Analytic Left Inversion of Multivariable Lotka–Volterra Models$^{\alpha}$

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Problem

Trajectory Generation Problem: *explicitly compute the input to drive a nonlinear system to produce some desired output.*

- **Fliess operators**: $F_c : u \mapsto y$ are analytic multivariable input-output maps, which are described by coefficients (c, η) and corresponding iterated integrals (M. Fliess, 1983).

- **Left inversion problem**: given a multivariable Fliess operator F_c and a function y in its range, determine an input u such that $y = F_c[u]$.

- **Hopf algebra antipode**: group $(G, \circ)$ of unital Fliess operators and its corresponding Hopf algebra H of coordinate functions; $G \ni F_c^{\circ -1} = F_c \circ S$, $\quad S : H \to H$

$$S \star \mathrm{id} = \mathrm{id} \star S = \epsilon$$

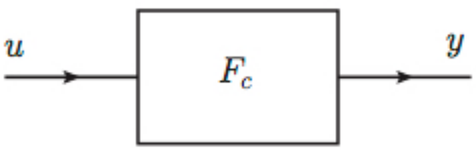
- **Lotka–Volterra Model**: $\dot{z}_i = \beta_i z_i + \sum_{j=1}^{n} \alpha_{ij} z_i z_j, \quad i = 1, 2, \ldots, n$
  Input-Output systems are obtained by introducing time dependence on the parameters $\beta_i(t)$ and $\alpha_{ij}(t)$ **(inputs u_k)**, and assuming that $y = h(z)$ **(outputs $y = F_c[u]$)**.

Setting

Fliess operator

$$y = F_c[u](t,t_0) = \sum_{\eta \in X^*} (c,\eta)\, E_\eta[u](t,t_0)$$

alphabet: $X = \{x_0, x_1, \ldots, x_m\}$



system: $c := \sum_{\eta \in X^*} \underbrace{(c,\eta)}_{\in \mathbb{R}^\ell} \eta \ \in \ \mathbb{R}^\ell\langle\langle X \rangle\rangle$

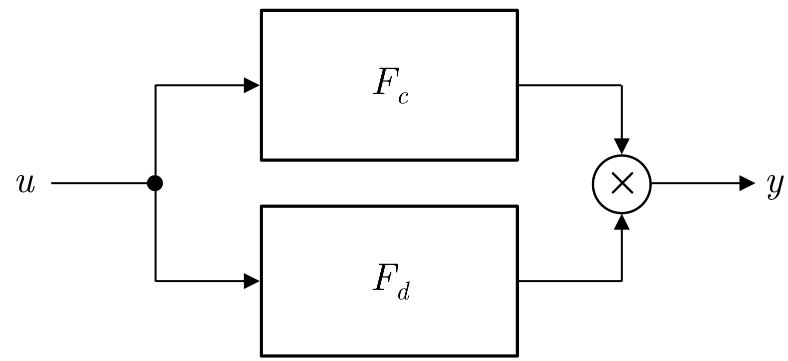
controls: $u : [t_0, t_1] \to \mathbb{R}^m, \quad u_0 := 1$

$x_i \longleftrightarrow u_i$

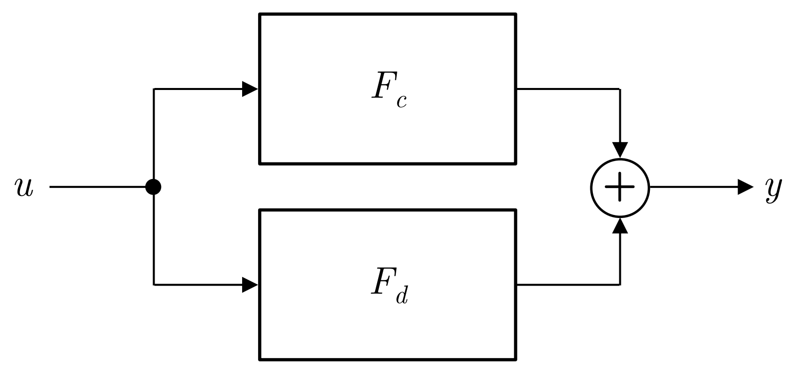
$$E_{x_i\bar{\eta}}[u](t,t_0) = \int_{t_0}^{t} u_i(s) E_{\bar{\eta}}[u](s,t_0)\, ds$$

