

Higgs bundles over elliptic curves



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Tesis presentada para la obtención del título de

Doctor en Matemáticas

por la Universidad Autónoma de Madrid.

Director: Óscar García-Prada.

12 de junio de 2012

A María

It may be normal, darling; but I'd rather be natural.

Truman Capote, *Breakfast at Tiffany's*.

Acknowledgements

First of all I want to thank Oscar Garcia-Prada and Peter Newstead, respectively, my official and unofficial co-supervisors. I am deeply grateful for the opportunity to work with them, for suggesting this problem and for their guidance and teaching. It has been a pleasure to learn from them along these years. This thesis would not have been possible without them.

I want to thank the *Consejo Superior de Investigaciones Científicas* (CSIC) for supporting this research through a *JAE-Pre* grant and for the financial help during the three internship I have enjoyed. These stays had been crucial for this research. I wish to acknowledge the research projects *Moduli spaces and geometric structures* (MEC, MTM2007-67623) and *Geometry of moduli spaces* (MEC MTM2010-17717) that allowed me to attend to numerous Conferences and Schools. I thank the *Instituto de Ciencias Matemáticas* (CSIC-UAM-UC3M-UCM) for his hospitality during these years and the *University of Liverpool* for my visits during winter 2009 and spring 2010, and *Isaac Newton Institute* (Cambridge) for my stay of spring 2011. I wish to express my gratitude to Ann Newstead for her kindly welcome to Liverpool and to her and Peter for making my stays much warmer.

I would like to thank Luis Alvarez-Consul, Tomas L. Gomez, Marina Logares, Ignasi Mundet, Andre G. Oliveira and Peter Gothen for several and very useful discussions about the topics of this thesis and for sharing with me mathematical ideas.

Agradezco la amistad, el cariño y el apoyo de mis queridísimos miluneros: Roberto, Ana, Marta, Mario y Álvaro. Compartir con vosotros un despacho tan mítico ha sido un verdadero honor y una alegría constante (*¡suave, suave, su su suave!*).

A base de no pocos congresos, se ha formado una pandilla a cuyos miembros les agradezco tantos y tan buenos ratos: Alfonso, David, Luis, Andratx, Laura, Florent, Francesco, Pietro, Udi (*Chaaaarlie*), Dario, Cristina y muchos más. Aquí toca incluir de nuevo a Luis A. Cónsul y a Marina, que, además de grandes matemáticos, son unos fenomenales compañeros de parranda.

Agradezco de todo corazón lo bien que se portaron conmigo Roberto y Luis en las semanas que compartí con ellos habitación en la residencia del CRM. Ambos consiguieron crear un ambiente más que propicio para trabajar en momentos cruciales de la escritura de esta tesis (*¡hasta me hacían la cena y la compra!*). Y lo más importante, crearon también el ambiente propicio para descansar y relajarse cuando no tocaba trabajar.

Grazie anche a Matteo per la loro meravigliosa compagnia a Cambridge, la sua conversazione meravigliosa e soprattutto, soprattutto, soprattutto, la sua cucina meravigliosa.

No puedo olvidarme de las meriendas de media tarde en la Sala Común del ICMAT, tan necesarias para que no le estalle a uno la cabeza. Leo, Javi, Jezabel, Rafa, Giovanni, Miguel, Marco... muchas gracias por vuestros ánimos durante estos últimos meses, han sido imprescindibles. Aunque ya lo hemos dejado atrás, agradezco las fantásticas comidas en el comedor de Serrano que he compartido con cédric, David Iglesias, Angel, Diego, Dan, Manolo, Ana... También agradezco los cafés de la cafetería de la Autónoma con Ana Zuma (por supuesto acompañados de un sencillito Pato al estilo Pekinés) y con David Torres cuya ayuda ha sido inestimable para tratar de comprender lo incomprendible (i.e. la misteriosa e inescrutable *Documentación para la Defensa*). No me olvido de las comidas con Javi y con Marquillos, seguidas de unas sobremesas al sol que eran pecata minuta. Ha sido un placer compartir con Javi una trayectoria académica plagada de intersecciones.

Quiero agradecer a David Martín, a Roberto y a diGo.Art el haberme acompañado durante la aventura que ha sido la organización de Graffiti y Mates.

Todo doctorando tiene un compañero de fatigas y el mío ha sido Fernando. Sin él, esta tesis seguramente hubiera sí que llegado a término, pero sin duda hubiera costado muchísimo más. Hemos compartido, entre los dos, el peso de la tesis y millones de cosas más: tardes de Jazz, charlas sobre el incierto futuro, opiniones, relatos, pelis, baloncesto y hasta una partida de póker. Con esta tesis he ganado un par de kilos, un par de dioptrías y un amigo para toda la vida. Gracias por haber estado ahí.

Ha sido maravilloso compartir con los físicos las comidas en psicología. Ana, Zon, Teresa, Javi Abajo y Estrada, Puri la Gris, Zuma, Dani, Paula, Dani Cuba, Fer, Paloma, Héctor... La risa es la mejor medicina para el estrés y gracias a vosotros he tomado una dosis diaria. Gracias también al resto de físicos no psicólogos porque son fenomenales: Ele, Elsa, Dario, Arturo, Robert, Carmen, Robertito... Gracias a la panda por los estupendos aperitivos de los domingos en Prospe y por ayudarme a crecer: Elena, Fer, Javi, Luis, Rodri, Mari, Laura, Gonzalo Velasco... Gracias en especial a Pablo, por estar ahí, aunque esté lejos. Y a Carmen y a Gaspar. Gracias también a los amigos de mis amigos que ya son mis amigos: Jaime, Peri, Diana, Diana, Noe, Noe, Jose, Ire, Majis, Óscar, Estefanía... Y gracias a mis compañeros del 15-M de Prospe, por compartir conmigo un sueño.

Aunque me repita, gracias a mis compis de piso (Gon incluido), que, durante dos años, me dieron motivos para olvidarme las llaves en la cerradura de la puerta al entrar en casa.

Gracias a toda mi familia por estar detrás. A mis tíos, a mis tías, a mis primos y primas. A mis vecinos de toda la vida que ya son familia. A mi abuela y a mis tíos Luis y Fran por la deliciosa rutina de los viernes.

Gracias a mi hermana y a mis padres por llenar de significado la palabra apoyo.

Y gracias a María.

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Chapter 1

Introducción

En esta tesis se estudian los fibrados de Higgs sobre una curva elíptica. En la primera parte estudiamos fibrados de Higgs con grupos de estructura complejos y clásicos. En la segunda parte los grupos de estructura son las formas reales de $GL(n, \mathbb{C})$ y en la tercera se trata el caso en que el grupo de estructura es un grupo de Lie complejo reductivo arbitrario.

Los fibrados de Higgs se han estudiado principalmente sobre superficies de Riemann compactas de género $g \geq 2$ y por ello en esta tesis queremos presentar un estudio sistemático del caso $g = 1$. En contraste con lo que sucede en género alto, en el caso elíptico puede obtenerse una descripción explícita de los espacios de móduli de fibrados de Higgs.

Espacios de móduli de fibrados sobre curva elípticas

El estudio de fibrados vectoriales comenzó en 1957 con el trabajo de Atiyah [A] donde se describe el conjunto de las clases de isomorfía de fibrados vectoriales indescomponibles. Tras el desarrollo de la teoría de invariantes geométricos (GIT), los resultados de Atiyah pudieron reinterpretarse como la identificación con $\text{Sym}^h X$ del espacio de móduli de fibrados vectoriales de rango n y grado d sobre la curva elíptica X , donde $h = \gcd(n, d)$. Aunque esta identificación era ampliamente conocida, fue, durante años, poco reflejada en la literatura. En 1993 Tu incluye en [Tu] la prueba de que, sobre curva elíptica, todo fibrado vectorial indescomponible es semistable.

La teoría de espacios de móduli de fibrados sobre superficies de Riemann fue desarrollada entre otros por Mumford, Seshadri, Narasimhan, Ramanan, Newstead y Ramanathan. En [Ra2] y [Ra3] Ramanathan construyó el espacio de móduli $M(G)_d$ de G -fibrados de grado $d \in \pi_1(G)$ sobre una superficie de Riemann compacta de género $g \geq 2$ cuando el grupo de estructura G es complejo reductivo. Probó que $M(G)_d$ es el cociente GIT de un esquema no singular R , que parametriza G -fibrados semiestables, por la acción de $GL(N, \mathbb{C})$, con N suficientemente grande. Es posible extender esta construcción al caso $g = 1$ (véase [LeP]) y, en particular, tenemos que $M(G)_d$ es normal, puesto R lo es.

Schweigert [S], Laszlo [La], Friedman y Morgan [FM1, FM2], junto con Witten [FMW] y Helmke y Slodowy [HS] estudiaron el espacio de móduli de fibrados principales holomorfos sobre una curva elíptica cuando el grupo de estructura G es complejo reductivo. Cuando G es simple y simplemente conexo, si Λ es la lattice de corraíces y W el grupo de

Weyl, tenemos que

$$M(G) \cong (X \otimes_{\mathbb{Z}} \Lambda) / W.$$

El método de [FM1] y [La] consiste en construir un morfismo biyectivo de $(X \otimes_{\mathbb{Z}} \Lambda) / W$ en $M(G)$. Como $M(G)$ es normal, por el Teorema Principal de Zariski, tendríamos que esta biyección es, de hecho, un isomorfismo.

Gracias a un teorema de Loojenga [Lo] (ver también [BS]), tenemos que $(X \otimes_{\mathbb{Z}} \Lambda) / W$ es isomorfo a un espacio proyectivo con pesos $\mathbb{W}\mathbb{P}(\bar{\lambda}_{\mathfrak{g}})$, cuyos pesos $\bar{\lambda}_G = (\lambda_1, \lambda_2, \dots)$ dependen sólo del grupo simplemente conexo G . En [FM2] encontramos una prueba directa de este isomorfismo. Un paso importante para esta prueba es la construcción de una familia de G -fibrados semiestables regulares parametrizada por un espacio vectorial $V_{G,d}$ menos el origen. Nótese que esta familia induce un morfismo sobreyectivo de $V_{G,d} - \{0\}$ en $M(G)_d$. El segundo paso consiste en probar que este morfismo factoriza a través de $(V_{G,d} - \{0\}) \rightarrow \mathbb{W}\mathbb{P}(\bar{\lambda}_{G,d})$, el cociente de $V_{G,d} - \{0\}$ por la acción de \mathbb{C}^* con pesos $\bar{\lambda}_{G,d}$, dando lugar a un morfismo biyectivo $\mathbb{W}\mathbb{P}(\bar{\lambda}_{G,d}) \rightarrow M(G)_d$. Como $M(G)_d$ es normal, gracias al Teorema Principal de Zariski, esta biyección es un isomorfismo.

Teoría general de fibrados de Higgs

Sea G un grupo de Lie reductivo (real o complejo) con compacto maximal K y sea $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ la descomposición de Cartan de su álgebra de Lie. Un G -fibrado de Higgs sobre la superficie de Riemann compacta X , es un par (E, Φ) , donde E es un fibrado principal holomorfo sobre X con grupo de estructura $K^{\mathbb{C}}$, y Φ , el campo de Higgs, es una sección del fibrado vectorial $E(\mathfrak{m}^{\mathbb{C}})$ tensorizado por Ω_X^1 , el fibrado canónico de X .

Cuando G es un grupo de Lie complejo reductivo, la complexificación $K^{\mathbb{C}}$ de su compacto maximal es igual a G , y la descomposición de Cartan es $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$. En ese caso, E es un G -fibrado y el campo de Higgs Φ es una sección del fibrado adjunto $E(\mathfrak{g})$ tensorizado por el fibrado de línea canónico.

Hitchin introdujo los G -fibrados de Higgs para $G = \mathrm{SL}(2, \mathbb{C})$ en [Hi1] donde construyó el espacio de módulos de estos fibrados. La existencia del espacio de módulos de G -fibrados de Higgs $\mathcal{M}(G)$ fue probada por Simpson [Si1] y Nitsure [Ni] para $G = \mathrm{GL}(n, \mathbb{C})$ y por Simpson [Si2, Si3] cuando G es un grupo complejo reductivo arbitrario. La existencia de $\mathcal{M}(G)$ cuando G es un grupo de Lie real y algebraico se deriva de la construcción general de Schmitt dada en [Sc].

Denotamos por Γ la extensión central universal del grupo fundamental $\pi_1(X)$ de una superficie de Riemann compacta por \mathbb{Z} . Definimos $\Gamma_{\mathbb{R}}$ como $\mathbb{R} \times_{\mathbb{Z}} \Gamma$. El espacio de módulos de representaciones centrales de $\Gamma_{\mathbb{R}}$ en G es

$$\mathcal{R}(G) = \mathrm{Hom}(\Gamma_{\mathbb{R}}, G) // G.$$

Como consecuencia de una serie de teoremas de Narasimhan y Seshadri [NS], Ramanathan [Ra1], Donaldson [D], Corlette [Co], Labourie [Lb], Hitchin [Hi1], Simpson [Si1, Si2, Si3] y Bradlow, Garcia-Prada, Gothen y Mundet-i-Riera [BGM, GGM] existe un homeomorfismo entre el espacio de módulos de G -fibrados de Higgs sobre una superficie de Riemann

compacta y el espacio de módulos de representaciones centrales de $\Gamma_{\mathbb{R}}$ en G

$$\mathfrak{M}(G) \stackrel{\text{homeo}}{\cong} \mathcal{R}(G).$$

En [Si3] Simpson prueba el Teorema de Isosingularidad que implica que $\mathfrak{M}(G)$ es normal si y sólo si $\mathcal{R}(G)$ es normal. A continuación consigue demostrar que $\mathcal{R}(\mathrm{GL}(n, \mathbb{C}))$ es normal sobre superficies de Riemann compactas de género $g \geq 2$, y por tanto, en ese caso sabemos que $\mathfrak{M}(G)$ es normal. Su prueba no puede aplicarse en género 1 y por ello la cuestión de la normalidad de $\mathfrak{M}(G)$ permanece abierta en el caso elíptico.

Consideramos $B(G) = \bigoplus H^0(X, K^{\otimes r_i})$, donde los r_i son los factores del grupo. Hitchin introdujo en [Hi2] un morfismo de $\mathfrak{M}(G)$ en $B(G)$ definido al evaluar los polinomios invariantes del álgebra de Lie sobre el campo de Higgs. Con este morfismo, llamado la aplicación de Hitchin, el espacio de módulos de G -fibrados de Higgs resulta ser un sistema completamente integrable algebraico. Para $g \geq 2$ y cuando G es un grupo de Lie complejo y clásico, Hitchin probó en [Hi2] que la fibra genérica de la aplicación de Hitchin es una variedad abeliana. Estos resultados fueron extendidos a grupos complejos reductivos arbitrarios por Faltings [Fa] y Donagi [Do]. Es importante remarcar que Donagi redefinió la base de Hitchin como el espacio de recubrimientos camerales de la curva.

Existe una dualidad entre aplicaciones de Hitchin para parejas de grupos duales de Langlands, G y G^L . Esta dualidad fue observada por primera vez por Hausel y Thaddeus [HT] para el par $\mathrm{SL}(n, \mathbb{C})$ y $\mathrm{PGL}(n, \mathbb{C})$ y fue extendida a pares de Langlands de grupos complejos reductivos arbitrarios por Donagi y Pantev [DP]. Como se tiene que las bases de Hitchin de grupos duales de Langlands son isomorfas $B(G) \cong B(G^L) \cong B$, vemos que $\mathfrak{M}(G)$ y $\mathfrak{M}(G^L)$ fibran sobre el mismo espacio. Entre otras cosas, esta dualidad afirma que las fibras sobre un punto genérico, $b \in B$, de las aplicaciones de Hitchin restringidas a la componente topológicamente trivial, $\mathfrak{M}(G)_0 \rightarrow B$ y $\mathfrak{M}(G^L)_0 \rightarrow B$, son variedades abelianas duales.

Resumen y resultados principales

Un resultado clave en nuestro estudio de fibrados de Higgs sobre una curva elíptica X es que un fibrado de Higgs con grupo de estructura $\mathrm{GL}(n, \mathbb{C})$ es (semi)estable si y sólo si el fibrado vectorial subyacente es (semi)estable [Proposiciones 4.2.1 y 4.2.3]. El hecho de que la semiestabilidad de un fibrado de Higgs depende sólo de la semiestabilidad del fibrado subyacente puede extenderse al resto de grupos de estructura tratados en esta tesis [Proposiciones 5.1.2, 6.2.1, 7.2.1 y 9.2.1] y esto implica la existencia de un morfismo sobreyectivo del espacio de módulos de G -fibrados de Higgs al espacio de módulos de $K^{\mathbb{C}}$ -fibrados principales [Proposiciones 4.3.9, 4.4.5, 4.5.7, 5.2.9, 5.3.11, 5.4.16, 6.3.9, 7.3.9, 7.4.9 y 9.6.1]

$$\mathfrak{M}(G) \rightarrow M(K^{\mathbb{C}}),$$

donde K es el subgrupo compacto maximal de G . Como las fibras de este morfismo sobreyectivo son conexas, las componentes conexas de $\mathfrak{M}(G)$ están determinadas por las componentes conexas de $M(K^{\mathbb{C}})$ (si $K^{\mathbb{C}}$ es conexo, por $d \in \pi_1(K^{\mathbb{C}})$).

Gracias a la equivalencia entre la estabilidad de un fibrado de Higgs y la de su fibrado subyacente, podemos usar la descripción de los fibrados vectoriales y de los G -fibrados principales dadas en [A] y [FM1] para describir los fibrados de Higgs poliestables sobre una curva elíptica [Corolarios 4.2.4, 7.2.5 y 9.2.3 y Proposiciones 5.1.3, 6.2.1 y 7.2.3]. Cuando G es un grupo clásico complejo reductivo, observamos, gracias a esta descripción, que todo G -fibrado de Higgs con un grado dado, reduce a un único (módulo conjugación) subgrupo de Levi de Jordan-Hölder. Este resultado de unicidad (módulo conjugación) del subgrupo de Levi de Jordan-Hölder para un grado fijo, puede extenderse a fibrados de Higgs con grupo de estructura complejo reductivo arbitrario [Proposición 9.4.1]. Cuando G es una forma real de $GL(n, \mathbb{C})$, la clase de conjugación del subgrupo de Levi de Jordan-Hölder no es única pero su número es finito. Usando familias de fibrados estables cuyo grupo de estructura es el subgrupo de Jordan-Hölder, podemos construir familias de fibrados de Higgs poliestables \mathcal{E} parametrizadas por Z de forma que cualquier fibrado de Higgs poliestable de grado d es isomorfo a \mathcal{E}_z para algún $z \in Z$. También hallamos un grupo finito Γ actuando sobre Z tal que $\mathcal{E}_{z_1} \cong \mathcal{E}_{z_2}$ si y sólo si existe $\gamma \in \Gamma$ que satisface que $z_2 = \gamma \cdot z_1$. Por la teoría de espacios de móduli sabemos que esta familia induce un morfismo biyectivo

$$Z / \Gamma \longrightarrow \mathfrak{M}(G)_d. \quad (1.1)$$

Si $\mathfrak{M}(G)_d$ es normal, esta biyección es, por el Teorema Principal de Zariski, un isomorfismo. Sin embargo, no podemos utilizar este método porque $\mathfrak{M}(U(p, q))$ y $\mathfrak{M}(GL(n, \mathbb{R}))$ no son normales en general. Ni siquiera podemos aplicarlo para grupos de estructura complejos reductivos porque la normalidad de $\mathfrak{M}(G)_d$ es una cuestión abierta.

En vista de esta situación, construimos un nuevo functor de móduli. El functor de móduli usual asocia a cualquier esquema T el conjunto de las familias de fibrados de Higgs parametrizadas por T . Tomaremos un nuevo functor [(3.11), (3.12), (3.15), (3.16), (3.17), (3.18), (6.2), (7.1), (7.2) y (9.1)] que asocie un conjunto familias más pequeño, concretamente el subconjunto de familias *localmente graduadas*. Gracias a esta nueva definición, las familias de fibrados de Higgs poliestables definidas previamente, tienen la propiedad universal local para el nuevo functor de móduli [Proposiciones 4.3.6, 5.2.3, 5.3.5, 5.4.5, 6.3.5, 7.3.4, 7.4.6 y 9.4.4]. Como consecuencia obtenemos una descripción explícita [Teoremas 4.3.7, 4.4.3, 4.5.4, 5.2.5, 5.3.7, 5.4.7, 5.4.14, 6.3.8, 7.3.6, 7.4.7 y 9.4.7] de los espacios de móduli de fibrados de Higgs $\mathcal{M}(G)_d$ asociados a los nuevos funtores de móduli,

$$\mathcal{M}(G)_d \cong Z / \Gamma.$$

Si recordamos (1.1) vemos que existe una biyección $\mathcal{M}(G)_d \rightarrow \mathfrak{M}(G)_d$, por tanto, nuestro espacio de móduli no clasifica nueva estructura. Cuando G es complejo reductivo, tenemos que $\mathcal{M}(G)_d$ es normal, y por tanto sabemos que $\mathcal{M}(G)_d$ es la normalización de $\mathfrak{M}(G)_d$ [Proposiciones 4.3.15, 4.4.12, 4.5.15, 5.2.14, 5.3.16, 5.4.23, 7.3.14 y 9.4.11].

Podemos estudiar la aplicación de Hitchin para este espacio de móduli y así observamos que la restricción de la aplicación de Hitchin a cada una de las componentes conexas del espacio de móduli de G -fibrados de Higgs $\mathcal{M}(G)_d \rightarrow \bigoplus H^0(X, \mathcal{O}^{\otimes r_i})$ no es sobreyectivo en general. Para preservar la sobreyectividad, redefinimos la base de Hitchin $B(G, d)$ como la imagen de $\mathcal{M}(G)_d$. Una vez que hemos descrito explícitamente el espacio de móduli

$\mathcal{M}(G)_d$ podemos describir con gran detalle las dos fibraciones

$$\begin{array}{ccc} & \mathcal{M}(G)_d & \\ & \swarrow \scriptstyle a & \searrow \scriptstyle b \\ M(K^{\mathbb{C}})_d & & B(G, d). \end{array}$$

En particular, describimos todas las fibras de la aplicación de Hitchin, no sólo las fibras genéricas [Corolarios 4.3.13, 4.4.10, 4.5.12, 5.2.13, 5.4.11, 6.3.13, 7.3.13, 7.4.13 y 9.5.4].

Nuestro estudio cubre dos parejas de grupos clásicos duales de Langlands, $SL(n, \mathbb{C})$ y $PGL(n, \mathbb{C})$ y $Sp(2m, \mathbb{C})$ and $SO(2m + 1, \mathbb{C})$. Para estos casos, observamos que las fibras sobre un punto no genérico son fibraciones de espacios proyectivos (en algunos casos cocientados por un grupo finito) sobre variedades abelianas duales [Comentarios 4.5.13 y 5.4.12].

Es importante remarcar que las fibras genéricas de b para los grupos de estructura reales $U^*(2m)$ y $GL(n, \mathbb{R})$ no son variedades abelianas duales, sino copias de espacios proyectivos [Corolarios 7.3.13 y 7.4.13].

Recordemos que $G = K^{\mathbb{C}}$ si G es complejo reductivo. Podemos dar una estructura de orbifold natural sobre $\mathcal{M}(G)_d$ y resulta que la proyección $\mathcal{M}(G) \xrightarrow{a} M(G)$ puede entenderse como el morfismo de variedades inducido por el fibrado cotangente en sentido orbifold de dicho orbifold [Teorema 9.6.2].

Esquema de la Parte I

En la Parte I estudiamos los fibrados de Higgs sobre una curva elíptica cuyos grupos de estructura son los grupos de Lie complejos reductivos clásicos.

El Capítulo 3 está dedicado a los preliminares necesarios para desarrollar nuestro trabajo. Comenzamos recordando en la Sección 3.1 algunas propiedades básicas de las curvas elípticas y repasando la teoría de espacios de móduli en la Sección 3.2. En la Sección 3.3 recordamos la descripción de los fibrados vectoriales sobre una curva elíptica dada por Atiyah.

En la Sección 3.4 planteamos el problema de móduli de la clasificación de fibrados de Higgs sobre una curva elíptica con grupo de estructura $GL(n, \mathbb{C})$. En la Sección 3.5 cuando los grupos de estructura son $SL(n, \mathbb{C})$ y $PGL(n, \mathbb{C})$ y en la Sección y 3.6 para $Sp(2m, \mathbb{C})$, $O(n, \mathbb{C})$ y $SO(n, \mathbb{C})$. En lugar de definir los fibrados de Higgs como un fibrado principal y una sección del fibrado adjunto, tomamos una definición equivalente en términos del fibrado vectorial obtenido a partir de la representación estándar con estructura adicional sobre el fibrado vectorial (si procede), y de una sección del fibrado de endomorfismos que satisfaga cierta compatibilidad con la estructura adicional. Damos las nociones de estabilidad en términos de la pendiente de los subfibrados invariantes bajo el campo de Higgs y compatibles con la estructura adicional, ya que estas nociones son más adecuadas para trabajar en este contexto. Definimos las familias localmente graduadas de fibrados de Higgs y los funtores de móduli asociados a este tipo de familias [(3.11), (3.12), (3.15), (3.16), (3.17) y (3.18)].

En el capítulo 4 damos una descripción de los espacios de móduli de fibrados de Higgs con grupos de estructura $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$ y $PGL(n, \mathbb{C})$.

En la Sección 4.1 estudiamos algunas propiedades de las curvas elípticas. Debido a la estructura de grupo de la curva elíptica X , los grupos finitos $X[h]$, formados por los puntos con h -torsión, actúan sobre copias de la curva $X \times \cdots \times X$. Probamos que el cociente de esta acción es de nuevo $X \times \cdots \times X$ [Lema 4.1.2]. Este hecho será usado para estudiar la relación entre las fibras de Hitchin para $SL(n, \mathbb{C})$ y $PGL(n, \mathbb{C})$ sobre el mismo punto de la base [Corolarios 4.4.10 y 4.5.12].

En la Sección 4.2 probamos lo siguiente: *sea (E, Φ) un fibrado de Higgs semistable (resp. estable) sobre una curva elíptica. Entonces E es un fibrado vectorial semistable (resp. estable). Si (E, Φ) es poliestable, entonces E es poliestable.* El resultado sobre semiestabilidad [Proposición 4.2.1] se deriva del estudio de la filtración de Harder-Narasimhan y del hecho de que el fibrado canónico de una curva elíptica es trivial. El resultado sobre estabilidad [Proposición 4.2.3] se obtiene a partir de los resultados de Atiyah [A] sobre el fibrado de endomorfismos de un fibrado vectorial sobre una curva elíptica.

Tras esto, en la Sección 4.3 utilizamos la clasificación de Atiyah de los fibrados vectoriales sobre una curva elíptica para construir una familia de fibrados de Higgs poliestables con la propiedad universal local entre las familias localmente graduadas. Gracias a ello, obtenemos que [Teorema 4.3.7]

$$\mathcal{M}(GL(n, \mathbb{C}))_d \cong \text{Sym}^h T^* X,$$

donde $h = \gcd(n, d)$. Como $T^* X \cong X \times \mathbb{C}$, podemos definir las siguientes proyecciones

$$\begin{array}{ccc} & \text{Sym}^h T^* X & \\ & \swarrow a & \searrow b \\ \text{Sym}^h X & & \text{Sym}^h \mathbb{C} \end{array}$$

donde a es la proyección al espacio de móduli de fibrados vectoriales [Proposición 4.3.9] y b es la aplicación de Hitchin [Lema 4.3.10]. Definimos tres involuciones sobre el espacio de móduli $\mathcal{M}(GL(n, \mathbb{C}))_d$ y las correspondientes involuciones sobre $\text{Sym}^h T^* X$ [Lema 4.3.14] que serán utilizadas para estudiar los fibrados de Higgs ortogonales, simplécticos y los fibrados de Higgs para formas reales de $GL(n, \mathbb{C})$. Terminamos dando una biyección entre $\mathcal{M}(GL(n, \mathbb{C}))_d$ y $\mathfrak{M}(GL(n, \mathbb{C}))_d$ y probando que $\mathcal{M}(GL(n, \mathbb{C}))_d$ es la normalización de $\mathfrak{M}(GL(n, \mathbb{C}))_d$ [Proposición 4.3.15].

Igualmente, en las Secciones 4.4 y 4.5 obtenemos descripciones explícitas de los espacios de móduli $\mathcal{M}(SL(n, \mathbb{C}))$ y $\mathcal{M}(PGL(n, \mathbb{C}))$ [Teoremas 4.4.3 y 4.5.4], de las aplicaciones de Hitchin asociadas [Lemas 4.4.7 y 4.5.8] y de las proyecciones a los espacios de móduli de fibrados vectoriales con determinante trivial y fibrados proyectivos [Proposiciones 4.4.5 y 4.5.7]. Así mismo, probamos la existencia de biyecciones entre $\mathcal{M}(SL(n, \mathbb{C}))$ y $\mathfrak{M}(SL(n, \mathbb{C}))$ y entre $\mathcal{M}(PGL(n, \mathbb{C}))$ y $\mathfrak{M}(PGL(n, \mathbb{C}))_{\bar{d}}$ [Proposiciones 4.4.12 y 4.5.15].

El Capítulo 5 está dedicado a dar una descripción explícita de los espacios de móduli de fibrados de Higgs con grupos de estructura $Sp(2m, \mathbb{C})$, $O(n, \mathbb{C})$ y $SO(n, \mathbb{C})$. En la Sección

5.1 probamos que la semiestabilidad (resp. poliestabilidad) de un fibrado de Higgs con alguno de estos grupos de estructura implica la semiestabilidad (resp. poliestabilidad) del fibrado de Higgs subyacente [Proposición 5.1.1] y la semiestabilidad (resp. poliestabilidad) del fibrado principal subyacente [Proposición 5.1.2]. Tras ello, describimos los fibrados de Higgs estables [Proposición 5.1.3 y Corolario 5.1.4] y poliestables [Proposición 5.1.6] con estos grupos de estructura.

En la Sección 5.2 construimos una familia de $\mathrm{Sp}(2m, \mathbb{C})$ -fibrados de Higgs poliestables con la propiedad universal local entre las familias localmente graduadas [Proposición 5.2.3]. Con esta familia obtenemos que [Teorema 5.2.5]

$$\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C})) \cong \mathrm{Sym}^m(T^*X/\mathbb{Z}_2).$$

Usando esta descripción explícita estudiamos la proyección a $M(\mathrm{Sp}(2m, \mathbb{C}))$ [Proposición 5.2.9], la aplicación de Hitchin [Lema 5.2.10] y sus fibras [Corolario 5.2.13]. Probamos la existencia de un morfismo biyectivo de $\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C}))$ en $\mathfrak{M}(\mathrm{Sp}(2m, \mathbb{C}))$ y vemos que $\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C}))$ es la normalización de $\mathfrak{M}(\mathrm{Sp}(2m, \mathbb{C}))$ [Proposición 5.2.14].

