Rectifiability, the Jones' β coefficients, and densities

Xavier Tolsa

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Rectifiability

$\mathcal{H}^n$ is the n-dimensional Hausdorff measure.

A set $E \subset \mathbb{R}^d$ is called *n-rectifiable* if there are n-dimensional C^1 (or Lipschitz) manifolds Γ_i such that $\mathcal{H}^n(E \setminus \bigcup_i \Gamma_i) = 0$.

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A Borel measure μ on $\mathbb{R}^d$ is called n-rectifiable if it is of the form

$$d\mu = g\, d\mathcal{H}^n|_E,$$

where E is n-rectifiable and $g \in L^1_{loc}(\mathcal{H}^n|_E)$.

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Characterization of n-rectifiable sets and measures in terms of:

- existence of approximate tangents,
- existence of densities,
- the size of projections.

(Besicovitch, Federer, Mattila, Preiss...).

Rectifiability and approximate tangents

Theorem (Besicovitch)

Let $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$. Then E is n-rectifiable if and only if at $\mathcal{H}^n$-a.e. $z \in E$ there exists an approximate tangent n-plane V. That is, for all $s > 0$,

$$\lim_{r \to 0} \frac{\mathcal{H}^n(E \cap B(z,r) \setminus X(z,V,s))}{r^n} = 0,$$

where $X(z,V,s)$ is the double cone with vertex at z and axis equal to V.

Rectifiability and densities

Upper n-dimensional density of μ: $\Theta^{n,*}(x,\mu) = \limsup_{r\to 0} \dfrac{\mu(B(x,r))}{r^n}$.

Lower n-dimensional density of μ: $\Theta^{n}_{*}(x,\mu) = \liminf_{r\to 0} \dfrac{\mu(B(x,r))}{r^n}$.

n-dimensional density of μ: $\Theta^{n}(x,\mu) = \lim_{r\to 0} \dfrac{\mu(B(x,r))}{r^n}$.

Theorem (Preiss)

Let μ be a Radon measure on $\mathbb{R}^d$. Then μ is n-rectifiable if and only if $\Theta^n(x,\mu)$ exists and is nonzero for μ-a.e. $x \in \mathbb{R}^d$.

AD-regular sets and uniform rectifiability

Uniform n-rectifiability is a quantitative version of n-rectifiability introduced by David and Semmes.

AD-regular sets and uniform rectifiability

Let $E \subset \mathbb{R}^d$. We say that it is n-AD-regular (or Ahlfors-David regular) if $\exists c > 0$ such that

$$c^{-1}\, r^n \leq \mathcal{H}^n(B(x,r) \cap E) \leq c\, r^n \qquad \text{for } x \in E,\ 0 < r \leq \operatorname{diam}(E).$$

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E is uniformly n-rectifiable if it is n-AD-regular and there are $M, \theta > 0$ such that for all $x \in E$, $0 < r \leq \operatorname{diam}(E)$, there exists a Lipschitz map

$$\varphi : \mathbb{R}^n \supset B_n(0,r) \to \mathbb{R}^d, \qquad \operatorname{Lip}(\varphi) \leq M,$$

such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap \varphi(B_n(0,r))\big) \geq \theta\, r^n.$$

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A measure μ is n-AD regular if it is of the form

$$d\mu = g\, d\mathcal{H}^n|_E,$$

where E is n-AD regular and $g \approx 1$.
μ is uniformly n-rectifiable if moreover E uniformly n-rectifiable.

Uniform rectifiability and Jones' β's

Given a ball $B(x,r)$ and a measure μ in $\mathbb{R}^d$, we denote

$$\beta_{\mu,p}(x,r) = \inf_L \left(\frac{1}{r^n} \int_{y\in B(x,r)} \left(\frac{\text{dist}(y,L)}{r} \right)^p \, d\mu(y) \right)^{1/p},$$

where the infimum is taken over all n-planes L.

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Theorem (David-Semmes)

Let μ be n-AD-regular. μ is uniformly n-rectifiable iff for all $x_0 \in \operatorname{supp}\mu$ and $R > 0$

$$\int_{x \in B(x_0,R)} \int_0^R \beta_{\mu,2}(x,r)^2 \, d\mu(x) \, \frac{dr}{r} \leq c\, R^n.$$

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- This can be thought of as an analog of the characterization of rectifiability in terms of existence of tangents.
- A previous L^∞ version of this result for $n = 1$ by Peter Jones.

