Representation for weak solutions of elliptic boundary value problems

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Workshop on harmonic analysis, PDE and geometric measure theory

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- based on two joint works with my student Sebastian Stahlhut and with Mihalis Mourgoglou, available on arXiv.
- development of Dirac operators for BVP from earlier works with Andreas Axelsson, Alan McIntosh, Steve Hofmann.
- Nothing could be done without the methodology of the solution of the Kato conjecture.

$\Omega = \mathbf{R}^{n+1}_+$. Same analysis works in unit ball and every domain obtained by bilipschitz change of variables.
Points in Ω: (t, x), $t > 0$, $x \in \mathbf{R}^n$.
Measurable, bounded, with $M_{m\times m}(\mathbf{C})$-valued coefficients $A_{i,j}$, $i, j = 0, \dots n$, $m \geq 1$. + Ellipticity (later)
Weak solution: $u \in W^{1,2}_{loc}(\Omega; \mathbf{C}^m)$ and $Lu = 0$ holds in $\mathcal{D}'(\Omega; \mathbf{C}^m)$: with summation convention

$$\mathrm{Re} \int_\Omega A^{\alpha,\beta}_{i,j} \partial_j u^\beta \, \overline{\partial_i \varphi^\alpha} dxdt = 0, \quad \forall \varphi \in C_0^\infty(\Omega; \mathbf{C}^m).$$

Short notation: $A^{\alpha,\beta}_{i,j} \partial_j u^\beta \, \overline{\partial_i \varphi^\alpha} = A\nabla u \cdot \nabla \overline{\varphi}$ and $Lu = \mathrm{div} A \nabla u$ in Ω.
$i = 0$ corresponds to the vertical direction, $i = 1, \dots, n$ to the horizontal directions.

Strongly elliptic real equations

• local regularity theory (Nash-Moser)

• Maximum principle: the classical Dirichlet problem with data $f \in C_c(\mathbf{R}^n)$ can be uniquely solved: $u \in C(\overline{\Omega})$ is bounded with $\|u\|_\infty \leq \|f\|_\infty$ and can be represented by applying the Riesz representation theorem:

$$u(t,x) = \int_{\mathbf{R}^n} f \, d\omega_L^{t,x}$$

Probability measure $\omega_L^{t,x}$ is the L-harmonic measure for L at pole (t,x).

• Possible ansatz by using layer potential methods from the fundamental solution.

• Many results starting in the late '70s for real symmetric equations: Dahlberg, Jerison, Kenig, Verchota, R. Fefferman, Pipher.... and recently for real non-symmetric equations: Hofmann, Kenig, Mayboroda, Pipher.

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Complex equations or systems

- no local regularity
- no maximum principle
- no fundamental solution

BVP problems in L^p, $1 < p < \infty$

Typical problems in harmonic analysis (for example for the Laplace equation).

• (Dir, A, p): Solve $Lu = 0$ with $\|\widetilde{N}(u)\|_p < \infty$ and $u_0 = f$ given in $L^p(\mathbf{R}^n; \mathbf{C}^m)$.

• (Reg, A, p): Solve $Lu = 0$ with $\|\widetilde{N}(\nabla u)\|_p < \infty$ and $\nabla_{\tan} u_0 = \nabla_{\tan} f$, f given in $\dot{W}^{1,p}(\mathbf{R}^n; \mathbf{C}^m)$.

• (Neu, A, p): Solve $Lu = 0$ with $\|\widetilde{N}(\nabla u)\|_p < \infty$ and $\partial_{\nu_A} u|_{t=0} = g$ given in $L^p(\mathbf{R}^n; \mathbf{C}^m)$.

$\widetilde{N}(h)$ is non-tangential maximal interior control of h defined in Ω: it comes up quite naturally.

Not always solvable nor well-posed. No comprehensive theory at this time.

Find trace and representation not on u but on its full gradient.

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non-tangential maximal function

Whitney ball:

$$W(t,x) := [(1-c_0)t,(1+c_0)t] \times B(x;c_1 t),$$

for fixed $c_0 \in (0,1)$, $c_1 > 0$.

$$\widetilde{N}(h)(x) := \sup_{t>0} t^{-(n+1)/2} \|h\|_{L_2(W(t,x))}$$

It is the L^2-variant of the usual pointwise maximal function

$$h^*(x) = \sup_{|x-y|<t} |h(t,y)|.$$

Theory for L^p, $1 < p < \infty$ well-known from Fatou type results. Fefferman-Stein extended this to $p \leq 1$ using the real Hardy space H^p (which agrees with L^p when $p > 1$).

