Workshop on Harmonic Analysis, Partial Differential Equations and Geometric Measure Theory
ICMAT, Campus de Cantoblanco, Madrid

Boundary value problems for elliptic operators with real non-symmetric coefficients

Svitlana Mayboroda
joint work with S. Hofmann, C. Kenig, J. Pipher

University of Minnesota

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Maximum principle, Positivity

What properties do harmonic functions have in rough domains?

Ω - arbitrary domain
Maximum principle:

- the maximum of a harmonic function is achieved on the boundary
- for positive data the solution is positive
- the Green function ($\Delta_x G(x,y) = \delta_y(x)$, $G|_{\partial\Omega} = 0$) is positive
- harmonic functions continuous up to the boundary satisfy

$$\|u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\partial\Omega)}$$

These results extend to general 2^{nd} order equations:
Stampacchia, 1962 divergence form equations
H. Berestycki, L. Nirenberg, S.R.S. Varadhan, 1994 non-divergence form elliptic equations

Estimates (well-posedness)

The maximum principle provides the sharp estimates for solutions with data in L^∞. What about L^p? What exactly is the dependence on the data (estimates)? Which data is allowed?

Well-posedness = existence + uniqueness + *sharp estimates*

Consider the solution to $\Delta u = 0$, $u|_{\partial\Omega} = f$, $f \in L^p(\partial\Omega)$ (Dirichlet problem)

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Well-posedness $=$ existence $+$ uniqueness $+$ *sharp estimates*

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"well-posed in L^p" means that there is a unique solution with $\|\mathcal{N}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p}$

$$\mathcal{N}u = \sup_{\Gamma(x)} |u|, \quad \Gamma(x) \text{ is a non-tangential cone}$$

Harmonic measure

The well-posedness in L^p for $-\Delta$ on Ω is equivalent to $\omega \in A^\infty$ – quantifiable absolute continuity of harmonic measure.

Recall: for $E \subset \partial\Omega$, $X \in \Omega$, $\omega^X(E)$ is a solution to

$$-\Delta u = 0 \quad \text{in } \Omega, \quad u\Big|_{\partial\Omega} = \mathbf{1}_E$$

evaluated at point X, that is, $u(X)$.

Equivalently, $\omega^X(E)$ is the probability for a Brownian motion starting at $X \in \Omega$ to exit through the set $E \subset \partial\Omega$.

We say that $\omega \in A^\infty$, or, more precisely, that for each cube $Q \subset \mathbb{R}^n$, the harmonic measure $\omega^{X_Q} \in A_\infty(Q)$, with constants that are uniform in Q if the following holds.

Harmonic measure

We say that $\omega \in A^\infty$, or, more precisely, that for each cube $Q \subset \mathbb{R}^n$, the harmonic measure $\omega^{X_Q} \in A_\infty(Q)$, with constants that are uniform in Q if the following holds.

$\forall\, Q \subseteq \partial\Omega$ and every Borel set $F \subset Q$, we have

$$\omega^{X_Q}(F) \leq C \left(\frac{|F|}{|Q|}\right)^\theta \omega^{X_Q}(Q), \tag{1}$$

where X_Q is the "corkscrew point" relative to Q.

In other words, Brownian travelers "see" portions of the boundary proportionally to their Lebesgue size.

A_∞ property is a qualitative version of the condition that ω is absolutely continuous with respect to Lebesgue measure

Variable coefficients

Laplacian $-\Delta = -\mathrm{div}\nabla$ corresponds to a *perfectly uniform material*. Real materials are inhomogeneous: $L = -\mathrm{div} A(x)\nabla$
A is an elliptic (in some sense, positive) matrix

Moreover, if
Ω – domain above the Lipschitz graph φ

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = f \in L^p \end{cases} \qquad \mapsto \qquad \begin{cases} Lu = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \\ u|_{\partial\mathbb{R}^{n+1}_+} = f \in L^p \end{cases}$$

using the mapping $(x,t) \mapsto (x, t-\varphi(x))$

$$L = -\mathrm{div}_{x,t}\, A(x)\nabla_{x,t}$$

Hence, considering such matrices accounts both for rough materials and rough domains

