Foreword

The last five years have witnessed a considerable amount of research activity around exceptional orthogonal polynomials, and it is with great pleasure that we welcome many of the experts that have contributed to these developments.

There is no doubt that this progress stems from a large body of knowledge in various fields, and it is exciting to participate in its development and see how a well shaped theory starts to build around the first few examples.

Some of us work around the globe and yet we share the same scientific interests. Until today many of us knew each other as names on the heading of a paper. We hope that this meeting will help us identify the many aspects of the theory that still need to be straightened out, and to foster progress in this field.

A big thank you goes to the Institute of Mathematical Sciences (ICMAT) for providing the main source of funding to make this happen, through the Severo Ochoa Excellence in Research Program of the Spanish Ministry of Science. The University of Valladolid and the Campus Maria Zambrano are very kindly letting us use the conference building with all its facilities. The Centre de Recherches Mathématiques in Montréal is co-sponsoring this event and we hope that this is just the first of many joint meetings in different areas of mathematical research. Many thanks also to SIAM for providing some grants that have allowed younger participants to participate in this conference.

We know many of you have travelled many miles to reach this place. Thank you for responding to our call and be willing to share your time with us. We hope that the charming city of Segovia will provide a stimulating atmosphere for our work and we will try our best to make you feel at home.

David Gómez-Ullate
Francisco Marcellán
Miguel A. Rodríguez

Segovia, September 8th 2014.
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Special polynomials associated with rational solutions of the Painlevé equations and soliton equations

Peter A. Clarkson
School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, UK

The Painlevé equations are six nonlinear ordinary differential equations that have been the subject of much interest in the past forty years, and have arisen in a variety of physical applications. Further the Painlevé equations may be thought of as nonlinear special functions. Rational solutions of the Painlevé equations are expressible in terms of the logarithmic derivative of certain special polynomials. The soliton equations, such as the Kortweg-de Vries, nonlinear Schrödinger and Boussinesq equations, are solvable by the inverse scattering method and have symmetry reductions which reduce them to the Painlevé equations.

Classical polynomials such as the Hermite and Laguerre polynomials have roots which describe vortex equilibria. Stationary vortex configurations with vortices of the same strength and positive or negative configurations are located at the roots of the Adler-Moser polynomials, which are associated with rational solutions of the Kortweg-de Vries equation.

"Rogue waves" are giant single waves appearing in the ocean. The average height of rogue waves is at least twice the height of the surrounding waves so they can be quite unexpected and mysterious. The focusing nonlinear Schrödinger and Boussinesq equations have been proposed as mathematical models for rogue waves.

In this talk I shall discuss special polynomials associated with rational solutions for the Painlevé equations and the Korteweg-de Vries, nonlinear Schrödinger and Boussinesq equations. I shall illustrate how these special polynomials associated with rational solutions for these equations arise in vortex dynamics and rogue waves.

A bispectral approach to exceptional polynomials

Antonio J. Durán
Universidad de Sevilla, Spain

Krall and exceptional polynomials are two of the more important extensions of the classical families of Hermite, Laguerre and Jacobi.

On the one hand, Krall or bispectral polynomials are orthogonal polynomials which are also eigenfunctions of a differential operator of order bigger than two (and polynomial coefficients). The first examples were introduced by H. Krall in 1940, and since the eighties a lot of effort has been devoted to this issue.

On the other hand, exceptional polynomials are orthogonal polynomials which are also eigenfunctions of a second order differential operator, but they differ from the classical polynomials in that their degree sequence contains a finite number of gaps, and hence the differential operator can have rational coefficients. In mathematical physics, these functions allow to write exact solutions to rational extensions of classical quantum potentials.

Taking into account these definitions, it is scarcely surprising that no connection has been yet found between bispectral and exceptional polynomials. However, if one considers difference operators instead
of differential ones (that is, the discrete level of Askey tableau), something very exciting happens: Duality interchanges Krall discrete and exceptional discrete polynomials. The purpose of this talk will be to show that this unexpected connection of bispectral discrete and exceptional polynomials can be used to enlighten or even to solve some of the most interesting questions concerning exceptional polynomials.

Exceptional Orthogonal Polynomials and Spectral Theory

Lance L. Littlejohn

Department of Mathematics, Baylor University, Waco, TX, USA

With $A = \{0\}$, Kamran, Milson, and Gómez-Ullate, in 2009, classified all sequences $\{p_n\}_{n \in \mathbb{N}_0 \setminus A}$ of polynomials with the following characteristic properties:

(a) $\deg(p_n) = n$ for $n \in \mathbb{N}_0 \setminus A$, where $A$ is a finite subset of $\mathbb{N}_0$;

(b) there exists an interval $I = (a, b)$ and a Lebesgue measurable weight $w > 0$ on $I$ such that

$$\int_I p_n p_m w = k_{n,m} \delta_{n,m} \quad (n, m \in \mathbb{N});$$

(c) there exists a second-order differential expression

$$\ell[y](x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x)$$

and, for each $n \in \mathbb{N}_0 \setminus A$, there exists a $\lambda_n \in \mathbb{C}$ such that $y = p_n(x)$ is a solution of

$$\ell[y](x) = \lambda_n y(x) \quad (x \in I);$$

(d) For $n \in A$, there does not exist a polynomial $p$ of degree $n$ such that $y = p(x)$ satisfies $\ell[y] = \lambda y$

for any choice of $\lambda \in \mathbb{C}$;

(e) All of the moments

$$\int_I x^n w(x) dx \quad (n \in \mathbb{N}_0)$$

of $w$ exist and are finite.

When $A = \{0\}$, these authors discovered two such polynomial sequences $\{p_n\}_{n=1}^\infty$ and named them the exceptional $X_1$-Jacobi and $X_1$-Laguerre polynomials because of their similarities to their classical cousins. The fact that these sequences omit a constant polynomial distinguishes their characterization from the classical Bochner classification of the Jacobi, Laguerre, and Hermite polynomials. [In early 2009, the authors sent their preprint work to both W. N. Everitt and Littlejohn asking our opinions on their results: both Everitt and Littlejohn were surprised and astonished with their findings!]. Since 2009, several authors have generalized the results of Kamran, Milson and Gómez-Ullate by finding other sequences of exceptional polynomials $\{p_n\}_{n \in \mathbb{N}_0 \setminus A}$, where $A$ is a finite subset of $\mathbb{N}_0$, which satisfy conditions (a)-(e). In this lecture, we discuss these general exceptional orthogonal polynomial sequences - including a new sequence - together with the spectral theory of their associated second-order differential expressions.
Multivariate orthogonal polynomials in the real space and Toda type integrable systems

Manuel Mañas
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Multivariate orthogonal polynomials in $D$ real dimensions are considered from the perspective of the Cholesky factorization of a moment matrix. The standard graded reversed lexicographical order is used in order to get an appropriate symmetric moment matrix whose block Cholesky factorization leads to multivariate orthogonal polynomials. The approach allows to construct the corresponding second kind functions, Jacobi type matrices and associated three term relations and also Christoffel–Darboux formulae, which involve quasi-determinants – and also Schur complements– of bordered truncations of the moment matrix. It is proven that the second kind functions are multivariate Cauchy transforms of the multivariate orthogonal polynomials.

