(a(0)\, e^{-i\omega t} + a^*(0)\, e^{i\omega t}\right). \tag{2.11}$$

The Quantum Harmonic Oscillator

The canonical quantization prescription proceeds by mapping the classical observables, position x and momentum p into operators in a Hilbert space $\mathcal{H}$ and replacing the Poisson bracket $\{\cdot,\cdot\}$ by a commutator of operators $i[\cdot,\cdot]$, i.e.

$$\{x, p\} = 1 \Rightarrow [\hat{x}, \hat{p}] = i\,\mathbb{I}, \tag{2.12}$$

where we assume that the Planck constant $\hbar = 1$.

The Hamiltonian is given by

$$\hat{H} = \frac{1}{2}\omega[\hat{p}^2 + \hat{x}^2],$$

which reads as

$$\hat{H} = \omega(a^\dagger a + \tfrac{1}{2}), \tag{2.13}$$

in terms of creation and annihilation operators

$$a = \sqrt{\frac{m\omega}{2}}\hat{x} + \frac{i}{\sqrt{2m\omega}}\hat{p}; \qquad a^\dagger = \sqrt{\frac{m\omega}{2}}\hat{x} - \frac{i}{\sqrt{2m\omega}}\hat{p}. \tag{2.14}$$

Now, because of the conmutation relations

$$[H, a^\dagger] = \omega a^\dagger \qquad\qquad [H, a] = -\omega a \qquad\qquad [a, a^\dagger] = 1,$$

we have that for any eigenstate $|E\rangle$ of the Hamiltonian $\hat{H}$, $\hat{H}|E\rangle = E|E\rangle$ there are two more eigenstates $a^\dagger|E\rangle$ and $a|E\rangle$, because

$$Ha^\dagger|E\rangle = a^\dagger H|E\rangle + \omega a^\dagger|E\rangle = (E+\omega)a^\dagger|E\rangle$$
$$H\,a|E\rangle = a\,H\,|E\rangle - \omega\,a\,|E\rangle = (E-\omega)a\,|E\rangle.$$

Since, $\langle\psi|H|\psi\rangle = \omega\langle\psi|a^\dagger a + \frac{1}{2}\mathbb{I}|\psi\rangle = (\omega + \frac{1}{2})||a|\psi\rangle||^2 \geq 0$ the ladder of quantum states have to end before the energy E becomes negative. The only possibility to avoid this is by the existence of a final ground state such that $a|E_0\rangle = 0$. But then, the energy of this ground state $H|E_0\rangle = \frac{1}{2}\omega|E_0\rangle$ is non-vanishing, unlike the energy of the classical vacuum configuration which vanishes. The non-trivial value of the quantum vacuum energy has remarkable consequences for the physics of the vacuum in the quantum field theory.

We shall denote from now on ground state $|E_0\rangle$ by $|0\rangle$. Higher energy states are obtained by applying the creation operator $a^\dagger$ to the ground state $|0\rangle$ by

$$|n\rangle = \frac{1}{\sqrt{(n+1)!}}(a^\dagger)^n|0\rangle; \qquad H|n\rangle = (n + \tfrac{1}{2})\,\omega|n\rangle \tag{2.15}$$

for any positive integer $n \in \mathbb{N}$. The state $|n\rangle$ has unit norm, i.e. $\|\,|n\rangle\|^2 = 1$ and satisfies that

$$a|n\rangle = \sqrt{n}|n-1\rangle.$$

In the harmonic oscillator the fundamental observables are the position x and the momentum p. Since by the quantization prescription $\hat{x}$ and $\hat{p}$ do not commute, the space of states has to be infinite-dimensional. Moreover, since any other operator O that commutes with both fundamental operators $[\hat{p}, O] = [\hat{x}, O] = 0$ has to be proportional to the identity $O = cI$, and the projector $\mathbb{P}$ to the subspace spanned by the states $|n\rangle$ commutes with $\hat{x}$ and $\hat{p}$ ($[\hat{p}, \mathbb{P}] = [\hat{x}, \mathbb{P}] = 0$) it follows that $\mathbb{P} = cI$. This implies that the subspace spanned by the vectors $|n\rangle$ is complete, i.e. does coincide with the whole Hilbert space $\mathcal{H}$.

Although the principles of quantum mechanics are identical for all quantum system and all separable Hilbert spaces are isomorphic, different systems can be distinguished by their algebra of observables. In the particular case of the harmonic oscillator the belonging of the position $\hat{x}$ and the momentum $\hat{p}$ operators to the algebra of observables not only implies that the Hilbert space is infinite dimensional, but also that it can be identified with the space of square integrable functions $L^2(\mathbb{R})$ of the position (Schrödinger representation) or momentum (Heisenberg representation).

In the Schrödinger representation the ground state reads

$$\langle x|0\rangle = \sqrt{\frac{\omega}{\pi}} e^{-\frac{1}{2}\omega x^2}.$$

The position and momentum operator are given by

$$\hat{x}\psi(x) = x\psi(x); \qquad \hat{p}\psi(x) = -i\partial_x\psi(x).$$

The Hamiltonian reads

$$\hat{H} = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2, \tag{2.16}$$

and the creation and destruction operators are

$$a = \frac{1}{\sqrt{2m\omega}}\left(\frac{d}{dx} + m\omega x\right); \qquad a^\dagger = \frac{1}{\sqrt{2m\omega}}\left(-\frac{d}{dx} + m\omega x\right). \tag{2.17}$$

It is easy to check that the excited states (2.15) are given by

$$\langle x|n\rangle = H_n(x) e^{-\frac{1}{2}\omega x^2}.$$

in terms of the Hermite polynomials

$$H_n(x) = e^{\frac{1}{2}\omega x^2} \frac{1}{\sqrt{(n+1)!}} (a^\dagger)^n e^{-\frac{1}{2}\omega x^2}.$$

The generalization for multidimensional harmonic oscillators is straightforward. If we have n-harmonic oscillators of frequencies ω_i and masses $m_i; i = 1, 2, \cdots n$. The position and momentum operators are given in the Schrödinger representation by

$$\hat{x_i}\psi(\mathbf{x}) = x_i\psi(\mathbf{x}); \qquad \hat{p_i}\psi(\mathbf{x}) = -i\partial_i\psi(\mathbf{x}),$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. They satisfy the canonical commutation relations

$$[\hat{x_i}, \hat{p_j}] = i\delta_{ij}.$$

The Hamiltonian is

$$\hat{H} = -\sum_{i=1}^{n} \frac{1}{2m_i}\left(\partial_i^2 + m\omega_i^2 x_i^2\right), \tag{2.18}$$

and in terms the creation and destruction operators

$$a_i = \frac{1}{\sqrt{2m_i\omega_i}}\left(\partial_i + m_i\omega_i x_i\right); \qquad a^\dagger = \frac{1}{\sqrt{2m_i\omega_i}}\left(-\partial_i + m_i\omega_i x_i\right).$$

reads

$$\hat{H} = \sum_{i=1}^{n} \omega_i (a_i^\dagger a_i + \tfrac{1}{2}), \tag{2.19}$$

The ground state

$$\langle \mathbf{x}|0\rangle = \left(\prod_{i=1}^{n} \sqrt{\frac{\omega_i}{\pi}}\right) e^{-\frac{1}{2}\sum_{i=1}^{n} \omega_i x_i^2},$$

has an energy given by the sum of the ground state energies of the different harmonic modes

$$E_0 = \frac{1}{2}\sum_{i=1}^{n} \omega_i.$$

It is easy to check that the excited states (2.15) are given by

$$\langle \mathbf{x}|n_1, n_2, \ldots, n_n\rangle = \left(\prod_{i=1}^{n} H_{n_i}(x_i)\right) e^{-\frac{1}{2}\sum_{i=1}^{n} \omega_i x_i^2}.$$

in terms of the Hermite polynomials $H_{n_i}, i = 1, 2, \ldots, n$.

2.2 Relativity and the Poincaré Group

The Einstein theory of Relativity is based on the unification of space and time into a four-dimensional Minkowski space-time $\mathbb{R}^4$ equiped with the Lorentzian metric of signature $(+, -, -, -)$

$$dx^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 = \sum_{\mu=0}^{3} \eta_{\mu\nu} dx_\mu dx_\nu, \tag{2.20}$$

where $x_0 = ct$ denotes the time-like coordinate and

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.21}$$

the Minkowski metric. From now on we shall assume that the speed of light c is normalized to unit.

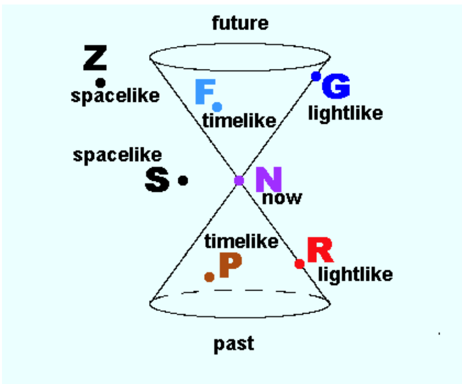

Figure 1. Light cone an causal structure of space-time.

The geodesic which connects two points x, y of Minkowski space-time is a straighline in $\mathbb{R}^4$ and the Minkowski distance between the two points is

$$d(x, y) = (y_0 - x_0)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2 - (y_3 - x_3)^2, \tag{2.22}$$

When this distance vanishes the geodesic line represents the trajectory of a light ray connecting the two points.

The line connecting two points x, y is time-like if $d(x,y) > 0$; space-like if $d(x,y) > 0$; light-like if $d(x,y) = 0$; Two points x, y are causally separated iff $d(x,y) \geq 0$; and spatially separated iff $d(x,y) < 0$; A causal line connecting x, y is future oriented if $y_0 - x_0 > 0$; past oriented if $y_0 - x_0 < 0$;

The space-time symmetries of a relativistic theory are space-time translations,

$$x' = x + a$$

space rotations, e.g.

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}, \qquad \Lambda^{\mu}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad 0 \leq \theta \leq 2\pi \tag{2.23}$$

and Lorentz transformations, e.g.

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}, \qquad \Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad \gamma = \sqrt{1 - v^2} \tag{2.24}$$

which are linear transformations which leave the Minkowski metric invariant

$$\eta_{\mu\nu} = \Lambda^{\sigma}_{\mu} \eta_{\sigma\tau} \Lambda^{\tau}_{\mu}.$$

There are some extra discrete symmetries which play an important role in field theory. They are generated by time reversal,

$$(\mathbf{x}, t) \to (\mathbf{x}, -t)$$

and parity

$$(\mathbf{x}, t) \to (-\mathbf{x}, t),$$

which differ from those of (2.23), (2.24). The whole group of space-time symmetries is the Poincar e group which contains all these continuous and discrete symmetries

$$\mathcal{P} = ISO(3,1) = \{(\Lambda, a); \Lambda \in O(3,1), a \in \mathbb{R}^4\}.$$

The first attempts to make compatible the quantum theory with the theory of relativity where based on covariant equations like the Maxwell equation of classical electrodynamics. This lead to the discovery of Klein-Gordon equation for scalar fields and the Dirac equation for spinorial fields. E. Wigner introduced a completely different approach. If the quantum theories are defined in a Hilbert space and the relativity is based on Poincaré group he conjectured that elementary quantum particles as the most elementary quantum system must support irreducible representations of the Poincaré group. This program was achieved by Wigner and Mackey and the results is the simplest irreducible representations are characterized by two numbers which represent the mass $m\mathbb{R}_+$ and the spin $s \in \mathbb{N}/2$ of the particle. The case of spin zero correspond to a scalar fields satisfying the Klein-Gordon equation.

3 Quantum Field Theory

The most interesting case of field theory is that which concerns relativistic fields. The compatibility of quantum theories with the theory of relativity is not immediate. The first attempts to formulate a quantum dynamics compatible with the theory of relativity lead to puzzling theories full of paradoxes, like the Klein paradox, which arises in the dynamics defined by Klein-Gordon or Dirac equations. The solution to those puzzles comes from the quantization of classical field theories.

A quantum field theory is a quantum theory which is relativistic invariant and where there is an special type of quantum operators which are associated to the classical fields.

In the case of a real scalar field ϕ a consistent theory should satisfy the following principles.

