Szegö Projections and Kerzman-Stein Formulas

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Outline

Let $\Omega = B(0,1) \subset \mathbb{C}$ and recall the Szegö projector,

$$S : L^2(\partial\Omega) \longrightarrow \mathcal{H}^2(\partial\Omega) \hookrightarrow L^2(\partial\Omega),$$

the orthogonal projection onto the closed subspace $\mathcal{H}^2(\partial\Omega)$ of $L^2(\partial\Omega)$.

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A higher-dimensional variant of the planar case: let $\Omega = B_n$, where

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Then for every $f \in L^2(\partial B_n)$

$$(Sf)(z) = \tfrac{1}{2} f(z) + \mathrm{PV} \int_{\partial B_n} \frac{f(\zeta)}{(1 - z \cdot \overline{\zeta})^n} \, d\sigma(\zeta), \quad z \in \partial B_n.$$

A variant of the theory of Calderón-Zygmund-type operators implies that S extends to a bounded operator on $L^p(\partial B_n)$ for every $p \in (1,\infty)$.

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Goals:

- work in the class of uniformly rectifiable subdomains (essentially optimal from the SIO theory point of view) of a Riemannian manifold
- replace the null space of $\overline{\partial}$ by the null space of a first order elliptic differential operator D which has coefficients exhibiting only a limited amount of smoothness.

Let M be a compact, connected, n-dimensional Riemannian manifold, of class $\mathscr{C}^2$, and assume that

$$D : \mathscr{C}^1(M, \mathcal{F}) \to \mathscr{C}^0(M, \mathcal{F}) \text{ a first-order elliptic differential operator}$$

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Assume that in each local coordinate chart U, and with respect to a trivialization of $\mathcal{F}$,

$$Du(x) = \sum A_j(x)\partial_j u(x) + B(x)u(x)$$

with

$$A_j \in \mathscr{C}^2\big(U, \mathbb{C}^{\kappa\times\kappa}\big), \quad B \in \mathscr{C}^1\big(U, \mathbb{C}^{\kappa\times\kappa}\big).$$

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In particular, the principal symbol of D is given in this representation by

$$\mathrm{Sym}(D, \xi) := i \sum \xi_j A_j(x) \quad \text{for} \quad \xi = (\xi_j)_j \in T_x^* M,$$

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The ellipticity of D amounts to having $\mathrm{Sym}(D, \xi)$ invertible if $\xi \neq 0$.

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- $D := \overline{\partial} + \overline{\partial}^*$ on a complex manifold M. Here $\overline{\partial} := \sum_{j=1}^{n} d\overline{z_j} \wedge \partial_{\overline{z_j}}$.

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  This time, $\overline{\partial}^2 = (\overline{\partial}^*)^2 = 0$ and $\overline{\Box} := -\overline{\partial}\overline{\partial}^* - \overline{\partial}^*\overline{\partial}$ has a scalar principal symbol. Since $D^* = D$ and $D^2 = -\overline{\Box}$, it follows that D is an operator of Dirac type.

Examples of Dirac type operators (continued)

- $M = \mathbb{R}^n$ and let $\mathscr{C}\ell(\mathbb{R}^n)$ be the Clifford algebra generated by the standard orthonormal basis $\{\mathbf{e}_j\}_{1\leq j\leq n}$ in $\mathbb{R}^n$. Consider

$$D := \sum_{j=1}^{n} \mathbf{e}_j \partial_j,$$

and note that $D^* = D$ and $D^2 = -\Delta$, the flat-space Laplacian.

D is the original flat space Dirac operator.

Let $\Omega \subset M$ be open and of finite perimeter. This implies

$$d\mathbf{1}_{\Omega} = -\nu\,\sigma \quad \text{in the sense of distributions,}$$

where $\nu \in T^*M$ is the outward pointing unit conormal to $\partial\Omega$ and

$$\sigma = \mathcal{H}^{n-1}\lfloor\partial\Omega$$

is the “surface area” on $\partial\Omega$, carried by the measure-theoretic boundary $\partial_*\Omega \subset \partial\Omega$. To avoid pathologies we assume

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Next, assume $\partial\Omega$ is Ahlfors-David regular (ADR) set, i.e., there exist $C_0, C_1 \in (0,\infty)$ such that if $x_0 \in \partial\Omega$ and $r \in (0, \operatorname{diam}\Omega)$ then

$$C_0 r^{n-1} \leq \mathcal{H}^{n-1}\big(\partial\Omega \cap B_r(x_0)\big) \leq C_1 r^{n-1}.$$