$$E_\emptyset[u] := 1$$

System interconnections I



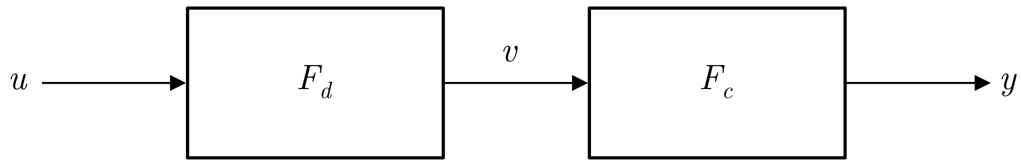
product connection: $F_cF_d = F_{c \,\sqcup\!\sqcup\, d}$



parallel connection: $F_c + F_d = F_{c+d}$

System interconnections II

Cascade connection



$$d := \sum_{\eta \in X^*} (d,\eta)\eta, \quad (d,\eta) \in \mathbb{R}^m,\ d_0 := 1$$

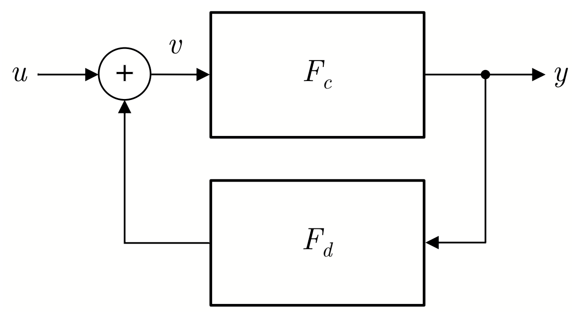
$$(F_c \circ F_d)[u](t,t_0) = \sum_{\eta \in X^*} (c,\eta)\, E_\eta\big[F_d[u]\big](t,t_0)$$

$$E_{x_i\bar{\eta}}[F_d[u]](t,t_0) = \int_{t_0}^{t} F_{d_i}[u](s,t_0) E_{\bar{\eta}}[F_d[u]](s,t_0)\, ds$$

$$(F_c \circ F_d)[u] = F_{c\circ' d}[u] \qquad x_i\eta \circ' d := x_0\big(d_i \sqcup\!\sqcup (\eta \circ' d)\big)$$

System interconnections III

Feedback loop



$$v = u + F_{d\circ' c}[v]$$

$$F_{c\bullet d}[u] = F_{c\circ'(\epsilon - d\circ' c)^{\circ -1}}[u]$$

Involves an extension of Fliess operators: $unital$ Fliess operators

$$F_{c_\epsilon}[u] := u + F_c[u] = (I + F_c)[u]$$

$c_\epsilon := \epsilon + c$

$$F_{c_\epsilon} \circ F_{d_\epsilon}[u] = F_{c_\epsilon \circ d_\epsilon}[u]$$

This composition defines a group $(G, \circ)$ with unit ϵ on $\mathbb{R}\langle\langle X_\epsilon \rangle\rangle$ [G-DE].

Coordinate functions I

Faà di Bruno type Hopf algebra

Coordinate functions: $a_\eta^i : G \to \mathbb{R},\ a_\eta^i(c_\epsilon) := \langle c_\epsilon, a_\eta^i \rangle = (c_\epsilon, \eta)_i \in \mathbb{R}$

$$\begin{aligned} \langle c_\epsilon \circ d_\epsilon, a_\eta^i \rangle &= \langle c_\epsilon \otimes d_\epsilon, \Delta(a_\eta^i) \rangle \\ &= \langle c_\epsilon \otimes d_\epsilon, \sum_{(\eta)} a_{\eta'}^i \otimes a_{\eta''}^j \rangle \end{aligned}$$

Theorem: Coordinate functions form a connected graded commutative non-cocommutative Hopf algebra $(H, \Delta, \epsilon, S, m, \iota)$.