A study of discrete and continuous deformations of the measure is presented and a Toda type integrable hierarchy is constructed, the corresponding flows are described through Lax and Zakharov–Shabat equations and bilinear equations are found. Matrix nonlinear partial difference and differential equations of the 2D Toda lattice type are found for several coefficients of the multivariate orthogonal polynomials. The discrete flows, which are shown to be connected with a Gauss–Borel factorization of the Jacobi type matrices and its quasi-determinants, lead to expressions for the multivariate orthogonal polynomials and its second kind functions which generalize to the multidimensional realm those that relate the Baker and adjoint Baker functions with ratios of Miwa shifted $\tau$-functions in the 1D scenario. In this context the multivariate extension of the elementary Darboux transformation is given in terms of quasi-determinants of matrices built up by the evaluation, at a poised set of nodes lying in an appropriate hyperplane in $\mathbb{R}^D$, of the multivariate orthogonal polynomials. Finally, using congruences in the space of semi-infinite matrices, it is shown that the discrete and continuous flows are intimately connected and determine nonlinear partial difference-differential equations that involve only one site in the integrable lattice behaving as a Kadomstev–Petviashvili type system.

Exceptional orthogonal polynomials: history, current trends, and future directions

Robert Milson
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Exceptional orthogonal polynomials are polynomial eigenfunctions of a second-order Sturm-Liouville Problem. They generalize the classical orthogonal polynomial families because we permit a finite number of degrees to be missing from the eigenpolynomial degree sequence. The origins of the subject are tied to exactly exactly solvable models of quantum mechanics, but a number of recent developments and applications point to a broader role in both mathematical physics and classical analysis. The talk is intended as a survey of the history and evolution of the exceptional polynomial idea, an enumeration of some current research directions, and some musings about future developments.

Exceptional and Multi-indexed Orthogonal Polynomials: an Overview

Ryu Sasaki
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The concept of exceptional and multi-indexed orthogonal polynomials [1, 2, 3, 4, 5, 6] are defined for the classical orthogonal polynomials satisfying second order differential or difference equations. The exceptional and multi-indexed orthogonal polynomials are obtained as the main part of the eigenfunctions of quantum mechanical systems deformed by Darboux transformations in terms of certain seed solutions (virtual state wavefunctions) of the original quantum mechanical systems corresponding
to the classical orthogonal polynomials. The seed solutions are derived from the eigenfunctions of the original quantum mechanical systems by discrete symmetry transformations. Darboux transformations in terms of another type of seed solutions (pseudo virtual state wavefunctions, obtained also by discrete symmetry) are equivalent to known type of rational deformations of classical orthogonal polynomials by using Kerin-Adler transformations. Various (Wronskian and Casoratian) identities are derived \[7, 8\] as embodiment of the above duality, which is equivalent to exact solvability of the classical orthogonal polynomials. The typical examples are the Jacobi, the Askey-Wilson and the \(q\)-Racah polynomials.

References


Around Bochner-Krall problem

Boris Shapiro

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In 1929 S. Bochner published his only paper [Bo] related to orthogonal polynomials. Although he left the area for good, the importance of his contribution is difficult to underestimate. It opened a new area of research in mathematics and at the moment has been cited 344 times.

Namely, the following classification problem was stated by S. Bochner for order \(N = 2\), and H. L. Krall for general order.

\textbf{Problem 1 ([Bo, Kr])} Classify all linear differential operators with real polynomial coefficients of the form:

\[ T = \sum_{i=1}^{k} Q_i(z) \frac{d^i}{dz^i}, \quad (1) \]

such that \(a)\) \(\deg Q_i(z) \leq i; b)\) there \(\exists a positive integer i_0 \leq k with \deg Q_{i_0}(z) = i_0,\) satisfying the condition that the set of polynomial solutions \(f\) of the formal spectral problem \(T f(z) = \lambda f(z), \ \lambda \in \mathbb{R},\) form a sequence of polynomials orthogonal with respect to some real bilinear form.
If a real bilinear form comes from a positive measure supported on $\mathbb{R}$, we say that we consider a positive Bochner-Krall problem. Following the terminology used in physics, we call linear differential operators given by (1) exactly solvable. Observe that any exactly solvable operator has a unique eigenpolynomial of any sufficiently large degree which makes Problem 1 well-posed.

Let us denote by $\{p_n^T(z)\}$ the sequence of eigenpolynomials of an exactly solvable operator $T$. (Here $\deg p_n^T(z) = n$ and $n$ runs from some positive integer to $+\infty$.) An exactly solvable operator which solves Problem 1 will be called a Bochner-Krall operator. If the corresponding real bilinear form comes from a positive measure supported on $\mathbb{R}$ we call this $T$ a positive Bochner-Krall operator.

Below we study the asymptotic root-counting measure for sequences of eigenpolynomials of arbitrary exactly solvable differential operators under the assumption that a conjecture of T. Bergkvist [Be] on the rate of growth of the maximal absolute value of their roots holds. We apply this information to draw conclusions about the Bochner-Krall problem.

References


Monodromy-free operators: classification and pole configurations

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A Schrödinger operator

$$L = -\frac{d^2}{dz^2} + u(z)$$

with meromorphic potential $u(z)$ in the complex plane $z \in \mathbb{C}$ is called monodromy-free if the corresponding Schrödinger equation $L\psi = E\psi$ has all the solutions meromorphic for all values of $E$.

Duistermaat and Grünbaum [1] found the conditions on the Laurent coefficients of the corresponding potentials $u(z)$ at each pole and classified all such potentials of the form

$$u(z) = \sum_{i=1}^{N} \frac{2}{(z - z_i)^2}$$

showing that all of them are the result of iterated Darboux transformations applied to $u = 0$. Oblomkov [2] has extended this result to the class of rational potentials with quadratic growth, proving that all of them have the Wronskian form

$$u(z) = -2 \frac{d^2}{dz^2} \log W(H_{k_1}, H_{k_2}, \ldots, H_{k_n}) + z^2$$

where $H_k$ are the classical Hermite polynomials, but for general polynomial growth the classification is largely open already in the sextic case [3].

I will give a review of some known results about monodromy-free operators, including the multidimensional case and the link with the Huygens principle and Hadamard problem [4]. I will also discuss the geometry of the corresponding pole configurations, including qualitative analysis of zeroes of the Wronskians of Hermite polynomials [5] and relation with the exceptional Hermite polynomials [6].
References


Group theoretical interpretation of families of discrete multivariate orthogonal polynomials and some applications

Luc Vinet

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A review of the recent group theoretical interpretations of the multivariate Krawtchouk, Meixner and Charlier polynomials respectively based on the orthogonal, pseudo-orthogonal and Euclidean groups will be offered. Applications to discrete superintegrable models and quantum state transfer will also be presented.