Rectifiability and β's in the non AD-regular case

Theorem

Let μ be a Radon measure in $\mathbb{R}^d$ such that $0 < \Theta^{n,*}(x,\mu) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. Then μ is n-rectifiable if and only if

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Corollary

Let $E \subset \mathbb{R}^d$ be an $\mathcal{H}^n$-measurable set with $\mathcal{H}^n(E) < \infty$. The set E is n-rectifiable if and only if

$$\int_0^1 \beta_{\mathcal{H}^n|_E,2}(x,r)^2 \,\frac{dr}{r} < \infty \qquad \text{for } \mathcal{H}^n\text{-a.e. } x \in E.$$

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- Pajot proved $\Leftarrow$ for $\mu = \mathcal{H}^n|E$ assuming $\Theta^n_*(x,\mu) > 0$ μ-a.e.

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- Pajot proved $\Leftarrow$ for $\mu = \mathcal{H}^n|E$ assuming $\Theta_*^n(x,\mu) > 0$ μ-a.e.
- Badger and Schul proved $\Leftarrow$ for μ such that $0 < \Theta_*^n(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$ μ-a.e.

Uniform rectifiability and densities

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What about uniform rectifiability?

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Denote

$$\Delta_\mu(x,r) = \frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n}.$$

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Theorem (Chousionis, Garnett, Le, T.)

Let μ be n-AD-regular. μ is uniformly n-rectifiable iff for all $x_0 \in \operatorname{supp}\mu$ and $R > 0$

$$\int_{x\in B(x_0,R)} \int_0^R \Delta_\mu(x,r)^2 \, d\mu(x) \, \frac{dr}{r} \leq c\, R^n.$$

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- Previous $C^{1+\varepsilon}$ versions of this result by Kenig-Toro, David-Kenig-Toro and Preiss-T.-Toro.

Δ coefficients in the non-AD-regular case (1)

Theorem (T., Toro)

Let μ be a Radon measure in $\mathbb{R}^d$ such that

$$0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty \quad \mu\text{-a.e.}$$

The following are equivalent:

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(b) $\displaystyle\int_0^1 \Delta_\mu(x,r)^2 \, \frac{dr}{r} < \infty$ *for μ-a.e. $x \in \mathbb{R}^d$.*

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(a) μ *is n-rectifiable.*

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(c) $\displaystyle\lim_{r\to 0} \Delta_\mu(x,r) = 0$ *for* μ*-a.e.* $x \in \mathbb{R}^d$.

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(c) $\displaystyle\lim_{r\to 0} \Delta_\mu(x,r) = 0$ *for μ-a.e. $x \in \mathbb{R}^d$.*

- (c)⇒(a) is false without the assumption $\Theta^n_*(x,\mu) > 0$ μ-a.e.

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Suppose $n = 1$.

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Suppose $n = 1$. Let μ be a Radon measure in $\mathbb{R}^d$ such that

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Then μ is 1-rectifiable if and only if

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- Other related results which compare the measure in close scales using Wasserstein distance by Azzam, David and Toro (assuming μ doubling).

Δ coefficients in the non-AD-regular case (3)

Corollary

Let $E \subset \mathbb{R}^d$, with $\mathcal{H}^n(E) < \infty$.
In the case $n = 1$, E is 1-rectifiable if and only if

$$\int_0^1 \Delta_{\mathcal{H}^1|_E}(x,r)^2 \, \frac{dr}{r} < \infty \quad \mathcal{H}^1\text{-a.e. } x \in E.$$

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In the case $n > 1$, E is n-rectifiable if and only if

$$\Theta_*^n(x, \mathcal{H}^n|_E) > 0 \quad \text{and} \quad \int_0^1 \Delta_{\mathcal{H}^n|_E}(x,r)^2 \, \frac{dr}{r} < \infty \quad \mathcal{H}^n\text{-a.e. } x \in E.$$

Some arguments

Proof that

$$\int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty \quad \mu\text{-a.e.}$$

$$\Longrightarrow \quad \mu \text{ rectifiable, } \text{ assuming } 0 < \Theta^{n,*}(x,\mu) < \infty \; \mu\text{-a.e.}$$

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On each $\mathcal{T}$ with root R, μ is "well approximated" by $g\,\mathcal{H}^n|\Gamma_{\mathcal{T}}$, where $\|g\|_\infty \lesssim \Theta(R)$ and $\Gamma_{\mathcal{T}}$ is a bilipschitz image of $\mathbb{R}^n$.