En las Secciones 5.3 y 5.4 obtenemos descripciones análogas de los espacios de móduli $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))$ [Teorema 5.3.7] y $\mathcal{M}(\mathrm{SO}(n, \mathbb{C}))$ [Teoremas 5.4.7 y 5.4.14], de sus proyecciones al espacio de móduli de fibrados principales [Proposiciones 5.3.11 y 5.4.16], de las correspondientes aplicaciones de Hitchin [Lemas 5.3.14, 5.4.10 y 5.4.17] y de las fibras de estas aplicaciones [Corolarios 5.3.15, 5.4.11 y 5.4.22]. Damos un morfismo biyectivo de $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))$ en $\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))$ y de $\mathcal{M}(\mathrm{SO}(n, \mathbb{C}))$ en $\mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))$ y observamos que $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))$ es la normalización de $\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))$, y $\mathcal{M}(\mathrm{SO}(n, \mathbb{C}))$ la de $\mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))$ [Proposiciones 5.3.16 y 5.4.23].

Esquema de la Parte II

La Parte II está dedicada al estudio de fibrados de Higgs para formas reales de $\mathrm{GL}(n, \mathbb{C})$, concretamente $\mathrm{U}(p, q)$, $\mathrm{GL}(n, \mathbb{R})$ y $\mathrm{U}^*(2m)$ cuando n es par.

En el Capítulo 6 estudiamos fibrados de Higgs con grupo de estructura $\mathrm{U}(p, q)$. Comenzamos recordando la definición de $\mathrm{U}(p, q)$ -fibrados de Higgs y sus nociones de estabilidad en la Sección 6.1. También definimos la noción de familias de $\mathrm{U}(p, q)$ -fibrados de Higgs localmente graduadas y planteamos el problema de móduli asociado a esta clase de familias [(6.2)].

En la Sección 6.2 exponemos lo siguiente: *un $\mathrm{U}(p, q)$ -fibrado de Higgs (V, W, β, γ) es semiestable si y sólo si V y W son fibrados vectoriales semiestables con igual pendiente. Es estable si $V \cong W$ son fibrados vectoriales estables y $\beta \circ \gamma$ es no nulo* [Proposición 6.2.1]. Una consecuencia inmediata de lo anterior es que la semiestabilidad implica que el invariante de Toledo es cero [Corolario 6.2.2].

En la Sección 6.3 estudiamos el espacio de móduli $\mathcal{M}(\mathrm{U}(p, q))_{(a,b)}$ asociado al functor de móduli definido en la Sección 6.1, donde (a, b) es el invariante topológico dado por los grados de los fibrados vectoriales subyacentes. Como el invariante de Toledo de un $\mathrm{U}(p, q)$ -fibrado de Higgs semiestable se anula, vemos que $\mathcal{M}(\mathrm{U}(p, q))_{(a,b)}$ es vacío si los invariantes (p, q, a, b) no son de la forma (nr, mr, nd, md) , con $\mathrm{gcd}(r, d) = 1$. Con la descripción de los $\mathrm{U}(p, q)$ -fibrados de Higgs estables podemos construir una familia de $\mathrm{U}(p, q)$ -fibrados

de Higgs poliestables con la propiedad universal local entre las familias localmente graduadas [Proposición 6.3.5] lo cual nos permite probar la existencia de un isomorfismo entre $\mathcal{M}(U(p, q))_{(a, b)}$ y una subvariedad de $\text{Sym}^n(X \times \mathbb{C}/\pm) \times \text{Sym}^m(X \times \mathbb{C}/\pm)$ [Teorema 6.3.8]. Estudiamos la aplicación de Hitchin en este contexto [Lema 6.3.11] y observamos que la dimensión de la fibra varía [Corolario 6.3.13]. Al final de la Sección 6.3 damos una descripción explícita de los espacios de móduli $\mathcal{M}(U(r, r))_{d, d}$ y $\mathcal{M}(U(r, 2r))_{d, 2d}$ y observamos que no son variedades normales [Comentarios 6.3.14 y 6.3.15].

Como los $U^*(2m)$ -fibrados de Higgs y los $GL(n, \mathbb{R})$ -fibrados de Higgs comparten una estructura común, los estudiamos juntos en el Capítulo 7. En la Sección 7.1 recordamos la definición de $U^*(2m)$ y $GL(n, \mathbb{R})$ -fibrados de Higgs, y las nociones de estabilidad para estos objetos. También definimos las familias localmente graduadas y los funtores de móduli asociados a este nuevo concepto de familia [(7.1) y (7.2)].

La Sección 7.2 contiene un estudio de las relaciones asociadas a la estabilidad: *el $Sp(2m, \mathbb{C})$ -fibrado (resp. $O(n, \mathbb{C})$ -fibrado) principal subyacente de un $U^*(2m)$ -fibrado de Higgs (resp. $GL(n, \mathbb{R})$ -fibrado de Higgs) semiestable es semiestable. Es poliestable si el $U^*(2m)$ -fibrado de Higgs (resp. $GL(n, \mathbb{R})$ -fibrado de Higgs) es poliestable* [Proposición 7.2.2].

En la Sección 7.3 construimos una familia de $U^*(2m)$ -fibrados de Higgs poliestables con la propiedad universal local entre familias localmente graduadas [Proposición 7.3.4]. Con esta familia obtenemos que [Teorema 7.3.6]

$$\mathcal{M}(U^*(2m)) \cong \text{Sym}^m(\mathbb{P}^1 \times \mathbb{C}).$$

Bajo esta descripción, la aplicación de Hitchin se corresponde con [Lema 7.3.10]

$$\text{Sym}^m(\mathbb{P}^1 \times \mathbb{C}) \longrightarrow \text{Sym}^m \mathbb{C}$$

y podemos observar que la fibra genérica es $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$ [Corolario 7.3.13].

En la Sección 7.4 probamos la existencia de un isomorfismo entre $\mathcal{M}(GL(n, \mathbb{R}))$ y la subvariedad de puntos fijos de una involución en $\text{Sym}^n T^*X$ [Teorema 7.4.7]. La fibra de la aplicación de Hitchin es una colección de copias de espacios proyectivos y su dimensión varía [Corolario 7.4.13]. Así mismo, damos una descripción explícita de $\mathcal{M}(GL(2, \mathbb{R}))$ y observamos que no es una variedad normal [Comentario 7.4.14].

Esquema de la Parte III

El homeomorfismo entre el espacio de móduli de G -fibrados sobre X y el espacio de móduli de representaciones unitarias de $\Gamma_{\mathbb{R}}$ se utiliza en [FM1] para obtener una descripción de los G -fibrados semistables. En la Parte III usamos esta descripción para estudiar los G -fibrados de Higgs sobre una curva elíptica cuando el grupo de estructura G es un grupo de Lie conexo y complejo reductivo.

El Capítulo 8 es un repaso de [FM1] y de [BFM]. En la Sección 8.1 incluimos algunos resultados a cerca de los grupos de Lie reductivos que necesarios para dar una descripción de los G -fibrados de Higgs estables, semistables y poliestables.

Como el grupo fundamental de una curva elíptica es isomorfo a $\mathbb{Z} \times \mathbb{Z}$, toda representación de $\Gamma_{\mathbb{R}}$ en un grupo semisimple G , está determinada por un par de elementos que casi-conmutan, es decir, dos elementos cuyo conmutador es un elemento del centro del grupo. En la Sección 8.2 repasamos el estudio de [BFM] sobre pares de elementos que casi-conmutan en grupos compactos y vemos que sus resultados extienden a grupos complejos reductivos.

En la Sección 8.3 recordamos la descripción de los G -fibrados holomorfos dada en [FM1].

El Capítulo 9 contiene un estudio de los G -fibrados de Higgs cuando G es complejo reductivo.

En la Sección 9.1 recordamos la definición de G -fibrado de Higgs y las nociones de estabilidad. Definimos las familias de G -fibrados de Higgs semiestables localmente graduadas y consideramos el functor de móduli que parametriza estas familias [(9.1)].

El resultado principal de la Sección 4.2 es el siguiente: *si (E, Φ) es un G -fibrado de Higgs semistable (resp. estable) sobre una curva elíptica, entonces E es un G -fibrado semistable (resp. estable). Si (E, Φ) es poliestable, entonces E es poliestable.* Como el fibrado canónico es trivial, el resultado sobre semiestabilidad [Proposición 9.2.1] se obtiene a partir de las propiedades de la reducción de Harder-Narasimhan. El resultado sobre la estabilidad [Proposición 9.2.2] se obtiene a partir de los resultados de [FM1] sobre el fibrado adjunto de un G -fibrado semiestable.

Si G es complejo reductivo y simple y $\tilde{d} \in \pi_1(G)$, observamos en la Sección 9.3 que no hay G -fibrados de Higgs estables de grado \tilde{d} a menos que $G = \mathrm{PGL}(n, \mathbb{C})$ y $\tilde{d} = d \pmod{n}$ con $\mathrm{gcd}(n, d) = 1$ [Proposición 9.3.3]. Esto permite determinar para qué grupos de estructura complejos reductivos y qué grados existen G -fibrados de Higgs estables [Corolario 9.3.5]. Describimos el espacio de móduli de G -fibrados de Higgs estables cuando no es vacío [Teorema 9.3.9] y construimos una familia universal [Comentario 9.3.10].

En la Sección 9.4 vemos que todos los G -fibrados de Higgs poliestables de grado d tienen el mismo (módulo conjugación) subgrupo de Levi de Jordan-Hölder $L_{G,d}$ [Proposición 9.4.1]. Tomando la extensión del grupo de estructura de $L_{G,d}$ a G de la familia universal de $L_{G,d}$ -fibrados de Higgs estable obtenemos una familia con la propiedad universal local entre familias localmente graduadas [Proposición 9.4.4]. Con esta familia obtenemos una descripción del espacio de móduli de G -fibrados de Higgs asociado al functor de móduli dado por las familias localmente graduadas [Teorema 9.4.7]. Cuando G es simple y simplemente conexo, Λ su lattice de corraíces y W su grupo de Weyl, por el Teorema 9.4.7 tenemos que

$$\mathcal{M}(G) \cong (T^*X \otimes_{\mathbb{Z}} \Lambda) / W.$$

Tenemos una biyección entre los espacios de móduli $\mathcal{M}(G)_d \rightarrow \mathfrak{M}(G)_d$ y, como $\mathcal{M}(G)_d$ es normal, $\mathcal{M}(G)_d$ es la normalización de $\mathfrak{M}(G)_d$ [Proposición 9.4.11].

En la Sección 9.5 usamos el enfoque de la aplicación de Hitchin basado en los cubrimientos camerales para describir la aplicación de Hitchin [Teorema 9.5.2]. Estudiamos también las fibras de dicha aplicación [Proposición 9.5.3]. En la Sección 9.6 estudiamos la fibrición $\mathcal{M}(G) \rightarrow M(G)$ y damos una interpretación de la misma en términos del fibrado cotangente en sentido orbifold de una estructura orbifold sobre $M(G)$ [Teorema 9.6.2].

Chapter 2

Introduction

In this thesis we study Higgs bundles over an elliptic curve. In the first part of the thesis, the structure groups of the Higgs bundles are classical complex reductive. In the second part the structure groups are real forms of $\mathrm{GL}(n, \mathbb{C})$ and in the third part the structure group is an arbitrary connected complex reductive Lie group .

Higgs bundles over compact Riemann surfaces have been studied mostly for genus $g \geq 2$ while in this thesis we present a systematic study of the $g = 1$ case. In contrast to Higgs bundles over Riemann surfaces of greater genus, in the elliptic case we can obtain an explicit description of the moduli spaces of Higgs bundles.

Moduli spaces of bundles over elliptic curves

The study of vector bundles over elliptic curves was started in 1957 by Atiyah, in [A], where he describes the set of isomorphism classes of indecomposable vector bundles. After GIT theory, Atiyah's results could be reinterpreted as the identification of the moduli space of semistable vector bundles with rank n and degree d over the elliptic curve X with $\mathrm{Sym}^h X$ (where $h = \mathrm{gcd}(n, d)$). Even though this was broadly known, there was, for many years, very little literature about it; in 1993 Tu included in [Tu] the proof of the fact that every indecomposable vector bundle over an elliptic curve is semistable.

The theory of moduli spaces of bundles over Riemann surfaces was developed by Mumford, Seshadri, Narasimhan, Ramanan, Newstead and Ramanathan among others. In [Ra2] and [Ra3] Ramanathan gave the construction of the moduli space $M(G)_d$ of G -bundles of topological class $d \in \pi_1(G)$ over a compact Riemann surface of genus $g \geq 2$, where G is a connected complex reductive Lie group. He proved that $M(G)_d$ is the GIT quotient of a non-singular scheme R parametrizing semistable G -bundles of topological class d by the action of $\mathrm{GL}(N, \mathbb{C})$ for N big enough. One can extend this construction to the special case of $g = 1$ (see for instance [LeP] in the case of $G = \mathrm{GL}(n, \mathbb{C})$ from which the general case follows) and in particular we have that $M(G)_d$ is normal since R is normal.

Schweigert [S], Laszlo [La], Friedman and Morgan [FM1, FM2], Witten [FMW] and Helmke and Slodowy [HS] studied the moduli space of holomorphic principal bundles over an elliptic curve whose structure group G is a complex reductive Lie group. When G is simple and simply connected, if Λ is the coroot lattice and W is the Weyl group, we have

$$M(G) \cong (X \otimes_{\mathbb{Z}} \Lambda) / W.$$

The method of [FM1] and [La] consists of constructing a bijective morphism from $(X \otimes_{\mathbb{Z}} \Lambda)/W$ to $M(G)$. Since $M(G)$ is normal, this is enough to prove that it is an isomorphism by Zariski's Main Theorem.

After a theorem of Looijenga [Lo] (see also [BS]), we have that $(X \otimes_{\mathbb{Z}} \Lambda)/W$ is isomorphic to a weighted projective space $\mathbb{WP}(\bar{\lambda}_{\mathfrak{g}})$ whose weights $\bar{\lambda}_G = (\lambda_1, \lambda_2, \dots)$ depend only on the simply connected group G . [FM2] contains a direct proof of this isomorphism. An important step in the proof is the construction of a universal family of regular semistable G -bundles of topological class d parametrized by a vector space $V_{G,d}$ without the zero point; note that this family induces a surjective morphism from $V_{G,d} - \{0\}$ to $M(G)_d$. In the second step they prove that this morphism factors through a bijective morphism from $\mathbb{WP}(\bar{\lambda}_{G,d})$ to $M(G)_d$, where $\mathbb{WP}(\bar{\lambda}_{G,d})$ is the quotient of $V_{G,d} - \{0\}$ by an action of \mathbb{C}^* with weights $\bar{\lambda}_{G,d}$. Since $M(G)_d$ is normal, this bijection is an isomorphism by Zariski's Main Theorem.

General theory of Higgs bundles

Let G be a reductive Lie group (real or complex) with maximal compact subgroup K and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the Cartan decomposition of its Lie algebra. A G -Higgs bundle over the compact Riemann surface X is a pair (E, Φ) where E is a holomorphic principal bundle over X with structure group $K^{\mathbb{C}}$, and Φ , called the *Higgs field*, is a section of the associated bundle $E(\mathfrak{m}^{\mathbb{C}})$ twisted by Ω_X^1 , the canonical line bundle of X .

When G is a complex reductive Lie group the complexification of the maximal compact subgroup $K^{\mathbb{C}}$ is the group itself and the Cartan decomposition is $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$. In that case, E is a G -bundle and the Higgs field Φ is a section of the adjoint bundle $E(\mathfrak{g})$ twisted by the canonical line bundle.

Hitchin introduced G -Higgs bundles for $G = \mathrm{SL}(2, \mathbb{C})$ in [Hi1] where he constructed their moduli space. The existence of the moduli space of G -Higgs bundles $\mathfrak{M}(G)$ was proved by Simpson [Si1] and Nitsure [Ni] for $G = \mathrm{GL}(n, \mathbb{C})$ and by Simpson [Si2, Si3] when G is an arbitrary complex reductive Lie group. The existence of $\mathfrak{M}(G)$ when G is a real algebraic Lie group follows from the general construction of Schmitt given in [Sc].

We denote by Γ the universal central extension by \mathbb{Z} of the fundamental group $\pi_1(X)$ of a compact Riemann surface, and we set $\Gamma_{\mathbb{R}}$ to be $\mathbb{R} \times_{\mathbb{Z}} \Gamma$. We define the *moduli space of central representations* of $\Gamma_{\mathbb{R}}$ in G as the GIT quotient

$$\mathcal{R}(G) = \mathrm{Hom}(\Gamma_{\mathbb{R}}, G) // G.$$

As a consequence of a chain of theorems by Narasimhan and Seshadri [NS], Ramanathan [Ra1], Donaldson [D], Corlette [Co], Labourie [Lb], Hitchin [Hi1], Simpson [Si1, Si2, Si3] and Bradlow, Garcia-Prada, Gothen and Mundet-i-Riera [BGM] and [GGM] there exists a homeomorphism between the moduli space of G -Higgs bundles over a compact Riemann surface X and the moduli space of central representations of $\Gamma_{\mathbb{R}}$ in G

$$\mathfrak{M}(G) \stackrel{\text{homeo}}{\cong} \mathcal{R}(G).$$

In [Si3] Simpson proved the Isosingularity Theorem which implies that $\mathfrak{M}(G)$ is normal if and only if $\mathcal{R}(G)$ is normal. He proves that $\mathcal{R}(\mathrm{GL}(n, \mathbb{C}))$ is normal for compact Riemann

surfaces of genus $g \geq 2$, and therefore, in that case, $\mathfrak{M}(G)$ is normal. His proof does not apply for the genus 1 case and the question of normality of $\mathfrak{M}(G)$ on elliptic curves remains open.

Consider $B(G) = \bigoplus H^0(X, (\Omega_X^1)^{\otimes r_i})$, where the r_i are the factors of the group. Hitchin introduced in [Hi2] a morphism from $\mathfrak{M}(G)$ to $B(G)$ defined by evaluating the invariant polynomials of the Lie algebra on the Higgs field. With this morphism, called the Hitchin map, the moduli space of G -Higgs bundles turns out to be an algebraically completely integrable system. For $g \geq 2$ and when G is a classical complex group, Hitchin proved in [Hi2] that the generic fibres of the Hitchin map are abelian varieties. These results were extended to arbitrary reductive groups by Faltings in [Fa] and Donagi in [Do]. It is important to note that Donagi redefined the Hitchin base as the space of cameral covers, i.e. certain Galois covers of the curve with the Weyl group as Galois group.

There is a duality between the Hitchin fibrations for two Langlands dual groups, G and G^L , which was first observed by Hausel and Thaddeus in [HT] for the case of $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{PGL}(n, \mathbb{C})$ and extended for arbitrary complex reductive Lie groups by Donagi and Pantev in [DP]. We have that the Hitchin bases for Langlands dual groups are isomorphic $B(G) \cong B(G^L) \cong B$. Among other things, this duality states that the fibres over a generic point, $b \in B$, of the Hitchin fibrations restricted to the topologically trivial component of the moduli space, $\mathfrak{M}(G)_0 \rightarrow B$ and $\mathfrak{M}(G^L)_0 \rightarrow B$, are dual abelian varieties.

Summary and main results

A key result on our study on Higgs bundles over an elliptic curve X is that a Higgs bundle for the group $\mathrm{GL}(n, \mathbb{C})$ is (semi)stable if and only if the underlying vector bundle is (semi)stable [Propositions 4.2.1 and 4.2.3]. The fact that the semistability of the Higgs bundle depends only on the semistability of the underlying bundle can be extended to the rest of the groups covered in this thesis [Propositions 5.1.2, 6.2.1, 7.2.1 and 9.2.1] and this implies the existence of a surjective morphism [Propositions 4.3.9, 4.4.5, 4.5.7, 5.2.9, 5.3.11, 5.4.16, 6.3.9, 7.3.9, 7.4.9 and 9.6.1] from the moduli space of G -Higgs bundles to the moduli space of principal $K^{\mathbb{C}}$ -bundles

$$\mathfrak{M}(G) \rightarrow M(K^{\mathbb{C}}).$$

where K is the maximal compact subgroup of G . Since the fibres of this surjective morphism are connected, the connected components of $\mathfrak{M}(G)$ are labelled by the connected components of $M(K^{\mathbb{C}})$ (if $K^{\mathbb{C}}$ is connected, by $d \in \pi_1(K^{\mathbb{C}})$).

Thanks to the equivalence between stability of Higgs bundles and the stability of the underlying bundle, we can make use of the description of vector bundles and G -bundles given in [A] and [FM1] to obtain an explicit description of polystable Higgs bundles over an elliptic curve [Corollaries 4.2.4, 7.2.5 and 9.2.3 and Propositions 5.1.3, 6.2.1 and 7.2.3]. When G is a classical complex group, we observe from this description that every polystable G -Higgs bundle with a given topological class reduces to a unique (up to conjugation) Jordan-Hölder Levi subgroup. This uniqueness result (up to conjugation) of the Jordan-Hölder Levi subgroup once we have fixed the topological class can be extended to arbitrary

complex reductive structure Lie groups [Proposition 9.4.1]. When G is a real form of $\mathrm{GL}(n, \mathbb{C})$ the conjugation class of the Jordan-Hölder Levi subgroup is not unique but there is a finite number of them. Using families of stable Higgs bundles for the Jordan-Hölder Levi subgroup we can construct families of polystable Higgs bundles \mathcal{E} parametrized by Z and a finite group Γ acting on Z such that for every polystable Higgs bundle of topological type d there exists $z \in Z$ such that \mathcal{E}_z is isomorphic to it and $\mathcal{E}_{z_1} \cong \mathcal{E}_{z_2}$ if and only if there exists $\gamma \in \Gamma$ giving $z_2 = \gamma \cdot z_1$. By moduli theory we know that this family induces a bijective morphism

$$Z / \Gamma \longrightarrow \mathfrak{M}(G)_d. \quad (2.1)$$

If $\mathfrak{M}(G)_d$ is normal, this bijection is in fact an isomorphism by Zariski's Main Theorem. However, we can not use this method in general to describe the moduli space since $\mathfrak{M}(U(p, q))$ and $\mathfrak{M}(\mathrm{GL}(n, \mathbb{R}))$ are not normal in general. We can not even apply it to the case of G complex reductive, since normality of $\mathfrak{M}(G)_d$ is an open question.

In view of this we construct a new moduli functor. The usual moduli functor associates to any scheme T the set of families of Higgs bundles parametrized by T . We will take a new moduli functor [(3.11), (3.12), (3.15), (3.16), (3.17), (3.18), (6.2), (7.1), (7.2) and (9.1)] that associates a smaller set of families of Higgs bundles, namely the set of *locally graded* families. From this modification of the moduli functor we gain that the previous families of polystable Higgs bundles have the local universal property for this new moduli problem [Propositions 4.3.6, 5.2.3, 5.3.5, 5.4.5, 6.3.5, 7.3.4, 7.4.6 and 9.4.4]. Then we obtain an explicit description [Theorems 4.3.7, 4.4.3, 4.5.4, 5.2.5, 5.3.7, 5.4.7, 5.4.14, 6.3.8, 7.3.6, 7.4.7 and 9.4.7] of the moduli spaces of Higgs bundles $\mathcal{M}(G)_d$ associated to this moduli functor

$$\mathcal{M}(G)_d \cong Z / \Gamma.$$

Recalling (2.1) we can observe that there exists a bijection $\mathcal{M}(G)_d \rightarrow \mathfrak{M}(G)_d$, thus our new moduli space is not classifying extra structure. Furthermore, when G is complex reductive, we have that $\mathcal{M}(G)_d$ is normal, and we know that $\mathcal{M}(G)_d$ is the normalization of $\mathfrak{M}(G)_d$ in these cases [Propositions 4.3.15, 4.4.12, 4.5.15, 5.2.14, 5.3.16, 5.4.23, 7.3.14 and 9.4.11].

We can study the Hitchin map for this moduli space and we observe that the restriction of the Hitchin map to the connected components of the moduli space of G -Higgs bundles $\mathcal{M}(G)_d \rightarrow \bigoplus H^0(X, \mathcal{O}^{\otimes r_i})$ is not surjective in general. In order to preserve the surjectivity of the Hitchin map we redefine for each d the Hitchin base $B(G, d)$ as the image of $\mathcal{M}(G)_d$. Once we have described explicitly the moduli space $\mathcal{M}(G)_d$ we can describe in great detail the two fibrations

$$\begin{array}{ccc} & \mathcal{M}(G)_d & \\ & \swarrow a & \searrow b \\ M(K^{\mathbb{C}})_d & & B(G, d). \end{array}$$

In particular we can describe all the fibres of the Hitchin fibration, not only the generic ones [Corollaries 4.3.13, 4.4.10, 4.5.12, 5.2.13, 5.4.11, 6.3.13, 7.3.13, 7.4.13 and 9.5.4].

In our study, we treat two pairs of classical Langlands dual groups, $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{PGL}(n, \mathbb{C})$, and $\mathrm{Sp}(2m, \mathbb{C})$ and $\mathrm{SO}(2m + 1, \mathbb{C})$. For these cases, we observe that the

fibres over a non-generic point are fibrations of projective spaces (in some cases quotiented by a finite group) over dual abelian varieties [Remarks 4.5.13 and 5.4.12].

It is important to remark that the generic fibres of b for the real groups $U^*(2m)$ and $GL(n, \mathbb{R})$ are not abelian varieties but copies of projective spaces [Corollaries 7.3.13 and 7.4.13].

Recall that $G = K^{\mathbb{C}}$ when G is complex reductive. We can define a natural orbifold structure on $M(G)_d$ and it turns out that the projection $\mathcal{M}(G) \xrightarrow{a} M(G)$ can be understood as the morphism of varieties induced by the orbifold cotangent bundle of this orbifold [Theorem 9.6.2].

Outline of Part I

In Part I we study Higgs bundles over an elliptic curve for classical groups.

Chapter 3 is devoted to the preliminaries that will be needed to develop our work. We start recalling in Section 3.1 some basic properties of elliptic curves and giving in Section 3.2 an overview of moduli theory. In Section 3.3 we recall the description of vector bundles given by Atiyah.

In Sections 3.4, 3.5 and 3.6 we state the moduli problem for the classification of Higgs bundles over an elliptic curve respectively for $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$ and $PGL(n, \mathbb{C})$ and for $Sp(2m, \mathbb{C})$, $O(n, \mathbb{C})$ and $SO(n, \mathbb{C})$. Instead of defining a Higgs bundle for one of these groups as a principal bundle together with a section of the adjoint bundle, we take an equivalent definition in terms of a vector bundle obtained from the standard representation with (if necessary) extra structure on it, and a section of the endomorphism bundle satisfying some compatibility conditions with the extra structure. We include the stability notions stated in terms of the slope of certain subbundles preserved by the Higgs field since they are more appropriate in this context. We also define locally graded families and the moduli functors associated to this new concept of families [(3.11), (3.12), (3.15), (3.16), (3.17) and (3.18)].

In Chapter 4 we give the description of the moduli spaces of Higgs bundles over elliptic curves for the groups $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$ and $PGL(n, \mathbb{C})$.

In Section 4.1 we study certain properties of an elliptic curve considered as an abelian variety. Using the group structure we are able to define a particular action of the finite groups of h -torsion elements, $X[h]$, on copies of the curve $X \times \cdots \times X$, proving that the quotient by this action is again $X \times \cdots \times X$ [Lemma 4.1.2]. This will be used to study the relation between the Hitchin fibre for $SL(n, \mathbb{C})$ and $PGL(n, \mathbb{C})$ over the same point [Corollaries 4.4.10 and 4.5.12].

In Section 4.2 we prove the following: *let (E, Φ) be a semistable (resp. stable) Higgs bundle over an elliptic curve. Then E is a semistable (resp. stable) vector bundle. If (E, Φ) is polystable, then E is polystable.* The statement for semistability [Proposition 4.2.1] follows from the Harder-Narasimhan filtration and the fact that the canonical bundle is trivial. The statement for stability [Proposition 4.2.3] follows from the description of endomorphism bundles of vector bundles over elliptic curves given by Atiyah in [A].

After this, in Section 4.3 we use Atiyah's classification of vector bundles over an elliptic curve to construct a family of polystable Higgs bundles with the local universal property

among locally graded families [Proposition 4.3.6]. After that we obtain [Theorem 4.3.7]

$$\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d \cong \mathrm{Sym}^h T^* X,$$

where $h = \mathrm{gcd}(n, d)$. Since $T^* X \cong X \times \mathbb{C}$ we can define the following projections

$$\begin{array}{ccc} & \mathrm{Sym}^h T^* X & \\ a \swarrow & & \searrow b \\ \mathrm{Sym}^h X & & \mathrm{Sym}^h \mathbb{C} \end{array}$$

where a is identified with the projection to the moduli space of vector bundles [Proposition 4.3.9] and b with the Hitchin map [Lemma 4.3.10]. We define three involutions on the moduli space $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d$ and the corresponding involutions on $\mathrm{Sym}^h T^* X$ [Lemma 4.3.14] that will be used in the study of orthogonal and symplectic Higgs bundles or Higgs bundles for real forms of $\mathrm{GL}(n, \mathbb{C})$. We finish the section proving the existence of a bijection between $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d$ and $\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d$ and showing that $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d$ is the normalization of $\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d$ [Proposition 4.3.15].

In similar terms, we obtain in Sections 4.4 and 4.5 explicit descriptions of $\mathcal{M}(\mathrm{SL}(n, \mathbb{C}))$ and $\mathcal{M}(\mathrm{PGL}(n, \mathbb{C}))$ [Theorems 4.4.3 and 4.5.4], their associated Hitchin maps [Lemmas 4.4.7 and 4.5.8] and their projections to the moduli spaces of special and projective vector bundles [Propositions 4.4.5 and 4.5.7]. We also prove the existence of bijections between $\mathcal{M}(\mathrm{SL}(n, \mathbb{C}))$ and $\mathcal{M}(\mathrm{PGL}(n, \mathbb{C}))$ and $\mathfrak{M}(\mathrm{SL}(n, \mathbb{C}))$ and $\mathfrak{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}}$ [Propositions 4.4.12 and 4.5.15].

Chapter 5 is dedicated to the explicit description of moduli spaces of Higgs bundles for the groups $\mathrm{Sp}(2m, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$. In Section 5.1 we prove that, if a $\mathrm{Sp}(2m, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$ or $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle is semistable (resp. polystable), then, its underlying Higgs bundle for $\mathrm{GL}(n, \mathbb{C})$ is semistable (resp. polystable) [Proposition 5.1.1] and its underlying principal bundle is semistable (resp. polystable) [Proposition 5.1.2]. We describe the stable [Proposition 5.1.3 and Corollary 5.1.4] and polystable Higgs bundles [Proposition 5.1.6] for these groups.