Based on work done in collaboration with V.X. Genest (Montreal), H. Miki (Kyoto), A. Zhedanov (Donetsk).
Communications

The exceptional Bessel polynomials

Mohamed Jalel Atia and Said Chneguir
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Gómez-Ullate, Kamran and Milson have found polynomial eigenfunctions of a Sturm-Liouville problem [1]. These polynomials, denoted by $X_1$-Laguerre and $X_1$-Jacobi and starting with degree one, are eigenfunctions of a second order linear differential operator. In this paper, we investigate the $X_1$-Bessel case which we denote by $\hat{B}^\alpha_n(x)$. We wrote these polynomials as explicit functions of $n$, decompose it for the basis $(x-b)^2 x^i$, and expand $\hat{B}^\alpha_n(x)$ in terms of Bessel orthogonal polynomials $B^\alpha_n(x)$, using generalized Carlitz formula. Finally, we give a non-hermitian orthogonality satisfied by these polynomials.

References


ODEs featuring many parameters and (possibly only) polynomial solutions

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Some such ODE will be reported and its solutions discussed.

KP hierarchy for a cyclic quiver and generalised Bessel operators

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We introduce a generalisation of the KP hierarchy, which is intimately related to the cyclic quiver with $m$ vertices; the case $m = 1$ corresponds to the usual KP hierarchy. Generalising the result of [1], we show that our hierarchy admits special solutions parameterised by suitable quiver varieties. Using the approach of [2], we identify the dynamics of the singularities for these solutions with the classical Calogero–Moser system for the complex reflection groups $G(m, 1, n)$. The constructed solutions are closely related to the bispectral operators from the work [3]. Namely, in [3] the authors consider generalised Bessel operators of the form

$$L = x^{-m} \left( x \frac{d}{dx} - c_1 \right) \cdots \left( x \frac{d}{dx} - c_m \right), \quad c_i \text{ - arbitrary parameters.}$$

The case $m = 2$ corresponds to the usual Bessel operator. Higher order rational Darboux transformations are then performed in [3] to produce from $L$ families of bispectral differential operators. Compared to the above $L$, these operators acquire additional apparent singularities away from $x = 0$. As it turns out, these singularities correspond to the singularities of the above solutions to the generalised KP hierarchy. As a result of our work, the bispectral operators obtained by Darboux transformations from $L$ are given a non-obvious parameterisation by the points of certain quiver varieties.
References


Asymptotics of orthogonal polynomials with respect to oscillatory weights

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We present recent results on the asymptotic behavior and asymptotic zero distribution (as the degree and/or other parameters tend to $\infty$) of polynomials $p_n(x)$ that are orthogonal with respect to a weight function $w(x)$ that is oscillatory on the real axis. The two main examples will be $w(x) = e^{i\omega x}$ on $[-1, 1]$, where $\omega > 0$ is a real parameter, potentially large, and $w(x) = J_\nu(x)$ on $[0, \infty)$, where $J_\nu(x)$ is the Bessel function of order $\nu$. Because of the oscillatory nature of the weight function, the sequence of orthogonal polynomials is not guaranteed to exist or could be lacunary. The tools used for the analysis are logarithmic potential theory in the complex plane and the $S$-property, together with the Riemann-Hilbert formulation and the Deift-Zhou steepest descent method.

This is based partially on joint work with Daan Huybrechs, Arno B. J. Kuijlaars (KU Leuven, Belgium) and Pablo Román (Universidad Nacional de Córdoba, Argentina).

References


Multivariate orthogonal polynomials as overlap coefficients for superintegrable systems

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In this talk, I shall explain how $d$-variable orthogonal polynomials arise as overlap coefficients in superintegrable systems in $d + 1$ dimensions. I shall moreover illustrate how this identification can be exploited to derive the main properties of the occurring polynomials.
Quasi-Exact Solvability of Heun and related equations

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The Heun Equation [1], i.e. a Fuchsian equation with four regular singular points, is analyzed in the context of Quasi-Exactly Solvable systems associated with the Lie algebra sl(2, R) [2], [3]. Under certain conditions of the parameters Heun equation becomes a QES system and as a consequence it is possible to find a finite dimensional invariant module of orthogonal polynomials as solutions.

The four confluent cases of Heun equation: Confluent, Bi-confluent, Double-confluent and Tri-confluent Heun equations inherit this property for different regimes of parameters.

All these equations present a wide range of applications to mathematical Physics, including as particular cases very well known equations as Generalized Spheroidal Equation, Razavy Eq., Whittaker-Hill Eq., etc. Several previously known “finite” solutions of these equations can be viewed in this perspective as the finite module arising from the QES property.

References

Exceptional orthogonal polynomials and generalized Schur polynomials

Yves Grandati

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We show that the exceptional orthogonal polynomials can be viewed as confluent limits of the generalized Schur polynomials introduced by Sergeev and Veselov.

Probability distributions connected to exceptional orthogonal polynomials

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In the present paper we establish a connection between certain exceptional orthogonal polynomials [1, 2] and linear combinations of probability density functions corresponding to noncentered, quasi correlated distributions [3]. Still, in order to do so, we do not follow the usual method of replacing the weight function with a probability density function. Instead, we start with the generating functions for the exceptional orthogonal polynomials. After obtaining them, we compare certain shifted series with the moment generating functions for the distributions [4]. We have to remark that the generating functions in our paper are different from the double generating functions obtained in [2]. For our purposes we need single generated functions. To serve our aim, we use the linearization coefficients corresponding to the undeformed (original) orthogonal polynomials.

References
The bispectral problem: the good, the bad and the ugly

F. Alberto Grünbaum
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I will mention a few open problems, some results on applications, and some comments on ways to reconsider the original question studied with Hans Duistermaat [1], particularly in a non-commutative context.

References

Prepotential approach to rational extensions of solvable potentials and exceptional orthogonal polynomials

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We would like to demonstrate how the recently discovered rational extensions of solvable potentials, including those related to the exceptional orthogonal polynomials, can be constructed in a direct and systematic way, which we called the prepotential approach. In this approach, the prepotential, the deforming function, the potential, the eigenfunctions and eigenvalues are all derivable within the same framework, without the need of supersymmetry, shape invariance, or Darboux-Crum transformations.

For those potentials related to the exceptional orthogonal polynomials, it is interesting to note that the main part of the eigenfunctions, which are the exceptional orthogonal polynomials, can be expressed as bilinear combination of the corresponding deformation function ξ(η) and its derivative ξ′(η).

References

Transformations of degenerate hypergeometric equations into degenerate Heun equations

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The hypergeometric equation $\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0$, giving $2F_1$ functions can be transformed into a peculiar Heun equation with appropriate singularities by a change of variable $x = R(t)$. K. Kuiken gave, in 1979, 6 solutions with $R(t)$ polynomials of second degree and, later, R. S. Maier obtained new solutions with polynomials of higher degrees. This work investigates similar transformations, giving exact solutions with parametric constraints, for confluent hypergeometric equations transformed into the 4 degenerate Heun equations, i.e. confluent, double confluent, biconfluent and triconfluent Heun equations.

