Quasi Riesz transforms, Hardy spaces and generalised sub-Gaussian heat kernel estimates

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Contents

Current Section

Two main topics:

- Riesz transforms on Riemannian manifolds
- Hardy spaces on metric measure spaces

Assumptions: Volume growth, heat kernel estimates

The doubling volume property: (M, d, μ): a metric measure space. Set $V(x, r) = \mu(B(x, r))$. There exists a constant $C > 0$ such that

$$V(x, 2r) \leq CV(x, r), \forall x \in M, r > 0. \tag{D}$$

A simple consequence of (D):

$$\frac{V(x, r)}{V(x, s)} \leq C \left(\frac{r}{s}\right)^{\nu}, \ \forall x \in M, \ r \geq s > 0.$$

If M is non-compact, we also have a reverse inequality.

Current Section

Background

Strichartz (1983): For which kind of non-compact Riemannian manifold M and for which $p \in (1, \infty)$, the two semi-norms $\||\nabla f|\|_p$ and $\|\Delta^{1/2} f\|_p$ are equivalent, $\forall f \in C_c^\infty(M)$?

- The Riesz transform $\nabla\Delta^{-1/2}$ is L^p bounded on M if

$$\||\nabla f|\|_p \leq C\|\Delta^{1/2} f\|_p, \forall f \in C_0^\infty(M). \tag{R_p}$$

- The reverse Riesz transform is L^p bounded on M if

$$\|\Delta^{1/2} f\|_p \leq C\|\nabla f\|_p, \forall f \in C_0^\infty(M). \tag{RR_p}$$

By duality, we have $(R_p) \Rightarrow (RR_{p'})$, where p' is the conjugate of p.
Well-known results: On $\mathbb{R}^n$, Riemannian manifolds with non-negative Ricci curvature, Lie groups with polynomial growth etc, Riesz transforms are L^p bounded for $1 < p < \infty$.

Gaussian heat kernel estimates on Riemannian manifolds

(M, d, μ): a complete non-compact Riemannian manifold.
$(e^{-t\Delta})_{t>0}$: heat semigroup; $p_t(x, y)$: the heat kernel.
Most familiar heat kernel estimates:

- On-diagonal upper estimate

$$p_t(x, x) \leq \frac{C}{V(x, \sqrt{t})}, \forall x \in M, t > 0. \tag{DUE}$$

- Off-diagonal upper estimate:

$$p_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-c\frac{d^2(x, y)}{t}\right), \forall x, y \in M, t > 0. \tag{UE}$$

- Gradient upper estimate:

$$|\nabla p_t(x, y)| \leq \frac{C}{\sqrt{t}V(y, \sqrt{t})}, \forall x \in M, t > 0. \tag{G}$$

Gaussian heat kernel estimates and Riesz transforms

Theorem (Coulhon-Duong 99)

Let M be a complete non-compact Riemannian manifold satisfying (D) and (DUE). Then the Riesz transform $\nabla\Delta^{-1/2}$ is of weak type $(1,1)$ and thus L^p bounded for $1 < p \leq 2$.

Remark: Under the same assumptions, (R_p) may not hold for $p > 2$. For example: on the connected sum of $\mathbb{R}^n$ (consisting of two copies of $\mathbb{R}^n \backslash \{B(0,1)\}$, $n \geq 2$), the Riesz transform is L^p bounded for $1 < p < n$, but not L^p bounded for $p \geq n$, see [Coulhon-Duong 99], [Carron-Coulhon-Hassell 06].

Theorem (Auscher-Coulhon-Duong-Hofmann 04, Coulhon-Sikora 10)

Let M be a complete non-compact Riemannian manifold satisfying (D) and (G). Then (R_p) and (RR_p) hold for all $1 < p < \infty$.

Questions

It is not known whether the two conditions (D) and (DUE) are necessary for the L^p $(1 < p < 2)$ boundedness of the Riesz transform. The are two natural questions:

1. Can we remove (one of) the two conditions?
2. Can we replace the Gaussian heat kernel estimate by some other natural heat kernel estimates?
   For example, on manifolds satisfying (D) and sub-Gaussian heat kernel estimates, are the Riesz transforms L^p bounded for $1 < p < 2$?