In Section 5.2 we construct a family of polystable $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles with the local universal property among locally graded families of $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles [Proposition 5.2.3]. With such a family we obtain that [Theorem 5.2.5]

$$\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C})) \cong \mathrm{Sym}^m(T^* X / \mathbb{Z}_2).$$

Using this explicit description we study the projection to $\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C}))$ [Proposition 5.2.9], the Hitchin map [Lemma 5.2.10] and its fibres [Corollary 5.2.13]. We prove the existence of a bijective morphism from $\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C}))$ to $\mathfrak{M}(\mathrm{Sp}(2m, \mathbb{C}))$ and we show that $\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C}))$ is the normalization of $\mathfrak{M}(\mathrm{Sp}(2m, \mathbb{C}))$ [Proposition 5.2.14].

In Sections 5.3 and 5.4 we obtain analogous descriptions of the spaces $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))$ [Theorem 5.3.7] and $\mathcal{M}(\mathrm{SO}(n, \mathbb{C}))$ [Theorems 5.4.7 and 5.4.14], their projections to the moduli spaces of principal bundles [Propositions 5.3.11 and 5.4.16], their associated Hitchin maps [Lemmas 5.3.14, 5.4.10 and 5.4.17] and the fibres of these maps [Corollaries 5.3.15,

5.4.11 and 5.4.22]. We give a bijective morphism from $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))$ and $\mathcal{M}(\mathrm{SO}(n, \mathbb{C}))$ to $\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))$ and $\mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))$ and we prove that $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))$ is the normalization of $\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))$, and $\mathcal{M}(\mathrm{SO}(n, \mathbb{C}))$ the normalization of $\mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))$ [Propositions 5.3.16 and 5.4.23].

Outline of Part II

Part II is devoted to the study of Higgs bundles for real forms of $\mathrm{GL}(n, \mathbb{C})$, namely $\mathrm{U}(p, q)$, $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{U}^*(2m)$ when n is even.

In Chapter 6 we study Higgs bundles for $\mathrm{U}(p, q)$. We start recalling in Section 6.1 the definition of $\mathrm{U}(p, q)$ -Higgs bundles and their stability notions. We define locally graded families of $\mathrm{U}(p, q)$ -Higgs bundles and we state the moduli problem associated to this new concept of families [(6.2)].

From Section 6.2 we have the following: *a $\mathrm{U}(p, q)$ -Higgs bundle (V, W, β, γ) is semistable if and only if V and W are semistable vector bundles of the same slope. It is stable if $V \cong W$ are stable vector bundles and $\beta \circ \gamma$ is non zero* [Proposition 6.2.1]. A straightforward consequence of the previous statement is that semistability implies that the Toledo invariant is zero [Corollary 6.2.2].

In Section 6.3 we study $\mathcal{M}(\mathrm{U}(p, q))_{(a,b)}$, where (a, b) is the topological invariant for $\mathrm{U}(p, q)$ -Higgs bundles (the degrees of the underlying vector bundles). By the vanishing of the Toledo invariant, we know that $\mathcal{M}(\mathrm{U}(p, q))_{(a,b)}$ is empty unless the invariants (p, q, a, b) are (nr, mr, nd, md) with $\gcd(r, d) = 1$. With the description of stable $\mathrm{U}(p, q)$ -Higgs bundles we can construct a family of polystable $\mathrm{U}(p, q)$ -Higgs bundles with the local universal property [Proposition 6.3.5] which allows us to prove the existence of an isomorphism between $\mathcal{M}(\mathrm{U}(p, q))_{(a,b)}$ and some subvariety of $\mathrm{Sym}^n(X \times \mathbb{C}) \times \mathrm{Sym}^m(X \times \mathbb{C})$ [Theorem 6.3.8]. We study the Hitchin fibration in this context [Lemma 6.3.11] and we observe that the dimension of the fibres varies [Corollary 6.3.13]. At the end of 6.3 we give an explicit description of the moduli spaces $\mathcal{M}(\mathrm{U}(r, r))_{d,d}$ and $\mathcal{M}(\mathrm{U}(r, 2r))_{d,2d}$ and we observe that they are not normal [Remarks 6.3.14 and 6.3.15].

Since $\mathrm{U}^*(2m)$ -Higgs bundles and $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles share a common structure we study them together in Chapter 7. In Section 7.1 we recall $\mathrm{U}^*(2m)$ and $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles, and the notions of stability for these objects. We also define locally graded families and the moduli functors associated to this new concept of families [(7.1) and (7.2)].

Section 7.2 contains the study of the stability relations: *the underlying $\mathrm{Sp}(2m, \mathbb{C})$ -bundle (resp. the underlying $\mathrm{O}(n, \mathbb{C})$ -bundle) of a semistable $\mathrm{U}^*(2m)$ -Higgs bundle (resp. a semistable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle) is semistable. It is polystable if the $\mathrm{U}^*(2m)$ -Higgs bundle (resp. $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle) is polystable* [Proposition 7.2.2].

In Section 7.3 we construct a family of polystable $\mathrm{U}^*(2m)$ -Higgs bundles with the local universal property among locally graded families [Proposition 7.3.4]. With such a family we obtain the following explicit description [Theorem 7.3.6]

$$\mathcal{M}(\mathrm{U}^*(2m)) \cong \mathrm{Sym}^m(\mathbb{P}^1 \times \mathbb{C}).$$

Under this explicit description, the Hitchin map corresponds to [Lemma 7.3.10]

$$\mathrm{Sym}^m(\mathbb{P}^1 \times \mathbb{C}) \longrightarrow \mathrm{Sym}^m \mathbb{C}$$

and we observe that the generic fibre is $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$ [Corollary 7.3.13].

In Section 7.4 we obtain an isomorphism between $\mathcal{M}(\mathrm{GL}(n, \mathbb{R}))$ and the fixed point set of a certain involution in $\mathrm{Sym}^n T^*X$ [Theorem 7.4.7]. The fibre of the Hitchin map is a collection of projective spaces and its dimension varies [Corollary 7.4.13]. We give an explicit description of $\mathcal{M}(\mathrm{GL}(2, \mathbb{R}))$ and we observe that it is not normal [Remark 7.4.14].

Outline of Part III

The homeomorphism between the moduli space of G -bundles over X and the moduli space of unitary representations of $\Gamma_{\mathbb{R}}$ is used in [FM1] to obtain a description of semistable G -bundles. In Part III we will make use of this description to study G -Higgs bundles over an elliptic curve when G is a connected complex reductive Lie group.

Chapter 8 is a review of [FM1] and [BFM]. In Section 8.1 we include some results on reductive Lie groups that will be necessary for the description of stable, semistable and polystable G -Higgs bundles.

Since the fundamental group of an elliptic curve is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, every representation of $\Gamma_{\mathbb{R}}$ into a semisimple group G is determined by an almost commuting pair, i.e. two elements of the group whose commutator is an element of the centre. In Section 8.2 we review the study of almost commuting pairs on compact Lie groups given in [BFM] and we check that their results extend to complex reductive Lie groups.

In Section 8.3 we recall the description of holomorphic G -bundles given in [FM1].

Chapter 9 contains the study of G -Higgs bundles when G is a connected complex reductive Lie group.

In Section 9.1 we recall the definitions of G -Higgs bundles and stability of G -Higgs bundles. We define locally graded families of semistable G -Higgs bundles and consider the moduli functor that parametrizes these families [(9.1)].

The main result of Section 4.2 is the following: *if (E, Φ) is a semistable (resp. stable) G -Higgs bundle over an elliptic curve, then E is a semistable (resp. stable) G -bundle. If (E, Φ) is polystable, then E is polystable.* Since the canonical bundle is trivial, the statement of semistability [Proposition 9.2.1] follows from the properties of the Harder-Narasimhan reduction. The statement for stability [Proposition 9.2.2] follows from the properties of the adjoint bundle of a semistable principal bundle given in [FM1] and reviewed in Section 8.3.

If G is complex and simple and $\tilde{d} \in \pi_1(G)$, we observe in Section 9.3 that there are no stable G -Higgs bundles of topological class \tilde{d} unless $G = \mathrm{PGL}(n, \mathbb{C})$ and $\tilde{d} = d \pmod{n}$ with $\mathrm{gcd}(n, d) = 1$ [Proposition 9.3.3]. This allows us to determine for which complex reductive structure groups and which topological classes there exist stable G -Higgs bundles [Corollary 9.3.5]. Whenever it is not empty we describe the moduli space of stable G -Higgs bundles [Theorem 9.3.9] and we construct a universal family [Remark 9.3.10].

In Section 9.4 we see that all the polystable G -Higgs bundles of topological class d have the same Jordan-Hölder Levi subgroup $L_{G,d}$ (up to conjugation) [Proposition 9.4.1]. Taking the extension of structure group of the universal family of stable $L_{G,d}$ -Higgs bundles gives us a family with the local universal property among locally graded families [Proposition 9.4.4]. With this family we obtain a description of the moduli space of G -Higgs bundles associated to the moduli functor of locally graded families [Theorem 9.4.7]. When G is simple and simply connected, Λ is the coroot lattice and W the Weyl group, by Theorem 9.4.7 we have

$$\mathcal{M}(G) \cong (T^*X \otimes_{\mathbb{Z}} \Lambda) / W.$$

We have a bijection between our two moduli spaces $\mathcal{M}(G)_d \rightarrow \mathfrak{M}(G)_d$ and, since $\mathcal{M}(G)_d$ is normal, it is the normalization of $\mathfrak{M}(G)_d$ [Proposition 9.4.11].

In Section 9.5 we use the cameral covers approach to study the Hitchin map [Theorem 9.5.2] for this moduli space. We study also the fibres of this map [Proposition 9.5.3]. In Section 9.6 we study the fibration $\mathcal{M}(G) \rightarrow M(G)$ and we give an interpretation of this map in terms of the cotangent orbifold bundle of a natural orbifold structure defined on $M(G)$ [Theorem 9.6.2].

Part I

Higgs bundles for classical complex Lie groups

Chapter 3

The moduli problem

3.1 Some preliminaries on elliptic curves

Let X be a compact Riemann surface of genus 1 and let x_0 be a distinguished point on it; we call the pair (X, x_0) an *elliptic curve*. However, by abuse of notation, we usually refer to the elliptic curve simply as X .

The Abel-Jacobi map $\text{aj}_h : \text{Sym}^h X \rightarrow \text{Pic}^h(X)$ sends the tuple $[x_1, \dots, x_h]_{\mathfrak{S}_h}$ to the line bundle $L(D)$, where D is the divisor associated to the tuple of points. For $h > 2g - 2 = 0$ the map is surjective and the inverse image of $L \in \text{Pic}^h(X)$ is given by the zeroes of the sections of L , i.e. it is the projective space

$$\text{aj}_h^{-1}(L) = \mathbb{P}H^0(X, L) \cong \mathbb{P}^{h-1}. \quad (3.1)$$

For $h = 1$ this inverse image is a point and then $\text{aj}_1 : X \xrightarrow{\cong} \text{Pic}^1(X)$ is an isomorphism. The distinguished point x_0 of the elliptic curve gives an isomorphism between $\text{Pic}^d(X)$ and $\text{Pic}^{d-h}(X)$,

$$t_h^{x_0} : \text{Pic}^d(X) \longrightarrow \text{Pic}^{d-h}(X) \quad (3.2)$$

$$L \longmapsto L \otimes \mathcal{O}(x_0)^{-h}.$$

For every d we define the isomorphism

$$\zeta_{1,d}^{x_0} : \text{Pic}^d(X) \longrightarrow X, \quad (3.3)$$

given by $\zeta_{1,d}^{x_0} = \text{aj}_1^{-1} \circ t_{d-1}^{x_0}$. In particular $\zeta_{1,0}^{x_0} : \text{Pic}^0(X) \xrightarrow{\cong} X$ defines an abelian group structure on X with x_0 as the identity. The elliptic curve (X, x_0) with this abelian group structure is an abelian variety.

The abelian group structure defined on X induces naturally an abelian group structure on T^*X and then, T^*X is a commutative algebraic group. Recall here that the canonical bundle is trivial, so

$$T^*X \cong X \times \mathbb{C}.$$

The isomorphism of abelian varieties $\varsigma_{1,0}^{x_0} : \text{Pic}^0(X) \rightarrow X$ induces an isomorphism of commutative algebraic groups

$$\xi_{1,0}^{x_0} : \text{Pic}^0(X) \times H^0(X, \mathcal{O}) \longrightarrow T^*X, \quad (3.4)$$

where we recall that $T^*\text{Pic}^0(X)$ is naturally identified with $\text{Pic}^0(X) \times H^0(X, \mathcal{O})$.

3.2 Review on moduli spaces

In this section we follow [Ne].

For every collection of geometrical objects A , we say that a *family* \mathcal{F} of objects in A parametrized by T is a collection of objects \mathcal{F}_t of A indexed by the points $t \in T$. We will ask \mathcal{F} to satisfy a certain algebraic property P that will be specified in each particular case. If a family satisfies the algebraic property P we say that it is a family of *P -type*.

Suppose we have \sim , an equivalence relation for the objects of A . We say that two families \mathcal{F} and \mathcal{F}' parametrized by T are *equivalent* if for every $t \in T$ we have that $\mathcal{F}_t \sim \mathcal{F}'_t$.

Given a collection of objects A , a definition of P -type families and an equivalence relation \sim among them we define an associated moduli functor

$$\text{Mod}(A, P, \sim) : (\text{ schemes }) \longrightarrow (\text{ sets })$$

such that for any scheme T we have that

$$\text{Mod}(A, P, \sim)(T) = \left\{ \begin{array}{l} \text{equivalence classes of } P\text{-type families} \\ \text{of objects of } A \text{ parametrized by } T \end{array} \right\} \quad (3.5)$$

We see that every family \mathcal{F} parametrized by T induces a map

$$\mu_{\mathcal{F}} : T \longrightarrow A/\sim$$

$$t \longmapsto [\mathcal{F}_t]_{\sim}.$$

The moduli construction appears to provide the previous map with an algebraic meaning.

A *coarse moduli space* (or simply a *moduli space*) for the moduli functor $\text{Mod}(A, P, \sim)$ is a scheme M and a bijection $\alpha : A/\sim \rightarrow M$ such that

- for any family \mathcal{F} parametrized by T ,

$$\nu_{\mathcal{F}} = \alpha \circ \mu_{\mathcal{F}} : T \longrightarrow M \quad (3.6)$$

is a morphism,

- for any scheme N and any natural transformation $\Psi : \text{Mod}(A, P, \sim) \rightarrow \text{Hom}(-, N)$, the map

$$\gamma = \Psi(\{pt\}) \circ \alpha^{-1} : M \longrightarrow N \quad (3.7)$$

is a morphism.

If we have another scheme M' and another $\alpha' : A/\sim \rightarrow M'$ satisfying the previous relations, then, the map

$$\alpha \circ (\alpha')^{-1} : M' \longrightarrow M$$

is bijective with inverse $\alpha' \circ \alpha^{-1}$, and then an isomorphism. It follows that the moduli space is unique up to isomorphism.

Given a morphism of schemes $f : U \rightarrow T$, and given a family \mathcal{F} parametrized by T , we define $f^*\mathcal{F}$ to be the family parametrized by U such that for every $u \in U$, the object $(f^*\mathcal{F})_u$ is $\mathcal{F}_{f(u)}$.

We say that a P -type family \mathcal{U} parametrized by M is *universal* for a given moduli problem if, for any other family \mathcal{F} parametrized by T there is a unique morphism $f : T \rightarrow M$ with $f^*\mathcal{U} \sim \mathcal{F}$. In that case we say that the pair (M, \mathcal{U}) is a *fine moduli space* for the moduli problem, and in particular M is a coarse moduli space.

A family \mathcal{E} of P -type parametrized by Z is said to have the *local universal property* among families of P -type if for any other family of P -type \mathcal{F} , parametrized by T and any point $t \in T$, there exists a neighbourhood U containing t and a (not necessarily unique) morphism $f : U \rightarrow Z$ such that $\mathcal{F}|_U \sim f^*\mathcal{E}$.

Families with the local universal property are very useful to describe moduli spaces as we can see in the following result.

Proposition 3.2.1. (Proposition 2.13 of [Ne]) *Let us suppose that for the moduli functor $\text{Mod}(A, P, \sim)$ there exists a family of P -type \mathcal{E} parametrized by Z with the local universal property among families of P -type. Suppose that there exists a group G acting on Z such that $\mathcal{E}_{z_1} \sim \mathcal{E}_{z_2}$ if and only if z_1 and z_2 lie in the same orbit of this action. Then, a categorical quotient of Z is a coarse moduli space for the functor $\text{Mod}(A, P, \sim)$ if and only if it is an orbit space.*

3.3 Moduli space of vector bundles over an elliptic curve

We define the *slope* of a vector bundle E as the quotient

$$\mu(E) := \frac{\deg(E)}{\text{rk}(E)}.$$

A vector bundle E is *semistable* if every subbundle F of E satisfies

$$\mu(F) \leq \mu(E).$$

The vector bundle is *stable* if the above inequality is strict for every proper subbundle and it is *polystable* if it decomposes into a direct sum of stable vector bundles.

If E is strictly semistable, then it has a proper subbundle E_1 with slope equal to $\mu(E)$. If we take E_1 to be minimal, it is stable. The quotient vector bundle E/E_1 is semistable and has slope equal to $\mu(E)$. Repeating this process we obtain a *Jordan-Hölder filtration* of E

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E,$$

where every quotient E_i/E_{i-1} is stable of slope $\mu(E_i/E_{i-1}) = \mu(E)$. Using this filtration, we define for every semistable vector bundle E its *associated graded vector bundle*

$$\mathrm{gr} E \cong \bigoplus_i (E_i/E_{i-1}).$$

Although the Jordan-Hölder filtration of a given semistable vector bundle E is not unique one can prove that the isomorphism class of $\mathrm{gr} E$ is unique.

A *family* of semistable vector bundles over X parametrized by T is a vector bundle \mathcal{V} over $X \times T$ such that the restriction to every slice $X \times \{t\}$, where t is any point of T , is a semistable vector bundle over $X \times \{t\}$. We call this algebraic property for families $P_{(n,d)}^0$.

For every semistable vector bundle E with graded object $\mathrm{gr} E$, using an appropriate extension of vector bundles one can construct a family \mathcal{F}_E parametrized by \mathbb{C} such that the restriction of this family to $X \times \{0\}$ is $\mathcal{F}_E|_{X \times \{0\}} \cong \mathrm{gr} E$ and for any $t \neq 0$ we have $\mathcal{F}_E|_{X \times \{t\}} \cong E$. The family \mathcal{F}_E illustrates the jump phenomenon.

Since \mathbb{C} is irreducible, in order to obtain a Hausdorff moduli space we need the following definition of S-equivalence. We say that two semistable vector bundles, E_1 and E_2 , are *S-equivalent* if $\mathrm{gr} E_1 \cong \mathrm{gr} E_2$.

Let $A_{(n,d)}$ denote the collection of semistable vector bundles over X of rank n and degree d . We define S-equivalence for families pointwise, i.e. we write $\mathcal{V} \sim_S \mathcal{V}'$ if and only if

$$\mathcal{V}|_{X \times \{t\}} \sim_S \mathcal{V}'|_{X \times \{t\}}$$

for every point $t \in T$. Finally we consider the functor $\mathrm{Mod}(A_{(n,d)}, P_{(n,d)}^0, S)$ defined in (3.5).

This functor possesses a moduli space, which we denote by $M(\mathrm{GL}(n, \mathbb{C}))_d$. There is a bijective correspondence between $\mathrm{GL}(n, \mathbb{C})$ -bundles and vector bundles of rank n , which justifies the use of the structure group to identify the moduli space of vector bundles.

Let $A_{(n,d)}^{st}$ denote the collection of stable vector bundles of rank n and degree d . Note that for E stable $E \cong \mathrm{gr} E$ and then S-equivalence of two families of stable vector bundles \mathcal{W} and \mathcal{W}' is equal to isomorphism pointwise; we write $\mathcal{W} \stackrel{pt}{\cong} \mathcal{W}'$ if for every $t \in T$ we have that $\mathcal{W}|_{X \times \{t\}} \cong \mathcal{W}'|_{X \times \{t\}}$. The moduli functor $\mathrm{Mod}(A_{(n,d)}^{st}, P_{(n,d)}^0, \stackrel{pt}{\cong})$ possesses a moduli space $M^{st}(\mathrm{GL}(n, \mathbb{C}))_d$. It can be proved that $M^{st}(\mathrm{GL}(n, \mathbb{C}))_d$ is a smooth Zariski open subset of $M(\mathrm{GL}(n, \mathbb{C}))_d$. Note also that when $\mathrm{gcd}(n, d) = 1$ one has $A_{(n,d)}^{st} = A_{(n,d)}$ and then $M^{st}(\mathrm{GL}(n, \mathbb{C}))_d = M(\mathrm{GL}(n, \mathbb{C}))_d$.

In [A], Atiyah studied holomorphic vector bundles over an elliptic curve. He based his description on the notion of indecomposable bundle instead of using the notions of stability since at that time GIT had not been developed. In 1991 Tu in [Tu] gave an interpretation of Atiyah's results for the construction of the moduli spaces of vector bundles over elliptic curves.

We list some properties of vector bundles over elliptic curves, all of which are contained in [A] or [Tu] (possibly with some changes of notation).

- If $\mathrm{gcd}(n, d) = 1$, the morphism given by the determinant

$$\det : M(\mathrm{GL}(n, \mathbb{C}))_d \xrightarrow{\cong} \mathrm{Pic}^d(X)$$

is an isomorphism. Furthermore, if E is a stable vector bundle of rank n and degree d coprime, we have that E is indecomposable and $E \otimes L \cong E$ if and only if L is a line bundle in $\text{Pic}^0(X)[n]$ (i.e. L is such that $L^{\otimes n} \cong \mathcal{O}$).

Recall the map $\zeta_{1,d}^{x_0}$ given in (3.3). We have that $\zeta_{n,d}^{x_0} = \zeta_{x_0,1,d} \circ \det$ gives a description of the moduli space of vector bundles with rank and degree coprime in terms of the curve

$$\zeta_{n,d}^{x_0} : M(\text{GL}(n, \mathbb{C}))_d \xrightarrow{\cong} X. \quad (3.8)$$

- There exists a unique indecomposable bundle F_n of degree 0 and rank n such that $H^0(X, F_n)$ is not 0. Moreover $\dim H^0(X, F_n) = 1$ and F_n is a multiple extension of copies of \mathcal{O} . In particular F_n is semistable.
- Every indecomposable bundle of degree 0 and rank n is of the form $F_n \otimes L$ for a unique line bundle L of zero degree.
- If $\gcd(n, d) = h > 1$,
 - every indecomposable bundle of rank n and degree d is of the form $E' \otimes F_h$ for a unique stable bundle E' of rank $n' = \frac{n}{h}$ and degree $d' = \frac{d}{h}$;
 - every semistable bundle of rank n and degree d is of the form $\bigoplus_{j=1}^s (E'_j \otimes F_{h_j})$, where each E'_j is stable of rank n' and degree d' and $\sum_{j=1}^s h_j = h$;
 - every polystable bundle of rank n and degree d is of the form $E'_1 \oplus \dots \oplus E'_h$, where each E'_i is stable of rank n' and degree d' ;
 - as a consequence $M^{st}(\text{GL}(n, \mathbb{C}))_d$ is empty and the map

$$\chi_{n,d}^0 : \text{Sym}^h M(\text{GL}(n', \mathbb{C}))_{d'} \longrightarrow M(\text{GL}(n, \mathbb{C}))_d, \quad (3.9)$$

$$[[E'_1]_S, \dots, [E'_h]_S]_{\mathfrak{S}_h} \longmapsto [E'_1 \oplus \dots \oplus E'_h]_S$$

is an isomorphism.

- We can give a description of the moduli space of vector bundles in terms of the curve. Take the isomorphism $\text{Sym}^h M(\text{GL}(n', \mathbb{C}))_{d'} \longrightarrow \text{Sym}^h X$ induced by the map $\zeta_{n',d'}^{x_0}$ given in (3.8). Composing with $(\chi^0)_{n,d}^{-1}$ gives the isomorphism

$$\zeta_{n,d}^{x_0} : M(\text{GL}(n, \mathbb{C}))_d \xrightarrow{\cong} \text{Sym}^h X. \quad (3.10)$$

- If E is stable, $\text{End } E \cong \bigoplus_{L_i \in \text{Pic}^0(X)[n]} L_i$.
- We have $F_n \cong F_n^*$ and $F_n \otimes F_m$ is a direct sum of various F_ℓ . In particular $\text{End } F_n \cong F_1 \oplus F_3 \oplus \dots \oplus F_{2n-1}$.

3.4 Higgs bundles

A *Higgs bundle* over an elliptic curve X is a pair (E, Φ) , where E is a vector bundle on X and Φ is an endomorphism of E called the *Higgs field*. Two Higgs bundles, (E, Φ) and (E', Φ') , are isomorphic if there exists an isomorphism of vector bundles $f : E \rightarrow E'$ that sends Φ to Φ' , i.e. such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow \Phi & & \downarrow \Phi' \\ E & \xrightarrow{f} & E' \end{array} .$$

Given the Higgs bundle (E, Φ) , we say that a subbundle $F \subset E$ is Φ -invariant if $\Phi(F)$ is contained in F .

A Higgs bundle (E, Φ) is *semistable* if the slope of any Φ -invariant subbundle F satisfies

$$\mu(F) \leq \mu(E).$$

The Higgs bundle is *stable* if the above inequality is strict for every proper Φ -invariant subbundle and *polystable* if it is semistable and isomorphic to a direct sum of stable Higgs bundles.

If (E, Φ) is strictly semistable, then it has a proper Φ -invariant subbundle E_1 with slope equal to $\mu(E)$. If we take E_1 to be minimal, the pair (E_1, Φ_1) is a stable Higgs bundle (Φ_1 is the restriction of Φ to E_1). We can easily see that the Higgs bundle given by E/E_1 and the induced Higgs field $\tilde{\Phi}$ is semistable and has slope equal to $\mu(E)$. Again, we take a $\tilde{\Phi}$ -invariant minimal subbundle with slope equal to $\mu(E/E_1)$. This gives a stable Higgs bundle $(E_2/E_1, \tilde{\Phi}_2)$, which corresponds to a Φ -invariant subbundle $E_2 \subset E$ containing E_1 . This process gives us a *Jordan-Hölder filtration* of Φ -invariant subbundles

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E$$

where every quotient $(E_i/E_{i-1}, \tilde{\Phi}_i)$ is stable with slope $\mu(E_i/E_{i-1}) = \mu(E)$.

For every semistable Higgs bundle (E, Φ) we define its *associated graded object*

$$\mathrm{gr}(E, \Phi) = \bigoplus_i (E_i/E_{i-1}, \tilde{\Phi}_i).$$

Although the Jordan-Hölder filtration may not be uniquely determined by (E, Φ) , the isomorphism class of $\mathrm{gr}(E, \Phi)$ is.

A *family of Higgs bundles* parametrized by T is a pair $\mathcal{E} = (\mathcal{V}^{\mathcal{E}}, \Phi^{\mathcal{E}})$, where $\mathcal{V}^{\mathcal{E}}$ is a holomorphic vector bundle over $X \times T$ (i.e. a family of vector bundles) and $\Phi^{\mathcal{E}}$ is a holomorphic section of $\mathrm{End} \mathcal{V}^{\mathcal{E}}$. For every $t \in T$, we will write $\mathcal{E}_t = (\mathcal{V}_t^{\mathcal{E}}, \Phi_t^{\mathcal{E}})$ for the Higgs bundle over X obtained by restricting \mathcal{E} to $X \times \{t\}$.

It is not difficult to see that for every semistable Higgs bundle (E, Φ) with graded object $\mathrm{gr}(E, \Phi)$, one can construct a family $\mathcal{F}_{(E, \Phi)}$ parametrized by \mathbb{C} that illustrates the jump phenomenon, i.e. for every $t \neq 0$ we have $\mathcal{F}_{(E, \Phi)}|_{X \times \{t\}} \cong E$ while $\mathcal{F}_{(E, \Phi)}|_{X \times \{0\}} \cong$

$\text{gr}(E, \Phi)$. If our definition of S-equivalence does not identify (E, Φ) and $\text{gr}(E, \Phi)$ we obtain a non-Hausdorff moduli space.

We say that two semistable Higgs bundles (E, Φ) and (E', Φ') are *S-equivalent* if $\text{gr}(E, \Phi) \cong \text{gr}(E', \Phi')$. We denote by $[(E, \Phi)]_S$ the S-equivalence class of (E, Φ) .

We say that \mathcal{E} is a family of semistable Higgs bundles if \mathcal{E}_t is a semistable Higgs bundle for every $t \in T$. Two families of semistable Higgs bundles \mathcal{E}_1 and \mathcal{E}_2 parametrized by the same variety T are S-equivalent if for every $t \in T$ the Higgs bundles $(\mathcal{E}_1)_t$ and $(\mathcal{E}_2)_t$ are S-equivalent, we denote this by $\mathcal{E}_1 \sim_S \mathcal{E}_2$.

We denote by $\mathcal{A}_{(n,d)}$ the collection of semistable Higgs bundles over X of rank n and degree d and by $\mathcal{A}_{(n,d)}^{st}$ the subcollection of stable ones. We denote by $P_{(n,d)}$ the algebraic condition defined above for the definition of families of semistable Higgs bundles. With these ingredients we state our moduli problem considering the moduli functor defined in (3.5)

$$\text{Mod}(\mathcal{A}_{(n,d)}, P_{(n,d)}, S).$$

By [Ni] and [Si1] there exists a coarse moduli space $\mathfrak{M}(\text{GL}(n, \mathbb{C}))_d$ of S-equivalence classes of semistable Higgs bundles of rank n and degree d associated to the moduli functor $\text{Mod}(\mathcal{A}_{(n,d)}, P_{(n,d)}, S)$. The points of $\mathfrak{M}(\text{GL}(n, \mathbb{C}))_d$ can be identified also with isomorphism classes of polystable Higgs bundles since in every S-equivalence class there is always a polystable Higgs bundle which is unique up to isomorphism.

Note that a stable Higgs bundle is equal to its graded object and then S-equivalence between stable Higgs bundles is equivalent to isomorphism; hence S-equivalence for families of stable Higgs bundles is the same as isomorphism pointwise (we denote it by $\stackrel{pt}{\cong}$). We can consider the functor

$$\text{Mod}(\mathcal{A}_{(n,d)}^{st}, P_{(n,d)}, \stackrel{pt}{\cong}).$$

It is also contained in [Ni] and [Si1] that the moduli space $\mathfrak{M}^{st}(\text{GL}(n, \mathbb{C}))_d$ parametrizing isomorphism (hence S-equivalence) classes of stable Higgs bundles of rank n and degree d associated to $\text{Mod}(\mathcal{A}_{(n,d)}^{st}, P_{(n,d)}, \stackrel{pt}{\cong})$ exists and is an open subvariety of $\mathfrak{M}(\text{GL}(n, \mathbb{C}))_d$.