- 1. *Quantum principle:* The space of quantum states is the space of rays a separable Hilbert space $\mathcal{H}$.
- 2. *Unitarity:* There is a (anti)unitary representation $U(\Lambda, a)$ of the Poincaré group in $\mathcal{H}$, where time reversal is represented as an anti-unitary operator $U(T)$.
- 3. *Spectral condition:* The spectrum of generators of space time translations P_μ is contained in the forward like cone
$$\bar{V}_+ = \{p_\mu; p^2 \geq 0, p_0 \geq 0\}. \tag{3.1}$$
- 4. *Vacuum state:* There is a unique vacuum state $\Psi_0 \in \mathcal{H}$, with $P_\mu \Psi_0 = 0$.
- 5. *Field Theory (real boson):* For any classical field f in the space $\mathcal{S}(\mathbb{R}^3)$ of *fast* decreasing smooth $C^\infty(\mathbb{R}^3)$ functions [1] there is field operator $\phi(f)$ in $\mathcal{H}$ which satisfies that $\phi(f) = \phi(f)^\dagger$. The field operator can be considered as the smearing by f of a fundamental field operator $\phi(x)$
$$\phi(f) = \int d^3\mathbf{x} f(\mathbf{x})\phi(x). \tag{3.2}$$
The subspace spanned by the vectors $\phi(f_1)\phi(f_2)\cdots\phi(f_n)|0\rangle$ for arbitrary test functions $f_1, f_2, \cdots f_n \in \mathcal{S}(\mathbb{R}^3)$ is a dense subspace of $\mathcal{H}$.
- 6. *Poincaré covariance:* Let $\tilde{f} \in \mathcal{S}(\mathbb{R}^4)$ be a test function defined in Minkowski space-time and
$$\phi(\tilde{f}) = \int_{\mathbb{R}^4} d^4x\, \phi(x)\tilde{f}(x), \tag{3.3}$$
where $\phi(x) = \phi(\mathbf{x}, t) = e^{itP_0}\phi(\mathbf{x})e^{-itP_0}$.Then,
$$U(\Lambda, a)\phi(\tilde{f})U(\Lambda, a)^\dagger = \phi(\tilde{f}_{(\Lambda,a)}) \tag{3.4}$$
[2],
where
$$\tilde{f}_{(\Lambda,a)}(x) = \tilde{f}(\Lambda^{-1}(x - a)) \tag{3.5}$$
- 7. *(Bosonic) Local Causality:* For any $f, g \in \mathcal{S}(\mathbb{R}^3)$ the corresponding field operators $\phi(f), \phi(g)$ commute, [3]
$$[\phi(f), \phi(g)] = 0. \tag{3.6}$$

[1] In the case of massless fields the test function has to be of compact support, i.e. $f \in \mathcal{D}(\mathbb{R}^3) = C_0^\infty(\mathbb{R}^3)$

[2] For higher spin fields the Poincaré representation satisfies $U(\Lambda, a)\phi(f)U(\Lambda, a)^\dagger = S(\Lambda)^{-1}\phi(f_{(\Lambda,a)})$, where S is a linear n-dimensional representation of Lorentz group and ϕ is a field with n-components

[3] In the fermionic case the commutator $[\cdot, \cdot]$ is replaced by an anticommutator $\{\cdot, \cdot\}$

3.1 Canonical Quantization

As in the case of quantum mechanics there are special cases where the quantum field theory arises from the quantization of a classical field theory.

Let us consider a scalar real field ϕ in Minkowski space-time $\mathbb{R}^4$. The classical field theory is given according to the variational principle from the stationary field configurations $\phi(x)$of the classical action functional

$$S[\phi(x)] \equiv \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) = \int d^4x \, \left(\tfrac{1}{2}\partial_\mu\phi \, \partial^\mu\phi - V(\phi)\right). \tag{3.7}$$

The equations of motion are obtained, thus, from the Euler-Lagrange equations

$$\partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right] - \frac{\delta \mathcal{L}}{\delta \phi} = 0 \qquad \Longrightarrow \qquad \Box \, \phi + \frac{\delta V}{\delta \phi} = 0, \tag{3.8}$$

where $\Box = \partial_\mu \partial^\mu$. Notice that the Poincaré invariance of the action implies the Poincaré invariance of the equations of motion.

The quantization is usually formulated in the Hamiltonian formalism. Thus, it is necessary to start from the classical canonical formalism. Let $\mathcal{M}$ be the configuration space of square integrable classical fields at any fixed time (e.g. $t = 0$),

$$\mathcal{M} = \left\{\phi(\mathbf{x}) = \phi(\mathbf{x}, 0); \parallel \phi \parallel^2 = \int d^3\mathbf{x} \, |\phi(\mathbf{x}, 0)|^2 < \infty \right\}. \tag{3.9}$$

The Legendre transformation maps the tangent space $T\mathcal{M}$ into the cotangent space $T^*\mathcal{M}$, fixing the value of the canonical momentum

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \dot{\phi}, \tag{3.10}$$

from the Lagrangian

$$L = \int d^3\mathbf{x} \left(\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2 - V(\phi)\right), \tag{3.11}$$

where $\dot{\phi} = \partial_t \phi$. The corresponding Hamiltonian is given by

$$H = \int d^3\mathbf{x} \left(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2 + V(\phi)\right), \tag{3.12}$$

In the case of a free massive theory with mass m, $V(\phi) = \frac{1}{2}m^2\phi^2$ and the Hamiltonian reads

$$H = \tfrac{1}{2}\left(\parallel \pi \parallel^2 + \parallel \boldsymbol{\nabla}\phi \parallel^2 + m^2 \parallel \phi \parallel^2\right), \tag{3.13}$$

where we have used the $L^2(\mathbb{R}^3)$ norm introduced in (3.9).

In this case the classical vacuum solution is unique $\phi = 0$. However, in the massless case $m = 0$ the vacuum is degenerated, because and constant configuration $\phi = \text{cte}$ is a solution with finite energy, although such configurations in the massive case have infinity energy.

The symplectic structure of $T^*\mathcal{M}$

$$\omega = \int d^3\, \mathbf{x} \; d\pi \wedge d\phi \tag{3.14}$$

induces a Poisson structure in the space of functionals of $T^*\mathcal{M}$. Given two local functionals $\mathcal{F}(\phi, \pi)$, $\mathcal{G}(\phi, \pi)$ of the canonical variables

$$\mathcal{F}(\phi, \pi) = \int d^3\mathbf{x} \; F(\phi, \pi), \qquad \mathcal{G}(\phi, \pi) = \int d^3\mathbf{x} \; G(\phi, \pi). \tag{3.15}$$

Their Poisson bracket is defined by

$$\{\mathcal{F},\mathcal{G}\} \equiv \int d^3\mathbf{x} \left[\frac{\boldsymbol{\delta} F}{\boldsymbol{\delta}\phi}\frac{\delta G}{\delta\pi} - \frac{\delta F}{\delta\pi}\frac{\boldsymbol{\delta} G}{\boldsymbol{\delta}\phi}\right], \tag{3.16}$$

where the functional derivative $\frac{\delta}{\delta\phi}$ is given by

$$\frac{\boldsymbol{\delta} F}{\boldsymbol{\delta}\phi} = \frac{\delta\mathcal{F}}{\delta\phi} - \partial_\mu\left[\frac{\delta\mathcal{F}}{\delta(\partial_\mu\phi)}\right]. \tag{3.17}$$

The Poisson brackets of fundamental fields are

$$\begin{aligned} \{\phi(\mathbf{x}_1),\phi(\mathbf{x}_2)\} &= \{\pi(\mathbf{x}_1),\pi(\mathbf{x}_2)\} = 0, \\ \{\phi(\mathbf{x}_1),\pi(\mathbf{x}_2)\} &= \delta^3(\mathbf{x}_1-\mathbf{x}_2). \end{aligned} \tag{3.18}$$

because of the basic rules of functional derivation

$$\frac{\delta\phi(\mathbf{x}_1)}{\delta\phi(\mathbf{x}_2)} = \delta^3(\mathbf{x}_1-\mathbf{x}_2); \qquad \frac{\delta\pi(\mathbf{x}_1)}{\delta\pi(\mathbf{x}_2)} = \delta^3(\mathbf{x}_1-\mathbf{x}_2). \tag{3.19}$$

The appearance of delta functions in the Poisson structure of the fields reflects the fact that a mathematically sound analysis of field theory requires the use of distributions. This will be even more necessary for the quantum fields. Thus, it is convenient to consider smeared field functionals. Given a classical function f which might be more regular that the $L^2(\mathbb{R}^3)$ fields (e.g. $f\in\mathcal{S}(\mathbb{R}^3)$ for massive fields, or $f\in\mathcal{D}(\mathbb{R}^3)$ for massless fields) the smeared fields are defined by the images of the linear functional

$$\phi(f) = \int d^3\mathbf{x} f(\mathbf{x})\phi(x); \qquad \pi(f) = \int d^3\mathbf{x} f(\mathbf{x})\pi(x) \tag{3.20}$$

in $L^2(\mathbb{R}^3)$.

The Poisson structure can be expressed in terms of smeared fields $\phi(f)$ as

$$\begin{aligned} \{\phi(f_1),\phi(f_2)\} &= \{\pi(f_1),\pi(f_2)\} = 0, \\ \{\phi(f_1),\pi(f_2)\} &= (f_1,f_2) \end{aligned} \tag{3.21}$$

where $(\cdot,\cdot)$ denotes the Hilbert product of $L^2(\mathbb{R}^3)$.

By choosing an orthonormal Hilbert basis of test functions f_n in $L^2(\mathbb{R}^3)$ we can get a discrete representation of the Poisson structure,

$$\{\phi_n,\phi_m\} = \{\phi_n,\phi_m\} = 0; \quad \{\phi_n,\pi_m\} = \delta_{mn} \tag{3.22}$$

where $\phi_n=\phi(f_n)$ and $\pi_n=\pi(f_n)$.

In that representation the Hamiltonian operator (3.13) becomes

$$H = \tfrac{1}{2}\sum_{n=0}^{\infty}{\pi_n}^2 - \tfrac{1}{2}\sum_{n,m=0}^{\infty}\Delta_{mn}\phi_m\phi_n + \tfrac{1}{2}m^2\sum_{n=0}^{\infty}{\phi_n}^2, \tag{3.23}$$

where

$$\Delta_{mn} = (f_m,\Delta f_n) = (f_m,\boldsymbol{\nabla}^2 f_n). \tag{3.24}$$

In this representation it is clear that the system describes an infinity of coupled harmonic oscillators. The way of disentangling the coupling is to find the normal modes, i.e. to choose of basis of test functions f_n where the interaction operator Δ is diagonal. The normal modes are plane waves which do not belong to $L^2(\mathbb{R}^3)$. For such a reason it is convenient to introduce an infrared regulator, i.e. to consider the system in a finite volume. There are physical reason why this method is sensible. In the quantum case there will appear a series of divergences of two types: ultraviolet (UV) divergences, which are due to short range singularities dues to to the local products of distributions, and infrared (IR) divergences which are dues to the infinity volume of space. Both need to be regularized and renormalized as we will see later in these lectures. In this perspective the introduction of a finite volume can be considered as a regulator of IR divergences. Poincaré invariance will be recovered in the limit of infinite volume at the very end.

We shall consider mostly the torus T^3 compactification of ($\mathbb{R}^3$), The normal modes in this case are normalizable plane waves,

$$f^+_{\mathbf{n}}(\mathbf{x}) = \frac{1}{2}(f_{\mathbf{n}}(\mathbf{x}) + f_{\mathbf{n}}(\mathbf{x})^*); \quad f^-_{\mathbf{n}}(\mathbf{x}) = -\frac{i}{2}(f_{\mathbf{n}}(\mathbf{x}) - f_{\mathbf{n}}(\mathbf{x})^*) \qquad \mathbf{n} \in \mathbb{Z}^3_+. \tag{3.25}$$

with

$$f_{\mathbf{n}}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{i2\pi\mathbf{n}\cdot x/L}. \tag{3.26}$$

where $\mathbf{n} \in \mathbb{Z}^3$, $\mathbf{n}\cdot\mathbf{x} = n_1x_1 + n_2x_2 + n_3x_3$ and L is the length of each side of the torus, i.e. $\mathbf{x} \in [0, L]^3$. The normal modes diagonalize the Hamiltonian because

$$\Delta f^{\pm}_{\mathbf{n}}(x) = -\tfrac{2\pi}{L}(\mathbf{n}\cdot\mathbf{n}) f^{\pm}_{\mathbf{n}}(x). \tag{3.27}$$

However, it is more convenient to use the complex modes f_n, provided that in the mode expansion the fields

$$\phi(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \phi_n f_{\mathbf{n}}(\mathbf{x}), \tag{3.28}$$

the coefficients $\phi_n = \phi(f)$ satisfy the reality conditions $\phi^*_{\mathbf{n}} = \phi_{-\mathbf{n}}$ in order guarantee the reality of the fields $\phi^* = \phi$

