Dimension of p-harmonic measure in space

Murat Akman

ICMAT
Instituto de Ciencias Matemáticas

Workshop on Harmonic Analysis
Partial Differential Equations and Geometric Measure Theory
January 12-16, 2015, Madrid

Joint work with John Lewis and Andrew Vogel

ODE TO THE P-LAPLACIAN

"I used to be in love with the Laplacian so worked hard to please her with beautiful theorems. However she often scorned me for the likes of Björn Dahlberg, Gene Fabes, Carlos Kenig, and Thomas Wolff. Gradually I became interested in her sister the p Laplacian, $1 < p < \infty$, $p \neq 2$*. I did not find her as pretty as the Laplacian and she was often difficult to handle because of her nonlinearity. However over many years I took a shine to her and eventually developed an understanding of her disposition. Today she is my girl and the Laplacian pales in comparison to her."*

— *John Lewis*

Outline

1. Part I: $\sigma-$finiteness of p-harmonic measure in space for $p \geq n$
2. Part II: Example of a domain for which $\mathcal{H} - \dim \mu < n - 1$ for $p \geq n$
3. Part III: Related Work

Part I: $\sigma-$finiteness of p-harmonic measure in space for $p \geq n$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let N be an open neighborhood of $\partial\Omega$.

Part I: $\sigma-$finiteness of p-harmonic measure in space for $p \geq n$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let N be an open neighborhood of $\partial\Omega$.

Fix p, $1 < p < \infty$ and suppose that u is p-harmonic in $\Omega \cap N$. That is, $u \in W^{1,p}(\Omega \cap N)$ and

$$\int \langle |\nabla u|^{p-2}\nabla u, \nabla \phi\rangle \,\mathrm{d}x = 0 \quad \text{for all } \phi \in W_0^{1,p}(\Omega \cap N).$$

Part I: $\sigma-$finiteness of p-harmonic measure in space for $p \geq n$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let N be an open neighborhood of $\partial\Omega$.

Fix p, $1 < p < \infty$ and suppose that u is p-harmonic in $\Omega \cap N$. That is, $u \in W^{1,p}(\Omega \cap N)$ and

$$\int \langle |\nabla u|^{p-2} \nabla u, \nabla \phi \rangle \, \mathrm{d}x = 0 \quad \text{for all } \phi \in W_0^{1,p}(\Omega \cap N).$$

If u has continuous second partials in $\Omega \cap N$ and $\nabla u \neq 0$ then u is a classical solution to the p-Laplace equation in $\Omega \cap N$:

$$\nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right) = 0$$

Assume that $u > 0$ in $\Omega \cap N$ and $u = 0$ on $\partial\Omega$ in the Sobolev sense.

Set $u \equiv 0$ in $N \setminus \Omega$. Then u is p-harmonic in N.

[HKM]: Juha Heinonen, Tero Kilpeläinen, Olli Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Dover Publications Inc (2006).

Assume that $u > 0$ in $\Omega \cap N$ and $u = 0$ on $\partial\Omega$ in the Sobolev sense.

Set $u \equiv 0$ in $N \setminus \Omega$. Then u is p-harmonic in N.

It is well know from [HKM, Chapter 21] that there is a finite, positive, Borel measure μ associated with u satisfying

$$\int \langle |\nabla u|^{p-2} \nabla u, \nabla \psi \rangle \, \mathrm{d}x = - \int \psi \, \mathrm{d}\mu \quad \text{for all nonnegative } \psi \in C_0^\infty(N).$$

μ has support contained in $\partial\Omega$ and is called p-harmonic measure.

[HKM]: Juha Heinonen, Tero Kilpeläinen, Olli Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations.* Dover Publications Inc (2006).

Let $\lambda > 0$ be a real valued, positive, and increasing function on $(0, r_0)$ with $\lim_{r \to 0} \lambda(r) = 0$.

Let $\lambda > 0$ be a real valued, positive, and increasing function on $(0, r_0)$ with $\lim_{r \to 0} \lambda(r) = 0$.

Let $\mathcal{H}^{\lambda}(E)$ denote the Hausdorff measure of $E \subset \mathbb{R}^n$ relative to λ defined in the following way;

for fixed $0 < \delta < r_0$ let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, \;\; i = 1, 2, \ldots$

Let $\lambda > 0$ be a real valued, positive, and increasing function on $(0, r_0)$ with $\lim_{r \to 0} \lambda(r) = 0$.

Let $\mathcal{H}^\lambda(E)$ denote the Hausdorff measure of $E \subset \mathbb{R}^n$ relative to λ defined in the following way;

for fixed $0 < \delta < r_0$ let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, \ \ i = 1, 2, \ldots$

Set $\phi_\delta^\lambda(E) := \inf_{L(\delta)} \sum \lambda(r_i)$. Then $\mathcal{H}^\lambda(E) := \lim_{\delta \to 0} \phi_\delta^\lambda(E)$.

When $\lambda(r) = r^\alpha$ we write $\mathcal{H}^\alpha$ for $\mathcal{H}^\lambda$. Define the Hausdorff dimension of a Borel measure ν by

Let $\lambda > 0$ be a real valued, positive, and increasing function on $(0, r_0)$ with $\lim_{r \to 0} \lambda(r) = 0$.

Let $\mathcal{H}^\lambda(E)$ denote the Hausdorff measure of $E \subset \mathbb{R}^n$ relative to λ defined in the following way;

for fixed $0 < \delta < r_0$ let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, \ \ i = 1, 2, \ldots$

$$\text{Set } \phi_\delta^\lambda(E) := \inf_{L(\delta)} \sum \lambda(r_i). \ \text{Then } \mathcal{H}^\lambda(E) := \lim_{\delta \to 0} \phi_\delta^\lambda(E).$$

When $\lambda(r) = r^\alpha$ we write $\mathcal{H}^\alpha$ for $\mathcal{H}^\lambda$. Define the Hausdorff dimension of a Borel measure ν by

$$\mathcal{H} - \dim \nu := \inf\{\alpha \,|\, \exists \text{ a Borel set } E \subset \partial\Omega; \ \mathcal{H}^\alpha(E) = 0, \ \nu(\mathbb{R}^n \setminus E) = 0\}.$$

i.e., *it is the “smallest dimension” of a set with full ν measure.*

Let $\lambda > 0$ be a real valued, positive, and increasing function on $(0, r_0)$ with $\lim_{r \to 0} \lambda(r) = 0$.

Let $\mathcal{H}^\lambda(E)$ denote the Hausdorff measure of $E \subset \mathbb{R}^n$ relative to λ defined in the following way;

for fixed $0 < \delta < r_0$ let $L(\delta) = \{B(z_i, r_i)\}$ be such that $E \subseteq \bigcup B(z_i, r_i)$ and $0 < r_i < \delta, \;\; i = 1, 2, \ldots$

$$\text{Set } \phi_\delta^\lambda(E) := \inf_{L(\delta)} \sum \lambda(r_i). \text{ Then } \mathcal{H}^\lambda(E) := \lim_{\delta \to 0} \phi_\delta^\lambda(E).$$

When $\lambda(r) = r^\alpha$ we write $\mathcal{H}^\alpha$ for $\mathcal{H}^\lambda$. Define the Hausdorff dimension of a Borel measure ν by

$$\mathcal{H} - \dim \nu := \inf\{\alpha \,|\, \exists \text{ a Borel set } E \subset \partial\Omega;\ \mathcal{H}^\alpha(E) = 0,\ \nu(\mathbb{R}^n \setminus E) = 0\}.$$

i.e., *it is the "smallest dimension" of a set with full ν measure.*

When everything is smooth,

$$\mathrm{d}\mu = |\nabla u|^{p-1} \, \mathrm{d}\mathcal{H}^{n-1}|_{\partial\Omega}.$$

Results of interest for harmonic measure, ω

When $p = 2$ and u is the Green's function with pole at $z \in \Omega$ then $\mu = \omega(z, \cdot)$ is harmonic measure with respect to $z \in \Omega$.

[C]: Lennart Carleson. On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn.*, 10:113123, 1985.

[JW]: Peter W. Jones and Thomas Wolff. Hausdorff dimension of harmonic measures in the plane. *Acta Math.*, 161(1-2):131144, 1988.

[W]: Thomas Wolff. Plane harmonic measures live on sets of -finite length. *Ark. Mat.*, 31(1):137-172, 1993.

Results of interest for harmonic measure, ω

When $p = 2$ and u is the Green's function with pole at $z \in \Omega$ then $\mu = \omega(z, \cdot)$ is harmonic measure with respect to $z \in \Omega$.

- **Carleson**: $\mathcal{H} - \dim \omega = 1$ when $\partial\Omega$ is a snowflake in the plane and $\mathcal{H} - \dim \omega \leq 1$ when Ω is the complement of a self similar Cantor set.

[C]: Lennart Carleson. On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn.*, 10:113123, 1985.

[JW]: Peter W. Jones and Thomas Wolff. Hausdorff dimension of harmonic measures in the plane. *Acta Math.*, 161(1-2):131144, 1988.

[W]: Thomas Wolff. Plane harmonic measures live on sets of -finite length. *Ark. Mat.*, 31(1):137-172, 1993.

