SCIENTIFIC REVIEW: Singular integrals in quantum euclidean spaces

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Authors: Adrián M. González-Pérez (KU Leuven, Bélgica), Marius Junge (University of Illinois at Urbana, EE.UU.) and Javier Parcet (ICMAT)
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One of the most dramatic shifts in our understanding of the physical world occurred in 1925, when Heisenberg proved that quantum phenomena could be deduced from the equations of Newtonian physics if we interpreted the variables that depend on time as infinite matrices. John von Neumann proposed to model Heisenberg matrices as adjoint operators over Hilbert spaces. The theory of von Neumann algebras is a *noncommutative or quantum form of measure theory* and provides a rigorous mathematical framework for matrix mechanics. The von Neumann Programme responds to the need of "quantizing mathematics" in order to complete the classical/relativistic notions of measure and geometry. It constitutes a challenge of extraordinary magnitude that has exceeded the contributions of von Neumann himself. Indeed, today we are able to speak of *noncommutative geometry*, *quantum probability*, *operator spaces*, *quantum groups* and so on. Apart from its importance for mathematics, connections exist with theoretical physics; in string theory, quantum field theory and quantum information. All of this explains the necessity to develop the von Neumann Programme in other directions, one of which is harmonic analysis.

In 1980, Alain Connes introduced *noncommutative geometry* [5] as an extension of differential geometry over "noncommutative manifolds" in the operator algebra language introduced by von Neumann. The archetypes of noncommutative manifolds are *quantum tori* and *quantum Euclidean spaces.* Given Θ a real antisymmetric matrix $n \times n$, the associated n-dimensional torus is defined (vaguely) as the algebra generated by unitary operators $u_1, u_2, \ldots, u_n$ that satisfy the canonical commutation relations $u_j u_k = \exp(2\pi i \Theta_{jk}) u_k u_j$. When $\Theta=0$, the u_j are the primary characters $x \to \exp(2\pi i x_j)$ and the associated algebra that of the functions (smooth, continuous, bounded...) over the classical torus **R**/**Z** x...x **R**/**Z**. When Θ does not cancel, the algebra is noncommutative and is known as a quantum torus. The quantum Euclidean spaces admit a similar definition; the Heisenberg-Weyl algebra is the most well-known model. Connes introduced in [4] pseudo-differential operators in quantum tori with the aim of extending the Atiyah-Singer index theorem in this context. The insight provided by Connes' approach was subsequently confirmed by the Gauss-Bonnet theorem for quantum tori [6, 7] and a solid elliptic operator theory. In its 40 years of history, noncommutative geometry has also developed connections with number theory and string theory [12].

Pseudo-differential operator theory emerged in the mid-1960s with the work conducted by Kohn, Nirenberg and Hörmander [9]. The idea is to use the Fourier transform to represent differential operators $L = \Sigma_{|\alpha|\leq m} a_\alpha(x) \partial_x^{\alpha}$ that can be inverted except for an error term. Pseudo-differential operators can be interpreted as singular integrals. Calderón-Zygmund theory [3] ––paradigm of modern harmonic analysis–– then provides p-Sobolev-type estimates of the approximation and error terms, which leads to the deepest results of the theory. Unfortunately, the work by Connes and his collaborators does not include this class of estimates due to deep obstructions to develop the theory of singular integrals over noncommutative L_p spaces, defined over von Neumann algebras.

In the work reported here, the core of singular integral theory and pseudo-differential calculus are established on the model algebras for noncommutative geometry: quantum forms of tori and Euclidean spaces. The latter –also known as Moyal deformations in theoretical physics or CCR algebras in quantum probability– include the Heisenberg-Weyl algebra determined by the position and the momentum in quantum mechanics. These results on pseudo-differential operators go beyond the work of Connes, thanks to a new form of the Calderón-Zygmund theory in these algebras, which are developed in the same work and which crucially includes general kerrnels which are not of convolution type. This enables to deduce L_p boundedness and p-Sobolev estimates for regular, exotic and forbidden symbols in the expected ranges. In L_2, the authors also generalize the Bourdaud and Calderón-Vaillancourt theorems [1, 2] for exotic and forbidden symbols. All the foregoing establishes the quantum forms of the most famous results of pseudo-differential operator theory [13]. As an application of these methods, L_p regularity of solutions to the first elliptic PDEs in von Neumann algebras are proved.

Finally, it is worth pointing out that noncommutative Calderón-Zygmund theory has precedents in the work of the authors with interesting connections in geometric group theory and Lipschitz operator functions. However, unlike in the previous results [8, 10], this is the first model that works in *purely noncommutative* algebras; that is, algebras that contain no copies of doubling metric spaces in the form of tensor products or crossed products. Recently, some of the authors have developed in [11] an *algebraic form of Calderón-Zygmund theory* that is valid in general von Neumann algebras equipped with a Markov process that satisfies strictly algebraic conditions.

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