Off-diagonal estimates and weighted elliptic operators

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- Background
  - Main motivators and instigators
  - Weighted elliptic operators
  - Extended Calderón-Zygmund theory
  - Operators defined by sesquilinear forms
  - Weighted Sobolev Spaces
  - Gaffney estimates
  - Kato for weighted ellipticity

- New results
  - Off Diagonal estimates
  - The functional calculus
  - Riesz transform bounds
  - Square function estimates
  - Kato estimates
  - Unweighted Kato estimates

Some of the main motivations

- Auscher, Hofmann, Lacey, McIntosh, Tchamitchian, "*The solution of the Kato square root problem for second order elliptic operators in* $\mathbb{R}^n$", Ann.Math. 2002.

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- Auscher and Martell, "*Weighted norm inequalities, off diagonal estimates and elliptic operators I,II, III, IV*", Adv.Math 2007, J.Evol.Eq. 2007, JFA 2006, Math Z 2008.

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- Cruz-Uribe, R. "*The Kato problem for operators with weighted ellipticity*", TAMS (to appear)

Auscher and Martell "*Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part III:Harmonic Analysis of elliptic operators,*" JFA 241 (2006) 703-746.

$$L = -\operatorname{div} \mathbf{A}(x)\nabla, \qquad \mathbf{A} \in \mathcal{E}(\lambda, \Lambda).$$

- Functional calculus for L, and weighted f.c.
- Riesz transform estimates (Auscher) $\left\| \nabla L^{-1/2} f \right\|_p \sim \|f\|_p$, $p_- < p < q_+$.
- RT weighted estimates $\left\| \nabla L^{-1/2} f \right\|_{L^p(u)} \lesssim \|f\|_{L^p(u)}$, $p_- r_w < p < q_+ / (s_w)'$.
- Reverse inequalities for $\sqrt{L}$. $\max\left\{1, \frac{np_-}{n+p_-}\right\} < p < p_+$,

$$\left\| L^{1/2} f \right\|_p \lesssim \|\nabla f\|_p$$

- Square function estimates.
- Commutators with bmo functions $\|[\varphi(L), b]\|_p \lesssim \|b\|_{\mathrm{BMO}} \|f\|_p$ (more).

Weights, A_p and reverse Hölder classes

A weight is any nonnegative locally integrable function u in $\mathbb{R}^n$.

- The A_p class $p > 1$

$$[u]_{A_p} = \sup_B \fint_B u(x)\ dx \left(\fint_B u^{-\frac{1}{p-1}} dx \right)^{p-1} < \infty.$$

- The A_1 class

$$[u]_{A_1} = \sup_B \operatorname*{ess\,sup}_{x \in B} u(x)^{-1} \fint_B u(x)\ dx < \infty.$$

- The RH_s class $s > 1$

$$[u]_{RH_s} = \sup_B \left(\fint_B u(x)\ dx \right)^{-1} \left(\fint_B u(x)^s\ dx \right)^{1/s} < \infty.$$

- The RH_∞ class

$$[u]_{RH_\infty} = \sup_B \operatorname*{ess\,sup}_{x \in B} u(x) \left(\fint_B u(x)\ dx \right)^{-1} < \infty.$$

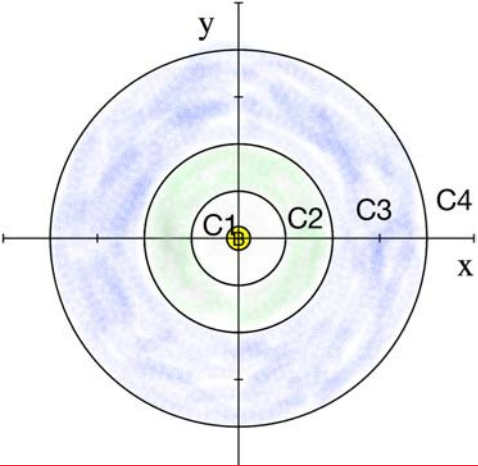
Some well known properties of A_p weights

