The Dirichlet boundary problem for second order parabolic operators satisfying Carleson condition

Martin Dindoš

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Admissible Domains

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Domains satisfying Lewis-Murray condition will be called *admissible*. We consider the following natural "surface measure" supported on boundary of such domain Ω.

For $A \subset \partial\Omega$ let

$$\sigma(A) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}\left(A \cap \{(X,t) \in \partial\Omega\}\right) dt.$$

Here $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure on the Lipschitz boundary $\partial\Omega_t = \{(X,t) \in \partial\Omega\}$.

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Let $\Gamma(.)$ be a collection of nontangential cones with vertices at boundary points $Q \in \partial\Omega$.

$\Gamma(Q) = \{(X,t) \in \Omega :\ d((X,t),Q) < (1+\alpha)\mathrm{dist}((X,t),\partial\Omega)\}$ for some $\alpha > 0$. Here d is the parabolic distance function

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$$N(u)(Q) = \sup_{X \in \Gamma(Q)} |u(X)|.$$

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The L^p Dirichlet problem

Definition

Let $1 < p \leq \infty$ and Ω be an admissible parabolic domain. Consider the parabolic Dirichlet boundary value problem

$$\begin{cases} v_t = \operatorname{div}(A\nabla v) & \text{in } \Omega, \\ v = f \in L^p & \text{on } \partial\Omega, \\ N(v) \in L^p(\partial\Omega, d\sigma). \end{cases} \tag{1}$$

where the matrix $A = [a_{ij}(X,t)]$ satisfies the uniform ellipticity condition and σ is the measure supported on $\partial\Omega$ defined above.
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$$\|N(v)\|_{L^p(\partial\Omega,d\sigma)} \lesssim \|f\|_{L^p(\partial\Omega,d\sigma)}. \tag{2}$$

The implied constant depends only the operator L, p, and the the domain Ω.

Remark. It is well-know that the parabolic PDE (1) with continuous boundary data is uniquely solvable. This can be established by considering approximation of bounded measurable coefficients of matrix A by a sequence of smooth matrices A_j and then taking the limit $j \to \infty$. This limit will exits in $L^\infty(\Omega) \cap W^{1,2}_{loc}(\Omega)$ using the the maximum principle and the L^2 theory. Uniqueness follows from the maximum principle.

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Parabolic measure

Thanks to the unique solvability of the continuous boundary value problem for each interior point $(X,t) \in \Omega$ we can define a unique measure ω^X supported on $\partial\Omega$ for which we have

$$u(X,t) = \int_{\partial\Omega} f(Z)\, d\omega^{(X,t)}(Z).$$

Here u is a solution of the Dirichlet boundary value problem with continuous data $f \in C(\partial\Omega)$.

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A_∞ condition

Let ω be doubling.

Recall that a measure ω is said to be A_∞ with respect to measure σ if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $E \subset \Delta$ and

$$\frac{\omega(E)}{\omega(\Delta)} < \epsilon, \qquad \text{then} \qquad \frac{\sigma(E)}{\sigma(\Delta)} < \delta.$$

The class A_∞ is related to another class of measures B_p, $p > 1$ which are classes of measures satisfying *Reverse Hölder* inequality.

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Consider the distance function δ of a point (X,t) to the boundary $\partial\Omega$

$$\delta(X,t) = \inf_{(Y,\tau)\in\partial\Omega} d[(X,t),(Y,\tau)].$$

If

$$\delta(X,t)^{-1}\left(\operatorname{osc}_{B_{\delta(X,t)/2}(X,t)} a_{ij}\right)^2$$

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A nonnegative measure $\mu : \Omega \to [0, \infty)$ is called Carleson if it is compatible with the "surface" measure σ we have defined. That is there exists a constant $C = C(r_0)$ such that for all $r \leq r_0$ and all surface balls $\Delta_r \subset \partial\Omega$ we have

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Observe that Rivera's result does not state for which p the L^p Dirichlet problem is solvable. Such $p < \infty$ can be potentially very large.

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L^p solvability for operators satisfying small Carleson condition

This is a joint result with Sukjung Hwang (Edinburgh).

Theorem

Let Ω be an admissible parabolic domain with character (L, N, C_0). Let $A = [a_{ij}]$ be a matrix with bounded measurable coefficients defined on Ω satisfying the uniform ellipticity and boundedness with constants λ and Λ and $\boldsymbol{B} = [b_i]$ be a vector with measurable coefficients defined on Ω. In addition, assume that

$$d\mu = \left[\delta(X,t)^{-1}\left(osc_{B_{\delta(X,t)/2}(X,t)} A\right)^2 + \delta(X,t) \sup_{B_{\delta(X,t)/2}(X,t)} |\boldsymbol{B}|^2\right] dX\,dt$$

is the density of a Carleson measure on Ω with Carleson norm $\|\mu\|_C$.

Then there exists $C(p) > 0$ such that if for some $r_0 > 0$ $\max\{L, \|\mu\|_{C,r_0}\} < C(p)$ then the L^p boundary value problem

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is solvable for all $2 \leq p < \infty$. Moreover, the estimate

$$\|N(v)\|_{L^p(\partial\Omega, d\sigma)} \leq C_p \|f\|_{L^p(\partial\Omega, d\sigma)},$$

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where the matrix $A = [a_{ij}(X,t)]$ satisfies the uniform ellipticity condition and σ is the measure supported on $\partial\Omega$ defined above. We say that Dirichlet problem with data in $BMO(\partial\Omega, d\sigma)$ is solvable if the (unique) solution u with continuous boundary data f satisfies the estimate

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for all parabolic surface balls $\Delta \subset \partial\Omega$.

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BMO solvability under A_∞ assumption

Theorem

Let Ω be an admissible parabolic domain and $L = \partial_t - div(A\nabla\cdot)$ a parabolic operator defined above. Assume that the parabolic measure for the operator L is in $A_\infty(d\sigma)$.

Then the BMO Dirichlet problem for the operator L is solvable and the estimate

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Observe that we have on the right hand side the L^∞ norm, not the BMO norm! Clearly

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Obtaining A_∞ for the parabolic measure

We are assuming that the estimate

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Our goal is to show that the measure is A_∞. That is, we want to show that for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $E \subset \Delta$ and

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$$E \subset \mathcal{O}_k \subset \mathcal{O}_{k-1} \subset \cdots \subset \mathcal{O}_0 \subset= \Delta.$$

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Key idea: Function f is taken of the form

$$f = \sum_{i=0}^{k} (-1)^i f_i,$$

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