Variable coefficients

$$L = -\mathrm{div}_{x,t}\, A(x,t)\nabla_{x,t} \ \text{ in } \ \mathbb{R}^{n+1}_+ = \{(x,t) : \ x \in \mathbb{R}^n, \ t > 0\}$$

For what A the boundary problems are well-posed in L^p?
Is smoothness an issue?
Recall that the maximum principle ($p = \infty$) holds for all elliptic A

Ω – domain above the Lipschitz graph φ

$$\left\{ \begin{array}{l} \Delta u = 0 \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = f \in L^p \end{array} \right. \qquad \mapsto \qquad \left\{ \begin{array}{l} Lu = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \\ u|_{\partial\mathbb{R}^{n+1}_+} = f \in L^p \end{array} \right.$$

using the mapping $(x,t) \mapsto (x, t - \varphi(x))$

$$L = -\mathrm{div}_{x,t}\, A(x)\nabla_{x,t}$$

the matrix of A has NO smoothness: bounded coefficients

Known results: REAL SYMMETRIC case

$$L = -\mathrm{div}_{x,t}\, A(x,t)\nabla_{x,t} \ \text{ in } \ \mathbb{R}^{n+1}_+ = \{(x,t) : \ x \in \mathbb{R}^n, \ t > 0\}$$

For what A the BVP's are well-posed?

Some smoothness in t is necessary: Caffarelli, Fabes, Kenig, '81

(recall: the change of variables from Δ gives a t-independent A)

If A is real and symmetric:

- Well-posedness for t-independent matrices:
  D. Jerison, C. Kenig, 1981 (Dirichlet);
  C. Kenig, J. Pipher, 1993 (Neumann)
- Perturbation: roughly, if $|A_1(x,t) - A_0(x,t)|^2 \, \frac{dxdt}{t}$ is Carleson and well-posedness holds for A_0 then it holds for A_1
  B. Dahlberg, 1986;
  R. Fefferman, C. Kenig, J. Pipher, 1991 (Dirichlet)
  C. Kenig, J. Pipher, 1993-95 (Regularity, Neumann+Regularity with *small* Carleson measure)

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$$\sup_Q \frac{1}{|Q|} \int_Q \int_0^{l(Q)} |A_1(x,t) - A_0(x,t)|^2 \frac{dxdt}{t} < \infty$$

  and well-posedness holds for A_0 then it holds for A_1
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What does it imply for a given matrix $A = A(x,t)$?
Note: $A(x,0)$ is t-independent. Thus, if $|A(x,t) - A(x,0)|^2 \frac{dxdt}{t}$ is Carleson then we have well-posedness. Carleson condition is sharp.

Real non-symmetric or complex case: obstacles

What if A is complex or even just real non-symmetric?

(Among applications: real non-symmetric - homogenization, living cells; complex - porous media; gateway to systems and higher order operators etc)

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(Among applications: real non-symmetric - homogenization, living cells; complex - porous media; gateway to systems and higher order operators etc)

Recall that for Δ on a Lipschitz domain the Dirichlet problem is well-posed for $2 - \varepsilon < p < \infty$

- $p = \infty$ – Maximum Principle
- $p = 2$ – integral identity (Hilbert space AND symmetry!)
- $2 < p < \infty$ – interpolation

Plus harmonic measure techniques or layer potentials

Similarly for the real symmetric case;
Neumann and regularity - "dual" $1 < p < 2 + \varepsilon$

Real non-symmetric or complex case: obstacles

General complex matrices:

- no positivity $\Longrightarrow$ no harmonic measure techniques
- no maximum principle (hence, no $p = \infty$)
  ($u \notin L^\infty$, even for $f \in C_0^\infty$; $e^{-tL}f$, $e^{-t\sqrt{L}}f$ are not bounded)
  $n \geq 5$ – V. G. Maz'ya, S. A. Nazarov and B. A. Plamenevskiĭ, 1982; P. Auscher, T. Coulhon, Ph. Tchamitchian, 1996; E.B. Davies, 1997; $n \geq 3$ – S. Hofmann, A. McIntosh, S.M., 2011 (based on an example of Frehse)
- no integral identity (because of lack of symmetry)
  hence, cannot approach L^2
- no well-posedness in L^2 – C. Kenig, H. Koch, J. Pipher, T. Toro, 2000
- the solutions, potentials, e^{-tL}, $e^{-t\sqrt{L}}$, Riesz transform $\nabla L^{-1/2}$ are beyond Calderón-Zygmund theory

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Real non-symmetric or complex case: results

$$L = -\mathrm{div}_{x,t}\, A(x,t)\nabla_{x,t} \ \text{ in } \ \mathbb{R}^{n+1}_+ = \{(x,t) :\, x \in \mathbb{R}^n,\ t > 0\}$$

For what A the BVP's are well-posed?