We say that a family of semistable Higgs bundles $\mathcal{F} \rightarrow X \times T$ is *locally graded* if for every point t of T there exists an open subset $U \subset T$ containing t and a set of families $\mathcal{E}_1, \dots, \mathcal{E}_\ell$ where each \mathcal{E}_i is a family of stable Higgs bundles parametrized by U such that for every point t' of U we have

$$\text{gr } \mathcal{F}|_{X \times \{t'\}} \cong \bigoplus_{i=1}^{\ell} \mathcal{E}_i|_{X \times \{t'\}},$$

and therefore $\mathcal{F}|_{X \times U} \sim_S \bigoplus_i \mathcal{E}_i$. We call this algebraic condition for families of semistable Higgs bundles $Q_{(n,d)}$. With this algebraic condition for families we construct the moduli functor

$$\text{Mod}(\mathcal{A}_{(n,d)}, Q_{(n,d)}, S), \tag{3.11}$$

and, if it exists, we will denote by $\mathcal{M}(\text{GL}(n, \mathbb{C}))_d$ the moduli space of S-equivalence classes of semistable Higgs bundles of rank n and degree d associated to this moduli functor.

3.5 $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundles

A $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundle over the elliptic curve X is a Higgs bundle (E, Φ) with trivial determinant and traceless Higgs field, i.e. $\det E \cong \mathcal{O}$ and $\mathrm{tr} \Phi = 0$. A $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundle (E, Φ) is *semistable*, *stable* or *polystable* if (E, Φ) is respectively a semistable, stable or polystable Higgs bundle.

A *Jordan-Hölder filtration* of the semistable $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundle (E, Φ) is a Jordan-Hölder filtration of (E, Φ) as a Higgs bundle and its *associated graded object* $\mathrm{gr}(E, \Phi)$ is the associated graded object of (E, Φ) as a Higgs bundle.

We denote by $\hat{\mathcal{A}}_n$ the collection of semistable $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundles, we see that $\hat{\mathcal{A}}_n$ is the subset of $\mathcal{A}_{n,0}$ given by the Higgs bundles with trivial determinant and traceless Higgs field. A *family of $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundles* $\mathcal{E} \rightarrow X \times T$ is a family of Higgs bundles $\mathcal{E} = (\mathcal{V}, \Phi)$ such that for every $t \in T$ we have $\det \mathcal{V}_t \cong \mathcal{O}_{X \times \{t\}}$ and $\mathrm{tr} \Phi_t = 0$, i.e. the algebraic condition is $P_{(n,0)}$.

Two semistable $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundles (E_1, Φ_1) and (E_2, Φ_2) are *S-equivalent* if we have that (E_1, Φ_1) and (E_2, Φ_2) are S-equivalent Higgs bundles. Again, we define S-equivalence for families pointwise. Once we have introduced all this notation we can consider the following moduli functor

$$\mathrm{Mod}(\hat{\mathcal{A}}_n, P_{(n,0)}, S).$$

We use $\mathfrak{M}(\mathrm{SL}(n, \mathbb{C}))$ to denote the coarse moduli space associated to this moduli problem, its existence will be discussed later.

We write $\hat{\mathcal{A}}_n^{st}$ for the subcollection of stable $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundles. If two stable Higgs bundles are S-equivalent then they are isomorphic. This implies that S-equivalence for families is equal to isomorphism pointwise and therefore we can define the following moduli functor

$$\mathrm{Mod}(\hat{\mathcal{A}}_n^{st}, P_{(n,0)}, \cong).$$

This moduli functor possesses a moduli space $\mathfrak{M}^{st}(\mathrm{SL}(n, \mathbb{C}))$, which is a Zariski open subset of $\mathfrak{M}(\mathrm{SL}(n, \mathbb{C}))$.

A family of $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundles is *locally graded* if it is a locally graded family of Higgs bundles, i.e. if it satisfies the algebraic condition $Q_{(n,0)}$. Associated to the condition of locally graded families we have the following moduli functor

$$\mathrm{Mod}(\hat{\mathcal{A}}_n, Q_{(n,0)}, S) \tag{3.12}$$

and we write $\mathcal{M}(\mathrm{SL}(n, \mathbb{C}))$ for the moduli space associated to this moduli functor if it exists.

For any family of Higgs bundles $\mathcal{E} = (\mathcal{V}, \Phi)$ we can construct the family of rank 1 and degree 0 Higgs bundles $(\det \mathcal{V}, \mathrm{tr} \Phi)$. Then, for the moduli spaces $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0$ or

$\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_0$ of zero degree Higgs bundles we have the following morphism

$$(\det, \mathrm{tr}) : \mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_0 \longrightarrow \mathrm{Pic}^0(X) \times H^0(X, \mathcal{O})$$

$$(\det, \mathrm{tr}) : \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0 \longrightarrow \mathrm{Pic}^0(X) \times H^0(X, \mathcal{O})$$

$$[(E, \Phi)]_S \longmapsto [(\det E, \mathrm{tr} \Phi)]_S,$$

and we can consider the closed subvarieties

$$\mathfrak{M}_{\substack{\det=\mathcal{O} \\ \mathrm{tr}=0}}(\mathrm{GL}(n, \mathbb{C})) = (\det, \mathrm{tr})^{-1}(\mathcal{O}, 0) \subset \mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_0$$

and

$$\mathcal{M}_{\substack{\det=\mathcal{O} \\ \mathrm{tr}=0}}(\mathrm{GL}(n, \mathbb{C})) = (\det, \mathrm{tr})^{-1}(\mathcal{O}, 0) \subset \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0.$$

Since $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0$ and $\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_0$ are coarse moduli spaces there exist two bijections $\alpha_P : \mathcal{A}_{(n,0)}/\sim \rightarrow \mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_0$ and $\alpha_Q : \mathcal{A}_{(n,0)}/\sim \rightarrow \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0$ satisfying that (3.6) and (3.7) are morphisms.

If we have two Higgs bundles (E_1, Φ_1) and (E_2, Φ_2) with $(E_1, \Phi_1) \sim_S (E_2, \Phi_2)$ then, certainly $\det E_1 \cong \det E_2$ and $\mathrm{tr} \Phi_1 = \mathrm{tr} \Phi_2$. This implies that $\hat{\mathcal{A}}_n/\sim_S$ injects into $\mathcal{A}_{(n,0)}/\sim_S$. We can take the restrictions of α_P and α_Q to $\hat{\mathcal{A}}_n/\sim_S$ which we denote by α'_P and α'_Q and we observe that $(\mathfrak{M}_{\substack{\det=\mathcal{O} \\ \mathrm{tr}=0}}(\mathrm{GL}(n, \mathbb{C})), \alpha'_P)$ and $(\mathcal{M}_{\substack{\det=\mathcal{O} \\ \mathrm{tr}=0}}(\mathrm{GL}(n, \mathbb{C})), \alpha'_Q)$ satisfy the moduli conditions (3.6) and (3.7) and therefore they are respectively moduli spaces for the functors $\mathrm{Mod}(\hat{\mathcal{A}}_n, P_{(n,0)}, S)$ and $\mathrm{Mod}(\hat{\mathcal{A}}_n, Q_{(n,0)}, S)$, i.e.

$$\mathfrak{M}(\mathrm{SL}(n, \mathbb{C})) = (\det, \mathrm{tr})^{-1}(\mathcal{O}, 0) \subset \mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_0 \quad (3.13)$$

and

$$\mathcal{M}(\mathrm{SL}(n, \mathbb{C})) = (\det, \mathrm{tr})^{-1}(\mathcal{O}, 0) \subset \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0. \quad (3.14)$$

After (3.13) we see that the existence of the coarse moduli space $\mathfrak{M}(\mathrm{SL}(n, \mathbb{C}))$ follows from [Ni] and [Si1].

A $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundle over the elliptic curve X is pair $(\mathbb{P}(E), \Phi)$ where E is a vector bundle, $\mathbb{P}(E)$ is the projective bundle given by E and $\Phi \in H^0(X, \mathrm{End} E)$ has $\mathrm{tr} \Phi = 0$. A $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundle has topological invariant $\tilde{d} \in \mathbb{Z}_n$ where $\tilde{d} = (d \bmod n)$.

Remark 3.5.1. Over a curve, a projective bundle comes always from a vector bundle, although this fact is no longer true in higher dimension.

A $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundle $(\mathbb{P}(E), \Phi)$ is *semistable*, *stable* or *polystable* if (E, Φ) is respectively a semistable, stable or polystable Higgs bundle. If $\mathrm{gr}(E, \Phi) = (E', \Phi')$, we define *the associated graded object* of the semistable $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundle $(\mathbb{P}(E), \Phi)$ as the pair $(\mathbb{P}(E'), \Phi')$.

We denote by $\hat{\mathcal{A}}_{n, \tilde{d}}$ the collection of semistable $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundles of degree $\tilde{d} \in \mathbb{Z}_n$. A family of $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundles $\mathcal{E} \rightarrow X \times T$ is a pair (\mathcal{P}, Φ) where \mathcal{P} is

a projective bundle over $X \times T$ and Φ is a holomorphic section of the adjoint bundle of \mathcal{P} . We denote by $\check{P}_{n,\tilde{d}}$ this algebraic condition for families. If we define S-equivalence for families pointwise we can consider the following moduli functor

$$\text{Mod}(\check{\mathcal{A}}_{n,\tilde{d}}, \check{P}_{n,\tilde{d}}, S).$$

From [Si3] we know that there exists a coarse moduli space $\mathfrak{M}(\text{PGL}(n, \mathbb{C}))_{\tilde{d}}$ parametrizing S-equivalence classes of semistable $\text{PGL}(n, \mathbb{C})$ -Higgs bundles of degree \tilde{d} .

We write $\check{\mathcal{A}}_{n,\tilde{d}}^{st}$ for the subcollection of stable $\text{PGL}(n, \mathbb{C})$ -Higgs bundles. Since two S-equivalent stable Higgs bundles are isomorphic, S-equivalence for families is equal to isomorphism pointwise. After these considerations we can define the following moduli functor

$$\text{Mod}(\check{\mathcal{A}}_{n,\tilde{d}}^{st}, \check{P}_{n,\tilde{d}}, S).$$

We use $\mathfrak{M}^{st}(\text{PGL}(n, \mathbb{C}))_{\tilde{d}}$ to denote the moduli space associated to this moduli functor.

A family of $\text{PGL}(n, \mathbb{C})$ -Higgs bundles $\mathcal{E} \rightarrow X \times T$ is *locally graded* if there exists a locally graded family of Higgs bundles (\mathcal{V}, Φ) such that $\mathcal{E} \sim_S (\mathbb{P}(\mathcal{V}), \Phi)$. We say that locally graded families of $\text{PGL}(n, \mathbb{C})$ -Higgs bundles satisfy the algebraic condition $\check{Q}_{n,\tilde{d}}$. Associated to this algebraic condition we define the following moduli functor

$$\text{Mod}(\check{\mathcal{A}}_{n,\tilde{d}}, \check{Q}_{n,\tilde{d}}, S). \quad (3.15)$$

If this moduli functor has an associated moduli space we denote it by $\mathcal{M}(\text{PGL}(n, \mathbb{C}))_{\tilde{d}}$.

Tensoring a vector bundle E by a line bundle $L \in \text{Pic}(X)$ preserves the endomorphism bundle $\text{End}(E) \cong \text{End}(L \otimes E)$ and so $\text{Pic}(X)$ acts on the collection of Higgs bundles $\mathcal{A}_{(n,d)}$. It is straightforward to check that $(L \otimes E, \Phi)$ is stable, semistable or polystable if and only if (E, Φ) is stable, semistable or polystable. If the graded object of a semistable Higgs bundle (E, Φ) is $\text{gr}(E, \Phi) = (E', \Phi')$ we have that $\text{gr}(L \otimes E, \Phi)$ is $(L \otimes E', \Phi')$. We can prove that two vector bundles E_1 and E_2 give the same projective bundle if and only if $E_2 = L \otimes E_1$. If E_1 and E_2 have the same degree, then $L \in \text{Pic}^0(X)$. As a consequence, if $\mathcal{A}_{n,\tilde{d}}^{\text{tr}=0}$ is the collection of rank n Higgs bundles with traceless Higgs field and $\check{\mathcal{A}}_{n,\tilde{d}}^{\text{st},\text{tr}=0}$ the subcollection of stable ones, we have

$$\check{\mathcal{A}}_{n,\tilde{d}} = \mathcal{A}_{(n,d)}^{\text{tr}=0} / \text{Pic}^0(X)$$

and

$$\check{\mathcal{A}}_{n,\tilde{d}}^{st} = \mathcal{A}_{(n,d)}^{\text{st},\text{tr}=0} / \text{Pic}^0(X).$$

3.6 Symplectic and orthogonal Higgs bundles

A $\text{Sp}(2m, \mathbb{C})$ -Higgs bundle over the elliptic curve X is a triple (E, Ω, Φ) , where E is a rank $2m$ holomorphic vector bundle over X , $\Omega \in H^0(X, \Lambda^2 E^*)$ is a non-degenerate symplectic form on E and $\Phi \in H^0(X, \text{End } E)$ is an endomorphism of E which anticommutes with Ω , i.e. for every $x \in X$ we have

$$\Omega(u, \Phi(v)) = -\Omega(\Phi(u), v), \quad \text{for every } u, v \in E_x.$$

Two $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles, (E, Ω, Φ) and (E', Ω', Φ') are isomorphic if there exists an isomorphism $f : E' \rightarrow E$ such that $\Omega' = f^t \Omega f$ and $\Phi' = f^{-1} \Phi f$.

In similar terms, we have that an $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle over X is a triple (E, Q, Φ) , where E is a rank n holomorphic vector bundle, $Q \in H^0(X, \mathrm{Sym}^2 E^*)$ is a non-degenerate symmetric form on E and $\Phi \in H^0(X, \mathrm{End} E)$ is an endomorphism of E which anticommutes with Q , i.e. for every $x \in X$ we have

$$Q(u, \Phi(v)) = -Q(\Phi(u), v), \quad \text{for every } u, v \in E_x.$$

Again, two $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles (E, Q, Φ) and (E', Q', Φ') are isomorphic if there exists an isomorphism $f : E' \rightarrow E$ such that $Q' = f^t Q f$ and $\Phi' = f^{-1} \Phi f$.

A $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle is a quadruple (E, Q, Φ, τ) such that (E, Q, Φ) is a $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle and τ is a trivialization of $\det E$ (a never vanishing section of $\det E$) compatible with Q , that is $\tau^2 = (\det Q)^{-1}$. An isomorphism between the $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles $(E_1, Q_1, \Phi_1, \tau_1)$ and $(E_2, Q_2, \Phi_2, \tau_2)$ is an isomorphism of the underlying $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles f that sends τ_1 to τ_2 . Note that the existence of a trivialization of $\det E$ implies that $\det E \cong \mathcal{O}$. A direct sum of various $\mathrm{SO}(n_i, \mathbb{C})$ -Higgs bundles is the $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle given by the $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle which is the direct sum of the underlying $\mathrm{O}(n_i, \mathbb{C})$ -Higgs bundles plus the trivialization of the determinant induced by the trivialization of the different $\mathrm{SO}(n_i, \mathbb{C})$ -Higgs bundles.

Symplectic and orthogonal Higgs bundles are particular cases of triples (E, Θ, Φ) , where (E, Φ) is a Higgs bundle and $\Theta : (E, \Phi) \rightarrow (E^*, -\Phi^t)$ is an isomorphism of Higgs bundles that satisfies $\Theta = b\Theta^t$. If $b = 1$, we have a $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle, and if $b = -1$ it is a $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundle. Since Θ is an isomorphism of Higgs bundles, we represent it as a commuting diagram

$$\begin{array}{ccc} E & \xrightarrow{\Theta} & E^* \\ \downarrow \Phi & & \downarrow -\Phi^t \\ E & \xrightarrow{\Theta} & E^*, \end{array} \quad \text{where } \Theta = b\Theta^t.$$

Given the isomorphism $\Theta : E \rightarrow E^*$, for every subbundle F of E we can define its orthogonal complement with respect to Θ ,

$$F^{\perp\Theta} = \{v \in E \text{ such that } \Theta(v, u) = 0 \text{ for every } u \in F\}.$$

A subbundle is *isotropic* with respect to Θ when $F \subset F^{\perp\Theta}$. It is *coisotropic* if $F^{\perp\Theta} \subseteq F$. Isotropic subbundles are crucial in the definition of stability for symplectic and orthogonal vector bundles.

The following notions of stability, semistability and polystability of $\mathrm{Sp}(2m, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles are the notions of stability worked out in [GGM] (see also [AG] for the stability of $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles).

Let (E, Θ, Φ) be a $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundle or a $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle. Note that $\mu(E) = 0$ since $E \cong E^*$. We say that (E, Θ, Φ) is *semistable* if and only if for any Φ -invariant isotropic subbundle F of E we have

$$\mu(F) \leq \mu(E) = 0.$$

and it is *stable* if the above inequality is strict for any proper Φ -invariant isotropic subbundle. Finally, (E, Θ, Φ) is *polystable* if it is semistable and for any non-zero, strict, Φ -invariant and isotropic subbundle F of degree 0 there is a coisotropic and Φ -invariant subbundle F' of degree 0 such that $E = F \oplus F'$.

Note that the criterion for semistability and stability of $\mathrm{Sp}(2m, \mathbb{C})$ and $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles coincides with the criterion for the vectorial case applied only to isotropic subbundles.

If (E, Θ, Φ) is a strictly semistable $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundle or $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle we can find an isotropic Φ -invariant subbundle E_1 with slope equal to zero. We denote by (E_1, Φ_1) the Higgs bundle induced by the restriction of the Higgs field to E_1 . Since every subbundle of an isotropic subbundle is isotropic, we can assume that (E_1, Φ_1) is stable. We set $E'_1 = E_1^{\perp\Theta}$ and we construct $(E/E'_1, \Phi'_1)$ where Φ'_1 is the induced Higgs field on the quotient. Since Θ is non-degenerate and anticommutes with the Higgs field, it induces an isomorphism $\theta_1 : (E_1, \Phi_1) \xrightarrow{\cong} ((E/E'_1)^*, -\Phi'_1)$.

Recall that E_1 is isotropic and then $E_1 \subset E'_1 = E_1^{\perp\Theta}$; we write \tilde{E}_1 for the quotient E'_1/E_1 . Since Θ anticommutes with Φ , the bundle E'_1 is Φ -invariant and the restriction of Φ to E'_1 induces a Higgs field for the quotient $\tilde{E}_1 = E'_1/E_1$ that we denote by $\tilde{\Phi}_1$. Through this quotient Θ induces $\tilde{\Theta}_1$ on \tilde{E}_1 non-degenerate and so we obtain an $\mathrm{Sp}(2m, \mathbb{C})$ or $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle $(\tilde{E}_1, \tilde{\Theta}_1, \tilde{\Phi}_1)$. Since (E, Θ, Φ) is semistable we have that $(\tilde{E}_1, \tilde{\Theta}_1, \tilde{\Phi}_1)$ is semistable as well.

If $(\tilde{E}_1, \tilde{\Theta}_1, \tilde{\Phi}_1)$ is not stable, we can find $E_1 \subset E_2 \subset E'_1$ isotropic and Φ -invariant, such that $(E_2/E_1, \bar{\Phi}_2)$, where $\bar{\Phi}_2$ is the restriction of $\tilde{\Phi}_1$ to E_2/E_1 , is a stable Higgs bundle. We define $E'_2 = E_2^{\perp\Theta}$. This bundle is contained in E'_1 since $E_1 \subset E_2$. We write $\bar{\Phi}'_2$ for the Higgs bundle induced by $\tilde{\Phi}_1$ in the quotient E'_1/E'_2 . Since $\tilde{\Theta}_1$ is non-degenerate and anticommutes with the Higgs field, it induces an isomorphism $\theta_2 : (E_2/E_1, \bar{\Phi}_2) \xrightarrow{\cong} ((E'_1/E'_2)^*, -\bar{\Phi}'_2)$.

We define \tilde{E}_2 as the quotient E'_2/E_2 . We have that Θ induces $\tilde{\Theta}_2$ on \tilde{E}_2 non-degenerate and so we obtain an $\mathrm{Sp}(2m, \mathbb{C})$ or $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle $(\tilde{E}_2, \tilde{\Theta}_2, \tilde{\Phi}_2)$. Since (E, Θ, Φ) is semistable we have that $(\tilde{E}_2, \tilde{\Theta}_2, \tilde{\Phi}_2)$ is again semistable.

We iterate this process until we obtain $(\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k)$ that is a stable triple or zero. This gives a *Jordan-Hölder filtration* of Φ -invariant subbundles, consisting of

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E'_2 \subset E'_1 \subset E'_0 = E$$

where $E_i^{\perp\Theta} = E'_i$, and for $i \neq k$ we have that $(E_i/E_{i-1}, \bar{\Phi}_i)$ and $(E'_{i-1}/E'_i, \bar{\Phi}'_i)$ are stable Higgs bundles satisfying $\theta_i : (E_i/E_{i-1}, \bar{\Phi}_i) \xrightarrow{\cong} ((E'_{i-1}/E'_i)^*, -\bar{\Phi}'_i)$.

For every semistable $\mathrm{Sp}(2m, \mathbb{C})$ or $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle (E, Θ, Φ) we define its *associated graded object*

$$\mathrm{gr}(E, \Theta, \Phi) = (\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k) \oplus \bigoplus_{i=1}^{k-1} \left((E_i/E_{i-1}) \oplus (E'_{i-1}/E'_i), \begin{pmatrix} 0 & b\theta_i^t \\ \theta_i & 0 \end{pmatrix}, \bar{\Phi}_i \oplus \bar{\Phi}'_i \right),$$

where $b = -1$ for $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles and $b = 1$ for $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles. Let us recall that $\theta_i : (E_i/E_{i-1}, \bar{\Phi}_i) \xrightarrow{\cong} ((E'_{i-1}/E'_i)^*, -\bar{\Phi}'_i)$, and then, $(\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k)$ is stable if there exists.

As happens in the vectorial case, the Jordan-Hölder filtration may not be unique but $\text{gr}(E, \Theta, \Phi)$ is unique up to isomorphism.

Since $\text{SO}(2, \mathbb{C}) \cong \mathbb{C}^*$ is abelian, every $\text{SO}(2, \mathbb{C})$ -Higgs bundle is stable, and then semistable and polystable. Whenever $n > 2$ we have that $Z(\text{SO}(n, \mathbb{C}))$ is a subgroup of $Z(\text{O}(n, \mathbb{C}))$ and then a $\text{SO}(n, \mathbb{C})$ -Higgs bundle is semistable, stable or polystable if it is semistable, stable or polystable as an $\text{O}(n, \mathbb{C})$ -Higgs bundle. The graded object of a $\text{SO}(n, \mathbb{C})$ -Higgs bundle is given by the graded object of the underlying $\text{O}(n, \mathbb{C})$ -Higgs bundle

$$\text{gr}(E, Q, \Phi, \tau) = (\text{gr}(E, Q, \Phi), \tau).$$

A family of $\text{Sp}(2m, \mathbb{C})$ -Higgs bundles (resp. a family of $\text{O}(n, \mathbb{C})$ -Higgs bundles) parametrized by T is a triple $(\mathcal{V}, \Theta, \Phi)$, where \mathcal{V} is a family of vector bundles parametrized by T , Θ is a non-degenerate holomorphic symplectic (resp. symmetric) form on \mathcal{V} and Φ is an endomorphism of \mathcal{V} such that anticommutes with Θ . A family of $\text{SO}(n, \mathbb{C})$ -Higgs bundles is a quadruple $(\mathcal{V}, \mathcal{Q}, \Phi, \mathcal{T})$ where $(\mathcal{V}, \mathcal{Q}, \Phi)$ is a family of $\text{O}(n, \mathbb{C})$ -Higgs bundles and \mathcal{T} is a section of $\det \mathcal{V}$ such that $\mathcal{T}^2 = (\det \mathcal{Q})^{-1}$. We denote by $\tilde{P}_m, \mathring{P}_n$ and \bar{P}_n the algebraic condition of the families of semistable $\text{Sp}(2m, \mathbb{C})$, $\text{O}(n, \mathbb{C})$ and $\text{SO}(n, \mathbb{C})$ -Higgs bundles defined above.

As in the vectorial case, for every semistable $\text{Sp}(2m, \mathbb{C})$, $\text{O}(n, \mathbb{C})$ or $\text{SO}(n, \mathbb{C})$ -Higgs bundle, one can construct (using a certain extension of vector bundles) a family parametrized by \mathbb{C} whose restriction to any slice $X \times \{t\}$ with $t \neq 0$ is isomorphic to our starting semistable $\text{Sp}(2m, \mathbb{C})$, $\text{O}(n, \mathbb{C})$ or $\text{SO}(n, \mathbb{C})$ -Higgs bundle while the restriction to $X \times \{0\}$ is isomorphic to the graded object. In order to obtain a separated moduli space we give the following definition of S-equivalence. Let (E, Θ, Φ) and (E', Θ', Φ') be two semistable $\text{Sp}(2m, \mathbb{C})$ -Higgs bundles or $\text{O}(n, \mathbb{C})$ -Higgs bundles, we say that they are *S-equivalent* if $\text{gr}(E, \Theta, \Phi) \cong \text{gr}(E', \Theta', \Phi')$. Analogously, two semistable $\text{SO}(n, \mathbb{C})$ -Higgs bundles $(E_1, Q_1, \Phi_1, \tau_1)$ and $(E_2, Q_2, \Phi_2, \tau_2)$ are *S-equivalent* if and only if $\text{gr}(E_1, Q_1, \Phi_1, \tau_1)$ and $\text{gr}(E_2, Q_2, \Phi_2, \tau_2)$ are isomorphic as $\text{SO}(n, \mathbb{C})$ -Higgs bundles.

We define stability and S-equivalence of families pointwise, as we did for families of Higgs bundle. S-equivalence between families of stable objects implies isomorphism pointwise.

Let us denote by $\tilde{\mathcal{A}}_m, \mathring{\mathcal{A}}_n$ and by $\bar{\mathcal{A}}_n$ the collections of semistable $\text{Sp}(2m, \mathbb{C})$, $\text{O}(n, \mathbb{C})$ and $\text{SO}(n, \mathbb{C})$ -Higgs bundles over X and by $\tilde{\mathcal{A}}_m^{st}, \mathring{\mathcal{A}}_n^{st}$ and $\bar{\mathcal{A}}_n^{st}$ the subcollections of stable ones. We consider the moduli functors defined in (3.5)

$$\begin{aligned} &\text{Mod}(\tilde{\mathcal{A}}_m, \tilde{P}_m, S), \\ &\text{Mod}(\mathring{\mathcal{A}}_n, \mathring{P}_n, S) \end{aligned}$$

and

$$\text{Mod}(\bar{\mathcal{A}}_n, \bar{P}_n, S).$$

It is proved in [Si3] that there exist the moduli spaces associated to these moduli functors that we denote respectively by $\mathfrak{M}(\text{Sp}(2m, \mathbb{C}))$, $\mathfrak{M}(\text{O}(n, \mathbb{C}))$ and by $\mathfrak{M}(\text{SO}(n, \mathbb{C}))$. The points of these moduli spaces either represent S-equivalence classes of semistable objects or isomorphism classes of polystable ones.

Note that a stable Higgs bundle is equal to its graded object and then, S-equivalence between stable Higgs bundles is equivalent to isomorphism. Since S-equivalence is defined pointwise, from the definition of the Jordan-Hölder filtration we know that S-equivalence for families of stable Higgs bundles is the same as isomorphism pointwise. We also consider the moduli functors

$$\begin{aligned} & \text{Mod}(\tilde{\mathcal{A}}_m^{st}, \tilde{P}_m, \overset{pt}{\cong}), \\ & \text{Mod}(\mathring{\mathcal{A}}_n^{st}, \mathring{P}_n, \overset{pt}{\cong}), \\ & \text{Mod}(\overline{\mathcal{A}}_n^{st}, \overline{P}_n, \overset{pt}{\cong}) \end{aligned}$$

and the associated moduli spaces $\mathfrak{M}^{st}(\text{Sp}(2m, \mathbb{C}))$, $\mathfrak{M}^{st}(\text{O}(n, \mathbb{C}))$ and $\mathfrak{M}^{st}(\text{SO}(n, \mathbb{C}))$.

We say that $\mathcal{E} \rightarrow X \times T$, a family of semistable $\text{Sp}(2m, \mathbb{C})$ -Higgs bundles (resp. semistable $\text{O}(n, \mathbb{C})$ -Higgs bundles), is *locally graded* if for every $t \in T$ there exists an open subset $U \subset T$ containing t and a set of families $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_\ell$ parametrized by U , such that \mathcal{F}_0 is a family of stable $\text{Sp}(2m_0, \mathbb{C})$ -Higgs bundles (resp. a family of stable $\text{O}(n_0, \mathbb{C})$ -Higgs bundles) and the \mathcal{F}_i , for $i > 0$, are families of the form

$$\mathcal{F}_i = \left(\mathcal{V}_i \oplus \mathcal{V}_i^*, \begin{pmatrix} 0 & b\vartheta_i^t \\ \vartheta_i & 0 \end{pmatrix}, \begin{pmatrix} \Phi_i & \\ & \Phi_i' \end{pmatrix} \right)$$

where $b = -1$ (resp. $b = 1$), \mathcal{V}_i is a family of stable vector bundles, $\vartheta_i : \mathcal{V}_i \rightarrow \mathcal{V}_i$ is an isomorphism, Φ_i and Φ_i' are, respectively, endomorphisms of \mathcal{V}_i and \mathcal{V}_i^* satisfying $\vartheta_i \Phi_i = -(\Phi_i')^t \vartheta_i$. The set of families is such that for every $t' \in U$ we have

$$\mathcal{E}|_{X \times \{t'\}} \sim_S \bigoplus_{i=0}^{\ell} \mathcal{F}_i|_{X \times \{t'\}}.$$

We say that the locally graded families of semistable $\text{Sp}(2m, \mathbb{C})$ -Higgs bundles (resp. $\text{O}(n, \mathbb{C})$ -Higgs bundles) satisfy the algebraic condition \tilde{Q}_m (resp. \mathring{Q}_n).

Analogously, we say that the family of semistable $\text{SO}(n, \mathbb{C})$ -Higgs bundles $\mathcal{E} \rightarrow X \times T$ is *locally graded* if for every $t \in T$ there exists an open subset $U \subset T$ containing t and families $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_\ell$ parametrized by U , where \mathcal{F}_0 is a family of stable $\text{SO}(n_0, \mathbb{C})$ -Higgs bundles and the \mathcal{F}_i , for $i > 0$, are families the form

$$\mathcal{F}_i = \left(\mathcal{V}_i \oplus \mathcal{V}_i^*, \begin{pmatrix} 0 & \vartheta_i^t \\ \vartheta_i & 0 \end{pmatrix}, \begin{pmatrix} \Phi_i & \\ & \Phi_i' \end{pmatrix}, \mathcal{T}_i \right)$$

where \mathcal{V}_i is a family of stable vector bundles, $\vartheta_i : \mathcal{V}_i \rightarrow \mathcal{V}_i$ an isomorphism, Φ_i and Φ_i' endomorphisms of \mathcal{V}_i and \mathcal{V}_i^* satisfying $\vartheta_i \Phi_i = -(\Phi_i')^t \vartheta_i$ and \mathcal{T}_i^2 a section of $-(\det \vartheta_i)^{-2}$. The family is locally graded if for every $t' \in U$ we have

$$\mathcal{E}|_{X \times \{t'\}} \sim_S \bigoplus_{i=0}^{\ell} \mathcal{F}_i|_{X \times \{t'\}}.$$

We and we write \overline{Q}_n for the algebraic condition of locally graded families of semistable $\text{SO}(n, \mathbb{C})$ -Higgs bundles.

