Quiver mutation loops and partition q-series

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September 18, 2015

Joint work with
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Plan

1. Motivation — Why quiver mutation loop ?
2. Partition q-series — definition & examples
3. Quantum dilogarithm and partition q-series

Based on papers with Y.Terashima:

- Comm. Math. Phys., **336** (2015) 811-830 [arXiv:1403.6569]
- Comm. Math. Phys., **338** (2015) 457-481 [arXiv:1408.0444]

Ubiquity of quiver mutations

Quiver mutations appear in many fields in different guise:
cluster algebras, 3-dimensional topology, gauge theory, Donaldson-Thomas theory, stability conditions, wall-crossing, WKB analysis, ...

Our Strategy

- Define a key mathematical object, à la partition function of statistical mechanics, by using only combinatorial data of quiver mutations.
- Every property of such object would be shared by various "realizations" of mutations.

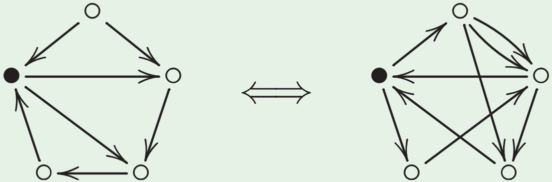
In this talk, we introduce partition q-series for mutations and explain some nice properties of them.

Mutation of a quiver

Mutation of a quiver Q at a vertex k:

1. For each path $i \to k \to j$, add a new arrow $i \to j$;
2. Reverse all arrows with source or target k.
3. Remove newly created 2-cycle, if any.

Example



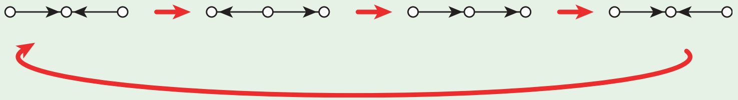
Mutation loop

A mutation loop is a triple $\gamma = (Q, m, \varphi)$, where

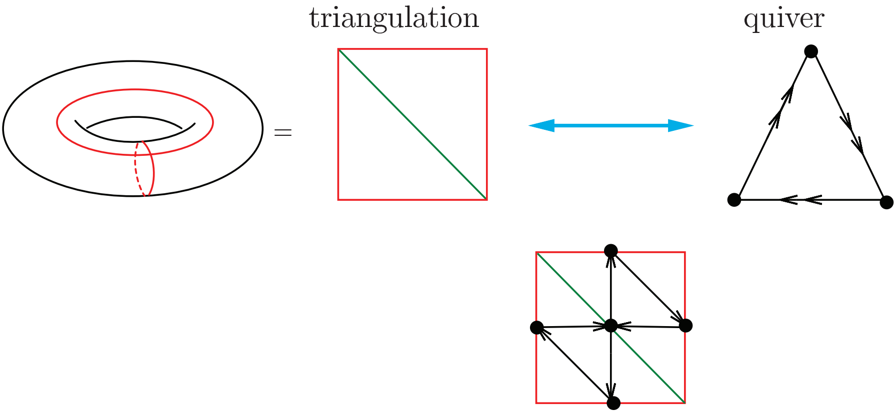
- an initial quiver Q
- a mutation sequence $m = (m_1, m_2, \ldots, m_T)$
- boundary condition = an isomorphism φ of the initial quiver Q_0 and the final quiver Q_T.

Example

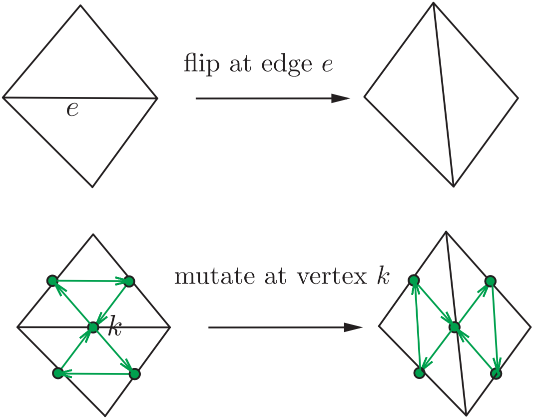
- $Q = \left(a \to b \leftarrow c\right)$
- Mutation sequence $m = (b, a, c)$
- Boundary condition $\varphi : a' = a,\ b' = b,\ c' = c$