More arguments

We prove the Carleson type condition

$$\sum_{R\in\text{Roots}} \Theta_\mu(R)\,\mu(R) \lesssim \|\mu\| + \iint_0^\infty \beta_{\mu,2}(x,r)^2\,\frac{dr}{r}\,d\mu(x) < \infty.$$

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Since $\displaystyle\int \sum_{R\in\text{Roots}:x\in R} \Theta_\mu(R)\,d\mu(x) = \sum_{R\in\text{Roots}} \Theta_\mu(R)\,\mu(R) < \infty,$

we get

$$\sum_{R\in\text{Roots}:x\in R} \Theta_\mu(R) < \infty \quad \mu\text{-a.e.}$$

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Since $\displaystyle\int \sum_{R\in\text{Roots}:x\in R} \Theta_\mu(R)\,d\mu(x) = \sum_{R\in\text{Roots}} \Theta_\mu(R)\,\mu(R) < \infty,$

we get

$$\sum_{R\in\text{Roots}:x\in R} \Theta_\mu(R) < \infty \quad \mu\text{-a.e.}$$

Thus if there are infinitely many roots $R_1 \supset R_2 \supset R_3 \supset \ldots$ that contain x,

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Similar arguments for $\Delta_\mu(x, r)$, $n = 1$

Proof that

$$\int_0^1 \Delta_\mu(x,r)^2 \, \frac{dr}{r} < \infty \quad \mu\text{-a.e.}$$

$\Longrightarrow$ μ rectifiable, assuming $0 < \Theta^{1,*}(x,\mu) < \infty$ μ-a.e.

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We can assume that $\iint_0^\infty \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x) < \infty$.

We use a geometric corona decomposition:
We split the dyadic lattice $\mathcal{D}$ into "trees" of cubes.
For $Q \in \mathcal{D}$, denote $\Theta_\mu(Q) = \dfrac{\mu(Q)}{\ell(Q)}$.
In each tree $\mathcal{T} \subset \mathcal{D}$, with root $R \in \mathcal{D}$, $\Theta_\mu(Q) \approx \Theta_\mu(R)$, for every $Q \in \mathcal{T}$.
On each $\mathcal{T}$ with root R, μ is "well approximated" by $g\,\mathcal{H}^1|\Gamma_{\mathcal{T}}$, where $\|g\|_\infty \lesssim \Theta(R)$ and $\Gamma_{\mathcal{T}}$ is a bilipschitz image of $\mathbb{R}$.

More arguments

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Denote

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Theorem (Azzam, T.)

Let μ be a Radon measure in $\mathbb{R}^d$ such that such that $0 < \Theta^{n,*}(x,\mu) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. Then μ is n-rectifiable if and only if

$$\int_0^1 \beta_{\mu,2}(x,r)^2\, \Theta_\mu(x,r)\, \frac{dr}{r} < \infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

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$p > 0$.

Curvature and β_2's

Curvature of μ:

$$c^2(\mu) = \iiint \frac{1}{R(x,y,z)^2}\, d\mu(x) d\mu(y) d\mu(z),$$

where $R(x,y,z)$ is the radius of the circumference that passes through x, y, z.

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Let $n = 1$. Let μ be a Radon measure in $\mathbb{R}^2$ such that $\mu(B(x,r)) \leq r$ for all $x \in \mathbb{R}^2$, $r > 0$. Then

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- When μ is AD-regular Mattila, Melnikov and Verdera, and Jones showed

$$c^2(\mu) \approx \iint_0^\infty \beta_{\mu,2}(x,r)^2\, \frac{dr}{r}\, d\mu(x).$$

A corollary regarding analytic capacity

Corollary

Let $E \subset \mathbb{C}$ be compact. Then

$$\gamma(E) \approx \sup \mu(E),$$

where μ satisfies

$$\sup_{r>0} \Theta^1_\mu(x,r) + \int_0^\infty \beta^1_{\mu,2}(x,r)^2\, \Theta^1_\mu(x,r)\, \frac{dr}{r} \leq 1 \quad \textit{for all } x \in \mathbb{C}.$$

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Conjecture

Let $E \subset \mathbb{R}^{n+1}$ be compact. Then

$$\kappa(E) \approx \sup \mu(E),$$

where μ satisfies

$$\sup_{r>0} \Theta^n_\mu(x,r) + \int_0^\infty \beta^n_{\mu,2}(x,r)^2\, \Theta^n_\mu(x,r)\, \frac{dr}{r} \leq 1 \quad \text{for all } x \in \mathbb{R}^{n+1}.$$

The end

Thank you.