Once we have set up this notation, we consider the moduli functors

$$\text{Mod}(\tilde{\mathcal{A}}_m, \tilde{\mathcal{Q}}_m, S), \quad (3.16)$$

$$\text{Mod}(\mathring{\mathcal{A}}_n, \mathring{\mathcal{Q}}_n, S), \quad (3.17)$$

$$\text{Mod}(\overline{\mathcal{A}}_n, \overline{\mathcal{Q}}_n, S). \quad (3.18)$$

Whenever they exist, we denote by $\mathcal{M}(\text{Sp}(2m, \mathbb{C}))$, $\mathcal{M}(\text{Sp}(2m, \mathbb{C}))$ and $\mathcal{M}(\text{Sp}(2m, \mathbb{C}))$ the moduli spaces associated to these moduli spaces.

Chapter 4

Moduli spaces of Higgs bundles

4.1 Some results for elliptic curves

Let (A, a_0) be a commutative abelian group. If A is a quasiprojective variety and the multiplication map $\mu : A \times A \rightarrow A$ is a morphism, we can define the following morphism

$$\alpha_{A,h} : \text{Sym}^h A \longrightarrow A \quad (4.1)$$

$$[a_1, \dots, a_h]_{\mathfrak{S}_h} \longmapsto \sum_{i=1}^h a_i.$$

Thanks to the group structure of A , the symmetric group \mathfrak{S}_h acts on the $(h-1)$ -th Cartesian product of A . Consider the following morphism between cartesian products of A

$$v_{A,h} : A \times \overset{h-1}{\times} A \longrightarrow A \times \overset{h}{\times} A \quad (4.2)$$

$$(a_1, \dots, a_{h-1}) \longmapsto (a_1, \dots, a_{h-1}, -\sum_{i=1}^{h-1} a_i),$$

and the projection on the first $h-1$ factors

$$q_{A,h} : A \times \overset{h}{\times} A \longrightarrow A \times \overset{h-1}{\times} A$$

$$(a_1, \dots, a_{h-1}, a_h) \longmapsto (a_1, \dots, a_{h-1}).$$

The action of $\sigma \in \mathfrak{S}_h$ on the $(h-1)$ -tuple $(a_1, \dots, a_{h-1}) \in A \times \overset{h-1}{\times} A$ is defined by

$$\sigma \cdot (a_1, \dots, a_{h-1}) = q_{A,h}(\sigma \cdot (v_{A,h}(a_1, \dots, a_{h-1}))). \quad (4.3)$$

Note that the morphism

$$u_{A,h} : (A \times \overset{h-1}{\times} A) / \mathfrak{S}_h \longrightarrow \ker \alpha_{A,h} \quad (4.4)$$

$$[a_1, \dots, a_{h-1}]_{\mathfrak{S}_h} \longmapsto [a_1, \dots, a_{h-1}, -\sum_{i=1}^{h-1} a_i]_{\mathfrak{S}_h}$$

is an isomorphism.

Let us denote by $A[h]$ the subgroup of h -torsion points of (A, a_0) , i.e.

$$A[h] = \{a \in A \text{ such that } a + \overset{.h}{.} + a = a_0\}.$$

The abelian group (A, a_0) acts on $\text{Sym}^h A$ with weight m by the group operation

$$a' \cdot [a_1, \dots, a_h]_{\mathfrak{S}_h} = [m \cdot a' + a_1, \dots, m \cdot a' + a_h]_{\mathfrak{S}_h}. \quad (4.5)$$

If m divides h this action induces an action of the finite subgroup $A[h] \subset A$ on $\text{Sym}^h A$. Note that the action of $A[h]$ preserves the fibres of the map $\alpha_{A,h}$, in particular its kernel.

For any tuple of integers (m_1, \dots, m_ℓ) , we can define a weighted (m_1, \dots, m_ℓ) -action of A on $A \times \dots \times A$. For every $a' \in A$ we define

$$a' \cdot (a_1, \dots, a_\ell) = (a_1 + m_1 a', \dots, a_\ell + m_\ell a').$$

Lemma 4.1.1. *Let (m_1, \dots, m_ℓ) be a tuple of integers and let h be a positive integer. Write r for $\gcd(h, m_1, \dots, m_\ell)$. The weighted (m_1, \dots, m_ℓ) -action of $A[h]$ is free if and only if $r = 1$.*

Proof. Suppose we have $a' \in X[h]$ such that $a' \cdot (a_0, \dots, a_0) = (a_0, \dots, a_0)$. Then $m_i a' = a_0$ for every i . This implies that a' is a m_i -torsion element for every i . On the other hand if there exists $a' \in A[h] \cap \bigcap_i A[m_i]$ different from a_0 , clearly we have that the (m_1, \dots, m_ℓ) -weighted action of a' is trivial and therefore the action of $A[h]$ is not free. We have seen that the action is free if and only if the subgroup $A[h] \cap \bigcap_i A[m_i]$ is trivial.

We can check that $A[n_1] \cap A[n_2] = A[r']$ where $r' = \gcd(n_1, n_2)$. First note that $A[r'] \subset A[n_1] \cap A[n_2]$. To see the other inclusion suppose we have $a' \in A[a] \cap A[b]$. There always exist two integers b_1 and b_2 such that $b_1 n_1 + b_2 n_2 = r'$. We have that $r' a' = b_1 n_1 a' + b_2 n_2 a'$ and since $n_1 a' = a_0$ and $n_2 a' = a_0$ we have $r' a' = a_0$.

It follows easily by induction that, if we set $r = \gcd(h, m_1, \dots, m_\ell)$, we have

$$A[h] \cap \bigcap_i A[m_i] = A[r].$$

□

Let us study the elliptic curve (X, x_0) as an abelian variety. We recall the exact sequence

$$0 \longrightarrow X[h] \longrightarrow X \xrightarrow{f_h} X \longrightarrow 0,$$

where $f_h(x) = hx = x + \overset{.h}{.} + x$. Due to the isomorphism $X / \ker f_h \cong \text{Im } f_h$ we have

$$\tilde{f}_h : X / X[h] \xrightarrow{\cong} X. \quad (4.6)$$

This result leads us to the following description of the quotient by the weighted action for the case of elliptic curves.

Lemma 4.1.2. *Let (X, x_0) be an elliptic curve and consider the weighted (m_1, \dots, m_ℓ) -action of $X[h]$ on $(X \times \dots \times X)$. Then we have an isomorphism of abelian varieties*

$$(X \times \dots \times X) / X[h] \cong X \times \dots \times X.$$

Proof. First we treat the case where one of the m_i is equal to 1; without loss of generality we take $m_1 = 1$. We have a morphism

$$(X \times \dots \times X) / X[h] \longrightarrow X / X[h] \times X \times \dots \times X$$

$$[(x_1, \dots, x_\ell)]_{X[h]} \longmapsto ([x_1]_{X[h]}, x_2 - m_2 x_1, \dots, x_\ell - m_\ell x_1).$$

This is in fact an isomorphism of abelian varieties since it has an inverse

$$X / X[h] \times X \times \dots \times X \longrightarrow (X \times \dots \times X) / X[h]$$

$$([z_1]_{X[h]}, z_2, \dots, z_\ell) \longmapsto [(z_1, m_2 z_1 + z_2, \dots, m_\ell z_1 + z_\ell)]_{X[h]}.$$

By (4.6), the lemma is true when $m_1 = 1$.

We will make use of induction to prove the lemma in general. By (4.6) the lemma is true for length 1. Let us suppose that for length $\ell - 1$ the lemma is true. Write r_1 for $\gcd(m_1, h)$ and note that the quotient of the weighted (m_1, \dots, m_ℓ) -action is

$$(X \times X \times \dots \times X) / X[h] \cong ((X \times X \times \dots \times X) / X[r_1]) / X[h/r_1].$$

Since r_1 divides m_1 , the action of $X[r_1]$ on the first factor is trivial, so

$$(X \times X \times \dots \times X) / X[r_1] \cong X \times (X \times \dots \times X) / X[r_1].$$

By induction, we have

$$X \times (X \times \dots \times X) / X[r_1] \cong X \times X \times \dots \times X.$$

This implies that our starting quotient by the weighted $(m_1, m_2, \dots, m_\ell)$ -action of $X[h]$ is isomorphic to the quotient by a weighted $(m_1, n'_2, \dots, n'_\ell)$ -action of $X[h/r_1]$, where n'_2, \dots, n'_ℓ are some integers. If $\gcd(m_1, (h/r_1))$ is not 1, we repeat the procedure until we obtain

$$(X \times X \times \dots \times X) / X[h] \cong (X \times X \times \dots \times X) / X[h']$$

where the weights of the second quotient are $(m_1, n_2, \dots, n_\ell)$, satisfying $\gcd(m_1, h') = 1$, and again n_2, \dots, n_ℓ are some integers that we do not need to compute. If we choose positive integers p, q such that $pm_1 + qh' = 1$, then this action is equivalent to the action of $X[h']$ with weights $(1, pn_2, \dots, pn_\ell)$. We have already proved that this quotient is isomorphic to $X \times \dots \times X$ and this completes the induction. \square

Since the abelian structure of (X, x_0) is given by the Abel-Jacobi map aj and the tensorization map $t_d^{x_0}$, the morphism $\alpha_{X,h} : \text{Sym}^h X \rightarrow X$ is equal to the composition $(\text{aj}_1)^{-1} \circ (t_{h-1}^{x_0}) \circ (\text{aj}_h)$. This, together with (3.1), implies that $\alpha_{X,h}^{-1}(x) \cong \mathbb{P}^{h-1}$ for every $x \in X$, in particular

$$\ker \alpha_{X,h} \cong \mathbb{P}^{h-1}. \quad (4.7)$$

We can give a description of $\mathrm{Sym}^h X$ different to that of the standard definition. If $\mathcal{P} \rightarrow X \times \mathrm{Pic}^0(X)$ is the Poincaré bundle, then $\mathcal{P} \otimes p_1^* \mathcal{O}(hx_0)$ is a universal family of line bundles of degree h , where p_1 is the projection from $X \times \mathrm{Pic}^h(X)$ to X . If we write p_2 for the other projection, we have that $\mathrm{Sym}^h X \xrightarrow{\mathrm{aj}_h} \mathrm{Pic}^h(X)$ is the projectivization of $(p_2)_*(\mathcal{P} \otimes p_1^* \mathcal{O}(hx_0))$, i.e. the projectivization of the Fourier-Mukai transform of $\mathcal{O}(hx_0)$. The Fourier-Mukai transform of $\mathcal{O}(hx_0)$ is a stable vector bundle of rank h and degree -1 . In Section 3.3 we will see that all the stable vector bundles over an elliptic curve are determined by the determinant, and then there is a unique projective bundle with coprime rank and degree $\mathrm{gcd}(n, d) = 1$. Let us call this projective bundle $P_{(n,d)}$. With this notation we have that

$$(\mathrm{Sym}^h X \rightarrow X) \cong P_{(h,-1)}. \quad (4.8)$$

Remark 4.1.3. Considered as a group, X acts on $\mathrm{Sym}^h X$ as described in (4.5). The action of the finite subgroup $X[h]$ preserves the fibres of $\alpha_{X,h}$ and therefore we have an action of $X[h]$ on $\ker \alpha_{X,h}$. Recalling (4.7), this action induces an action of $X[h]$ on \mathbb{P}^{h-1} . If r divides h we have $X[r] \subset X[h]$ and then $X[r]$ acts on \mathbb{P}^{h-1} as well.

Since $T^*X \cong X \times \mathbb{C}$, when (A, a_0) is $(T^*X, (x_0, 0))$ the elements of h -torsion of T^*X must have the form $(x, 0)$. Then we have

$$(T^*X)[h] \cong X[h],$$

and then $X[h]$ acts on $\ker \alpha_{T^*X,h}$.

4.2 Stability of Higgs bundles in terms of the underlying vector bundle

The triviality of the canonical line bundle simplifies the study of the semistability of Higgs bundles over elliptic curves.

Proposition 4.2.1. *The Higgs bundle (E, Φ) is semistable if and only if E is semistable.*

Proof. If the vector bundle E is semistable, every subbundle F satisfies $\mu(F) \leq \mu(E)$, so the Higgs bundle (E, Φ) is semistable too.

Suppose E is not semistable and take its Harder-Narasimhan filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{s-1} \subset F_s = E,$$

where the $E_i = F_i/F_{i-1}$ are semistable with $\mu(E_i) > \mu(E_j)$ if $i < j$. In particular we have $H^0(X, \mathrm{Hom}(E_i, E_j)) = 0$ if $i < j$.

The subbundle $F_1 = E_1$ has $\mu(F_1) > \mu(E)$. Suppose $\Phi(F_1)$ is non-zero and take F_ℓ such that $\Phi(F_1) \subseteq F_\ell$ but $\Phi(F_1) \not\subseteq F_{\ell-1}$. Then Φ induces a nonzero morphism from $E_1 = F_1$ to $E_\ell = F_\ell/F_{\ell-1}$, but there are no non-zero morphisms unless $\ell = 1$. Then F_1 is Φ -invariant and (E, Φ) is unstable. \square

Corollary 4.2.2. *If $\mathrm{gcd}(n, d) = 1$, then (E, Φ) is stable if and only if E is stable.*

The only possible endomorphisms of a stable vector bundle are multiples of the identity. This allows us to extend Corollary 4.2.2 to the non-coprime case.

Proposition 4.2.3. *(E, Φ) is stable if and only if E is stable.*

Proof. By Proposition 4.2.1 the semistability of a Higgs bundle is equivalent to the semistability of the underlying vector bundle. If (E, Φ) is stable, it is semistable and by Proposition 4.2.1 E is semistable.

Take first E strictly semistable and indecomposable. We have $E \cong E' \otimes F_h$, where $h = \gcd(n, d) > 1$ and E' is stable of rank $n' = \frac{n}{h}$ and degree $d' = \frac{d}{h}$. The endomorphism bundle satisfies

$$H^0(X, \text{End } E) \cong H^0(X, \text{End } E' \otimes \text{End } F_h) \cong \bigoplus_{L_i \in \text{Pic}^0(X)[n']} H^0(X, L_i \otimes \text{End } F_h).$$

We have $\text{End } F_h \cong F_1 \oplus \dots \oplus F_{2h-1}$, and $H^0(X, L_i \otimes F_j) = 0$ for every F_j and every non-trivial $L_i \in \text{Pic}^0(X)[n]$, so $H^0(X, L_i \otimes \text{End } F_h) = 0$ if $L_i \not\cong \mathcal{O}$. This implies that $H^0(X, \text{End } E) \cong H^0(X, \text{End } F_h)$ and every endomorphism of E has the form $\Phi = \text{id}_{E'} \otimes \phi$ for some $\phi \in \text{End } F_h$.

We know that F_h has a unique subbundle isomorphic to \mathcal{O} . Let $\phi \in \text{End } F_h$. If ϕ is non-zero, $\phi(\mathcal{O})$ is a subbundle of F_h with a non-zero holomorphic section, so $\phi(\mathcal{O}) = \mathcal{O}$ and \mathcal{O} is ϕ -invariant. If ϕ is zero, \mathcal{O} is again ϕ -invariant. It follows that $E' \otimes \mathcal{O}$ is a Φ -invariant subbundle of E which contradicts the stability of (E, Φ) .

Now we take E strictly semistable and decomposable, and take (E, Φ) with an arbitrary choice of the Higgs field. Let us fix a stable bundle E' of rank n' and degree d' . We can write

$$E \cong \bigoplus_{j=1}^s (E'_j \otimes F_{h_j}) \cong \bigoplus_{j=1}^s (E' \otimes L'_j \otimes F_{h_j})$$

for some line bundles L'_j of degree 0. If $L'_j \not\cong L'_k$ for some j, k , then

$$\text{Hom}(L'_j \otimes F_{h_j}, L'_k \otimes F_{h_k}) = 0$$

since every Jordan-Hölder factor of $\text{Hom}(L'_j \otimes F_{h_j}, L'_k \otimes F_{h_k})$ is isomorphic to the line bundle $L'^{-1}_j \otimes L'_k$, which is a non-trivial line bundle of degree 0. It follows that (E, Φ) is decomposable.

If all L'_j are isomorphic, we can take $L'_j = \mathcal{O}$ by making a different choice of E' if necessary, so we can take $E = E' \otimes \bigoplus_{j=1}^s F_{h_j}$ and then

$$\text{End } E \cong \left(\bigoplus L_i \right) \otimes \text{End} \left(\bigoplus_{j=1}^s F_{h_j} \right), \quad L'_i \cong \mathcal{O}.$$

Now $\text{End}(\bigoplus_{j=1}^s F_{h_j})$ is a direct sum of various F_ℓ , so $H^0(L_i \otimes \text{End}(\bigoplus_{j=1}^s F_{h_j})) = 0$ unless $L_i \cong \mathcal{O}$. It follows that

$$\text{End } E \cong \text{End} \left(\bigoplus_{j=1}^s F_{h_j} \right), \quad \Phi = \text{id} \otimes \phi, \quad \phi \in H^0 \left(X, \text{End} \left(\bigoplus_{j=1}^s F_{h_j} \right) \right).$$

Now $\bigoplus_{j=1}^s F_{h_j}$ has a unique subbundle \mathcal{O}^s and $H^0(\mathcal{O}^s) \xrightarrow{\cong} H^0(\bigoplus_{j=1}^s F_{h_j})$. So \mathcal{O}^s is ϕ -invariant and $E' \otimes \mathcal{O}^s$ is Φ -invariant. This implies that (E, Φ) is strictly semistable unless all $h_j = 1$ and $E \cong E' \otimes \mathcal{O}^h$. In this case $\phi \in \text{End } \mathcal{O}^s = \{s \times s \text{ matrices}\}$. Now choose an eigenvector v for ϕ . Then $E' \otimes v$ contradicts stability of (E, Φ) and (E, Φ) is strictly semistable. \square

We end this section with a description of polystable Higgs bundles which follows immediately from the previous proposition and the characterization of stable vector bundles and their endomorphism bundles given by Atiyah.

Corollary 4.2.4. *Let (E, Φ) be polystable of rank n and degree d , and $h = \gcd(n, d)$. Then*

$$(E, \Phi) = \bigoplus_{i=1}^h (E_i, \Phi_i)$$

where E_i is a stable bundle of rank $n' = \frac{n}{h}$ and degree $d' = \frac{d}{h}$ and

$$\Phi_i \in H^0(X, \text{End } E_i) \cong H^0(X, \mathcal{O}) \otimes \text{id}_{E_i}.$$

Furthermore (E, Φ) is polystable only if E is polystable.

Remark 4.2.5. Although the polystability of the underlying vector bundle is a necessary condition for the polystability of the Higgs bundle, it is not sufficient.

To illustrate this fact, take (E, Φ) such that $E \cong \mathcal{O} \oplus \mathcal{O}$ and $\Phi = A \otimes 1_X$, where A is non-diagonalizable. It follows that (E, Φ) is an indecomposable Higgs bundle. Since E is a strictly polystable vector bundle, (E, Φ) is not stable. Also, (E, Φ) is indecomposable so it is not possible to express (E, Φ) as a direct sum of stable Higgs bundles.

4.3 Moduli spaces of Higgs bundles

Theorem 4.3.1. *Let n and d be two integers and write $h = \gcd(n, d)$. If $h > 1$, we have*

$$\mathfrak{M}^{\text{st}}(\text{GL}(n, \mathbb{C}))_d = \emptyset.$$

Proof. By Atiyah's results, if $h > 1$, every polystable vector bundle of rank n and degree d is a direct sum of stable vector bundles of rank $n' = n/h$ and degree $d' = d/h$ and therefore is not stable. So there are no stable vector bundles with rank n and degree d , and by Proposition 4.2.3 there are no stable Higgs bundles. \square

Proposition 4.3.2. *Let $\gcd(n, d) = 1$, then the morphism given by the determinant and the trace*

$$(\det, \text{tr}) : \mathfrak{M}(\text{GL}(n, \mathbb{C}))_d \longrightarrow \mathfrak{M}(\text{GL}(1, \mathbb{C}))_d$$

is an isomorphism.

Proof. Recall that, if the rank and degree are coprime, the determinant gives an isomorphism

$$\det : M(\text{GL}(n, \mathbb{C}))_d \xrightarrow{\cong} M(\text{GL}(1, \mathbb{C}))_d.$$

Taking differentials we obtain the following isomorphism

$$(\det, \text{tr}) : T^*M(\text{GL}(n, \mathbb{C}))_d \longrightarrow T^*M(\text{GL}(1, \mathbb{C}))_d.$$

If $M^{st}(\text{GL}(n, \mathbb{C}))$ is non-empty, we know that $T^*M^{st}(\text{GL}(n, \mathbb{C}))_d$ is an open subscheme of $\mathfrak{M}^{st}(\text{GL}(n, \mathbb{C}))_d$. Due to Proposition 4.2.3, every stable Higgs bundle has a stable underlying vector bundle, so the previous inclusion is an equality, $T^*M^{st}(\text{GL}(n, \mathbb{C}))_d = \mathfrak{M}^{st}(\text{GL}(n, \mathbb{C}))_d$. Finally we have that $M^{st}(\text{GL}(n, \mathbb{C}))_d$ is equal to $M(\text{GL}(n, \mathbb{C}))_d$ and $\mathfrak{M}^{st}(\text{GL}(n, \mathbb{C}))_d$ equal to $\mathfrak{M}(\text{GL}(n, \mathbb{C}))_d$. \square

The moduli space of rank 1 Higgs bundles is naturally isomorphic to

$$\mathfrak{M}(\text{GL}(1, \mathbb{C}))_d \cong T^* \text{Pic}^d(X).$$

In particular $\xi_{1,0}^{x_0}$ defined in (3.4) gives $\mathfrak{M}(\text{GL}(1, \mathbb{C}))_0 \cong T^*X$. Let $\widetilde{t}_d^{x_0}$ be the isomorphism between $T^* \text{Pic}^d(X)$ and $T^* \text{Pic}^0(X)$ induced by the translation morphism $t_d^{x_0}$ defined in (3.2). We set for every d the following isomorphism

$$\xi_{1,d}^{x_0} : \mathfrak{M}(\text{GL}(1, \mathbb{C}))_d \xrightarrow{\cong} T^*X$$

defined by $\xi_{1,d}^{x_0} = \xi_{1,0}^{x_0} \circ \widetilde{t}_d^{x_0}$. For pairs of integers (n, d) such that $\gcd(n, d) = 1$ we define $\xi_{n,d}^{x_0} = \xi_{1,d}^{x_0} \circ (\det, \text{tr})$.

Proposition 4.3.3. *If $\gcd(n, d) = 1$, then*

$$\xi_{n,d}^{x_0} : \mathfrak{M}(\text{GL}(n, \mathbb{C}))_d \longrightarrow T^*X \tag{4.9}$$

is an isomorphism.

Proof. This is a straightforward consequence of Proposition 4.3.2 and the existence of the isomorphism $\xi_{1,d}^{x_0}$. \square

Proposition 4.3.4. *Let $\gcd(n, d) = 1$. There exists a universal family $\mathcal{E}_{(n,d)} \rightarrow X \times T^*X$ of stable Higgs bundles of rank n and degree d .*

Proof. It is a standard fact that, when the rank and the degree are coprime, there exists a universal family $\mathcal{V}_{(n,d)}$ of stable vector bundles parametrized by X such that for every $x \in X$ we have $\zeta_{n,d}^{x_0}((\mathcal{V}_{(n,d)})_x) = x$. Now we take a family of Higgs bundles over $X \times \mathbb{C} \cong T^*X$ such that the restriction to $X \times (x, \lambda)$ is given by the pair $((\mathcal{V}_{(n,d)})_x, \frac{1}{n}\lambda \otimes \text{id}_{(\mathcal{V}_{(n,d)})_x})$. We can check that for any $(x, \lambda) \in T^*X$ one has

$$\xi_{n,d}^{x_0}((\mathcal{E}_{(n,d)})_{(x,\lambda)}) = (x, \lambda).$$

To see that $\mathcal{E}_{(n,d)}$ is a universal family take any other family $\mathcal{F} \rightarrow X \times T$, and the induced family of vector bundles \mathcal{W} given by taking the underlying vector bundle of \mathcal{F}_t , for every $t \in T$. Since $\mathcal{V}_{(n,d)}$ is a universal family, there exists a morphism $f_1 : T \rightarrow X$ such that $\mathcal{W} \sim_S \gamma^* \mathcal{V}_{(n,d)}$. The Higgs field of every \mathcal{F}_t is of the form $\frac{1}{n}\lambda_t \otimes \text{id}_{\mathcal{V}_t}$, and so we get a morphism $f_2 : T \rightarrow \mathbb{C}$ that to every point $t \in T$ associates λ_t . Taking $f = (f_1, f_2)$ we obtain a morphism $f : T \rightarrow T^*X$ such that \mathcal{F} is S-equivalent to $(\text{id}_X \times f)^* \mathcal{E}_{(n,d)}$ and this morphism is canonical. \square

Let us study the non-coprime case. Until the end of this section we will consider Higgs bundles of rank n and degree d and we set $h = \gcd(n, d)$, $n' = \frac{n}{h}$ and $d' = \frac{d}{h}$.

Note that the fibre product $\mathcal{E}_{(n',d')} \times_X \mathcal{E}_{(n',d')}$ is parametrized over X by $T^*X \times T^*X$. We write $\mathcal{E}'_{(n,d)}$ for the family $\mathcal{E}_{(n',d')} \times_X \cdot^h \cdot \times_X \mathcal{E}_{(n',d')}$. Thanks to the Proposition 4.3.4 we see that $\mathcal{E}'_{(n,d)}$ is a universal family of stable $(\mathrm{GL}(n', \mathbb{C}) \times \cdot^h \cdot \times \mathrm{GL}(n', \mathbb{C}))$ -Higgs bundles with topological invariant $(d', \cdot^h \cdot, d')$ parametrized by $T^*X \times \cdot^h \cdot \times T^*X$.

Let us consider the injection $i : \mathrm{GL}(n', \mathbb{C}) \times \cdot^h \cdot \times \mathrm{GL}(n', \mathbb{C}) \hookrightarrow \mathrm{GL}(n, \mathbb{C})$. We denote by $\mathcal{E}_{(n,d)}$ the extension of structure group $i_* \mathcal{E}'_{(n,d)}$.

Remark 4.3.5. The family $\mathcal{E}_{(n,d)}$ is parametrized by $Z_h = T^*X \times \cdot^h \cdot \times T^*X$, and if z_1 and z_2 are two points of Z , the Higgs bundles $(\mathcal{E}_{(n,d)})_{z_1}$ and $(\mathcal{E}_{(n,d)})_{z_2}$ are S-equivalent if and only if z_1 is a permutation of z_2 (i.e. if they are related by the natural action of \mathfrak{S}_h on Z_h).

Proposition 4.3.6. *The family $\mathcal{E}_{(n,d)}$ has the local universal property among locally graded families of semistable Higgs bundles.*

Proof. Take \mathcal{F} to be any locally graded family of semistable Higgs bundles of rank n and degree d parametrized by T . Set $h = \gcd(n, d)$, $n' = n/h$ and $d' = d/h$. Since \mathcal{F} is locally graded, for every $t \in T$ there exists an open neighborhood U and a set of families $\mathcal{E}_1, \dots, \mathcal{E}_h$ of stable Higgs bundles of rank n' and degree d' parametrized by U and such that

$$\mathcal{F}|_{X \times U} \sim_S \bigoplus_{i=1}^h \mathcal{E}_i.$$

Since $\mathcal{E}_{(n',d')}$ is a universal family, for every \mathcal{E}_i there exists $f_i : U \rightarrow T^*X$ such that $\mathcal{E}_i \sim_S f_i^* \mathcal{E}_{(n',d')}$. Setting $f = (f_1, \dots, f_h)$, we have

$$\mathcal{F}|_{X \times U} \sim_S f^* \mathcal{E}_{(n,d)}.$$

□

Theorem 4.3.7. *There exists a coarse moduli space of S-equivalence classes of semistable Higgs bundles $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d$ associated to the moduli functor $\mathrm{Mod}(\mathcal{A}_{(n,d)}, Q_{(n,d)}, S)$. We have an isomorphism*

$$\xi_{n,d}^{x_0} : \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d \xrightarrow{\cong} \mathrm{Sym}^h T^*X \quad (4.10)$$

$$[(E, \Phi)]_S \longmapsto [\xi_{n',d'}^{x_0}(E_1, \Phi_1), \dots, \xi_{n',d'}^{x_0}(E_h, \Phi_h)]_{\mathfrak{S}_h}$$

where $\mathrm{gr}(E, \Phi) \cong \bigoplus_{i=1}^h (E_i, \Phi_i)$.

Proof. Since $\mathrm{Sym}^h T^*X = Z_h / \mathfrak{S}_h$ is an orbit space, the theorem is a straightforward consequence of Proposition 4.3.6 and Remark 4.3.5. □

Remark 4.3.8. We can check that the map $\xi_{n,d}^{x_0} : \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d \xrightarrow{\cong} \mathrm{Sym}^h X$ defined in (3.10) is the restriction of $\xi_{n,d}^{x_0}$ to $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d \subset \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d$.

Take the natural projection $p : T^*X \rightarrow X$ and let us define the induced map

$$p^{(h)} : \text{Sym}^h T^*X \longrightarrow \text{Sym}^h X \quad (4.11)$$

$$[(x_1, \lambda_1), \dots, (x_h, \lambda_h)]_{\mathfrak{S}_h} \longmapsto [x_1, \dots, x_h]_{\mathfrak{S}_h}$$

Proposition 4.3.9. *There is a surjective morphism*

$$a_{(n,d)} : \mathcal{M}(\text{GL}(n, \mathbb{C}))_d \longrightarrow M(\text{GL}(n, \mathbb{C}))_d$$

$$[(E, \Phi)]_S \longmapsto [E]_S.$$

Furthermore the diagram

$$\begin{array}{ccc} \mathcal{M}(\text{GL}(n, \mathbb{C}))_d & \xrightarrow{a_{(n,d)}} & M(\text{GL}(n, \mathbb{C}))_d \\ \xi_{n,d}^{x_0} \downarrow \cong & & \cong \downarrow \xi_{n,d}^{x_0} \\ \text{Sym}^h T^*X & \xrightarrow{p^{(h)}} & \text{Sym}^h X \end{array}$$

commutes.