Antipode: $S : H \to H \quad \langle c_\epsilon^{\circ -1}, a_\eta^i \rangle = \langle c_\epsilon, S(a_\eta^i) \rangle$

$$S(a_\eta^i) = -a_\eta^i - {\sum_{(\eta)}}' S(a_{\eta'}^i) a_{\eta''}^j = -a_\eta^i - {\sum_{(\eta)}}' a_{\eta'}^i S(a_{\eta''}^j)$$

Coordinate functions II

Coproduct and antipode calculations

$$\Delta : H \to H \otimes H$$

$$\Delta(a^i_\emptyset) = a^i_\emptyset \otimes \mathbf{1} + \mathbf{1} \otimes a^i_\emptyset$$

$$\Delta(a^i_{x_j}) = a^i_{x_j} \otimes \mathbf{1} + \mathbf{1} \otimes a^i_{x_j}$$

$$\Delta(a^i_{x_0}) = a^i_{x_0} \otimes \mathbf{1} + \mathbf{1} \otimes a^i_{x_0} + a^i_{x_\ell} \otimes a^\ell_\emptyset$$

$$\Delta(a^i_{x_j x_k}) = a^i_{x_j x_k} \otimes \mathbf{1} + \mathbf{1} \otimes a^i_{x_j x_k}$$

$$S : H \to H$$

$$S(a^i_\emptyset) = -a^i_\emptyset$$

$$S(a^i_{x_j}) = -a^i_{x_j}$$

$$S(a^i_{x_0}) = -a^i_{x_0} + a^i_{x_\ell} a^\ell_\emptyset$$

$$S(a^i_{x_j x_k}) = -a^i_{x_j x_k}$$

$$\langle c_\epsilon \circ d_\epsilon, a^i_{x_j x_k} \rangle = (c_\epsilon \circ d_\epsilon, x_j x_k)_i$$

$$= a^i_{x_0}(c_\epsilon) + a^i_{x_0}(d_\epsilon) + a^i_{x_\ell}(c_\epsilon) a^\ell_\emptyset(d_\epsilon)$$

$$= (c_\epsilon, x_0)_i + (d_\epsilon, x_0)_i + (c_\epsilon, x_\ell)_i (d_\epsilon, \emptyset)_\ell$$

$$\langle c^{\circ -1}_\epsilon, a^i_{x_j x_k} \rangle = (c^{\circ -1}_\epsilon, x_j x_k)_i$$

$$= -a^i_{x_0}(c_\epsilon) + a^i_{x_\ell}(c_\epsilon) a^\ell_\emptyset(c_\epsilon)$$

$$= -(c_\epsilon, x_0)_i + (c_\epsilon, x_\ell)_i (c_\epsilon, \emptyset)_\ell$$

Left Inversion of MIMO Fliess operators I

Observe: $c \in \mathbb{R}\langle\langle X \rangle\rangle$ can be written as $c = c_N + c_F$, where $c_N := \sum_{k\geq 0}(c, x_0^k)x_0^k$ and $c_F := c - c_N$.

Definition: Given $c \in \mathbb{R}^k\langle\langle X \rangle\rangle$, let $r_i \geq 1$ be the largest integer such that $\mathrm{supp}(c_{F,i}) \subseteq x_0^{r_i-1}X^*$, $i = 1, 2, \ldots, m$. Then c_i has relative degree r_i if $x_0^{r_i-1}x_j \in \mathrm{supp}(c_i)$, for $j \in \{1, \ldots, m\}$, otherwise it is not well defined. In addition, c has vector relative degree $r = [r_1\ r_2\ \cdots\ r_m]$ if each c_i has relative degree r_i and the $m \times m$ matrix

$$A = \begin{bmatrix} (c_1, x_0^{r_1-1}x_1) & (c_1, x_0^{r_1-1}x_2) & \cdots & (c_1, x_0^{r_1-1}x_m) \\ \vdots & \vdots & \vdots & \vdots \\ (c_m, x_0^{r_m-1}x_1) & (c_m, x_0^{r_m-1}x_2) & \cdots & (c_m, x_0^{r_m-1}x_m) \end{bmatrix}$$

has full rank. Otherwise, c does not have vector relative degree.

This definition coincides with the usual definition of relative degree given in a state space setting. But this definition is independent of the state space setting.