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That is, $\exists\ \varepsilon, M \in (0, \infty)$ such that for each $x \in \partial\Omega$ and $0 < R < \operatorname{diam}\Omega$ one can find a Lipschitz map $\varphi : B'_R \to \mathbb{R}^n$ (where B'_R is a ball of radius R in $\mathbb{R}^{n-1}$) with Lipschitz constant $\leq M$, and such that

$$\mathcal{H}^{n-1}\big(B(x,R) \cap \partial\Omega \cap \varphi(B'_R)\big) \geq \varepsilon R^{n-1}.$$

Definition: Ω *satisfies an interior corkscrew condition if exist* $M > 1$ *and* $r_* > 0$ *s.t. for each* $x \in \partial\Omega$ *and* $r \in (0, r_*)$ *there exists*

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$$\begin{aligned}
\{\mathscr{C}^1 \text{ domains}\} &\subsetneq \{\text{domains locally given as upper-graphs} \\
&\qquad\qquad \text{of functions with gradients in } \mathrm{VMO} \cap L^\infty\} \\
&= \{\text{Lipschitz domains with } \mathrm{VMO} \text{ normals}\} \\
&= \{\text{Lipschitz domains}\} \cap \{\text{regular } \mathrm{SKT} \text{ domains}\} \\
&\subsetneq \{\text{regular } \mathrm{SKT} \text{ domains}\} \\
&\subsetneq \{\text{two-sided } \mathrm{NTA} \text{ domains}\} \cap \{\text{Ahlfors regular domains}\} \\
&\subsetneq \{\mathrm{UR} \text{ domains}\}.
\end{aligned}$$

For each $p \in (1, \infty)$ it is natural to consider the Hardy space associated with a Dirac type operator D in a UR domain $\Omega \subset M$ as

$$\mathcal{H}^p(\Omega, D) := \Big\{ u \in \mathscr{C}^0(\Omega, \mathcal{F}) : Du = 0 \text{ in } \Omega, \ \ \mathcal{N}u \in L^p(\partial\Omega), \qquad \text{and } \ u\Big|_{\partial\Omega}^{\rm n.t.} \text{ exists } \sigma\text{-a.e. on } \partial\Omega \Big\},$$

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and equip it with the norm $\|u\|_{\mathcal{H}^p(\Omega,D)} := \|\mathcal{N}u\|_{L^p(\partial\Omega)}$. Here, $\mathcal{N}$ and $u\big|_{\partial\Omega}^{\text{n.t.}}$ are suitably defined relative to Ω. Also introduce the boundary Hardy spaces

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Later we shall see that $\mathcal{H}^p(\partial\Omega, D)$ is a closed subspace of $L^p(\partial\Omega)$ if Ω is a UR domain. Assuming that this is the case, consider what would be the natural notion of Szegö operator in this context, i.e., the orthogonal projection

$$S_D : L^2(\partial\Omega) \longrightarrow \mathcal{H}^2(\partial\Omega, D) \hookrightarrow L^2(\partial\Omega).$$

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Basic question: *To what extent is this the case for more general M, D, Ω?*

Theorem

Let $\Omega \subset M$ be a UR domain and assume that D is a Dirac type operator with top coefficients of class $\mathscr{C}^2$ and lower coefficients of class $\mathscr{C}^1$.

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$$S_D : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad \forall\, p \in (q,q').$$

If, moreover, Ω is a regular SKT domain, then we may take $q = 1$, i.e., the above result is valid for every $p \in (1,\infty)$.

The Szegö projector may then be used to represent $L^p(\partial\Omega)$ as a direct twisted sum of boundary Hardy spaces.

Theorem

Let D be a Dirac type operator with top coefficients of class $\mathscr{C}^2$, lower coefficients of class $\mathscr{C}^1$, and assume that $\Omega \subset M$ is a UR domain, with geometric measure theoretic outward unit conormal ν.

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$$L^p(\partial\Omega) = \mathcal{H}^p(\partial\Omega, D) \oplus i\mathrm{Sym}(D^*, \nu)\mathcal{H}^p(\partial\Omega, D^*)$$

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where the direct sum is topological, and also orthogonal when $p = 2$. Moreover, if Ω is a regular SKT domain, then we may take $q = 1$, i.e., the above result is valid for every $p \in (1,\infty)$.