Localisation of the Riesz transform

The Riesz transform $\nabla\Delta^{-1/2}$ is L^p bounded on M if and only if the local Riesz transform $\nabla(I+\Delta)^{-1/2}$ and the Riesz transform at infinity $\nabla e^{-\Delta}\Delta^{-1/2}$ are L^p bounded.
[Coulhon-Duong 99]: Under local doubling property and local Gaussian heat kernel upper bound (very weak), the local Riesz transform is L^p bounded for $1 < p \leq 2$.
Quasi Riesz transforms: $\nabla(I+\Delta)^{-1/2} + \nabla e^{-\Delta}\Delta^{-\alpha}$ with $\alpha \in (0, 1/2)$.

Quasi Riesz transforms on general Riemannian manifolds

Proposition

Let M be a complete manifold. Then, for any fixed $\alpha \in (0, 1/2)$, the operator $\nabla e^{-\Delta}\Delta^{-\alpha}$ is bounded on L^p for all $1 < p \leq 2$.

The proof easily follows from the fact below:

Proposition

Let M be a complete Riemannian manifold. Then for $1 < p \leq 2$, we have

$$\||\nabla e^{-t\Delta}|\|_{p\to p} \leq Ct^{-1/2}. \tag{G_p}$$

Note that (G_p) is also equivalent to the multiplicative inequality

$$\||\nabla f|\|_p^2 \leq C\|f\|_p\|\Delta f\|_p,$$

see [Coulhon-Duong 03, Coulhon-Sikora 10].
A simple proof: using Stein's approach to show the L^p boundedness of the Littlewood-Paley-Stein function.

Sub-Gaussian heat kernel estimates

Let $m > 2$.
Sub-Gaussian heat kernel upper estimate on a Riemannian manifold:

$$p_t(x,y) \leq \frac{C}{V(x,\rho^{-1}(t))} \exp\left(-cG(d(x,y),t)\right), \qquad (UE_{2,m})$$

where $\rho(t) = \begin{cases} t^2, & 0 < t < 1, \\ t^m, & t \geq 1; \end{cases}$ and $G(r,t) = \begin{cases} \dfrac{r^2}{t}, & t \leq r, \\ \left(\dfrac{r^m}{t}\right)^{1/(m-1)}, & t \geq r. \end{cases}$

Examples: fractal manifolds.
Construction of Vicsek manifolds from Vicsek graphs: replacing the edges with tubes, and gluing them smoothly at the vertices.

A typical examples

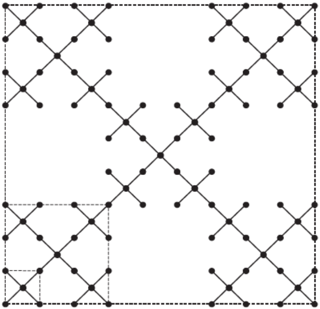

Figure: **A fragment of the Vicsek graph in $\mathbb{R}^2$**

Generally in $\mathbb{R}^n$, let $D = \log_3(2^n + 1)$. The Vicsek manifold satisfies $\mu(B(x,r)) \simeq r^D$ and $(UE_{2,m})$ with $m = D + 1$.

Comparing Sub-Gaussian and Gaussian heat kernel estimates

The Gaussian heat kernel upper bound coincides with $(UE_{2,2})$.
Let $m > 2$. For $t > 1$,

$$V(x, t^{1/2}) > V(x, t^{1/m}).$$

That means $p_t(x, x)$ decays with t more slowly in the sub-Gaussian case than in the Gaussian case.
For $t \geq \max\{1, d(x, y)\}$,

$$\left(\frac{d^m(x, y)}{t}\right)^{1/(m-1)} \geq \frac{d^2(x, y)}{t},$$

then $p_t(x, y)$ decays with $d(x, y)$ faster in the sub-Gaussian case than in the Gaussian case.
But on the whole, the two kinds of pointwise estimates are incomparable.

Sub-Gaussian heat kernel upper estimates and quasi Riesz transforms

Theorem

Let M be a complete manifold satisfying (D) and $(UE_{2,m})$, then the quasi Riesz transform $\nabla e^{-\Delta}\Delta^{-\alpha} + \nabla(I+\Delta)^{-1/2}$ is weak $(1,1)$ bounded and L^p bounded for $1 < p \leq 2$.

