Characterizing Lyapunov domains via Riesz transforms on Hölder spaces

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joint work with Marius Mitrea and Joan Verdera

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Setting

$\Omega \subseteq \mathbb{R}^n$ open set of locally finite perimeter

ν outward unit normal to Ω (in the GMT sense)

$\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ "surface measure"

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^n$

Ahlfors regular and UR sets

Definition

$\partial\Omega$ is called Ahlfors regular if at all scales and locations behaves like an $(n-1)$-dimensional surface, i.e., there exists $C \geq 1$ such that

$$C^{-1} R^{n-1} \leq \mathcal{H}^{n-1}\big(B(x,R) \cap \partial\Omega\big) \leq C\, R^{n-1},$$

for each $x \in \partial\Omega$ and $R \in (0, \operatorname{diam} \partial\Omega)$.

Definition

$\partial\Omega$ is called a UR set if it is Ahlfors regular and at all scales and locations contains big pieces of Lipschitz images, i.e., there exist ε, $M \in (0,\infty)$ such that for each $x \in \partial\Omega$ and $R \in (0, \operatorname{diam} \partial\Omega)$, there is a Lipschitz map $\Phi : B_R^{n-1} \to \mathbb{R}^n$ (B_R^{n-1} = ball of radius R in $\mathbb{R}^{n-1}$) with Lipschitz constant $\leq M$, such that

$$\mathcal{H}^{n-1}\big(\partial\Omega \cap B(x,R) \cap \Phi(B_R^{n-1})\big) \geq \varepsilon R^{n-1}.$$

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The case when $\partial\Omega$ is a UR set

Theorem (G. David and S. Semmes, 1991)

Assume that $\partial\Omega$ is Ahlfors regular. Then all "reasonable" Singular Integral Operators have truncated versions bounded on $L^2(\partial\Omega)$ uniform w.r.t. the truncation parameter $\Leftrightarrow$ $\partial\Omega$ is a UR set.

Truncated version of reasonable SIO's:

$$(T_\varepsilon f)(x) := \int_{y\in\partial\Omega\setminus B(x,\varepsilon)} k(x-y)f(y)\,d\sigma(y), \quad x\in\partial\Omega,$$

k is odd, smooth in $\mathbb{R}^n\setminus\{0\}$, and $|\nabla^\ell k(x)| \lesssim |x|^{-(n-1+\ell)}$, $\forall\,\ell\in\mathbb{N}_0$.

For homogeneous kernels, $\partial\Omega$ UR $\Rightarrow \lim_{\varepsilon\to 0^+}(T_\varepsilon f)(x)$ exists for σ-a.e. $x\in\partial\Omega$ and $f\in L^p(\partial\Omega)$, $p\in(1,\infty)$ [Hofmann, Mitrea, Taylor, 2010].

Moral: For the study of SIO's on L^p spaces, the class of UR sets is the optimal environment.

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The role of Riesz transforms

Question: Can one streamline the class of SIO's in the David-Semmes theorem to just Riesz transforms?

Truncated Riesz transforms: given $\varepsilon > 0$, for $j = 1, \ldots, n$,

$$(R_{j,\varepsilon} f)(x) := \frac{1}{\omega_{n-1}} \int_{y \in \partial\Omega \setminus B(x,\varepsilon)} \frac{x_j - y_j}{|x-y|^n} f(y)\, d\sigma(y), \quad x \in \partial\Omega.$$

Note: $k_j(x) := \dfrac{x_j}{|x|^n}$ is of the type considered earlier and is homogeneous.

Difficult question! Answer: YES
$n = 2$ proved by P. Mattila, M.S. Melnikov, and J. Verdera [Ann. of Math., 1996] and the higher dimensionl case by F. Nazarov, X. Tolsa, and A. Volberg [Acta Math., 2014].