Spectral theory for exceptional Laguerre polynomials

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The Bochner Classification Theorem (1929) characterizes the polynomial sequences $\{p_n\}_{n=0}^{\infty}$, with $\deg p_n = n$ that simultaneously form a complete set of eigenstates for a second-order differential operator and are orthogonal with respect to a positive Borel measure having finite moments of all orders. Indeed, up to a complex linear change of variable, only the classical Hermite, Laguerre, and Jacobi polynomials satisfy these conditions.

In 2009, Gómez-Ullate, Kamran, and Milson generalized this theorem and found that for sequences $\{p_n\}_{n=1}^{\infty}$, $\deg p_n = n$ (without the constant polynomial), the only such sequences are the exceptional $X_1$-Laguerre and $X_1$-Jacobi polynomials. More precisely, with $A = \{0\}$, Kamran, Milson, and Gómez-Ullate \cite{1}, in 2009, classified all sequences $\{p_n\}_{n \in \mathbb{N}_0 \setminus A}$ of polynomials with the following characteristic properties:

(a) $\deg(p_n) = n$ for $n \in \mathbb{N}_0 \setminus A$, where $A$ is a finite subset of $\mathbb{N}_0$;

(b) there exists an interval $I = (a, b)$ and a Lebesgue measurable weight $w > 0$ on $I$ such that

$$\int_I p_n p_m w = k_n \delta_{n,m} \quad (n, m \in \mathbb{N})$$

for some $k_n > 0$;

(c) there exists a second-order differential expression

$$\ell[y](x) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x)$$

and, for each $n \in \mathbb{N}_0 \setminus A$, there exists a $\lambda_n \in \mathbb{C}$ such that $y = p_n(x)$ is a solution of

$$\ell[y](x) = \lambda_n y(x) \quad (x \in I);$$

(d) For $n \in A$, there does not exist a polynomial $p$ of degree $n$ such that $y = p(x)$ satisfies $\ell[y] = \lambda y$ for any choice of $\lambda \in \mathbb{C}$;
(e) All of the moments
\[ \int_I x^n w(x) \, dx \quad (n \in \mathbb{N}_0) \]
of \( w \) exist and are finite.

If \(|A|\) denotes the cardinality of the set \( A \), we call a sequence \( \{p_n\}_{n \in \mathbb{N}_0 \setminus A} \) satisfying conditions (a)–(e) above an \textit{exceptional sequence of codimension} \(|A|\). Since 2009, several authors have generalized the results of Kamran, Milson and Gómez-Ullate by finding other sequences of exceptional polynomials \( \{p_n\}_{n \in \mathbb{N}_0 \setminus A} \), where \( A \) is a finite subset of \( \mathbb{N}_0 \), satisfying each of the conditions in (a)–(e).

We discuss three families of exceptional Laguerre polynomials, each spanning a flag of codimension \( m \). Specifically, we deal with two such exceptional Laguerre sequences associated with
\[ A = \{0, 1, \ldots, m - 1\} \tag{2} \]
and a new exceptional Laguerre sequence when
\[ A = \{1, 2, \ldots, m\}. \tag{3} \]

Associated with (2), the two exceptional Laguerre sequences are known as the Type I and Type II exceptional Laguerre polynomials; properties of these two sets have been studied at length and can be found in, among others, the contributions [2], [3] and [4]. After a brief review of their properties, we develop the spectral theory of the two second-order exceptional Laguerre differential equations having these sequences as eigenfunctions. As mentioned above, we also reveal a new sequence of exceptional Laguerre polynomials, naturally named the Type III exceptional \( X_m \)-Laguerre polynomials; these polynomials are associated with the set \( A \) given in (3).

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\section*{Bilinear identities for the KP hierarchy with self-consistent sources}

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In this talk, we will show how to construct a soliton hierarchy with self-consistent sources (SHwSCSs) by using the squared eigenfunction symmetry of a soliton hierarchy. The Wronskian solutions (including soliton solutions) of the SHwSCSs can be obtained by using the Darboux transformation. We will also construct the bilinear identities for the wave functions of the KP hierarchy with self-consistent sources (KPHwSCSs). By introducing an auxiliary parameter, whose flow corresponds to the squared eigenfunction symmetry of the KP hierarchy, we find the tau-function for the KPHwSCSs. It is shown that
the bilinear identities will generate all the Hirota’s bilinear equations for the zero-curvature forms of the KPHwSCSs, which includes two types of KP equation with self-consistent sources (KPEwSCS). The Hirota’s bilinear equations obtained here are in a simpler form by comparing with the existing results.

References


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Darboux-Dressing transformations and the search for rogue waves

Sara Lombardo

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The quest for exact solutions of nonlinear partial differential equations (PDEs), which are ubiquitous in the description of physical phenomena is a central theme in mathematics.

In this talk I will report on the search for rational solutions of nonlinear dispersive equations, focusing on integrable models which describe the resonant interaction of two or more waves in 1+1 dimensions. In particular, I will consider a system of three coupled wave equations, which includes as special cases, the vector Nonlinear Schrödinger equation (or Manakov System) with both self- and cross-focusing/defocusing interaction terms, and the equations describing the resonant interaction of three waves. The Darboux-Dressing transformation is applied under the condition that the solutions have rational, or mixed rational-exponential, dependence on coordinates, leading to an algebraic construction which relies on nilpotent matrices and their Jordan form and it allows for a systematic search of all bounded rational (mixed rational-exponential) solutions.

This research, done in collaboration (see references below) in the last two years, shows interesting, novel features on rogue wave solutions, including the role played by modulation instability in the defocusing regime.

References


Some Exceptional Orthogonal Polynomials and Related Stochastic Process

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It is well-known, from the theory of Karlin and McGregor [1], that the three term recurrence relations for ordinary orthogonal polynomials (OPs) are linked with the probability matrix or birth and death process. In [2], it is shown that $j$-th exceptional Laguerre and Jacobi polynomials hold the $4j+1$ recurrence relations instead of three term recurrence relations, although their precise form and the corresponding stochastic process was not yet given. In this talk, the recurrence relations for several exceptional OPs are investigated and their connections with the probability matrices are also discussed.

References


Some Properties of the Multi-indexed Orthogonal Polynomials

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The exceptional orthogonal polynomials satisfying second order differential or difference equations are main theme of this workshop. We present some properties of the multi-indexed orthogonal polynomials of Laguerre, Jacobi, Wilson and Askey-Wilson types: recurrence relations and equivalences [1, 2].