Results of interest for harmonic measure, ω

When $p = 2$ and u is the Green's function with pole at $z \in \Omega$ then $\mu = \omega(z, \cdot)$ is harmonic measure with respect to $z \in \Omega$.

- **Carleson**: $\mathcal{H} - \dim \omega = 1$ when $\partial\Omega$ is a snowflake in the plane and $\mathcal{H} - \dim \omega \leq 1$ when Ω is the complement of a self similar Cantor set.

- **Jones-Wolff**: Let $\Omega \subset \mathbb{C}^\star$ be a domain whose complement has positive capacity. Then there is a set $F \subset \partial\Omega$ with Hausdorff dimension ≤ 1, such that $\omega(z, F) = 1$ for $z \in \Omega$.

[C]: Lennart Carleson. On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn.*, 10:113123, 1985.

[JW]: Peter W. Jones and Thomas Wolff. Hausdorff dimension of harmonic measures in the plane. *Acta Math.*, 161(1-2):131144, 1988.

[W]: Thomas Wolff. Plane harmonic measures live on sets of -finite length. *Ark. Mat.*, 31(1):137-172, 1993.

Results of interest for harmonic measure, ω

When $p = 2$ and u is the Green's function with pole at $z \in \Omega$ then $\mu = \omega(z, \cdot)$ is harmonic measure with respect to $z \in \Omega$.

- **Carleson**: $\mathcal{H} - \dim \omega = 1$ when $\partial\Omega$ is a snowflake in the plane and $\mathcal{H} - \dim \omega \leq 1$ when Ω is the complement of a self similar Cantor set.

- **Jones-Wolff**: Let $\Omega \subset \mathbb{C}^\star$ be a domain whose complement has positive capacity. Then there is a set $F \subset \partial\Omega$ with Hausdorff dimension ≤ 1, such that $\omega(z, F) = 1$ for $z \in \Omega$.

- **Wolff**: Let $\Omega = \mathbb{C}^\star \setminus E$ where E is a compact set. Then there is a set $F \subset \partial\Omega$ satisfying $\omega(z, F) = 1$ with $\sigma-$finite one-dimensional Hausdorff measure.

[C]: Lennart Carleson. On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn.*, 10:113123, 1985.

[JW]: Peter W. Jones and Thomas Wolff. Hausdorff dimension of harmonic measures in the plane. *Acta Math.*, 161(1-2):131144, 1988.

[W]: Thomas Wolff. Plane harmonic measures live on sets of -finite length. *Ark. Mat.*, 31(1):137-172, 1993.

- **Bourgain**: $\mathcal{H} - \dim \omega \leq n - \tau$ whenever $\Omega \subset \mathbb{R}^n$ where $\tau = \tau(n) > 0$.

[B]: Jean Bourgain. On the Hausdorff dimension of harmonic measure in higher dimension. *Inv. Math.*, 87:477-483, 1987.

[W]: Thomas Wolff, Counterexamples with harmonic gradients in $\mathbb{R}^3$, *In Essays on Fourier analysis in honor of Elias M. Stein*, 42:321-384, 1995.

[LVV]: John L. Lewis, Gregory C. Verchota, and Andrew L. Vogel. Wolff snowflakes. *Pacific J. Math.*, 218 (2005), no. 1, 139166.

- **Bourgain**: $\mathcal{H}-\dim\omega \leq n-\tau$ whenever $\Omega\subset\mathbb{R}^n$ where $\tau=\tau(n)>0$.

- **Wolff**: There exists a *Wolff snowflake* in $\mathbb{R}^3$ for which $\mathcal{H}-\dim\omega<2$,
and there is another one for which $\mathcal{H}-\dim\omega>2$.

[B]: Jean Bourgain. On the Hausdorff dimension of harmonic measure in higher dimension. *Inv. Math.*, 87:477-483, 1987.

[W]: Thomas Wolff, Counterexamples with harmonic gradients in $\mathbb{R}^3$, *In Essays on Fourier analysis in honor of Elias M. Stein*, 42:321-384, 1995.

[LVV]: John L. Lewis, Gregory C. Verchota, and Andrew L. Vogel. Wolff snowflakes. *Pacific J. Math.*, 218 (2005), no. 1, 139166.

- **Bourgain**: $\mathcal{H}-\dim\omega \leq n-\tau$ whenever $\Omega \subset \mathbb{R}^n$ where $\tau=\tau(n)>0$.

- **Wolff**: There exists a *Wolff snowflake* in $\mathbb{R}^3$ for which $\mathcal{H}-\dim\omega<2$, and there is another one for which $\mathcal{H}-\dim\omega>2$.

- **Lewis-Verchota-Vogel**: Wolff's result holds in $\mathbb{R}^n$; Harmonic measure on both sides of a Wolff snowflake, say ω_+,ω_- could have

$$\max(\mathcal{H}-\dim\omega_+,\mathcal{H}-\dim\omega_-)<n-1$$

or

$$\min(\mathcal{H}-\dim\omega_+,\mathcal{H}-\dim\omega_-)>n-1.$$

[B]: Jean Bourgain. On the Hausdorff dimension of harmonic measure in higher dimension. *Inv. Math.*, 87:477-483, 1987.

[W]: Thomas Wolff, Counterexamples with harmonic gradients in $\mathbb{R}^3$, *In Essays on Fourier analysis in honor of Elias M. Stein*, 42:321-384, 1995.

[LVV]: John L. Lewis, Gregory C. Verchota, and Andrew L. Vogel. Wolff snowflakes. *Pacific J. Math.*, 218 (2005), no. 1, 139166.

Results of interest for p-harmonic measure

For general $p \neq 2$, we call μ as p-harmonic measure associated with a p-harmonic function.

[BL]: Björn Bennewitz and John Lewis. On the dimension of p-harmonic measure. *Ann. Acad. Sci. Fenn. Math.*, 30(2):459505, 2005.

[LNV]: John Lewis, Kaj Nyström, and Andrew Vogel. On the dimension of p-harmonic measure in space. *J. Eur. Math. Soc. (JEMS)* 15 (2013), no. 6, 21972256.

Results of interest for p-harmonic measure

For general $p \neq 2$, we call μ as p-harmonic measure associated with a p-harmonic function.

Bennewitz-Lewis: If $\partial\Omega \subset \mathbb{R}^2$ is a quasi circle in the plane then $\mathcal{H} - \dim\, \mu \geq 1$ when $1 < p < 2$ while $\mathcal{H} - \dim\, \mu \leq 1$ if $2 < p < \infty$. Moreover, strict inequality holds for $\mathcal{H} - \dim\, \mu$ when $\partial\Omega$ is the Von Koch snowflake.

[BL]: Björn Bennewitz and John Lewis. On the dimension of p-harmonic measure. *Ann. Acad. Sci. Fenn. Math.*, 30(2):459505, 2005.

[LNV]: John Lewis, Kaj Nyström, and Andrew Vogel. On the dimension of p-harmonic measure in space. *J. Eur. Math. Soc. (JEMS)* 15 (2013), no. 6, 21972256.

Results of interest for p-harmonic measure

For general $p \neq 2$, we call μ as p-harmonic measure associated with a p-harmonic function.

Bennewitz-Lewis: If $\partial\Omega \subset \mathbb{R}^2$ is a quasi circle in the plane then $\mathcal{H} - \dim \mu \geq 1$ when $1 < p < 2$ while $\mathcal{H} - \dim \mu \leq 1$ if $2 < p < \infty$. Moreover, strict inequality holds for $\mathcal{H} - \dim \mu$ when $\partial\Omega$ is the Von Koch snowflake.

Lewis-Nyström-Vogel:

- μ is concentrated on a set of $\sigma-$finite $\mathcal{H}^{n-1}$ measure when $\partial\Omega$ is sufficiently "flat" in the sense of Reifenberg and $p \geq n$.
- All examples produced by Wolff snowflake has $\mathcal{H} - \dim \mu < n - 1$ when $p \geq n$.
- There is a Wolff snowflake for which $\mathcal{H} - \dim \mu > n - 1$ when $p > 2$, near enough 2

[BL]: Björn Bennewitz and John Lewis. On the dimension of p-harmonic measure. *Ann. Acad. Sci. Fenn. Math.*, 30(2):459505, 2005.

[LNV]: John Lewis, Kaj Nyström, and Andrew Vogel. On the dimension of p-harmonic measure in space. *J. Eur. Math. Soc. (JEMS)* 15 (2013), no. 6, 21972256.

To state our recent work we need a notion of n capacity. If $K \subset \overline{B}(x, r)$ is a compact set, define n capacity of K as

$$Cap(K, B(x, 2r)) = \inf \int_{\mathbb{R}^n} |\nabla \psi|^n \mathrm{d}x$$

where the infimum is taken over all infinitely differentiable ψ with compact support in $B(x, 2r)$ and $\psi \equiv 1$ on K.

To state our recent work we need a notion of n capacity. If $K \subset \overline{B}(x, r)$ is a compact set, define n capacity of K as

$$Cap(K, B(x, 2r)) = \inf \int_{\mathbb{R}^n} |\nabla \psi|^n \mathrm{d}x$$

where the infimum is taken over all infinitely differentiable ψ with compact support in $B(x, 2r)$ and $\psi \equiv 1$ on K.