- $A_1 \subset A_p \subset A_q$ for $1 \leq p \leq q < \infty$.
- $RH_\infty \subset RH_q \subset RH_p$ for $1 < p \leq q \leq \infty$.
- $$A_\infty = \bigcup_{1 \leq p < \infty} A_p = \bigcup_{1 < s \leq \infty} RH_s.$$
- A_p is left open
$$u \in A_p,\ p > 1 \Longrightarrow \exists \varepsilon > 0 : u \in A_{p-\varepsilon}.$$
- RH_s is right open
$$u \in RH_s,\ s < \infty \Longrightarrow \exists \varepsilon > 0 : u \in RH_{s+\varepsilon}.$$
- $1 < p < \infty, u \in A_p \iff w^{-\frac{1}{p-1}} \in A_{p'}, p' = p/\left(p-1\right).$
- If $w \in A_\infty$ then dw is doubling.

Extended Calderón-Zygmund theory

Some notation: Given an Euclidean ball $B = B_r(x) \subset \mathbb{R}^n$ denote by

- $C_1(B) = 4B$
- $C_j(B) = 2^{j+1}B/2^jB, j \geq 2.$



Extended Calderón-Zygmund theory

Theorem 1 (Auscher and Martell (II-III))

Given $w \in A_2$ with doubling order D, $1 \leq p_0 < q_0 \leq \infty$, $T : L^{q_0}(w) \longrightarrow L^{q_0}(w)$ (bounded) sublinear, $\{\mathcal{A}_r\}_{r>0}$ linear from L_c^∞ into $L^{q_0}(w)$. Suppose that $\forall B = B_r$, $f \in L_c^\infty$ with support$(f) \subset B$ and $j \geq 2$,

$$\left(\fint_{C_j(B)} |T(I - \mathcal{A}_r) f|^{p_0} \, dw \right)^{1/p_0} \leq g(j) \left(\fint_B |f|^{p_0} \, dw \right)^{1/p_0},$$

and for $j \geq 1$,

$$\left(\fint_{C_j(B)} |\mathcal{A}_r f|^{q_0} \, dw \right)^{1/q_0} \leq g(j) \left(\fint_B |f|^{p_0} \, dw \right)^{1/p_0},$$

where $\sum g(j) < \infty$. Then for all $p_0 < p < q_0$, there is a constant C such that for all $f \in L_c^\infty$,

$$\|Tf\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$

Extended Calderón-Zygmund theory

Theorem 2 (Auscher and Martell (II-III))

Given $w \in A_2$, $1 \le p_0 < q_0 \le \infty$, T sublinear and bounded on $L^{p_0}(w)$, $\{\mathcal{A}_r\}_{r>0}$ linear and bounded from $\mathcal{D} \subset L^{p_0}(w)$ into $L^{p_0}(w)$, and S linear from $\mathcal{D}$ into measurable functions on $\mathbb{R}^n$. Suppose that $\forall f \in \mathcal{D}$, $B = B_r$,

$$\left(\fint_B |T(I - \mathcal{A}_r) f|^{p_0} \, dw\right)^{1/p_0} \le \sum_{j\ge1} g(j) \left(\fint_{2^{j+1}B} |Sf|^{p_0} \, dw\right)^{1/p_0},$$

$$\left(\fint_B |T \mathcal{A}_r f|^{q_0} \, dw\right)^{1/q_0} \le \sum_{j\ge1} g(j) \left(\fint_{2^{j+1}B} |Tf|^{p_0} \, dw\right)^{1/p_0},$$

where $\sum g(j) < \infty$. Then for all $p_0 < p < q_0$, and weights $v \in A_{p/p_0}(w) \bigcap RH_{(q_0/p)'}(w)$, there is a constant C such that for all $f \in \mathcal{D}$,

$$\|Tf\|_{L^p(v\,dw)} \le C \|Sf\|_{L^p(v\,dw)}.$$

Operators given by sesquilinear forms

Let $\mathfrak{a}$ be a sesquilinear form with dense domain $\mathcal{D}(\mathfrak{a}) \subset H$ in a Hilbert space H such that

- $\operatorname{Re} \mathfrak{a}(u,u) \geq 0$, (accretive)
- $|\mathfrak{a}(u,v)| \leq M \, \|u\|_{\mathfrak{a}} \|v\|_{\mathfrak{a}}$, with $\|f\|_{\mathfrak{a}} = \left(\operatorname{Re} \mathfrak{a}(f,f) + \langle f,f \rangle_H\right)^{1/2}$, (continuous).
- $\left(\mathcal{D}(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}}\right)$ is complete (closed),

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- $(\mathcal{D}(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$ is complete (closed),

then there exists an associated operator $L_{\mathfrak{a}}$ such that

$$\mathfrak{a}(u,v) = \langle L_{\mathfrak{a}} u, v\rangle_H, \qquad \forall u \in \mathcal{D}(L_{\mathfrak{a}}),\ v \in \mathcal{D}(\mathfrak{a}),$$

with $\mathcal{D}(L_{\mathfrak{a}})$ dense in H.

