Spherical double flag varieties

Evgeny Smirnov

Higher School of Economics
Department of Mathematics

Independent University of Moscow

International Conference on Geometry and Quantization
September 17, 2015

Outline

1. General definitions
   - Flag varieties
   - Schubert varieties
2. Spherical double flags of type A
   - Double Grassmannians
   - Combinatorics of B-orbits in double Grassmannians
3. Cominuscule flag varieties
   - Definition
   - Combinatorial and geometric results

Flag varieties

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  - $P = P_{max}^{(k)}$: *Grassmannian* of k-planes $G/P_{max} = Gr(k, n)$.

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  - normal;
  - Cohen–Macaulay;
  - have rational singularities...

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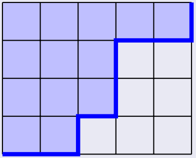

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- Types A and C: interpretation in terms of quivers by P. Magyar, J. Weyman, A. Zelevinsky.

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Combinatorics becomes more involved!

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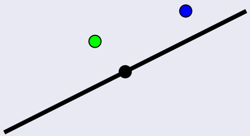

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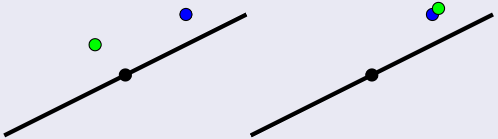

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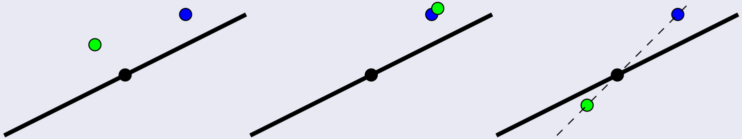

One $B \times B$-orbit splits into three B-orbits!

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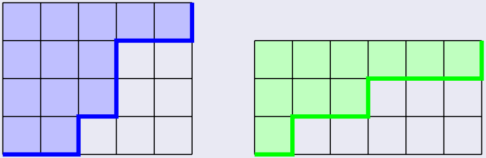

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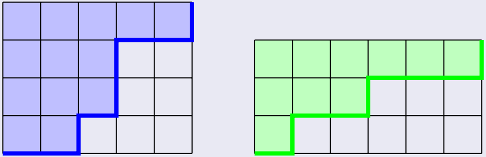

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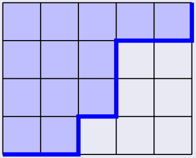

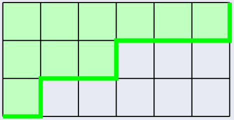

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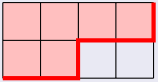

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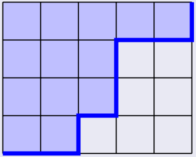

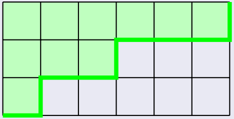

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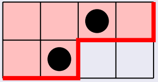

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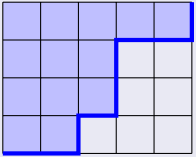

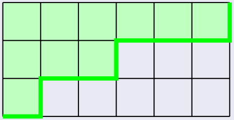

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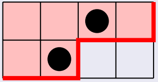

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- This allows to construct resolutions of singularities of orbit closures à la Bott–Samelson–Demazure–Hansen.

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We will consider *double cominuscule flag varieties*
(they are all spherical).

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- In type A was proved by methods of quiver theory (G.Bobiński, G.Zwara, 2001)
- Normality can fail for nonsimply laced G.

Eso es todo...

¡Gracias por su atención!