In terms of the complex modes the Hamiltonian is diagonal

$$H = \tfrac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^3} \left(|\pi_{\mathbf{n}}|^2 + (|\tfrac{2\pi\mathbf{n}}{L}|^2 + m^2)|\phi_{\mathbf{n}}|^2\right), \tag{3.29}$$

and it is evident that the system (4.4) describes an infinity of harmonic oscillators with frequencies

$$\omega_{\mathbf{n}} = \sqrt{|\tfrac{2\pi\mathbf{n}}{L}|^2 + m^2}. \tag{3.30}$$

Canonical quantization maps classical fields into operators in a Hilbert space $\mathcal{H}$ satisfying the commutation relations obtained by replacing Poisson brackets by commutators

$$\{\cdot,\cdot\} \Longrightarrow i\,[\cdot,\cdot], \tag{3.31}$$

which can be realized in the Schrödinger representation on space of functionals of $\mathcal{M}$ by

$$\hat{\pi}(\mathbf{x}) = -i\,\frac{\delta}{\delta\phi(\mathbf{x})}; \qquad \hat{\phi}(\mathbf{x}) = \phi(\mathbf{x}). \tag{3.32}$$

The corresponding quantum Hamiltonian is

$$\hat{H} = \tfrac{1}{2}\left(\parallel \hat{\pi} \parallel^2 + \parallel \boldsymbol{\nabla}\phi \parallel^2 + m^2 \parallel \phi \parallel^2\right), \tag{3.33}$$

In terms of smeared functions the Schrödinger representation of the momentum operator

$$\hat{\pi}(f) = -i \int d^3\mathbf{x}\, f(\mathbf{x}) \frac{\delta}{\delta\phi(\mathbf{x})}, \tag{3.34}$$

becomes just a Gateaux derivative operator (see P. Michor lectures)

$$\hat{\pi}(f)\mathcal{F}(\phi) = -i \lim_{s\to 0} \frac{1}{s} \left(\mathcal{F}(\phi + sf) - \mathcal{F}(\phi)\right). \tag{3.35}$$

For any orthonormal basis of test functions f_n in $L^2(\mathbb{R}^3)$ we have that for classical fields $\phi \in L^2(\mathbb{R}^3)$

$$\phi(\mathbf{x}) = \sum_{n=0}^{\infty} \phi_n f_n(\mathbf{x}), \tag{3.36}$$

where $\phi_n = \phi(f_n)$. Moreover, since by linearity $\phi(f) = \sum_{n=0}^{\infty} \phi_n f_n$

$$\hat{\pi}(\mathbf{x})\mathcal{F}(\phi) = -i \frac{\delta}{\delta\phi(\mathbf{x})} \mathcal{F}\left(\sum_{n=0}^{\infty} \phi_n f_n(\mathbf{x})\right) \tag{3.37}$$

and from (3.35) it follows that

$$\hat{\pi}(f_n) = -i\, f_n \frac{\delta}{\delta\phi_n}. \tag{3.38}$$

In the plane wave basis the quantum Hamiltonian

$$\hat{H} = \tfrac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^3} \left(\frac{\delta}{\delta\phi_{-\mathbf{n}}} \frac{\delta}{\delta\phi_{\mathbf{n}}} + \omega_{\mathbf{n}}^2 \,|\phi_{\mathbf{n}}|^2 \right), \tag{3.39}$$

again corresponds to an infinity of harmonic oscillators with frequencies $\omega_{\mathbf{n}}$.

4 The quantum vacuum

The advantage of the diagonal structure of the quantum Hamiltonian in the plane wave basis is that it facilitates the analysis of its spectrum.

In particular the ground state known in QFT as vacuum state is given by

$$\Psi_0 = \prod_{\mathbf{n}\in\mathbb{Z}^3} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\omega_{\mathbf{n}}|\phi_{\mathbf{n}}|^2} = \exp\left\{ -\tfrac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^3} \left(\omega_{\mathbf{n}}|\phi_{\mathbf{n}}|^2 + \log 2\pi\right) \right\} \tag{4.1}$$

Indeed,

$$\hat{H}\Psi_0 = E_0\Psi_0, \tag{4.2}$$

where the vacuum energy

$$E_0 = \frac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^3} \omega_{\mathbf{n}} \tag{4.3}$$

is half the divergent sum of all normal modes frequencies. This appearance of this divergence is a genuine quantum effect which is induced by the fact the lowest energy (*zero-point energy*) of each quantum oscillator is non-vanishing. The divergence is generated by the large momentum $\mathbf{n}$ (*ultraviolet*) modes. The vacuum energy is the first quantity of the quantum theory which presents UV divergences.

4.1 Renormalization of Vacuum Energy

The appearance of UV divergences postponed the formulation of quantum field theories for two decades. The solution of the UV puzzle came from the renormalization program. The main idea behind the renormalization program is the disassociation of the fundamental observables like the quantum Hamiltonian and the observed quantities.

To implement the renormalization program we need a previous step which is known a regularization. In this step we introduce a modification of all fundamental (*bare*) operators depending on a UV scale parameter Λ in a way the it becomes a well defined operator with a finite spectrum, e.g. by cutting the infinite sum in (4.6) to a finite sum The modification disappears in the limit $\Lambda \to \infty$ where we recover the original divergent expressions. The second step consist in *physical* modification of the fundamental observables by absorbing the sources of divergences into the physical parameters like masses, charges, couplings or energy scales of the theory in a way that the they remain finite in the $\Lambda \to \infty$.

To illustrate the implementation of the renormalization mechanism let us consider the case of the Hamiltonian operator of the free field theory (3.33).

The regularization can be introduced in different ways. Let us consider two different methods

- Sharp momentum cut-off

$$\hat{H}_\Lambda = \tfrac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^3}^{\omega_\mathbf{n}<\Lambda} \left(\frac{\delta}{\delta\phi_{-\mathbf{n}}} \frac{\delta}{\delta\phi_{\mathbf{n}}} + \omega_\mathbf{n}^2 \, |\phi_\mathbf{n}|^2 \right), \tag{4.4}$$

- Heat kernel regularization

$$\hat{H}_\epsilon = \tfrac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^3} \left(\frac{\delta}{\delta\phi_{-\mathbf{n}}} \frac{\delta}{\delta\phi_{\mathbf{n}}} + \omega_\mathbf{n}^2 \, e^{-\epsilon\omega_\mathbf{n}^2} \, |\phi_\mathbf{n}|^2 \right), \tag{4.5}$$

which can be related by choosing $\epsilon = \frac{\sqrt{2}}{\Lambda^2}$.

There are other methods which include higher derivative terms or lattice discretization of the continuum space, but for simplicity we shall not discuss them in this course.

The renormalization of the fundamental Hamiltonian is obtained by substracting an *unobservable* constant quantity E_0 in such a way that the observable (*renormalized*) quantum Hamiltonian

$$\hat{H}_{\rm ren} = \lim_{\Lambda\to\infty} (\hat{H}_\Lambda - E_0(\Lambda)), \tag{4.6}$$

is a well defined quantum operator with a finite energy spectrum.

Even if the renormalization of the Hamiltonian solves the divergency problem, one might wonder about the physical meaning. First, let us analyse the structure of the divergences of vacuum energy.

In the large L limit the vacuum energy E_0 becomes a good approximation to the Riemann integral

$$E_0(\Lambda, L) = \frac{1}{2} L^3 \int_{|k|\le\Lambda} \frac{d^3k}{(2\pi)^3}\, \omega(k) + \mathcal{O}(L\Lambda) \tag{4.7}$$

Now it becomes clear that the infrared divergence is just due to the infinite volume of the system and translation invariance. However the vacuum energy density

$$\mathcal{E}_0(\Lambda) = \lim_{L\to\infty} \frac{E_0(\Lambda, L)}{L^3} = \frac{1}{2} \int_{|k|\le\Lambda} \frac{d^3k}{(2\pi)^3}\, \omega(k) \tag{4.8}$$

is free of IR divergences. However, the integral (4.8) is UV divergent. In the sharp momentum cut-off regularization

$$\mathcal{E}_0(\Lambda) = \frac{\Lambda^4}{16\pi^2} + \frac{m^2\Lambda^2}{16\pi^2} + \frac{m^4}{64\pi^2} \log\frac{m^2}{\Lambda^2} + \frac{m^4(1-\log 16)}{128\pi^2} + \mathcal{O}(\tfrac{1}{\Lambda^2}) \tag{4.9}$$

in the large Λ limit. Whereas in the heat kernel regularization

$$\mathcal{E}_0(\Lambda) = \frac{1}{8\pi^2\epsilon^2} - \frac{m^2}{16\pi^2\epsilon} + \frac{m^4}{64\pi^2}(2 + \gamma + \log\frac{\epsilon m^2}{4}) + \mathcal{O}(\epsilon) \tag{4.10}$$

for small values of $\epsilon = \frac{\sqrt{2}}{\Lambda^2}$. The leading quartic and logarithmic divergent terms are the same in both regularizations whereas the quadratically divergent term is different.

The source of divergence is of ultraviolet origin because it comes from the integration of $\omega(\mathbf{k})$ at large values of the momentum. The quantum field theory of free scalar fields is a infinite set of harmonic oscillators, each one labelled by $\mathbf{k}$. Each of these oscillators contribute to the vacuum energy with their zero-point energy, $\frac{1}{2}\omega(\mathbf{k})$. This total contribution of zero-points energies to the vacuum energy density gives infinity, since even then there are modes with arbitrary high momentum. This is the ultraviolet origin of this divergence. It appears in any quantum field theory and not only in the free scalar quantum field. It is something intrinsic to the theory of quantum fields.

4.2 Momentum Operator

The generator of space translations is the momentum operator $\hat{P}_i$. In the free scalar field theory it is given by

$$\hat{P}_i = \int d^3\mathbf{x}\,(\hat{\pi}\partial_i\phi)\,, \qquad i = 1, 2, 3, \tag{4.11}$$

Since the vacuum state is invariant,

$$\hat{P}_i\Psi_0 = i\int d^3\mathbf{x}\;\phi\sqrt{\mathbf{\nabla}^2 + m^2}\,\partial_i\phi\Psi_0 = \frac{i}{2}\int d^3\mathbf{x}\,\partial_i\left(\phi\sqrt{\mathbf{\nabla}^2 + m^2}\,\phi\right)\Psi_0 = 0, \tag{4.12}$$

which apparently does not require renormalization as the vacuum energy, However, this is too naive. If we write (4.13) in terms of the Fourier modes of the field,

$$\hat{P}_i\Psi_0 = \frac{2\pi}{L}\sum_{\mathbf{n}\in\mathbb{Z}^3}(n_i\,\omega_{\mathbf{n}})\Psi_0 = 0, \tag{4.13}$$

we realized that the sum is divergent. However, in the cut-off or heat kernel regularizations the regularized eigenvalues vanish

$$\hat{P}_i\Psi_0 = \frac{2\pi}{L}\sum_{\mathbf{n}\in\mathbb{Z}^3}^{|\omega_{\mathbf{n}}|<\Lambda}(n_i\,\omega_{\mathbf{n}})\Psi_0 = \frac{2\pi}{L}\sum_{\mathbf{n}\in\mathbb{Z}^3}(n_i\,\omega_{\mathbf{n}}e^{-\epsilon\omega_{\mathbf{n}}^2})\Psi_0 = 0, \tag{4.14}$$

and, thus, the renormalized value of the vacuum eigenvalue of $\hat{P}_i$ vanish. This is due to the spherical symmetry of both regularizations. What is remarkable is that in the case of the vacuum energy any choice of the regularization provides a non-vanishing value. In this sense there is a different between the quantum generators of space and time translations, which seems to be not in agreement with the Lorentz symmetry. This is a genuine characteristic of canonical quantization as will be emphasized later in the course.

4.3 Casimir effect

The existence of UV divergences in vacuum energy is not a special property of the scalar field. Any quantum field theory faces the same problem, e.g. the electromagnetic field in quantum electrodynamics or the fields of the Standard Model have UV divergent vacuum energies. We have renormalized the divergences by removing the whole contribution of vacuum energy. However, this does not mean that it is an unphysical quantity. The fact that the vacuum energy can have observable consequences was first pointed out by H. Casimir in 1948. He remarked that although we can remove a fixed vacuum energy for the free fields, the variation of the vacuum energy under external conditions could be detected and observed.

Consider a pair of infinite, perfectly conducting plates placed parallel to each other at a distance d. The conducting character of the plates implies that the electromagnetic forces vanishes at both plate surfaces. The presence of the plates modifies the vacuum energy in a d-dependent way. If the perturbation increases the vacuum energy with the distance it will induce an attractive force between the plates and this force will be repulsive if the energy decreases with the distance.

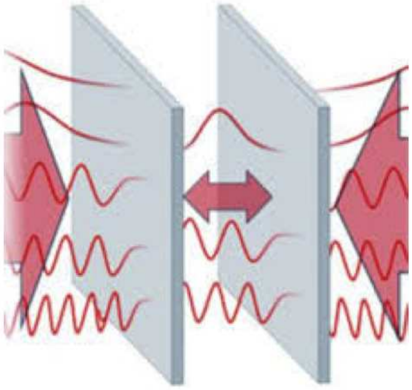

Figure 2. Three different domains in the Casimir effect.