Operators given by sectorial sesquilinear forms

Let $L_{\mathfrak{a}}$ be the operator associated to a densely defined, accretive, continuous, closed sesquilinear form in a Hilbert space H. If for some $0 \leq \vartheta < \frac{\pi}{2}$,

- $|\operatorname{Im} \mathfrak{a}(u,u)| \leq \tan(\vartheta) \operatorname{Re} \mathfrak{a}(u,u)$ (sectorial of angle ϑ)

then $L_{\mathfrak{a}}$ is sectorial of angle $\vartheta + \frac{\pi}{4}$, i.e.:
(i) $\sigma(L_{\mathfrak{a}}) \subset \overline{\Sigma_{\vartheta+\frac{\pi}{4}}}$,
(ii) $\sup \left\{ \|z\, R(z, L_{\mathfrak{a}})\|_{\mathrm{op}} \mid z \in \mathbb{C} \backslash \overline{\Sigma_{\omega'}} \right\} < \infty$ for all $\omega' > \vartheta + \frac{\pi}{4}$,
$R(z, L_{\mathfrak{a}}) = (z - L_{\mathfrak{a}})^{-1}$.

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$R(z, L_{\mathfrak{a}}) = (z - L_{\mathfrak{a}})^{-1}$.
A consequence of (ii) is that $L_{\mathfrak{a}}$ has a bounded holomorphic calculus in H.
(iii) If φ is a bounded holomorphic function in $\Sigma_{\omega'}$ then

$$\left\|\varphi(L_{\mathfrak{a}})\right\|_{\mathrm{op}} \leq \left\|\varphi\right\|_{\infty}.$$

In particular, $\left\|e^{-tL_{\mathfrak{a}}}u\right\|_H \leq \left\|u\right\|_H$ for all $t > 0$.

Weighted Sobolev spaces

Given a weight w we let

$$L^p(w) = \left\{ f \text{ measurable, } \|f\|_{L^p(w)} < \infty \right\}$$

with $\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dw \right)^{1/p}$. Similarly, for integers $k \geq 0$, we let

$$W^{k,p}(w) = \{ f \in L^p(w) : |D^\alpha f| \in L^p(w) \quad |\alpha| \leq k \}$$

where $D^\alpha f$ is distributional. The norm is defined as

$$\|f\|_{W^{k,p}(w)} = \left(\sum_{j=0}^{k} \sum_{|\alpha|=j} \int_{\mathbb{R}^n} |D^\alpha f(x)|^p \, dw \right)^{1/p}.$$

Weighted Sobolev spaces

Let Λ^s be the pseudodifferential operator with symbol $\left(1+4\pi^2\left|\xi\right|^2\right)^{-s/2}$ **(Bessel potential)**.

Theorem 3 (Miller (TAMS 82))

If $w \in A_p$ then for all integers $k \geq 0$

$$W^{k,p}(w) = \Lambda^{-k}\left(L^p(w)\right)$$

with equivalence of norms, i.e.:

$$\|u\|_{W^{k,p}(w)} \approx \left\|\Lambda^k u\right\|_{L^p(w)}.$$

Moreover, $W^{k,p}(w)$ is a Banach space and, if $p=2$, $H^k(w) := W^{k,2}(w)$ is a Hilbert space with inner product

$$\langle u, v\rangle_{H^k(w)} = \sum_{j=0}^{k} \sum_{|\alpha|=j} \int_{\mathbb{R}^n} D^\alpha u(x)\, \overline{D^\alpha v(x)}\, dw.$$