A compact set $E \subset \mathbb{R}^n$ is said to be locally (n, r_0) uniformly fat or locally uniformly (n, r_0) thick provided there exists $r_0, \beta > 0$ such that whenever $x \in E$, $0 < r \leq r_0$

$$Cap(E \cap \overline{B}(x, r), B(x, 2r)) \geq \beta.$$

Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$, $\rho > 0$.

Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$, $\rho > 0$.

Let $u > 0$ be p-harmonic in $O \cap B(\hat{z}, \rho)$ with continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$.

Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$, $\rho > 0$.

Let $u > 0$ be p-harmonic in $O \cap B(\hat{z}, \rho)$ with continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$.

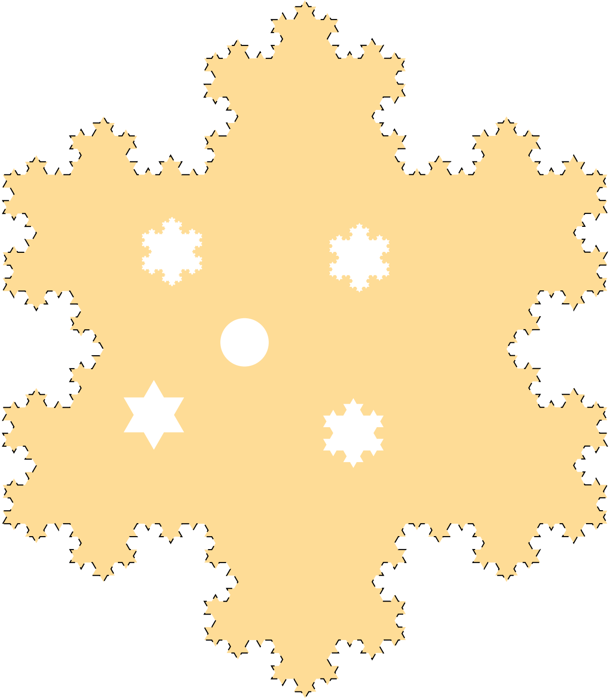
Extend u to all $B(\hat{z}, \rho)$ by defining $u \equiv 0$ on $B(\hat{z}, \rho) \setminus O$. Then u is p-harmonic in $B(\hat{z}, \rho)$.

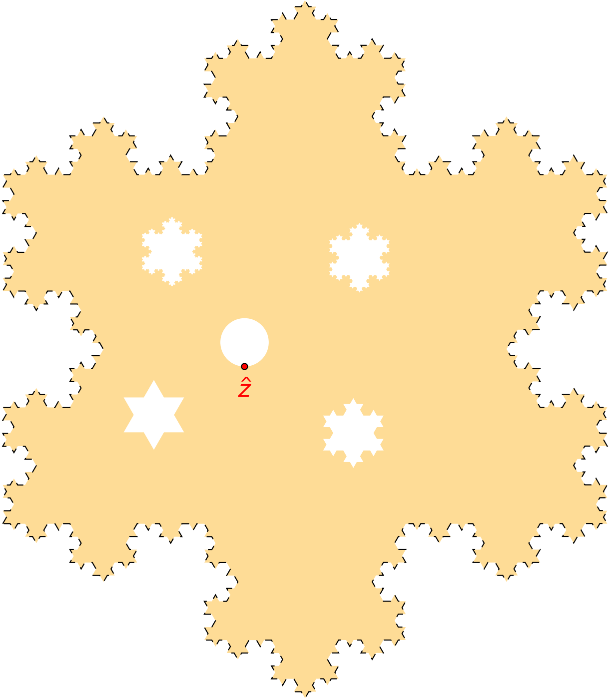
Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$, $\rho > 0$.

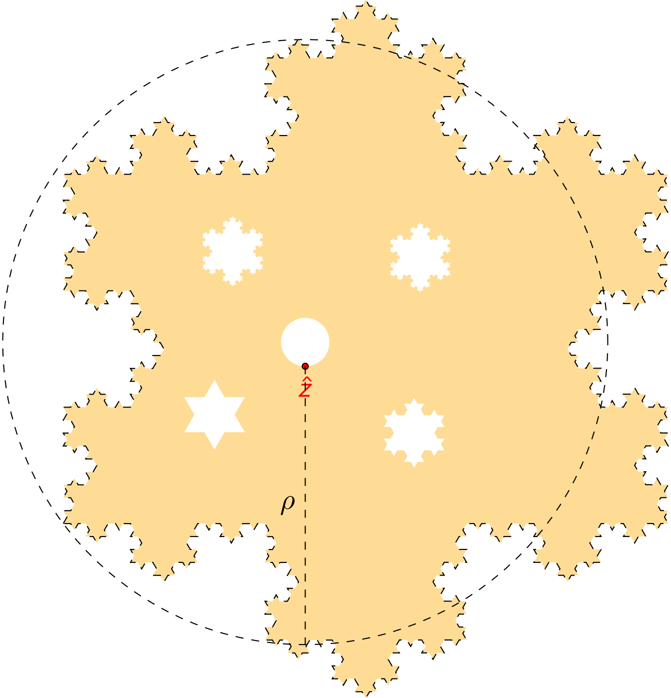
Let $u > 0$ be p-harmonic in $O \cap B(\hat{z}, \rho)$ with continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$.

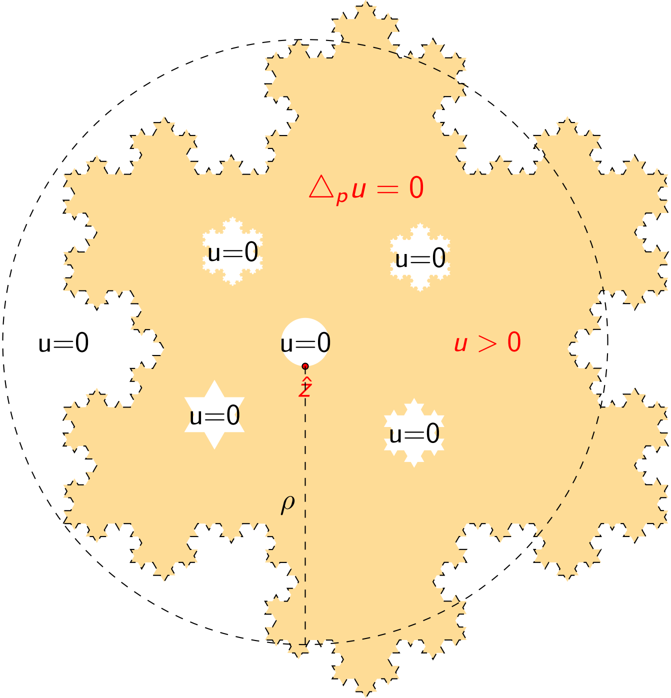
Extend u to all $B(\hat{z}, \rho)$ by defining $u \equiv 0$ on $B(\hat{z}, \rho) \setminus O$. Then u is p-harmonic in $B(\hat{z}, \rho)$.

Let μ be the p-harmonic measure associated with u.

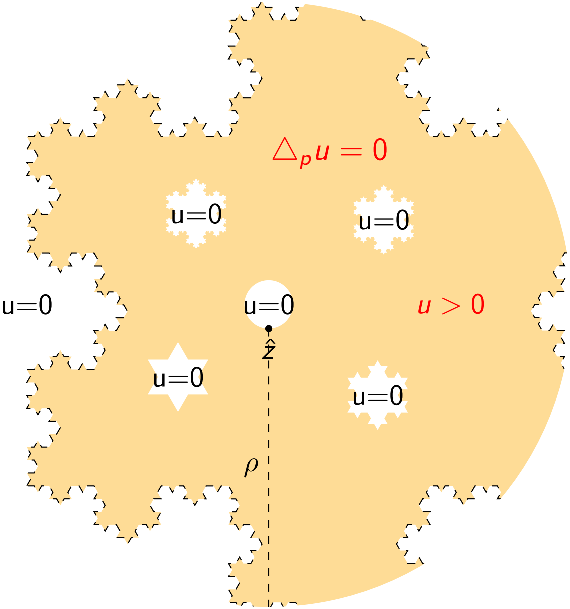



$\hat{Z}$


$\hat{z}$
ρ



$\triangle_p u = 0$
u=0
u=0
u=0
u=0
$u > 0$
$\hat{z}$
u=0
u=0
ρ



$\triangle_p u = 0$
u=0
u=0
u=0
u=0
$u > 0$
$\hat{z}$
u=0
u=0
ρ

New result for p-harmonic measure in space

Theorem A (A.-Lewis-Vogel)

If $p > n$ then μ is concentrated on a set of $\sigma-$finite $\mathcal{H}^{n-1}$ measure.

Same result holds when $p = n$ provided that $\partial O \cap B(\hat{z}, \rho)$ is locally uniformly fat in the sense of $n-$capacity.

[ALV]: Murat Akman, John Lewis, and Andrew Vogel, Hausdorff dimension and $\sigma-$finiteness of $p-$ harmonic measures in space when $p \geq n$. arXiv:1306.5617, submitted.

New result for p-harmonic measure in space

Theorem A (A.-Lewis-Vogel)

If $p > n$ then μ is concentrated on a set of $\sigma-$finite $\mathcal{H}^{n-1}$ measure.

Same result holds when $p = n$ provided that $\partial O \cap B(\hat{z}, \rho)$ is locally uniformly fat in the sense of $n-$capacity.