Proof. By Proposition 4.2.1, we know that the Higgs bundle (E, Φ) is semistable if and only if E is a semistable vector bundle. Suppose that the semistable Higgs bundle (E', Φ') is S -equivalent to (E, Φ) . Then we have that $\text{gr}(E', \Phi') \cong \text{gr}(E, \Phi)$. This implies that $\text{gr}(E') \cong \text{gr}(E)$, and then E' belongs to the S -equivalence class $[E]_S$. This proves the existence of the morphism $a_{(n,d)}$. It is straightforward to see that this map is surjective.

The commutativity of the diagram follows from Remark 4.3.8. \square

Let $q_{n,1}, \dots, q_{n,n}$ be the invariant polynomials for a rank n matrix. The Hitchin map is defined evaluating the Higgs field on them

$$b_{(n,d)} : \mathcal{M}(\text{GL}(n, \mathbb{C}))_d \longrightarrow B_n = \bigoplus_{i=1}^n H^0(X, \mathcal{O}) \quad (4.12)$$

$$[(E, \Phi)]_S \longrightarrow (q_{n,1}(\Phi), \dots, q_{n,n}(\Phi)).$$

Since $H^0(X, \mathcal{O}) \cong \mathbb{C}$ we have that $B_n \cong \mathbb{C}^n$. Also, thanks to the invariant polynomials $q_{n,1}, \dots, q_{n,n}$ we can construct a morphism

$$q_n : \mathbb{C}^n \longrightarrow \mathbb{C}^n (\cong B_n) \quad (4.13)$$

$$(\lambda_1, \dots, \lambda_n) \longmapsto (q_{n,1}(D_{\lambda_1, \dots, \lambda_n}), \dots, q_{n,n}(D_{\lambda_1, \dots, \lambda_n})),$$

where $D_{\lambda_1, \dots, \lambda_n}$ is the diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. We can see that this morphism descends to the following isomorphism

$$\bar{q}_n : \text{Sym}^n \mathbb{C} \xrightarrow{\cong} \mathbb{C}^n (\cong B_n) \quad (4.14)$$

$$[\lambda_1, \dots, \lambda_n]_{\mathfrak{S}_n} \longmapsto (q_{n,1}(D_{\lambda_1, \dots, \lambda_n}), \dots, q_{n,n}(D_{\lambda_1, \dots, \lambda_n})).$$

Take $h = \gcd(n, d)$ and set $n' = n/h$ and $d' = d/h$. If (E, Φ) is a rank n polystable Higgs bundle of degree d , we have $(E, \Phi) \cong \bigoplus_{i=1}^h (E_i, \Phi_i)$ where the E_i are stable and $\Phi_i = \frac{1}{n'} \lambda_i \text{id}_{E_i}$; in that case, one has $\xi_{n', d'}^{x_0}([(E_i, \Phi_i)]_S) = (x_i, \lambda_i)$. When we apply $q_{n,i}$ to Φ we obtain polynomials in $\lambda_1, \dots, \lambda_h$

$$\begin{aligned} q_{n,1}(\Phi) &= \sum_{i=1}^h n' \cdot \frac{1}{n'} \lambda_i, \\ &\vdots \\ q_{n,n}(\Phi) &= \left(\frac{1}{n'} \lambda_1 \right)^{n'} \dots \left(\frac{1}{n'} \lambda_h \right)^{n'}. \end{aligned}$$

The image of the Hitchin map is always contained in a submanifold of dimension h that we denote by $B_{(n,d)}$. Suppose that $V_{n,h}$ is the following subspace of \mathbb{C}^n

$$V_{n,h} = \{v \in \mathbb{C}^n : v = (v_1, \dots, v_1, \dots, v_h, \dots, v_h)\}, \quad (4.15)$$

we can easily check that $B_{(n,d)} = q_n(V_{n,h})$. This implies that $B_{(n,d)}$ is smooth.

Setting $h = \gcd(n, d)$, we can consider the following map

$$i_{(n,d)} : \text{Sym}^h \mathbb{C} \xrightarrow{\quad} \text{Sym}^n \mathbb{C} \quad (4.16)$$

$$[\lambda_1, \dots, \lambda_h]_{\mathfrak{S}_h} \longmapsto \left[\frac{1}{n'} \lambda_1, \dots, \frac{1}{n'} \lambda_1, \dots, \frac{1}{n'} \lambda_h, \dots, \frac{1}{n'} \lambda_h \right]_{\mathfrak{S}_n}.$$

Since $B_{(n,d)}$ is smooth, we see that $\beta_{(n,d)} = \bar{q}_n \circ i_{(n,d)}$ is an isomorphism

$$\beta_{(n,d)} : \text{Sym}^h \mathbb{C} \xrightarrow{\cong} B_{(n,d)}. \quad (4.17)$$

Let us define the projection

$$\pi^{(h)} : \text{Sym}^h(T^*X) \xrightarrow{\quad} \text{Sym}^h(\mathbb{C}) \quad (4.18)$$

$$[(x_1, \lambda_1), \dots, (x_h, \lambda_h)]_{\mathfrak{S}_h} \longmapsto [\lambda_1, \dots, \lambda_h]_{\mathfrak{S}_h}.$$

We have all the ingredients to give an explicit description of the Hitchin map.

Lemma 4.3.10. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d & \xrightarrow{b_{(n,d)}} & B_{(n,d)} \\ \xi_{n,d}^{x_0} \downarrow \cong & & \cong \downarrow \beta_{(n,d)}^{-1} \\ \mathrm{Sym}^h(T^*X) & \xrightarrow{\pi^{(h)}} & \mathrm{Sym}^h(\mathbb{C}). \end{array} \quad (4.19)$$

Proof. Let $h = \gcd(n, d)$ and set $n' = n/h$ and $d' = d/h$. By Corollary 4.2.4, given any polystable Higgs bundle (E, Φ) , we have that

$$(E, \Phi) \cong \bigoplus_{i=1}^h (E_i, \lambda_i \cdot \mathrm{id}_{E_i}),$$

where $(E_i, \lambda_i \cdot \mathrm{id}_{E_i})$ are stable Higgs bundles of rank n' and degree d' . For every $i \in \{1, \dots, h\}$ we take $(x_i, \frac{1}{n'} \lambda_i) = \xi_{n',d'}^{x_0}((E_i, \Phi_i))$. This implies that

$$(\pi^{(h)} \circ \xi_{n,d}^{x_0})([(E, \Phi)]_S) = \left[\frac{1}{n'} \lambda_1, \dots, \frac{1}{n'} \lambda_h \right]_{\mathfrak{S}_h}.$$

Recalling $b_{(n,d)}$ and \bar{q}_n defined in (4.12) and (4.14) we see that

$$(\bar{q}_n^{-1} \circ b_{(n,d)})([(E, \Phi)]_S) = [\lambda_1, \dots, \lambda_1, \dots, \lambda_{n'}, \dots, \lambda_{n'}]_{\mathfrak{S}_n}.$$

This tuple lies in the image of $i_{(n,d)}$, we have

$$(i_{(n,d)}^{-1} \circ \bar{q}_n^{-1} \circ b_{(n,d)})([(E, \Phi)]_S) = \left[\frac{1}{n'} \lambda_1, \dots, \frac{1}{n'} \lambda_h \right]_{\mathfrak{S}_h},$$

which is precisely $(\beta_{(n,d)}^{-1} \circ b_{(n,d)})([(E, \Phi)]_S)$. □

The generic element of $B_{(n,d)}$ is given by the \mathfrak{S}_h -orbit of the following h -tuple

$$\bar{\lambda}_{gen} = (\lambda_1, \dots, \lambda_h), \quad (4.20)$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$.

Lemma 4.3.11.

$$(\pi^{(h)})^{-1}([\bar{\lambda}_{gen}]_{\mathfrak{S}_h}) \cong X \times \dots \times X.$$

Proof. The centralizer $Z_{\mathfrak{S}_h}(\bar{\lambda}_{gen})$ of $\bar{\lambda}_{gen}$ in \mathfrak{S}_h is trivial. Hence the centralizer of any element of $T^*X \times \dots \times T^*X$ of the form

$$((x_1, \lambda_1), \dots, (x_h, \lambda_h)).$$

is also trivial. If two tuples $((x_1, \lambda_1), \dots, (x_h, \lambda_h))$ and $((x'_1, \lambda_1), \dots, (x'_h, \lambda_h))$ lie in the same \mathfrak{S}_h -orbit, then they are related by the action of an element of $Z_{\mathfrak{S}_h}(\bar{\lambda}_{gen})$. Since this group is trivial, we have that $(\pi^{(h)})^{-1}([\bar{\lambda}_{gen}]_{\mathfrak{S}_h})$ is given by the subset of $T^*X \times \dots \times T^*X$ which projects to $(\lambda_1, \dots, \lambda_h)$, which is isomorphic to $X \times \dots \times X$. □

An arbitrary point of $B_{(n,d)}$ is given by a h -tuple of the following form

$$\bar{\lambda}_{arb} = (\lambda_1, \dots, \lambda_1, \dots, \lambda_\ell, \dots, \lambda_\ell),$$

where $h = m_1 + \dots + m_\ell$.

Recall that $P_{(m,-1)}$ is the unique stable projective bundle over X with rank m and degree -1 .

Lemma 4.3.12.

$$(\pi^{(h)})^{-1}([\bar{\lambda}_{arb}]_{\mathfrak{S}_h}) = P_{(m_1,-1)} \times \dots \times P_{(m_\ell,-1)}.$$

Proof. The centralizer of $\bar{\lambda}_{arb}$ in \mathfrak{S}_h is

$$Z_{\mathfrak{S}_h}(\bar{\lambda}_{arb}) = \mathfrak{S}_{m_1} \times \dots \times \mathfrak{S}_{m_\ell},$$

where the factor \mathfrak{S}_{m_i} acts only on the entries of $\bar{\lambda}_{arb}$ equal to λ_i .

Two tuples of $T^*X \times \dots \times T^*X$ that project to $\bar{\lambda}_{arb}$ lie in the same \mathfrak{S}_h -orbit if and only if some element of $Z_{\mathfrak{S}_h}(\bar{\lambda}_{arb})$ sends one tuple to the other. Let us write $(T^*X \times \dots \times T^*X)_{\bar{\lambda}_{arb}}$ for the set of tuples as above that project to $\bar{\lambda}_{arb}$.

We have

$$(\pi^{(h)})^{-1}([\bar{\lambda}_{arb}]_{\mathfrak{S}_h}) \cong (T^*X \times \dots \times T^*X)_{\bar{\lambda}_{arb}} / Z_{\mathfrak{S}_h}(\bar{\lambda}_{arb}),$$

where the action of \mathfrak{S}_{m_i} permutes the entries of a tuple that are pairs of the form (x_{ij}, λ_i) .

We can easily see that $(T^*X \times \dots \times T^*X)_{\bar{\lambda}_{arb}} \cong X \times \dots \times X$ and then

$$\begin{aligned} (\pi^{(h)})^{-1}([\bar{\lambda}_{arb}]_{\mathfrak{S}_h}) &\cong (X \times \dots \times X) / \mathfrak{S}_{m_1} \times \dots \times \mathfrak{S}_{m_\ell} \\ &\cong \text{Sym}^{m_1} X \times \dots \times \text{Sym}^{m_\ell} X. \end{aligned}$$

□

Corollary 4.3.13. *The generic Hitchin fibre of $\mathcal{M}(\text{GL}(n, \mathbb{C}))_d \rightarrow B_{(n,d)}$ is the abelian variety $X \times \dots \times X$, while the arbitrary Hitchin fibre is a holomorphic fibration over the abelian variety $X \times \dots \times X$ with fibre $\mathbb{P}^{m_1-1} \times \dots \times \mathbb{P}^{m_\ell-1}$.*

Let us consider the involution of the moduli space of Higgs bundles

$$\ell_{n,d} : \mathcal{M}(\text{GL}(n, \mathbb{C}))_d \longrightarrow \mathcal{M}(\text{GL}(n, \mathbb{C}))_d \quad (4.21)$$

$$[(E, \Phi)]_S \longmapsto [(E, -\Phi)]_S.$$

When we restrict to degree zero, we have the following involutions

$$\iota_n : \mathcal{M}(\text{GL}(n, \mathbb{C}))_0 \longrightarrow \mathcal{M}(\text{GL}(n, \mathbb{C}))_0 \quad (4.22)$$

$$[(E, \Phi)]_S \longmapsto [(E^*, -\Phi^t)]_S,$$

and

$$j_n : \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0 \longrightarrow \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0 \quad (4.23)$$

$$[(E, \Phi)]_S \longmapsto [(E^*, \Phi^t)]_S.$$

We also consider the involutions

$$l_h : \mathrm{Sym}^h T^* X \longrightarrow \mathrm{Sym}^h T^* X \quad (4.24)$$

$$[(x_1, \lambda_1), \dots, (x_h, \lambda_h)]_{\mathfrak{S}_h} \longmapsto [(x_1, -\lambda_1), \dots, (x_h, -\lambda_h)]_{\mathfrak{S}_h},$$

$$i_n : \mathrm{Sym}^n T^* X \longrightarrow \mathrm{Sym}^n T^* X \quad (4.25)$$

$$[(x_1, \lambda_1), \dots, (x_n, \lambda_n)]_{\mathfrak{S}_n} \longmapsto [(-x_1, -\lambda_1), \dots, (-x_n, -\lambda_n)]_{\mathfrak{S}_n},$$

and

$$j_n : \mathrm{Sym}^n T^* X \longrightarrow \mathrm{Sym}^n T^* X \quad (4.26)$$

$$[(x_1, \lambda_1), \dots, (x_n, \lambda_n)]_{\mathfrak{S}_n} \longmapsto [(-x_1, \lambda_1), \dots, (-x_n, \lambda_n)]_{\mathfrak{S}_n}.$$

The following result gives an explicit description of the involutions $\ell_{n,d}$, i_n and j_n .

Lemma 4.3.14. *Let $h = \mathrm{gcd}(n, d)$. The following diagrams commute*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d & \xrightarrow[\cong]{\xi_{n,d}^{x_0}} & \mathrm{Sym}^h T^* X \\ \ell_{n,d} \downarrow & & \downarrow l_h \\ \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d & \xrightarrow[\cong]{\xi_{n,d}^{x_0}} & \mathrm{Sym}^h T^* X, \end{array}$$

$$\begin{array}{ccc} \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0 & \xrightarrow[\cong]{\xi_{n,0}^{x_0}} & \mathrm{Sym}^n T^* X \\ i_n \downarrow & & \downarrow i_n \\ \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0 & \xrightarrow[\cong]{\xi_{n,0}^{x_0}} & \mathrm{Sym}^n T^* X. \end{array}$$

and

$$\begin{array}{ccc} \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0 & \xrightarrow[\cong]{\xi_{n,0}^{x_0}} & \mathrm{Sym}^n T^* X \\ j_n \downarrow & & \downarrow j_n \\ \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0 & \xrightarrow[\cong]{\xi_{n,0}^{x_0}} & \mathrm{Sym}^n T^* X. \end{array}$$

Proof. Set $n' = n/h$ and $d' = d/h$. Take $(x, \lambda) = \xi_{n',d'}^{x_0}(E_i, \Phi_i)$. The first diagram is commutative since $\xi_{n',d'}^{x_0}(E_i, -\Phi_i) = (x_i, -\lambda_i)$.

If the image under $\xi_{1,0}^{x_0}$ of (L, ϕ) is (x, λ) , we see that the image of $(L^*, -\phi)$ is $(-x, -\lambda)$ and the image of (L^*, ϕ) is $(-x, \lambda)$. The commutativity of the last two diagrams is a straightforward consequence of this fact. \square

We finish the section studying $\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d$, the moduli space associated to the moduli functor $\mathrm{Mod}(\mathcal{A}_{(n,d)}, P_{(n,d)}, S)$.

Proposition 4.3.15. *We have a bijective morphism $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d \rightarrow \mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d$, hence $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d$ is the normalization of $\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d$.*

Proof. The family $\mathcal{E}_{(n,d)}$ induces a morphism

$$\nu_{\mathcal{E}_{(n,d)}} : T^*X \times \dots \times T^*X \longrightarrow \mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d,$$

and by Remark 4.3.5 it factors through

$$\nu'_{\mathcal{E}_{(n,d)}} : \mathrm{Sym}^h(T^*X) \longrightarrow \mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d.$$

Let us denote by $\overline{\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d}$ the normalization of $\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d$. We have that $\mathrm{Sym}^h(T^*X)$ is normal. Then, by the universal property of the normalization, $\nu'_{\mathcal{E}_{(n,d)}}$ factors through

$$\nu''_{\mathcal{E}_{(n,d)}} : \mathrm{Sym}^h(T^*X) \longrightarrow \overline{\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d}.$$

This map is an isomorphism since it is generically bijective and $\overline{\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d}$ is normal. Then $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_d$ is the normalization of $\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d$. \square

Remark 4.3.16. Both moduli spaces would be isomorphic if $\mathfrak{M}(\mathrm{GL}(n, \mathbb{C}))_d$ is normal, but normality in this case is an open question.

4.4 Moduli spaces of $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundles

Theorem 4.4.1.

$$\mathfrak{M}^{st}(\mathrm{SL}(1, \mathbb{C})) = \mathcal{M}(\mathrm{SL}(1, \mathbb{C})) = \{[(\mathcal{O}, 0)]_S\},$$

and for $n > 1$

$$\mathfrak{M}^{st}(\mathrm{SL}(n, \mathbb{C})) = \emptyset.$$

Proof. When the determinant is trivial the vector bundle has zero degree. The only stable Higgs bundles with zero degree are the Higgs line bundles and therefore the only stable $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundle is the unique element of rank 1. \square

Recall the map $\alpha_{T^*X, n} : \mathrm{Sym}^n T^*X \rightarrow T^*X$ defined in (4.1).

Proposition 4.4.2. *The diagram*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0 & \xrightarrow[\cong]{\xi_{n,0}^{x_0}} & \mathrm{Sym}^n T^*X \\ \downarrow (\det, \mathrm{tr}) & & \downarrow \alpha_{T^*X, n} \\ \mathfrak{M}(\mathrm{GL}(1, \mathbb{C}))_0 & \xrightarrow[\cong]{\xi_{1,0}^{x_0}} & T^*X \end{array}$$

commutes.

Proof. Take an arbitrary $[(E, \Phi)]_S \in \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0$. We know that $\mathrm{gr}(E, \Phi)$ is isomorphic to $\bigoplus_{i=1}^n (L_i, \lambda_i \mathrm{id}_{L_i})$. We have that $(\det, \mathrm{tr}) \mathrm{gr}(E, \Phi) = (\bigotimes_{i=1}^n L_i, (\sum_{i=1}^n \lambda_i) \cdot \mathrm{id}_{\det E})$ and since $\xi_{1,0}^{x_0}$ is a morphism of quasiprojective varieties preserving the group structure

$$\xi_{1,0}^{x_0} \left(\bigotimes_{i=1}^n L_i, \left(\sum_{i=1}^n \lambda_i \right) \cdot \mathrm{id}_{\det E} \right) = \left(\sum_{i=1}^n x_i, \sum_{i=1}^n \lambda_i \right) = \sum_{i=1}^n (x_i, \lambda_i),$$

where $x_i = \xi_{1,0}^{x_0}(L_i)$. We also have $\alpha_{T^*X,n} \circ \xi_{n,0}^{x_0}(\mathrm{gr}(E, \Phi)) = \sum_{i=1}^n (x_i, \lambda_i)$ and this proves the commutativity of the diagram. \square

Recall the action defined in (4.3).

Theorem 4.4.3. *There exists a coarse moduli space $\mathcal{M}(\mathrm{SL}(n, \mathbb{C}))$ of S -equivalence classes of semistable $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundles associated to the moduli functor $\mathrm{Mod}(\hat{\mathcal{A}}_n, Q_{(n,0)}, S)$. There is an isomorphism*

$$\hat{\xi}_n^{x_0} : \mathcal{M}(\mathrm{SL}(n, \mathbb{C})) \xrightarrow{\cong} (T^*X \times {}^{n-1}T^*X) / \mathfrak{S}_n. \quad (4.27)$$

Proof. The existence of $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0$ follows by Theorem 4.3.7. Then we have by (3.14) that $\mathcal{M}(\mathrm{SL}(n, \mathbb{C}))$ exists and it is isomorphic to $(\det, \mathrm{tr})^{-1}(\mathcal{O}, 0)$.

We write $(\xi_n^{x_0})'$ for the restriction of the map $\xi_{n,0}^{x_0}$ to the fibre $(\det, \mathrm{tr})^{-1}(\mathcal{O}, 0)$. Recall that from Proposition 4.4.2 we have that $(\xi_n^{x_0})'$ is an isomorphism

$$(\xi_n^{x_0})' : \mathcal{M}(\mathrm{SL}(n, \mathbb{C})) \xrightarrow{\cong} \ker \alpha_{T^*X,n}.$$

Let us write $\hat{\xi}_n^{x_0}$ for the isomorphism $u_{T^*X,n}^{-1} \circ (\xi_n^{x_0})'$ where $u_{T^*X,n}$ is the isomorphism between $\ker \alpha_{T^*X,n}$ and $(T^*X \times {}^{n-1}T^*X) / \mathfrak{S}_n$ defined in (4.4). \square

We can describe the moduli space of $\mathrm{SL}(n, \mathbb{C})$ -bundles using the fact that $M(\mathrm{SL}(n, \mathbb{C}))$ is a subvariety of $\mathcal{M}(\mathrm{SL}(n, \mathbb{C}))$.

Remark 4.4.4. Thanks to Remark 4.3.8 and Proposition 4.4.2 we have a commutative diagram

$$\begin{array}{ccc} M(\mathrm{GL}(n, \mathbb{C}))_0 & \xrightarrow[\cong]{\xi_{n,0}^{x_0}} & \mathrm{Sym}^n X \\ \det \downarrow & & \downarrow \alpha_{X,n} \\ \mathrm{Pic}^0 X & \xrightarrow[\cong]{\xi_{1,0}^{x_0}} & X. \end{array}$$

As a consequence

$$M(\mathrm{SL}(n, \mathbb{C})) \cong \ker \alpha_{X,n} \cong \mathbb{P}^{n-1},$$

as is described in [Tu].

If we define $\hat{\xi}_n^{x_0}$ to be the restriction of $\hat{\xi}_n^{x_0}$ to $M(\mathrm{SL}(n, \mathbb{C})) \subset \mathcal{M}(\mathrm{SL}(n, \mathbb{C}))$ we have

$$\hat{\xi}_n^{x_0} : M(\mathrm{SL}(n, \mathbb{C})) \xrightarrow{\cong} (X \times {}^{n-1}X) / \mathfrak{S}_n. \quad (4.28)$$

The quotient above corresponds with the description given in [FM1] of the moduli space $M(\mathrm{SL}(n, \mathbb{C})) \cong (X \otimes_{\mathbb{Z}} \Lambda) / W$ where Λ and W are respectively the coroot lattice and the Weyl group of $\mathfrak{sl}(n, \mathbb{C})$.

Consider the following map

$$\hat{p}_n : (T^*X \times {}^{n-1}T^*X) / \mathfrak{S}_n \longrightarrow (X \times {}^{n-1}X) / \mathfrak{S}_n$$

$$[(x_1, \lambda_1), \dots, (x_n, \lambda_n)]_{\mathfrak{S}_n} \longmapsto [x_1, \dots, x_n]_{\mathfrak{S}_n}$$

Proposition 4.4.5. *There is a surjective morphism*

$$\hat{a}_n : \mathcal{M}(\mathrm{SL}(n, \mathbb{C})) \longrightarrow M(\mathrm{SL}(n, \mathbb{C}))$$

$$[(E, \Phi)]_S \longmapsto [E]_S.$$

Furthermore the diagram

$$\begin{array}{ccc} \mathcal{M}(\mathrm{SL}(n, \mathbb{C})) & \xrightarrow{\hat{a}_n} & M(\mathrm{SL}(n, \mathbb{C}))_d \\ \xi_n^{x_0} \downarrow \cong & & \cong \downarrow \xi_n^{x_0} \\ \ker \alpha_{T^*X, h} & \xrightarrow{\hat{p}_n} & \ker \alpha_{X, h}. \end{array}$$

commutes

Proof. This follows from Proposition 4.3.9 and the fact that $\mathcal{M}(\mathrm{SL}(n, \mathbb{C}))$ is a subvariety of $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0$. \square

Remark 4.4.6. In the case of rank $n = 2$, \mathfrak{S}_2 is \mathbb{Z}_2 , and the non-trivial element of \mathbb{Z}_2 sends (x, λ) to $(-x, -\lambda)$. Then

$$\mathcal{M}(\mathrm{SL}(2, \mathbb{C})) \cong T^*X / \mathbb{Z}_2, \quad (4.29)$$

and

$$M(\mathrm{SL}(2, \mathbb{C})) \cong X / \mathbb{Z}_2 \cong \mathbb{P}^1. \quad (4.30)$$

The Hitchin map \hat{b}_n for $\mathrm{SL}(n, \mathbb{C})$ -Higgs bundles is defined as the restriction to the subset $(\det, \mathrm{tr})^{-1}([\mathcal{O}, 0]_S)$ of $\mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0$ of the Hitchin map $b_{(n,0)}$, which was defined in (4.12). We denote by \hat{B}_n the subset of $B_n = \bigoplus_{i=1}^n H^0(X, \mathcal{O})$ given by the the image of \hat{b}_n (i.e. the image of $b_{(n,d)}$ restricted to traceless Higgs fields). We can see that $\hat{B}_n \cong \mathbb{C}^{n-1}$, and, using the isomorphism $\bar{q}_n : \mathrm{Sym}^n \mathbb{C} \rightarrow B_n$ defined in (4.14), we have that $(\bar{q}_n)^{-1}(\hat{B}_n) = \ker \alpha_{\mathbb{C}, n} \subset \mathrm{Sym}^n \mathbb{C}$, where $\alpha_{\mathbb{C}, n}$ is defined in (4.1). Thanks to (4.4) we know that $u_{\mathbb{C}, n} : (\mathbb{C} \times {}^{n-1}\mathbb{C}) / \mathfrak{S}_n \rightarrow \ker \alpha_{\mathbb{C}, n}$ is an isomorphism. Then, when we take the restriction to $\hat{B}_n \subset B_n$ of the composition $\hat{\beta}_n = \bar{q}_n \circ u_{\mathbb{C}, n}$ we obtain an isomorphism

$$\hat{\beta}_n : \mathbb{C} \times {}^{n-1}\mathbb{C} / \mathfrak{S}_n \xrightarrow{\cong} \hat{B}_n.$$

We define the map

$$\hat{\pi}_n : T^*X \times {}^{n-1}T^*X / \mathfrak{S}_n \longrightarrow \mathbb{C} \times {}^{n-1}\mathbb{C} / \mathfrak{S}_n$$

$$[(x_1, \lambda_1), \dots, (x_{n-1}, \lambda_{n-1})]_{\mathfrak{S}_n} \longmapsto [\lambda_1, \dots, \lambda_{n-1}]_{\mathfrak{S}_n}.$$

Lemma 4.4.7. *The diagram*

$$\begin{array}{ccc}
\mathcal{M}(\mathrm{SL}(n, \mathbb{C})) & \xrightarrow{\hat{b}_n} & \hat{B}_n \\
\hat{\xi}_n^{x_0} \downarrow \cong & & \cong \downarrow \hat{\beta}_n^{-1} \\
T^*X \times \overset{n-1}{\times} T^*X / \mathfrak{S}_n & \xrightarrow{\hat{\pi}_n} & \mathbb{C} \times \overset{n-1}{\times} \mathbb{C} / \mathfrak{S}_n
\end{array} \tag{4.31}$$

commutes.

Proof. Recall the map $\pi^{(n)}$ defined in (4.18). Let us write π'_n for the restriction of $\pi^{(n)}$ to $\ker \alpha_{T^*X, n}$, so that the image of π'_n is contained in $\ker \alpha_{X, n}$.

We see that $\hat{\pi}_n = u_{\mathbb{C}, n}^{-1} \circ \pi'_n \circ u_{T^*X, n}$ and we recall that $\hat{\xi}_n^{x_0} = u_{T^*X, n}^{-1} \circ (\xi_{n, 0}^{x_0})'$. The commutativity of the diagram follows from the commutativity of (4.19). \square

The generic element of \hat{B}_n comes from a $(n-1)$ -tuple of the form

$$\bar{\lambda}_{gen} = (\lambda_1, \dots, \lambda_{n-1}), \tag{4.32}$$

such that $\lambda_i \neq \lambda_j$ if $i \neq j$ and $\lambda_i \neq \lambda_n$ where $\lambda_n = -\sum_{i=1}^{n-1} \lambda_i$.

Lemma 4.4.8.

$$\hat{\pi}_n^{-1}([\bar{\lambda}_{gen}]_{\mathfrak{S}_n}) \cong X \times \overset{n-1}{\times} X.$$

Proof. We see that the centralizer of $\bar{\lambda}_{gen}$ in \mathfrak{S}_n is trivial. Since

$$\hat{\pi}_n^{-1}([\bar{\lambda}_{gen}]_{\mathfrak{S}_n}) \cong ((X, \lambda_1) \times \overset{n-1}{\times} (X, \lambda_{n-1})) / Z_{\mathfrak{S}_n}(\bar{\lambda}_{gen}),$$

we obtain the result. \square

An arbitrary point of \hat{B}_n is given by a h -tuple of the form

$$\bar{\lambda}_{arb} = (\lambda_1, \overset{m_1}{\dots}, \lambda_1, \dots, \lambda_{\ell-1}, \overset{m_{\ell-1}}{\dots}, \lambda_{\ell-1}, \lambda_{\ell}, \overset{m_{\ell}-1}{\dots}, \lambda_{\ell}), \tag{4.33}$$

where ℓ is the number of different λ_i when we include λ_n , and m_{ℓ} is the multiplicity of λ_{ℓ} when we include λ_n .