The Fabes-Kenig-Serapioni Poincaré inequality

Given $w \in A_\infty$ we define

$$r_w = \inf \{p : w \in A_p\}, \qquad s_w = \sup \{q : w \in RH_q\}.$$

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Theorem 4 (Fabes, Kenig, Serapioni, Comm. PDE 1982)

$p \geq 1$, $w \in A_p$, $p_w^* = p\frac{nr_w}{nr_w - p}$ if $p < nr_w$, $p_w^* = \infty$ otherwise. Then for every $p \leq q < p_w^*$, $f \in C_0^\infty(B_r)$,

$$\left(\fint_{B_r} |f(x)|^q \, dw \right)^{1/q} \leq Cr \left(\fint_{B_r} |\nabla f(x)|^p \, dw \right)^{1/p}.$$

If $f \in C^\infty(B_r)$, then

$$\left(\fint_{B_r} |f(x) - f_{B_r,w}|^q \, dw \right)^{1/q} \leq Cr \left(\fint_{B_r} |\nabla f(x)|^p \, dw \right)^{1/p}.$$

Weighted elliptic operators

Given $0 < \lambda \leq \Lambda < \infty$, we let $\mathcal{E}_n(\lambda, \Lambda)$ be the set of complex $n \times n$ matrices $\mathbf{A}(x)$ such that

$$\begin{aligned} \lambda |\xi|^2 &\leq \operatorname{Re}\langle \mathbf{A}(x)\xi, \xi\rangle, \qquad \forall \xi \in \mathbb{C}^n, \quad \text{(ellipticity)}, \\ \Lambda |\xi|\,|\eta| &\geq |\langle \mathbf{A}(x)\xi, \eta\rangle|, \qquad \forall \xi, \eta \in \mathbb{C}^n, \quad \text{(boundedness)}. \end{aligned}$$

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For $w \in A_2$, and $\mathbf{A} \in \mathcal{E}_n(\lambda, \Lambda)$, we define the sesquilinear form in $H^1(w)$

$$\mathfrak{a}_w(u, v) = \int_{\mathbb{R}^n} \mathbf{A}(x)\nabla u \cdot \overline{\nabla v}\, dw.$$

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It easily follows that the form $\mathfrak{a}_w$ is densely defined in $L^2(w)$, accretive, continuous, closed, and sectorial of angle $\vartheta = \arctan\left(\sqrt{\frac{\Lambda^2}{\lambda^2} - 1}\right)$.

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It easily follows that the form $\mathfrak{a}_w$ is densely defined in $L^2(w)$, accretive, continuous, closed, and sectorial of angle $\vartheta = \arctan\left(\sqrt{\frac{\Lambda^2}{\lambda^2} - 1}\right)$. The associated operator L_w has dense domain $\mathcal{D}(L_w)$ in $L^2(w)$ and it is formally given by

$$L_w u(x) = -\frac{1}{w(x)} \operatorname{div} w(x)\mathbf{A}(x)\nabla u(x) =: -\frac{1}{w(x)} \operatorname{div} \mathbf{A}_w(x)\nabla u(x).$$

Gaffney type estimates

The sectoriality of L_w and the techniques in *"The solution of the Kato square root problem for second order operators on $\mathbb{R}^n$" (Lemma 2.1), Auscher, Hofmann, Lacey, McIntosh, Tchamitchian, (Annals 2002)*, provide:

Theorem 5 (Cruz-Uribe, R., JFA 2008)

$w \in A_2$, $\mathbf{A} \in \mathcal{E}(\lambda, \Lambda)$, *E, F closed in* $\mathbb{R}^n$, $z \in \Sigma_\nu$, $0 < \nu < \arctan\left(\frac{\lambda}{\sqrt{\Lambda^2 - \lambda^2}}\right)$,

$$\begin{aligned}
\left\| e^{-zL_w}(\mathbf{1}_E f)\,\mathbf{1}_F \right\|_{L^2(w)} &\leq Ce^{-\frac{cd^2(E,F)}{|z|}} \|\mathbf{1}_E f\|_{L^2(w)}, \\
\left\| \sqrt{|z|}\nabla e^{-zL_w}(\mathbf{1}_E f)\,\mathbf{1}_F \right\|_{L^2(w)} &\leq Ce^{-\frac{cd^2(E,F)}{|z|}} \|\mathbf{1}_E f\|_{L^2(w)}, \\
\left\| zL_w e^{-zL_w}(\mathbf{1}_E f)\,\mathbf{1}_F \right\|_{L^2(w)} &\leq Ce^{-\frac{cd^2(E,F)}{|z|}} \|\mathbf{1}_E f\|_{L^2(w)}.
\end{aligned}$$

The Kato estimate for weighted elliptic operators

Theorem 6 (Cruz-Uribe, R, TAMS 2013?)

