Recent developments on the Strominger System

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The Strominger System

- (X, Ω) Calabi-Yau n-fold: X complex n-dimensional smooth manifold with holomorphic volume form $\Omega \in \Omega^{n,0}(X)$
- G compact semi-simple Lie group
- $P_G \to X$ principal G-bundle

Unknowns: g hermitian metric ($\omega = g(J\cdot, \cdot)$), A connection on P_G, ∇ unitary connection on (TX, g).

Strominger System (ST)

$$F_A \wedge \omega^{n-1} = 0, \qquad F_A^{0,2} = 0, \tag{1}$$

$$R_\nabla \wedge \omega^{n-1} = 0, \qquad R_\nabla^{0,2} = 0, \tag{2}$$

$$d^*\omega - i(\overline{\partial} - \partial) \log \|\Omega\|_\omega = 0, \tag{3}$$

$$i\partial\overline{\partial}\omega - \alpha'(\operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A) = 0, \tag{4}$$

Remarks: 1) typically ω non Kähler, 2) topological constraints

$$c_1(P_G) = c_1(X) = 0, \; c_2(P_G) = c_2(X).$$

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Today, $n = 3$.

Basic example

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- Ivanov-Papadopoulos' no-go Theorem [IP '01], implies that solutions with this ansatz are precisely metrics with $SU(3)$ holonomy (X needs to be Kählerian in this case).

By Yau's solution of the Calabi Conjecture '78, such metrics are in one to one correspondence with the Kähler cone of X

$$\mathcal{K} \subset H^{1,1}(X, \mathbb{R}).$$

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A glimpse on the geometry ...

- (1) HYM: polystable holomorphic vector bundle over (X, Ω) (degree measured with respect to $\|\Omega\|_\omega \omega^2$).
- (2) HYM: **non-standard** holomorphic structure on TX (also polystable).
- (3) Dilatino: torsion connection with $SU(3)$ holonomy (Ricci-flat)

$$\nabla^+ = \nabla^{LC} + 1/2 g^{-1} d^c \omega \qquad (Bismut)$$

- (4) Bianchi identity: Flat connection on a line bundle over $C^\infty(\Sigma^2, \mathbb{C}^2 \times X)$ [Freed '86], Twisted string structures [SSS '12], Quantum sheaf cohomology [DGKS '11].

Physics: Bianchi $\sim$ *Green-Schwarz mechanism* '84 for anomaly cancellation in heterotic string. ST $\sim$ supersymmetric compatification with non-trivial NS 3-form flux $H = d^c\omega$ (4D, $N = 1$).

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In a nutshell.

The Strominger system provides:

- a natural generalization of the condition of $SU(3)$-holonomy for a metric (Kähler-Ricci flat) in a non-Kähler Calabi-Yau 3-fold.

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Three problems

- ***Existence problem***: characterize existence of solutions $\sim$ Yau's Conjecture.
- ***Moduli problem***: construct a moduli space and endowe it with a natural geometry.
- $(0, 2)$-***mirror symmetry***: identify pairs of solutions which are physically equivalent.

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The existence problem: Yau's Conjecture

The existence problem for the Strominger system is widely open, mainly due to our lack of understanding of the Bianchi Identity.

Conjecture (Yau '06)

Let $((X,\Omega),V)$ be a pair given by a bundle V over a balanced CY3 (X,Ω,ω_0) such that

$$c_1(V) = 0, \qquad c_2(V) = c_2(X) \text{ (Bott–Chern).}$$

Then, $((X,\Omega),V)$ admits a solution of the Strominger system provided that V is $[\omega_0 \wedge \omega_0]$-polystable.

Remark: Does not specify 'where' does the metric-solution live. General expectation: $[\|\Omega\|_\omega \omega^2] = [\omega_0^2]$ (Bott-Chern).

Evidence: provided by following examples (*ad hoc* constructions, lack of general method).

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The existence problem: Kähler case

Assume (X, Ω) Kählerian.