The main modification introduced by the plates is the presence of boundary conditions on the classical fields. Since the free electromagnetic field corresponds to photons with two polarizations the electromagnetic vacuum energy is twice the vacuum energy of a massless scalar field with vanishing boundary conditions on the plates. The physical $\mathbb{R}^3$ space is split into three disjoint domains

$$\begin{aligned} \Omega_{I} &= \{\mathbf{x} \in \mathbb{R}^3; -\infty < x_3 \leq \tfrac{d}{2}\} \\ \Omega_{II} &= \{\mathbf{x} \in \mathbb{R}^3; -\frac{d}{2} < x_3 \leq \frac{d}{2}\} \\ \Omega_{III} &= \{\mathbf{x} \in \mathbb{R}^3; -\infty < x_3 \leq \frac{d}{2}\}. \end{aligned}$$

The physical vacuum is the product of the vacua of the different sectors

$$\psi_0(\phi) = \psi_{I}(\phi_{I})\, \psi_{II}(\phi_{II})\, \psi_{III}(\phi_{III}) \tag{4.15}$$

and the vacuum energy the sum of the vacuum energies of the three domains

$$E_0 = E_{I} + E_{II} + E_{III} \tag{4.16}$$

The calculation of vacuum energy density $\mathcal{E}_{II} = E_{II}/V_{II}$ can be seen in appendix A and gives the following result using the heat kernel regularization

$$\mathcal{E}_{II} = \frac{1}{8\pi^2\epsilon^2} + \frac{1}{16\sqrt{\pi}\epsilon^{\frac{3}{2}}}\frac{1}{d} - \frac{\pi^2}{1440 d^4} + \mathcal{O}(\epsilon^{\frac{1}{2}}) \tag{4.17}$$

In the other cases we only get

$$\mathcal{E}_{I} = \frac{1}{8\pi^2\epsilon^2}, \qquad \mathcal{E}_{III} = \frac{1}{8\pi^2\epsilon^2} \tag{4.18}$$

because of their infinite transversal size. The common divergent term corresponds to the vacuum density in infinite volume. Thus, it disappears under vacuum energy renormalization. The $\epsilon^{\frac{3}{2}}$ divergent term between the plates correspond to the selfenergy of the plates and has to be renormalized as well. The remaining renormalized vacuum energy density between the plates is

$$\mathcal{E}_{_{II}}^{\rm ren} = -\frac{\pi^2}{1440 d^4} \tag{4.19}$$

is negative which leads to an attractive force between the plates.

5 Fields versus Particles

We have assumed that the field operators act on a separable Hilbert space $\mathcal{H}$ without any special properties. In canonical quantization we assumed that the space of quantum states is given by functionals of the configuration space of square integrable classical fields $\mathcal{M}$ (3.9). However, such a space has not a well defined Hilbert product. The reason being that in quantum mechanics the equivalent space of functions of the configuration space is given by the L^2 product in terms of the Lebesgue measure of $\mathbb{R}^n$, $d^n\,x$. However in infinite dimensional Hilbert spaces the equivalent Lebesgue measure is not well defined, because the basic building blocks of hypercubes of size L have infinity volume if $L > 1$ or zero if $L < 1$. Thus, although all operators: the fields $\phi(f)$, the Hamiltonian $\hat{H}$ and momentum operator $\hat{\mathbf{P}}$ are formally selfadjoint with respect to the naive generalization of Lebesgue measure $\delta\phi$, the definition of the quantum field theory require a rigourous definition of the the Hilbert product and a redefinition of the physical observables.

The key ingredient is that the naive vacuum state (4.1) defines a good measure in the space of functionals on space of classical fields $\mathcal{M}$. Indeed, the measure defined by

$$\delta\mu(\phi) = \mathcal{N} e^{-(\phi,\sqrt{-\boldsymbol{\nabla}^2+m^2}\phi)}\delta\phi = \prod_{\mathbf{n}\in\mathbb{Z}^3} \tfrac{1}{\sqrt{\pi}} e^{-\omega_{\mathbf{n}}\phi_n^2}\delta\phi_n, \tag{5.1}$$

where $\mathcal{N}$ is the normalization factor, which guarantees that the volume of the configuration space is unit. According to Minlos' theorem (see Appendix B)the Gaussian measure $\delta\mu$ is supported on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^3)$ in the massive case and on the space of generalized distributions $\mathcal{D}'$ in the massless case.

The above definition requires a redefinition of all physical physical states and operators by a similarity transformation

$$\Psi(\phi) \Rightarrow e^{\frac{1}{2}(\phi,\sqrt{-\boldsymbol{\nabla}^2+m^2}\phi)}\Psi(\phi); \qquad O \Rightarrow e^{\frac{1}{2}(\phi,\sqrt{-\boldsymbol{\nabla}^2+m^2}\phi)} O e^{-\frac{1}{2}(\phi,\sqrt{-\boldsymbol{\nabla}^2+m^2}\phi)}, \tag{5.2}$$

The field operator $\phi(f)$ remains unchanged whereas the canonical momentum operator $\hat{\pi}(f)$ becomes

$$\hat{\pi}(f) = -i \int d^3\mathbf{x}\, f(\mathbf{x}) \left(\frac{\delta}{\delta\phi(\mathbf{x})} - \sqrt{-\boldsymbol{\nabla}^2+m^2}\phi(x) \right), \tag{5.3}$$

and are now selfadjoint with respect to the Hilbert product

$$(\mathcal{F},\mathcal{G}) = \int_{\mathcal{S}'(\mathbb{R}^3)} \delta\mu(\phi)\, \mathcal{F}(\phi)^* \mathcal{G}(\phi) \tag{5.4}$$

of $\mathcal{H} = L^2(\mathcal{S}'(\mathbb{R}^3), \delta\mu)$.

The vacuum state becomes trivial

$$\Psi_0 = 1, \tag{5.5}$$

which now is normalizable with respect to the Gaussian measure (5.1), i.e $(\Psi_0, \Psi_0) = 1$.

The new renormalized Hamiltonian

$$\hat{H}_{\rm ren} = -\int d^3\mathbf{x} \left(\frac{\delta}{\delta\phi(\mathbf{x})} - 2\sqrt{-\boldsymbol{\nabla}^2+m^2}\phi(x) \right) \frac{\delta}{\delta\phi(\mathbf{x})}, \tag{5.6}$$

and the excited states are just field polynomials.

5.1 Fock space

The space of physical states is generated by polynomials of field operators, e.g.

$$\mathcal{F}_{(f_1,f_2,\cdots,f_n)} = \phi(f_1)\phi(f_2)\cdots\phi(f_n). \tag{5.7}$$

The simplest state is a degree zero polynomial: the vacuum state. The degree one monomials

$$\phi(f) \tag{5.8}$$

correspond to one-particle states, where f is the quantum wave packet state of the particle. In mathematical terms one-particle states constitute the dual space of the configuration space of classical fields [4]. Indeed, the functional

$$\mathcal{F}_f(\phi) = \phi(f) \tag{5.9}$$

associated to one-particle states is linear on the space of quantum field $\phi \in \mathcal{S}'(\mathbb{R}^3)$. Higher order monomials correspond to linear combinations of quantum states with different number of particles. To pick up only states with a defined number of particles one has to proceed as in the harmonic oscillator case where the eigenstates of the Hamiltonian are given by Hermite polynomials which involve suitable combinations of monomials.

For such a reason to identify physical states with a simple interpretation in terms of particles it is convenient to introduce a coherent state basis. This can be achieved in terms of creation and annihilation operators,

$$a(f) = \phi(\sqrt{-\boldsymbol{\nabla}^2+m^2}f) + i\hat{\pi}(f) \qquad a(f)^\dagger = \phi(\sqrt{-\boldsymbol{\nabla}^2+m^2}\,f) - i\hat{\pi}(f)^\dagger. \tag{5.10}$$

It is easy to show that

$$[a(f),a(g)] \;=\; [a(f)^\dagger,a(g)^\dagger] = 0 \tag{5.11}$$

$$\left[a(f),a(g)^\dagger\right] \;=\; 2(f,\sqrt{-\boldsymbol{\nabla}^2+m^2}\,g) \tag{5.12}$$

Using the basis of plane waves (3.26) we have

$$\hat{H}_{\mathbf{ren}} = \frac{1}{4}\sum_{\mathbf{n}\in\mathbb{Z}^3}\left(a^\dagger_{\mathbf{n}}a_{\mathbf{n}} + a_{\mathbf{n}}a^\dagger_{\mathbf{n}}\right) - E_0 = \frac{1}{2}\sum_{\mathbf{n}\in\mathbb{Z}^3} a^\dagger_{\mathbf{n}}a_{\mathbf{n}} \tag{5.13}$$

where $a(f_{\mathbf{n}}) = a_{\mathbf{n}}$,

$$\hat{\mathbf{P}}_{\mathbf{ren}} = \frac{1}{4}\sum_{\mathbf{n}\in\mathbb{Z}^3}\frac{\mathbf{n}}{\omega_{\mathbf{n}}}\left(a^\dagger_{\mathbf{n}}a_{\mathbf{n}} + a_{\mathbf{n}}a^\dagger_{\mathbf{n}}\right), \tag{5.14}$$

and we can define the number operator as

$$\hat{N} = \frac{1}{2}\sum_{\mathbf{n}\in\mathbb{Z}^3}\frac{1}{\omega_{\mathbf{n}}}a^\dagger_{\mathbf{n}}a_{\mathbf{n}}. \tag{5.15}$$

The main property of the number operator are its commutation relations with the creation and annihilation operators,

$$[\hat{N},a_{\mathbf{n}}] = -a_{\mathbf{n}}, \quad [\hat{N},a^\dagger_{\mathbf{n}}] = a^\dagger_{\mathbf{n}}. \tag{5.16}$$

The creation operators $a(f)$ can generate by iterative actions on the vacuum a basis of physical states. In particular, the state[5]

$$|f\rangle = a^\dagger(f)|0\rangle \tag{5.17}$$

[4] This is explains why in the case of gauge fields, where the configuration space of classical gauge fields modulo gauge transformations is a curved manifold, the particle interpretation of quantum states is so difficult

[5] We use the Dirac notation, where $|0\rangle = \Psi_0$ is the vacuum state, and the ket $|f\rangle$ denotes the state $a(f)^\dagger\Psi_0 = a(f)^\dagger|0\rangle$.

can be considered as an one particle state with wave packet f. Indeed, it easy to check that

$$\hat{N}|f\rangle = |f\rangle \tag{5.18}$$

The normalization of the creation operators simplifies the identification of the norm of one-particle states,

$$\langle f|f\rangle = \parallel f \parallel^2 = \int d^3\mathbf{x}\, |f|^2. \tag{5.19}$$

The completion of the space of one-particle states with this norm is then

$$\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}) = \{f, \parallel f \parallel^2 < \infty\}, \tag{5.20}$$

which again can be identified with the dual space of the configuration space of classical gauge fields.

The next step are the two particle states. They are of the form

$$|f_1, f_2\rangle = \frac{1}{\sqrt{2}} a^\dagger(f_1) a^\dagger(f_2)|0\rangle = \frac{1}{\sqrt{2}} a^\dagger(f_2)|f_1\rangle, \tag{5.21}$$

and satisfy that

$$\hat{N}|f_1, f_2\rangle = 2|f_1, f_2\rangle. \tag{5.22}$$

The n-particle states can be identified with

$$|f_1, f_2, \cdots, f_n\rangle = \frac{1}{\sqrt{n!}} a^\dagger(f_1) a^\dagger(f_2) \cdots a^\dagger(f_n)|0\rangle, \tag{5.23}$$

because they satisfy that

$$\hat{N}|f_1, f_2, \cdots, f_n\rangle = n|f_1, f_2, \cdots, f_n\rangle. \tag{5.24}$$

Now because of the commutation properties of the bosonic field operators, the space of states with n-particles is not the tensor product $\mathcal{H}^{\otimes n}$ of n Hilbert spaces of one-particle states $\mathcal{H}$. Instead it can be identified with the subspace of symmetric states involving n particles,

$$^s\mathcal{H}^{\otimes \mathbf{n}} \subset \mathcal{H} \otimes \mathcal{H} \otimes \overset{\mathbf{n}}{\cdots} \otimes \mathcal{H}, \tag{5.25}$$

of the space of quantum states of n distinguishable particles $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \mathcal{H} \otimes \overset{\mathbf{n}}{\cdots} \otimes \mathcal{H}$. The subspace of symmetric states $^s\mathcal{H}^{\otimes \mathbf{n}}$ is the Hilbert space of n identical bosonic particles.

In this sense the bosonic nature of the commutation relations implies the bosonic statistics of the corresponding particles. To some extent this example illustrates the existence of a link between the spin of the fields and the statistics of the corresponding particles. In general, for any field theory the spin-statistics connection follows from the fundamental principles (spin-statistics theorem).