References


Special functions and Lie groups revisited

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In this communication we revisit the deep relationship between Lie groups and orthogonal polynomials. Differential recurrence relations of orthogonal polynomials allow us to realize, via ladder operators, a symmetry Lie algebra. The corresponding orthogonal polynomials support an infinite-dimensional irreducible representation of this non-compact Lie algebra, whose second order Casimir $C$ gives rise to the second order differential equation that defines these orthogonal polynomials.
In such a way we obtain the Weyl-Heisenberg algebra \( h(1) \) with \( C = 0 \) for Hermite polynomials and the \( su(1, 1) \) Lie algebra with \( C = -1/4 \) for Laguerre and Legendre polynomials. The associated Legendre polynomials and the spherical harmonics belong to the same irreducible representation of the Lie algebra \( so(3, 2) \) with quadratic Casimir equals to \(-5/4\) and the Jacobi polynomials present a \( su(2, 2) \) symmetry.

The existence of such a ladder structure suggests to be the general condition to obtain a Lie algebra representation and allows us to define the “algebraic special functions”. These functions are proposed to be the connection between Lie algebras and square-integrable functions so that the space of linear operators on the \( L^2 \) functions is homomorphic to the universal enveloping algebra of the symmetry algebra.

References

Ladder Operators for Solvable Potentials Connected with Exceptional Orthogonal Polynomials
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Exceptional orthogonal polynomials constitute the main part of the bound-state wavefunctions of some solvable quantum potentials, which are rational extensions of well-known shape-invariant ones. The former potentials are most easily built from the latter by using higher-order supersymmetric quantum mechanics (SUSYQM) or Darboux method. They may in general belong to three different types (or a mixture of them): types I and II, which are strictly isospectral, and type III, for which \( k \) extra bound states are created below the starting potential spectrum. A well-known SUSYQM method enables one to construct ladder operators for the extended potentials by combining the supercharges with the ladder operators of the starting potential. The resulting ladder operators close a polynomial Heisenberg algebra (PHA) with the corresponding Hamiltonian. In the special case of type III extended potentials, for this PHA the \( k \) extra bound states form \( k \) singlets isolated from the higher excited states. Some alternative constructions of ladder operators will be reviewed. Among them, there is one that combines the state-adding and state-deleting approaches to type III extended potentials (or so-called Darboux-Crum and Krein-Adler transformations) and mixes the \( k \) extra bound states with the higher excited states. Such a novel approach is of interest for building integrals of motion for two-dimensional superintegrable systems constructed from rationally-extended potentials.

The fifth Painlevé equation and continuous orthogonal polynomials
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Here we investigate applications of some continuous orthogonal polynomials. It can be shown that the coefficients of the three-term recurrence relation, which the polynomials satisfy, can be expressed in terms of Hankel determinants and various modified Wronskian determinants. These specific types of Wronskians are very closely related to some special function solutions of the fifth Painlevé equation. We shall be discussing this connection in detail.

We will highlight the connection between the coefficients of the three-term recurrence relations of various orthogonal polynomials. The polynomials discussed include various deformed Laguerre polynomials, Pollaczek-Jacobi polynomials and time-dependent Jacobi polynomials. Once the logarithmic derivative of Hankel and Wronskian determinants are taken these results can be compared directly to special function solutions of the fifth Painlevé equation \( (P_5) \)

\[
\frac{d^2w}{dz^2} = \left( \frac{1}{2w} + \frac{1}{w-1} \right) \left( \frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{(w-1)^2}{z^2} \left( Aw + \frac{B}{w} \right) + \frac{Cw}{z} + \frac{ Dw(w+1) }{w-1},
\]

where \( A, B, C, \) and \( D \) are all arbitrary constants, and its associated Hamiltonian equation \( (S_5) \)

\[
\left( z \frac{d^2\sigma}{dz^2} \right)^2 = \left[ 2 \left( z \frac{d\sigma}{dz} \right)^2 - z \frac{d\sigma}{dz} + \sigma \right]^2 - 4 \prod_{j=0}^{3} \left( \frac{d\sigma}{dz} + \kappa_j \right), \tag{4}
\]

where \( \sum_{j=0}^{3} \kappa_j = 0. \)

A generalized class of orthogonal polynomials related to Gaussian hypergeometric functions

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In this work, we consider a generalized class of orthogonal polynomials given by

\[
R_n(\beta, \gamma; z) = \frac{\gamma^n}{\beta^n} \frac{\Gamma(n)}{\Gamma(\gamma-n)} \frac{2F_1}{2F_1} \left[ -n, \beta; \gamma; 1 - z \right], \quad n \geq 0.
\]

where \( 2F_1(a, b; c; z) \) is the well-known Gauss hypergeometric function. Using the approach of three term recurrence relation, formulæ for moments and weight function are obtained explicitly. The chain sequences related to this polynomial are also investigated.

Multi-indexed Jacobi polynomials and Maya diagrams

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Multi-indexed Jacobi polynomials are defined by the Wronskian of four types of eigenfunctions of a deformed Pöschl-Teller Hamiltonian. We give a correspondence between multi-indexed Jacobi polynomials and pairs of Maya diagrams, and we show that any multi-indexed Jacobi polynomial is essentially equal to some multi-indexed Jacobi polynomial of two types of eigenfunction. As an application, we show a Wronskian-type formula of some special eigenstates of the deformed Pöschl-Teller Hamiltonian.

References

Asymptotic zero distribution of polynomial solutions to degenerate exactly-solvable equations

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We study polynomial solutions to $Tf = \lambda f$, where $T = \sum_{j=1}^{k} Q_j D^j$, $D := d/dz$, and the $Q_j$ are complex polynomials in a complex variable satisfying the condition $\deg Q_j \leq j$ with equality for at least one $j$. One can easily observe that every exactly solvable operator has an eigenpolynomial of any sufficiently large degree. This class of operators enters the famous Bochner-Krall problem asking to describe all exactly solvable operators whose sequence of eigenpolynomials consists of orthogonal polynomials with respect to a real (positive) measure supported on $\mathbb{R}$. In the degenerate case, i.e. if $\deg Q_k < k$, the root of maximal modulus of the $n$th degree eigenpolynomial $f_n$ tends to infinity when $n \to \infty$. It has been conjectured that the root of maximal modulus $z_{\text{max}}$ grows as a rational power of $n$, i.e.

$$\lim_{n \to \infty} z_{\text{max}}^n = c_T n^d, c_T > 0.$$  

The value of $d$ is determined by a particular form of $T$, it depends on $\deg Q_j$, $j = 1, \ldots, k$. Studying the generic case we are able to derive explicit expressions for $d$ and $c_T$, formerly only estimated.

