Dynamics of the horocycle flow for homogeneous and non-homogeneous foliations by hyperbolic surfaces*

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GeoDynApp - ECSING group

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*Joint work with Françoise Dal'Bo, Matilde Martínez, and Alberto Verjovsky.

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- The foliation point of view - the homogeneous case.
- Some progress in the non-homogeneous case

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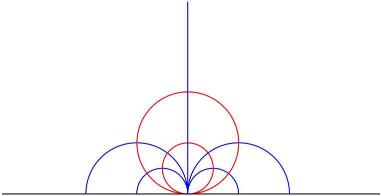

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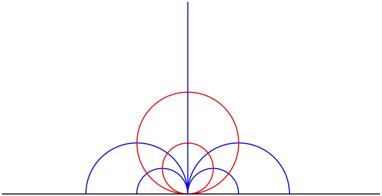

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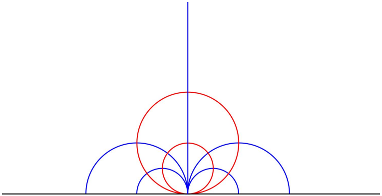

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  with the Riemannian metric

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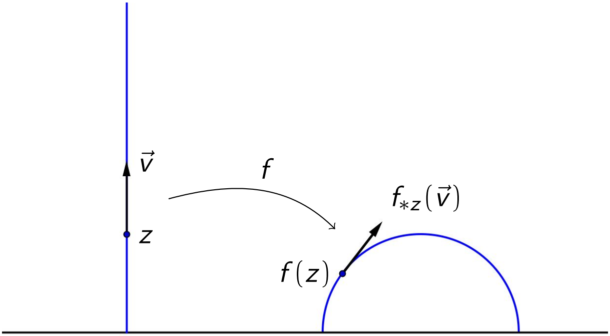
  – Γ cocompact discrete (torsion-free) subgroup of

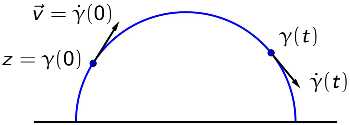
$$PSL(2, \mathbb{R}) = \{ f(z) = \frac{az+b}{cz+d} \,|\, a, b, c, d \in \mathbb{R},\ ad - bc = 1 \}.$$

- Unitary tangent bundle $T^1S = \Gamma\backslash T^1\mathbb{H} = \Gamma\backslash PSL(2,\mathbb{R})$



$PSL(2,\mathbb{R}) \curvearrowright T^1\mathbb{H}$ is free and transitive

- **Geodesic flow**:



$g_t(v) = (\gamma(t), \dot{\gamma}(t))$

$$\Gamma \backslash PSL(2,\mathbb{R}) \curvearrowleft D = \{ \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \mid \lambda > 0 \}$$

$$g_t(\Gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \Gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

right action of the diagonal group D

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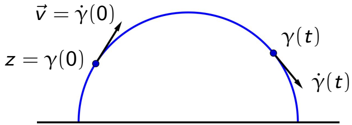

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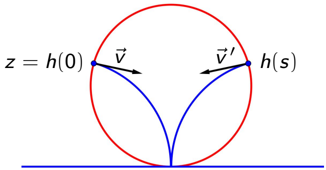
- **Horocycle flow**:



$h_s(v) = (h(s), \vec{v}')$

$\Gamma \backslash PSL(2,\mathbb{R}) \curvearrowleft U = \{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{R} \}$

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right action of the unipotent group U

- **Affine action**: $\Gamma\backslash PSL(2,\mathbb{R}) \curvearrowleft B = DU =$ affine group

$$g_t \circ h_s = h_{-se^t} \circ g_t$$

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Theorem (Hedlund, 1936)

Let $S = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. Then the U-action

$$X = T^1 S = \Gamma\backslash PSL(2,\mathbb{R}) \curvearrowleft U$$

is minimal (with dense orbits).

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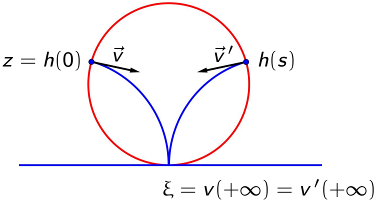

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- Γ non-elementary $\Rightarrow \Gamma \curvearrowright \Lambda(\Gamma)$ minimal.
- Γ cocompact $\Rightarrow \Lambda(\Gamma) = \partial\mathbb{H}$.