The Fock space is the Hilbert space of all multiparticle bosonic states,

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} {}^s\mathcal{H}^{\otimes n}. \tag{5.26}$$

It is the Hilbert space of the (bosonic) quantum field theory. In the free theory the Hamiltonian $\hat{H}$ and the number of particles operator $\hat{N}$ commute. Thus, the energy levels have a definite number of particles. However, in the presence of interactions this is not longer true, e.g. for

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$$

we have that

$$[\hat{N}, V] \neq 0$$

which means that the number of particles might change by time evolution. This is one of the novel characteristics of quantum field theory. New process like, decaying of particles, pair creation and photon emission in atoms can occur in the theory. Quantum field theory in a natural way describe such a processes in an accurate manner.

Although we have identified the Fock structure of the space of quantum states with a number of particles stratification, the notion of particle is secondary and it is just the quantum field who is fundamental. The explanation of why two particles are identical come from the fact that they are generated by the same field, which permeates the whole Universe. In this way we understand why the particles coming in cosmic rays from far Galaxies are identical to the same particles on earth. The link is the quantum field. Moreover, there are field theories where the particle composition is unclear. For example in gauge theories, the fundamental fields are quark and gluon fields, but the physical particles are mesons, baryons and glueballs which are bounded composites of quarks and gluons.

5.2 Wick theorem

An important consequence of the Gaussian nature of the ground state of a free field theory is the clustering property of the vacuum expectation values of the product of field operators

$$\langle 0|\phi(f_1)\phi(f_1)\cdots\phi(f_n)|0\rangle = \langle\phi(f_1)\phi(f_1)\cdots\phi(f_n)\rangle = \int \delta\mu\, \phi(f_1)\phi(f_1)\cdots\phi(f_n).$$

The cluster property is a fundamental characteric of Gaussian measures which gives rise to the Wick theorem which states that

$$\langle\phi(f_1)\phi(f_2)\cdots\phi(f_n)\rangle = \begin{cases} 0 & \text{for } n=2m+1 \\ \dfrac{1}{2!m!}\displaystyle\sum_{\sigma\in S_n}\langle\phi(f_{\sigma(1)})\phi(f_{\sigma(2)})\rangle\cdots\langle\phi(f_{\sigma(2m-1)})\phi(f_{\sigma(2m)})\rangle & \text{for } n=2m \end{cases}$$

6 Fields in Interaction

The free theory quantized in the previous section shows the basic properties of a relativistic quantum field theory, but the goal is to quantize theories of interacting fields. The procedure is basically the same, but the main difference is that the interacting Hamiltonian is not exactly solvable. For example, let us consider the $\frac{\lambda}{4!}\phi^4$ theory Hamiltonian

$$\hat{H} = \tfrac{1}{2}\left(\parallel \hat{\pi} \parallel^2 + \parallel \boldsymbol{\nabla}\phi \parallel^2 + m^2 \parallel \phi \parallel^2 + \frac{\lambda}{4!} \parallel \phi^2 \parallel^2\right), \tag{6.1}$$

Using the same quantization rules as in (3.32) we get a formal quantum Hamiltonian $\hat{H}$ which is defined in the space of functionals in the space of classical fields $\mathcal{M}$. However, as we have seen in the case of free fields, the theory need a renormalization.

Using the plane wave basis on a finite torus, we have

$$\hat{H} = \tfrac{1}{2}\sum_{\mathbf{n}\in\mathbb{Z}^3}\left(\frac{\delta}{\delta\phi_{-\mathbf{n}}}\frac{\delta}{\delta\phi_{\mathbf{n}}} + \omega_{\mathbf{n}}^2\,|\phi_{\mathbf{n}}|^2 + \frac{\lambda}{12}\sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^3}\phi_{\mathbf{n}}\phi_{\mathbf{n}_1}\phi_{\mathbf{n}_2}\phi_{-\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2}\right) \tag{6.2}$$

Again the regularization of UV divergences requires the introduction of a regularization, e.g.

$$\hat{H}_\Lambda = \tfrac{1}{2}\sum_{\mathbf{n}\in\mathbb{Z}^3}^{\omega_{\mathbf{n}}<\Lambda}\left(\frac{\delta}{\delta\phi_{-\mathbf{n}}}\frac{\delta}{\delta\phi_{\mathbf{n}}} + \omega_{\mathbf{n}}^2\,|\phi_{\mathbf{n}}|^2 + \frac{\lambda}{12}\sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^3}^{\omega_{\mathbf{n}_1},\omega_{\mathbf{n}_2}<\Lambda}\phi_{\mathbf{n}}\phi_{\mathbf{n}_1}\phi_{\mathbf{n}_2}\phi_{-\mathbf{n}-\mathbf{n}_1-\mathbf{n}_2}\right). \tag{6.3}$$

The Hamiltonian (6.3) can be split into two terms

$$\hat{H}_\Lambda = H_0 + \hat{H}_\Lambda^{\rm int}. \tag{6.4}$$

The first term

$$\hat{H}_0 = \tfrac{1}{2} \sum_{\mathbf{n}\in\mathbb{Z}^3}^{\omega_{\mathbf{n}}<\Lambda} \left(\frac{\delta}{\delta\phi_{-\mathbf{n}}} \frac{\delta}{\delta\phi_{\mathbf{n}}} + \omega_{\mathbf{n}}^2 \, |\phi_{\mathbf{n}}|^2 \right) \tag{6.5}$$

is just the Hamiltonian of the free bosonic theory, whereas the second term

$$\hat{H}_\Lambda^{\rm int} = \frac{\lambda}{4!} \sum_{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\in\mathbb{Z}^3}^{\omega_{\mathbf{n}_1},\omega_{\mathbf{n}_2},\omega_{\mathbf{n}_3}<\Lambda} \phi_{\mathbf{n}_1}\phi_{\mathbf{n}_2}\phi_{\mathbf{n}_3}\phi_{-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3}. \tag{6.6}$$

contains the interaction terms. The renormalization of H_0 can be performed as in previous section by substracting the vacuum energy of the free theory,

$$\hat{H}_\Lambda^{\rm ren} = H_0^{\rm ren} + \hat{H}_\Lambda^{\rm int}. \tag{6.7}$$

But there are new divergences generated by the interacting terms which require an extra renormalization.

6.1 Renormalization of excited states

The easiest way of dealing with the interacting theory is to consider the interacting term $\hat{H}_\Lambda^{\rm int}$ as a perturbation. In first order of perturbation theory the vacuum energy gets an additional contribution

$$\Delta E_0 = \langle 0|\hat{H}_{\rm int}|0\rangle, \tag{6.8}$$

which by Wick's theorem

$$\Delta E_0 = \frac{\lambda}{8} \sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^3}^{\omega_{\mathbf{n}_1},\omega_{\mathbf{n}_2}<\Lambda} \langle 0|\phi_{-\mathbf{n}_1}\phi_{\mathbf{n}_1}|0\rangle\langle 0|\phi_{-\mathbf{n}_2}\phi_{\mathbf{n}_2}|0\rangle, \tag{6.9}$$

gives an extra divergent contribution to the vacuum energy,

$$\Delta E_0 = \tfrac{\lambda}{128\pi^4} \left(\Lambda^4 - 2\Lambda^2 m^2 (\log\tfrac{m^2}{2\Lambda^2} - \tfrac{1}{2}) + m^4 (\log\tfrac{m^2}{2\Lambda^2} - \tfrac{1}{2})^2 \right) + \mathcal{O}\left(\tfrac{m^2}{\Lambda^2}\right), \tag{6.10}$$

which has to be substracted to renormalize the vacuum energy to zero.

The vacuum energy is not the only divergent quantity of the theory. The energy of one-particle states gets a perturbative correction which is also UV divergent. The energy of the state $|f_{\mathbf{n}}\rangle = a_{\mathbf{n}}|0\rangle$ in the free theory is $\omega_{\mathbf{n}}$. The first order correction to the excited state energy is

$$\Delta E_{\mathbf{n}} = \langle f_{\mathbf{n}}|\hat{H}_{\rm int}|f_{\mathbf{n}}\rangle. \tag{6.11}$$

Using Wick's theorem, a simple calculation shows that

$$\Delta E_{\mathbf{n}} = \Delta E_0 + \frac{\lambda}{8} \sum_{\mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^3}^{\omega_{\mathbf{n}_1},\omega_{\mathbf{n}_2}<\Lambda} \langle f_{\mathbf{n}}|\phi_{-\mathbf{n}_1}\phi_{\mathbf{n}_1}|f_{\mathbf{n}}\rangle\langle 0|\phi_{-\mathbf{n}_2}\phi_{\mathbf{n}_2}|0\rangle. \tag{6.12}$$

Both terms are divergent. The first term corresponds the vacuum energy correction, which is removed by the previous renormalization of vacuum energy (6.10). The second term is a new type of UV quadratic divergence. In the sharp momentum cutoff it is given by

$$\frac{\lambda}{32\pi^2\omega_{\mathbf{n}}} \left(\Lambda^2 - m^2 \left(\log\frac{m^2}{2\Lambda^2} - \frac{1}{2} \right) \right). \tag{6.13}$$

The renormalization of the divergence can be absorbed by a renormalization of the mass of the theory. Indeed if we redefine the Hamiltonian of the theory as

$$\hat{H}_{\rm int}^{\rm ren} = \frac{1}{2}\Delta m^2 \sum_{\mathbf{n}\in\mathbb{Z}^3} |\phi_{\mathbf{n}}|^2 + \frac{\lambda}{4!} \sum_{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\in\mathbb{Z}^3}^{\omega_{\mathbf{n}_1},\omega_{\mathbf{n}_2},\omega_{\mathbf{n}_3}<\Lambda} \phi_{\mathbf{n}_1}\phi_{\mathbf{n}_2}\phi_{\mathbf{n}_3}\phi_{-\mathbf{n}_1-\mathbf{n}_2-\mathbf{n}_3}, \tag{6.14}$$

where

$$\Delta m^2 = -\frac{\lambda}{32\pi^2}\left(\Lambda^2 - m^2\left(\log\frac{m^2}{2\Lambda^2} - \frac{1}{2}\right)\right), \tag{6.15}$$

the first order correction to the energy of all one-particle levels is finite. With the above prescription there is no correction to the free value $\omega_{\mathbf{n}}$, but we could have renormalized the mass of the theory by a different substraction: $m^2 \to m^2 - \Delta m^2 + a^2$. In that case the *renormalized* value of the energy one-particle states will be after resummation of the perturbative series,

$$\omega_n^r = \sqrt{\mathbf{n}^2 + m^2 + a^2}.$$

With the above renormalizations of the vacuum energy and mass the theory is finite at first order of perturbation theory. This means that the corrections to the higher energy levels are finite at first order in λ.

We can understand now the physical meaning of the renormalization program. The physical parameters which appear in the classical Lagrangian do not necessary coincide with the corresponding quantum physical parameters. This includes the constant term which can always be added to the classical Lagrangian without changing the dynamics, but determines the vacuum energy of the quantum theory, the mass of the theory m and the coupling constant λ.

Until now we have only renormalized the mass and the vacuum energy. However, in higher orders of perturbation theory new UV divergences appear. They can be absorbed by new renormalizations of the vacuum energy E_0, the mass m^2, the coupling constant λ and the field operators $\hat{\phi}(f)$.

However, the proof of consistency of the resulting theory is quite involved and required few decades to be completely achieved. One of the main problems is that in the canonical approach the preservation of the relativistic invariance is not guaranteed. Among other things the use of UV cutoff break Lorentz invariance and one has to prove that the renormalization prescriptions do preserve the relativistic symmetries. In general, it is not obvious that the interacting theory satisfies the general quantum field principles of section 3.

For such a reasons it is convenient to develop a new approach to quantization based on a covariant formalism, where time and space are treated on the same footing.

7 Covariant approach

If we consider the Heisenberg representation of quantum operators the field operator evolves according to the Heisenberg law

$$\phi(\mathbf{x},t) = U(t)\phi(\mathbf{x})U(t)^\dagger, \tag{7.1}$$

and if we consider test functions $\tilde{f} \in \mathcal{S}(\mathbb{R}^4)$ the smeared operators

$$\phi(\tilde{f}) = \int_{\mathbb{R}^4} d^4x \phi(\mathbf{x},t)\tilde{f}(x,t) \tag{7.2}$$

satisfy that

$$U(t)\phi(\tilde{f})U(t)^\dagger = \phi(\tilde{f}_{(\mathbb{I},t)}), \tag{7.3}$$

where

$$\tilde{f}_{(\mathbb{I},a)}(x,t) = f(x, t-a). \tag{7.4}$$

In terms of the new covariant field operators the principles of quantum field theory are similar to the ones introduced in section 3. The only changes affect to the last three principles which read:

- 5. *Field Theory (real boson) :* For any $f \in \mathcal{S}(\mathbb{R}^3)$ there is field operator $\phi(\tilde{f})$ in $\mathcal{H}$ which satisfies that $\phi(\tilde{f}) = \phi(\tilde{f})^*$. The subspace spanned by the vectors $\phi(f_1)\phi(f_2)\cdots\phi(f_n)|0\rangle$ for arbitrary test functions $f_1, f_2, \cdots f_n \in \mathcal{S}(\mathbb{R}^4)$ is a dense subspace of $\mathcal{H}$.