A key tool is a certain type of Kerzman-Stein formula for a Cauchy type operator associated to D. In its original format for the Cauchy-Riemann operator $\overline{\partial}$ in the complex plane, this reads

$$S_{\overline{\partial}} = (\tfrac{1}{2}I + C_{\overline{\partial}})(I + C_{\overline{\partial}} - C^*_{\overline{\partial}})^{-1}$$

where $C_{\overline{\partial}}$ denotes the classical Cauchy operator

$$C_{\overline{\partial}}f(z) := \mathrm{PV}\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}\,d\zeta, \qquad z \in \partial\Omega.$$

When $\Omega \subset \mathbb{C}$ is a bounded $\mathscr{C}^1$ domain (the context considered by Kerzman-Stein) it turns out that $C_{\overline{\partial}}$ is "*almost self adjoint*". This ensures the existence of the inverse and also gives that $S_{\overline{\partial}}$ behaves essentially like $C_{\overline{\partial}}$. In particular, the boundedness of $C_{\overline{\partial}}$ in $L^p(\partial\Omega)$ implies the boundedness of $S_{\overline{\partial}}$ in $L^p(\partial\Omega)$.

To proceed along similar lines in this more general case, need a Cauchy operator C_D associated with D much as $C_{\overline{\partial}}$ was associated with $\overline{\partial}$. To set the stage write $C_{\overline{\partial}}$ in a manner minimizing the involvement of $\mathbb{C}$, i.e.:

$$C_{\overline{\partial}} f(z) = i \int_{\partial\Omega} E(z-\zeta)\mathrm{Sym}(\overline{\partial}, \nu(\zeta)) f(\zeta)\, d\sigma(\zeta), \qquad z \in \mathbb{C} \setminus \partial\Omega,$$

where $E(z) := \frac{1}{2\pi}\frac{1}{z}$ is the fundamental solution of the $\overline{\partial}$ operator in $\mathbb{C}$.

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where $E(z) := \frac{1}{2\pi}\frac{1}{z}$ is the fundamental solution of the $\overline{\partial}$ operator in $\mathbb{C}$. When D replaces $\overline{\partial}$ we need to construct $E(x,y)$ fundamental solution for D, i.e., $D_x[E(x,y)] = \delta_y(x)$. In a first stage, it is useful to identify such a fundamental solution under the assumption that

$$D : H^{1,2}(M,\mathcal{F}) \to L^2(M,\mathcal{F}) \ \text{ is invertible.}$$

Assuming that this is the case, D has an inverse,

$$D^{-1} : L^2(M,\mathcal{F}) \to H^{1,2}(M,\mathcal{F}).$$

Then the celebrated Schwartz Kernel Theorem yields the existence of a “double" distribution $E(x,y) \in \mathcal{D}'(M \times M, \mathcal{F} \otimes \mathcal{F})$ with the property that if dV is the volume element on M then for any reasonable section v in $\mathcal{F}$,

$$D^{-1}v(x) = \int_M E(x,y)v(y)\,dV(y), \qquad x \in M.$$

In particular, applying D to both sides gives

$$v(x) = DD^{-1}v(x) = \int_M D_x E(x,y)v(y)\,dV(y),$$

which shows that $D_x[E(x,y)]$ is indeed a Dirac distribution with mass at x, as wanted.

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The bottom line is that we may take as a fundamental solution for D the Schwartz kernel $E(x,y) \in \mathcal{D}'(M \times M, \mathcal{F} \otimes \mathcal{F})$ of the operator $D^{-1} : L^2(M,\mathcal{F}) \to H^{1,2}(M,\mathcal{F})$, provided this inverse exists.

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Definition: D has UCP provided if $u \in H^{1,2}(M, \mathcal{F})$ is such that $Du = 0$ on M and $u\big|_{\mathcal{O}} = 0$ for some nonempty open set $\mathcal{O} \subset M$ then $u = 0$ on M.

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Key: even in the case of structures with limited regularity, Dirac type operators have UCP.

Assume in what follows that

D is an elliptic 1^{st}-order operator such that D and D^* have UCP.

A different route (compared with what was done when D^{-1} is known to exist) is called for. We are motivated to consider

$$\mathbb{D} := \begin{pmatrix} iM_a & D^* \\ D & iM_a \end{pmatrix} : \mathcal{F} \oplus \mathcal{F} \to \mathcal{F} \oplus \mathcal{F}$$

where M_a denotes the operator of pointwise multiplication by a nonnegative scalar function $a \in \mathscr{C}^1$ (not identically zero).

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In addition, $\mathbb{D}$ differs by a compact operator from what one gets by taking $a \equiv 0$, so

$$\text{index}\,\mathbb{D} = \text{index}\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} = \text{index}\,D + \text{index}\,D^* = 0.$$

Thus, $\mathbb{D}$ is invertible iff has a trivial kernel.