Theorem

Let $0 < p < \infty$. Let u be harmonic in Ω. The following are equivalent

1. $\|u^*\|_p < \infty$.
2. $\|S(t\nabla u)\|_p < \infty$ *and u vanishes as* $t \to \infty$.
3. *There exists a unique* $f \in H^p$ *such that* $u(t,x) = P_t * f(x)$, *where P_t is the Poisson kernel.*

Moreover, $\|f\|_{H^p} \sim \|u^*\|_p \sim \|S(t\nabla u)\|_p$.

Lusin area functional: $S(F)(x) = \Big(\iint_{|x-y|<t} \big|F(t,y)\big|^2 \, \frac{dtdy}{t^{n+1}} \Big)^{\frac{1}{2}}$

Solving via Cauchy-Riemann

- $\Delta u = 0$ on $\Omega = \{(t,x); t > 0, x \in \mathbf{R}\}$, $u_0 = g \in L^2(\mathbf{R})$ via Hardy spaces.
- $\Delta u = 0 \iff u = \mathrm{Re}\, v, \partial_{\bar z} v = 0$ (Cauchy-Riemann)
- Write $v = a + ib = \begin{bmatrix} a \\ b \end{bmatrix}$. Then

$$\partial_{\bar z} v = 0 \iff \partial_t v + Dv = 0, \quad D = \begin{bmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{bmatrix}.$$

$D = D^*$ and $sp(D) = \mathbf{R}$. To solve for v, need initial value v_0 to be in $\mathrm{R}(\chi^+(D)) = \tilde{\mathcal{H}}^+$ where $\chi^+ = 1_{(0,\infty)}$. Now

$$v_0 \in \tilde{\mathcal{H}}^+ \iff v_0 = \begin{bmatrix} a \\ H(a) \end{bmatrix}$$

where H is the Hilbert transform: $H(a)$ is the conjugate function of a.

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• The space $\tilde{\mathcal{H}}^+$ is thus the classical (holomorphic) Hardy space: L^2 functions that have a (controlled) holomorphic extension to the upper half-space. If $v_0 \in \tilde{\mathcal{H}}^+$, then

$$v_t(x) = (e^{-tD} v_0)(x)$$

is nothing but the Cauchy extension formula.

• Conversely $\tilde{\mathcal{H}}^+$ is the trace space of holomorphic functions v in the upper half-space with control $\|v^*\|_2 < \infty$ (Fatou theory).

• Scheme for solving the Dirichlet problem is

$$g(= Dir.data) \to v_0 = \begin{bmatrix} g \\ H(g) \end{bmatrix} \in \tilde{\mathcal{H}}^+ \to v_t = e^{-tD} v_0 \to u = \mathrm{Re}\, v$$

• In higher dimensions and for various spaces of data, similar strategy following Stein and Weiss: introduction of the real Hardy spaces on Euclidean spaces.

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Assumptions

1. A measurable, bounded, with $M_{m\times m}(\mathbf{C})$-valued coefficients
2. We assume A be t-independent: t-dependence can be considered but some regularity in t is required (otherwise, counterexamples to solvability exist: Cwikel-Fabes-Kenig)
3. strict accretivity in the sense of Gårding: there exists $\kappa > 0$ s.t. $\forall\, u \in C^1_c(\mathbf{R}^{n+1}; \mathbf{C}^m)$

$$\operatorname{Re} \int_{\mathbf{R}^n} A(x)\nabla_{t,x}u \cdot \overline{\nabla_{t,x}u}\, dx \geq \kappa \int_{\mathbf{R}^n} |\nabla_{t,x}u|^2\, dx.$$

From now on $m = 1$ (same results): ellipticity is equivalent to the usual pointwise lower estimate $\operatorname{Re} A(x)\xi \cdot \bar{\xi} \geq \kappa|\xi|^2$.