- 6. *Poincaré covariance:* For any Poincaré transformation (Λ, a) and classical field test function defined in Minkowski space-time $\tilde{f} \in \mathcal{S}(\mathbb{R}^4)$

$$U(\Lambda, a)\phi(\tilde{f})U(\Lambda, a)^\dagger = \phi(\tilde{f}_{(\Lambda,a)}), \tag{7.5}$$

  where

$$\tilde{f}_{(\Lambda,a)}(x) = \tilde{f}(\Lambda^{-1}(x-a)). \tag{7.6}$$

- 7. *(Bosonic) Local Causality:* For any $\tilde{f}, \tilde{g} \in \mathcal{S}(\mathbb{R}^4)$ whose domains are space-like separated[6] the corresponding field operators $\phi(\tilde{f}), \phi(\tilde{f})$ commute[7]

$$[\phi(f), \phi(g)] = 0. \tag{7.7}$$

It is not difficult to show that these principles are satisfied by the free field theory. The only non-trivial test is the calculation of the commutator of free fields $[\phi(f), \phi(g)]$. After some simple algebra it can be shown that it is an operator proportional to the identity operator times a real function of f and g which can be estimated by the vacuum expectation value of the operator,

$$[\phi(f), \phi(g)] = \mathbb{I} \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y\, f(x)\, \Delta(x-y)\, g(y) = \mathbb{I}\, \langle 0|[\phi(f), \phi(g)]|0\rangle \tag{7.8}$$

where

$$\Delta(x-y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{k}^2 + m^2}} \left(e^{ik\cdot(x-y)} - e^{-ik\cdot(x-y)} \right), \tag{7.9}$$

and $k \cdot (x-y) = \mathbf{k} \cdot (\mathbf{x} - \mathbf{y}) - \sqrt{\mathbf{k} + m^2}(x_0 - y_0)$. The local causality property (7.7) follows from the fact that the causal propagator kernel $\Delta(x-y)$ vanish for equal times $x_0 = y_0$, since the two terms in

$$\Delta(\mathbf{x} - \mathbf{y}, 0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{k}^2 + m^2}} \left(e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \right) = 0, \tag{7.10}$$

give the same contributions by flipping the sign of k in one of them. Although the expression of causal propagator kernel $\Delta(x-y)$ is relativistic invariant, it seems to be non-covariant. However, it can be written in a covariant form

$$\Delta(x-y) = D(x-y) - D(y-x), \tag{7.11}$$

where

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \theta(k_0)\delta(k^2 + m^2) e^{ik\cdot(x-y)}. \tag{7.12}$$

This result shows that the covariant approach to quantization could also be derived from the Peierls covariant classical approach to field theory, replacing Peierls brackets by commutators.

To check that fundamental principles are satisfied in an interacting theory is more difficult, but it can be seen that in perturbation theory it is satisfied even after renormalization.

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[6] $\tilde{f}, \tilde{g}$ are space-like separated if for any $x, y \in \mathbb{R}^4$ such that $\tilde{f}(x) \neq 0$ and $\tilde{g}(y) \neq 0$, $d(x, y) < 0$.

[7] In the fermionic case the commutator $[\cdot, \cdot]$ is replaced by an anticommutator $\{\cdot, \cdot\}$

7.1 Euclidean approach

Working with field operators in the Fock space is hard because they are unbounded operators. For such a reason it is more convenient to consider its expectation values on the different states. Since the full Fock space is generated by the completeness principle by the field operators, it is enough to consider the expectation values of the products of field operators on the vacuum state.

These expectation values are known as Wightman functions

$$W(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \langle 0|\phi(\tilde{f}_1)\phi(\tilde{f}_2)\ldots\phi(\tilde{f}_n)|0\rangle. \tag{7.13}$$

However the unbounded character of the field operators $\phi(\tilde{f})$ reflects in the oscillating behavior of the Wightman functions. For such a reason is much more convenient to introduce the Euclidean time analytic extensions of the quantum fields and the corresponding Wightman functions which after analytic continuation become Schwinger functions. Indeed if the consider a Euclidean time $\tau = it$ the τ-evolution of the field operators becomes

$$\phi_E(\mathbf{x}, \tau) = e^{\tau H}\phi(\mathbf{x}, 0)e^{-\tau H}. \tag{7.14}$$

The smearing by $\tilde{f}$ of $\phi_E(\mathbf{x}, t)$ by a test function defines the Euclidean field operators

$$\phi_E(\tilde{f}) = \int_{\mathbb{R}^4} d^4x\, \phi_E(\mathbf{x}, t)\tilde{f}(\mathbf{x}, t). \tag{7.15}$$

Now the vacuum expectation values of products of field operators $\phi_E(\tilde{f})$ is not always well defined because the Euclidean time evolution is given by hermitian operators $U_E(\tau) = U(it)$ which define a semigroup instead of a group unlike the case of real time evolution. The hermitian operators $U_E(\tau)$ are only bounded for positive values of the Euclidean time $\tau < 0$. For such a reason the vacuum expectation values of products of field operators $\phi_E(\tilde{f})$ require some time-ordering of the domains of the test functions. If the supports of the family of functions $\tilde{f}_i \in \mathcal{S}^{\mathbb{R}^4}$ are ordered, i.e. for any x_i where $f(x_i) \neq 0$, $\tau_1 > \tau_2 > \cdots > \tau_n$, then

$$S_n(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \langle 0|\phi_E(\tilde{f}_1)\phi_E(\tilde{f}_2)\ldots\phi_E(\tilde{f}_n)|0\rangle \tag{7.16}$$

is a well defined function and does coincide in that case with the analytic extension of the corresponding Minkowskian vacuum expectation values.

Moreover, it can be extended for multivariable test functions $\tilde{\mathbf{f}}_\mathbf{n} \in \mathcal{S}(\mathbb{R}^{4n})$ defined in $\mathbb{R}^{4n}$ by

$$S_n(\tilde{\mathbf{f}}_\mathbf{n}) = \int_{\mathbb{R}^4} d^4x_1 \int_{\mathbb{R}^4} d^4x_2 \ldots \int_{\mathbb{R}^4} d^4x_n\, \phi_E(x_1)\phi_E(x_2)\ldots\phi_E(x_n)\tilde{f}_n(x_1, x_2, \ldots, x_n), \tag{7.17}$$

when, $\tilde{\mathbf{f}}_\mathbf{n}$ has support in a time ordered subset of $\mathbb{R}^{4n}$, i.e. $\tilde{f}_n(x_1, x_2, \ldots, x_n) = 0$ if $x \in \mathbb{R}^{4n}$ does not satisfies one of the inequalities $\tau_1 > \tau_2 > \cdots > \tau_n$. In the particular case of $\tilde{\mathbf{f}}_\mathbf{n} = \tilde{f}_1\tilde{f}_2\ldots\tilde{f}_n$ the expectation value (7.17) reduces to (7.16). But $S_n(\tilde{\mathbf{f}}_\mathbf{n})$ can be extended for multivariables functions with more general support by analytic extension from the Mikowskian definition.

The interesting thing is that this analytic extension also provides a finite value for the case where the supports are not time-ordered. These analytically extended functions, known as Schwinger functions, although can only be expressed as vacuum expectation values of products of Euclidean fields (7.16) when the supports of the the test functions are time-ordered, in practice, because of their symmetry under permutations, can only be calculated in that way.

The quantum field theory can be completely formulated in terms of Schwinger functions and the fundamental principles reformulated in the following way.

Let θ be the Euclidean-time reflection symmetry defined by $\theta(\mathbf{x}, \tau) = \theta(\mathbf{x}, -\tau)$. The action of θ on $\mathbb{R}^4$ induces a transformation on the classical fields test functions defined by $\theta\tilde{f}(x) = \tilde{f}(\theta x)$ and in the multivariable test functions $\tilde{\mathbf{f}}_\mathbf{n} \in \mathcal{S}$ in a similar way $\theta\tilde{\mathbf{f}}_\mathbf{n}(x_1, x_2, \ldots, x_n) = \tilde{\mathbf{f}}_\mathbf{n}(\theta x_1, \theta x_2, \ldots, \theta x_n)$

- 1. Regularity. The Schwinger functions S_n are tempered distributions in $\mathcal{S}$, satisfying the reflection reality condition

$$S_n(\tilde{\mathbf{f}}_{\mathbf{n}})^* = S_n(\theta\tilde{\mathbf{f}}^*_{\mathbf{n}}) \tag{7.18}$$

- 2. Permutation Symmetry. The Schwinger functions are symmetric under permutations, i.e.

$$S_n(\tilde{f}_{\sigma(1)}, \tilde{f}_{\sigma(2)}, \ldots, \tilde{f}_{\sigma(n)}) = S_n(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) \tag{7.19}$$

  for any permutation $\sigma \in \mathcal{S}_n$.

- 3. Euclidean invariance. The Schwinger functions are covariant under Euclidean transformations, i.e.

$$S_n(\tilde{\mathbf{f}}_{(\mathbf{\Lambda},\mathbf{a}),\mathbf{n}}) = S_n(\tilde{\mathbf{f}}_{\mathbf{n}}) \tag{7.20}$$

  for any Euclidean transformation $(\Lambda, a) \in E_4 = ISO(4)$.

- 4. Reflection positivity. For any family of multivariable functions test functions $\tilde{\mathbf{f}}_{n_i} \in S(R_+^{4n_i}), i = 0, 1, 2, \ldots, n$ the following inequality [8]

$$\sum_{i,j=0}^{n} S_{n_i+n_j}(\theta\tilde{\mathbf{f}}^*_{\mathbf{n_i}}.\tilde{\mathbf{f}}_{\mathbf{n_j}}) \geq 0 \tag{7.21}$$

  holds.

- 5. Cluster property. For any pair of multivariable functions test functions $\tilde{\mathbf{f}}_n \in S(R^{4n}), \tilde{\mathbf{f}}_m \in S(R^{4m})$, we have that

$$\lim_{\sigma\to\infty} S_{n+m}(\tilde{\mathbf{f}}_{\mathbf{n}}.\tilde{\mathbf{f}}_{(\mathbb{I},\sigma),\mathbf{m}}) = S_n(\tilde{\mathbf{f}}_{\mathbf{n}})S_m(\tilde{\mathbf{f}}_{\mathbf{m}}), \tag{7.22}$$

  where $(\mathbb{I}, \tau)$ is the Euclidean time translation $(\mathbb{I}, \sigma)(\mathbf{x}, \tau) = (\mathbf{x}, \tau + \sigma)$.

  These Euclidean principles follow from the field theory principles introduced in Section 3. The Euclidean principles 1-3 are a straightforward consequence of the Minkowskian principles. The third Euclidean principle follows from the positivity of the norm of the state

$$\sum_{i=0}^{n} \phi(\tilde{\mathbf{f}}_{n_i})|0\rangle \tag{7.23}$$

  Finally the cluster property is a consequence of the uniqueness of the vacuum assumed in the fourth Minkowskian principle.

  What is not so evident is to show that from the Euclidean principles one can reconstruct a quantum field theory satisfying the Minkowskian principles. The proof was achieved by Osterwalder and Schrader in the early seventies. We will not elaborate in the proof that can be seen in the books by B. Simon and J. Glimm and A. Jaffe included in the bibliography.

7.2 Functional integral approach

The major advantage of the Euclidean approach is that the Schwinger functions are better behaved that the corresponding Wightman distributions and what is more important they can be derived in most of the cases from functional integration with respect to a probability measure. This also allows by introducing a suitable regularization a systematic numerical approach.

[8] $R^4_+ = \{(x, \tau) \in \mathbb{R}^4, \tau \geq 0\}$

The result which was first suggested by K. Symanzik and E. Nelson is that formally speaking the Schwinger functions can be considered as the momentum operators of a functional measure defined in the space of distributions $\mathcal{S}(R^4)$ defined by the exponential of the Euclidean classical action S_E, i.e.

$$S_n(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \int_{\mathcal{S}'(R^4)} \delta\phi \; e^{-S_E(\phi)} \, \phi(\tilde{f}_1)\phi(\tilde{f}_2)\ldots\phi(\tilde{f}_n) = \tag{7.24}$$

In the case of free field theory

$$S_E(\phi) = \frac{1}{2} \parallel \nabla\phi \parallel^2 + \frac{m^2}{2} \parallel \phi \parallel^2 = \frac{1}{2}(\phi, (-\nabla^2 + m^2)\phi) \tag{7.25}$$

and we have that

$$S_2(\tilde{f}, \tilde{g}) = \int_{\mathcal{S}'(R^4)} \delta\phi \; e^{-S_E(\phi)} \phi(\tilde{f})\phi(\tilde{g}) = \frac{1}{2}(\tilde{f}, (-\nabla^2 + m^2)^{-1}\tilde{g}) = \int_{\mathcal{S}'(R^4)} \delta\mu_m \phi(\tilde{f})\phi(\tilde{g}) \tag{7.26}$$

where $\delta\mu_m$ is the Gaussian measure defined on $\mathcal{S}'(R^4)$ with vanishing mean and covariance the operator $(-\nabla^2 + m^2)^{-1}$.