Denote by s_{ℓ} the morphism

$$s_{\ell} : X \times \overset{\ell}{\times} X \longrightarrow X \tag{4.34}$$

$$(x_1, \dots, x_{\ell}) \longmapsto \sum_{i=1}^{\ell} x_i.$$

We have an isomorphism

$$w_{\ell} : X \times \overset{\ell-1}{\times} X \longrightarrow \ker s_{\ell} \tag{4.35}$$

$$(x_1, \dots, x_{\ell-1}) \longmapsto (x_1, \dots, x_{\ell-1}, -\sum x_i).$$

Lemma 4.4.9. *Let $\overline{\lambda}_{arb}$ be as in (4.33), then*

$$\hat{\pi}_n^{-1}([\overline{\lambda}_{arb}]_{\mathfrak{S}_n}) \cong w_\ell^*((P_{(m_1, -1)} \times \cdots \times P_{(m_\ell, -1)})|_{\ker s_\ell}).$$

Proof. Since $\hat{\pi}_n$ is the restriction of $\pi^{(n)}$ to $\alpha_{T^*X, n}^{-1}(x_0, 0) \subset \text{Sym}^n T^*X$, then the fibre of an arbitrary element is the restriction of the fibre of $\pi^{(n)}$ to the subset of tuples whose factors sum to x_0 , that is, the kernel of s_ℓ . Thanks to the description of $(\pi^{(n)})^{-1}([\overline{\lambda}_{arb}]_{\mathfrak{S}_n})$ given in Lemma 4.3.12 we have

$$\hat{\pi}_n^{-1}([\overline{\lambda}_{arb}]_{\mathfrak{S}_n}) \cong (P_{m_1} \times \cdots \times P_{m_\ell})|_{\ker s_\ell}.$$

Finally we take the pull-back by w_ℓ to see $\hat{\pi}_n^{-1}([\overline{\lambda}_{arb}]_{\mathfrak{S}_n})$ as a fibration over $X \times \ell\text{-!} \times X$. \square

Corollary 4.4.10. *The generic Hitchin fibre of $\mathcal{M}(\text{SL}(n, \mathbb{C})) \rightarrow \hat{B}_n$ is the abelian variety $X \times n\text{-!} \times X$. The arbitrary Hitchin fibre is a holomorphic fibration over the abelian variety $X \times \ell\text{-!} \times X$ with fibre $\mathbb{P}^{m_1-1} \times \ell\text{-!} \times \mathbb{P}^{m_\ell-1}$.*

We finish the section studying $\mathfrak{M}(\text{SL}(n, \mathbb{C}))$, the moduli space associated to the moduli functor $\text{Mod}(\hat{\mathcal{A}}_n, P_{(n,0)}, S)$. We need to define a family of $\text{SL}(n, \mathbb{C})$ -Higgs bundles.

Let us recall the map $v_{T^*X, n}$ defined in (4.2)

$$v_{T^*X, n} : (T^*X \times n\text{-!} \times T^*) \longrightarrow (T^*X \times n\text{-!} \times T^*).$$

We denote by $\mathcal{E}'_{(n,0)}$ the restriction to $X \times \text{Im}(v_{T^*X, n})$ of the family of Higgs bundles $\mathcal{E}_{(n,0)}$. We see that $\mathcal{E}'_{(n,0)}$ is a family of $\text{SL}(n, \mathbb{C})$ -Higgs bundles. We set

$$\hat{\mathcal{E}}_n = (\text{id}_X \times v_{T^*X, n})^* \mathcal{E}'_{(n,0)}.$$

Remark 4.4.11. For two points $z_1, z_2 \in T^*X \times n\text{-!} \times T^*X$ we have $\hat{\mathcal{E}}_n|_{X \times \{z_1\}} \cong \hat{\mathcal{E}}_n|_{X \times \{z_2\}}$ if and only if there exists $\sigma \in \mathfrak{S}_n$ such that $z_2 = \sigma \cdot z_1$. Since all the $\text{SL}(n, \mathbb{C})$ -Higgs bundles parametrized by $\hat{\mathcal{E}}_n$ are polystable, isomorphism implies S -equivalence.

Proposition 4.4.12. *We have a bijective morphism $\mathcal{M}(\text{SL}(n, \mathbb{C})) \rightarrow \mathfrak{M}(\text{SL}(n, \mathbb{C}))$, hence $\mathcal{M}(\text{SL}(n, \mathbb{C}))$ is the normalization of $\mathfrak{M}(\text{SL}(n, \mathbb{C}))$.*

Proof. The family $\hat{\mathcal{E}}_n$ induces a morphism

$$\nu_{\hat{\mathcal{E}}_n} : T^*X \times n\text{-!} \times T^*X \longrightarrow \mathfrak{M}(\text{SL}(n, \mathbb{C})),$$

and by Remark 4.4.11 it factors through

$$\nu'_{\hat{\mathcal{E}}_n} : (T^*X \times n\text{-!} \times T^*X) / \mathfrak{S}_n \longrightarrow \mathfrak{M}(\text{SL}(n, \mathbb{C}))$$

which is bijective.

Let us denote by $\overline{\mathfrak{M}(\text{SL}(n, \mathbb{C}))}$ the normalization of $\mathfrak{M}(\text{SL}(n, \mathbb{C}))$. The quasiprojective variety $(T^*X \times n\text{-!} \times T^*X) / \mathfrak{S}_n$ is normal since it is the quotient of a smooth (and

therefore normal) variety by a discrete group. Then, by the universal property of the normalization, $\nu'_{\hat{\mathcal{E}}_n}$ factors through

$$\nu''_{\hat{\mathcal{E}}_n} : (T^*X \times \overset{n-1}{\dots} \times T^*X) / \mathfrak{S}_n \longrightarrow \overline{\mathfrak{M}(\mathrm{SL}(n, \mathbb{C}))}.$$

This map is an isomorphism since it is a bijection and $\overline{\mathfrak{M}(\mathrm{SL}(n, \mathbb{C}))}$ is normal. Then $\mathcal{M}(\mathrm{SL}(n, \mathbb{C}))$ is the normalization of $\mathfrak{M}(\mathrm{SL}(n, \mathbb{C}))$. \square

Remark 4.4.13. Both moduli spaces would be isomorphic if $\mathfrak{M}(\mathrm{SL}(n, \mathbb{C}))$ is normal, but normality in this case is an open question.

4.5 Moduli spaces of $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundles

Since a $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundle $(\mathbb{P}(E), \Phi)$ is stable if and only if the Higgs bundle (E, Φ) is stable, we have the following.

Theorem 4.5.1.

$$\mathfrak{M}^{st}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} = \emptyset$$

unless $n \in \mathbb{Z}^+$ and $\tilde{d} \in \mathbb{Z}_n$ are coprime. In that case

$$\mathfrak{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} = \mathfrak{M}^{st}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} \cong \{pt\}.$$

Proof. We recall from Theorems 4.3.1 and 4.3.3 that there exist stable Higgs bundles of rank n and degree d if and only if $\mathrm{gcd}(n, d) = 1$.

Let $(\mathbb{P}(E), \Phi)$ be a stable $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundle, then (E, Φ) is a stable Higgs bundle and by Proposition 4.2.3 E is stable. Since E is stable $H^0(X, \mathrm{End} E) = \{\lambda \cdot \mathrm{id}_E : \lambda \in \mathbb{C}\}$ and so, the only $\Phi \in H^0(X, \mathrm{End} E)$ with $\mathrm{tr} \Phi = 0$ is $\Phi = 0$. Also, by [A] we know that $\det : M(\mathrm{GL}(n, \mathbb{C}))_d \xrightarrow{\cong} M(\mathrm{GL}(n, \mathbb{C}))_d$. Then, every stable vector bundle of rank n and degree d isomorphic to $L \otimes E$.

From the previous discussion we know that every $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundle is isomorphic to

$$(\mathbb{P}(L \otimes E), 0) \cong (\mathbb{P}(E), 0).$$

It follows that the moduli space is a point. \square

Let $h = \mathrm{gcd}(n, d)$. Recall that $T^*X \cong X \times \mathbb{C}$ and then we have the map $\pi : T^*X \rightarrow \mathbb{C}$. We take

$$\pi^h : T^*X \times \overset{h}{\dots} \times T^*X \longrightarrow \mathbb{C} \times \overset{h}{\dots} \times \mathbb{C}$$

and we compose it with the addition map

$$w_{\mathbb{C}, h} : \mathbb{C} \times \overset{h}{\dots} \times \mathbb{C} \longrightarrow \mathbb{C}$$

$$(\lambda_1, \dots, \lambda_h) \longmapsto \sum_{i=1}^h \lambda_i$$

We denote by $\mathcal{E}'_{(n,d)} = (\mathcal{V}'_{(n,d)}, \Phi'_{(n,d)})$ the restriction to $X \times \ker(w_{\mathbb{C}, h} \circ \pi^h)$ of the family $\mathcal{E}_{(n,d)}$. We observe that $\mathcal{E}'_{(n,d)}$ is a family of polystable Higgs bundles with trace-less Higgs

field which is parametrized by $\ker(w_{\mathbb{C},h} \circ \pi^h)$. This family induces a family of polystable $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundles

$$\check{\mathcal{E}}_{n,\tilde{d}} = (\mathbb{P}(\mathcal{V}'_{(n,d)}), \Phi'_{(n,d)})$$

We consider the action of \mathfrak{S}_h and X on $T^*X \times \mathbb{A}^1 \times T^*X$. We see that the action of these two groups commute and preserve $\ker(w_{\mathbb{C},h} \circ \pi^h)$.

Remark 4.5.2. Let $z_1, z_2 \in T^*X \times \mathbb{A}^1 \times T^*X$, we have that $\check{\mathcal{E}}_{n,\tilde{d}}|_{X \times \{z_1\}} \cong \check{\mathcal{E}}_{n,\tilde{d}}|_{X \times \{z_2\}}$ if and only if there exists $x' \in X$ and $\sigma \in \mathfrak{S}_h$ such that $z_2 = x' + \sigma \cdot z_1$.

Proposition 4.5.3. *The family of polystable $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundles $\check{\mathcal{E}}_{n,\tilde{d}}$ has the local universal property among locally graded families.*

Proof. Let $\mathcal{F} \rightarrow X \times T$ be a locally graded family of $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundles. For every $t \in T$ there exists $U \subset T$ containing t and a family of Higgs bundles $\mathcal{E} = (\mathcal{V}, \Phi)$ parametrized by U with $\mathrm{tr} \Phi = 0$ and such that $t' \in U$ one has

$$\mathcal{F}|_{X \times \{t'\}} \sim_S (\mathbb{P}(\mathcal{V}), \Phi)|_{X \times \{t'\}}.$$

The family of polystable Higgs bundles $\mathcal{E}_{(n,d)}$ has the local universal property for \mathcal{E} , then there exists $U' \subset T$ containing t and a morphism $f : U' \rightarrow T^*X \times \mathbb{A}^1 \times T^*X$ such that

$$f^* \mathcal{E}_{(n,d)} \sim_S \mathcal{E}.$$

We see that the image of f is entirely contained in $\ker(w_{\mathbb{C},h} \circ \pi^h)$, then we have that the pull-back $f^* \check{\mathcal{E}}_{n,\tilde{d}}$ is well defined. Furthermore, if $\mathcal{E}_{(n,d)} = (\mathcal{V}_{(n,d)}, \Phi_{(n,d)})$, we have that

$$f^* \check{\mathcal{E}}_{n,\tilde{d}} \cong (\mathbb{P}(f^* \mathcal{V}_{(n,d)}), f^* \Phi_{(n,d)}).$$

Then, for every t we always have an open subset $U'' = U \cap U'$ containing t and such that

$$\mathcal{F}|_{X \times U''} \sim_S f^* \check{\mathcal{E}}_{n,\tilde{d}}.$$

□

We study the quotient of $\ker(w_{\mathbb{C},h} \circ \pi^h)$ by the action of $X \times \mathfrak{S}_h$. Let us consider the map $\alpha_{T^*X,h}$ defined in (4.1), we can easily check that

$$\ker(w_{\mathbb{C},h} \circ \pi^h) / (X \times \mathfrak{S}_h) \cong \alpha_{T^*X,h} / X[h]. \quad (4.36)$$

Theorem 4.5.4. *Let $h = \mathrm{gcd}(n, d)$, where d is a representative of $\tilde{d} \in \mathbb{Z}_n$, and write $n' = \frac{n}{h}$. There exists a moduli space $\mathcal{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}}$ associated to the moduli functor $\mathrm{Mod}(\check{\mathcal{A}}_{n,\tilde{d}}, \check{P}_{n,\tilde{d}}, S)$. We have the following isomorphism*

$$\check{\mathcal{E}}_{n,d}^{x_0} : \mathcal{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} \xrightarrow{\cong} (T^*X \times \mathbb{A}^1 \times T^*X) / (\mathfrak{S}_h \times X[h]). \quad (4.37)$$

Proof. After Proposition 4.5.3 the theorem follows from Remark 4.5.2, Proposition 3.2.1 and (4.36) where we can see that $\ker(w_{\mathbb{C},h} \circ \pi^h)/(X \times \mathfrak{S}_h)$ is an orbit space. \square

Since $Z_h^0 = (X \times \cdot^h \times X)$ embeds into $Z_h = (T^*X \times \cdot^h \times T^*X)$, the restriction of $\check{\mathcal{E}}_{n,\tilde{d}}$ to $X \times Z_m^0$ gives a family of polystable $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundles with zero Higgs field and we denote by $\check{\mathcal{E}}_{n,\tilde{d}}^0 \rightarrow X \times Z_m^0$ the underlying family of $\mathrm{PGL}(n, \mathbb{C})$ -bundles. This family induces a morphism from Z_h^0 the moduli space of projective bundles

$$\nu_{\check{\mathcal{E}}_{n,\tilde{d}}^0} : X \times \cdot^h \times X \longrightarrow M(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}}.$$

By Remark 5.2.4 we see that $\nu_{\check{\mathcal{E}}_m^0}$ factors through the following bijective morphism

$$\nu'_{\check{\mathcal{E}}_{n,\tilde{d}}^0} : X \times \cdot^h \times X / \mathfrak{S}_h \times X \longrightarrow M(\mathrm{PGL}(n, \mathbb{C})).$$

Remark 4.5.5. Since $\nu'_{\check{\mathcal{E}}_{n,\tilde{d}}^0}$ is a bijective morphism and $M(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}}$ is a normal algebraic variety, it is an isomorphism. Taking the inverse gives the following isomorphism

$$\zeta_{n,\tilde{d}}^{x_0} : M(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} \xrightarrow{\cong} X \times \cdot^h \times X / \mathfrak{S}_h \times X \quad (4.38)$$

$$[(E, \Omega)]_S \longmapsto [[\zeta_{1,0}^{x_0}(L_1)]_{\mathbb{Z}_2}, \dots, [\zeta_{1,0}^{x_0}(L_m)]_{\mathbb{Z}_2}]_{\mathfrak{S}_m}.$$

Using (4.7) this gives an isomorphism

$$M(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} \cong \mathbb{P}^{h-1}/X[h].$$

where the action of $X[h]$ is described in Remark 4.1.3. Let us define $\check{\zeta}_{x_0,n,d}$ to be the restriction of $\check{\zeta}_{n,d}^{x_0}$ to $M(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}}$. This is an isomorphism.

Remark 4.5.6. When $n = 2$ and $\tilde{d} = 0$, we have the action of $\mathfrak{S}_2 \cong \mathbb{Z}_2$ sending a point to its inverse. Taking the second power we obtain the isomorphism $X \cong X/X[2]$, and we see that under this isomorphism the action of \mathbb{Z}_2 is again defined by sending a point to its inverse. Then

$$\mathcal{M}(\mathrm{PGL}(2, \mathbb{C}))_0 \cong (T^*X/X[2])/\mathbb{Z}_2 \cong T^*X/\mathbb{Z}_2, \quad (4.39)$$

and

$$M(\mathrm{PGL}(2, \mathbb{C}))_0 \cong (X/X[2])/\mathbb{Z}_2 \cong X/\mathbb{Z}_2 \cong \mathbb{P}^1. \quad (4.40)$$

Let us consider the map

$$\check{p}_h : (T^*X \times \cdot^{h-1} \times T^*X)/(\mathfrak{S}_h \times X[h]) \longrightarrow (X \times \cdot^{h-1} \times X)/(\mathfrak{S}_h \times X[h])$$

$$[(x_1, \lambda_1), \dots, (x_h, \lambda_h)]_{\mathfrak{S}_h \times X[h]} \longmapsto [x_1, \dots, x_h]_{\mathfrak{S}_h \times X[h]}$$

Proposition 4.5.7. *There is a surjective morphism*

$$\begin{aligned} \check{a}_{n,\tilde{d}} : \mathcal{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} &\longrightarrow M(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} \\ [(\mathbb{P}(E), \Phi)]_S &\longmapsto [\mathbb{P}(E)]_S. \end{aligned}$$

Furthermore the diagram

$$\begin{array}{ccc} \mathcal{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} & \xrightarrow{\check{a}_{n,\tilde{d}}} & M(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} \\ \xi_{n,\tilde{d}}^{x_0} \Big\downarrow \cong & & \cong \Big\downarrow \xi_{x_0,n,\tilde{d}} \\ \ker \alpha_{T^*X,h}/X[h] & \xrightarrow{\check{p}_h} & \ker \alpha_{X,h}/X[h] \end{array}$$

commutes.

Proof. This follows from Proposition 4.3.9 and Remark 4.5.5. \square

We have that $\{q_{n,2}, \dots, q_{n,n}\}$ is a basis of the invariant polynomials of the adjoint representation of $\mathrm{PGL}(n, \mathbb{C})$ on its Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. We define the Hitchin map for $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundles evaluating them on the Higgs field

$$\check{b}_{n,\tilde{d}} : \mathcal{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} \longrightarrow \check{B}_n = \bigoplus_{i=2}^n H^0(X, \mathcal{O})$$

$$[(\mathbb{P}(E), \Phi)]_S \longmapsto (q_{n,2}(\Phi), \dots, q_{n,n}(\Phi))$$

We see that $\check{B}_n = \hat{B}_n \subset B_n$. In fact, if d is such that $\tilde{d} = d \pmod{n}$, the image of $\check{b}_{n,\tilde{d}}$ is the image under $b_{(n,d)}$ of the set Higgs bundles with trace-less Higgs field. Recalling $\beta_{(n,d)}$ defined in (4.17) and $u_{\mathbb{C},h}$ in (4.4) we see that $\check{\beta}_{n,\tilde{d}} = \beta_{(n,d)} \circ u_{\mathbb{C},h}$ gives an isomorphism when we restrict to $\check{B}_{n,\tilde{d}} = \check{B}_n \cap B_{(n,d)}$

$$\check{\beta}_{n,\tilde{d}} : \mathbb{C} \times \mathbb{C}^{h-1} \times \mathbb{C} / \mathfrak{S}_h \xrightarrow{\cong} \check{B}_{n,\tilde{d}}.$$

Let us consider the map

$$\check{\pi}_h : T^*X \times \mathbb{C}^{h-1} \times T^*X / (\mathfrak{S}_h \times X[h]) \longrightarrow \mathbb{C} \times \mathbb{C}^{h-1} \times \mathbb{C} / \mathfrak{S}_h$$

$$[(x_1, \lambda_1), \dots, (x_{h-1}, \lambda_{h-1})]_{\mathfrak{S}_h \times X[h]} \longmapsto [\lambda_1, \dots, \lambda_{h-1}]_{\mathfrak{S}_h}.$$

Lemma 4.5.8. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}} & \xrightarrow{\check{b}_{n,\tilde{d}}} & \check{B}_{n,\tilde{d}} \\ \xi_{n,\tilde{d}}^{x_0} \Big\downarrow \cong & & \cong \Big\downarrow \check{\beta}_{n,\tilde{d}}^{-1} \\ (T^*X \times \mathbb{C}^{h-1} \times T^*X) / (\mathfrak{S}_h \times X[h]) & \xrightarrow{\check{\pi}_h} & \mathbb{C} \times \mathbb{C}^{h-1} \times \mathbb{C} / \mathfrak{S}_h. \end{array} \quad (4.41)$$

Proof. Since the image under $\check{b}_{n,\tilde{d}}$ of $(\mathbb{P}(E), \Phi)$ is equal to the image under $b_{(n,d)}$ of (E, Φ) , the lemma follows from Lemma 4.3.10. \square

Remark 4.5.9. Note that $\check{B}_{n,\tilde{d}} = \hat{B}_h$. If we write \tilde{q}_h for the projection

$$\tilde{q}_h : T^*X \times {}^{h-1} \times T^*X / \mathfrak{S}_h \rightarrow (T^*X \times {}^{h-1} \times T^*X) / (\mathfrak{S}_h \times X[h]),$$

we have that $\hat{\pi}_h$, defined in (4.31), factors through \tilde{q}_h and

$$\hat{\pi}_h = \check{\pi}_h \circ \tilde{q}_h.$$

Lemma 4.5.10. *Let $\bar{\lambda}_{gen}$ be defined as in (4.32). Then*

$$\check{\pi}_h^{-1}([\bar{\lambda}_{gen}]_{\mathfrak{S}_h}) \cong X \times {}^{h-1} \times X.$$

Proof. From Remark 4.5.9 and Lemma 4.4.8 we have

$$\check{\pi}_h^{-1}([\bar{\lambda}_{gen}]_{\mathfrak{S}_h}) \cong (X \times {}^{h-1} \times X) / X[h],$$

where $X[h]$ acts on $X \times {}^{h-1} \times X$ with a weighted $(1, \dots, 1)$ -action. Thanks to Lemma 4.1.2, we know that $(X \times {}^{h-1} \times X) / X[h]$ is $X \times {}^{h-1} \times X$. \square

An arbitrary point of $\check{B}_{n,0} = \hat{B}_n$ is given in (4.33). Recall that m_i denotes the multiplicity of λ_i and $h = m_1 + \dots + m_\ell$. We set $r = \gcd(h, m_1, \dots, m_\ell)$.

Recall the morphism s_ℓ given by (4.34) and the isomorphism w_ℓ given by (4.35).

Lemma 4.5.11. *Let $\bar{\lambda}_{arb}$ be defined as in (4.33). Then*

$$\check{\pi}_h^{-1}([\bar{\lambda}_{arb}]_{\mathfrak{S}_h}) \cong w_\ell^*((P_{m_1} \times \dots \times P_{m_\ell})|_{\ker s_\ell}) / X[h].$$

This is a holomorphic fibration with fibre $(\mathbb{P}^{m_1-1} \times \dots \times \mathbb{P}^{m_\ell-1}) / X[r]$ over $X \times {}^{h-1} \times X$.

Proof. By Remark 4.5.9 and Lemma 4.4.9, we have

$$\check{\pi}_h^{-1}([\bar{\lambda}_{arb}]_{\mathfrak{S}_h}) \cong w_\ell^*((P_{m_1} \times \dots \times P_{m_\ell})|_{\ker s_\ell}) / X[h],$$

where the action of $X[h]$ on $w_\ell^*((P_{m_1} \times \dots \times P_{m_\ell})|_{\ker s_\ell})$ is in fact the action of $X[h]$ on $\text{Sym}^{m_1} X \times \dots \times \text{Sym}^{m_\ell} X$. This gives a weighted (m_1, \dots, m_ℓ) -action of $X[h]$ on the base of the fibration $\text{Sym}^{m_1} X \times \dots \times \text{Sym}^{m_\ell} X \rightarrow X \times {}^{h-1} \times X$ given by the direct sum of various $\text{Sym}^{m_i} X \rightarrow X$. Since $h = m_1 + \dots + m_\ell$, it is straightforward to check that the action of $X[h]$ restricts to $\ker s_\ell$.

Take $r = \gcd(h, m_1, \dots, m_\ell)$. The subgroup $X[r] \subset X[h]$ acts on $\text{Sym}^{m_1} X \times \dots \times \text{Sym}^{m_\ell} X$ and it acts trivially on the base $X \times {}^{h-1} \times X$. The quotient of $\text{Sym}^{m_1} X \times \dots \times \text{Sym}^{m_\ell} X$ by the action of $X[r]$ is a holomorphic fibration over $X \times {}^{h-1} \times X$ with fibre $(\mathbb{P}^{m_1-1} \times \dots \times \mathbb{P}^{m_\ell-1}) / X[r]$, where the action of $X[r]$ is described in Remark 4.1.3.

The quotient of $\text{Sym}^{m_1} X \times \dots \times \text{Sym}^{m_\ell} X$ by $X[h]$ is equivalent to the quotient of $(\text{Sym}^{m_1} X \times \dots \times \text{Sym}^{m_\ell} X) / X[r]$ by $X[h] / X[r]$. Since $X[h] / X[r] \cong X[h/r]$ and the weighted action of $X[r]$ is trivial, we can check that the weighted (m_1, \dots, m_ℓ) -action of $X[h] / X[r]$ on $X \times {}^{h-1} \times X$ is equivalent to the weighted $(m_1/r, \dots, m_\ell/r)$ -action of $X[h/r]$ on $X \times {}^{h-1} \times X$. By Lemma 4.1.1, this action is free since we have that

$\gcd(h/r, m_1/r, \dots, m_\ell/r) = 1$. As a consequence, $(\text{Sym}^{m_1} X \times \dots \times \text{Sym}^{m_\ell} X)/X[h]$ is a holomorphic fibration with fibre $(\mathbb{P}^{m_1-1} \times \dots \times \mathbb{P}^{m_\ell-1})/X[r]$ over $(X \times \dots \times X)/X[h/r]$.

The weighted $(m_1/r, \dots, m_\ell/r)$ -action of $X[h/r]$ restricted to $\ker s_\ell$ is equivariant under w_ℓ to the weighted $(m_1/r, \dots, m_{\ell-1}/r)$ -action of $X[h/r]$ on $X \times \dots \times X$, then $\tilde{\pi}_h^{-1}([\bar{\lambda}_{arb}]_{\mathfrak{S}_h})$ is a holomorphic fibration with fibre $(\mathbb{P}^{m_1-1} \times \dots \times \mathbb{P}^{m_{\ell-1}-1})/X[r]$ over $(X \times \dots \times X)/X[h/r]$. By Lemma 4.1.2 this is isomorphic to $X \times \dots \times X$. \square

Let us recall that the abelian variety $X \times \dots \times X$ is self dual, i.e.

$$X \times \dots \times X \cong X \times \dots \times X.$$

Corollary 4.5.12. *The generic fibre of the Hitchin fibration $\mathcal{M}(\text{PGL}(n, \mathbb{C}))_0 \rightarrow \check{B}_{n,0}$ is the abelian variety $X \times \dots \times X$.*

The arbitrary fibre of the Hitchin fibration $\mathcal{M}(\text{PGL}(n, \mathbb{C}))_0 \rightarrow \check{B}_{n,0}$ is a fibration over $X \times \dots \times X$ with fibre $(\mathbb{P}^{m_1-1} \times \dots \times \mathbb{P}^{m_\ell-1})/X[r]$.

Remark 4.5.13. The generic fibre of the Hitchin fibration $\mathcal{M}(\text{PGL}(n, \mathbb{C}))_0 \rightarrow \check{B}_{n,0}$ is an abelian variety dual to the abelian variety of the corresponding fibre of the Hitchin fibration $\mathcal{M}(\text{SL}(n, \mathbb{C})) \rightarrow \hat{B}_n$.

The arbitrary fibre of the Hitchin fibration $\mathcal{M}(\text{PGL}(n, \mathbb{C}))_0 \rightarrow \check{B}_{n,0}$ is a fibration over an abelian variety. This abelian variety is dual to the base of the fibration of the corresponding fibre of the Hitchin fibration $\mathcal{M}(\text{SL}(n, \mathbb{C})) \rightarrow \hat{B}_n$.

We finish the section studying $\mathfrak{M}(\text{PGL}(n, \mathbb{C}))_{\check{d}}$, the moduli space associated to the moduli functor $\text{Mod}(\check{\mathcal{A}}_{n,\check{d}}, \check{P}_{n,\check{d}}, S)$.

Using the family of Higgs bundles $\mathcal{E}_{(n,d)} = (\mathcal{V}_{(n,d)}, \Phi_{(n,d)})$ we define $\check{\mathcal{E}}_{n,\check{d}}$ as the family of $\text{PGL}(n, \mathbb{C})$ -Higgs bundles $(\mathbb{P}(\mathcal{V}_{(n,d)}), \Phi_{(n,d)})$.

Remark 4.5.14. The family $\check{\mathcal{E}}_{n,\check{d}}$ is parametrized by $Z_h = T^*X \times \dots \times T^*X$ and given two points $z_1, z_2 \in Z_h$ we have $\check{\mathcal{E}}_{n,\check{d}}|_{X \times \{z_1\}} \cong \check{\mathcal{E}}_{n,\check{d}}|_{X \times \{z_2\}}$ if and only if there exists $\sigma \in \mathfrak{S}_n$ and $z \in T^*X$ such that $z \cdot z_2 = \sigma \cdot z_1$. Since all the $\text{PGL}(n, \mathbb{C})$ -Higgs bundles parametrized by $\check{\mathcal{E}}_{n,\check{d}}$ are polystable isomorphism implies S -equivalence.

Proposition 4.5.15. *We have a bijective morphism $\mathcal{M}(\text{PGL}(n, \mathbb{C}))_{\check{d}} \rightarrow \mathfrak{M}(\text{PGL}(n, \mathbb{C}))_{\check{d}}$, hence $\mathcal{M}(\text{PGL}(n, \mathbb{C}))_{\check{d}}$ is the normalization of $\mathfrak{M}(\text{PGL}(n, \mathbb{C}))_{\check{d}}$.*

Proof. The family $\check{\mathcal{E}}_{n,\check{d}}$ is parametrized by $T^*X \times \dots \times T^*X$ and induces the following morphism

$$\nu_{\check{\mathcal{E}}_{n,\check{d}}} : T^*X \times \dots \times T^*X \longrightarrow \mathfrak{M}(\text{PGL}(n, \mathbb{C}))_{\check{d}},$$

and by Remark 4.5.14 it factors through

$$\nu'_{\check{\mathcal{E}}_{n,\check{d}}} : \text{Sym}^h(T^*X) / T^*X \longrightarrow \mathfrak{M}(\text{PGL}(n, \mathbb{C}))_{\check{d}}$$

which is bijective.

Let us denote by $\overline{\mathfrak{M}(\text{PGL}(n, \mathbb{C}))_{\check{d}}}$ the normalization of $\mathfrak{M}(\text{PGL}(n, \mathbb{C}))_{\check{d}}$. Since

$$\text{Sym}^h T^*X / T^*X \cong (T^*X \times \dots \times T^*X) / \mathfrak{S}_h \times X[h],$$

it is a normal variety normal because it is the quotient of a smooth (and therefore normal) variety by a discrete group. Then by the universal property of the normalization, $\nu'_{\hat{\mathcal{E}}_n}$ factors through

$$\nu''_{\hat{\mathcal{E}}_{n,\tilde{d}}} : (T^*X \times^{h^{-1}} T^*X) / \mathfrak{S}_h \times X[h] \longrightarrow \overline{\mathfrak{M}(\mathrm{PGL}(n, \mathbb{C}))}.$$

This map is an isomorphism since it is a bijection and $\overline{\mathfrak{M}(\mathrm{PGL}(n, \mathbb{C}))}_{\tilde{d}}$ is normal. Then $\mathcal{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}}$ is the normalization of $\mathfrak{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}}$. \square

Remark 4.5.16. Both moduli spaces would be isomorphic if $\mathfrak{M}(\mathrm{PGL}(n, \mathbb{C}))_{\tilde{d}}$ is normal, but normality in this case is an open question.

Chapter 5

Moduli spaces of symplectic and orthogonal Higgs bundles

5.1 Stability of symplectic and orthogonal Higgs bundles

We start this section with a well known result for symplectic and orthogonal Higgs bundles over compact Riemann surfaces of arbitrary genus.