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Conormal gradient

Write

$$Lu = \partial_t(a\partial_t u + b\cdot\nabla_x u) + \mathrm{div}_x(c\partial_t + d\nabla_x u)$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\partial_{\nu_A} u(t,x) := a(x)\partial_t u(t,x) + b(x)\cdot\nabla_x u(t,x)$$

Conormal gradient of u :

$$\nabla_A u(t,x) = \begin{bmatrix} \partial_{\nu_A} u(t,x) \\ \nabla_x u(t,x) \end{bmatrix}$$

Dirac operator

$$\nabla_A u(t,x) = \begin{bmatrix} \partial_{\nu_A} u(t,x) \\ \nabla_x u(t,x) \end{bmatrix}$$

$$Lu = 0 \iff \partial_t \nabla_A u + DB\nabla_A u = 0$$

$$D = \begin{bmatrix} 0 & \mathrm{div}_x \\ -\nabla_x & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ 0 & I \end{bmatrix}^{-1}.$$

Lemma (A., Axelsson, McIntosh)

$u \mapsto \nabla_A u$ correspondence between weak solutions of $Lu = 0$ and distributional solutions $F \in L^2_{loc}(\Omega; \mathbf{C}^{n+1})$ of

$$\partial_t F + DBF = 0, \quad \mathrm{curl}_x F_\parallel = 0.$$

Notation: $F = \begin{bmatrix} F_\perp \\ F_\parallel \end{bmatrix}$, with $F_\perp$ $\mathbf{C}$-valued, $F_\parallel$ $\mathbf{C}^n$-valued.

Functional calculus

D self-adjoint on $L^2(\mathbf{R}^n;\mathbf{C}^{n+1})$, $\overline{\mathsf{R}(D)} = \{F;\mathrm{curl}_x F_\parallel = 0\}$.

Theorem

- *(AAM) B is bounded and accretive iff A is bounded and accretive.*
- *(Classical) DB bi-sectorial operator of type $\omega < \pi/2$ on L^2 with $\overline{R(DB)} = \overline{R(D)}$: hence $L^2 = \overline{R(D)} \oplus N(DB)$.*
- *(Axelsson-Keith-McIntosh) DB has $H^\infty(S_\mu)$-functional calculus on $\overline{R(D)}$ for bi-sectors S_μ, $\omega < \mu < \pi/2$.*

$$\int_0^\infty \|\psi_t(DB)h\|_2^2 \frac{dt}{t} \sim \|h\|_2^2, \quad \forall h \in \overline{\mathsf{R}(D)}, \forall \psi \in \Psi^{\neq 0}(S_\mu)$$

$$\|b(DB)h\|_2 \lesssim \|b\|_\infty \|h\|_2, \quad \forall b \in H^\infty(S_\mu), \forall h \in \overline{\mathsf{R}(D)}.$$

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- *(Classical) DB bi-sectorial operator of type $\omega < \pi/2$ on L^2 with $\overline{R(DB)} = \overline{R(D)}$: hence $L^2 = \overline{R(D)} \oplus N(DB)$.*
- *(Axelsson-Keith-McIntosh) DB has $H^\infty(S_\mu)$-functional calculus on $\overline{R(D)}$ for bi-sectors S_μ, $\omega < \mu < \pi/2$.*

$$\int_0^\infty \|\psi_t(DB)h\|_2^2 \frac{dt}{t} \sim \|h\|_2^2, \quad \forall h \in \overline{\mathrm{R}(D)}, \forall \psi \in \Psi^{\neq 0}(S_\mu)$$

$$\|b(DB)h\|_2 \lesssim \|b\|_\infty \|h\|_2, \quad \forall b \in H^\infty(S_\mu), \forall h \in \overline{\mathrm{R}(D)}.$$

Functional calculus

D self-adjoint on $L^2(\mathbf{R}^n;\mathbf{C}^{n+1})$, $\overline{\mathsf{R}(D)} = \{F; \mathrm{curl}_x F_{\parallel} = 0\}$.

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L^p analog: Hardy space

Let $0 < p < \infty$. Define H^p_{DB} as the completion of the space of those $h \in \overline{\mathsf{R}(D)}$ for which $S(F) \in L^p$ with $F(t,x) = \psi_t(DB)h(x)$.

For $p = 2$, $H^2_{DB} = \overline{\mathsf{R}(D)}$ (by H^∞-fc). For $p \neq 2$, this space depends on $\psi \in \Psi(S_\mu)$ but one can show that for fixed p, there is a large class of ψ for which these spaces are all the same with equivalent (quasi-)norms $\|S(F)\|_p < \infty$.