Thus, $\mathbb{D}$ is invertible iff has a trivial kernel. In this regard, first note that for each $u = (v, w) \in H^{1,2}(M, \mathcal{F} \oplus \mathcal{F})$ we have

$$(\mathbb{D}u, u)_{L^2(M)} = i \int_M a|u|^2 \, d\mathrm{V} + 2 \,\mathrm{Re} \int_M \langle Dv, w \rangle \, d\mathrm{V}.$$

Consequently, if $u \in \mathrm{Ker}\,\mathbb{D}$ it follows that

$$0 = \mathrm{Im}\,(\mathbb{D}u, u)_{L^2(M)} = \int_M a|u|^2 \, d\mathrm{V}.$$

Hence $u = (v, w) \in \mathrm{Ker}\,\mathbb{D}$ satisfies $u = 0$ on $\mathcal{O} := \{x : \, a(x) \neq 0\}$. Thus, $v = 0$ on $\mathcal{O}$ and $w = 0$ on $\mathcal{O}$ so ultimately $av = 0$ on M and $aw = 0$ on M. Given that on M we also have

$$0 = \mathbb{D}u = \begin{pmatrix} iav + D^* w \\ Dv + iaw \end{pmatrix},$$

this also forces $Dv = 0$ and $D^* w = 0$ on M.

At this stage, we may conclude that if D is an elliptic $1^{\rm st}$-order operator such that

$$D \text{ and } D^* \text{ have UCP}$$

then

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is both Fredholm with index zero and one-to-one, thus an invertible operator. Then the Schwartz kernel $\mathbb{E}(x, y)$ of the inverse $\mathbb{D}^{-1}$ is a fundamental solution for the operator $\mathbb{D}$.

Based on this fundamental solution, we then proceed to associate to the operator $\mathbb{D}$ the following Cauchy-type integral operator

$$C_{\mathbb{D}}f(x) := \mathrm{PV}\, i \int_{\partial\Omega} \mathbb{E}(x,y)\mathrm{Sym}(\mathbb{D},\nu(y))f(y)\,d\sigma(y), \quad x \in \partial\Omega.$$

When $M = \mathbb{R}^n$ and D is homogeneous with constant coefficients (and $a = 0$), then $\mathbb{E}(x,y)$ is of the form $k(x-y)$ with $k \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ odd and homogeneous of degree $-(n-1)$. When Ω is a UR domain in $\mathbb{R}^n$, fundamental work of G. David and S. Semmes yields bounds on $L^p(\partial\Omega)$ with $p \in (1,\infty)$ for SIO's of the form

$$Bf(x) := \mathrm{PV} \int_{\partial\Omega} k(x-y)f(y)\,d\sigma(y), \quad x \in \partial\Omega.$$

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Recall that $\mathbb{D}$ plays only an auxiliary role in this business, since we are primarily interested in the original (unperturbed) operator D.

To attempt to remedy this, keep in mind that D is a “piece" of $\mathbb{D}$. Idea: work componentwise, and write $\mathbb{E}(x,y) \in \mathrm{Hom}\,(\mathcal{F}_y \oplus \mathcal{F}_y, \mathcal{F}_x \oplus \mathcal{F}_x)$ as

$$\mathbb{E}(x,y) = \begin{pmatrix} E_{00}(x,y) & E_{01}(x,y) \\ E_{10}(x,y) & E_{11}(x,y) \end{pmatrix}, \qquad x,y \in M, \quad x \neq y.$$

where

$$E_{00}(x,y) \in \mathrm{Hom}\,(\mathcal{F}_y, \mathcal{F}_x), \quad E_{01}(x,y) \in \mathrm{Hom}\,(\mathcal{F}_y, \mathcal{F}_x),$$

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Then the fact that

$$\mathbb{D}_x[\mathbb{E}(x,y)] = \delta_y(x) \cdot I_{2\times 2}$$

becomes equivalent to

$$ia(x)E_{00}(x,y) + D_x^*[E_{10}(x,y)] = \delta_y(x),$$

$$ia(x)E_{01}(x,y) + D_x^*[E_{11}(x,y)] = 0,$$

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In particular, the last equality implies that

$E_{01}(\cdot, y)$ is a fundamental solution (with pole at y)

for the operator D outside of the support of a.