Motivation : quiver $\Longleftrightarrow$ surface triangulation

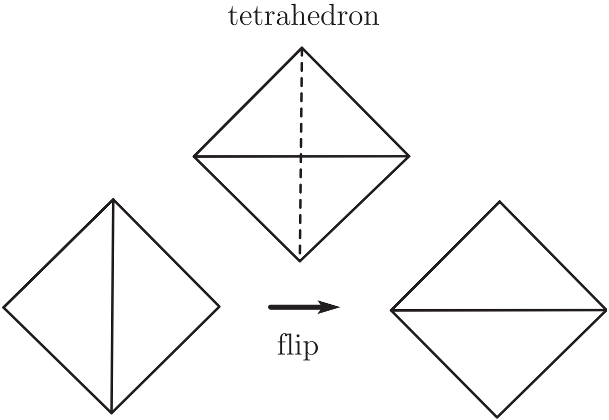


Motivation : mutation $\Longleftrightarrow$ flip



mutation sequence $\Longleftrightarrow$ surface diffeomorphism

Motivation : flip $\Longleftrightarrow$ tetrahedron

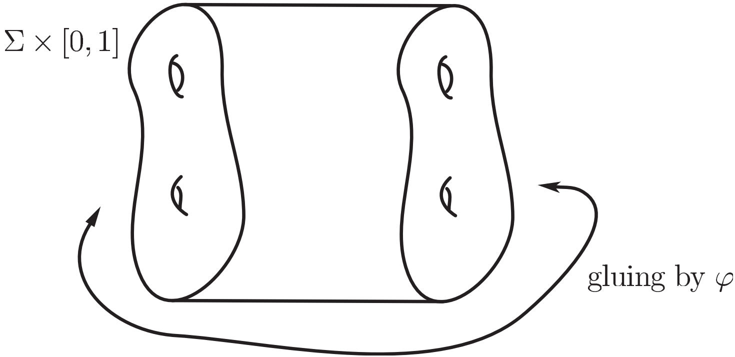


mutation sequence $\Longleftrightarrow$ mapping cylinder

Motivation : Triangulation of 3-manifolds

Surface bundle with a mapping class φ of a surface Σ

$$M := (\Sigma \times [0,1])/(x,0) \sim (\varphi(x),1)$$



Motivation : Mutations $\Longleftrightarrow$ 3-dim. topology

Combinatorial	Geometric
Quiver	Triangulation of a surface
Mutation	Tetrahedron
Mutation sequence	Mapping class
Mutation network	Triangulation of a 3-manifold
Mutation loop	Surface bundle over S^1

Cluster transformations $\Longleftrightarrow$ Hyperbolic geometry

Penner, Fock, Teschner, Gekhtman-Shapiro-Vainshtein, Fock-Goncharov, Fomin-Shapiro-Thurston, Nagao-Terashima-Yamazaki, Hikami-Inoue, ...

??? $\Longleftrightarrow$ Quantum field theories

Quantization via path-integral / state sum

Various approach to quantization:

- Algebraic : Non-commutative deformation, D-modules, ...
- Geometric : Geometric quantization, Poisson structure, ...
- Combinatorial : Partition function ($\Leftarrow$ This talk)

Partition function = "generating function of all possible states "

- path-integral : quantum field theory
  = state-sum : statistical machanics
- Statistical model on a lattice Λ with local variables s_i taking their values in S.
- Partition function

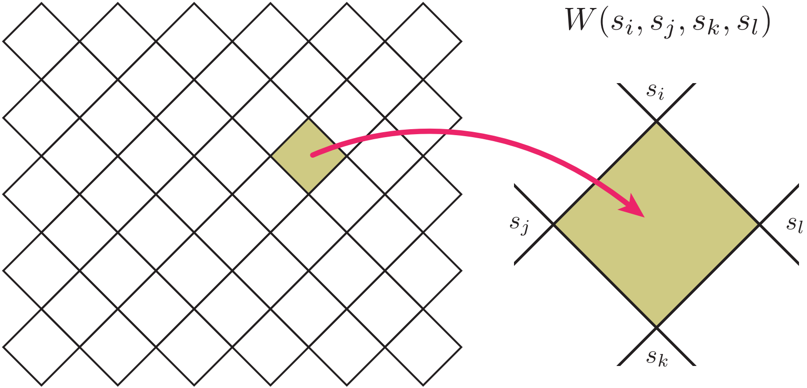
$$Z = \sum_{s_i : i \in \Lambda} \prod W(\{s_i\})$$

s_i : "state", "spin", "color", "field" variable at vertex i
W = Boltzmann weight (defined locally in Λ) $\propto \exp(-E/kT)$

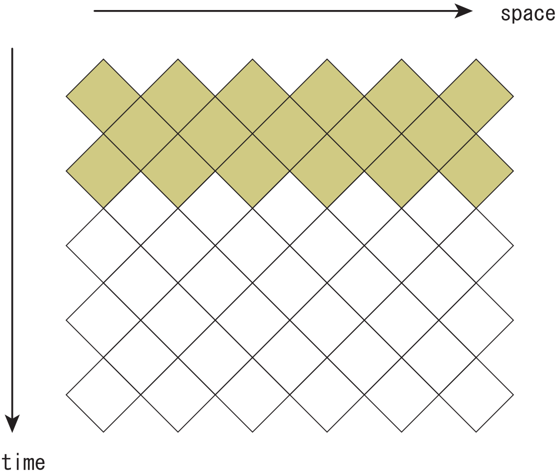
Quantization via path-integral / state sum

Partition function — IRF (interaction-around-a-face) model

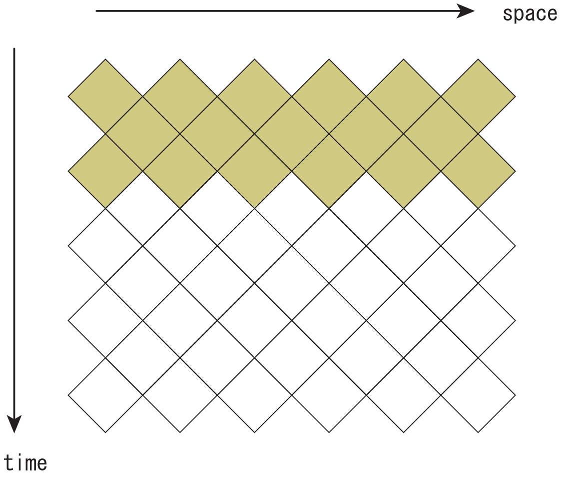
$$Z = \sum_{s_i : i \in \Lambda} \prod_{\text{faces}} W(\{s_i\})$$