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In fact, by using the $T(1)$ theorem for SIO's with odd kernels (in spaces of homogeneous type), the L^p-boundedness of $R_{j,\varepsilon}$'s (uniform with respect to ε) reduces to just

$$R_j 1 \in \mathrm{BMO}(\partial\Omega), \qquad 1 \leq j \leq n \qquad (*)$$

where $\mathrm{BMO}(\partial\Omega)$ is the John-Nirenberg space of functions of bounded mean oscillations on $\partial\Omega$ and $R_j : \mathcal{C}^\alpha(\partial\Omega) \to \big(\mathcal{C}^\alpha(\partial\Omega)\big)^*$ is the linear mapping given by

$$\langle R_j f, g\rangle := \tfrac{1}{2}\int_{\partial\Omega}\int_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n}[f(y)g(x) - f(x)g(y)]\, d\sigma(y)d\sigma(x)$$

for every $f, g \in \mathcal{C}^\alpha(\partial\Omega)$.

As such, the Nazarov-Tolsa-Volberg result may be rephrased as:

Under the background assumption $\partial\Omega$ is Ahlfors regular,
$\partial\Omega$ is a UR set $\Longleftrightarrow$ $(*)$ holds

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So far, looked at SIO's on L^p spaces.

Next natural question: Will changing $\mathrm{BMO}(\partial\Omega)$ to a more regular space in the equivalence

$$\partial\Omega \text{ is a UR set } \Longleftrightarrow R_j 1 \in \mathrm{BMO}(\partial\Omega), \qquad 1 \leq j \leq n,$$

yield an equivalence where Ω is correspondingly more regular?

We are interested in replacing $\mathrm{BMO}(\partial\Omega)$ with the Hölder space $\mathcal{C}^\alpha(\partial\Omega)$ for any $\alpha \in (0, 1)$.

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A bit more GMT

Recall that the measure-theoretic boundary $\partial_*\Omega$ of $\Omega \subseteq \mathbb{R}^n$ is defined by

$$\partial_*\Omega := \Big\{ x \in \partial\Omega : \limsup_{r\to 0^+} \frac{\mathcal{H}^n(B(x,r)\cap\Omega)}{r^n} > 0 \quad \text{and} \quad \limsup_{r\to 0^+} \frac{\mathcal{H}^n(B(x,r)\setminus\Omega)}{r^n} > 0 \Big\}.$$

GMT Fact: If Ω has locally finite perimeter, the outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. In particular, the condition

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$$

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Theorem (D.M., M. Mitrea, J. Verdera, 2014)

If $\alpha \in (0,1)$, $\partial\Omega$ *compact Ahlfors regular*, $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$, $\partial\Omega = \partial(\overline{\Omega})$, *then* $R_j 1 \in \mathcal{C}^\alpha(\partial\Omega)$, $1 \leq j \leq n \iff \Omega$ *is a* $\mathcal{C}^{1,\alpha}$ *domain.*

Moreover, if Ω *is a* $\mathcal{C}^{1,\alpha}$ *domain, then for every odd, homogeneous polynomial* P *in* $\mathbb{R}^n$ *the generalized Riesz transform* $T = T_P$ *given by*

$$Tf(x) := \text{p.v.} \int_{\partial\Omega} \frac{P(x-y)}{|x-y|^{n-1+\deg P}} f(y)\, d\sigma(y), \quad x \in \partial\Omega$$

is bounded from $\mathcal{C}^\alpha(\partial\Omega)$ *into* $\mathcal{C}^\alpha(\partial\Omega)$, *and if*

$$\mathbb{T}f(x) := \int_{\partial\Omega} \frac{P(x-y)}{|x-y|^{n-1+\deg P}} f(y)\, d\sigma(y), \quad x \in \Omega,$$

then $\mathbb{T} : \mathcal{C}^\alpha(\partial\Omega) \to \mathcal{C}^\alpha(\overline{\Omega})$ *is also bounded.*

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A motivating example

The PDE modeling elastic deformation phenomena is described via the Lamé system in $\mathbb{R}^n$ (where $\lambda, \mu \in \mathbb{R}$ are the Lamé moduli)

$$Lu := \mu\Delta u + (\lambda + \mu)\nabla \mathrm{div}\, u, \quad u = (u_1, ..., u_n) \in C^2.$$