Perturbation: roughly, $|A_1(x,t) - A_0(x)|^2 \, \frac{dxdt}{t}$ has a small Carleson norm. Then well-posedness for L^0 implies the well-posedness for L^1

I will not discuss the questions of perturbation today

S. Hofmann, S.M., M. Mourgoglou, 2010-11

P. Auscher, A. Axelsson, 2010

The problem is: we don't know well-posedness for a *t-independent* $A(x,0)$, aside from the real symmetric case

Real non-symmetric or complex case: results

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For what A the BVP's are well-posed?

Well-posedness for t-independent matrices:

real non-symmetric, $\mathbb{R}^2$ (only $\mathbb{R}^2$!):
C. Kenig, H. Koch, J. Pipher, T. Toro, 2000 (Dirichlet);
C. Kenig, D. Rule, 2009 (Neumann, regularity)

real non-symmetric, $\mathbb{R}^n$ (any $n \geq 2$):
S. Hofmann, C. Kenig, S.M., J. Pipher (Dirichlet), 2013
S. Hofmann, C. Kenig, S.M., J. Pipher (Regularity), 2014
A. Barton, S.M. (fractional Sobolev/Besov spaces), 2014

Real non-symmetric case: Dirichlet problem

Theorem (S. Hofmann, C. Kenig, S.M., J. Pipher, 2013)

Let $A = A(x)$ be an elliptic matrix with real bounded measurable coefficients (possibly non-symmetric), and $L = -\mathrm{div}_{x,t} A(x) \nabla_{x,t}$.

Then there is a $p < \infty$ such that the Dirichlet problem with the data in L^p is well-posed.
Equivalently, for each cube $Q \subset \mathbb{R}^n$, the L-harmonic measure $\omega_L^{X_Q} \in A_\infty(Q)$, with constants that are uniform in Q.

Here $X_Q := (x_Q, \ell(Q))$ is the "Corkscrew point" relative to Q and a non-negative Borel measure $\omega \in A_\infty(Q_0)$, if there are $C, \theta > 0$ such that $\forall\, Q \subseteq Q_0$ and every Borel set $F \subset Q$, we have

$$\omega(F) \leq C \left(\frac{|F|}{|Q|} \right)^{\theta} \omega(Q). \qquad (2)$$

A_∞ property is a qualitative version of the condition that ω is absolutely continuous with respect to Lebesgue measure.

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Equivalently, for each cube $Q \subset \mathbb{R}^n$, the L-harmonic measure $\omega_L^{X_Q} \in A_\infty(Q)$, with constants that are uniform in Q.

Note: the result is sharp, in the sense that $\forall p_0 > 0$ there is an L such that the Dirichlet problem is not well-posed in L^{p_0}
(C. Kenig, H. Koch, J. Pipher, T. Toro, 2000)

$$L = \mathrm{div} \begin{pmatrix} 1 & m(x) \\ -m(x) & 1 \end{pmatrix} \nabla, \qquad m(x) = \begin{cases} k, & x > 0 \\ -k, & x < 0 \end{cases}$$

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The result is also sharp in the sense that it cannot be generalized to all complex matrices [H. Koch, S.M., 2014]
– more about this later

Strategy

Recall: the Dirichlet problem is well-posed in L^p if there is a unique solution with $\|\mathcal{N}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p}$,
$\mathcal{N}$ is the non-tangential maximal function

$$\mathcal{N}u = \sup_{\Gamma(x)} |u|, \quad \Gamma(x) \text{ is a non-tangential cone}$$

Recall also: the square function

$$S(u)(x) := \left(\iint_{\Gamma(x)} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2}$$

Strategy:

1. "$S < N$" (in L^q, $0 < q < \infty$, and localized)
2. "$N < S$" (in L^q, $0 < q < \infty$, and localized)
3. $S \approx N$ implies $\omega_L \in A^\infty$ (localized)

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1. "$S < N$" (in L^q, $0 < q < \infty$, and localized)
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Note 1: According to [Dahlberg, Jerison, Kenig, 84], if the Lebesgue measure is A^∞ with respect to the L-harmonic measure, then $S \approx N$. But A^∞ is exactly what we are trying to prove!