$w \in A_2$, $\mathbf{A} \in \mathcal{E}(\lambda, \Lambda)$, there exists $C = C\left(n, \lambda, \Lambda, [w]_{A_2}\right)$ such that

$$C^{-1} \|\nabla f\|_{L^2(w)} \leq \left\| L_w^{\frac{1}{2}} f \right\|_{L^2(w)} \leq C \|\nabla f\|_{L^2(w)}$$

for all $f \in H^1(w)$.

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Auscher, Rosén, Rule, *Boundary value problems for degenerate elliptic equations and systems*, (2014).
Extended Kato square root estimates to more general operators and systems.

Off diagonal estimates for e^{-tL_w}

Definition 7 (Full off diagonal estimates)

Given $1 \leq p \leq q \leq \infty$, a family of sublinear operators $\{T_t\}$ satisfies full off-diagonal estimates from $L^p(w)$ to $L^q(w)$, denoted by

$$T_t \in \mathcal{F}\left(L^p(w) \longrightarrow L^q(w)\right)$$

if there exists constants $C, c, \theta > 0$ such that for all closed E and F

$$\left\|T_t\left(f\mathbf{1}_E\right)\mathbf{1}_F\right\|_{L^q(w)} \leq Ct^{-\theta}e^{-\frac{d^2(E,F)}{t}}\left\|f\mathbf{1}_E\right\|_{L^p(w)}.$$

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Note: By the Gaffney estimates for e^{-tL_w} we have that

$$\begin{aligned} e^{-tL_w} &\in \mathcal{F}\left(L^2(w) \longrightarrow L^2(w)\right), \\ \sqrt{t}\nabla e^{-tL_w} &\in \mathcal{F}\left(L^2(w) \longrightarrow L^2(w)\right), \\ tL_w e^{-tL_w} &\in \mathcal{F}\left(L^2(w) \longrightarrow L^2(w)\right). \end{aligned}$$

Definition 8 (Off diagonal estimates on balls)

Given $1 \leq p \leq q \leq \infty$, a family of sublinear operators $\{T_t\}$ satisfies off-diagonal estimates on ball from $L^p(w)$ to $L^q(w)$, denoted by

$$T_t \in \mathcal{O}\left(L^p(w) \longrightarrow L^q(w)\right)$$

if there exists constants $c, \theta_1, \theta_2 > 0$ such that for all balls B,

$$\left(\fint_B |T_t(\mathbf{1}_B f)|^q \, dw\right)^{\frac{1}{q}} \lesssim \Upsilon\left(\frac{r}{\sqrt{t}}\right)^{\theta_2} \left(\fint_B |f|^p \, dw\right)^{\frac{1}{p}},$$

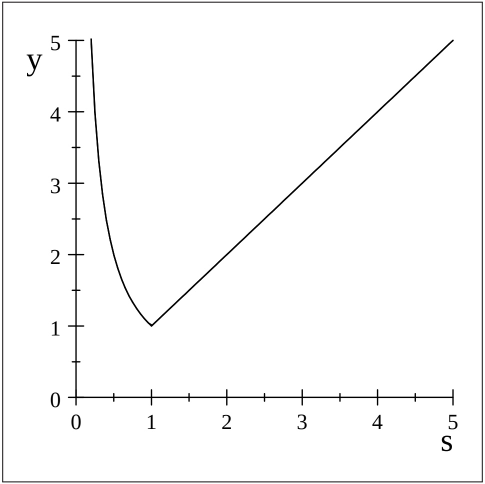
where $r = r(B)$, and for all $j \geq 2$