- [Li–Yau ('05)]: Non-Kähler solutions via perturbation of [CHSW '85] solution on $TX \oplus \mathcal{O}_X$. Examples on $SU(4)$, $SU(5)$ bundles over generic quintic in $\mathbb{P}^4$.
- [Andreas & GF '10]: Any stable holomorphic vector bundle V over a Kähler manifold with holonomy $= SU(3)$, such that $c_1(V) = 0$, $c_2(V) = c_2(X)$, can be perturbed to a solution of ST.

Remarks:

- AGF solves Yau's conjecture for X Kählerian. However, method has no control on $[\|\Omega\|_\omega \omega^2]$.
- Recovers Li-Yau. **Huybrechts' Theorem**: 'For a generic quintic, $TX \oplus \mathcal{O}_X$ admits stable deformations with non-trivial restriction to embedded rational curves' ($\sim$ superpotential).
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The existence problem: non Kähler case

Some existence results:

- [Fu & Yau '08]: Goldstein–Prokushkin non Kählerian fibration (elliptic fibrations over $K3$, Monge-Ampère eq).
- [Fernandez-Ivanov-Ugarte-Villacampa '08]: Invariant solutions on nilmanifolds.
- [Fei-Yau '14]: invariant solutions on parallelizable manifolds $X = G/\Gamma$ (Wang Th.).
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- [Fei-2 '15]: Solution on complement of a fibre in twistor space.

Remark: for $G = SL(2, \mathbb{C})$, considered by Biswas–Mukerjee '13 and Andreas–GF '14 (parameterised by real hyperbolic 3-manifolds).

Uniqueness? How many solutions are there?

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The Moduli Problem

- *The moduli space of solutions of the Strominger system* $\mathcal{M}_{ST}$ *is a fundamental gadget in string theory, where it describes basic pieces (scalar massless fields) of the* $4D$ *effective field theory induced by a* $N = 1$ *supersymmetric heterotic string compactification.*

- *Mathematically,* $\mathcal{M}_{ST}$ *has to be constructed. Conjectured (physics prediction) to be endowed with an interesting Kähler-Hodge metric.*

- *Understading the Kähler geometry of* $\mathcal{M}_{ST}$ *should lead us to new insight for the existence and uniqueness problem (*$\sim$ *Yau's Conjecture).*

- $\mathcal{M}_{ST}$ *provides a potential ground where to study a generalization of classical mirror symmetry.*

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- *Mathematically,* $\mathcal{M}_{ST}$ *has to be constructed. Conjectured (physics prediction) to be endowed with an interesting Kähler-Hodge metric.*
- *Understading the Kähler geometry of* $\mathcal{M}_{ST}$ *should lead us to new insight for the existence and uniqueness problem (*$\sim$ *Yau's Conjecture).*
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The moduli problem: Elliptic curves

On an elliptic curve $E = (X, \Omega)$, the Strominger system reduces to $(\nabla = \nabla_g^{LC})$

$$F_A = 0, \qquad S_g = 0.$$

Theorem (Atiyah '57, Tu '93): Moduli of degree 0, rank r, polystable holomorphic vector bundles over E, isomorphic to $S^r E$.

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Observation (GF–Shahbazi): dimension $2r + 5 + 1$ equal to dimension of $\mathbb{R}_{>0} \times SO(2, r+2)/SO(2) \times SO(2+r)$ ($N = 1$, $8D$-sugra).

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- (Becker–Tseng '06, Cyrier-Lapan '07, Becker–Tseng–Yau '07) partial results on infinitesimal structure. Fixed complex structure on X
- (Melnikov–Sharpe '11, De la Ossa–Svanes / Anderson–Gray–Sharpe '14) calculation of infinitesimal variations of Strominger and relation with first Dolbeault cohomology $H^1(\mathcal{Q})$ of a holomorphic double extension

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- Consider a holomorphic double extension for a suitable holomorphic Atiyah Lie algebroid V over X, constructed from a solution of ST.

$$0 \to T^*X \to \mathcal{Q} \to V \to 0$$
$$0 \to \mathrm{ad}\, P^c \to V \to TX \to 0$$

- Calculate infinitesimal variations of ST using the stronger anomaly cancellation condition (vs Bianchi identity) and prove they give co-cycles for the Dolbeault complex of $\mathcal{Q}$

$$H := d^c\omega = dB - \alpha'(CS(\nabla) - CS(A)), \qquad (\textit{locally})$$

- postulate $H^1(\mathcal{Q})$ as an 'approximation' to tangent of moduli.