Note 2: $N < S$ in L^2 only was proved by Auscher and Axelsson

Note 3: the fact that $S \approx N$ on all Lipschitz domains implies $\omega_L \in A^\infty$ was known [Kenig, Koch, Pipher, Toro]. We will not be able to use that literally (orientation matters!) but still... Also, again in [Kenig, Koch, Pipher, Toro] the entire scheme was successfully used in dimension 2.

Note 4: How can one possibly approach $S < N$ in general???

$S < N$ estimates: dream case

Let u be a solution to $Lu = -\mathrm{div}_{x,t} A(x) \nabla_{x,t} u = 0$ in $\mathbb{R}^{n+1}$. Then

$$\|Su\|^2_{L^2(\mathbb{R}^n)} \overset{Fubini}{=} \iint_{\mathbb{R}^{n+1}_+} |\nabla u(x,t)|^2 \, t \, dtdx$$

$$\overset{ellipticity}{\approx} 2 \iint_{\mathbb{R}^{n+1}_+} \langle A \nabla u, \nabla u \rangle \, t \, dtdx \; = \; \iint_{\mathbb{R}^{n+1}_+} L(u^2) \, t \, dtdx$$

$$\overset{Int\ by\ parts}{=} \iint_{\mathbb{R}^{n+1}_+} u^2 \, L^*(t) \, dtdx + \int_{\mathbb{R}^n} |u(x,0)|^2 A_{n+1,n+1} \, dx$$

$$\approx \int_{\mathbb{R}^n} |u(x,0)|^2 dx \lesssim \|\mathcal{N} u\|^2_{L^2(\mathbb{R}^n)}.$$

IF $L^*(t) = 0$!!! (as it is in the case of the Laplacian)

$S < N$ estimates: reality

Need $L^*(t) = 0$ (as it is in the case of the Laplacian)

What is $L^*(t)$? Let us write $A = \left[\begin{array}{c|c} A_{\|} & \mathbf{b} \\ \hline \mathbf{c} & d \end{array}\right]$, where $A_{\|}$ is $n \times n$.

<u>Observation 1:</u> $L^*(t) = -\mathrm{div}_{x,t} A^*(x) \nabla_{x,t}(t) = -\sum_{i,j} \partial_i A_{ji} \partial_j (t) = -\sum_i \partial_i A_{n+1,i} = -\mathrm{div}_x \mathbf{c} - \partial_t A_{n+1,n+1} = -\mathrm{div}_x \mathbf{c}$

Hence, div-free part is harmless.

<u>Observation 2:</u> Let us map $\mathbb{R}^{n+1}_+$ into the graph domain $\Omega_\varphi := \{(x,t) : t > \varphi(x)\}$, via the mapping $t \to t - \varphi(x)$. Then $Lu = 0$ in $\mathbb{R}^{n+1}_+$ iff $L_\varphi v = 0$ in Ω_φ, $v(x,t) := u(x, t - \varphi(x))$, with

$$A_\varphi = \left[\begin{array}{c|c} A_{\|} & \mathbf{b} + A_{\|} \nabla_x \varphi \\ \hline \mathbf{c} + A_{\|}^* \nabla_x \varphi & \langle A\mathbf{p}, \mathbf{p} \rangle \end{array}\right], \quad \mathbf{p} := (\nabla_x \varphi, 1)$$

$S < N$ estimates: reality

Need $L^*(t) = 0$ (as it is in the case of the Laplacian)
Observation 1: $L^*(t) = -\mathrm{div}_x \mathbf{c}$
Observation 2: Let us map $\mathbb{R}^{n+1}_+$ into the graph domain $\Omega_\varphi := \{(x,t) : t > \varphi(x)\}$, via the mapping $t \to t - \varphi(x)$. Then $Lu = 0$ in $\mathbb{R}^{n+1}_+$ iff $L_\varphi v = 0$ in Ω_φ, $v(x,t) := u(x, t - \varphi(x))$, with

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Recall: if $\mathbf{c} \in L^2$, then it has an adapted Hodge decomposition: $\mathbf{c} = A_{\|}^*\nabla_x f + \mathbf{h}$, with $\mathrm{div}\,\mathbf{h} = 0$. Hence, taking $\varphi = -f$ above we are left with div-free h only!