$$\left(\fint_B \left|T_t\left(\mathbf{1}_{C_j(B)} f\right)\right|^q dw\right)^{\frac{1}{q}} \lesssim 2^{j\theta_1} \Upsilon\left(\frac{2^j r}{\sqrt{t}}\right)^{\theta_2} e^{-\frac{c 4^j r^2}{t}} \left(\fint_{C_j(B)} |f|^p \, dw\right)^{\frac{1}{p}},$$

and

$$\left(\fint_{C_j(B)} |T_t(\mathbf{1}_B f)|^q \, dw\right)^{\frac{1}{q}} \lesssim 2^{j\theta_1} \Upsilon\left(\frac{2^j r}{\sqrt{t}}\right)^{\theta_2} e^{-\frac{c 4^j r^2}{t}} \left(\fint_B |f|^p \, dw\right)^{\frac{1}{p}}$$

The function Y



$$Y(s) = \max\{s, 1/s\} = \exp(|\ln s|)$$

Off diagonal estimates for the semigroup e^{-tL_w}

Theorem 9 (Cruz-Uribe, Martell, R.)

There exist $p_- = p_-(L_w) < p_+(L_w) = p_+$ *with*

$$1 \leq p_- \leq (2_w^*)' < 2 < 2_w^* \leq p_+ \leq \infty$$

such that if $p_- < p \leq q < p_+$ *then* $e^{-tL_w} \in \mathcal{O}(L^p(w) \longrightarrow L^q(w))$.

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Proof (hint).

$e^{-tL_w}, \sqrt{t}\nabla e^{-tL_w} \in \mathcal{F}\left(L^2(w) \longrightarrow L^2(w)\right) \subset \mathcal{O}\left(L^2(w) \longrightarrow L^2(w)\right)$, then

$$\begin{aligned}
&\left(\fint_B \left|e^{-tL_w}(\mathbf{1}_B f)\right|^q dw\right)^{1/q} \\
\le\ &\left(\fint_B \left|e^{-tL_w}(\mathbf{1}_B f)\right|^2 dw\right)^{1/2} + r\left(\fint_B \left|\nabla e^{-tL_w}(\mathbf{1}_B f)\right|^2 dw\right)^{1/2} \\
\lesssim\ &\Upsilon\left(\frac{r}{\sqrt{t}}\right)^{1+\theta_2}\left(\fint_B |f|^2\, dw\right)^{1/2}.
\end{aligned}$$

Weighted off diagonal estimates for the semigroup e^{-tL_w}

Theorem 10

$w \in A_2$, $p_-(L_w) = p_- < p \leq q < p_+ = p_+(L_w)$, and $u \in A_{p/p_-}(w) \bigcap RH_{(p+/q)'}(w)$ we have that

$$e^{-tL_w} \in \mathcal{O}\left(L^p(udw) \longrightarrow L^q(udw)\right).$$

Weighted off diagonal estimates for the semigroup e^{-tL_w}

Theorem 10

$w \in A_2$, $p_-(L_w) = p_- < p \leq q < p_+ = p_+(L_w)$, and $u \in A_{p/p_-}(w) \bigcap RH_{(p+/q)'}(w)$ we have that

$$e^{-tL_w} \in \mathcal{O}\left(L^p(udw) \longrightarrow L^q(udw)\right).$$

Corollary 11

If w is a weight such that $1 \leq r_w < 1 + \frac{2}{n}$ and $s_w > \frac{n}{2} r_w + 1$, then

$$e^{-tL_w} \in \mathcal{O}\left(L^2 \longrightarrow L^2\right).$$

In particular, it suffices that $w \in A_{\frac{n}{n-1}} \bigcap RH_{n+1}$.

Off diagonal estimates for $\sqrt{t}\nabla e^{-tL_w}$

Theorem 12 (Cruz-Uribe, Martell, R.)

There exist $q_- = q_-(L_w) < q_+(L_w) = q_+$ *with*

$$1 \leq q_- \leq (2_w^*)' < 2 < q_+ \leq \infty$$

such that if $q_- < p \leq q < q_+$ *then* $\sqrt{t}\nabla e^{-tL_w} \in \mathcal{O}\left(L^p(w) \longrightarrow L^q(w)\right)$.
Moreover, $q_-(L_w) = p_-(L_w)$.

Off diagonal estimates for $\sqrt{t}\nabla e^{-tL_w}$

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Note: The proof that $q_+ > 2$ is nontrivial.