Functions $b(DB)$ with $b \in H^\infty(S_\mu)$ have bounded extensions to H^p_{DB}. In particular, for $\chi^+ = 1_{\Re z > 0}$, we have a natural closed spectral subspace $H^{p,+}_{DB}$ defined as the range of χ^+_p = bounded extension of $\chi^+(DB)$. For $\chi^- = 1_{\Re z < 0}$, we obtain $H^{p,-}_{DB}$. In fact, (χ^+_p, χ^-_p) forms a pair of bounded complementary projections on H^p_{DB}.

Also, analytic semigroup $(e^{-t|DB|})_{t\geq 0}$ extends to H^p_{DB}, with

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What is the Hardy space?

Difficulty: A completion is an abstract object.
The space H^p_D (when $A = I$) is naturally identified to a subspace of H^p with the restriction $\frac{n}{n+1} < p$. What about H^p_{DB}?

Theorem (A., Stahlhut)

There is an open interval $I_L \subset (\frac{n}{n+1}, \infty)$ of values p for which $H^p_{DB} = H^p_D$ with equivalent norms. This interval contains $[\frac{2n}{n+2}, 2]$.

Remark: for real equations, one can show that I_L contains $[1, 2]$. For dimensions $n = 1$ or for constant systems, one can show that $I_L = (\frac{n}{n+1}, \infty)$.

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Classification thm for Reg and Neu

Theorem (A., Mourgoglou)

Let $p \in I_L$. Then, for any weak solution u to $Lu = 0$ on Ω, the followings are equivalent:

(i) $\|\widetilde{N}(\nabla u)\|_p < \infty$.

(ii) $\|S(t\partial_t \nabla u)\|_p < \infty$ *and* $\nabla_A u(t, \cdot)$ *converges to 0 in the sense of distributions as* $t \to \infty$.

(iii) $\exists! F_0 \in H^{p,+}_{DB}$, *called the conormal gradient of u at $t = 0$ and denoted by* $\nabla_A u|_{t=0}$, *such that* $\nabla_A u(t, \,.\,) = S_p(t)F_0$.

(iv) $\exists F_0 \in H^p_D$ *such that* $\nabla_A u(t, \,.\,) = S_p(t)\chi^+_p F_0$.

Here, $S_p(t)$ is the bounded extension to $H^p_D = H^p_{DB}$ of $e^{-t|DB|}$. Moreover,

$$\|\widetilde{N}(\nabla u)\|_p \sim \|S(t\partial_t \nabla u)\|_p \sim \|\nabla_A u|_{t=0}\|_{H^p} \sim \|\chi^+_p F_0\|_{H^p}.$$

Consequences

The previous theorem shows semigroup representation for conormal gradients of weak solutions u with conditions (i) or (ii) and that $H^{p,+}_{DB}$ is the trace space of those conormal gradients.

Proofs are complicated: one can not apply the Fatou type results based on the maximum principle. They are independent of well-posedness of the BVPs.

Well-posedness of the BVPs can be shown to be equivalent to invertibility of boundary maps. Fix $p \in I_L$.

1. The regularity problem (Reg, A, p) is well-posed iff the map $H^{p,+}_{DB} \to H^p_{\nabla} : \nabla_A u|_{t=0} \to \nabla_x u|_{t=0}$ is invertible.
2. The Neumann problem (Neu, A, p) is well-posed iff the map $H^{p,+}_{DB} \to H^p : \nabla_A u|_{t=0} \to \partial_{\nu_A} u|_{t=0}$ is invertible.

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Establishing invertibility is another story.

Theory of Sobolev spaces adapted to DB. Define $\dot W^{-1,p}_{DB}$ as the completion of the space of those $h \in \overline{\mathsf{R}(D)}$ for which $S(tF) \in L^p$ with $F(t,x) = \psi_t(DB)h(x)$.

Independence on ψ in a large subclass of $\Psi(S_\mu)$.

Functions $b(DB)$ with $b \in H^\infty(S_\mu)$ have bounded extensions on $\dot W^{-1,p}_{DB}$. In particular, for $\chi^+ = 1_{\Re z>0}$, we have a natural closed spectral subspace $\dot W^{-1,p,+}_{DB}$ defined as the range of $\widetilde{\chi}^+_p =$ extension of $\chi^+(DB)$. For $\chi^- = 1_{\Re z<0}$, we obtain $\dot W^{-1,p,-}_{DB}$. In fact, $\widetilde{\chi}^+_p, \widetilde{\chi}^-_p$ form a pair of bounded complementary projections on $\dot W^{-1,p}_{DB}$.