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Lemma

$H_{\mathcal{M}}$ is a closed U-bi-invariant subset of $PSL(2,\mathbb{R})$.

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$\Downarrow$ Γ has no parabolic elements

$$[\gamma_{n_k}, \gamma_{n_{k'}}] = I \qquad \forall\, k, k' \in \mathbb{N}.$$

$$\Downarrow$$

$$f^{-1}\gamma_{n_k} f = u \begin{pmatrix} \lambda_{n_k} & 0 \\ 0 & \lambda_{n_k}^{-1} \end{pmatrix} u^{-1} \text{ with } u \in U \text{ and } \lambda_{n_k} \to \infty$$

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$$\Downarrow$$

$$f_{n_k}(e_1) = (\lambda_{n_k}, 0) \to e_1 = (1, 0),$$

which is not possible. □

Put

$$f_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

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$$u_n' = \begin{pmatrix} 1 & \frac{\alpha - a_n}{c_n} \\ 0 & 1 \end{pmatrix} \quad u_n'' = \begin{pmatrix} 1 & -\frac{1}{\alpha}(b_n + d_n \frac{\alpha - a_n}{c_n}) \\ 0 & 1 \end{pmatrix}$$

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$$\Downarrow \quad c_n \to 0$$

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So $\mathcal{M}$ is B-invariant and hence $\mathcal{M} = X$. □

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Theorem (Hedlund, 1936)

Let $S = \Gamma\backslash\mathbb{H}$ be a compact hyperbolic surface. Then the U-action $X = T^1S = \Gamma\backslash PSL(2,\mathbb{R}) \curvearrowleft U$ is minimal.

Theorem (A.-Dal'Bo)

Let G be a connected Lie group. Let $X = \Gamma\backslash PSL(2,\mathbb{R}) \times G$ be the quotient of the Lie group $PSL(2,\mathbb{R}) \times G$ by a cocompact discrete subgroup Γ.

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$$PSL(2,\mathbb{R}) \xleftarrow{p_1} PSL(2,\mathbb{R}) \times G \xrightarrow{p_2} G$$

Examples

- **A. Borel**: $\exists \Gamma \subset PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ cocompact discrete

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Proposition

Let G be a connected Lie group and let Γ be a discrete subgroup of $PSL(2,\mathbb{R}) \times G$. The action $X \curvearrowleft B$ is minimal if and only if $\overline{p_2(\Gamma)} = G$ and $p_1(\Gamma) \curvearrowright \partial\mathbb{H}$ is minimal.

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Aim

If Γ is cocompact, then $X \curvearrowleft B$ is minimal if and only if $\overline{p_2(\Gamma)} = G$.

Definition

An element $(f, g) \in PSL(2,\mathbb{R}) \times G$ is said to be *semi-parabolic* if $f \neq I$ belongs to U up to conjugation.

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Proof sketch.

$$\exists\,(fuf^{-1}, g) \in \Gamma \text{ semi-parabolic } \Rightarrow xD^{+} \text{ diverges for } x = \Gamma(f, g')$$

Proposition

If $X = \Gamma\backslash PSL(2,\mathbb{R}) \times G$ is compact, then

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$$X \curvearrowleft PSL(2,\mathbb{R}) \text{ is minimal.}$$

▸ Proof

Proof. Similar to the case where G is trivial:

- $\exists\,(f_n, g_n)$ in $H_{\mathcal{M}} = \{\, h \in PSL(2,\mathbb{R}) \times G \,|\, \mathcal{M}h \cap \mathcal{M} \neq \emptyset \,\}$ converging to (I, e):

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$$\overline{xU} = \overline{\Gamma(f,g)U} = \mathcal{M} \;\Rightarrow\; \exists\,\gamma_n = (\gamma_{1n}, \gamma_{2n}) \in \Gamma \;\exists\, u_n \in U \;:$$

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- Bi-invariance of $H_{\mathcal{M}}$:

Lemma

$H_{\mathcal{M}}$ is a closed U-bi-invariant subset of $PSL(2,\mathbb{R}) \times G$.