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such that if $q_- < p \le q < q_+$ *then* $\sqrt{t}\nabla e^{-tL_w} \in \mathcal{O}\left(L^p(w) \longrightarrow L^q(w)\right)$. *Moreover,* $q_-(L_w) = p_-(L_w)$.

Note: The proof that $q_+ > 2$ is nontrivial.Just use this: *Caccioppoli, Poincaré, Ghering, Hodge projection (Auscher-Martell estimates), Riesz transform estimates, Functional calculus, Semigroup estimates.*

The functional calculus

Denote by $\mathcal{H}_0^\infty(\Sigma_\nu)$ the set of holomorphic functions φ in the sector $\Sigma_\nu = \{|\arg z| < \nu\}$ which satisfy

$$|\varphi(z)| \le c\frac{|z|^s}{1+|z|^{2s}} \text{ for some } c, s > 0.$$

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Proposition 12.1 (Cruz-Uribe, Martell, R.)

For $w \in A_2$, $\mathbf{A} \in \mathcal{E}(\lambda, \Lambda)$ fix ν such that $\arctan\left(\sqrt{\frac{\Lambda^2}{\lambda^2} - 1}\right) < \nu < \pi$. *Then for $p_-(L_w) < p < p_+(L_w)$ and any $\varphi \in \mathcal{H}_0^\infty(\Sigma_\nu)$,*

$$\|\varphi(L_w) f\|_{L^p(w)} \le C \|\varphi\|_\infty \|f\|_{L^p(w)}.$$

with C independent of f and φ.
Furthermore, if $v \in A_{p/p_-}(w) \bigcap RH_{(p_+/p)'}(w)$ then L_w also has a bounded holomorphic calculus on $L^p(v\,dw)$:

$$\|\varphi(L_w) f\|_{L^p(v\,dw)} \le C \|\varphi\|_\infty \|f\|_{L^p(v\,dw)}.$$

The functional calculus, unweighted space

Corollary 13 (Cruz-Uribe, Martell, R.)

For $\mathbf{A} \in \mathcal{E}(\lambda, \Lambda)$ *fix* ν *such that* $\arctan\left(\sqrt{\frac{\Lambda^2}{\lambda^2} - 1}\right) < \nu < \pi$. *Then if* $w \in A_2$ *is such that* $1 < r_w < 1 + \frac{2}{n}$ *and* $s_w > \frac{n}{2} r_w + 1$, *then for any* $\varphi \in \mathcal{H}_0^\infty(\Sigma_\nu)$,

$$\|\varphi(L_w) f\|_{L^2} \leq C \|\varphi\|_\infty \|f\|_{L^2}.$$

In particular, it suffices to take $w \in A_{\frac{n}{n-1}} \bigcap RH_{n+1}$.

Riesz transform estimates

Proposition 13.1

For each $p_-(L_w) < p < q_+(L_w)$, there exists C such that

$$\left\| \nabla L_w^{-1/2} f \right\|_{L^p(w)} \leq C \|f\|_{L^p(w)} .$$

Furthermore, if $v \in A_{p/p_-}(w) \bigcap RH_{(q_+/p)'}(w)$ then

$$\left\| \nabla L_w^{-1/2} f \right\|_{L^p(v\,dw)} \leq C \|f\|_{L^p(v\,dw)} .$$

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Furthermore, if $v \in A_{p/p_-}(w) \bigcap RH_{(q_+/p)'}(w)$ then

$$\left\|\nabla L_w^{-1/2} f\right\|_{L^p(v\,dw)} \leq C \|f\|_{L^p(v\,dw)}.$$

Corollary 14

If $w \in A_2$, then for all weights v and exponents q such that $p_- r_v(w) < q < q_+/(s_v(w))'$,

$$\left\|\nabla L_w^{-1/2} f\right\|_{L^2} \leq C \|f\|_{L^2}.$$

Square function estimates

$$g_{L_w} f(x) = \left(\int_0^\infty \left| (tL_w)^{1/2} e^{-tL_w} f(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

$$G_{L_w} f(x) = \left(\int_0^\infty \left| t^{1/2} \nabla e^{-tL_w} f(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