Proof: the Calderón-Zygmund theory, the weighted estimate for the gradient of the heat kernel, similarly as in [Coulhon-Duong 99].

Counterexample for $p > 2$

For $p > 2$, the Riesz transform is not L^p bounded on Vicsek manifolds.

Proposition

Let M be a Vicsek manifolds, then the Riesz transform is not L^p bounded for $p > 2$.

This is an improvement of the result in [Coulhon-Duong 03], where (RR_p) was shown to be false for $1 < p < \frac{2D}{D+1}$.
Idea of the Proof: show that (RR_p) is not true for $1 < p < 2$.
Take $D' = \frac{2D}{D+1}$. If (RR_p) holds, the heat kernel estimate $p_t(x,x) \leq Ct^{-\frac{D}{D+1}}$ $(t \geq 1)$ implies that (see [Coulhon 92]) for all $f \in \mathcal{C}_0^\infty(M)$ such that $\|f\|_p/\|f\|_1 \leq 1$,

$$\|f\|_p^{1+\frac{p}{(p-1)D'}} \leq C\|f\|_1^{\frac{p}{(p-1)D'}} \|\Delta^{1/2} f\|_p \leq C\|f\|_1^{\frac{p}{(p-1)D'}} \||\nabla f|\|_p.$$

Choose $\{F_n\}$ to contradict the above inequality.

Construction for $\{F_n\}$

[Barlow-Coulhon-Grigor'yan 2001]: Let $\Omega_n = \Gamma \bigcap [0, 3^n]^N$ and $q = 2^N + 1 = 3^D$. Denote by z_0 the centre of Ω_n and by $z_i, i \geq 1$ its corners. Define F_n as follows: $F_n(z_0) = 1, F_n(z_i) = 0, i \geq 1$, and extend F_n as a harmonic function in the rest of Ω_n. If z belongs to some γ_{z_0, z_i}, then $F_n(z) = 3^{-n} d(z_i, z)$. If not, then $F_n(z) = F_n(z')$, where z' is the nearest vertex in certain line of z_0 and z_i.

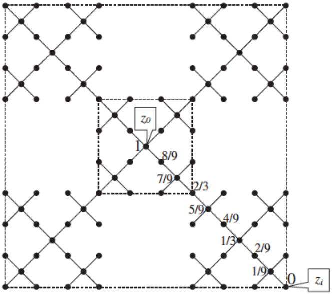

Figure: **The function F_2**

Current Section

Hardy spaces associated with the Laplacian

References:
[Auscher-McIntosh-Russ 08]: Hardy spaces of differential forms of all degrees on complete Riemannian manifolds satisfying the doubling volume property.
[Hofmann-Lu-Mitrea-Mitrea-Yan 11]: metric measure space with doubling measure, and with a non-negative self-adjoint operator satisfying the Davies-Gaffney estimate:

$$|< e^{-tL} f_1, f_2 >| \leq C \exp\Big(- \frac{d^2(U_1, U_2)}{ct} \Big) \|f_1\|_2 \|f_2\|_2, \ \forall t > 0,$$

[Uhl 11]: metric measure space with doubling measure, and with an injective non-negative self-adjoint operator satisfying the generalised Davies-Gaffney estimate.

$L^{p_0} - L^{p_0'}$ off-diagonal heat kernel estimates on metric measure spaces

Let $1 \le p_0 < 2$. Let $\beta_1 < \beta_2$. We say that M satisfies the generalised $L^{p_0} - L^{p_0'}$ off-diagonal estimate if for $x, y \in M$ and $t > 0$,

$$\begin{aligned}
&\|\mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)}\|_{p_0 \to p_0'} \\
&\le \begin{cases} \dfrac{C}{V^{\frac{1}{p_0} - \frac{1}{p_0'}}(x,t)} \exp\left(- c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_1}{\beta_1 - 1}}\right) & 0 < t < 1, \\ \dfrac{C}{V^{\frac{1}{p_0} - \frac{1}{p_0'}}(x,t)} \exp\left(- c\left(\dfrac{d(x,y)}{t}\right)^{\frac{\beta_2}{\beta_2 - 1}}\right), & t \ge 1, \end{cases}
\end{aligned} \qquad (DG^{p_0, p_0'}_{\beta_1, \beta_2})$$

where

$$\rho(t) = \begin{cases} t^{\beta_1}, & 0 < t < 1, \\ t^{\beta_2}, & t \ge 1; \end{cases}$$