On the exceptional Bannai-Ito polynomials

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We will discuss the exceptional polynomials derived from the Bannai-Ito polynomials. First we introduce the exceptional Bannai-Ito operator:

$$H^{(2)} = \lambda_n B^{(2)}_n,$$

$$H = F(x)(RT^+ - \mathbb{I}) + G_1(x)R - G_2(x)\mathbb{I},$$

$$F(x) = -\frac{p_2(x)}{q_2(x)} \frac{(y - 1 - \rho_1)(y - 1 - \rho_2)}{y - 1/2},$$

$$G_1(x) = \frac{q_2(x)}{p_2(x)} \frac{(y - r_1)(y - r_2)}{y},$$

$$G_2(x) = \frac{q_2(x)}{p_2(x)} \frac{(y + r_1)(y + r_2)}{y},$$

where $p_2(x) = x^2 + s_3/s_1$, $q_2(x) = p_2(x - 1/2; r_1 - 1/2, r_2 - 1/2, \rho_1 + 1/2, \rho_2 + 1/2)$ and $s_1 = r_1 + r_2 + \rho_1 + \rho_2$, $s_3 = r_2 \rho_1 \rho_2 + r_1 \rho_1 \rho_2 + r_1 r_2 \rho_1 + r_1 r_2 \rho_2$. Then the corresponding polynomial eigenfunctions and their orthogonality relations are explicitly given. We also mention some related topics.

References


Binomial orthogonal polynomials

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30
Given a symmetric positive measure $\sigma$ on an interval $[-a,a]$, $0 < a \leq +\infty$, one can construct two one-parametric families of orthogonal polynomials by real and pure imaginary one-dimensional perturbations of the tridiagonal matrix corresponding to the measure $\sigma$. In the present talk, we consider an example of such perturbations of a given finite discrete measure.

Namely, we construct an interesting example of non-positive moment functional $L_N$ on the real line such that the corresponding Stiltjes rational function

$$R_{L_N}(z) = \sum_{k=0}^{+\infty} \frac{s_k}{z^{k+1}},$$

where $s_k = (L_N, x^k)$, $k = 0, 1, 2, \ldots$, are the moments, has the form

$$R_{L_N}(z) = -\left(\frac{N+1}{1}\right) a z^N - \left(\frac{N+1}{3}\right) a^3 z^{N-2} - \left(\frac{N+1}{5}\right) a^5 z^{N-4} - \ldots \frac{z-a}{N+1}.$$

We construct the corresponding orthogonal (w.r.t. $L_N$) polynomials and discuss their properties. Namely, I consider a positive symmetric moment functional $L_N$ such that the corresponding Stiltjes rational function

$$R_{L_N}(z) = \sum_{k=0}^{+\infty} \frac{s_k}{z^{k+1}},$$

where $s_k = (L_N, x^k)$, $k = 0, 1, 2, \ldots$, are the moments, has the form

$$R_{L_N}(z) = \tan\left(N\arccot\left(\frac{z}{a}\right)\right).$$

The corresponding orthogonal polynomials are connected to Chebyshev and discrete Chebyshev orthogonal polynomials.
Posters

Operational rules for Brenke polynomial sets: Dunkl Appell $\delta$-orthogonal polynomials as special case

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For a given polynomial set $\{P_n\}_{n\geq0}$, we define three operators, not depending on $n$, by
\[ \lambda P_n = nP_{n-1}, \quad \rho P_n = P_{n+1} \quad \text{and} \quad \tau B_n = P_n, \quad n = 0, 1, \ldots, \]
where $P_{-1} = 0$ and $\{B_n\}_{n\geq0}$ verifies
\[ \lambda B_n = nB_{n-1}, \quad B_0(0) = 1 \quad \text{and} \quad B_n(0) = 0, \quad n = 1, 2, \ldots. \]

$\lambda$, $\rho$ and $\tau$ are called respectively the lowering, the raising and the transfer operators associated to the polynomial set $\{P_n\}_{n\geq0}$ while $\{B_n\}_{n\geq0}$ is called basic sequence associated to $\lambda$.

The operational rules associated to these two operators are known as quasi-monomiality principle.

In this work, we use quasi-monomiality technics to Brenke polynomial sets. In particular, we consider the Generalized Gould-Hopper polynomials, we give new proofs of some known corresponding properties as well as new expansions formula related to this family.

References


Test criterion for finding the global minimum of a function using exclusion algorithm

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The problem of finding the global minimum of a vector function is very common in science, economics and engineering. One of the most notable approaches to find the global minimum of a function is that based on interval analysis. In this area, the exclusion algorithms (EAs) are a well-known tool for finding the global minimum of a function over a compact domain. There are several choices for the minimization condition. In this paper, we introduce a new exclusion test and analyze the efficiency and computational complexity of exclusion algorithms based on this approach. We consider Lipschitz functions and give a new minimization condition for the exclusion algorithm. Then we study the convergence and complexity of the method.

On the Christoffel-Darboux formula for generalized matrix orthogonal polynomials

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We obtain an extension of the Christoffel-Darboux formula for matrix orthogonal polynomials with a generalized Hankel symmetry, including the Adler-van Moerbeke generalized orthogonal polynomials.
Co-polynomials on the unit circle

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The purpose of the present contribution is to investigate the effects of finite modifications of Verblunsky coefficients on Szegő recurrences (see [1]). More precisely, we study the structural relations and the corresponding C–functions of the orthogonal polynomials with respect to these modifications from the initial ones.

References


On Impulse Switching Functions of Inverters as an Orthogonal System

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Impulse functions are widely used in signal theory. Switching functions of voltages and currents quantities are useful in electrical engineering. Using Clarke transform on three phase system one can obtain complex time function. In fact, it deals with the complex phasors rotating in Gauss complex plane with certain angular velocity. Obviously, those functions are harmonic but using power converters they can be strongly non-harmonic, sometimes piecewise constants with zero spaces between them. Then, it deals with power series of time pulses. From those series the impulse switching functions can be derived, which are orthogonal ones. Derived relations of voltages can be used for currents calculations in electrical engineering system using impulse transfer function and its time discretization. Similarly, one can derive relation for continuous but non-harmonic time waveforms. The contribution deals with the complex Fourier transform that has been considered for both three-phase and two-phase orthogonal voltages and currents. The investigated systems are power electronic converters supplying alternating motors. Output voltages of them are strongly non-harmonic ones, so they must be pulse-modulated due to requested nearly sinusoidal currents with low total harmonic distortion.

References

On the first eigensurface for the third order spectrum of p-biharmonic operator with weight

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We consider the following eigenvalue problem

\[ \begin{cases} 
\text{Find } (\beta, \Gamma, u) \in \mathbb{R}^N \times \mathbb{R}_+ \times X \setminus \{0\} \text{ such that} \\
\Delta_p^2 u + 2\beta \nabla (|\Delta u|^{p-2} \Delta u) + |\beta|^2 |\Delta u|^{p-2} \Delta u = \Gamma m|u|^{p-2} u \quad \text{in } \Omega, \\
u = \Delta u = 0 \\
\end{cases} \tag{7} \]

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \) \((N \geq 1)\), \( \beta \in \mathbb{R}^N \), \( \Delta_p^2 \) denotes the p-biharmonic operator defined by \( \Delta_p^2 u = \Delta(|\Delta u|^{p-2} \Delta u) \), \( X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \), and \( m \in M = \{ m \in L^\infty(\Omega)/\text{meas}\{x \in \Omega/m(x) > 0\} \neq 0 \} \). For \( p = 2 \), \( \Delta^2 = \Delta, \Delta \) is the iterated Laplacian which have been studied by many authors. For example, Gupta and Kwong [2] studied the existence of and \( L^p \)-estimates for the solutions of certain Biharmonic boundary value problems which arises in the study of static equilibrium of an elastic body.