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Theorem 15

Let $p_- (L_w) < p < p_+ (L_w)$ and $p_- (L_w) < q < q_+ (L_w)$, then

$$\|g_{L_w} f\|_{L^p(w)} \sim \|f\|_{L^p(w)}, \quad \forall f \in L^p(w) \bigcap L^2(w)$$

and

$$\|G_{L_w} f\|_{L^q(w)} \sim \|f\|_{L^q(w)}, \quad \forall f \in L^q(w) \bigcap L^2(w).$$

Weighted square function estimates

Theorem 16

For all weights u and exponents p such that $p_- r_u(w) < p < p_+/(s_u(w))'$,

$$\|g_{L_w} f\|_{L^p(u\,dw)} \sim \|f\|_{L^p(u\,dw)},$$

and for all weights v and exponents q such that $p_- r_v(w) < q < q_+/(s_v(w))'$,

$$\|G_{L_w} f\|_{L^q(v\,dw)} \sim \|f\|_{L^q(v\,dw)}.$$

Finally, the inequality $\|f\|_{L^q(v\,dw)} \lesssim \|G_{L_w} f\|_{L^p(v\,dw)}$ holds for $p_- < q < \infty$ whenever $v \in A_p(w)$.

Kato estimates

Theorem 17 (Cruz-Uribe, Martell, R.)

$w \in A_2$, $\max\{r_w, p_-\} < p < q_+$, then

$$\left\| L_w^{1/2} f \right\|_{L^p(w)} \sim \|\nabla f\|_{L^p(w)}$$

and if $v \in A_{\frac{p}{\max\{r_w, p_-\}}}(w) \bigcap RH_{(q_+/p)'}(w)$, then

$$\left\| L_w^{1/2} f \right\|_{L^p(v\, dw)} \sim \|\nabla f\|_{L^p(v\, dw)}.$$

Unweighted Kato estimates

Theorem 18 (Cruz-Uribe, Martell, R.)

Suppose that $w \in A_2$ and $p_- r_{\frac{1}{w}}(w) < 2 < q_+ / \left(s_{\frac{1}{w}}(w)\right)'$, then

$$\left\| L_w^{1/2} f \right\|_{L^2} \sim \|\nabla f\|_{L^2}.$$

In particular, this holds if $w \in A_1 \bigcap RH_{1+\frac{n}{2}}$.

Unweighted Kato estimates

Note on the weight conditions

$$r_{\frac{1}{w}}(w) < p \iff w \in RH_{p'}, \qquad \text{and} \qquad s_{\frac{1}{w}}(w) > p \iff w \in A_{p'}.$$

Unweighted Kato estimates

Note on the weight conditions

$$r_{\frac{1}{w}}(w) < p \iff w \in RH_{p'}, \qquad \text{and} \qquad s_{\frac{1}{w}}(w) > p \iff w \in A_{p'}.$$

- Hence $p_- r_{\frac{1}{w}}(w) \leq 2$ holds if

$$2\frac{nr_w}{nr_w + 2}(s_w)' < 2;$$

Unweighted Kato estimates

Note on the weight conditions

$$r_{\frac{1}{w}}(w) < p \iff w \in RH_{p'}, \qquad \text{and} \qquad s_{\frac{1}{w}}(w) > p \iff w \in A_{p'}.$$

- Hence $p_- r_{\frac{1}{w}}(w) \leq 2$ holds if

$$2\frac{nr_w}{nr_w + 2}(s_w)' < 2;$$

- and $2 < q_+ / \left(s_{\frac{1}{w}}(w)\right)'$ requires

$$r_w < (r_w')' = \left(s_{\frac{1}{w}}(w)\right)' < \frac{q_+}{2}.$$

Unweighted Kato estimates

Note on the weight conditions

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$$2\frac{nr_w}{nr_w + 2}(s_w)' < 2;$$

- and $2 < q_+ / \left(s_{\frac{1}{w}}(w)\right)'$ requires

$$r_w < (r'_w)' = \left(s_{\frac{1}{w}}(w)\right)' < \frac{q_+}{2}.$$

- In particular, if $r_w = 1$ ($w \in A_1$), the second condition is satisfied. For the first, we also need $\frac{n}{n+2}(s_w)' < 1$. i.e.:

$$\frac{n+2}{2} < s_w.$$

Thank you!