One basic approach to solving BVPs for this system uses boundary SIO, such as the single layer associated with the Lamé system

$$\mathcal{S}_{Lame} f(x) := \Big(\int_{\partial\Omega} \sum_{\beta=1}^{n} E_{\alpha\beta}(x-y) f_\beta(y)\, d\sigma(y) \Big)_{1\leq\alpha\leq n} \quad x \in \Omega,$$

where (assuming $n \geq 3$)

$$E_{\alpha\beta}(x) := \frac{-1}{2\mu(2\mu+\lambda)\omega_{n-1}} \left[\frac{3\mu+\lambda}{n-2} \frac{\delta_{\alpha\beta}}{|x|^{n-2}} + \frac{(\mu+\lambda)x_\alpha x_\beta}{|x|^n} \right].$$

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When such BVPs are considered in the context of Hölder spaces in Lyapunov domains the issue becomes whether this SIO behaves naturally on such a scale. Our previous theorem implies that the operator

$$\mathcal{S}_{Lame} : \mathcal{C}^{\alpha}(\partial\Omega) \to \mathcal{C}^{1,\alpha}(\overline{\Omega})$$

is well-defined and bounded.

A motivating example

To justify that the single layer for the Lamé system is smoothing of order one on the Hölder scale observe that for each $j = 1, \dots, n$ the previous theorem applies to the integral operator $\mathbb{T} := \partial_j \mathcal{S}_{Lame}$ and gives that $\partial_j \mathcal{S}_{Lame} : \mathcal{C}^\alpha(\partial\Omega) \to \mathcal{C}^\alpha(\overline{\Omega})$ boundedly. Indeed,

$$(\partial_j \mathcal{S}_{Lame} f)(x) = \Big(\int_{\partial\Omega} \sum_{\beta=1}^{n} (\partial_j E_{\alpha\beta})(x-y) f_\beta(y)\, d\sigma(y) \Big)_{1 \le \alpha \le n} \quad x \in \Omega$$

and its integral kernel is a matrix in which the (α, β) entry is given by $P(x-y)/|x-y|^{n-1+\deg P}$ with P the homogeneous, odd, polynomial of degree 3:

$$P(x) = \frac{(3\mu+\lambda)\delta_{\alpha\beta} x_j |x|^2 - (\mu+\lambda)(\delta_{\alpha j} x_\beta |x|^2 + \delta_{\beta j} x_\alpha |x|^2 - n x_j x_\alpha x_\beta)}{2\mu(2\mu+\lambda)\omega_{n-1}}.$$

From this the desired conclusion follows using

$$\|\mathcal{S}_{Lame} f\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} = \|\mathcal{S}_{Lame} f\|_{L^\infty(\Omega)} + \sum_{j=1}^{n} \|\partial_j \mathcal{S}_{Lame} f\|_{\mathcal{C}^\alpha(\overline{\Omega})}.$$

Some ingredients in the proof

$R_j 1 \in \mathcal{C}^\alpha(\partial\Omega)$, $1 \le j \le n \Longrightarrow \Omega$ is a $\mathcal{C}^{1,\alpha}$ domain

- $\partial\Omega$ compact Ahlfors regular and $\partial\Omega = \partial(\overline{\Omega})$, then $\nu \in \mathcal{C}^\alpha(\partial\Omega) \Longleftrightarrow \Omega$ a $\mathcal{C}^{1,\alpha}$ domain [Hofmann, Mitrea, Taylor, 2007]

- Clifford algebra $(\mathcal{C}\ell_n, +, \odot)$ which is the minimal enlargement of $\mathbb{R}^n$ to a unitary real algebra such that $x \odot x = -|x|^2$ for any $x \in \mathbb{R}^n$. Note that by identifying the canonical basis $\{e_j\}_{1\le j\le n}$ from $\mathbb{R}^n$ with the imaginary units in $\mathcal{C}\ell_n$, we have $\mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n$ via

$x = (x_1, \dots, x_n) \equiv \sum_{j=1}^n x_j e_j \in \mathcal{C}\ell_n$. Also, $u = \sum_{l=0}^n \sum_{|I|=l} u_I\, e_I$ with $u_I \in \mathbb{C}$, for each $u \in \mathcal{C}\ell_n$, where $e_I = e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_l}$ if $I = (i_1, i_2, \dots, i_l)$ for $1 \le i_1 < i_2 < \cdots < i_l \le n$ and $e_0 := e_\emptyset := 1$.