Drawback: Needs X to be a $\partial\bar{\partial}$-manifold.

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To discuss the method in arXiv:1503.07562 (GF, Rubio, Tipler), we simplify ST to an 'Abelianized Strominger system' (AST).

M compact, oriented, $6d$ manifold, $L \to M$ a hermitian line bundle, $\mathcal{A}$ space of unitary connections on L

Unknowns: $\mathcal{P} = \{(\Omega, A, \omega) : \text{satisfying } (1),(2)\} \subset \Omega^3(\mathbb{C}) \times \mathcal{A} \times \Omega^2$

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2. ω is $J_\Omega -$ compatible, positive

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$$T_p\mathcal{P} \subset \Omega^{3,0+2,1} \oplus \Omega^1(M, i\mathbb{R}) \oplus \Omega^2$$

The gauge group, induces infinitesimal action of $\mathrm{Lie}\,\tilde{\mathcal{G}}$

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Linearisation of AST induces a complex of differential operators

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Linearisation of AST induces a complex of differential operators (3rd arrow $L = \oplus_{i=0}^{4} L_i$)

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Space of infinitesimal deformations/obstructions

Theorem (_____,Rubio,Tipler)

The complex is elliptic. A suitable modification of L leads to an extension of the complex S^* with identical first cohomology group

$$0 \to \operatorname{Lie} \tilde{\mathcal{G}} \to T_0\mathcal{P} \to S^3 \to S^4 \to \dots$$

Definition: the space of infinitesimal deformations of ST is defined as $H^1(S^*)$. The space of obstruction for ST is defined as $H^2(S^*)$.

To keep in mind! There is a well-defined map

$$H^1(S^*) \to H^3(X, \mathbb{R})$$
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Heuristics of the flux map

Suppose $p_0 = (\Omega, A, \omega) \in \mathcal{P}$ is a solution with unobstructed deformations: a neighbourhood $0 \in U \subset H^1(S^*)$ provides smooth coordinates around $[p_0]$ in the moduli space

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The exterior derivative $\delta = dflux$ defines a closed $H^3(M, \mathbb{R})$-valued 1-form

$$\delta \in \Omega^1(\mathcal{M}_{AST}, H^3(M, \mathbb{R})),$$

and hence natural foliation on the moduli space integrating $\operatorname{Ker} \delta$ (the *flux* only defined up to equivariant-mapping class group of L).

Anomaly cancellation and flux quantization

Want to understand the leaves of this natural foliation: try infinitesimally

$$0 \longrightarrow H^1(\mathring{S}^*) \longrightarrow H^1(S^*) \xrightarrow{\delta} H^3(X, \mathbb{R}).$$

The *refined space of variations* $H^1(\mathring{S}^*)$ comes closer to the physics approach (De la Ossa–Svanes/Anderson–Gray–Sharpe).

- require *flux quantization.*
- versus the Bianchi identity, they use *the anomaly equation: local, Green-Schwarz mechanism,*

$$H := d^c\omega = dB - \alpha'(CS(\nabla) - CS(A))$$

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Down the rabbit hole ...