It is obvious that this Schwinger function does coincide with the analytic extension of the Wightman function of the free theory. In fact, from the functional integral formulation it is easy to check that the Schwinger functions satisfy the regularity, symmetry and Euclidean covariance principles. Concerning the reflection positivity property it is not so evident. Let us check that this is the case to illustrate the subtleties of the very special Osterwalder-Schrader's property. Let us consider the case of a one-particle states with a complex function $\tilde{f} \in \mathcal{S}(R^4_+)$

$$S_2(\theta\tilde{f}^*, \tilde{f}) = \int_{\mathcal{S}'(R^4)} \delta\phi \, e^{-S_E(\phi)} \theta\phi(\tilde{f})^*\phi(\tilde{f}) = \frac{1}{2}(\theta\tilde{f}, (-\nabla^2 + m^2)^{-1}\tilde{f}). \tag{7.27}$$

Let us define $\varphi = (-\nabla^2 + m^2)^{-1}\tilde{f}$. Since θ commutes with $(-\nabla^2 + m^2)$ we have

$$(\theta\tilde{f}, (-\nabla^2 + m^2)^{-1}\tilde{f}) = (\theta\tilde{f}, \varphi) = ((-\nabla^2 + m^2)\theta\varphi, \varphi). \tag{7.28}$$

and since the support of $\tilde{f}$ is contained in $\mathcal{S}(\mathbb{R}^4_+)$, that of $\theta\tilde{f}$ is in $\mathcal{S}(\mathbb{R}^4_-)$, thus, we can restrict the integral in (7.28) to $\mathcal{S}(\mathbb{R}^4_+)$,

$$(\theta\tilde{f}, \varphi) = (\theta\tilde{f}, \varphi)_- = ((-\nabla^2 + m^2)\theta\varphi, \varphi)_- \tag{7.29}$$

By the same reason, $(\theta\varphi, (-\nabla^2 + m^2)\varphi)_- = (\theta\varphi, \tilde{f})_- = 0$ and

$$(\theta\tilde{f}, \varphi) = ((-\nabla^2 + m^2)\theta\varphi, \varphi)_- - (\theta\varphi, (-\nabla^2 + m^2)\varphi)_-. \tag{7.30}$$

Integrating by parts one gets

$$(\theta\tilde{f}, \varphi) = -\int_{\mathbb{R}^3} d^3\mathbf{x} \, \partial_n\theta\varphi^*\varphi + \int_{\mathbb{R}^3} d^3\mathbf{x} \, \theta\varphi^*\partial_n\varphi \tag{7.31}$$

where $\partial_n\varphi$ denotes the normal derivative of φ at the boundary $\partial\mathbb{R}_+ = \mathbb{R}^3$ at Euclidean time $\tau = 0$ of $\mathbb{R}^4_+$. Now, at the boundary $\tau = 0$, $\theta\varphi = \varphi$ and $\partial_n\theta\varphi = -\partial_n\varphi$, thus,

$$(\theta\tilde{f}, \varphi) = 2\text{Re} \int_{\mathbb{R}^3} d^3\mathbf{x} \, \theta\varphi^*\partial_n\varphi. \tag{7.32}$$

Finally, by integrating by parts back we get

$$\text{Re} \int_{\mathbb{R}^3} d^3\mathbf{x} \, \theta\varphi^*\partial_n\varphi = (\nabla\varphi, \nabla\varphi)_- - (\varphi, -\nabla^2\varphi)_-. \tag{7.33}$$

and since $-\nabla^2\varphi = -\nabla^2(-\nabla^2 + m^2)^{-1}\tilde{f} = \tilde{f} - m^2\varphi$,

$$(\theta\tilde{f}, \varphi) = 2 \parallel \nabla\varphi \parallel_-^2 + 2m^2 \parallel \varphi \parallel_-^2 \geq 0. \tag{7.34}$$

The proof of reflection positivity for higher order Schwinger functions follows from Wick theorem in a similar way.

The cluster property of the two-point Schwinger formula follows from the fact that the kernel $(-\nabla^2 + m^2)^{-1}(x, y)$ vanish in the limit $\parallel x - y \parallel \to \infty$.

In this formalism the functional integral of the interacting theory can be understood as a Riemann-Stieltjes measure with respect to the Gaussian measure of the free theory $\delta\mu_m$, i.e.

$$S_n(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \int_{\mathcal{S}'(R^4)} \delta\mu_m(\phi)\ e^{-V(\phi)}\, \phi(\tilde{f}_1)\phi(\tilde{f}_2), \ldots, \phi(\tilde{f}_n). \tag{7.35}$$

Perturbation theory is defined just by the Taylor expansion of $e^{-V(\phi)}$ in power series and the formal commutation of the Gaussian integration with the Taylor sum. In the $\lambda\phi^4$ case the perturbation theory is defined by

$$S_n(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\lambda^n}{4!^n} \int_{\mathcal{S}'(R^4)} \delta\mu_m(\phi)\ \parallel \phi^2 \parallel^{2n}\ \phi(\tilde{f}_1)\phi(\tilde{f}_2), \ldots, \phi(\tilde{f}_n). \tag{7.36}$$

In this formalism the UV divergent appear when computing the different terms of (7.36) by using Wick's theorem. But the advantage of the covariant formalism is that there the preservation of Poincaré symmetries under renormalization is more transparent.

8 What is beyond

From the Euclidean formulation it follows that the functional integral approach is a constructive way of quantizing a field theory. The perturbative expansion (7.36) provides a very explicit way of computing Schwinger functions. The ultraviolet divergences that arise there, can be renormalized by absorbing the divergences in the bare parameters of the theory in some cases.

From this point of view the quantum field theories are classified in two classes: theories where the set of parameters of the Lagrangian are enough to absorb all UV divergences and theories where they are not enough. Theories of the first family are called renormalizable whereas those of the second class are unrenormalizable. Of course, only theories of the first type are sensible since with a finite number of parameters they can predict the behavior of all quantum states.

To distinguish between both cases one has to work out the perturbation theory and find a good prescription scheme to renormalize all UV divergences. The best renormalization scheme is the BPHZ scheme, developed by Bogolibov, Parasiuk, Hepp and Zimmerman to provide a rigorous proof to the perturbative renormalization program.

In a heuristic way, one can distinguish the renormalizable theory just by a power counting algorithm. This consist in assigning a physical dimension to the fields under scale transformations in a way that leave the kinetic term of the action scale invariant (dimensionless). In the scalar theory this means that the scalar field ϕ has dimension $d_\phi = 1$, the same as the space-time derivatives ∂_x operator. In that way the kinetic term of the action

$$\frac{1}{2} \parallel \nabla\phi \parallel^2$$

become dimensionless. The theory is renormalizable by power counting if all the terms of the action have non-positive dimensions. This constraint only allows Poincaré invariant terms like

$$\frac{m^2}{2} \parallel \phi \parallel^2,$$

which has dimension $d = -2$,

$$\frac{\sigma}{3!}\int_{\mathbb{R}^4} d^4x\ \phi(x)^3,$$

which has dimension $d = -2$, or

$$\frac{\lambda}{4!}\int_{\mathbb{R}^4} d^4x\ \phi(x)^4$$

which is dimensionless. No other selfinteracting terms give rise to a renormalizable theory. This limitation became very important in model building because it introduces very stringent limitations. The remarkable thing is that Nature has chosen renormalizable models to build the theory of fundamental interactions.

The only fundamental theory which does not satisfy the renormalizability criterium is Einstein theory of Gravitation. In that theory due to diffeomorphism invariance the Einstein term contains and infinity of terms with positive dimensions.

One of the advantages of the covariant approach is that does not requires the existence of a classical Lagrangian. It is enough to have a complete set of Schwinger functions satisfying the fundamental properties 1-5 of a QFT. This opens the possibility of quantum systems which are not defined by quantization of a classical system. There are few examples of that. But also it opens the possibility of having different field theories with the same Schwinger functions. In that case they are quantum-mechanically equivalent although their classical theories are completely different.

In two space-time dimensions, theories which in addition to the fundamental properties 1-5 are conformally invariant, have been analysed and classified without any reference to the corresponding classical systems. In three dimension there has been a recent breakthrough which open the possibility of having similar results. However, in four space-time dimensions the problem is far from a solution.

In the early seventies WIlson developed an interpretation of the renormalization procedure as a non-linear representation of the one-dimensional group of dilations. In the Euclidean formalism using a space-time lattice regularization Wilson mapped the quantum field theory system into a statistical mechanical one. Then, using the properties of second order phase transitions he interpreted the renormalization of a field theory as a limit process near a critical point of the renormalization group associated to the second order phase transition. The Wilson method provided a new non-perturbative approach to quantum field theory which allows a numerical treatment and has been intensively used in quantum chromodynamics.

However, with the Wilson's approach it was also born the possibility to considering QFT as the ultimate theory of Nature. It can be considered as just a successful approximation to the intimate structure of Nature. This approach, known as effective field theory, considers that the range of validity of the quantum field theory has an energy upper bound beyond which the theory does not hold. The limit scale is sometimes associated to the Planck energy scale, but for some theories might be smaller.

In the last three decades to solve the problem of quantizing gravity there have been many attempts which searched for theories with Beyond field theory. From one way or another all these attempts consider the possibility of non-local interactions. The most popular approach the superstring theory. The connection of all the non-local approach with fundamental aspects of Nature has not yet been confirmed by experiments.

A Casimir Effect

In the domain Ω_{II} between two parallel plates the normal modes of the field ϕ become discrete under Dirichlet boundary conditions

$$k_3 = \frac{n\pi}{d} \qquad n \in \mathbb{Z}$$

whereas k_1 and k_2 remain continuous in $\mathbb{R}^2$. For the other two domains Ω_I, Ω_{III} the Fourier modes of the classical field ϕ are continuous, i.e. $\mathbf{k} \in \mathbb{R}^3$

In the heat kernel regularization the vacuum energy density between the plates is

$$\mathcal{E}_{II} = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-\epsilon \mathbf{k}^2}\sqrt{\mathbf{k}^2} \tag{A.1}$$

$$+ \frac{1}{2d}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1 dk_2}{(2\pi)^2} e^{-\epsilon(k_1^2+k_2^2)}\sqrt{k_1^2+k_2^2} \tag{A.2}$$

$$+ \frac{1}{2d}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1 dk_2}{(2\pi)^2} e^{-\epsilon(k_1^2+k_2^2+(\frac{n\pi}{d})^2)}\sqrt{k_1^2+k_2^2+\left(\frac{n\pi}{d}\right)^2}.$$

The first term is the vacuum energy of the free field and gives

$$\mathcal{E}^{(1)} = \frac{1}{8\pi^2\epsilon^2}. \tag{A.3}$$

The second term corresponds to the selfenergy of the plates and gives

$$\mathcal{E}^{(2)} = \frac{1}{16\sqrt{\pi}\epsilon^{\frac{3}{2}}}\frac{1}{d}. \tag{A.4}$$

Finally, to calculate the third term we define the function

$$f(z) = \frac{1}{2\pi}\int_0^{\infty} k\, dk\, e^{-\epsilon\sqrt{k^2+(\frac{z\pi}{d})^2}}\sqrt{k^2+\left(\frac{z\pi}{d}\right)^2} = \frac{1}{4\pi}\int_{(\frac{z\pi}{d})^2}^{\infty} d\kappa\, e^{-\epsilon\sqrt{\kappa}}\sqrt{\kappa}. \tag{A.5}$$

The contribution of the third terms to the vacuum energy can be written as

$$\mathcal{E}_{II}^{(3)} = \frac{1}{d}\left[\frac{1}{2}f(0) + \sum_{n=1}^{\infty} f(n) - \int_0^{\infty} dz\, f(z)\right] \tag{A.6}$$

Using the Euler-MacLaurin formula

$$\sum_{n=1}^{\infty} f(n) - \int_0^{\infty} dz\, f(z) = -\frac{1}{2}\left[f(0)+f(\infty)\right] + \frac{1}{12}\left[f'(\infty)-f'(0)\right]$$

$$-\frac{1}{720}\left[f'''(\infty)-f'''(0)\right] + \ldots \tag{A.7}$$

Now the function (A.5) satisfies that $f(\infty) = f'(\infty) = f'''(\infty) = 0$ and $f'(0) = 0$, the value of $\mathcal{E}_{II}$ is determined by $f'''(0)$. Computing this term and removing the ultraviolet cutoff, $\epsilon \to 0$ we find the result

$$\mathcal{E}^{(3)} = \frac{1}{1440}f'''(0) = -\frac{\pi^2 S}{1440 d^4}. \tag{A.8}$$

B Gaussian Measures

Let us consider in $\mathbb{R}$ the following Gaussian measure

$$d\mu_c = \frac{e^{-\frac{x^2}{2c}}}{\sqrt{2\pi c}}dx. \tag{B.1}$$

The integral of any $L^1(\mathbb{R})$ function is defined by

$$\langle f\rangle = \int_{\mathbb{R}} d\mu_c(x)\, f(x) \tag{B.2}$$

gets its main contribution from the interval $(-c, c)$. The main properties of the Gaussian measure are given by the average of its momenta,

- $\langle 1\rangle_c = 1$
- $\langle x^{2m+1}\rangle_c = 0$
- $\langle x^{2m}\rangle_c = (2m-1)!!\langle x^2\rangle^m = (2m-1)!!\, c^{2m}$

The last results is known as Wick's theorem.