- $\mathcal{H} - \dim \mu \leq n - 1$ when $p \geq n$.

[ALV]: Murat Akman, John Lewis, and Andrew Vogel, Hausdorff dimension and $\sigma-$finiteness of $p-$ harmonic measures in space when $p \geq n$. arXiv:1306.5617, submitted.

New result for p-harmonic measure in space

Theorem A (A.-Lewis-Vogel)

If $p > n$ then μ is concentrated on a set of $\sigma-$finite $\mathcal{H}^{n-1}$ measure.

Same result holds when $p = n$ provided that $\partial O \cap B(\hat{z}, \rho)$ is locally uniformly fat in the sense of $n-$capacity.

- $\mathcal{H} - \dim \mu \leq n - 1$ when $p \geq n$.

The main idea for our proof comes from the 1993 paper of Wolff mentioned earlier.

[ALV]: Murat Akman, John Lewis, and Andrew Vogel, Hausdorff dimension and $\sigma-$finiteness of $p-$ harmonic measures in space when $p \geq n$. arXiv:1306.5617, submitted.

Some Remarks

- If $w \in \partial O$ and $B(w, 4r) \subset B(\hat{z}, \rho)$ then there exists $c = c(p, n) \geq 1$ with

$$\frac{1}{c} r^{p-n} \mu(B(w, r/2)) \leq \max_{B(w,r)} u^{p-1} \leq c r^{p-n} \mu(B(w, 2r)).$$

Some Remarks

- If $w \in \partial O$ and $B(w, 4r) \subset B(\hat{z}, \rho)$ then there exists $c = c(p, n) \geq 1$ with

$$\boxed{\frac{1}{c} r^{p-n} \mu(B(w, r/2)) \leq \max_{B(w,r)} u^{p-1} \leq c r^{p-n} \mu(B(w, 2r)).}$$

- The left-hand side is true for any open set O and $p \geq n$.

Some Remarks

- If $w \in \partial O$ and $B(w, 4r) \subset B(\hat{z}, \rho)$ then there exists $c = c(p, n) \geq 1$ with

$$\boxed{\frac{1}{c} r^{p-n} \mu(B(w, r/2)) \leq \max_{B(w,r)} u^{p-1} \leq c r^{p-n} \mu(B(w, 2r)).}$$

- The left-hand side is true for any open set O and $p \geq n$.

- The right-hand side requires uniform fatness assumption when $p = n$ and it is the only place this assumption is used.

Some Remarks

- If $w \in \partial O$ and $B(w, 4r) \subset B(\hat{z}, \rho)$ then there exists $c = c(p, n) \geq 1$ with

$$\boxed{\frac{1}{c} r^{p-n} \mu(B(w, r/2)) \leq \max_{B(w,r)} u^{p-1} \leq c r^{p-n} \mu(B(w, 2r)).}$$

- The left-hand side is true for any open set O and $p \geq n$.

- The right-hand side requires uniform fatness assumption when $p = n$ and it is the only place this assumption is used.

- **Conjecture**: Theorem A holds without uniform fatness assumption when $p = n$.

The tools we have used requires to find a PDE in divergence form for which u, u_{x_k} are both solutions and $\log|\nabla u|$ is a sub solution for $p \geq n$ at points where $\nabla u \neq 0$.

The tools we have used requires to find a PDE in divergence form for which u, u_{x_k} are both solutions and $\log|\nabla u|$ is a sub solution for $p \geq n$ at points where $\nabla u \neq 0$.

It is known that if

$$L\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}\zeta_j\right) \text{ where } b_{ij} = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2]$$

then

The tools we have used requires to find a PDE in divergence form for which u, u_{x_k} are both solutions and $\log|\nabla u|$ is a sub solution for $p \geq n$ at points where $\nabla u \neq 0$.

It is known that if

$$L\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (b_{ij}\zeta_j) \text{ where } b_{ij} = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2]$$

then

$$\min(p-1,1)|\xi|^2|\nabla u|^{p-2} \leq \sum_{i,k=1}^{n} b_{ik}\xi_i\xi_k \leq \max(1,p-1)|\nabla u|^{p-2}|\xi|^2$$

The tools we have used requires to find a PDE in divergence form for which u, u_{x_k} are both solutions and $\log|\nabla u|$ is a sub solution for $p \geq n$ at points where $\nabla u \neq 0$.

It is known that if

$$L\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}\zeta_j\right) \text{ where } b_{ij} = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2]$$

then

$$\min(p-1,1)|\xi|^2|\nabla u|^{p-2} \leq \sum_{i,k=1}^{n} b_{ik}\xi_i\xi_k \leq \max(1,p-1)|\nabla u|^{p-2}|\xi|^2$$

- $\zeta = u$ and $\zeta = u_{x_k}$ are both solutions for $k = 1, \ldots, n$ to $L\zeta = 0$.

The tools we have used requires to find a PDE in divergence form for which u, u_{x_k} are both solutions and $\log|\nabla u|$ is a sub solution for $p \geq n$ at points where $\nabla u \neq 0$.

It is known that if

$$L\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}\zeta_j) \text{ where } b_{ij} = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2]$$

then

$$\min(p-1,1)|\xi|^2|\nabla u|^{p-2} \leq \sum_{i,k=1}^{n} b_{ik}\xi_i\xi_k \leq \max(1,p-1)|\nabla u|^{p-2}|\xi|^2$$

- $\zeta = u$ and $\zeta = u_{x_k}$ are both solutions for $k = 1, \dots, n$ to $L\zeta = 0$.
- $\zeta = \log|\nabla u|$ is a sub solution to $L\zeta = 0$ when $p \geq n$ and $\nabla u \neq 0$.

The tools we have used requires to find a PDE in divergence form for which u, u_{x_k} are both solutions and $\log|\nabla u|$ is a sub solution for $p \geq n$ at points where $\nabla u \neq 0$.

It is known that if

$$L\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}\zeta_j) \text{ where } b_{ij} = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2]$$

then

$$\min(p-1,1)|\xi|^2|\nabla u|^{p-2} \leq \sum_{i,k=1}^{n} b_{ik}\xi_i\xi_k \leq \max(1,p-1)|\nabla u|^{p-2}|\xi|^2$$

- $\zeta = u$ and $\zeta = u_{x_k}$ are both solutions for $k = 1, \ldots, n$ to $L\zeta = 0$.
- $\zeta = \log|\nabla u|$ is a sub solution to $L\zeta = 0$ when $p \geq n$ and $\nabla u \neq 0$.
- Is $\log|\nabla u|$ a super solution when $p < n$ and $|\nabla u| \neq 0$?

The tools we have used requires to find a PDE in divergence form for which u, u_{x_k} are both solutions and $\log|\nabla u|$ is a sub solution for $p \geq n$ at points where $\nabla u \neq 0$.

It is known that if

$$L\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}\zeta_j\right) \text{ where } b_{ij} = |\nabla u|^{p-4}[(p-2)u_{x_i}u_{x_j} + \delta_{ij}|\nabla u|^2]$$

then

$$\min(p-1,1)|\xi|^2|\nabla u|^{p-2} \leq \sum_{i,k=1}^{n} b_{ik}\xi_i\xi_k \leq \max(1,p-1)|\nabla u|^{p-2}|\xi|^2$$

- $\zeta = u$ and $\zeta = u_{x_k}$ are both solutions for $k = 1, \ldots, n$ to $L\zeta = 0$.
- $\zeta = \log|\nabla u|$ is a sub solution to $L\zeta = 0$ when $p \geq n$ and $\nabla u \neq 0$.
- Is $\log|\nabla u|$ a super solution when $p < n$ and $|\nabla u| \neq 0$?
  - **Conjecture**: There is p_0, $2 < p_0 < n$, such that if $p_0 \leq p$ then $\mathcal{H} - \dim \mu \leq n - 1$.

Sketch of the Proof of Theorem A

Sketch of the Proof of Theorem A

Proposition

Let λ *be a non decreasing function on* $[0,1]$ *with*

$$\lim_{t\to 0}\frac{\lambda(t)}{t^{n-1}}=0.$$

There exists $c=c(p,n)$ *and a set* $Q\subset\partial O\cap B(\hat{z},\rho)$ *such that*

$$\boxed{\mu(\partial O\cap B(\hat{z},\rho)\setminus Q)=0}$$

and for every $w\in Q$ *there exists arbitrarily small* $r=r(w)>0$ *and a compact set* $F=F(w,r)$*such that*

$$\boxed{\mathcal{H}^{\lambda}(F)=0 \text{ and } \mu(B(w,100r))\leq c\mu(F).}$$

Sketch of the Proof of Theorem A

Proposition

Let λ *be a non decreasing function on* $[0,1]$ *with*

$$\lim_{t\to 0}\frac{\lambda(t)}{t^{n-1}}=0.$$

There exists $c=c(p,n)$ *and a set* $Q\subset\partial O\cap B(\hat{z},\rho)$ *such that*

$$\boxed{\mu(\partial O\cap B(\hat{z},\rho)\setminus Q)=0}$$

and for every $w\in Q$ *there exists arbitrarily small* $r=r(w)>0$ *and a compact set* $F=F(w,r)$*such that*

$$\boxed{\mathcal{H}^{\lambda}(F)=0 \text{ and } \mu(B(w,100r))\leq c\mu(F).}$$

We first show how our result follows from this proposition.