Left Inversion of MIMO Fliess operators II

Lemma: The set of series $\mathbb{R}^{m\times m}\langle\langle X\rangle\rangle$ having invertible constant terms is a group under the shuffle product. In particular, the shuffle inverse of any such series C is

$$C^{\,\sqcup\!\sqcup\,-1} = ((C,\emptyset)(I-C'))^{\,\sqcup\!\sqcup\,-1} = (C,\emptyset)^{-1}(C')^{\,\sqcup\!\sqcup\,*},$$

where $C' = I-(C,\emptyset)^{-1}C$ is proper, i.e., $(C',\emptyset)=0$, and $(C')^{\,\sqcup\!\sqcup\,*} := \sum_{k\geq 0}(C')^{\,\sqcup\!\sqcup\,k}$.

Lemma: For any $C \in \mathbb{R}^{m\times m}\langle\langle X\rangle\rangle$ with an invertible constant term, F_C, which is defined componentwise by $[F_C]_{i,j} = F_{C_{i,j}}$, has a well defined multiplicative inverse given by $(F_C)^{-1} = F_{C^{\,\sqcup\!\sqcup\,-1}}$.

Notation: Let $\mathbb{R}[[X_0]]$ be all commutative series over $X_0 := \{x_0\}$. When $c \in \mathbb{R}[[X_0]]$, $F_c[u](t) = \sum_{k\geq 0}(c,x_0^k)E_{x_0^k}[u](t) = \sum_{k\geq 0}(c,x_0^k)t^k/k!$.

Left Inversion of MIMO Fliess operators III

$$y^{(r)} = F_{(x_0^r)^{-1}(c)}[u] + F_C[u]u \in \mathbb{R}^m$$

$$u = -(F_C[u])^{-1} F_{(x_0^r)^{-1}(c-c_y)}[u], \quad y(t) = F_{c_y}[u](t) = \sum_{k\geq 0} (c_y, x_0^k) t^k / k!$$

$$u = -F_d[u], \qquad d = C^{\sqcup\!\sqcup\, -1} \sqcup\!\sqcup\, (x_0^r)^{-1}(c - c_y)$$

$(x_0^r)^{-1}(c - c_y)_i = (x_0^{r_i})^{-1}(c_i - c_{y_i})$ and $C_{i,j} = (x_0^{r_i - 1} x_j)^{-1}(c_i)$

Theorem: Suppose $c \in \mathbb{R}^m\langle\langle X \rangle\rangle$ has vector relative degree r. Let y be analytic at $t = 0$ with generating series $c_y \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$ satisfying $(c_y, x_0^{(r)}) \overset{*}{=} (c, x_0^{(r)})$. Then the input

$$u(t) = \sum_{k=0}^{\infty} (c_u, x_0^k) \frac{t^k}{k!} \quad \text{with} \quad c_u = ((C^{\sqcup\!\sqcup\, -1} \sqcup\!\sqcup\, (x_0^r)^{-1}(c - c_y))^{\circ -1})_{|N},$$

is the unique solution to $F_c[u] = y$ on $[0, T]$ for some $T > 0$.

Note: the condition $*$ on c_y ensures that y is in the range of F_c.

Multivariable I/O Lotka–Volterra Models I

$\dot{z}_i = \beta_i z_i + \sum_{j=1}^{n} \alpha_{ij} z_i z_j, \quad i = 1, 2, \ldots, n$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} \beta_1 z_1 - \alpha_{12} z_1 z_2 \\ -\beta_2 z_2 + \alpha_{21} z_1 z_2 - \alpha_{23} z_2 z_3 \\ -\beta_3 z_3 + \alpha_{32} z_3 z_2 \end{pmatrix}$$

2 Predators - 1 Prey

The systems within the first octant have:

- periodic orbits around $(\beta_2/\alpha_{21}, \beta_1/\alpha_{12}, 0)$ if $\beta_1\alpha_{32} = \beta_3\alpha_{12}$
- extinction of one population if $\beta_1\alpha_{32} < \beta_3\alpha_{12}$
- unbounded growing if $\beta_1\alpha_{32} > \beta_3\alpha_{12}$.