In the case where \( \beta = 0 \), we obtain the eigenvalue problem

\[ \begin{cases} 
\text{Find } (\Gamma, u) \in \mathbb{R}_+ \times X \setminus \{0\} \text{ such that} \\
\Delta_p^2 u = \Gamma m|u|^{p-2} u \quad \text{in } \Omega, \\
u = \Delta u = 0 \\
\end{cases} \tag{8} \]

This problem was considered by P. Drábeš and M. Štani [3] for \( m = 1 \), the authors showed that the problem (8) has a principal positive eigenvalue which is simple and isolated. In [4], A. El Khalil, S. Kellati and A. Touzani, have studied the spectrum of the p-biharmonic operator with weight and with Dirichlet boundary conditions, they showed that this spectrum contains at least one non-decreasing sequence of positive eigenvalues. In [5] M. Talbi and N. Tsouli considered the spectrum of the weighted p-biharmonic operator with weight and showed that the following eigenvalue problem

\[ \begin{cases} 
\Delta(\rho)|\Delta u|^{p-2} \Delta u = \lambda m|u|^{p-2} u \quad \text{in } \Omega, \\
u = \Delta u = 0 \\
\end{cases} \tag{9} \]

where \( \rho \in C(\overline{\Omega}) \) and \( \rho > 0 \), contains at least one non-decreasing sequence of eigenvalues and studied the one dimensional case. The authors, in the same reference gave the first eigenvalue \( \lambda_1 \) and showed that if \( m \geq 0 \) a.e, \( \lambda_1 \) is simple and associated with positive eigenfunction. Also they showed that if \( m \in C(\overline{\Omega}) \), \( \lambda_1 \) is isolated and every positive or negative eigenfunction is associated with \( \lambda_1 \). Recently the authors [1], showed that the spectrum of problem (7) contains at least one sequence of positive eigensurfaces \( (\Gamma_n^p(\cdot, m))_n \) defined by

\[
(\forall \beta \in \mathbb{R}^N) \quad \Gamma_n^p(\beta, m) = \inf_{K \in \mathcal{B}_n} \sup_{u \in K} \int_\Omega e^{\beta x} |\Delta u|^p dx,
\]

and

\[
\Gamma_n^p(\beta, m) \to +\infty \quad \text{as} \quad n \to +\infty,
\]

where

\[
\mathcal{B}_n = \{ K \subset \mathcal{N}_\beta : K \text{ is compact, symmetric and } \gamma(K) \geq n \}.
\]
\[
\mathcal{N}_\beta = \{ u \in X; \int_\Omega m e^{\beta x} |u|^p dx = 1 \}.
\]

The main goal of this work is to show that \( \Gamma_{p,1}^\beta(.,m) \) is the first eigensurface and if \( m \geq 0 \) a.e., \( \Gamma_{p,1}^\beta(.,m) \) is simple and associated with positive eigenfunction. Furthermore if \( m \in C(\overline{\Omega}) \), \( \Gamma_{p,1}^\beta(.,m) \) is isolated and every positive eigenfunction is associated with \( \Gamma_{p,1}^\beta(.,m) \).

References


The \( d \)-symmetric classical polynomials

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An extension of symmetric classical orthogonal polynomials is \( d \)-symmetric classical \( d \)-orthogonal polynomials, \( d \) being a positive number. These polynomials and their derivatives satisfy particular \((d+1)\)-order recurrence relations. Many works deals with these families. Our purpose is to derive some more results for this class of polynomials. In fact, in terms of hypergeometric functions, we express their generating functions, the explicit representation, and the corresponding component sets. Moreover, we state the inversion formula, which is used to obtain the moments of the corresponding \( d \)-dimensional vector of the linear functionals and the corresponding vectors of weights, providing of integral representations for these moments.

Oscillation theorems for the Wronskian of an arbitrary sequence of orthogonal polynomials

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The work of Adler provides necessary and sufficient conditions for the Wronskian of a given sequence of eigenfunctions of Schrödinger’s equation to have constant sign in its domain of definition. We extend this result by giving explicit formulas for the number of real zeros of the Wronskian of an
arbitrary sequence of eigenfunctions. Our results apply in particular to Wronskians of classical orthogonal polynomials, thus generalizing classical results by Karlin and Szegő. Our formulas hold under very mild conditions that are believed to hold for generic values of the parameters. In the Hermite case, our results allow to prove some conjectures recently formulated by Felder et al. Numerical evidence shows that our results are also valid for Wronskians of orthogonal polynomials with respect to an arbitrary measure.

**Generalized Rayleigh and Jacobi processes and exceptional orthogonal polynomials**

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The Fokker-Planck equation (FPE) is one of the basic equations widely used for studying the effect of fluctuations in macroscopic systems. Because of its broad applicability, it is therefore of great interest to obtain solutions of the FPE for various physical situations.

As with any other equation, exactly solvable FPEs are hard to come by. Thus any new addition to the stock of exactly solvable FPEs is always welcome. One of the methods of solving FPE with time-independent diffusion and drift coefficients is to transform the FPE into a time-independent Schrödinger equation, and then solve the eigenvalue problem of the latter. Owing to this connection, an exactly solvable Schrödinger equation may lead to a corresponding exactly solvable FPE. In fact, the well-known solvable FPE describing the Ornstein–Uhlenbeck process is related to the Schrödinger equation of the harmonic oscillator. Another interesting process, called the Rayleigh process, is described by a FPE transformable to the Schrödinger equation of the radial oscillator.

Based on this relation, we would like to present four types of infinitely many exactly solvable FPEs related to the newly discovered exceptional orthogonal polynomials. The discoveries of these new kinds of polynomials, and the quantal systems related to them, have been among the most interesting developments in mathematical physics in recent years. The quantal systems that involve these new polynomials are the deformed radial oscillator and the trigonometric Darboux-Pöschl-Teller potential. Using the transformation between FPE and the Schrödinger equation, we obtained the corresponding FPEs, which are deformations of the Rayleigh process and the Jacobi process. We have shown numerically how the deformation modifies the drift coefficient of the FPE, and hence the evolution of the probability density function. It gives the possibility to manipulate evolution of stochastic processes by suitable choice of a deforming function.

**References**


**Analysis of the symmetrization for multiple orthogonal polynomials**

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For a symmetric sequence of type II multiple orthogonal polynomials satisfying a high-term recurrence relation, we provide the relationship between the Weyl function associated to the corresponding block Jacobi matrix and the Stieltjes matrix function. Next, from an arbitrary sequence of type II multiple orthogonal polynomials with respect to a set of $d$ linear functionals, we obtain a total of $d + 1$ sequences of type II multiple orthogonal polynomials, which can be used to construct a new sequence
of symmetric type II multiple orthogonal polynomials. Finally, we prove a Favard-type result for certain sequences of matrix multiple orthogonal polynomials satisfying a matrix four-term recurrence relation with matrix coefficients. This is a joint work with A. Branquinho.