- First observation: $\mathcal{H}^{n-1}(P_m) < \infty$ for each positive integer m where

$$P_m := \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{t \to 0} \frac{\mu(B(x,t))}{t^{n-1}} > \frac{1}{m} \right\}.$$

- First observation: $\mathcal{H}^{n-1}(P_m) < \infty$ for each positive integer m where

$$P_m := \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{t \to 0} \frac{\mu(B(x,t))}{t^{n-1}} > \frac{1}{m} \right\}.$$

Therefore, this set has

$$P_= \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{t \to 0} \frac{\mu(B(x,t))}{t^{n-1}} > 0 \right\}.$$

has $\sigma-$finite $\mathcal{H}^{n-1}$ measure.

- First observation: $\mathcal{H}^{n-1}(P_m) < \infty$ for each positive integer m where

$$P_m := \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{t \to 0} \frac{\mu(B(x,t))}{t^{n-1}} > \frac{1}{m} \right\}.$$

Therefore, this set has

$$P_= \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{t \to 0} \frac{\mu(B(x,t))}{t^{n-1}} > 0 \right\}.$$

has $\sigma-$finite $\mathcal{H}^{n-1}$ measure.

- Second observation: From Proposition and measure theoretic arguments there exists a Borel set $Q_1 \subset Q$ with

$$\mu(\partial O \cap B(\hat{z}, \rho) \setminus Q_1) = 0 \text{ and } \mathcal{H}^{\lambda}(Q_1) = 0.$$

- Third observation: $\mu(Q \setminus P) = 0$.

- Third observation: $\mu(Q \setminus P) = 0$.

Otherwise, there is a compact set $K \subset Q \setminus P$ and a positive non decreasing λ_0 with $\lim_{t \to 0} \frac{\lambda_0(t)}{t^{n-1}} = 0$ satisfying

$$\boxed{\mu(K) > 0 \text{ and } \lim_{t \to 0} \frac{\mu(B(x,t))}{\lambda_0(t)} = 0 \text{ uniformly for } x \in K.}$$

- Third observation: $\mu(Q \setminus P) = 0$.

Otherwise, there is a compact set $K \subset Q \setminus P$ and a positive non decreasing λ_0 with $\lim_{t \to 0} \frac{\lambda_0(t)}{t^{n-1}} = 0$ satisfying

$$\mu(K) > 0 \text{ and } \lim_{t \to 0} \frac{\mu(B(x,t))}{\lambda_0(t)} = 0 \text{ uniformly for } x \in K.$$

This tells us that $\mu \ll \mathcal{H}^{\lambda_0}$ on K. Choose Q_1 relative to λ_0 to conclude that $\mathcal{H}^{\lambda_0}(K \cap Q_1) = 0$ implies $\mu(K \cap Q_1) = \mu(K) = 0$ $\lightning$.

- Third observation: $\mu(Q \setminus P) = 0$.

Otherwise, there is a compact set $K \subset Q \setminus P$ and a positive non decreasing λ_0 with $\lim_{t \to 0} \frac{\lambda_0(t)}{t^{n-1}} = 0$ satisfying

$$\mu(K) > 0 \text{ and } \lim_{t \to 0} \frac{\mu(B(x,t))}{\lambda_0(t)} = 0 \,\text{uniformly for}\, x \in K.$$

This tells us that $\mu \ll \mathcal{H}^{\lambda_0}$ on K. Choose Q_1 relative to λ_0 to conclude that $\mathcal{H}^{\lambda_0}(K \cap Q_1) = 0$ implies $\mu(K \cap Q_1) = \mu(K) = 0$ $\lightning$.

- μ is concentrated on P which has $\sigma-$finite $\mathcal{H}^{n-1}$ measure. This finishes the proof of our result assuming Proposition.

Sketch of the Proof of Proposition

- Translation, dilation invariance of the p-Laplacian and a measure theoretic argument to reduce the proof of Proposition to the situation when $w = 0$, $B(0, 100) \subset B(\hat{z}, \rho)$.

Sketch of the Proof of Proposition

- Translation, dilation invariance of the p-Laplacian and a measure theoretic argument to reduce the proof of Proposition to the situation when $w = 0,\ B(0, 100) \subset B(\hat{z}, \rho)$.

- There is some c = c(p, n) and $2 \leq t \leq 50$ such that

$$\boxed{\frac{1}{c} \leq \mu(B(0,1)) \leq \max_{B(0,2)} u \leq \max_{B(0,t)} u \leq c\mu(B(0,100)) \leq c^2.}$$

Sketch of the Proof of Proposition

- Translation, dilation invariance of the p-Laplacian and a measure theoretic argument to reduce the proof of Proposition to the situation when $w = 0$, $B(0, 100) \subset B(\hat{z}, \rho)$.

- There is some c = c(p, n) and $2 \leq t \leq 50$ such that

$$\frac{1}{c} \leq \mu(B(0,1)) \leq \max_{B(0,2)} u \leq \max_{B(0,t)} u \leq c\mu(B(0,100)) \leq c^2.$$

To finish the proof of Proposition, it suffices to show for given small $\epsilon, \tau > 0$ that there exists a Borel set $E \subset \partial O \cap B(0, 20)$ and $c = c(p, n) \geq 1$ with

$$\phi_\tau^\lambda(E) \leq \epsilon \text{ and } \mu(E) \geq \frac{1}{c}.$$

A stopping time argument

- Let M a large positive number and $s < e^{-M}$.

For each $z \in \partial O \cap B(0, 15)$ there is $t = t(z)$, $0 < t < 1$ with either

(α) $\mu(B(z, t)) = Mt^{n-1}, t > s$

or

(β) $t = s$.

A stopping time argument

- Let M a large positive number and $s < e^{-M}$.

For each $z \in \partial O \cap B(0, 15)$ there is $t = t(z)$, $0 < t < 1$ with either

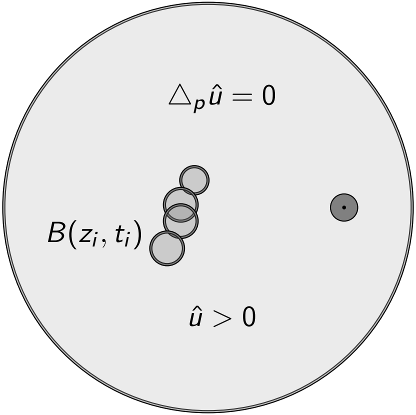
(α) $\mu(B(z, t)) = Mt^{n-1}, t > s$

or

(β) $t = s$.

- Use the Besicovitch covering theorem to get a covering $B(z_j, t_j)_1^N$ of $\partial O \cap B(0, 15)$ where $t_j = t(z_j)$ is the maximal for which either (α) or (β) holds.

$$\Omega := O \cap B(0,15) \setminus \bigcup_{i=1}^{N} \overline{B}(z_i, t_i) \text{ and } D := \Omega \setminus \overline{B}(\tilde{z}, 2r_1)$$



- Let $\hat{u}$ be the p-harmonic function in D with continuous boundary values, $\hat{u} = \min_{\overline{B}(\tilde{z}, 2r_1)} u$ on $\partial \overline{B}(\tilde{z}, 2r_1)$ and $\hat{u} = 0$ on $\partial\Omega$. Let $\hat{\mu}$ be the p-harmonic measure associated with $\hat{u}$.

- $\hat{u} \le u$ in D.

- $\hat{u} \leq u$ in D.

- $\partial\Omega$ is smooth except for a set of finite $\mathcal{H}^{n-2}$

- $\hat{u} \leq u$ in D.

- $\partial\Omega$ is smooth except for a set of finite $\mathcal{H}^{n-2}$

Using some barrier type estimate one can also show

$$\boxed{|\nabla \hat{u}| \leq cM^{\frac{1}{p-1}} \text{ in } D}$$

- $\hat{u} \leq u$ in D.