ANSATZ: **Input-output models are obtained by introducing time dependence on the parameters $\beta_i(t)$'s or $\alpha_{ij}(t)$'s (inputs), and assuming $y = h(z)$ (outputs).**

Multivariable I/O Lotka–Volterra Models II

Vector relative degree r for three LV systems with $y_1 = z_2$ and $y_2 = z_3$:

I/O map	r	range restrictions
$F_c : \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$	not defined	–
$F_c : \begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$	$[1\ 1]$	$(c_{y_1}, \emptyset) = (c_1, \emptyset)$ $(c_{y_2}, \emptyset) = (c_2, \emptyset)$
$F_c : \begin{bmatrix} \beta_1 \\ \beta_3 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$	$[2\ 1]$ (full)	$(c_{y_1}, \emptyset) = (c_1, \emptyset)$ $(c_{y_1}, x_0) = (c_1, x_0)$ $(c_{y_2}, \emptyset) = (c_2, \emptyset)$

Consider case 2: $r = [1\ 1]$, $u_1 := \beta_2$, $u_2 := \beta_3$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} \beta_1 z_1 - \alpha_{12} z_1 z_2 \\ \alpha_{21} z_1 z_2 - \alpha_{23} z_2 z_3 \\ \alpha_{32} z_3 z_2 \end{pmatrix} - \begin{pmatrix} 0 \\ z_2 \\ 0 \end{pmatrix} u_1 - \begin{pmatrix} 0 \\ 0 \\ z_3 \end{pmatrix} u_2, \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \end{pmatrix}$$

with $z_i(0) = z_{i,0} > 0$, $i = 1, 2, 3$. Normalizing all parameters to 1.

Multivariable I/O Lotka–Volterra Models III

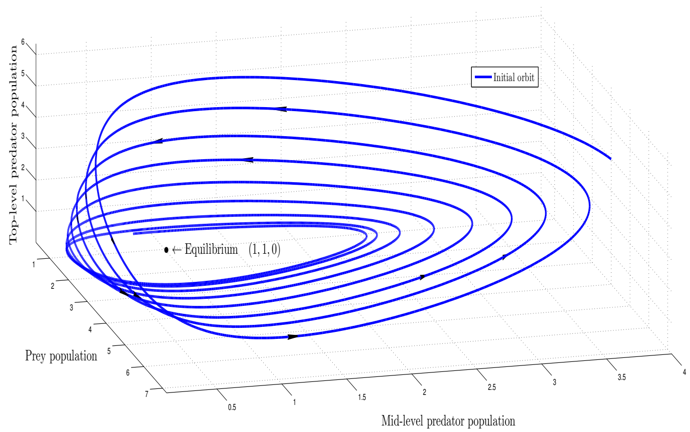
Inputs: $u_1 = \beta_2$, $u_2 = \beta_3$

Vector relative degree $r = [1 \;\; 1]$.

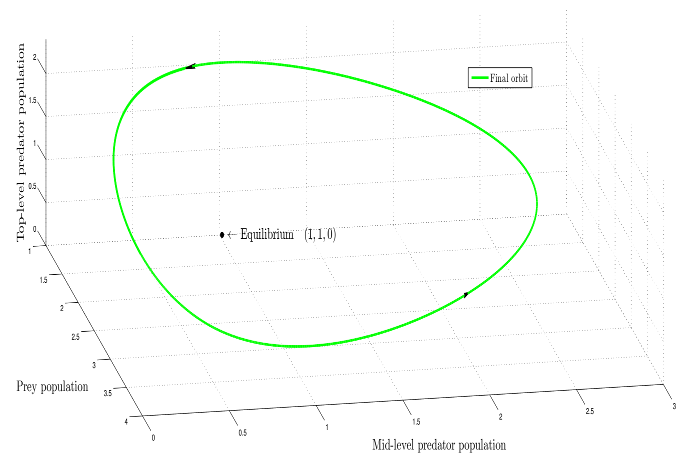
$$c_1 = z_{2,0} + (\alpha_{21} z_{1,0} z_{2,0} - \alpha_{23} z_{2,0} z_{3,0}) x_0 - (z_{2,0}) x_1 + 0 x_2 + \cdots ,$$

$$c_2 = z_{3,0} + (\alpha_{32} z_{2,0} z_{3,0}) x_0 + 0 x_1 - (z_{3,0}) x_2 + \cdots .$$

Extinction vs. periodic orbit



$u_1 = 1$, $u_2 = 1.2$.