There is one special function $f(x) = e^{ixy}$ whose average is the generating function $g_c(y)$ of all momenta of the measure

$$g_c(y) = \langle g_c(y)\rangle_c = e^{-\frac{c}{2}y^2}. \tag{B.3}$$

Indeed, all momenta of the measure can be obtained from the derivatives of the generating function at the origin

$$\langle x^m\rangle = (-i)^m \frac{d^m}{dy^m} g_c\Big|_{y=0}. \tag{B.4}$$

The multidimensional generalization is straightforward. Let C be a positive, symmetric matrix, i.e.

$$(x, Cy) = (Cx, y), \qquad (x, Cx) > 0). \tag{B.5}$$

Positivity implies the non-degenerate character of C, $\det C \neq 0$, which guarantees the existence of the inverse matrix C^{-1}.

The Gaussian measure is defined by

$$d\mu_C = \frac{e^{-\frac{1}{2}\,(\mathbf{x}, C^{-1}\mathbf{x})}}{\sqrt{2\pi \det C}} d^n\mathbf{x}. \tag{B.6}$$

The momenta of the multidimensional Gaussian measure are obtained in terms of the covariance matrix C. Indeed

- $\langle 1\rangle_C = 1$
- $\langle x^{i_1}x^{i_2}\dots x^{i_{2m-1}}\rangle_C = 0$
- $\langle x^{i_1}x^{i_2}\dots x^{i_{2m}}\rangle_C = \dfrac{1}{2^m m!}\displaystyle\sum_{\sigma\in S_{2m}} C_{\sigma(i_1)\sigma(i_2)}C_{\sigma(i_3)\sigma(i_4)}\dots C_{\sigma(i_{2m-1}\sigma(i_{2m}))}$

The last formula is Wick's theorem. The derivatives of the generating function i

$$g_C(\mathbf{y}) = \langle e^{i(\mathbf{x},\mathbf{y})}\rangle_C = e^{-\frac{1}{2}(\mathbf{y}, C\mathbf{y})}. \tag{B.7}$$

generate the momenta of the Gaussian measure,

$$\langle x^{i_1}x^{i_2}\dots x^{i_m}\rangle_C = (-i)^m \frac{\partial^m g_C}{\partial_{x^{i_1}}\partial_{x^{i_2}}\dots\partial_{x^{i_m}}}\Big|_{\mathbf{y}=0}. \tag{B.8}$$

- **Gaussian measures in Hilbert Spaces**

The above measures can be generalized for infinite dimensional topological vector spaces. The simplest case is the Hilbert space case. Let us consider a be a positive, symmetric, trace class operator C in a Hilbert space $\mathcal{H}$:

- $(x, Cy) = (Cx, y)$, for any $x, y \in \mathcal{H}$
- $(x, Cx) > 0$, for any $x \in \mathcal{H}$
- $\mathrm{tr}\, C < \infty$.

Positivity implies the non-degenerate character of C, which guarantees the existence of the inverse matrix C^{-1}. Let us assume for concreteness that $\mathcal{H} = L^2(\mathbb{R}^n, \mathbb{C})$ and

$$C_s = (-\Delta + m^2)^{-s}$$

in $\mathbb{R}^4$ with $s > 2$. It easy to check that C_s is positive, symmetric, trace class operator in $L^2(\mathbb{R}^n, \mathbb{C})$.

The Minlos theorem establishes that the measure defined by C_s is a Borelian probability measure in $\mathcal{H} = L^2(\mathbb{R}^n, \mathbb{C})$. The momenta of this Gaussian measure are again obtained in terms of the covariance matrix C,

- $\langle 1 \rangle_C = 1$
- $\langle (g, f_1)(g, f_2) \dots (g, f_{2m-1}) \rangle_C = 0$
- $\langle (g, f_1)(g, f_2) \dots (g, f_{2m-1} \rangle_C = \dfrac{1}{2^m m!} \sum_{\sigma \in S_{2m}} (f_{\sigma(1)}, C f_{\sigma(2)})(f_{\sigma(3)}, C f_{\sigma(4)}) \dots (f_{\sigma(2m-1)}, C f_{\sigma(2m)})$.

The last formula is the infinite-dimensional version of Wick's theorem. The generating functional is

$$G_C(f) = \langle e^{i(\mathbf{x}, \mathbf{y})} \rangle_C = \int_{L^2(\mathbb{R}^n, \mathbb{C})} d\mu_C(g) e^{i(f, Cg)} = e^{-\frac{1}{2}(f, Cf)}, \tag{B.9}$$

because its functional derivatives generate the momenta of the measure,

$$\langle (g, f_1)(g, f_2) \dots (g, f_m) \rangle_C = (-i)^m \frac{\partial^m G_C}{\partial_{f_1} \partial_{f_2} \dots \partial_{f_m}} \bigg|_{f=0}. \tag{B.10}$$

However, in field theory the natural covariance are not so regular so one needs to make appeal to another version of Minlos' theorem which applies to covariances which are not of trace class. In this case the space of test functions is *dual* in some sense to the distributions of the space where the measure is supported.

Let $\mathcal{S}(\mathbb{R}^n)$ the space of fast decreasing smooth $C^\infty(\mathbb{R}^n)$ functions and C a positive, symmetric, bounded operator in $\mathcal{S}(\mathbb{R}^n)$. The Minlos theorem states that there is a unique Borelian Gaussian measure with C covariance in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$, which is the dual of $\mathcal{S}(\mathbb{R}^n)$. The same holds for the space of smooth functions of compact support $\mathcal{D}(\mathbb{R}^n)$ and its dual $\mathcal{D}'(\mathbb{R}^n)$, the space of distributions.

In the first case the generating function

$$G_C(f) = \langle e^{i(\mathbf{x}, \mathbf{y})} \rangle_C = \int_{\mathcal{S}'(\mathbb{R}^n, \mathbb{C})} d\mu_C(g) e^{i(f, g)} = e^{-\frac{1}{2}(f, Cf)}, \tag{B.11}$$

is defined only for test functions $f \in \mathcal{S}(\mathbb{R}^n)$, whereas in the second case they are on defined for for functions of compact support $\mathcal{D}(\mathbb{R}^n)$.

There are two special Gaussian measures which arise in quantum field theory. One defined by the covariance operator

$$C_0 = (-\nabla^2 + m^2)^{-1/2}$$

in $\mathbb{R}^3$ that corresponds to the measure defined by the ground state of a free bosonic field theory. The second is defined by the covariance operators

$$C_E = (-\Delta + m^2)^{-1}$$

in $\mathbb{R}^4$, which corresponds to the measure given by the Euclidean functional integral of a free bosonic theory. In physical terms C_E is known as the Euclidean propagator of a scalar field.

C Peierls brackets

There is an alternative canonical approach to classical dynamics developed by R. Peierls which allows a relativistic covariant description of classical field theory.

The standard approach uses the Poisson structure based on equal time commutators (2.6) (3.16). However, in relativistic theories the simultaneity of space separated points is not a relativistic invariant notion. For such a reason R. Peierls introduced an equivalent dynamical approach which explicitly preserves relativistic covariance.

The phase space in classical mechanics T^*M contains all Cauchy data $(x,p) \in T^*M$ and a canonical symplectic structure

$$\omega_0 = \sum_{i=1}^{n} dx^i \wedge dp^i$$

that determine the time evolution of the system for any kind of Hamiltonian function $H(p,q)$ following the Hamilton motion equations

$$\dot{x} = \{x, H\}, \qquad \dot{p} = \{p, H\}, \tag{C.1}$$

where $\{,\cdot,\cdot\}$ is the Poisson bracket defined by the symplectic form ω_0.

Peierls remarks that the phase space can be identified with the trajectories of the system in M induced by any given Lagrangian L_0. In this sense, the Cauchy data are not fixed at a given initial time but by the trajectories themselves, which illustrates why it is the suitable framework for a covariant formulation.

Next, Peierls introduced a Poisson structure in the space of trajectories in the following way. Given two time-dependent functions A and B in $TM \times \mathbb{R}$ we can consider two new dynamical systems with Lagrangians $L_A = L_0 + \lambda A$ and $L_B = L_0 + \lambda B$. The trajectories of the new dynamics differ from those of that governed by L_0. If we compare the deviation of the trajectories with the same asymptotic values at $t = -\infty$ in the limit $\lambda \to 0$ we get two new functions on the space of trajectories

$$\mathrm{D}_A B(t) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left[\int_{-\infty}^{t} ds\, B(x_\lambda(s), s) - \int_{-\infty}^{t} ds\, B(x_0(s), s) \right], \tag{C.2}$$

In a similar way one can define two new functions by comparing the deviation of the trajectories with the same asymptotic values at $t = +\infty$

$$\text{ᗡ}_A B(t) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left[\int_{t}^{\infty} ds\, B(x_\lambda(s), s) - \int_{t}^{\infty} ds\, B(x_0(s), s) \right], \tag{C.3}$$

Then, the Peierls brackets $\{\cdot,\cdot\}_{L_0}$ are given by

$$\{A, B\}_{L_0} = \mathrm{D}_A B - \text{ᗡ}_A B. \tag{C.4}$$

It has been proved by Peierls in 1952 that, when the Lagrangian L_0 is regular it defines a Hamiltonian system, and the above bracket satisfies the following properties

$$\{A, B + C\}_{L_0} = \{A, B\}_{L_0} + \{A, C\}_{L_0} \qquad \text{Distributive}$$
$$\{A, BC\}_{L_0} = \{A, B\}_{L_0} C + B\{A, C\}_{L_0} \qquad \text{Leibnitz rule}$$
$$\{A, B\}_{L_0} = -\{B, A\}_{L_0} \qquad \text{Antisymmetry}$$
$$\{A, \{B, C\}_{L_0}\}_{L_0} = \{\{C, A\}_{L_0}, B\}_{L_0} + \{\{A, B\}_{L_0}, C\}_{L_0} \qquad \text{Jacobi}$$

and, thus, defines a Poisson structure in the space of classical trajectories.

In the case of the one-dimensional harmonic oscillator the Poisson bracket of position operator is

$$\{x(t_1), x(t_2)\} = \frac{1}{\omega}\sin\omega(t_1 - t_2). \tag{C.5}$$

The construction of Peierls brackets depends on the Lagrangian L_0 of the theory. In this sense is not as universal as the Poisson brackets induced by the symplectic structure of T^*M.

The generalization to field theory is straightforward and the result for a free scalar theory. In the case of the scalar field, a general solution of the field equations

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0 \tag{C.6}$$

can be obtained via its Fourier transform

$$(-k^2 + m^2)\widetilde{\phi}(k) = 0, \tag{C.7}$$

whose general solution can be written as $\widetilde{\phi}(p) = 2\pi\hat{\alpha}(k)\delta(k^2 - m^2)$, where $\hat{\alpha}(k)$ a completely general function of k^μ. The solution in position space obtained by inverse Fourier transform is

$$\begin{aligned} \phi(x) &= \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta(k^2 - m^2)\theta(k^0)\left[\alpha(p)e^{-ik\cdot x} + \alpha(k)^* e^{ik\cdot x}\right] \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left[\alpha(\mathbf{k})e^{-i\omega_k t + \mathbf{k}\cdot\mathbf{x}} + \alpha(\mathbf{k})^* e^{i\omega_k t - \mathbf{k}\cdot\mathbf{x}}\right]. \end{aligned} \tag{C.8}$$

The corresponding Peierls brackets are given by

$$\{\phi(x), \phi(y)\} = \Delta(x - y), \tag{C.9}$$

where $\Delta(x - y)$ is the causal propagator (7.9). In the case of equal time the Peierls bracket reduce to the Poisson bracket, and vanishes.

The covariant quantization rule is a generalization of the canonical one, to replace the Peierls brackets by operator commutators,

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