As notation suggests, $H^1(\mathring{S}^*)$ is the cohomology of a complex, such that

$$\underset{here!\nearrow}{\mathring{S}^0} \subset S^0 \oplus \Omega^2, \qquad \mathring{S}^1 = S^1 \oplus \Omega^2.$$

2-forms: play role of symmetries, but also as additional data for the objects parameterised

Sustitute operator $L_4 \colon S^1 \to \Omega^4$ by $\mathring{L}_4 \colon \mathring{S}^1 \to \Omega^3$ (*linearized Green-Schwarz*)

$$\mathring{L}_4(\dot{\Omega}, \dot{a}, \dot{\omega}, b) = db - \frac{1}{2}\left(J_0 d\dot{\omega} - J_0(d\omega)^{\dot{J}J_0} + 4\alpha' \dot{a} \wedge F\right)$$

Observation: $\mathring{P}(V, r, B) = (P(V, r), B)$ and $\mathring{L} = L_0 \oplus \ldots \oplus L_3 \oplus \mathring{L}_4$ define a complex with first coholomogy $H^1(\mathring{S}^*)$, provided that

$$\mathring{L}_5 \circ \mathring{P}(V, r, B) = dB + L_V(d^c\omega) + 2\alpha'((dr + \iota_V F) \wedge F) = 0$$

What is the meaning of this condition?

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Strominger meets generalized geometry

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Defining equation for **infinitesimal symmetries of smooth transitive Courant algebroid** (Baraglia, Rubio, Hitchin) constructed from fixed solution of AST

$$E = T \oplus i\mathbb{R} \oplus T^*.$$

$$\begin{aligned}
\langle V + r + \xi, W + t + \eta \rangle &= \frac{1}{2}(\eta(V) + \xi(W)) - \alpha' rt \\
[V + r + \xi, W + t + \eta] &= [V, W] + L_V\eta - i_W d\xi + i_W i_V(d^c\omega) \\
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$$\mathring{S}^0 = \operatorname{Lie}\mathring{\operatorname{Aut}} E \to \mathring{S}^1 = S^1 \oplus \Omega^2 \to \dots$$

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Remark: a priori, no analytical tools to prove that $H^1(\mathring{S}^*)$ is finite due to defining differential equation $(*)$ for $\operatorname{Lie}\mathring{\operatorname{Aut}} E$, but

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Remark: the *refined space of variations* $H^1(\mathring{S}^*)$ comes closer to the physics of the heterotic string, but it is not the right space: constains a single copy of $H^{1,1}(X, \mathbb{R})$ (potentially odd dimensional).

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Complexification of the 'Kähler moduli'

Using *inner symmetries of E*, we construct an **elliptic complex** $\widehat{S}^*$ **of differential operators of degree** 1, such that

$$0 \to H^2(M,\mathbb{R}) \to H^1(\widehat{S}^*) \to H^1(\mathring{S}^*) \to 0$$

We expect $H^1(\widehat{S}^*)$ to be **even-dimensional**, providing natural complexification of the 'Kähler moduli' (done by hand for moduli of Calabi-Yau metrics).

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The AGSOS map

Provided that $X = (M, \Omega)$ is $\partial\overline{\partial}$-manifold, we interpret the physical construction of Anderson–Gray–Sharpe–De la Ossa–Svanes as

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & H^2(M,\mathbb{R}) & \longrightarrow & H^1(\widehat{S}^*) & \longrightarrow & H^1(\mathring{S}^*) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle AGSOS} & & \downarrow & & \\
 & & & & H^1(\mathcal{Q}) & & H^1(S^*) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & H^3(M,\mathbb{R}) & &
\end{array}
$$

The holomorphic Courant algebroid $\mathcal{Q}$

Observation: Every solution of the following system determines a (transitive) holomorphic Courant algebroid. In particular, every solution of the Strominger system $(\Omega, \nabla, A, \omega)$ determines a holomorphic Courant algebroid over (X, Ω).

$$dd^c\tau^{1,1} = \alpha'(\operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F), \qquad F^{0,2} = 0, \qquad R^{0,2} = 0.$$

Definition: Complexify E, and consider the integrable lift $L = e^{-i\omega}T^{0,1}$ of $T^{0,1}$. Then, Gualtieri proves that the following defines a holomorphic Courant algebroid

$$\mathcal{Q} = L^{\perp}/L, \qquad i_{V^{0,1}}\overline{\partial}_{\mathcal{Q}}s = [e^{-i\omega}V^{0,1}, \tilde{s}] \mod L$$