$u_1 = u_2 = 1$.

Multivariable I/O Lotka–Volterra Models IV

Output function

One must select an output function

$$y(t) = \sum_{k=0}^{\infty} (c_y, x_0^k) \frac{t^k}{k!},$$

where $c_y = [c_{y_1}, c_{y_2}]^T$ is the generating series of y.

Consider a polynomial of degree 4: $(c_{y_j}, x_0^i) = v_{ij},\ i = 1, 2, 3, 4,\ j = 1, 2$.

$$A = (C, \emptyset) = \begin{bmatrix} (c_1, x_0^{r_1-1} x_1) & (c_1, x_0^{r_2-1} x_2) \\ (c_2, x_0^{r_1-1} x_1) & (c_2, x_0^{r_2-1} x_2) \end{bmatrix} = \mathrm{diag}\{-z_{2,0}, -z_{3,0}\} \text{ has full rank}$$

$$d = C^{\sqcup\!\sqcup\, -1} \sqcup\!\sqcup\, (x_0^r)^{-1}(c - c_y) = [d_1 \ \ d_2]^T.$$

$$c_u = (d^{\circ -1})_N$$

Multivariable I/O Lotka–Volterra Models V

Numerical Simulation

Note that $u_1(t_2) = u_2(t_2) = 1$ and $y_1(t_2) = z_2(t_2) = 1$ and $y_2(t_2) = 2$,

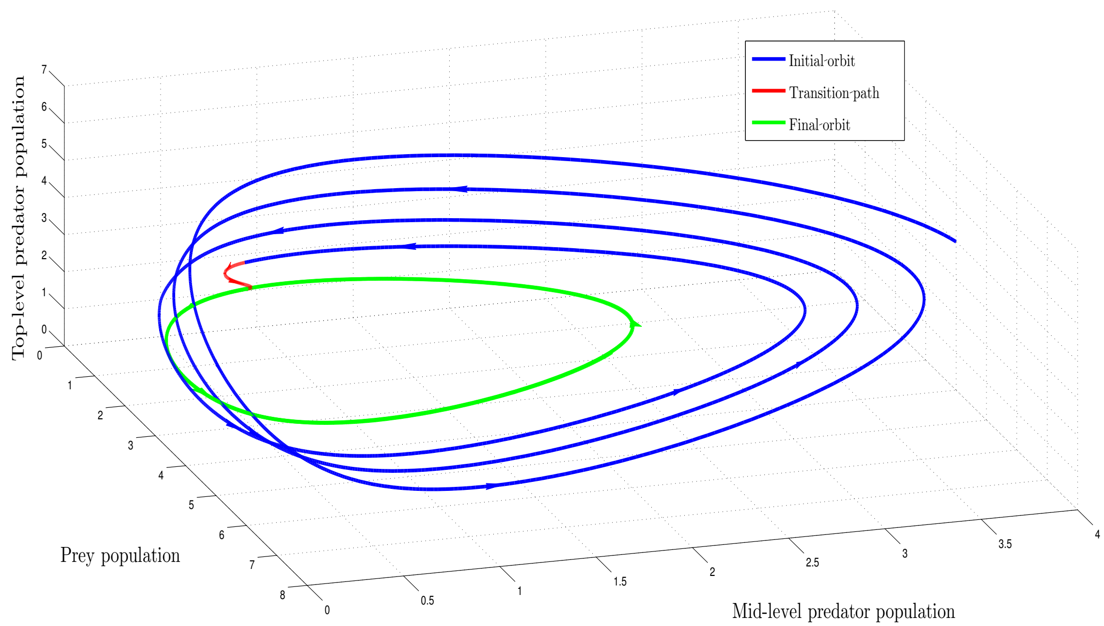


Fig. 5.2: Orbit transfer.

Conclusions

- The general multivariable left inverse problem for input-output systems represented as Fliess operators was solved explicitly via methods from combinatorial Hopf algebras: cancellation-free antipode formula.

- The technique was then illustrated for an orbit transfer problem in a three species Lotka–Volterra system: orbit transfer in order to avoid the extinction of the top-predator: System parameters β, α become controls.

- Efficiency of the software used for calculations/simulations is currently being improved.

Thank you for your attention!