Some consequences

For $p_0 = 2$,

$$\|\mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)}\|_{2\to 2} \leq \begin{cases} C \exp\Big(-c\Big(\frac{d(x,y)}{t}\Big)^{\frac{\beta_1}{\beta_1-1}}\Big) & 0 < t < 1, \\ C \exp\Big(-c\Big(\frac{d(x,y)}{t}\Big)^{\frac{\beta_2}{\beta_2-1}}\Big), & t \geq 1, \end{cases} . \qquad (DG_{\beta_1,\beta_2})$$

For $p_0 = 1$,

$$p_{\rho(t)}(x,y) \leq \begin{cases} \frac{C}{V(x,t)} \exp\Big(-c\Big(\frac{d(x,y)}{t}\Big)^{\frac{\beta_1}{\beta_1-1}}\Big) & 0 < t < 1, \\ \frac{C}{V(x,t)} \exp\Big(-c\Big(\frac{d(x,y)}{t}\Big)^{\frac{\beta_2}{\beta_2-1}}\Big), & t \geq 1. \end{cases} . \qquad (UE_{\beta_1,\beta_2})$$

$(UE_{\beta_1,\beta_2}) \Rightarrow (DG^{p_0,p_0'}_{\beta_1,\beta_2}) \Rightarrow (DG_{\beta_1,\beta_2})$.

Examples

- Euclidean spaces with higher order divergence form operators;
- Some fractals. For example, Sierpinski carpets, Sierpinski gaskets, Vicsek sets etc.
- Riemannian manifolds. For any $D \geq 1$ and any $2 \leq m \leq D+1$, there exists Riemannian manifold satisfying the polynomial volume growth $V(x,r) \simeq r^D$, $r \geq 1$, and $(UE_{2,m})$.

Outline

1. Define Hardy spaces via molecules $H^1_{L,\rho,mol}(M)$ and via square functions $H^p_{L,S^\rho_h}(M)$ which are adapted to the heat kernel estimate.
2. The two H^1 spaces defined via molecules and via square function are the same: $H^1_{L,\rho,mol}(M) = H^1_{L,S^\rho_h}(M)$.
3. The comparison between Hardy spaces $H^p_{L,S^\rho_h}(M)$, $H^p_{L,S_h}(M)$ and $L^p(M)$.
4. Application: the $H^1 - L^1$ boundedness of (quasi) Riesz transforms on Riemannian manifolds with sub-Gaussian heat kernel estimates.

Hardy spaces defined via molecules

Let M be a metric measure space satisfying (D) and (DG_{β_1,β_2}).

Definition

Let $\varepsilon > 0$ and $K > \frac{\nu}{2\beta_1}$. A function $a \in L^2(M)$ is called a $(1, 2, \varepsilon)-$molecule associated to L if there exist a function $b \in \mathcal{D}(L)$ and a ball B with radius r_B such that

1. $a = L^K b$;
2. It holds for every $k = 0, 1, \cdots, K$ and $i = 0, 1, 2, \cdots$, we have

$$\|(\rho(r_B)L)^k b\|_{L^2(C_i(B))} \leq \rho(r_B)2^{-i\varepsilon}V(2^i B)^{-1/2},$$

where $C_0(B) = B$, and $C_i(B) = 2^i B \backslash 2^{i-1} B$ for $i = 1, 2, \cdots$.

Hardy spaces defined via molecules

Definition

We say that $f = \sum_{n=0}^{\infty} \lambda_n a_n$ is a molecular $(1, 2, \varepsilon)-$representation of f if $(\lambda_n)_{n\in\mathbb{N}} \in l^1$, each a_n is a molecule, and the sum converges in the L^2 sense. We denote by $\mathbb{H}^1_{L,\rho,mol}$ the collection of all the functions with a molecular representation, where the norm of $\|f\|_{\mathbb{H}^1_{L,\rho,mol}(M)}$ is given by

$$\inf\Big\{\sum_{n=0}^{\infty}|\lambda_n| : f = \sum_{n=0}^{\infty}\lambda_n a_n \text{ is a molecular } (1,2,\varepsilon) - \text{representation}\Big\}.$$

The Hardy space $H^1_{L,\rho,mol}(M)$ is defined as the completion of $\mathbb{H}^1_{L,\rho,mol}(M)$ with respect to this norm.