THERE IS A MILLION OF PROBLEMS WITH THIS ARGUMENT

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Problem 1 (huge): you are now not on $\mathbb{R}^{n+1}_+$, but on Ω_φ, and if you calculate what it means in the above integration by parts, it means that you gained nothing (of course!)

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If you come back to $\mathbb{R}^{n+1}_+$ using the same change of variables, it will again show that you gained nothing (of course!)

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BUT you can maybe pull back using a smarter change of variables
Adapted pull-back: $L_\| := -\operatorname{div}_x A_\| \nabla_x$, $P_t = e^{-t^2 L_\|}$. Then

$$\rho(x,t) := (x, t + P^*_{\epsilon t}\,\varphi(x))$$

is a bijective map from the upper half space onto Ω_φ for ϵ small.

Why is it any better? A toy thought: if $L = -\Delta$, then $P_t\varphi$ is smooth, even for bad φ, it decays as $t \to \infty$... but there is more

$S < N$ estimates: what we actually do

Consider the pullback of L under the mapping

$$\rho(x,t) := \big(x, t - \varphi(x) + \mathcal{P}^*_{\eta t}\varphi(x)\big) \ : \mathbb{R}^{n+1}_+ \longrightarrow \mathbb{R}^{n+1}_+$$

where $\eta > 0$ small, and φ from the Hodge decomposition of $\mathbf{c}$.
Then $Lu = 0$ in $\mathbb{R}^{n+1}_+$ iff $L_1 u_1 = 0$, $u_1 := u \circ \rho$, where

$$A_1 := \left[\begin{array}{c|c} J A_{\|} & \mathbf{b} + A_{\|}\nabla_x\varphi - A_{\|}\nabla_x\mathcal{P}^*_{\eta t}\varphi \\ \hline \mathbf{h} - A^*_{\|}\nabla_x\mathcal{P}^*_{\eta t}\varphi & \frac{\langle A\,\mathbf{p},\mathbf{p}\rangle}{J} \end{array}\right].$$

Here, $\mathrm{div}\mathbf{h} = 0$ and $\mathbf{p}(x,t) = (\nabla_x\mathcal{P}^*_{\eta t}\varphi(x) - \nabla_x\varphi(x), -1)$.

$S < N$ estimates: what we actually do

After the pull-back dictated by Hodge decomposition... $Lu = 0$ in $\mathbb{R}^{n+1}_+$ iff $L_1 u_1 = 0$, $u_1 := u \circ \rho$, where

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Here, $\mathrm{div}\mathbf{h} = 0$ and $\mathbf{p}(x,t) = (\nabla_x \mathcal{P}^*_{\eta t}\varphi(x) - \nabla_x \varphi(x), -1)$.

Why $-A^*_\| \nabla_x \mathcal{P}^*_{\eta t}\varphi = -A^*_\| \nabla_x e^{-(\eta t)^2 L^*_\|}\varphi$ does not ruin everything (as opposed to $-A^*_\| \nabla_x \varphi$)?

$S < N$ estimates: what we actually do

Why $-A_{\|}^* \nabla_x \mathcal{P}_{\eta t}^* \varphi = -A_{\|}^* \nabla_x e^{-(\eta t)^2 L_{\|}^*} \varphi$ does not ruin everything (as opposed to $-A_{\|}^* \nabla_x \varphi$)?

- both the adapted Hodge decomposition (where $A_{\|}^*$ appears) and $\mathcal{P}_{\eta t}^* = e^{-(\eta t)^2 L_{\|}^*}$ "talk" to the operator L, hence, to the solutions
- by the solution of the Kato problem [Auscher, Hofmann, Lacey, McIntosh, Tchamitchian, 2002], it satisfies the square function estimates itself:

$$\|S(t\mathcal{P}_{\eta t} \mathrm{div}_x \varphi)\|_{L^2} \lesssim \|\varphi\|_{L^2}$$

  (and a variety of similar estimates holds)

More generally, the solution of the Kato problem plays a major role in the argument

$S < N$ estimates: what we actually do

THERE IS HALF A MILLION OF PROBLEMS WITH THE REMAINING ARGUMENT

Problem 2 (also big): φ coming from the adapted Hodge decomposition is $W^{1,2}$ $(\nabla\varphi \in L^2)$ and we need it to be Lipschitz!