- Geometric consequence of the Γ-cocompactness:

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There exists $k \in \mathbb{N}$ such that $f_n \notin B$ for $n \geq k$.

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- Using $H_{\mathcal{M}}$ is U-bi-invariant and passing to the limit, we have:

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, e\right) \in H_{\mathcal{M}} \quad \Leftrightarrow \quad \mathcal{M}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \cap \mathcal{M} \neq \emptyset \quad \forall \alpha \in \mathbb{R}_+^*$$

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By minimality, this implies

$$\mathcal{M}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \mathcal{M} \qquad \forall \alpha \in \mathbb{R}_+^*.$$

Remark

The two examples

$$\begin{array}{ccccc} PSL(2,\mathbb{R}) & \longrightarrow & PSL(2,\mathbb{R}) \times G & \longrightarrow & G = \begin{cases} PSL(2,\mathbb{R}) \\ SO(3) \end{cases} \\ & & \Big\downarrow \pi & & \\ & & X = \Gamma \backslash PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}) & & \end{array}$$

are G-Lie foliations.

In fact, X is the unitary tangent bundle of the G-Lie foliation

$$\begin{array}{ccc} \mathbb{H} \longrightarrow & PSL(2,\mathbb{R})/PSO(2,\mathbb{R}) \times G \longrightarrow G = \left\{ \begin{array}{l} PSL(2,\mathbb{R}) \\ SO(3) \end{array} \right. \\ & \Big\downarrow \pi \\ & M = \Gamma \backslash PSL(2,\mathbb{R})/PSO(2,\mathbb{R}) \times PSL(2,\mathbb{R}) \end{array}$$

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according to Fedida's description with holonomy representation given by

$$h = p_2|_\Gamma : \Gamma \to p_2(\Gamma) \subset G$$

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Examples

There are non-homogeneous examples constructed by G. Hector, S. Matsumoto and G. Meigniez whose leaves are hyperbolic surfaces.

Proposition

Let $\mathcal{F}$ be G-Lie foliation by hyperbolic surfaces of a compact connected manifold M, and let $X = T^1\mathcal{F}$ be its unitary tangent bundle.

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Theorem

Let $X = T^1\mathcal{F}$ be the unitary tangent bundle of a homogeneous G-Lie foliation $\mathcal{F}$ by hyperbolic surfaces of a compact manifold. If $\mathcal{F}$ is minimal, then $X \curvearrowleft U$ is minimal.

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Theorem

Let $X = T^1\mathcal{F}$ be the unitary tangent bundle of a homogeneous Riemannian foliation $\mathcal{F}$ by hyperbolic surfaces of a compact manifold. If $\mathcal{F}$ is minimal, then $X \curvearrowleft U$ is minimal (by using Molino's theory).

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Let Γ be a cocompact Fuchsian group. Consider the diagonal action

$$\Gamma \curvearrowright PSL(2,\mathbb{R}) \times \partial\mathbb{H}$$

given by

$$\gamma(f,\xi) = (\gamma f, \gamma(\xi))$$

and the quotient $X = \Gamma\backslash PSL(2,\mathbb{R}) \times \partial\mathbb{H}$.

Then X is the unitary tangent bundle of the transversely homographic foliation $\mathcal{F}$ induced by

$$\begin{array}{ccccc} \mathbb{H} & \longrightarrow & \mathbb{H} \times \partial\mathbb{H} & \xrightarrow{p_2} & \partial\mathbb{H} \\ & & \downarrow{\scriptstyle \pi} & & \\ & & M = \Gamma\backslash\mathbb{H} \times \partial\mathbb{H} & & \end{array}$$

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- $X \curvearrowleft B$ is not minimal as dual to the diagonal action

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  $$\gamma(\xi_1, \xi_2) = (\gamma(\xi_1), \gamma(\xi_2))$$

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Theorem (Martínez-Verjovsky)

Let $(M, \mathcal{F})$ be a compact lamination by hyperbolic surfaces, and let $X = T^1\mathcal{F}$ its unitary tangent bundle. Assume that $\mathcal{F}$ is minimal.