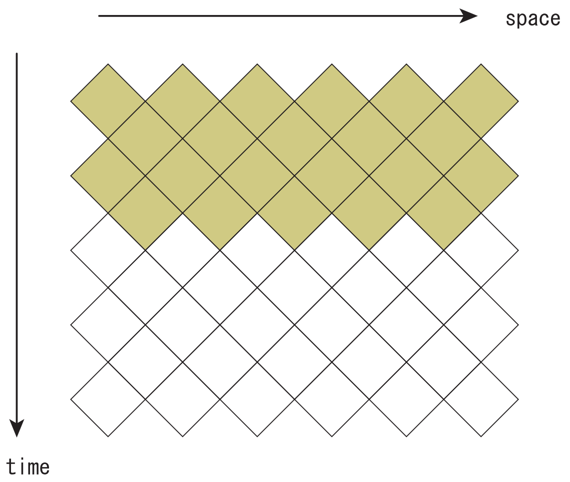
Quantization via path-integral / state sum



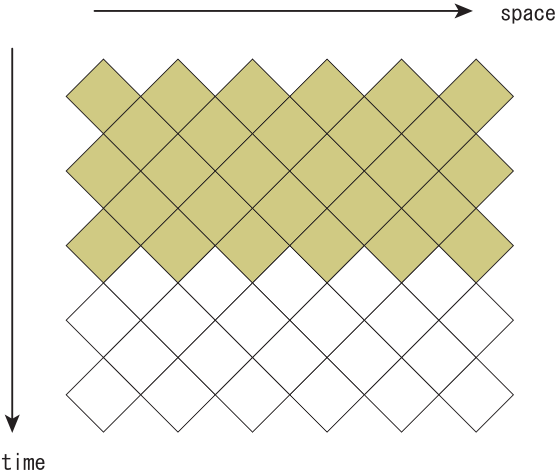
Quantization via path-integral / state sum



Quantization via path-integral / state sum



Quantization via path-integral / state sum



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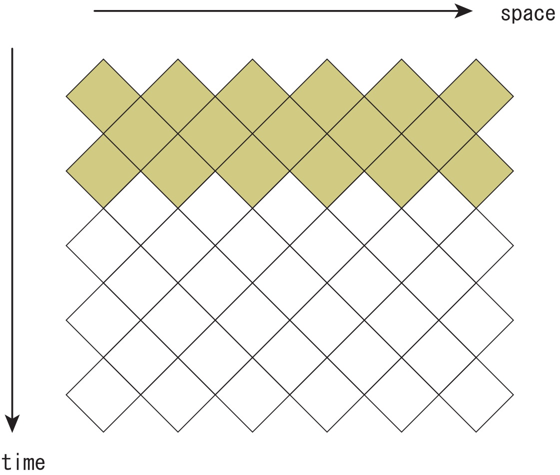

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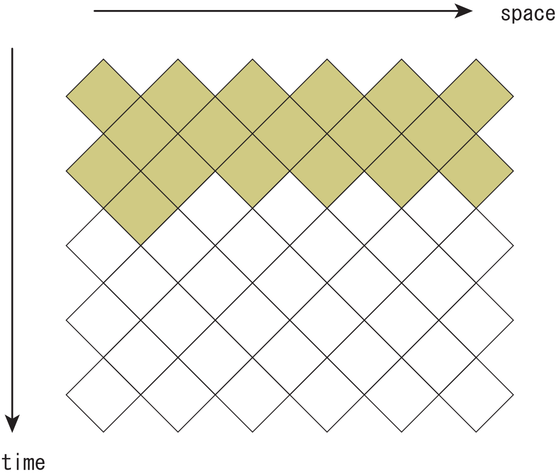

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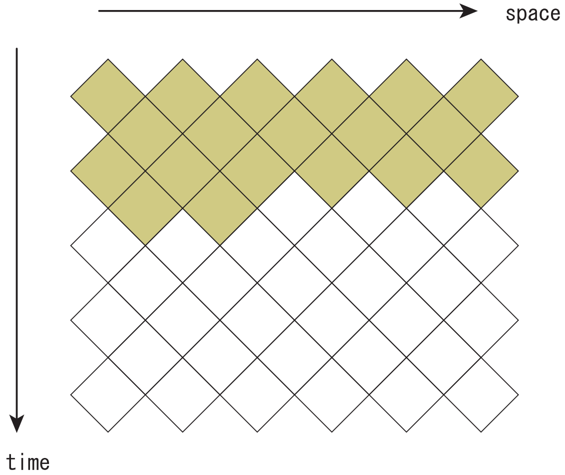

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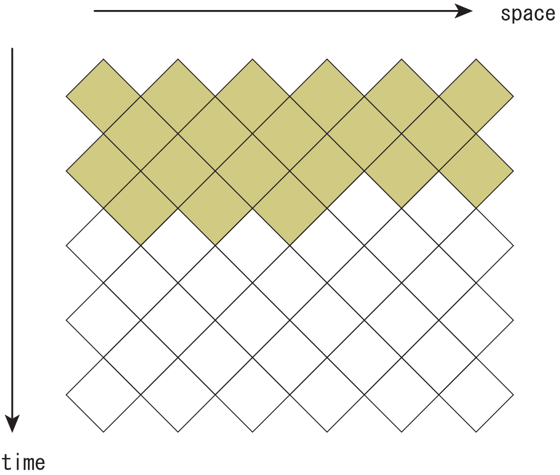
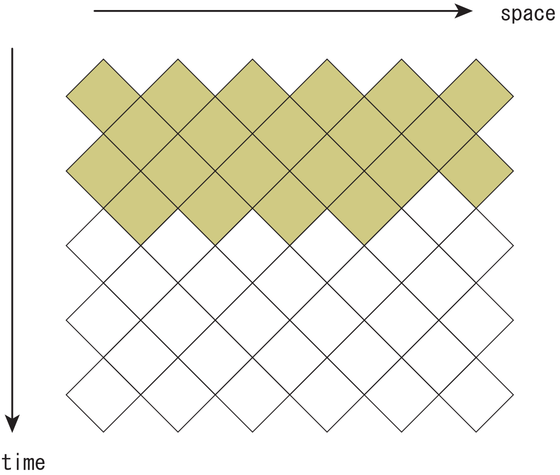