- $\partial\Omega$ is smooth except for a set of finite $\mathcal{H}^{n-2}$

Using some barrier type estimate one can also show

$$\boxed{|\nabla \hat{u}| \leq c M^{\frac{1}{p-1}} \text{ in } D}$$

and

$$\boxed{t_j^{1-n} \hat{\mu}(\overline{B}(z_j, t_j)) \leq c t_j^{1-p} \max_{B(z_j, 2t_j)} u^{p-1} \leq c^2 t_j^{1-n} \mu(B(z_j, 4t_j)).}$$

For a given $A >> 1$, $\{1, \ldots, N\}$ can be divided into disjoint subsets $\mathcal{G}, \mathcal{B}, \mathcal{U}$ as

$$\begin{cases} \mathcal{G} := \{j : \ t_j > s\} \\ \mathcal{B} := \{j : \ t_j = s \text{ and } |\nabla \hat{u}|^{p-1} \geq M^{-A} \text{ for some } x \in \partial\Omega \cap \partial B(z_j, t_j)\} \\ \mathcal{U} := \{j : \ j \text{ is not in } \mathcal{G} \text{ or } \mathcal{B}\} \end{cases}$$

For a given $A >> 1$, $\{1, \dots, N\}$ can be divided into disjoint subsets $\mathcal{G}, \mathcal{B}, \mathcal{U}$ as

$$\begin{cases} \mathcal{G} := \{j : \ t_j > s\} \\ \mathcal{B} := \{j : \ t_j = s \text{ and } |\nabla \hat{u}|^{p-1} \geq M^{-A} \text{ for some } x \in \partial\Omega \cap \partial B(z_j, t_j)\} \\ \mathcal{U} := \{j : \ j \text{ is not in } \mathcal{G} \text{ or } \mathcal{B}\} \end{cases}$$

We define

$$E := \partial O \cap \bigcup_{j \in \mathcal{G} \cup \mathcal{B}} B(z_j, t_j)$$

For a given $A >> 1$, $\{1, \ldots, N\}$ can be divided into disjoint subsets $\mathcal{G}, \mathcal{B}, \mathcal{U}$ as

$$\begin{cases} \mathcal{G} := \{j : \ t_j > s\} \\ \mathcal{B} := \{j : \ t_j = s \text{ and } |\nabla \hat{u}|^{p-1} \geq M^{-A} \text{ for some } x \in \partial\Omega \cap \partial B(z_j, t_j)\} \\ \mathcal{U} := \{j : \ j \text{ is not in } \mathcal{G} \text{ or } \mathcal{B}\} \end{cases}$$

We define

$$E := \partial O \cap \bigcup_{j \in \mathcal{G} \cup \mathcal{B}} B(z_j, t_j)$$

- Easy to show $\phi_\tau^\lambda(E) \leq \epsilon$

We show

$$\boxed{\int\limits_{\partial\Omega} |\nabla \hat{u}|^{p-1} \left|\log |\nabla \hat{u}|\right| \mathrm{d}\mathcal{H}^{n-1} \leq c' \log M}$$

We show

$$\boxed{\int_{\partial\Omega} |\nabla \hat{u}|^{p-1} \left|\log |\nabla \hat{u}|\right| \mathrm{d}\mathcal{H}^{n-1} \le c' \log M}$$

- We use this to show

$$\begin{aligned}\hat{\mu}(\partial\Omega \cap \bigcup_{j\in\mathcal{U}} \overline{B}(z_j, t_j)) &\le \hat{\mu}(\{x \in \partial\Omega : \, |\nabla \hat{u}(x)|^{p-1} \le M^{-A}\}) \\ &\le \frac{(p-1)}{(AlogM)} \int_{\partial\Omega} |\nabla \hat{u}|^{p-1} \left|\log |\nabla \hat{u}|\right| \mathrm{d}\mathcal{H}^{n-1} \le \frac{c}{A}\end{aligned}$$

We show

$$\boxed{\int_{\partial\Omega} |\nabla \hat{u}|^{p-1} \left|\log |\nabla \hat{u}|\right| \mathrm{d}\mathcal{H}^{n-1} \leq c' \log M}$$

- We use this to show

$$\begin{aligned}\hat{\mu}(\partial\Omega \cap \bigcup_{j\in\mathcal{U}} \overline{B}(z_j, t_j)) &\leq \hat{\mu}(\{x \in \partial\Omega : \, |\nabla \hat{u}(x)|^{p-1} \leq M^{-A}\}) \\ &\leq \frac{(p-1)}{(AlogM)} \int_{\partial\Omega} |\nabla \hat{u}|^{p-1} \left|\log |\nabla \hat{u}|\right| \mathrm{d}\mathcal{H}^{n-1} \leq \frac{c}{A}\end{aligned}$$

- A is ours to choose, and we choose it very large to make this set small.

We show

$$\boxed{\int_{\partial\Omega} |\nabla \hat{u}|^{p-1} \left|\log |\nabla \hat{u}|\right| \mathrm{d}\mathcal{H}^{n-1} \leq c' \log M}$$

- We use this to show

$$\boxed{\begin{aligned} \hat{\mu}(\partial\Omega \cap \bigcup_{j\in\mathcal{U}} \overline{B}(z_j, t_j)) &\leq \hat{\mu}(\{x \in \partial\Omega : \, |\nabla \hat{u}(x)|^{p-1} \leq M^{-A}\}) \\ &\leq \frac{(p-1)}{(AlogM)} \int_{\partial\Omega} |\nabla \hat{u}|^{p-1} \left|\log |\nabla \hat{u}|\right| \mathrm{d}\mathcal{H}^{n-1} \leq \frac{c}{A} \end{aligned}}$$

- A is ours to choose, and we choose it very large to make this set small.

- Use this to prove $\mu(E) \geq 1/c$.

Part II: Example of domain in $\mathbb{R}^n$ for which $\mathcal{H} - \dim \mu < n - 1$

[GM]: John B. Garnett and Donald E. Marshall, Harmonic Measure, volume 2 of New Mathematical Monographs. *Cambridge University Press*, Cambridge, 2008.

Part II: Example of domain in $\mathbb{R}^n$ for which $\mathcal{H} - \dim \mu < n-1$

There is an unpublished result of Jones-Wolff in [GM, Chapter IX, Theorem 3.1];

- **Jones-Wolff**: Let $\Omega = \mathbb{C} \cup \{\infty\} \setminus C$ where C is a certain compact set. Then $\mathcal{H} - \dim \omega < 1$.

[GM]: John B. Garnett and Donald E. Marshall, Harmonic Measure, volume 2 of New Mathematical Monographs. *Cambridge University Press*, Cambridge, 2008.

Part II: Example of domain in $\mathbb{R}^n$ for which $\mathcal{H} - \dim \mu < n - 1$

There is an unpublished result of Jones-Wolff in [GM, Chapter IX, Theorem 3.1];

- **Jones-Wolff**: Let $\Omega = \mathbb{C} \cup \{\infty\} \setminus C$ where C is a certain compact set. Then $\mathcal{H} - \dim \omega < 1$.

We generalized this result to p-harmonic measure, μ, in $\mathbb{R}^n$ for $p \geq n \geq 2$ and for a certain domain.

[GM]: John B. Garnett and Donald E. Marshall, Harmonic Measure, volume 2 of New Mathematical Monographs. *Cambridge University Press*, Cambridge, 2008.

Let S' be the square with side length $1/2$ and center 0 in $\mathbb{R}^n$. Let $\mathcal{C}_0 = S'$.

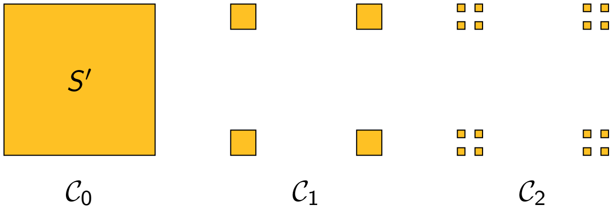
Let S' be the square with side length $1/2$ and center 0 in $\mathbb{R}^n$. Let $\mathcal{C}_0 = S'$.

Let $Q_{11}, \ldots, Q_{14}$ be the squares of the four corners of $\mathcal{C}_0$ of side length a_1, $0 < \alpha < a_1 < \beta < 1/4$, and let $\mathcal{C}_1 = \bigcup_{i=1}^{4} Q_{1i}$.

Let S' be the square with side length $1/2$ and center 0 in $\mathbb{R}^n$. Let $\mathcal{C}_0 = S'$.

Let $Q_{11}, \ldots, Q_{14}$ be the squares of the four corners of $\mathcal{C}_0$ of side length a_1, $0 < \alpha < a_1 < \beta < 1/4$, and let $\mathcal{C}_1 = \bigcup_{i=1}^{4} Q_{1i}$.

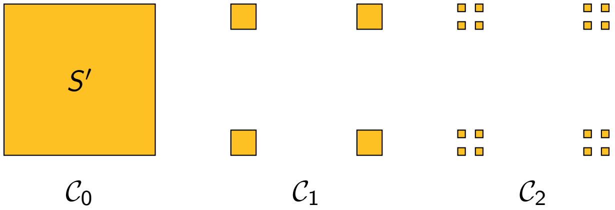
Let $\{Q_{2j}\}$, $j = 1, \ldots, 16$ be the square of corners of each Q_{1i}, $i = 1, \ldots, 4$ of side length $a_1 a_2$, $\alpha < a_2 < \beta$. Let $\mathcal{C}_2 = \bigcup_{j=1}^{16} Q_{2j}$.



Let S' be the square with side length $1/2$ and center 0 in $\mathbb{R}^n$. Let $\mathcal{C}_0 = S'$.

Let $Q_{11}, \ldots, Q_{14}$ be the squares of the four corners of $\mathcal{C}_0$ of side length a_1, $0 < \alpha < a_1 < \beta < 1/4$, and let $\mathcal{C}_1 = \bigcup_{i=1}^{4} Q_{1i}$.

Let $\{Q_{2j}\}$, $j = 1, \ldots, 16$ be the square of corners of each Q_{1i}, $i = 1, \ldots, 4$ of side length $a_1 a_2$, $\alpha < a_2 < \beta$. Let $\mathcal{C}_2 = \bigcup_{j=1}^{16} Q_{2j}$.



Continuing recursively, at the mth step we get 4^m squares Q_{mj}, $1 \leq j \leq 4^m$ of side length $a_1 a_2 \ldots a_m$, $\alpha < a_m < \beta$ and let $\mathcal{C}_m = \bigcup_{j=1}^{4^m} Q_{mj}$.