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▸ Proof sketch

Example

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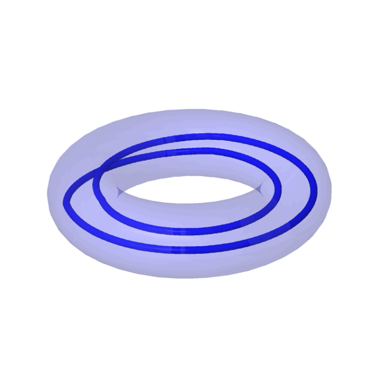

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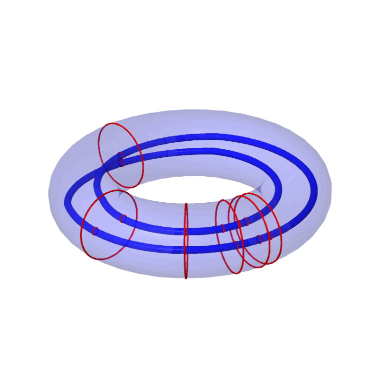
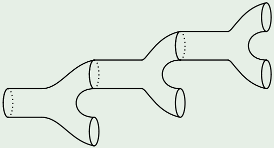


leaf without holonomy



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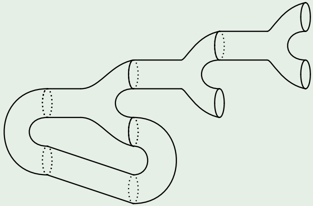
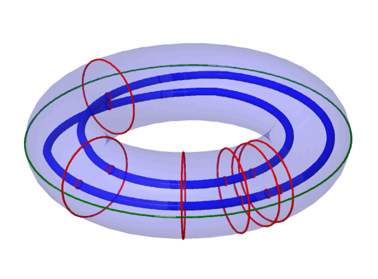


- Transverse dynamics of $\mathcal{F}$ is represented by the doubling map

$$f(x) = 2x \pmod 1$$



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$$\Downarrow \text{ Martínez-Verjovsky theorem}$$

$$X = T^1\mathcal{F} \curvearrowleft B \text{ is minimal}$$

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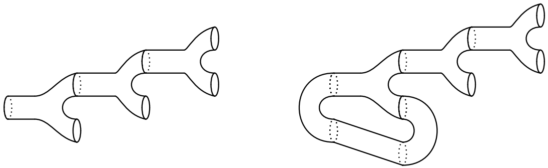
Remark [S. Matsumoto]

The horocyclic flow on each leaf of $\mathcal{F}$ has no minimal sets.



Theorem (A.-Dal'Bo-Martínez-Verjovsky)

Let $(M, \mathcal{F})$ be a minimal compact lamination by hyperbolic surfaces. If $\mathcal{F}$ admits a leaf without holonomy which is neither homeomorphic to the plane nor to the cylinder, then the horocycle flow on the unitary tangent bundle $X = T^1\mathcal{F}$ is minimal.

Theorem (A.-Dal'Bo-Martínez-Verjovsky)

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Corollary

For the non-homogeneous Lie foliations constructed by G. Hector, S. Matsumoto and G. Meigniez, the horocycle flow is minimal.

Proof skecht. Before we start, we need the following definition:

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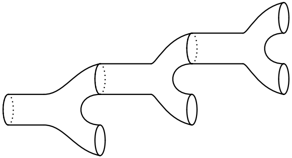
Definition

Consider a complete hyperbolic surface $L = \Gamma\backslash\mathbb{H}$ which can be partitioned into countably many pair of pants (*pants decomposition*) whose boundary components have uniformly bounded lengths (*good pants decomposition*). Its fundamental group Γ (which is purely hyperbolic of the first kind) is called *tight* by S. Matsumoto.



- **First Step**: Obtaining good pants decompositions of the leaves.

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Let $(M, \mathcal{F})$ be a minimal compact lamination such that

i) *the affine action $X = T^1\mathcal{F} \curvearrowleft B$ is minimal,*

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Let $\mathcal{M} \neq \emptyset$ be a U-minimal set in $T^1\mathcal{F}$. If there is a real number $t_0 > 0$ such that $g_{t_0}(\mathcal{M}) = \mathcal{M}$, then $\mathcal{M} = X$.