Quantization via path-integral / state sum



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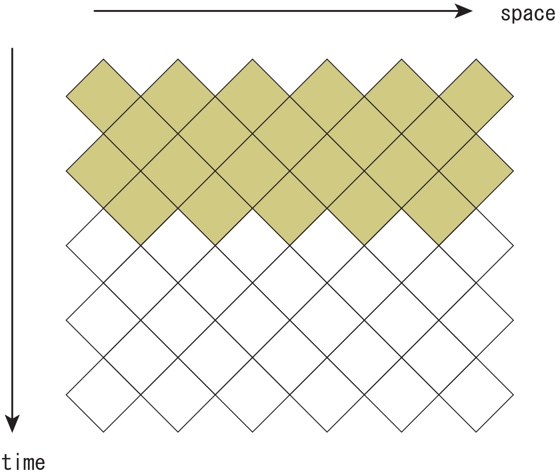
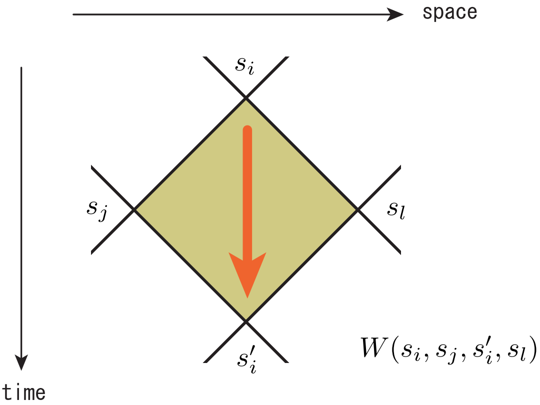


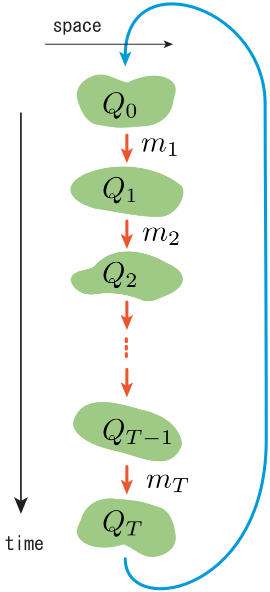
Quantization via path-integral / state sum

- Attaching one square at vertex i = time evolution "localized" at i.
- Only one spin variable s_i can change its value : $s_i \to s_i'$ at vertex i.
- All the other spins s_j $(j \neq i)$ remain the same value.



State sum from quiver mutation sequence

- Space direction $=$ quiver
- Time direction $=$ mutation sequence
- Periodic boundary condition in time direction $=$ mutation loop
- Attach a variable s_j at each vertex j.
- The change of spin $s_i \to s_i'$ is possible only when vertex i is mutated.
- The Boltzmann weight should be a local function.
- Mutation sequence is an *inhomogeneous* time-evolution of space (=quiver). The space-time graph Λ is dynamically generated from the initial quiver Q and the mutation sequence m.



Weights of mutations

- Weight of a mutation at vertex i:

$$\frac{q^{\frac{1}{2}(s_i+s'_i-\sum_{j:j\to i} s_j)(s_i+s'_i-\sum_{l:i\to l} s_l)}}{(q)_{s_i+s'_i-\sum_{j:j\to i} s_j}}$$

- k-variable with a mutation m_i at a vertex i:

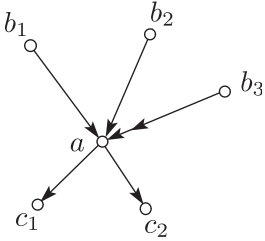
$$k_i := s_i + s'_i - \sum_{j:j\to i} s_j$$

q-factorial

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$$

Weights of mutations — Example

mutation at vertex a:



Weight of the mutation

$$W = \frac{q^{\frac{1}{2}(a+a'-b_1-b_2-2b_3)(a+a'-c_1-c_2)}}{(q)_{a+a'-b_1-b_2-2b_3}}$$

k-varaible:

$$k = a + a' - b_1 - b_2 - 2b_3$$

Partition q-series

Definition

For a mutation loop $\gamma = (Q, m, \varphi)$ with $m = (m_1, \cdots, m_T)$, we define partition q-series by

$$Z(\gamma) := \sum_{k \in \mathbb{N}^T} \prod_{t=1}^{T} W(m_t),$$

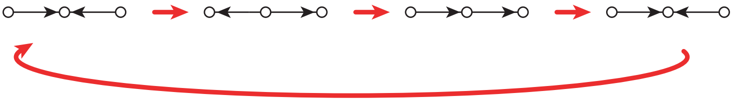
where $\mathbb{N} = \{0, 1, 2, \cdots\}$.

The relation between k- and s- variables depends on the global topology, especially on the boundary condition φ.