Hence, if we now consider the Cauchy-type integral operator

$$C_D f(x) := \mathrm{PV}\, i \int_{\partial\Omega} E_{01}(x,y)\mathrm{Sym}(D,\nu(y))f(y)\,d\sigma(y), \quad x \in \partial\Omega,$$

it follows that for every $f \in L^1(\partial\Omega, \mathcal{F})$,

$$C_{\mathbb{D}} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} C_D f \\ \dots \end{pmatrix}.$$

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This allows us to transfer the entire Calderón-Zygmund theory developed for the Cauchy operator $C_{\mathbb{D}}$ associated with the auxiliary operator $\mathbb{D}$ to the Cauchy operator C_D associated with the original operator D, in arbitrary UR subdomains of the manifold M.

Moreover the integral kernel of the aforementioned Cauchy operator plays a key role in the following

Theorem (Generalized Cauchy-Pompeiu Formula)

Let $\Omega \subset M$ be an Ahlfors regular domain. Also, let $D : \mathcal{F} \to \mathcal{F}$ be a $1^{\rm st}$-order elliptic operator such that both D and D^* have UCP and assume

$$u \in \mathscr{C}^0(\Omega, \mathcal{F}) \ \text{ is such that } \ Du \in L^1(\Omega), \quad \mathcal{N}u \in L^1(\partial\Omega),$$

$$\text{and } \ u\big|_{\partial\Omega}^{\rm n.t.} \ \text{ exists } \sigma\text{-a.e. on } \ \partial\Omega.$$

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$$\textit{and} \ \ u\big|_{\partial\Omega}^{\rm n.t.} \ \ \textit{exists } \sigma\textit{-a.e. on} \ \ \partial\Omega.$$

Then for every $x \in \Omega$,

$$u(x) = i \int\limits_{\partial\Omega} E_{01}(x,y)\mathrm{Sym}(D, \nu(y))\big(u\big|_{\partial\Omega}^{\rm n.t.}\big)(y)\, d\sigma(y)$$
$$+ \int\limits_{\Omega} E_{01}(x,y)(Du)(y)\, dV(y).$$

This theorem immediately yields a Cauchy reproducing formula for functions $u \in \mathscr{H}^p(\Omega, D)$ which, further, may be used to show that $\mathscr{H}^p(\partial\Omega, D)$ is a closed subspace of $L^p(\partial\Omega)$ if $\Omega \subset M$ is a UR domain.

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Note that u satisfies all conditions listed in the statement ($Du = 0$ in Ω). However, the corresponding Cauchy-Pompeiu formula fails since it reduces to

$$u(z) = \frac{1}{2\pi i} \int_{\partial B(0,1)} \frac{u(\zeta)}{\zeta - z}\, d\zeta \qquad \forall\, z \in \Omega.$$

This is false since $\int_{|\zeta|=1} \frac{d\zeta}{\zeta(\zeta - z)} = 0$ whenever $0 < |z| < 1$.

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Root of this failure: near the point $0 \in \partial\Omega$ there is no "boundary mass". One may attempt to prevent such pathologies from happening by requiring that

$$\partial\Omega \text{ is Ahlfors-David regular,}$$

which, in the present context, amounts to

$$\mathcal{H}^1(B_r(z) \cap \partial\Omega) \approx r, \quad \text{uniformly for } z \in \partial\Omega \text{ and } r \in (0, 1].$$

Nonetheless, problems persist since we can take a slit disk, say

$$\Omega := B(0, 1) \setminus \{(x, 0) : \ x \geq 0\} \subset \mathbb{C},$$

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Then $\partial\Omega$ is ADR and yet the Cauchy-Pompeiu formula does not hold in this case.

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where $\partial_*\Omega$, the measure theoretic boundary of the finite perimeter set Ω, is the support of the measure σ. Thus, in order to exclude this type of anomalies, we also need:

$$\mathcal{H}^1(\partial\Omega \setminus \partial_*\Omega) = 0,$$

a condition incorporated into the definition of an Ahlfors regular domain.

Another basic ingredient in the proof of the generalized Cauchy-Pompeiu formula is the following optimal version of the Divergence Formula on Ahlfors regular domains on manifolds.

Theorem (Sharp Divergence Theorem)

Let $\Omega \subset M$ be an Ahlfors regular domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In particular, Ω is a set of finite perimeter, and its outward unit conormal $\nu : \partial\Omega \to T^*M$ is defined σ-a.e. on $\partial\Omega$.

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Then

$$\int_\Omega \mathrm{div}\,\vec{F}\,dV = \int_{\partial\Omega} {}_{T^*M}\Big(\nu\,,\,\vec{F}\,\big|^{\mathrm{n.t.}}_{\partial\Omega}\Big)_{TM}\,d\sigma.$$