Hardy spaces defined via square functions

Consider the quadratic operator associated with the heat kernel defined by the following conical square function

$$S_h^\rho f(x) = \Big(\iint_{\Gamma(x)} |\rho(t) L e^{-\rho(t)L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \Big)^{1/2},$$

where the cone $\Gamma(x) = \{(y,t) \in M \times (0,\infty) : d(y,x) < t\}$.

Definition

The Hardy space $H^p_{L,S_h^\rho}(M)$, $p \geq 1$ is defined as the completion of the set $\{f \in \overline{R(L)} : \|S_h^\rho f\|_{L^p} < \infty\}$ with respect to the norm $\|S_h^\rho f\|_{L^p}$. The $H^p_{L,S_h^\rho}(M)$ norm is defined by $\|f\|_{H^p_{L,S_h^\rho}(M)} := \|S_h^\rho f\|_{L^p(M)}$.

$H^1_{L,\rho,mol}(M) = H^1_{L,S_h^\rho}(M)$

Theorem

Let M be a metric measure space satisfying the doubling volume property (D) and the heat kernel estimate (DG_{β_1,β_2}), $\beta_1 \leq \beta_2$. Then $H^1_{L,\rho,mol}(M) = H^1_{L,S_h^\rho}(M)$. Moreover, $\|f\|_{H^1_{L,\rho,mol}(M)} \simeq \|f\|_{H^1_{L,S_h^\rho}(M)}$.

Comparison of H^p and L^p

Theorem

Let M be a non-compact metric measure space satisfying the doubling volume property (D) and the heat kernel estimate $(DG^{p_0,p_0'}_{\beta_1,\beta_2})$. Then $H^p_{L,S_h^\rho}(M) = \overline{R(L)\cap L^p(M)}^{L^p(M)}$ for $p_0 < p < p_0'$.

Show that the adapted conical square functions is weak L^{p_0} bounded. The tools include the Calderón-Zygmund decomposition, functional calculus, and the $L^p - L^q$ theory for operators.

Corollary

Let M be a non-compact metric measure space satisfying the doubling volume property (D) and the following pointwise heat kernel estimate (UE_{β_1,β_2}). Then $H^1_{L,\rho,mol}(M) = H^1_{L,S_h^\rho}(M)$, and $H^p_{L,S_h^\rho}(M) = L^p(M)$ for $1 < p < \infty$.

$L^p(M) \neq H^p_{\Delta,S_h}(M)$

Theorem

Let M be a Riemannian manifold with polynomial volume growth $V(x,r) \simeq r^d$, $r \geq 1$, as well as two-sided sub-Gaussian heat kernel estimate $(HK_{2,m})$ with $2 < m < d/2$, that is, $(UE_{2,m})$ and the matching lower bound. Then $L^p(M) \subset H^p_{\Delta,S_h}(M)$ doesn't hold for $p \in \left(\frac{d}{d-m}, 2\right)$.

Remark 1 Vicsek manifolds satisfy $(HK_{2,m})$ with $m = d+1$.
Remark 2 The Hardy space $H^p_{\Delta,S_h}(M)$ is defined via

$$S_h f(x) = \left(\iint_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}.$$

Idea for the proof: Using obolev inequality, Green operator and the lower estimate of the heat kernel to prove by contradiction.

Application

Theorem

Let M be a manifold satisfying the doubling volume property (D) and the heat kernel estimate $(UE_{2,m})$, $m > 2$. Then for any fixed $\alpha \in (0, 1/2)$, the operator $\nabla e^{-\Delta}\Delta^{-\alpha}$ is $H^1_{\Delta,m} - L^1$ bounded.

We can recover again the L^p boundedness of quasi Riesz transforms from the complex interpolation theorem over Hardy spaces $H^p_{\Delta,S_h^m}(M)$

Thanks for your attention!