▸ Proof

- **Condition (i)**: All the leaves are 'topologically non-trivial'

$$\Downarrow \text{ Martínez-Verjovsky theorem}$$

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Aim

If L has a good pants decomposition, then for every $x \in T^1L$, either $\overline{xU} = T^1L$ or there is a real number $t_0 > 0$ such that $g_{t_0}(x) \in \overline{xU}$ (so any proper U-minimal set $\mathcal{M} \neq \emptyset$ in T^1L verifies $g_{t_0}(\mathcal{M}) = \mathcal{M}$).

- **Third Step**: Using Matsumoto's idea to prove U-minimality.

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Fundamental lemma

Let $L = \Gamma\backslash\mathbb{H}$ be a non-compact hyperbolic surface. Assume there are sequences of

- closed geodesics C_n with uniformly bounded lengths
- real numbers $t_n \to +\infty$ such that

$$g_{t_n}(x) \in C_n$$

for some $x \in T^1 L$. Then there is a real number $t_0 > 0$ such that

$$g_{t_0}(x) \in \overline{xU}.$$

Proposition

Assume L has a good pants decomposition. Then, for any tangent vector $x \in T^1L$, *either* $\overline{xU} = T^1L$ *or there is a real number* $t_0 > 0$ *such that* $g_{t_0}(x) \in \overline{xU}$.

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Using the duality between the U-action and the linear Γ-action and the classification of the limit points of Γ, we can deduce:

Theorem (S. Matsumoto)

For any hyperbolic surface with a good pants decomposition, the horocycle flow has no minimal sets.

▸ Proof

Theorem (A.-Dal'Bo-Martínez-Verjovsky)

Let $(M, \mathcal{F})$ be a minimal compact lamination by hyperbolic surfaces. If $\mathcal{F}$ admits a leaf without holonomy which is neither homeomorphic to the plane nor to the cylinder, then the horocycle flow on the unitary tangent bundle $X = T^1\mathcal{F}$ is minimal.

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GRACIAS

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- If $\Delta = p_1(\Gamma)$ is non-discrete dense, then $\Delta \curvearrowright \Lambda(\Delta) = \partial\mathbb{H}$ is minimal again.

- If Δ is neither discrete nor dense, then there is $f \in PSL(2, \mathbb{R})$ such that

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which also contradicts the compactness of X. $\square$

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The leaves of $\mathcal{F}$ are the orbits of a continuous locally free B-action if and only if there is a B-minimal set $\mathcal{M}$ in X such that

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Lemma

Let $\mathcal{M}$ be a B-minimal set of X. If

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- By hypothesis, C is cyclic and hence the B-invariant set

$$\mathcal{M}D = \bigcup_{t\in\mathbb{R}} g_t(\mathcal{M}) = \bigcup_{t\in[0,t_0]} g_t(\mathcal{M})$$

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- There is $t \in [0, t_0]$ such that $g_t(x) \in \mathcal{M}$. Since $g_t(xU) = g_t(x)U$ and $\mathcal{M}$ is U-invariant, $g_t(\overline{xU}) \subset \mathcal{M}$ and therefore $X = \mathcal{M}$. □

▸ Back

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Definition

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Theorem

For each $v \in T^1\mathbb{H}$, the following conditions are equivalent:

i) *the point $\xi = v(+\infty)$ is horocyclic,*

ii) *the horocycle orbit of the projected point $x \in T^1L$ is dense,*

iii) 0 *belongs to $\overline{\Gamma\upsilon}$ where υ is the element of E corresponding to the horocycle passing through v.*

Proof.

- From the previous theorem, any proper U-minimal set $\mathcal{M} \neq \emptyset$ in T^1L contains a horocycle centered at non-horocyclic limit point ξ of Γ.

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- By duality, we deduce that

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- By duality, we deduce that

$$e^{t_0/2}(\overline{\Gamma\upsilon}) = \overline{\Gamma\upsilon}$$

and therefore $0 \in \overline{\Gamma\upsilon}$, which contradits the assumption that ξ is non-horocyclic. □

Back