Typical example

The alternating quiver Q of type A_3:

- Mutation sequence $m = (b, a, c)$
- Boundary condition $\varphi = \mathrm{id} : a' = a,\ b' = b,\ c' = c$



Typical example

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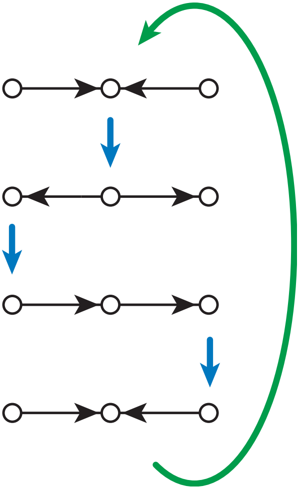


$$\begin{array}{ccccccc}
Q(0) & & a & \longrightarrow & b & \longleftarrow & c \\
\mu_2 \downarrow & & \| & & \vdots\, k_2 & & \| \\
Q(1) & & a & \longleftarrow & b' & \longrightarrow & c \\
\mu_1 \downarrow & & \vdots\, k_1 & & \| & & \| \\
Q(2) & & a' & \longrightarrow & b' & \longrightarrow & c \\
\mu_3 \downarrow & & \| & & \| & & \vdots\, k_3 \\
Q(3) & & a' & \longrightarrow & b' & \longleftarrow & c' \\
\mathrm{id} \downarrow & & \| & & \| & & \| \\
Q(0) & & a & \longrightarrow & b & \longleftarrow & c
\end{array}$$

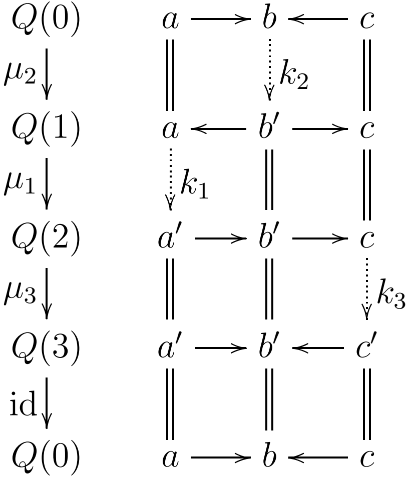
Typical example

Product of weights:

$$\frac{q^{\frac{1}{2}((2b-a-c)\cdot 2b+(2a-b)\cdot 2a+(2c-b)\cdot 2c)}}{(q)_{2b-a-c}(q)_{2a-b}(q)_{2c-b}}$$

k-variables from s-variables:

$$k_2 = 2b-a-c,$$
$$k_1 = 2a-b,$$
$$k_3 = 2c-b.$$



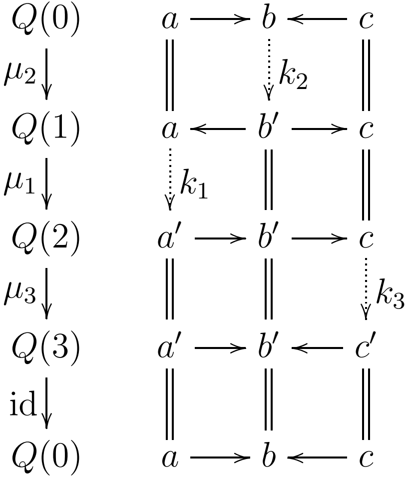
Typical example

Product of weights:

$$\frac{q^{\frac{1}{2}((2b-a-c)\cdot 2b+(2a-b)\cdot 2a+(2c-b)\cdot 2c)}}{(q)_{2b-a-c}(q)_{2a-b}(q)_{2c-b}}$$

s-variables from k-variables:

$$a = \frac{1}{4}(3k_1 + 2k_2 + k_3),$$
$$b = \frac{1}{2}(k_1 + 2k_2 + k_3),$$
$$c = \frac{1}{4}(k_1 + 2k_2 + 3k_3).$$



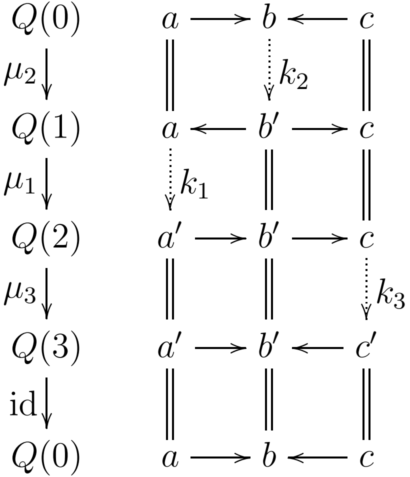
Typical example

Product of weights:

$$\frac{q^{\frac{3}{4}k_1^2+k_1k_2+k_2^2+k_2k_3+\frac{3}{4}k_3^2+\frac{k_3k_1}{2}}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}}$$

s-variables from k-variables:

$$a = \frac{1}{4}\left(3k_1 + 2k_2 + k_3\right),$$
$$b = \frac{1}{2}\left(k_1 + 2k_2 + k_3\right),$$
$$c = \frac{1}{4}\left(k_1 + 2k_2 + 3k_3\right).$$



Typical example

Partition q-series:

$$Z(\gamma) = \sum_{k\in\mathbb{N}^3} \frac{q^{k^\top D k}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}}$$

$$D = \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

D is the inverse of the Cartan matrix of type A_3!

$$D^{-1} = C_{A_3} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Typical example