Moreover, we consider $\mathcal{X} \otimes \mathcal{C}\ell_n$ by allowing $u_I \in \mathcal{X}$ some Banach space, such as $\mathcal{X} = \mathcal{C}^\alpha(\partial\Omega)$ or $\mathcal{X} = L^p(\partial\Omega)$, etc.

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Moreover, we consider $\mathcal{X} \otimes \mathcal{C}\ell_n$ by allowing $u_I \in \mathcal{X}$ some Banach space, such as $\mathcal{X} = \mathcal{C}^\alpha(\partial\Omega)$ or $\mathcal{X} = L^p(\partial\Omega)$, etc.

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$R_j 1 \in \mathcal{C}^\alpha(\partial\Omega)$, $1 \le j \le n \Longrightarrow \Omega$ is a $\mathcal{C}^{1,\alpha}$ domain

- $\partial\Omega$ compact Ahlfors regular and $\partial\Omega = \partial(\overline{\Omega})$, then $\nu \in \mathcal{C}^\alpha(\partial\Omega) \Longleftrightarrow \Omega$ a $\mathcal{C}^{1,\alpha}$ domain [Hofmann, Mitrea, Taylor, 2007]

- Clifford algebra $(\mathcal{C}\!\ell_n, +, \odot)$ which is the minimal enlargement of $\mathbb{R}^n$ to a unitary real algebra such that $x \odot x = -|x|^2$ for any $x \in \mathbb{R}^n$. Note that by identifying the canonical basis $\{e_j\}_{1\le j\le n}$ from $\mathbb{R}^n$ with the imaginary units in $\mathcal{C}\!\ell_n$, we have $\mathbb{R}^n \hookrightarrow \mathcal{C}\!\ell_n$ via

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- The Cauchy-Clifford operator

$$\mathbf{C}f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{n-1}} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, d\sigma(y), \quad x \in \partial\Omega,$$

where f is a $\mathcal{C}\ell_n$-valued function defined on $\partial\Omega$.

- Mapping properties for $\mathbf{C}$ under minimal smoothness assumptions on $\partial\Omega$. While for mapping properties of SIO's on L^p the class of UR sets is the optimal environment, the Cauchy-Clifford operator behaves naturally on Hölder spaces on a much larger class of sets (essentially, Ahlfors regular will do). This is surprising since the boundedness of all Riesz transforms on Hölder spaces forces the domain to be Lyapunov. More precisely, we prove the following result.

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Theorem (D. M., M. Mitrea, J. Verdera, 2014)

Fix $\alpha \in (0,1)$*. Suppose* $\Omega \subset \mathbb{R}^n$ *is such that* $\partial\Omega$ *is compact Ahlfors regular with* $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$*. Then for every* $f \in \mathcal{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n$ *the limit*

$$\mathbf{C}f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{n-1}} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, d\sigma(y)$$

exists for σ*-a.e.* $x \in \partial\Omega$*, the operator*

$$\mathbf{C} : \mathcal{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n \to \mathcal{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n$$

is bounded and

$$\mathbf{C}^2 = \tfrac{1}{4} I \quad \text{on} \ \ \mathcal{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n.$$

Theorem (cont.)

Moreover, if also $R_j 1 \in \mathrm{BMO}(\partial\Omega)$ *for* $1 \le j \le n$*, then for every* $p \in (1,\infty)$ *the pointwise limit above for* $\mathbf{C}f(x)$ *exists for* σ*-a.e.* $x \in \partial\Omega$ *whenever* $f \in L^p(\partial\Omega) \otimes \mathcal{C}\ell_n$ *and the operator* $\mathbf{C}$*, originally defined on* $\mathcal{C}^\alpha(\partial\Omega) \otimes \mathcal{C}\ell_n$*, extends boundedly to an operator*

$$\mathbf{C} : L^p(\partial\Omega) \otimes \mathcal{C}\ell_n \to L^p(\partial\Omega) \otimes \mathcal{C}\ell_n$$

with the property that $\mathbf{C}^2 = \frac{1}{4} I$ *in* $L^p(\partial\Omega) \otimes \mathcal{C}\ell_n$*, and it also extends to a bounded operator*

$$\mathbf{C} : \mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n \to \mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$$

with the property that $\mathbf{C}^2 = \frac{1}{4} I$ *in* $\mathrm{BMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$*.*

Some ingredients in the proof

- There exists

$$\text{p.v.} \int_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n}\, d\sigma(y) \quad \sigma\text{-a.e. } x \in \partial\Omega.$$

This can be seen either by relying on the fact that $\partial\Omega$ is $(n-1)$-rectifiable (given that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$) and invoking [X. Tolsa, 2008], or, alternatively, assuming that $R_j 1 \in \mathrm{BMO}(\partial\Omega)$, make use of the $T(1)$ Theorem and the pointwise existence of the p.v. Cauchy-Clifford operator on Hölder functions.