Otherwise, there are too many L^2 functions under one integral... and even worse, our change of variables $\rho(x,t) := \big(x, t - \varphi(x) + \mathcal{P}^*_{\eta t}\varphi(x)\big)$ is not 1-1.

$\nabla\varphi, \nabla\mathcal{P}^*_{\eta t}\varphi$, etc. $\in L^2$, so we can extract big sets where they are (almost) L^∞, but we still have to get to those sets!

If φ is Lipschitz, $|\varphi(x) - \varphi(x_0)| \leq M|x - x_0|$ for x bad and x_0 good.

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Magically, PDE helps!

$S < N$ estimates: what we actually do

Magically, PDE helps! φ is a $W^{1,2}$ weak solution of

$$L_{\|}^{*}\varphi = \operatorname{div}(\mathbf{c}), \quad (\text{since } \operatorname{div}(\mathbf{c}) = \operatorname{div}(\mathbf{h} - A^{*}\nabla\varphi) = -\operatorname{div} A^{*}\nabla\varphi)$$

and the same is true with φ replaced by $\varphi - \varphi(x_0)$, for a fixed x_0. Thus, by Moser-type interior estimates,

$$\sup_{Q(x_0)} |\varphi - \varphi(x_0)| \lesssim \left(\fint_{2Q(x_0)} |\varphi(z) - \varphi(x_0)|^2 \, dz \right)^{1/2} + l(Q(x_0)) \, \|\mathbf{c}\|_\infty$$

roughly, bounded for $\varphi \in W^{1,2}$.

$S < N$ estimates: what we actually do

The remaining 1/4 million of problems include: recall

$$A_1 := \left[\begin{array}{c|c} J\,A_\| & \mathbf{b} + A_\|\nabla_x\varphi - A_\|\nabla_x\mathcal{P}^*_{\eta t}\varphi \\ \hline \mathbf{h} - A^*_\|\nabla_x\mathcal{P}^*_{\eta t}\varphi & \frac{\langle A\,\mathbf{p},\mathbf{p}\rangle}{J} \end{array}\right],$$

with $\mathbf{p}(x,t) = (\nabla_x\mathcal{P}^*_{\eta t}\varphi(x) - \nabla_x\varphi(x), -1)$

- φ is still not quite Lipschitz and so the new matrix, A_1, is not elliptic
- $\mathbf{c}$ is not L^2 (needed for Hodge), but only L^∞, hence, L^2_{loc}
- $A_1^{n+1,n+1}$ is not t-independent any more (hence, will contribute to $L^*(t)$)
- localize $\Rightarrow$ introduce cutoff Φ $\Rightarrow$ handle the entire A_1 interacting with $\nabla\Phi$

Somehow, in the end, it all works

$N < S$

- P. Auscher, A. Axelsson, 2011

$$\|\mathcal{N}(u)\|_{L^2(\mathbb{R}^n)} \lesssim \|S(u)\|_{L^2(\mathbb{R}^n)} .$$

- we use a localization procedure AND $S < N$ to show that for each cube Q, and each $0 < \theta < 1$, there is a set $K_Q = K_Q(\theta) \subset\subset R_Q$, $R_Q = Q \times (0, l(Q)/2)$, with $\mathrm{dist}(K_Q, \partial R_Q) \approx \ell(Q)$ (depending upon θ), such that

$$\fint_{\theta Q} |u(x)|^2 \, dx \leq C_\theta \left(\frac{1}{|Q|} \iint_{R_Q} |\nabla u(x,t)|^2 t dt dx + \sup_{K_Q} |u|^2 \right)$$

- then, in particular, using a good-lambda argument,

$$\|\mathcal{N}(u)\|_{L^q(\mathbb{R}^n)} \lesssim \|S(u)\|_{L^q(\mathbb{R}^n)} \quad 0 < q < \infty.$$