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$d$-Orthogonality and $q$-Askey-scheme

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In this work, by solving a $d$-Geronimus problem, we introduce new examples of basic hypergeometric $d$-orthogonal polynomials which are useful to construct similar table to the $q$-Askey-scheme in the context of $d$-orthogonality. That, for $d = 1$, leads to a characterization theorem involving all polynomials belonging to the $q$-Askey-scheme, except the continuous $q$-Hermite ones. From some limit relations, we show that, the obtained $q$-polynomials represent the $q$-analogs of some known $d$-orthogonal polynomials.

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Scattering Amplitudes for Multi-indexed Extensions of Solvable Potentials

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New solvable one-dimensional quantum mechanical scattering problems are presented. They are obtained from known solvable potentials by multiple Darboux transformation in terms of virtual and pseudo virtual wavefunctions. The same method applied to confining potentials, e.g. Poschl-Teller and the radial oscillator potentials, has generated the multi-indexed Jacobi and Laguerre polynomials. Simple multi-indexed formulas are derived for the transmission and reflection amplitudes of several solvable potentials.

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Relations Between Classical and Generalized Jacobi Polynomials Orthogonal with Different Weight Functions

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It is well-known that Jacobi polynomials $\{P_n(x;\alpha,\beta)\}_{n=0}^{\infty}$ are orthogonal in the interval $I = (-1, 1)$ with respect to the weight function

$$J(x) = (1 - x)^{\alpha}(1 + x)^{\beta}, \quad x \in (-1, 1),$$

where $\alpha > -1$, $\beta > -1$. Very important special case of Jacobi polynomials are classical Legendre polynomials $\{P_n(x;0,0)\}_{n=0}^{\infty}$, for which $\alpha = \beta = 0$ in the weight function $J(x)$. Legendre polynomials and related functions are useful in physical geodesy (cf. [1]).

In [2] we introduced the system of polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ orthonormal in $I$ with respect to the weight function

$$Q(x) = (x^2)^{\gamma},$$

where $\gamma > 0$ and $Q_0(+\infty) > 0$. It is clear that these polynomials are generalization of the classical Legendre polynomials, which can be obtained by substituting $\gamma = 0$ in the weight function $Q(x)$. Further in [1] we established some relations between the polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ and two classes of
the classical Jacobi polynomials: the polynomials \( \{ P_n(x; 0, \frac{1}{2}) \}_{n=0}^{\infty} \) orthonormal in \( I \) with respect to the weight function \( J_1(x) = (1 + x)^{\gamma - \frac{3}{2}} \) and the polynomials \( \{ P_n(x; 0, \gamma) \}_{n=0}^{\infty} \) orthonormal in \( I \) with respect to the weight function \( J_2(x) = (1 + x)^{\gamma} \).

In this contribution we generalize the relations derived in [2] for generalized ultraspherical polynomials taking into account polynomials orthonormal in \( I \) with respect to the weight function \( \tilde{Q}(x) = (1 - x^2)^{\alpha} x^\gamma \) instead of the weight (2) with relation to the polynomials \( \{ P_n(x; \alpha, \gamma - \frac{1}{2}) \}_{n=0}^{\infty} \) and the polynomials \( \{ P_n(x; \alpha, \gamma) \}_{n=0}^{\infty} \) with \( \alpha > -1, \gamma > 0 \).

References


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Energy-dependent Schrödinger equations: inverse scattering

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In 1967 American mathematicians Gardner, Greene, Kruskal and Miura demonstrated that the initial-value problem for the KdV equation can be solved via the inverse scattering (IS) transform associated with the Schrödinger equation. This was a fundamental paper in the theory of solitons, IS transforms, and nonlinear completely integrable systems, since it led to the discovery that certain nonlinear PDEs are completely integrable: they can be solved via an IS transform and have some interesting common properties such as possessing soliton solutions, Lax pairs and infinitely many conserved quantities. In my poster I will discuss some generalizations of this connection, namely I will discuss an IS for energy-dependent Schrödinger equations with singular potentials and its applications in solving some PDEs. To be more specific, we develop the IS theory for one-dimensional energy-dependent Schrödinger equations

\[-y'' + q(x)y + 2kp(x)y = k^2y\]

on the half-line with highly singular potentials \( q \), namely, we consider potentials \( q \) of the form \( q = u' + u^2 \) for some \( u \in L^2(\mathbb{R}_+) \) (such potentials are called Miura potentials). Under some additional assumptions this Riccati representation of \( q \) is unique, and we study IS problem for the above equation along with the energy-dependent boundary condition at the origin. We show that the mapping that to every problem determined by \( (u; p) \) puts into correspondence its scattering function \( S \) is continuous with continuous inverse. We also obtain an explicit reconstruction formula for \( (u; p) \) in terms of \( S \). The work is based on a joint project with R. Hryniv (Lviv, Ukraine).
On Connection Sequence for Equivalent Polynomial Sets

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Given two polynomial sets \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \). The so-called connection coefficients from \( \{Q_n\}_{n \geq 0} \) to \( \{P_n\}_{n \geq 0} \) are the coefficients \( C_m(n) \) given in the expression:

\[
Q_n(x) = \sum_{m=0}^{n} C_m(n) P_m(x).
\]

(10)

The uniquely determined polynomial set \( \{R_n\}_{n \geq 0} \) given by

\[
R_n(x) = \sum_{m=0}^{n} C_m(n) x^m.
\]

(11)

is called the connection sequence from \( \{Q_n\}_{n \geq 0} \) to \( \{P_n\}_{n \geq 0} \).

In this work, we give some properties of the connection sequence between two polynomial sets with equivalent corresponding lowering operators. We illustrate with some examples.

References


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<tr>
<td>Time</td>
<td>Monday</td>
<td>Tuesday</td>
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<tr>
<td>9-10</td>
<td>Registration Opening</td>
<td>Odake</td>
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<tr>
<td>10-11</td>
<td>Milson</td>
<td>Durán</td>
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<tr>
<td>11-12</td>
<td>Coffee</td>
<td>Grünbaum</td>
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<td>12-13</td>
<td>Quesne</td>
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<tr>
<td>13-14</td>
<td>Odake</td>
<td>del Olmo</td>
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<tr>
<td>14-15</td>
<td>Free discussion</td>
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<tr>
<td>15-16</td>
<td>García-Ferrero</td>
<td>Castillo</td>
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<td>16-17</td>
<td>Veselev</td>
<td>Menco</td>
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<tr>
<td>17-18</td>
<td>Grandati</td>
<td>Salgiero</td>
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<tr>
<td>18-19</td>
<td>Break</td>
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Post Session