- Assuming that $R_j 1 \in \mathrm{BMO}(\partial\Omega)$, with $R_j 1$ originally defined as a functional in $\left(\mathcal{C}^\alpha(\partial\Omega)\right)^*$, we have

$$(R_j 1)(x) = \text{p.v.} \int_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n}\, d\sigma(y) \quad \sigma\text{-a.e. } x \in \partial\Omega.$$

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$$\mathbf{C}\nu(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot \nu(y)\, d\sigma(y)$$

$$= -\sum_{j=1}^{n} \Big(\text{p.v.} \int_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n}\, d\sigma(y)\Big)\mathbf{e}_j = -\sum_{j=1}^{n} (R_j 1)(x)\mathbf{e}_j$$

for σ-a.e. $x \in \partial\Omega$.

- The previous formula and $\mathbf{C}^2 = \frac{1}{4}I$ in $L^2(\partial\Omega) \otimes \mathcal{C}\ell_n$ yield

$$\nu = 4\mathbf{C}^2(\nu) = 4\mathbf{C}\Big(-\sum_{j=1}^{n} (R_j 1)\mathbf{e}_j\Big) \quad \text{in } L^2(\partial\Omega) \otimes \mathcal{C}\ell_n.$$

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Ω is a $\mathcal{C}^{1,\alpha}$ domain $\Longrightarrow \mathbb{T} : \mathcal{C}^{\alpha}(\partial\Omega) \to \mathcal{C}^{\alpha}(\overline{\Omega})$ is bounded

Main step: Assuming the odd, homogeneous polynomial P is also harmonic, show that $\exists\, C = C(n, \alpha, \Omega) > 1$ such that $\forall\, f \in \mathcal{C}^{\alpha}(\partial\Omega)$ we have, with $\rho(x) := \operatorname{dist}(x, \partial\Omega)$,

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Achieved via an induction over the degree $l \in 2\mathbb{N} - 1$ of P.

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- In the inductive step we use elements of Clifford Algebra. For $l \geq 3$ write (refining work of S.W. Semmes)

$$\frac{P(x)}{|x|^{n-1+l}} = \sum_{r,s=1}^{n} [k_{rs}(x)]_s \quad \forall\, x \in \mathbb{R}^n \setminus \{0\},$$

where $k_{rs} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n$ are odd, $\mathcal{C}^\infty$, homogeneous of degree $-(n-1)$, and

$$(Dk_{rs})(x) = \frac{l-1}{n+l-3} \frac{\partial}{\partial x_r} \left(\frac{P_{rs}(x)}{|x|^{n+l-3}} \right), \quad 1 \leq r, s \leq n,$$

for some family $\{P_{rs}\}_{r,s}$ of harmonic, homogeneous polynomials of degree $l-2$, where $D = \sum_{j=1}^{n} e_j \partial_j$ is the Dirac operator.

Some ingredients in the proof

If f is a Clifford-valued function set for each $r, s \in \{1, \dots, n\}$

$$\mathbb{T}_{rs} f(x) := \int_{\partial\Omega} k_{rs}(x-y) \odot f(y)\, d\sigma(y), \quad x \in \Omega,$$

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$$\leq c_n\, C^{l-2} 2^{(l-2)^2} 2^l \|P\|_{L^1(S^{n-1})} \|f\|_{\mathcal{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n}$$

for every $f \in \mathcal{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n$, where C is the constant in the estimate given by the induction hypothesis.