$\mathcal{N} \approx S$ on Lipschitz graph domains with transversal direction t

$$\Downarrow$$

ε-approximability: Given $\varepsilon > 0$, we say that u, $\|u\|_\infty \leq 1$, is ε-approximable if for every cube $Q_0 \subset \mathbb{R}^n$, there is a $\varphi = \varphi_{Q_0} \in W^{1,1}(T_{Q_0})$ such that $\|u - \varphi\|_{L^\infty(T_{Q_0})} < \varepsilon$ and $|\nabla \varphi|\, dxdt$ is a Carleson measure in Q_0.

$$\Downarrow$$

$$\omega \in A^\infty$$

$\mathcal{N} \approx S$ on Lipschitz graph domains with transversal direction $t \Rightarrow \varepsilon$-approximability $\Rightarrow \omega \in A^\infty$

Known: if $\Delta u = 0$ and u is bounded, then $|\nabla u|^2\, t dx dt$ is Carleson.
Question: is $|\nabla u|\, dx dt$ Carleson?
Answer: No. But it can be approximated arbitrarily well...

- Garnett; Varopoulos – harmonic function in $\mathbb{R}^{n+1}_+$ is ε-approximable
- Dahlberg – harmonic function in a Lipschitz domain is ε-approximable; $S \approx \mathcal{N}$ on all bounded Lipschitz domains implies ε-approximability
- Kenig, Koch, Pipher, Toro, 2000 – $S \approx \mathcal{N}$ on all bounded Lipschitz domains implies ε-approximability for general elliptic operators, which implies $\omega_L \in A^\infty$

In contrast to the above, our approach does not require S/N estimates on Lipschitz sub-domains of arbitrary orientation, but rather only local S/N estimates on Lipschitz graph domains, for which the t direction is transverse to $\partial\Omega$.

What about complex coefficients?

Theorem (H. Koch, S.M., 2014)

There exists an elliptic operator with complex t-independent bounded measurable coefficients such that the Dirichlet problem is not well-posed for any $1 < p < \infty$.

This uses a certain “combination” of counterexamples from [Frehse, 2008], [S.M., 2010], and [Kenig, Koch, Pipher, Toro, 2000]

Word of caution: the Dirichlet problem is defined in the same way as throughout this talk, while using a different maximal function (averaging?) might change the situation.

Extended summary

We proved that for any elliptic operator with real t-independent coefficients on any graph Lipschitz domain the following holds:

- The Dirichlet problem is well posed in L^p for some p
- L-harmonic measure is A^∞, in particular, absolutely continuous w.r.t. $d\sigma$
- ε-approximability for solutions
- $S \approx N$ estimates for solutions
- Carleson measure estimates for solutions
- Rellich: boundedness of the Dirichlet-to-Neumann operator in L^p (a posteriori)
- Regularity problem (S. Hofmann, C. Kenig, S.M., J. Pipher, 2014) and all intermediate problems in Besov/Sobolev spaces (A. Barton, S.M., 2015)

How far can this be pushed beyond Lipschitz domains?

Extended summary

How far can this be pushed beyond Lipschitz domains?

(e.g., for harmonic functions) – a few highlights

- F.&M. Riesz, 1916 – rectifiable, simply connected domain in $\mathbb{C}$
  One says that the set E is n-rectifiable, if there is a countable family of n-dimensional C^1 submanifolds $\{M_i\}_{i\geq 1}$ such that $H^n\left(E \setminus \cup_i M_i\right) = 0$
  Quantitative version: Lavrent'ev

Extended summary

How far can this be pushed beyond Lipschitz domains?

(e.g., for harmonic functions) – a few highlights

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- Jerison, Kenig, 1982 – non-tangentially accessible domains in $\mathbb{R}^n$: have corkscrew points (openness) and Harnack chains (connectivity)
  $\omega \in A^\infty$, $S \approx N$, ε-approximability, etc. all hold

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- Bishop, Jones, 1990 – NOT to uniformly rectifiable domains: A^∞ fails, even for harmonic measure, even in $\mathbb{C}$
  – one needs connectivity!
- Hofmann, Martell, S.M., 2014 – to uniformly rectifiable domains: square function estimates, ε-approximability, Carleson measure estimates all hold (while A^∞ fails)
  In fact, for general operators they carry over from Lipschitz domains