Version with area functional replaced by Carleson measures spaces $T^\infty_{2,\alpha}$ leads to BMO^{-1} and Hölder Λ^{-s} spaces: $\dot\Lambda^{\alpha-1}_{DB}$ for $0 \le \alpha < 1$.

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Theorem (A., Mourgoglou)

Let $q \in I_{L^*}$, assume $q > 1$ and let $p = q'$. Let u be a weak solution to $Lu = 0$ on Ω. The followings are equivalent:

(α) $\|S(t\nabla u)\|_p < \infty$ *and, if* $p \geq 2^*$, $u(t,\cdot)$ *converges to* 0 *in* $\mathcal{D}'$ *modulo constants as* $t \to \infty$.

(β) $\exists! F_0 \in \dot{W}_{DB}^{-1,p,+}$, *called the conormal gradient of u at $t = 0$ and denoted by* $\nabla_A u|_{t=0}$, *such that* $\nabla_A u(t,\,.\,) = \widetilde{S}_p(t) F_0$.

(γ) $\exists F_0 \in \dot{W}_D^{-1,p}$ *such that* $\nabla_A u(t,\,.\,) = \widetilde{S}_p(t)\widetilde{\chi}_p^+ F_0$.

Here, $\widetilde{S}_p(t)$ *is the extension of* $e^{-t|DB|}$ *on* $\dot{W}_{DB}^{-1,p} = \dot{W}_D^{-1,p}$. *Moreover,*

$$\|S(t\nabla u)\|_p \sim \|\nabla_A u|_{t=0}\|_{\dot{W}^{-1,p}} \sim \|\widetilde{\chi}_p^+ F_0\|_{\dot{W}^{-1,p}}.$$

Theorem (A., Mourgoglou)

Let $q \in I_{L^*}$, assume $q \leq 1$ and let $\alpha = n(\frac{1}{q} - 1)$. Let u be a weak solution to $Lu = 0$ on Ω. The followings are equivalent:

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Here, $\widetilde{S}_\alpha(t)$, χ_α, are the extensions of $e^{-t|DB|}$ on $\dot\Lambda^{\alpha-1}_{DB} = \dot\Lambda^{\alpha-1}_{D}$ and $\chi^+(DB)$ (obtained by duality). Moreover,

$$\|t\nabla u\|_{T^\infty_{2,\alpha}} \sim \|\nabla_A u|_{t=0}\|_{\dot\Lambda^{\alpha-1}} \sim \|\widetilde{\chi}^+_p F_0\|_{\dot\Lambda^{\alpha-1}}.$$

Well-posedness of the Dirichlet problem can be shown to be equivalent to invertibility of boundary maps provided we change $\|\widetilde{N}(u)\|_p < \infty$ to $\|S(t\nabla u)\|_p < \infty$. We dub this new problem (Dir', A, p). Fix $q \in I_{L^*}$, $q > 1$ and $p = q'$.

1. The modified problem (Dir', A, p) is well-posed iff the map $\dot{W}_{DB}^{-1,p,+} \to W_{\nabla}^{-1,p} : \nabla_A u|_{t=0} \to \nabla_x u|_{t=0}$ is invertible.
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Result: for p as above, one has $\|\widetilde{N}(u)\|_p \lesssim \|S(t\nabla u)\|_p$ provided we pick the solution that vanishes when $t \to \infty$ (in a certain sense). So this new Dirichlet problem is a priori more restrictive. The converse inequality in unclear.

In case of real equations, this result applies with $p \in [2, \infty)$. There is a version for $u|_{t=0} \in BMO$, and $u|_{t=0} \in \Lambda^\alpha$ for $\alpha < \alpha_0$.

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Conclusions

Elimination of DGN conditions

Clarification of a number of issues concerning representation and trace, not for solutions themselves but their conormal gradients

Characterisation of uniqueness and existence, separately

Refining of results on duality principles

All solutions in natural classes have (abstract) layer potential representation: this follows from the fact proved by A. Rosén that in classical situation, semigroups for DB and BD give layer potential formulæ

Consistency with the theory of energy solutions. Energy solutions are representable by our semigroups. Leads to results on compatible well-posedness (as in Barton-Mayboroda). Example: solvability with energy solutions implies compatible well-posedness (except, maybe, for Hardy data)

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Thank you!