Then $\mathcal{C}$ is obtained as the limit in the Hausdorff metric of $\mathcal{C}_m$ as $m \to \infty$

Let $\mathcal{S} = 2S' \subset \mathbb{R}^n$ and let u be a p-harmonic function in $\mathcal{S} \setminus \mathcal{C}$ with boundary values $u = 1$ on $\partial\mathcal{S}$ and $u = 0$ on $\mathcal{C}$. Let μ be the p-harmonic measure associated to u.

Let $\mathcal{S} = 2S' \subset \mathbb{R}^n$ and let u be a p-harmonic function in $\mathcal{S} \setminus \mathcal{C}$ with boundary values $u = 1$ on $\partial\mathcal{S}$ and $u = 0$ on $\mathcal{C}$. Let μ be the p-harmonic measure associated to u.

Following Jones-Wolff argument, using sub solution estimates, a stopping time argument similar to the one we have used we show that

Let $\mathcal{S} = 2S' \subset \mathbb{R}^n$ and let u be a p-harmonic function in $\mathcal{S} \setminus \mathcal{C}$ with boundary values $u = 1$ on $\partial\mathcal{S}$ and $u = 0$ on $\mathcal{C}$. Let μ be the p-harmonic measure associated to u.

Following Jones-Wolff argument, using sub solution estimates, a stopping time argument similar to the one we have used we show that

Theorem B (A.-Lewis-Vogel)

$\mathcal{H} - dim\, \mu < n - 1 \;\; when \;\; p \geq n.$

Is there any other measure or PDE that one can study the same problem?

[BL]: Björn Bennewitz and John Lewis. On the dimension of p-harmonic measure. *Ann. Acad. Sci. Fenn. Math.*, 30(2):459505, 2005.

Is there any other measure or PDE that one can study the same problem?

In [HKM, Chapter 21], it was shown that the measure associated with a positive weak solution u with 0 boundary values for a larger class of qusailinear elliptic PDEs exists;

$$\boxed{\text{div } \mathcal{A}(x, \nabla u) = 0}$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies certain structural assumptions.

The measure is so called $\mathcal{A}$-harmonic measure.

[BL]: Björn Bennewitz and John Lewis. On the dimension of p-harmonic measure. *Ann. Acad. Sci. Fenn. Math.*, 30(2):459505, 2005.

Is there any other measure or PDE that one can study the same problem?

In [HKM, Chapter 21], it was shown that the measure associated with a positive weak solution u with 0 boundary values for a larger class of qusailinear elliptic PDEs exists;

$$\boxed{\text{div } \mathcal{A}(x, \nabla u) = 0}$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies certain structural assumptions.

The measure is so called $\mathcal{A}$-harmonic measure.

If $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$, then the above PDE becomes the usual p-Laplace equation.

[BL]: Björn Bennewitz and John Lewis. On the dimension of p-harmonic measure. *Ann. Acad. Sci. Fenn. Math.*, 30(2):459505, 2005.

Is there any other measure or PDE that one can study the same problem?

In [HKM, Chapter 21], it was shown that the measure associated with a positive weak solution u with 0 boundary values for a larger class of qusailinear elliptic PDEs exists;

$$\boxed{\operatorname{div} \mathcal{A}(x, \nabla u) = 0}$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies certain structural assumptions.

The measure is so called $\mathcal{A}$-harmonic measure.

If $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$, then the above PDE becomes the usual p-Laplace equation.

In [BL, Closing remarks 10], the authors pointed out this fact and asked for what PDE one can obtain dimension estimates on the associated measure.

[BL]: Björn Bennewitz and John Lewis. On the dimension of p-harmonic measure. *Ann. Acad. Sci. Fenn. Math.*, 30(2):459505, 2005.

Is there any other measure or PDE that one can study the same problem?

In [HKM, Chapter 21], it was shown that the measure associated with a positive weak solution u with 0 boundary values for a larger class of qusailinear elliptic PDEs exists;

$$\boxed{\operatorname{div} \mathcal{A}(x, \nabla u) = 0}$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies certain structural assumptions.

The measure is so called $\mathcal{A}$-harmonic measure.

If $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$, then the above PDE becomes the usual p-Laplace equation.

In [BL, Closing remarks 10], the authors pointed out this fact and asked for what PDE one can obtain dimension estimates on the associated measure.

$$\{\text{Laplace}\} \subseteq \{\text{p-Laplace}\} \qquad\qquad \subseteq \{\mathcal{A} - \text{Harmonic PDEs}\}.$$

[BL]: Björn Bennewitz and John Lewis. On the dimension of p-harmonic measure. *Ann. Acad. Sci. Fenn. Math.*, 30(2):459505, 2005.

Is there any other measure or PDE that one can study the same problem?

In [HKM, Chapter 21], it was shown that the measure associated with a positive weak solution u with 0 boundary values for a larger class of qusailinear elliptic PDEs exists;

$$\boxed{\text{div } \mathcal{A}(x, \nabla u) = 0}$$

where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies certain structural assumptions.

The measure is so called $\mathcal{A}$-harmonic measure.

If $\mathcal{A}(\xi) = |\xi|^{p-2}\xi$, then the above PDE becomes the usual p-Laplace equation.

In [BL, Closing remarks 10], the authors pointed out this fact and asked for what PDE one can obtain dimension estimates on the associated measure.

$$\{\text{Laplace}\} \subseteq \{\text{p-Laplace}\} \subseteq \{\triangle_f u = 0\} \subseteq \{\mathcal{A} - \text{Harmonic PDEs}\}.$$

[BL]: Björn Bennewitz and John Lewis. On the dimension of p-harmonic measure. *Ann. Acad. Sci. Fenn. Math.*, 30(2):459505, 2005.

Introduction

Let p be fixed, $1 < p < \infty$. Let f be a function with following properties;

Introduction

Let p be fixed, $1 < p < \infty$. Let f be a function with following properties;

(a) $f : \mathbb{R}^n \to (0, \infty)$ is homogeneous of degree p.

That is, $f(\eta) = |\eta|^p f(\frac{\eta}{|\eta|}) > 0$ when $\eta \in \mathbb{R}^n \setminus \{0\}$.

Introduction

Let p be fixed, $1 < p < \infty$. Let f be a function with following properties;

(a) $f : \mathbb{R}^n \to (0, \infty)$ is homogeneous of degree p.

That is, $f(\eta) = |\eta|^p f(\frac{\eta}{|\eta|}) > 0$ when $\eta \in \mathbb{R}^n \setminus \{0\}$.

(b) f is uniformly convex in $B(0,1) \setminus B(0,1/2)$.

That is, Df is Lipschitz and $\exists c \geq 1$ such that for a.e. $\eta \in \mathbb{R}^n$,
$\frac{1}{2} < |\eta| < 1$ and all $\xi \in \mathbb{R}^n$ we have $c^{-1}|\xi|^2 \leq \sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial \eta_j \eta_k}(\eta)\xi_j\xi_k \leq c|\xi|^2$.

We consider weak solutions, u, to the Euler Lagrange equation;

$$\boxed{\triangle_f u := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial f(\nabla u)}{\partial \eta_i}\right) = 0}$$

We consider weak solutions, u, to the Euler Lagrange equation;

$$\triangle_f u := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial f(\nabla u)}{\partial \eta_i} \right) = 0$$

in $\Omega \cap N$ where N is an open neighborhood of $\partial\Omega$. Assume also that $u > 0$ in $N \cap \Omega$ with continuous boundary values on $\partial\Omega$. Set $u \equiv 0$ in $N \setminus \Omega$ to have $u \in W^{1,p}(N)$ and $\triangle_f u = 0$ weakly in N. Then, there exists a unique finite positive Borel measure μ_f associated with u having support contained in $\partial\Omega$ satisfying

$$\int \langle \nabla_\eta f(\nabla u), \nabla \phi \rangle \mathrm{d}x = -\int \phi \, \mathrm{d}\mu_f \text{ whenever } \phi \in C_0^\infty(N).$$

We consider weak solutions, u, to the Euler Lagrange equation;

$$\boxed{\triangle_f u := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial f(\nabla u)}{\partial \eta_i}\right) = 0}$$

in $\Omega \cap N$ where N is an open neighborhood of $\partial\Omega$. Assume also that $u > 0$ in $N \cap \Omega$ with continuous boundary values on $\partial\Omega$. Set $u \equiv 0$ in $N \setminus \Omega$ to have $u \in W^{1,p}(N)$ and $\triangle_f u = 0$ weakly in N. Then, there exists a unique finite positive Borel measure μ_f associated with u having support contained in $\partial\Omega$ satisfying

$$\boxed{\int \langle \nabla_\eta f(\nabla u), \nabla \phi \rangle \mathrm{d}x = -\int \phi \, \mathrm{d}\mu_f \text{ whenever } \phi \in C_0^\infty(N).}$$

- $f(\eta) = |\eta|^2 \to$ Laplace equation, $\triangle u = 0$.
- $f(\eta) = |\eta|^p$, $1 < p < \infty \to$ p-Laplace equation, $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$.