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$$(\mathbb{T}_{rs}\nu)(x) = \int_{\partial\Omega} k_{rs}(x-y)\odot\nu(y)\,d\sigma(y) = -\int_{\Omega}(Dk_{rs})(x-y)\,dy$$
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For $f : \partial\Omega \to \mathcal{C}\ell_n$ with Hölder scalar components consider

$$\widetilde{\mathbb{T}}_{rs} f(x) := \int_{\partial\Omega} \big(k_{rs}(x-y) \odot \nu(y)\big) \odot f(y)\, d\sigma(y), \qquad x \in \Omega.$$

Recall that

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for every $f \in \mathcal{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n$.

Key observation: $\nu\odot\nu = -1$. Two consequences of interest: first $f \in \mathcal{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n$ implies $\nu\odot f \in \mathcal{C}^\alpha(\partial\Omega)\otimes\mathcal{C}\ell_n$ with comparable norm, and second $\widetilde{\mathbb{T}}_{rs}(\nu\odot f) = -\mathbb{T}_{rs}f$. In the context of the above inequality these yield a similar estimate for $\mathbb{T}_{rs}f$.

Recalling that $\mathbb{T}f(x) = \sum_{r,s=1}^{n}\big[\mathbb{T}_{rs}f(x)\big]_s$ we may further combine all these to arrive at the following estimate for $\mathbb{T}$:

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$$\times 2\|\nu\|_{\mathcal{C}^\alpha(\partial\Omega)}\|P\|_{L^1(S^{n-1})}\|f\|_{\mathcal{C}^\alpha(\partial\Omega)},\quad f\in\mathcal{C}^\alpha(\partial\Omega)$$

Keeping careful tabs on the dependence of the degree l (to ensure that the above structural constant has the desired format) then completes the induction on l of the estimate

$$\sup_{x\in\Omega}|\mathbb{T}f(x)|+\sup_{x\in\Omega}\Big\{\rho(x)^{1-\alpha}\big|\nabla(\mathbb{T}f)(x)\big|\Big\}\leq C^l2^{l^2}\|P\|_{L^1(S^{n-1})}\|f\|_{\mathcal{C}^\alpha(\partial\Omega)}$$

when the polynomial P is also harmonic.

Some ingredients in the proof

- To remove the assumption that P is harmonic write $P(x) = \sum_{j=1}^{N+1} |x|^{2(j-1)} P_j(x)$ in $\mathbb{R}^n$, where each P_j is a harmonic homogeneous polynomial of degree $l - 2(j-1)$.

- Recall that the original goal was to show $\mathbb{T} : \mathcal{C}^\alpha(\partial\Omega) \to \mathcal{C}^\alpha(\overline{\Omega})$ is well-defined and bounded. To arrive at this conclusion from what we have just established, as a final step we use a general real-variable result to the effect that, in the current geometric setting, for every $\alpha \in (0,1)$ there exists $C = C(\Omega, \alpha) \in (0, \infty)$ such that

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Theorem (D. M., M. Mitrea, J. Verdera, 2014)

If $\Omega \subseteq \mathbb{R}^n$ *is open with* $\partial\Omega$ *compact Ahlfors regular and* $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$, *then*

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The fact that $R_j 1 \in \mathrm{VMO}(\partial\Omega)\big(\subseteq \mathrm{BMO}(\partial\Omega)\big)$ implies $\partial\Omega$ is a UR set is a consequence of the $T(1)$ theorem and the Nazarov-Tolsa-Volberg theorem.
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Theorem

Suppose $\Omega \subset \mathbb{R}^n$ *is such that* $\partial\Omega$ *is compact Ahlfors regular with* $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ *and* $R_j 1 \in \mathrm{BMO}(\partial\Omega)$ *for* $1 \leq j \leq n$*. Then for every* $f \in \mathrm{VMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$ *the limit*

$$\mathbf{C}f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{n-1}} \int_{\substack{y \in \partial\Omega \\ |x-y| > \varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, d\sigma(y)$$

exists for σ*-a.e.* $x \in \partial\Omega$*, the operator*

$$\mathbf{C} : \mathrm{VMO}(\partial\Omega) \otimes \mathcal{C}\ell_n \to \mathrm{VMO}(\partial\Omega) \otimes \mathcal{C}\ell_n$$

is bounded and

$$\mathbf{C}^2 = \tfrac{1}{4} I \quad \text{on } \mathrm{VMO}(\partial\Omega) \otimes \mathcal{C}\ell_n.$$