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SCIENTIFIC REVIEW: Fourier series in BMO with number theoretical implications

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Summary

Our ears break down any sounds into pure tones and each tone corresponds to a certain frequency and intensity. Mathematically speaking, the sound is represented by a function, and the pure tones by sinusoidal waves whose size is known as amplitude and is related to its intensity. In this paper, the authors study functions that also oscillate and whose frequencies grow polynomially. This oscillation depends on how the corresponding amplitudes decay.

The BMO (Bounded Mean Oscillation) space contains the functions whose mean oscillation over any interval is bounded by a constant, which does not depend on the interval. The least of these constants would be its norm, which measures how much the function oscillates. Thus, any bounded function is found in this space, but it also contains non-bounded functions, such as the logarithm. In fact, Fritz John and Louis Nirenberg demonstrated that this is the maximum permitted growth; that is, any function in this space must have peaks that are at most logarithmic.

This space plays an important role in the field of analysis, because on certain occasions it is the natural substitute for bounded function spaces, which help to understand the behaviour of solutions to partial differential equations. On the basis of an inequality proposed by Godfrey Harold Hardy, it was known that if a function possesses as frequencies w, all the naturals are then in BMO when its amplitudes decay as w^{-1}. It was also known that the lacunary series (whose frequencies grow exponentially) are found in BMO if they are in L^1, and therefore in L^2, which is equivalent to the sum of the squares of the absolute values being finite.

The intermediate case, which includes series with frequencies that grow polynomially, was analyzed by William Tazwell Sledd and David Allan Stegenga from a result by Charles Fefferman that characterizes when a series is found in BMO. In order to prove this theorem, an inequality similar to those known as large sieve inequalities (much used in number theory) and two deep results on BMO are employed. The first result is that BMO is the dual space of the so-called Hardy space H^1, which are analytic functions $u(z)$ on the unit disk, such that the integrals of its absolute value in circles of radius less than one, centered on the origin, are uniformly bounded. The second result states that any function of H^1 on the edge of the disk may decompose into *atoms*, which are zero-mean functions that take non-null values only in one interval and are bounded by the inverse of the longitude of the said interval.

Now, Chamizo, Córdoba and Ubis essentially prove the same result, although their argument is elementary and their bound on the norm is more precise. In particular, they deduce that if the frequencies w grow as a polynomial of degree d, then the amplitudes must decay as $dw^{-1/d}$ in order for the function to be in BMO.

The difficulty of proving this statement resides in the fact that the function is typically much more chaotic than in the Hardy case, but unlike what occurs in lacunary series, its frequencies continue to interact. This new proof starts by decomposing the function into two parts; one with low frequencies and the other with high. This division depends on the interval in which the oscillation of the function is being analyzed. The low frequency part is a smooth function in the interval selected and therefore its oscillation is always small.

While the part with high frequencies is small, on average it is more irregular, so it is more difficult to control its mean oscillation. What is done in this case is to employ the well-known Cauchy inequality in order to evaluate the mean oscillation of the square of the function in the said interval. This is simpler because its Fourier series can be used to expand the square and average each term of the sum separately.

Consequently, the authors arrive at a bilinear oscillatory sum that it is necessary to bound. Here, they apply a large sieve type of inequality that converts this oscillation into decay. Perhaps the simplest way to understand this inequality is to return to the average of the square; it is possible to bound this by means of a smoothed average, and after expanding the series again, this smoothing will have converted the oscillation into decay.

In the second part of the paper, these researchers study the particular case of the function whose frequencies w traverse the squared integers and whose amplitudes are $w^{-1/2}$. From the previous result, the said function is in BMO, but now they are able to state much more about it. Given that its frequencies are just the squares, its behaviour is more arithmetical than that of a generic function space.

In fact, at rational points it has a special behaviour, and at any irrational its size is determined by the ease with which it can be approximated by rationals. On the basis of this information, it can be demonstrated that it is impossible to draw a graph of the said function, since it possesses an uncountable number of logarithmic peaks. Furthermore, the authors are able to estimate, in any interval, both the mean oscillation and its precise growth on the dominant logarithmic peak. Both quantities will depend on the rational with the least denominator in the interval. The greater the said denominator, the less the oscillation and growth the function will have in that zone.