If we define

$$\boxed{L\zeta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(f_{\eta_i \eta_j} \zeta_j\right)}$$

Then

If we define

$$\boxed{L\zeta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(f_{\eta_i \eta_j} \zeta_j \right)}$$

Then

- $\zeta = u$ is a weak solution to $\mathcal{L}\zeta = 0$

If we define

$$\boxed{L\zeta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(f_{\eta_i\eta_j}\zeta_j\right)}$$

Then

- $\zeta = u$ is a weak solution to $\mathcal{L}\zeta = 0$
- $\zeta = u_{x_k}$ for $k = 1, \ldots, n$ is weak solution to $\mathcal{L}\zeta = 0$

If we define

$$\boxed{L\zeta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(f_{\eta_i\eta_j}\zeta_j\right)}$$

Then

- $\zeta = u$ is a weak solution to $\mathcal{L}\zeta = 0$
- $\zeta = u_{x_k}$ for $k = 1, \ldots, n$ is weak solution to $\mathcal{L}\zeta = 0$
- $\zeta = \log f(\nabla u)$ is a weak sub solution and weak solution to $\mathcal{L}\zeta = 0$

respectively when $p > n$ and $p = n$.

If we define

$$\boxed{L\zeta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(f_{\eta_i \eta_j} \zeta_j \right)}$$

Then

- $\zeta = u$ is a weak solution to $\mathcal{L}\zeta = 0$
- $\zeta = u_{x_k}$ for $k = 1, \dots, n$ is weak solution to $\mathcal{L}\zeta = 0$
- $\zeta = \log f(\nabla u)$ is a weak sub solution and weak solution to $\mathcal{L}\zeta = 0$

respectively when $p > n$ and $p = n$.

Using this sub solution estimate and following arguments we have used for p harmonic measure we show that

Theorem C (A.-Lewis-Vogel)

Theorem A and Theorem B hold for μ_f.

When $n = 2$ and $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain then

---

[M]: Nikolai Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc.*, 51(2):369384, 1985.

[LNP]: John Lewis, Kaj Nyström, and Pietro Poggi-Corradini. p-harmonic measure in simply connected domains. *Ann. Inst. Fourier Grenoble*, 61(2):689715, 2011.

When $n = 2$ and $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain then

- **Makarov**: $\omega \ll \mathcal{H}^\lambda$ where $\lambda(r) := r\,\exp\{A\sqrt{\log 1/r \log\log\log 1/r}\}$ if A is large.

[M]: Nikolai Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc.*, 51(2):369384, 1985.

[LNP]: John Lewis, Kaj Nyström, and Pietro Poggi-Corradini. p-harmonic measure in simply connected domains. *Ann. Inst. Fourier Grenoble*, 61(2):689715, 2011.

When $n = 2$ and $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain then

- **Makarov**: $\omega \ll \mathcal{H}^\lambda$ where $\lambda(r) := r \exp\{A\sqrt{\log 1/r \log\log\log 1/r}\}$ if A is large.

$$\boxed{\mathcal{H} - \dim \omega = 1.}$$

[M]: Nikolai Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc.*, 51(2):369384, 1985.

[LNP]: John Lewis, Kaj Nyström, and Pietro Poggi-Corradini. p-harmonic measure in simply connected domains. *Ann. Inst. Fourier Grenoble*, 61(2):689715, 2011.

When $n = 2$ and $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain then

- **Makarov**: $\omega \ll \mathcal{H}^{\lambda}$ where $\lambda(r) := r \exp\{A\sqrt{\log 1/r \log\log\log 1/r}\}$ if A is large.

$$\boxed{\mathcal{H} - \dim \omega = 1.}$$

- **Lewis-Nyström-Poggi Corradini**: Let $\hat{\lambda}(r) := r \exp\{A\sqrt{\log 1/r \ \log\log 1/r}\}$.
  a) $\mu_p \ll \mathcal{H}^{\hat{\lambda}}$ when $1 < p < 2$ for some $A = A(p) \geq 1$.
  b) μ_p is concentrated on a set of $\sigma-$finite $\mathcal{H}^{\hat{\lambda}}$ when $2 < p < \infty$ for some $A = A(p) \leq -1$.

[M]: Nikolai Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc.*, 51(2):369384, 1985.

[LNP]: John Lewis, Kaj Nyström, and Pietro Poggi-Corradini. p-harmonic measure in simply connected domains. *Ann. Inst. Fourier Grenoble*, 61(2):689715, 2011.

When $n = 2$ and $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain then

- **Makarov**: $\omega \ll \mathcal{H}^{\lambda}$ where $\lambda(r) := r \exp\{A\sqrt{\log 1/r \log\log\log 1/r}\}$ if A is large.

$$\boxed{\mathcal{H} - \dim \omega = 1.}$$

- **Lewis-Nyström-Poggi Corradini**: Let $\hat{\lambda}(r) := r \exp\{A\sqrt{\log 1/r \, \log\log 1/r}\}$.

  a) $\mu_p \ll \mathcal{H}^{\hat{\lambda}}$ when $1 < p < 2$ for some $A = A(p) \geq 1$.

  b) μ_p is concentrated on a set of $\sigma-$finite $\mathcal{H}^{\hat{\lambda}}$ when $2 < p < \infty$ for some $A = A(p) \leq -1$.

[M]: Nikolai Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc.*, 51(2):369384, 1985.

[LNP]: John Lewis, Kaj Nyström, and Pietro Poggi-Corradini. p-harmonic measure in simply connected domains. *Ann. Inst. Fourier Grenoble*, 61(2):689715, 2011.

- **Lewis**:

a) If $1 < p < 2$, then $\mu \ll \mathcal{H}^{\lambda}$ for $A = A(p)$ sufficiently large.

b) If $2 < p < \infty$, then μ is concentrated on a set of $\sigma-$finite $\mathcal{H}^{1}$.

[L]: John Lewis. p-harmonic measure in simply connected domains revisited. *Trans. Amer. Math. Soc.*, 367 (2015), no. 3, 15431583.

- **Lewis**:

a) If $1 < p < 2$, then $\mu \ll \mathcal{H}^{\lambda}$ for $A = A(p)$ sufficiently large.

b) If $2 < p < \infty$, then μ is concentrated on a set of $\sigma-$finite $\mathcal{H}^1$.

$$\mathcal{H} - \dim \mu \begin{cases} \geq 1 & \text{when } 1 < p < 2, \\ \leq 1 & \text{when } 2 < p < \infty. \end{cases}$$

[L]: John Lewis. p-harmonic measure in simply connected domains revisited. *Trans. Amer. Math. Soc.*, 367 (2015), no. 3, 15431583.

If we define

$$\boxed{L\zeta = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(f_{\eta_i \eta_j} \zeta_j \right)}$$

Then

If we define

$$\boxed{L\zeta = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(f_{\eta_i \eta_j} \zeta_j\right)}$$

Then

- $\zeta = u$ and $\zeta = u_{x_k}$ for $k = 1, 2$ are both weak solutions to $\mathcal{L}\zeta = 0$

If we define

$$\boxed{L\zeta = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(f_{\eta_i \eta_j} \zeta_j \right)}$$

Then

- $\zeta = u$ and $\zeta = u_{x_k}$ for $k = 1, 2$ are both weak solutions to $\mathcal{L}\zeta = 0$
- $\zeta = \log f(\nabla u)$ is a weak super solution, solution, and sub solution to

$\mathcal{L}\zeta = 0$ respectively when $1 < p < 2$, $p = 2$, and $2 < p < \infty$.

If we define

$$L\zeta = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(f_{\eta_i \eta_j} \zeta_j \right)$$

Then

- $\zeta = u$ and $\zeta = u_{x_k}$ for $k = 1, 2$ are both weak solutions to $\mathcal{L}\zeta = 0$
- $\zeta = \log f(\nabla u)$ is a weak super solution, solution, and sub solution to

$\mathcal{L}\zeta = 0$ respectively when $1 < p < 2$, $p = 2$, and $2 < p < \infty$.

Theorem D (A.)

a) *If* $1 < p \leq 2$, *there exists* $A = A(p, f) \geq 1$ *such that* $\mu_f \ll \mathcal{H}^{\lambda}$.

b) *If* $2 \leq p < \infty$, *then* μ_f *is concentrated on a set of* $\sigma-$*finite* $\mathcal{H}^1$.

If we define

$$L\zeta = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(f_{\eta_i \eta_j} \zeta_j \right)$$

Then

- $\zeta = u$ and $\zeta = u_{x_k}$ for $k = 1, 2$ are both weak solutions to $\mathcal{L}\zeta = 0$
- $\zeta = \log f(\nabla u)$ is a weak super solution, solution, and sub solution to

$\mathcal{L}\zeta = 0$ respectively when $1 < p < 2$, $p = 2$, and $2 < p < \infty$.

Theorem D (A.)

a) *If* $1 < p \leq 2$, *there exists* $A = A(p, f) \geq 1$ *such that* $\mu_f \ll \mathcal{H}^{\lambda}$.

b) *If* $2 \leq p < \infty$, *then* μ_f *is concentrated on a set of* $\sigma-$*finite* $\mathcal{H}^1$.

$$\mathcal{H} - \dim \mu_f \begin{cases} \geq 1 & \text{when } 1 < p < 2, \\ = 1 & \text{when } p = 2, \\ \leq 1 & \text{when } 2 < p < \infty. \end{cases}$$

THANK YOU!