Partition q-series:

$$Z(\gamma) = \sum_{k \in \mathbb{N}^3} \frac{q^{k^\top D k}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}} = \frac{1}{(q)_\infty} \sum_{n \in \mathbb{Z}} q^{\frac{3}{4}n^2}$$

$$D = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

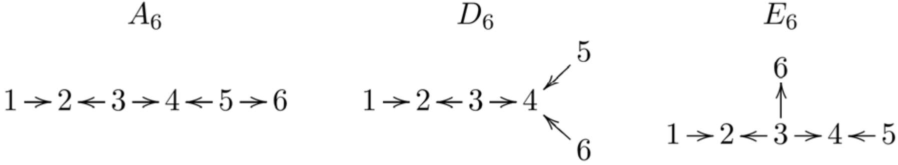
D is the inverse of the Cartan matrix of type A_3!

$$D^{-1} = C_{A_3} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

In particular, $Z(\gamma)$ is a modular function — a character of a module associated with an affine Lie algebras. (Lepowsky-Primc, Terhoeven)

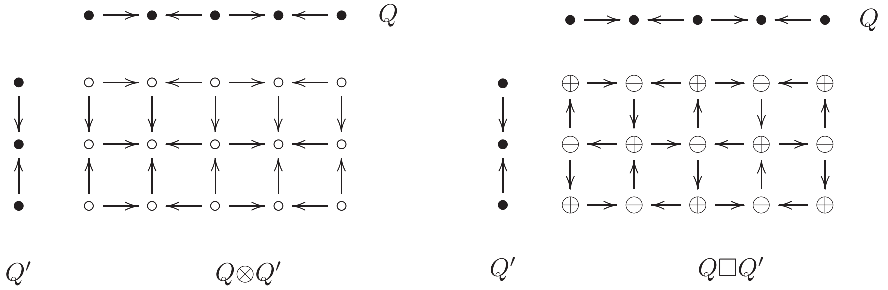
A quiver is alternating if each vertex is source or sink.

Alternating quiver of type ADE:



Product Quivers

Product of quivers:



A special mutation sequence $m = m_+ m_-$ of $Q \square Q'$.

Fermionic formula

Theorem (K.-Terashima)

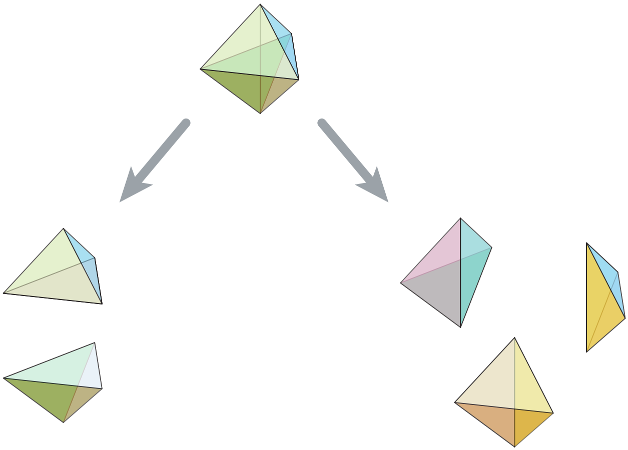
Consider the product $Q\square Q'$ of alternating quivers of ADE type. Let $\gamma = (Q\square Q', m, id)$ be the mutation loop with $m = m_+ m_-$. Then we have

$$Z(\gamma) = \sum_{k\in\mathbb{N}^n} \frac{q^{\frac{1}{2}k(C_Q\otimes C_{Q'}^{-1})k}}{(q)_k}$$

In particular, when $Q' = A_n$ type, $Z(\gamma)$ coincides with a fermionic character formula in the Kuniba-Nakanishi-Suzuki conjecture!

"Parafermionic" system associated with affine Lie algebra of type Q.
(conformal field theories associated with subquotients of $U(\hat{\mathfrak{g}})$-modules)

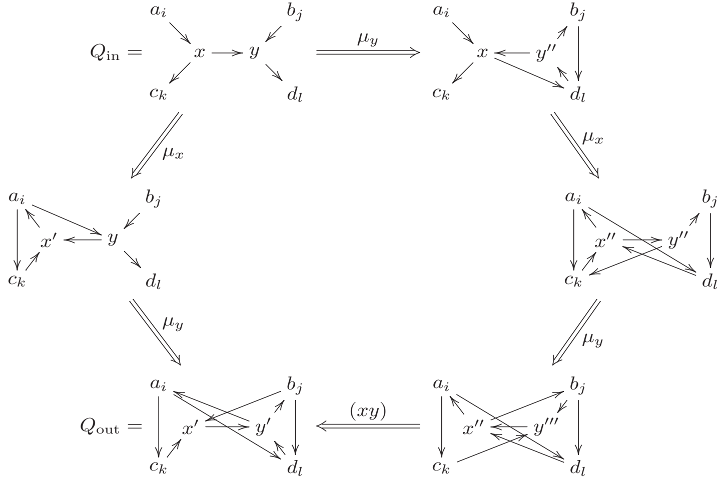
Pentagon move (Pachner 2-3 move)



2 tetrahedra $\longleftrightarrow$ 3 tetrahedra

Generalized pentagon move

$$\gamma = (\cdots, x, y, \cdots) \longleftrightarrow \gamma' = (\cdots, y, x, y, (xy), \cdots)$$



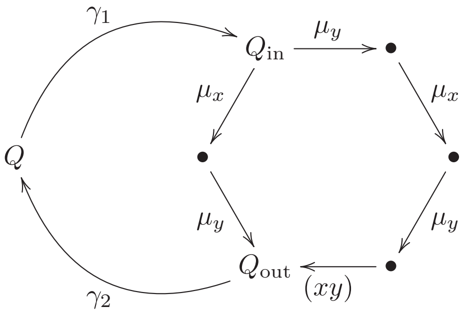
Generalized pentagon move

Theorem (K.-Terashima)

