SCIENTIFIC REVIEW: On the solvability of the Dirichlet problem for elliptic operators

Título original: "Perturbations of elliptic operators in 1-sided chord-arc domains. Part I: Small and large perturbation for symmetric operators".
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Partial differential equations (PDEs) enable the modelling of phenomena arising from natural, social and economic sciences. One example is provided by the equations that define the evolution of diffusion (such as the heating of a solid body to which a heat source is applied). In this type of problems, one seeks for solutions to an equation associated with a differential operator and defined in a certain region, prescribing the values taken by the solution at the boundary of that region. Finding the solution to a differential equation, when the boundary value is known, is what is called solving the *Dirichlet problem* associated to the differential operator under consideration.

The Dirichlet problem was originally posed for the *Laplace equation* – paradigm of the second-order elliptic partial differential equations – and it can be solved nowadays for many differential operators. The class of data that we consider in the boundary is crucial for solving the Dirichlet problem. For the Laplace equation, with continuous boundary data and with solutions that are continuous up to the boundary of the region, the well-known *Wiener criterion* characterizes the class of regions in which the Dirichlet problem can be solved.

Apart from the Laplace operator, however, it is interesting to study the same problem for more general operators, such as the second order *divergence form* elliptic operators, where the coefficients may vary according to the position. Furthermore, the situation becomes more complicated when studying problems in which the boundary values show worse behaviours. For example, we can pose the L^p-Dirichlet problem (with $1 < p < \infty$) where the boundary data belong to L^p, the space of functions whose pth-power is integrable, and in which the solutions coincide with their boundary values, in the sense of approximation to the boundary in regions that are generalized cones (that is, *non-tangential* regions). Thus, in this formulation, boundary values that are not necessarily continuous are allowed, and may even possess singularities. In such cases, is it possible to establish what conditions are required in order to ensure that the L^p-Dirichlet problem is solvable in the previous sense?

Different methods exist for approaching this question. In the article "Perturbations of elliptic operators in 1-sided chord-arc domains", Juan Cavero (ICMAT), Steve Hofmann (University of Missouri, USA) and José María Martell (ICMAT) start from a given operator for which the solvability of the L^p-Dirichlet problem is known (with a certain p, $1 < p < \infty$) , and study to what extent it is possible to modify the operator in question so that one is able to solve a similar Dirichlet problem for the new operator. In other words, it involves determining which *perturbations* of the original operator still admit solutions to the Dirichlet problem.

In their article, Cavero, Hofmann and Martell work in rough domains that are *open and path-connected* and whose boundary has *codimension* 1. Assuming that, in a quantitative or scale-invariant sense, they determine certain conditions that the operators should satisfy in order to ensure the *solvability* desired. Specifically, the conditions are about the difference in the coefficients of the original and the perturbed operator, where both are real symmetric. In the case where the discrepancy between the coefficients is sufficiently small (in the sense of a certain *Carleson measure*), the authors have proved that solving the L^p-Dirichlet problem for the original operator translates into solving the same problem (with the same value p) for the perturbed operator. More generally, for large discrepancies between the coefficients, what they show is that solving the L^p-Dirichlet problem for the original operator enables the solvability of the L^q-Dirichlet problem for the perturbed operator for another value $q \in (1,\infty)$ that is not necessarily p.

These results extend the work of Fefferman-Kenig-Pipher in [3] and Milakis-Pipher-Toro in [5, 6], who considered domains with a certain degree of regularity. Now, Cavero, Hofmann and Martell employ an alternative method that enables more general domains to be taken into account.

In the work in question, some applications are provided in which it is possible to establish geometric consequences from the solvability of the L^p-Dirichlet problem. For example, if a symmetric perturbation of an operator with coefficients having certain regularity, such as the *Laplacian,* possesses the property that the L^p-Dirichlet problem is solvable for some $p \in (1,\infty)$, then the perturbation theory establishes that it is possible to solve the L^q-Dirichlet problem for some other $q \in (1,\infty)$. This, together with the work by J. Azzam, S. Hofmann, J. M. Martell, K. Nyström and T. Toro [2], shows that the domain must have an open exterior in a quantitative sense, and therefore its boundary has a certain degree of regularity (it is *uniformly rectifiable*). As one may see, geometric information is thereby obtained about the region in which it is possible to solve a Dirichlet problem.

The techniques used in the foregoing questions require the inclusion as a hypothesis that the region should satisfy a certain notion of strong connectivity. Without this hypothesis, it is not known if the previous perturbation results are valid, which throws up an interesting challenge in the field. J. Azzam, S. Hofmann, J. M. Martell, M. Mourgoglou and X. Tolsa have recently presented a characterization of the regions in which it is possible to solve the Dirichlet problem-L^p for some $p \in (1,\infty)$ for the Laplace operator. This geometric characterization requires the boundary to be regular; that is, it should be *uniformly rectifiable* (as may be seen in [4]) as well as a condition of access to portions of the boundary by means of non-tangential paths. In this article [1], which is still in the process of peer-review, the proposed characterization is optimal, since none of the imposed conditions can be eliminated.

References:

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TELL ME ABOUT YOUR THESIS: Bruno Vergara

Thesis title: "Weighted inequalities in Fluid Mechanics and General Relativity: Carleman estimates and cusped traveling waves".
Author: Bruno Vergara (ICMAT-UAM)
Director: Alberto Enciso (ICMAT)
Date of submission: July 16th, 2019

Raquel G. Molina. **Bruno Vergara (February 18th, 1991, Valparaíso, Chile) says that the four years he has spent completing his doctoral thesis on partial differential equations (PDEs) have been extremely satisfying, especially because it has enabled him to collaborate with first-rank researchers. It is thanks to them, and particularly to his thesis supervisor, Alberto Enciso (ICMAT), that in this young researcher's own words: "I've been able to learn what research activity is all about, above and beyond mathematical techniques in themselves". He defines research work as a process full of difficult moments, of setbacks and ups and downs, but which nevertheless proceeds onwards.**

His thesis, entitled "Weighted inequalities in Fluid Mechanics and General Relativity: Carleman estimates and cusped traveling waves", is divided into two parts in which singular solutions to partial differential equations are analyzed in two different contexts: General relativity and Fluid Mechanics. In the first part, he seeks to generalize a known result in General Relativity – classical Morawetz estimates – to the case of waves with critically singular potentials and thereby to build new Carleman inequalities. The second part is focused on fluid mechanics equations; specifically, the Whitham equation for waves on the surface of a fluid. Together with his thesis supervisor, Vergara has managed to prove the convexity of singular solutions to this equation, for which no partial results previously existed.

Vergara states that "each problem has required the combination of both known and new ideas". For example, he has used estimates with singular weights and *ad-hoc* growth in order to control a critically singular potential. Another new idea consists of a strategy for finding singular solutions to PEDs with low regularity. To that end, he resorted to the construction of an approximate solution by using a combination of fine asymptotic analysis close to the singularities and computer-assisted estimates.

At present, this mathematician is at the University of Zürich (Switzerland), where he holds a position of postdoctoral researcher and says that in the future he would like to continue in the academic world.