($\tilde{s}$ lift of s to $L^{\perp}$) given by a double holomorphic extension

$$0 \to T^*X \to \mathcal{Q} \to W \to 0$$

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Explicitely: (abelian case) (Ω, A, ω) a solution. Take

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Remark: Unlike $H^1(\hat{S}^*)$, $H^1(\mathcal{Q})$ has a natural complex structure

$$AGSOS([\dot{\Omega}, \dot{a}, \dot{\omega}, b]) := [(\dot{J}^{1,0}, \dot{a}^{0,1}, i\dot{\omega}^{1,1} - b^{1,1} + \partial\sigma^{0,1})]$$

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Testing the new framework

From the point of view of physics, solutions of the Strominger system and metrics with $SU(3)$-holonomy are two different incarnations of the same phenomenon: a $N = 1$ supersymmetric compactification of the heterotic string to 4 dimensions.

Generalized Killing spinors

M compact 6-dimensional spin manifold and E **transitive** Courant algebroid. For E obtained from reduction, $E \cong T \oplus \operatorname{ad} P \oplus T^*$.

Generalized metric

$$E = V_+ \oplus V_-$$

Gualtieri's (generalized) connection

$$D^G_e e' = [e_-, e_+]_+ + [e_+, e_-]_- + [Ce_-, e_-]_- + [Ce_+, e_+]_+$$

where $V_+ \cong \{X + r + gX\}$ and $C(X + r + gX) = X - gX$, g Riemannian

Levi-Civita connection

$$D^{LC} = D^G - \frac{1}{3} T_{D^G}$$

Given $\phi \in C^\infty(M)$, D^{LC} modified canonically to D^ϕ, comp., torsion-free.

Killing spinor equations

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Theorem (_____,Rubio,Tipler)

On a trasitive Courant algebroid obtained from reduction, the Killing spinor equations are equivalent to the Strominger system. When E is exact, a solution of the Killing spinor equations is equivalent to a metric with $SU(3)$-holonomy. In particular, leaves of the flux foliation can be interpreted as moduli spaces of solutions of the killing spinor equations.

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Generalized geometry provides a unifying framework for the theory of the Strominger system and the well-stablished theory for metrics with $SU(3)$-holonomy. Bringing in techniques from the latter, by embedding the theory into generalized geometry, is a promising approach to the existence, uniqueness and moduli problem for the Strominger system.

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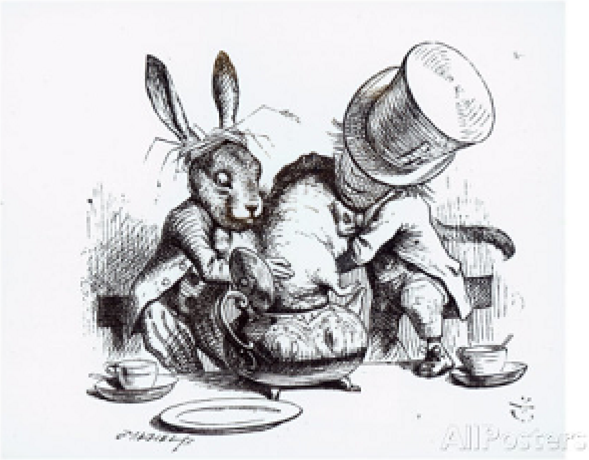
Theorem (_____,Rubio,Tipler)

On a trasitive Courant algebroid obtained from reduction, the Killing spinor equations are equivalent to the Strominger system. When E is exact, a solution of the Killing spinor equations is equivalent to a metric with $SU(3)$-holonomy. In particular, leaves of the flux foliation can be interpreted as moduli spaces of solutions of the killing spinor equations.

$$D_+^\phi \eta = 0, \qquad \not{D}_-^\phi \eta = 0, \quad \textit{for} \;\; \eta \in S_+(V_-)$$

Generalized geometry provides a unifying framework for the theory of the Strominger system and the well-stablished theory for metrics with $SU(3)$-holonomy. Bringing in techniques from the latter, by embedding the theory into generalized geometry, is a promising approach to the existence, uniqueness and moduli problem for the Strominger system.



Thank you!

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