The partition q-series are invariant under generalized pentagon moves:

$$Z(\gamma) = Z(\gamma')$$

- $Z(\gamma)$ is invariant under local deformation of the loop γ in the quiver exchange graph Γ:
  $V(\Gamma) = \{\text{quivers}\}$,
  $E(\Gamma) = \{\text{mutations}\}$
- $Z(\gamma) =$ "holonomy" or "Wilson loop" along $\gamma \subset \Gamma$



(Combinatorial) Donaldson-Thomas invariant

- Donaldson-Thomas invariant
  = (a generating function of) the "virtual number of points of moduli stack of semistable objects" in Calabi-Yau 3 category $\mathscr{C}$.

$$\mathrm{Hom}_{\mathscr{C}}(E,F) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(F,E[3])^*$$

  Bridgeland, Joyce, Song, Kontsevich, Soibelman, Nagao, Reineke, Szendrői, ...

- A "generic" Calabi-Yau 3 category $\mathscr{C}$ is realized as

$$\mathscr{C} \simeq D^b(\Gamma_3 Q\text{-mod})$$

  where $\Gamma_3 Q$ is a CY3 Ginzburg algebra associated with a quiver Q.

- B. Keller introduced Combinatorial DT invariant in terms of "path-ordered" product of quantum dilogarithms.

Quantum Dilogarithm

Definition

$$E(y) = \prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}}y)^{-1} \qquad \in \mathbb{Q}(q^{1/2})[[y]]$$
$$= 1 + \frac{q^{1/2}}{q-1}y + \cdots + \frac{q^{n^2/2}y^n}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} + \cdots$$
$$= \exp\left(\sum_{k=1}^{\infty}\frac{(-y)^k}{k(q^{-k/2}-q^{k/2})}\right)$$

Theorem (Fadeev-Kashaev-Volkov 1993)

$$y_1y_2 = qy_2y_1 \quad \Longrightarrow \quad E(y_1)E(y_2) = E(y_2)E(q^{-1/2}y_1y_2)E(y_1)$$

Non-commutative algebra

Definition

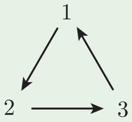
The non-commutative algbera $\mathbb{A}_Q$

- generators: $y_1, y_2, \cdots, y_n$
- relations: $y^{\alpha} y^{\beta} = q^{\frac{1}{2}\langle \alpha, \beta \rangle} y^{\alpha+\beta}, \quad (\alpha, \beta \in (\mathbb{Z}_{\geq 0})^n)$

where $\langle e_i, e_j \rangle = \#(i \to j) - \#(j \to i)$.

Example

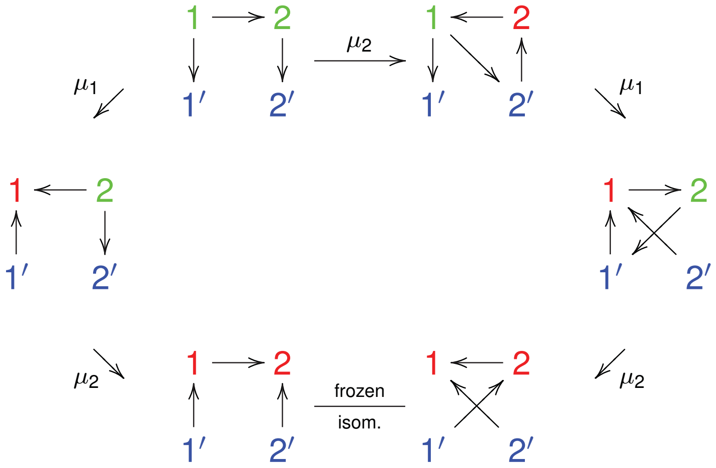
$Q =$



$$\mathbb{A}_Q = \langle y_1, y_2, y_3 \mid y_1 y_2 = q y_2 y_1,\ y_2 y_3 = q y_3 y_2,\ y_1 y_3 = q^{-1} y_3 y_1 \rangle$$

$$y^{(i_1, i_2, i_3)} = q^{-\frac{1}{2}(i_1 i_2 + i_2 i_3 - i_1 i_3)} y_1^{i_1} y_2^{i_2} y_3^{i_3}$$

Ice-quivers, c-vectors, and reddening sequence



Dilogarithm product along reddening sequence

Definition (Keller)

Reddning sequence

$$m = (m_1, m_2, \cdots, m_T)$$

c-vectors at the mutated vertices are

$$\varepsilon_1 \alpha_1, \varepsilon_2 \alpha_2, \cdots, \varepsilon_T \alpha_T \in \mathbb{Z}^n, \qquad \varepsilon_t = \pm 1,\ \alpha_t \in \mathbb{N}^n$$

$$E(m) := E(y^{\alpha_1})^{\varepsilon_1} E(y^{\alpha_2})^{\varepsilon_2} \cdots E(y^{\alpha_T})^{\varepsilon_T}$$

Theorem (Keller; Nagao, Reineke)

If m and m' are reddening sequence starting from Q. Then $E(m) = E(m')$.

combinatorial DT-invariant

Donaldson-Thomas (DT) invariant (Kontsevich-Soibelman) is the generating function which counts the number of "stable objects".

Definition

$E_Q := E(m)$ is well-defined as a formal power series intrinsically associated with Q. This is called the combinatorial DT-invariant of Q.

Example

$Q = (1)$: A_1-quiver

$$E_Q = E(y_1)$$

Example

$Q = (1 \to 2)$: A_2-quiver

$$y_1 y_2 = q y_2 y_1 \quad \Longrightarrow \quad E_Q = E(y_1)E(y_2) = E(y_2)E(q^{-1/2} y_1 y_2)E(y_1)$$

Relation with quantum dilogarithm

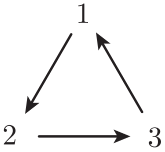
Theorem (K.-Terashima)

Let γ be a mutation loop associated with a reddening mutation sequence starting from Q.
Then the non-commutative partition q-series $Z(\gamma; y)$ coincides with the combinatorial Donaldson-Thomas invaraint $E_Q(y)$:

$$Z(\gamma; y) = E_Q(y)|_{q \to q^{-1}} \quad \in \mathbb{A}_Q$$

Example

$Q =$



The mutation sequence $m = (1,2,3,1)$ is reddening.

The non-commutative partition q-series

$$Z(\gamma) = \sum_{k_1,k_2,k_3,k_4 \geq 0} \frac{q^{\frac{1}{2}(k_1^2+k_2^2+k_3^2+k_4^2-k_1k_2+k_1k_3+k_1k_4-k_2k_4+k_3k_4)}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}(q)_{k_4}} y^{(k_1+k_3,k_2,k_3+k_4)}$$

Combinatorial Donaldson-Thomas invariant of Q:

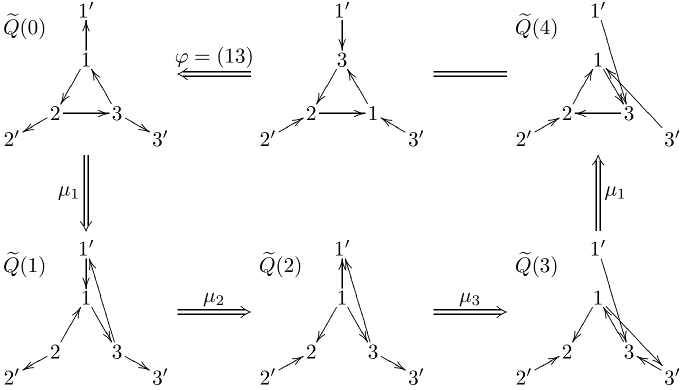
$$E_Q(y) = E(y^{(1,0,0)})E(y^{(0,1,0)})E(y^{(1,0,1)})E(y^{(0,0,1)})$$

quantum dilogarithm

$$E(x) = \prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}}y)^{-1}$$

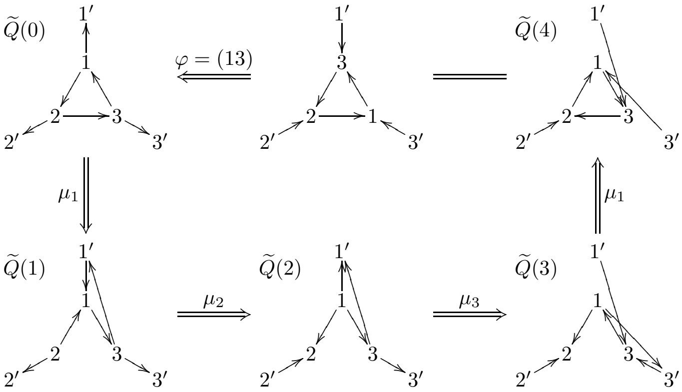
Reddening mutation sequence

The mutation sequence $m = (1,2,3,1)$ applied to "ice-quiver"



- Only one out-going arrow on every frozen vertex in the final quiver.
- The associated boundary condition φ : $1 \mapsto 3,\ 2 \mapsto 2,\ 3 \mapsto 1$

Non-commutative partition q-series



1. The associated boundary condition φ: $1 \mapsto 3,\ 2 \mapsto 2,\ 3 \mapsto 1$
2. The non-commutative grading: $y^{(k_1+k_3, k_2, k_3+k_4)} \in \mathbb{A}_Q$
3. The orientation: $(+,+,+,+)$.

Donaldson-Thomas invariant

The combinatorial Donaldson-Thomas invariant $E_Q(y)$ is defined as an ordered product of quantum dilogarithms along a maximal green sequence. (Keller; following Kontsevich-Soibelman, Nagao, Reineke)

$$E_Q(y) = E(y^{(1,0,0)})E(y^{(0,1,0)})E(y^{(1,0,1)})E(y^{(0,0,1)}) \in \mathbb{A}_Q$$

where

$$E(y) = \prod_{n=0}^{\infty}(1+q^{n+\frac{1}{2}}y)^{-1}$$

Theorem (K.-Terashima)

For any reddening mutation sequence m,

$$Z(\gamma;y) = E_Q(y)|_{q\to q^{-1}} \quad \in \mathbb{A}_Q$$

Summary

Partition q-series enjoy the following nice properties:

- Invariant under generalized pentagon moves.
- For (product of) Dynkin quivers and special mutation sequences, they coincide with characters of appropriate conformal field theories. Kuniba-Nakanishi-Suzuki formula
- For reddening mutation sequences, they coincide with the combinatorial Donaldson-Thomas invariants.
- Path integral formalism v.s. operator formalism

Various "realizations"

- 3-dimensional quantum topology
- Gauge theories Cecotti-Neitzke-Vafa, Terashima-Yamazaki, Dimofte-Gaiotto-Gukov, ...
- Categorification of cluster transformations Nagao, Reineke, Iyama, ...