Proposition 5.1.1. *Let (E, Θ, Φ) be a semistable $\mathrm{Sp}(2m, \mathbb{C})$ or $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle. Then (E, Φ) is semistable. If further (E, Θ, Φ) is polystable, then (E, Φ) is polystable too.*

Let $n > 2$ and let (E, Q, Φ, τ) be a semistable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle. Then (E, Φ) is semistable. If further (E, Q, Φ, τ) is polystable, then (E, Φ) is polystable too.

Proof. Suppose that (E, Φ) is unstable and take the first term of its Harder-Narasimhan filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = E.$$

It is a Φ -invariant subbundle F_1 that satisfies $\mu(F_1) > \mu(E) = 0$ and (F_1, Φ_{F_1}) is semistable. If the bundle F_1 is not isotropic then $\Theta_{F_1, F_1} \in H^0(X, F_1^* \otimes F_1^*)$ must be non-zero. Note that $(F_1^* \otimes F_1^*, \Phi_{F_1}^t \otimes \Phi_{F_1}^t)$ is semistable of negative degree and the existence of the non-zero holomorphic section Θ_{F_1, F_1} will imply the existence of a line subbundle of $F_1^* \otimes F_1^*$ of degree ≥ 0 . Due to the fact that Θ and Φ commute, we have that the bundle generated by Θ_{F_1, F_1} is $(\Phi_{F_1}^t \otimes \Phi_{F_1}^t)$ -invariant and this contradicts the semistability of $(F_1^* \otimes F_1^*, \Phi_{F_1}^t \otimes \Phi_{F_1}^t)$.

Then $\Theta_{F_1, F_1} = 0$ and therefore F_1 is isotropic. Since $\mu(F_1) > 0$ and it is Φ -invariant, (E, Θ, Φ) is unstable too. Hence, if (E, Θ, Φ) is semistable, (E, Φ) is semistable.

If the semistable bundle (E, Θ, Φ) is polystable then $(E, \Theta, \Phi) \cong \mathrm{gr}(E, \Theta, \Phi)$, so (E, Φ) is a direct sum of stable Higgs bundles plus (only in the case it does exist) the underlying Higgs bundle of the stable factor $(\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k)$. We see that it suffices to prove the lemma for stable $\mathrm{Sp}(2m, \mathbb{C})$ or $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles.

Take (E, Θ, Φ) stable and let F be a Φ -invariant subbundle of E . Write $W = F \cap F^{\perp\Theta}$ and suppose $W \neq 0$. Clearly W is isotropic and has negative slope due to the stability of (E, Θ, Φ) . But $\Theta(W) \cong W^* \subset E$ is isotropic too, and since $\mu(W^*) = -\mu(W) > 0$ it contradicts the stability of (E, Θ, Φ) , so $W = 0$.

Since $F \cap F^{\perp\Theta} = 0$, we see that Θ is non-degenerate when restricted to every Φ -invariant subbundle F of E . This restriction, which we call Θ_F , induces an isomorphism between F and F^* and so $\mu(F) = 0$. Thus (E, Φ) is semistable since every Φ -invariant subbundle $F \subset E$ has zero degree.

Assuming that (E, Φ) is not stable (otherwise the proof will be complete), we take (E_1, Φ_1) to be the first term of a Jordan-Hölder filtration. We have seen that $E_1 \cap E_1^{\perp\Theta} = 0$, so $E = E_1 \oplus E_1^{\perp\Theta}$. Since E_1 is Φ -invariant so is $E_1^{\perp\Theta}$, hence

$$(E, \Phi) = (E_1, \Phi_1) \oplus (E_1^{\perp\Theta}, \Phi_2),$$

where Φ_2 is the restriction of Φ to $E_1^{\perp\Theta}$. Note that (E_1, Φ_1) is a stable Higgs bundle and $(E_1^{\perp\Theta}, \Phi_2)$ is semistable. By induction, we decompose $(E_1^{\perp\Theta}, \Phi_2)$ into stable factors proving that (E, Φ) is polystable.

The statement for $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles follows from the fact that (when $n > 2$) a $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle (E, Q, Φ, τ) is stable, semistable or polystable if and only if the underlying $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle (E, Q, Φ) is stable, semistable or polystable. \square

Over an elliptic curve we can study the semistability and the polystability of $\mathrm{Sp}(2m, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$ or $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles in terms of their underlying $\mathrm{Sp}(2m, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$ or $\mathrm{SO}(n, \mathbb{C})$ -bundles.

Proposition 5.1.2. *Let (E, Θ, Φ) be a semistable $\mathrm{O}(n, \mathbb{C})$ or $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundle. Then (E, Θ) is semistable. If further (E, Θ, Φ) is polystable, (E, Θ) is polystable too.*

Let (E, Q, Φ, τ) be a semistable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle. Then (E, Q, τ) is a semistable $\mathrm{SO}(n, \mathbb{C})$ -bundle. If further (E, Q, Φ, τ) is polystable, (E, Q, τ) is polystable too.

Proof. Suppose that (E, Θ, Φ) is semistable. By Proposition 5.1.1 we have that (E, Φ) is a semistable Higgs bundle, and by Proposition 4.2.1 E is semistable.

By definition of semistability every subbundle F of E satisfies $\mu(F) \leq \mu(E)$, where $\mu(E) = 0$. In particular every isotropic subbundle satisfies the slope condition and then (E, Θ) is semistable.

Since a polystable bundle (E, Θ, Φ) is isomorphic to $\mathrm{gr}(E, \Theta, \Phi)$, we have the decomposition

$$(E, \Theta, \Phi) \cong (\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k) \oplus \bigoplus_{i=1}^{k-1} \left((E_i/E_{i-1}) \oplus (E'_{i-1}/E'_i), \begin{pmatrix} 0 & b\theta_i^t \\ \theta_i & 0 \end{pmatrix}, \bar{\Phi}_i \oplus \bar{\Phi}'_i \right),$$

where $b = -1$ for $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles and $b = 1$ for $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles. Note that the stable factor $(\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k)$ may not exist. The factors $((E_i/E_{i-1}), \bar{\Phi}_i)$ and $((E'_{i-1}/E'_i), \bar{\Phi}'_i)$ are stable Higgs bundles.

If it exists, the term $(\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k)$ is stable and then it is described in Proposition 5.1.3. We can check that $(\tilde{E}_k, \tilde{\Theta}_k)$ is stable. By Proposition 4.2.3 we see that (E_i/E_{i-1}) and (E'_{i-1}/E'_i) are stable vector bundles, then (E, Θ) is polystable.

The statement for $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles follows from the statement for $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles. \square

We denote the elements of $\mathrm{Pic}^0(X)[2]$ by $\mathcal{O}(= J_0)$, J_1 , J_2 and J_3 .

Proposition 5.1.3. *There are no stable $\mathrm{Sp}(2m)$ -Higgs bundles. Any stable $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle is isomorphic to one of the following*

1. *four $\mathrm{O}(1, \mathbb{C})$ -Higgs bundles, $(J_a, 1, 0)$, where $a = 0, 1, 2$ and 3 ,*
2. *six $\mathrm{O}(2, \mathbb{C})$ -Higgs bundles of the form $(J_a, 1, 0) \oplus (J_b, 1, 0)$, where $a \neq b$,*
3. *four $\mathrm{O}(3, \mathbb{C})$ -Higgs bundles of the form $(J_a, 1, 0) \oplus (J_b, 1, 0) \oplus (J_c, 1, 0)$ with a, b and c different,*
4. *the $\mathrm{O}(4, \mathbb{C})$ -Higgs bundle $(\mathcal{O}, 1, 0) \oplus (J_1, 1, 0) \oplus (J_2, 1, 0) \oplus (J_3, 1, 0)$.*

Proof. Suppose (E, Θ, Φ) is stable. By Lemma 5.1.1, the Higgs bundle (E, Φ) is polystable and by Corollary 4.2.4 decomposes as a direct sum of degree zero Higgs line bundles (recall that $d = \deg(E) = 0$, so $h = n$)

$$(E, \Phi) = \bigoplus_i (L_i, \phi_i), \quad \phi_i = \lambda_i \otimes \mathrm{id}_{L_i}, \quad \lambda_i \in \mathbb{C}.$$

Recall that Θ anticommutes with the Higgs field. If Θ is antisymmetric (i.e. (E, Θ, Φ) is a $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundle), Θ sends every factor to a different one. Thus every Φ -invariant line bundle is isotropic so there are no stable $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles over an elliptic curve.

We study the case where Θ is symmetric (i.e. (E, Θ, Φ) is a $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle). Since Θ anticommutes with the Higgs field, the image under Θ of the factor (L, ϕ_i) is $(L_i^*, -\phi_i)$. If L_i is not selfdual or ϕ_i is not zero, Θ sends it to another factor and L_i is a Φ -invariant isotropic subbundle that contradicts the stability of (E, Θ, Φ) . Thus a stable $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle decomposes into factors of the form $(J_a, 1, 0)$. Suppose that two of these factors coincide and so $J_a \oplus J_a$ is a subbundle of E . There is always a linear combination of these subbundles that gives an isotropic subbundle. Since the Higgs field is zero, the isotropic subbundle would be Φ -invariant and the $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle can not be stable.

The only possible stable $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles are the ones listed in the statement. It is straightforward to check that they are all stable since they do not have isotropic subbundles of degree 0. \square

Corollary 5.1.4. *Let $n \neq 2$. Any stable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle is isomorphic to*

1. *the $\mathrm{SO}(1, \mathbb{C})$ -Higgs bundle $(E_1^{st}, Q_1^{st}, \Phi_1^{st}, \tau_1^{st}) = (\mathcal{O}, 1, 0, 1)$,*
2. *the $\mathrm{SO}(3, \mathbb{C})$ -Higgs bundle*

$$(E_3^{st}, Q_3^{st}, \Phi_3^{st}, \tau_3^{st}) = \left(J_1 \oplus J_2 \oplus J_3, \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, 0, 1 \right),$$

3. the $\mathrm{SO}(4, \mathbb{C})$ -Higgs bundle

$$(E_4^{st}, Q_4^{st}, \Phi_4^{st}, \tau_4^{st}) = \left(\mathcal{O} \oplus J_1 \oplus J_2 \oplus J_3, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, 0, 1 \right).$$

Lemma 5.1.5. *Every $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundle (E, Q, Φ, τ) is stable and*

$$(E, Q, \Phi, \tau) \cong \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}, \sqrt{-1} \right), \quad (5.1)$$

where $L \in \mathrm{Pic}^d(X)$.

Proof. Since $\mathrm{SO}(2, \mathbb{C}) \cong \mathbb{C}^*$, every $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundle is stable. By Proposition 5.1.3 the $\mathrm{O}(2, \mathbb{C})$ -Higgs bundle (E, Q, Φ) underlying the $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundle (E, Q, Φ, τ) can not be stable. It therefore possesses an isotropic Φ -invariant subbundle L with $\deg L \geq 0$. Then

$$(E, Q, \Phi) \cong \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix} \right).$$

Since $\det Q^t Q$ is 1, we have that every $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundle with this underlying $\mathrm{O}(2, \mathbb{C})$ -Higgs bundle has the form given in (5.1) or

$$(E, Q, \Phi, \tau) \cong \left(L \oplus L^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & \\ & -\lambda \end{pmatrix}, -\sqrt{-1} \right).$$

Permuting L and L^* we construct an isomorphism associated to the matrix $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, that has determinant -1 , sending the previous $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundle to

$$(E, Q, \Phi, \tau) \cong \left(L^* \oplus L, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -\lambda & \\ & \lambda \end{pmatrix}, \sqrt{-1} \right).$$

□

Note that the degree of a $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundle written as in (5.1) is the degree of the first summand of the underlying vector bundle.

Once we have described the stable $\mathrm{Sp}(2m, \mathbb{C})$, $\mathrm{O}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles, we can give a description of the polystable ones.

Proposition 5.1.6. *A $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundle over an elliptic curve is polystable if and only if it is isomorphic to a direct sum of polystable $\mathrm{Sp}(2, \mathbb{C})$ -Higgs bundles.*

An $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle is polystable if and only if it is isomorphic to a direct sum of polystable $\mathrm{O}(2, \mathbb{C})$ -Higgs bundles or it is isomorphic to a direct sum of polystable $\mathrm{O}(2, \mathbb{C})$ -Higgs bundles and a stable $\mathrm{O}(m, \mathbb{C})$ -Higgs bundle.

Let $n > 2$. A $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle is polystable if and only if it is isomorphic to a direct sum of (stable) $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles of trivial degree or it is isomorphic to a direct sum of (stable) $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles of trivial degree and a stable $\mathrm{SO}(m, \mathbb{C})$ -Higgs bundle (where $m = 1, 3$ or 4).

Proof. By definition (E, Θ, Φ) is polystable if and only if it is isomorphic to $\text{gr}(E, \Theta, \Phi)$. One can rewrite this fact saying that (E, Θ, Φ) is polystable if and only if decomposes as follows,

$$(E, \Theta, \Phi) \cong (\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k) \oplus \bigoplus_{i=1}^{k-1} \left((E_i/E_{i-1}) \oplus (E'_{i-1}/E'_i), \begin{pmatrix} 0 & b\theta_i^t \\ \theta_i & 0 \end{pmatrix}, \bar{\Phi}_i \oplus \bar{\Phi}'_i \right),$$

where $(\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k)$ is stable (if it is not zero) and $((E_i/E_{i-1}), \bar{\Phi}_i)$ and $((E'_{i-1}/E'_i), \bar{\Phi}'_i)$ are stable Higgs bundles of degree 0. The only stable Higgs bundles with degree 0 are those of rank 1 so the factors

$$\left((E_i/E_{i-1}) \oplus (E'_{i-1}/E'_i), \begin{pmatrix} 0 & b\theta_i^t \\ \theta_i & 0 \end{pmatrix}, \bar{\Phi}_i \oplus \bar{\Phi}'_i \right)$$

are $\text{Sp}(2, \mathbb{C})$ or $\text{O}(2, \mathbb{C})$ -Higgs bundles. They are polystable since they are isomorphic to their associated graded objects. This proves the statement for $\text{O}(n, \mathbb{C})$ -Higgs bundles.

By Proposition 5.1.3, there are no stable $\text{Sp}(2m', \mathbb{C})$ -Higgs bundles for any m' . This implies that every $\text{Sp}(2m, \mathbb{C})$ -Higgs bundle decomposes only into $\text{Sp}(2, \mathbb{C})$ -Higgs bundles.

Every $\text{SO}(2, \mathbb{C})$ -Higgs bundle is stable. Recall that for $n > 2$, a $\text{SO}(n, \mathbb{C})$ -Higgs bundle is polystable if and only if the underlying $\text{O}(n, \mathbb{C})$ -Higgs bundle is polystable.

Let us take $n > 2$. By the description of polystable $\text{O}(n, \mathbb{C})$ -Higgs bundles we have given above, a $\text{SO}(n, \mathbb{C})$ -bundle is polystable if and only if it is a direct sum of stable $\text{SO}(2, \mathbb{C})$ -Higgs bundles of trivial degree and perhaps a $\text{SO}(m, \mathbb{C})$ -Higgs bundle with stable underlying $\text{O}(m, \mathbb{C})$ -Higgs bundle. From Proposition 5.1.3 we see that the only possible $\text{SO}(m, \mathbb{C})$ -Higgs bundles with stable underlying $\text{O}(m, \mathbb{C})$ -Higgs bundles are the stable $\text{SO}(m, \mathbb{C})$ -Higgs bundles given in Corollary 5.1.4. \square

We give two results about sufficient conditions for the existence of isomorphisms.

Proposition 5.1.7. *Let (E_1, Θ_1, Φ_1) and (E_2, Θ_2, Φ_2) be two polystable $\text{Sp}(2m, \mathbb{C})$ or $\text{O}(n, \mathbb{C})$ -Higgs bundles. If $(E_1, \Phi_1) \cong (E_2, \Phi_2)$, then $(E_1, \Theta_1, \Phi_1) \cong (E_2, \Theta_2, \Phi_2)$.*

Proof. If (E_j, Θ_j, Φ_j) are polystable $\text{O}(n, \mathbb{C})$ -Higgs bundles then, by Proposition 5.1.6 they have, up to isomorphism, the following form

$$(E_j, \Theta_j, \Phi_j) \cong \bigoplus_i (J_{a_{j,i}}, 0) \oplus \bigoplus_k \left(L_{j,k} \oplus L_{j,k}^*, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} \phi_{j,k} & \\ & -\phi_{j,k} \end{pmatrix} \right).$$

If $(E_1, \Phi_1) \cong (E_2, \Phi_2)$, we have for a certain ordering that

$$(J_{a_{1,i}}, 0) = (J_{a_{2,i}}, 0)$$

and

$$(L_{1,k}, \phi_{1,k}) = (L_{2,k}, \phi_{2,k}).$$

It follows immediately that (E_1, Θ_1, Φ_1) and (E_2, Θ_2, Φ_2) are isomorphic $\text{O}(n, \mathbb{C})$ -Higgs bundles.

The statement for $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles follows from the discussion above and the fact that

$$\left(L_{j,k} \oplus L_{j,k}^*, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \begin{pmatrix} \phi_{j,k} & \\ & -\phi_{j,k} \end{pmatrix} \right) \cong \left(L_{j,k}^* \oplus L_{j,k}, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \begin{pmatrix} -\phi_{j,k} & \\ & \phi_{j,k} \end{pmatrix} \right). \quad (5.2)$$

To prove (5.2) we see that the matrix

$$\begin{pmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{pmatrix}$$

gives the isomorphism we are looking for. \square

Proposition 5.1.8. *Let $(E_1, Q_1, \Phi_1, \tau_1)$ and $(E_2, Q_2, \Phi_2, \tau_2)$ be two polystable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles of the form*

$$(E_j, Q_j, \Phi_j, \tau_j) \cong (E_a^{st}, Q_a^{st}, \Phi_a^{st}, \tau_a^{st}) \oplus \bigoplus_i (E_{i,j}, Q_{i,j}, \Phi_{i,j}, \tau_{i,j}),$$

where $a = 1, 3$ or 4 and the $(E_{i,j}, Q_{i,j}, \Phi_{i,j}, \tau_{i,j})$ are stable $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles of trivial degree. If $(E_1, Q_1, \Phi_1) \cong (E_2, Q_2, \Phi_2)$, then $(E_1, Q_1, \Phi_1, \tau_1) \cong (E_2, Q_2, \Phi_2, \tau_2)$.

Proof. We recall that an isomorphism of $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles is an isomorphism of $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles that preserves the trivialization τ . We have seen that a $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundle has the following form

$$(E_{i,j}, Q_{i,j}, \Phi_{i,j}, \tau_{i,j}) = \left(L_{i,j} \oplus L_{i,j}^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda_{i,j} & \\ & -\lambda_{i,j} \end{pmatrix}, \sqrt{-1} \right).$$

We suppose now that $(E_1, Q_1, \Phi_1) \cong (E_2, Q_2, \Phi_2)$. This implies that $(E_{1,j}, Q_{1,j}, \Phi_{1,j})$ and $(E_{2,j}, Q_{2,j}, \Phi_{2,j})$ are isomorphic $\mathrm{O}(2, \mathbb{C})$ -Higgs bundles after a certain reordering of the factors. For this order, either

$$(E_{1,j}, Q_{1,j}, \Phi_{1,j}, \tau_{1,j}) \cong (E_{2,j}, Q_{2,j}, \Phi_{2,j}, \tau_{2,j}),$$

or

$$(E_{1,j}, Q_{1,j}, \Phi_{1,j}, \tau_{1,j}) \cong (E_{2,j}, Q_{2,j}, \Phi_{2,j}, -\tau_{2,j}).$$

On the other hand, for $a = 1, 3$ and 4 , we have

$$(E_a^{st}, Q_a^{st}, \Phi_a^{st}, \tau_a^{st}) \cong (E_a^{st}, Q_a^{st}, \Phi_a^{st}, -\tau_a^{st})$$

since both $\mathrm{SO}(a, \mathbb{C})$ -Higgs bundles are stable and by Corollary 5.1.4 there is only one stable $\mathrm{SO}(a, \mathbb{C})$ -Higgs bundle up to isomorphism.

As a consequence, we can construct a morphism that inverts the trivialization τ_j combined with the morphism that inverts the trivialization τ_a^{st} of the stable factor. Doing that, the trivialization τ of the total $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle remains unchanged. \square

5.2 Moduli spaces of $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles

Theorem 5.2.1. *For every $m > 0$*

$$\mathfrak{M}^{\mathrm{st}}(\mathrm{Sp}(2m, \mathbb{C})) = \emptyset.$$

Proof. This is included in Proposition 5.1.3. \square

Recall the universal family of Higgs line bundles $\mathcal{E}_{(1,0)} = (\mathcal{V}_{(1,0)}, \Phi_{(1,0)})$ with zero degree. We note that $\Lambda^2(\mathcal{V}_{(1,0)} \oplus \mathcal{V}_{(1,0)}^*) \cong \mathcal{V}_{(1,0)} \otimes (\mathcal{V}_{(1,0)})^* \cong \mathcal{O}_{X \times T^*X}$, we take the non-vanishing section of $\Lambda^2(\mathcal{V}_{(1,0)} \oplus \mathcal{V}_{(1,0)}^*)$

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can check that Ω anticommutes with $\Phi_{(1,0)} \oplus (-\Phi_{(1,0)})$ and then

$$\tilde{\mathcal{E}}_2 = (\mathcal{V}_{(1,0)} \oplus \mathcal{V}_{(1,0)}^*, \Omega, \Phi_{(1,0)} \oplus (-\Phi_{(1,0)}))$$

is a family of $\mathrm{Sp}(2, \mathbb{C})$ -Higgs bundles parametrized by T^*X . We note that the restriction of $\tilde{\mathcal{E}}_2$ to two different points of T^*X , z_1 and z_2 , give equivalent (isomorphic) bundles $\tilde{\mathcal{E}}_{z_1} \sim_S \tilde{\mathcal{E}}_{z_2}$ if and only if $z_1 = -z_2$.

Now we define $\tilde{\mathcal{E}}'_{2m}$ to be the family of $(\mathrm{Sp}(2, \mathbb{C}) \times \dots \times \mathrm{Sp}(2, \mathbb{C}))$ -Higgs bundles $\tilde{\mathcal{E}}_2 \times_X \dots \times_X \tilde{\mathcal{E}}_2$ parametrized by $T^*X \times \dots \times T^*X$. Let $i : \mathrm{Sp}(2, \mathbb{C}) \times \dots \times \mathrm{Sp}(2, \mathbb{C}) \hookrightarrow \mathrm{Sp}(2m, \mathbb{C})$ be the natural injection. We write $\tilde{\mathcal{E}}_{2m}$ for the extension of structure group $i_* \tilde{\mathcal{E}}'_{2m}$.

Take a family $\mathcal{F} \rightarrow X \times T$ of semistable $\mathrm{Sp}(2, \mathbb{C})$ -Higgs bundles of the form

$$\mathcal{F} = \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & -\vartheta^t \\ \vartheta & 0 \end{pmatrix}, \Phi \oplus \Phi' \right)$$

where \mathcal{V} is a family of stable vector bundles of degree zero (therefore line bundles), $\vartheta : \mathcal{V} \rightarrow \mathcal{V}$ is an isomorphism, and Φ and Φ' are endomorphisms of \mathcal{V} and \mathcal{V}^* satisfying $\vartheta\Phi = -(\Phi')^t\vartheta$. Since \mathcal{V} is a family of line bundles, we have that $\mathcal{V}^* \otimes \mathcal{V} \cong \mathcal{O}_{X \times T}$ and then $\vartheta \in H^0(X \times T, \mathcal{O}_{X \times T})$. Taking $\vartheta^{-1/2} : \mathcal{V} \rightarrow \mathcal{V}$ we can construct an isomorphism

$$\left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & -\vartheta^t \\ \vartheta & 0 \end{pmatrix}, \Phi \oplus \Phi' \right) \cong \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \Phi \oplus (-\Phi) \right).$$

Remark 5.2.2. Since by Proposition 5.1.3 there are no stable $\mathrm{Sp}(2m', \mathbb{C})$ -Higgs bundles for any value of m' , every locally graded family $\mathcal{E} \rightarrow X \times T$ of semistable $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles is such that for every $t \in T$ there exists an open subset $U \subset T$ containing t and a set of families $(\mathcal{V}_1, \Phi_1), \dots, (\mathcal{V}_m, \Phi_m)$ of Higgs bundles of rank 1 and degree 0 such that

$$\mathcal{E}|_{X \times U} \sim_S \bigoplus_{i=1}^m \left(\mathcal{V}_i \oplus \mathcal{V}_i^*, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \Phi_i \oplus (-\Phi_i) \right). \quad (5.3)$$

Proposition 5.2.3. *The family $\tilde{\mathcal{E}}_{2m}$ has the local universal property among locally graded families of semistable $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles.*

Proof. Take any locally graded family $\mathcal{E} \rightarrow X \times T$ of semistable $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles. By Remark 5.2.2 for every $t \in T$ there exists an open subset $U \subset T$ containing t and families $(\mathcal{V}_1, \Phi_1), \dots, (\mathcal{V}_m, \Phi_m)$ of Higgs bundles of rank 1 and degree 0 satisfying (5.3). We take the universal family of zero degree and rank 1 Higgs bundles $\mathcal{E}_{(1,0)} = (\mathcal{V}_{(1,0)}, \Phi_{(1,0)})$ and we know that for every (\mathcal{V}_i, Φ_i) there exists $f_i : U \rightarrow T^*X$ such that $(\mathcal{V}_i, \Phi_i) \sim_S f_i^* \mathcal{E}_{(1,0)}$.

Setting $f = (f_1, \dots, f_m)$ we observe that

$$\mathcal{E}|_{X \times U} \sim_S f^* \tilde{\mathcal{E}}_m.$$

□

The symmetric group \mathfrak{S}_m acts naturally on $(\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2)$ permuting the factors. Using this action we define Γ_m as the semidirect product

$$\Gamma_m = (\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2) \rtimes \mathfrak{S}_m \quad (5.4)$$

determined by the following commutation relations

$$\sigma \bar{c} = (\sigma \cdot \bar{c}) \sigma,$$

for any $\sigma \in \mathfrak{S}_m$ and any $\bar{c} \in (\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2)$.

Let us consider the action of Γ_m on $T^*X \times \dots \times T^*X$ induced by the permutation action of the symmetric group and the following action of group $(\mathbb{Z}_2 \times \dots \times \mathbb{Z}_2)$ on $T^*X \times \dots \times T^*X$

$$\begin{aligned} (1, \dots, 1, -1, 1, \dots, 1) \cdot ((x_1, \lambda_1), \dots, (x_i, \lambda_i), \dots, (x_m, \lambda_m)) &= \\ &= ((x_1, \lambda_1), \dots, (-x_i, -\lambda_i), \dots, (x_m, \lambda_m)). \end{aligned} \quad (5.5)$$

The quotient of this space by Γ_m under the action defined in (5.5) is

$$\mathrm{Sym}^m(T^*X/\mathbb{Z}_2) = T^*X \times \dots \times T^*X / \Gamma_m.$$

Note that (5.5) induces naturally an action of Γ_m on $X \times \dots \times X$ whose quotient is

$$\mathrm{Sym}^m(X/\mathbb{Z}_2) = X \times \dots \times X / \Gamma_m.$$

Remark 5.2.4. The family $\tilde{\mathcal{E}}_{2m}$ is parametrized by $Z_m = T^*X \times \dots \times T^*X$. We know that Γ_m acts on Z_m and we can check that for any two points $z_1, z_2 \in Z_m$ we have $(\tilde{\mathcal{E}}_{2m})_{z_1} \simeq (\tilde{\mathcal{E}}_{2m})_{z_2}$ if and only if $z_1 = \gamma \cdot z_2$ for some $\gamma \in \Gamma_m$ acting as in (5.5).

Theorem 5.2.5. *There exists a moduli space $\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C}))$ associated to the moduli functor $\mathrm{Mod}(\tilde{\mathcal{A}}_m, \tilde{\mathcal{Q}}_m, S)$. We have an isomorphism*

$$\tilde{\xi}_m^{x_0} : \mathcal{M}(\mathrm{Sp}(2m, \mathbb{C})) \xrightarrow{\cong} \mathrm{Sym}^m(T^*X/\mathbb{Z}_2) \quad (5.6)$$

$$[(E, \Omega, \Phi)]_S \longmapsto [[\xi_{1,0}^{x_0}(L_1, \phi_1)]_{\mathbb{Z}_2}, \dots, [\xi_{1,0}^{x_0}(L_m, \phi_m)]_{\mathbb{Z}_2}]_{\mathfrak{S}_m}$$

where $\mathrm{gr}(E, \Omega, \Phi) = \bigoplus_{i=1}^m (L_i \oplus L_i^*, \Omega_i, \phi_i \oplus (-\phi_i))$.

Proof. Since $Z_m/\Gamma_m = \text{Sym}^m(T^*X/\mathbb{Z}_2)$ is an orbit space, the theorem follows from Proposition 5.2.3 and Remark 5.2.4. \square

We study the relation between $\mathcal{M}(\text{Sp}(2m, \mathbb{C}))$ and $\mathcal{M}(\text{GL}(n, \mathbb{C}))$. Let us recall the involution ι_n on $\mathcal{M}(\text{GL}(n, \mathbb{C}))_0$ defined in (4.22). Let (E, Ω, Φ) be a semistable $\text{Sp}(2m, \mathbb{C})$ -Higgs bundle and recall that Ω is an antisymmetric isomorphism between (E, Φ) and $(E^*, -\Phi^t)$. By Proposition 5.1.1, (E, Φ) is semistable, and then, its S-equivalence class is a fixed point of the involution ι_{2m} . Taking the underlying Higgs bundle, we construct the following morphism

$$\tilde{\delta}_m : \mathcal{M}(\text{Sp}(2m, \mathbb{C})) \longrightarrow (\mathcal{M}(\text{GL}(2m, \mathbb{C}))_0)^{\iota_n}$$

$$[(E, \Omega, \Phi)]_S \longmapsto [(E, \Phi)]_S.$$

Recall that, by Lemma 4.3.14, the fixed point subvariety $\mathcal{M}(\text{GL}(n, \mathbb{C}))_0^{\iota_n}$ is isomorphic to the subvariety $(\text{Sym}^n T^*X)^{i_n}$, where the involution i_n is defined in (4.25). When $n = 2m$, we can define the following morphism of quasiprojective varieties

$$\tilde{d}_m : \text{Sym}^m(T^*X/\mathbb{Z}_2) \longrightarrow (\text{Sym}^{2m} T^*X)^{i_{2m}}$$

$$[[x_1, \lambda_1]_{\mathbb{Z}_2}, \dots, [x_m, \lambda_m]_{\mathbb{Z}_2}]_{\mathfrak{S}_m} \longmapsto [(x_1, \lambda_1), (-x_1, -\lambda_1), \dots, (x_m, \lambda_m), (-x_m, -\lambda_m)]_{\mathfrak{S}_{2m}}. \quad (5.7)$$

By Lemma 4.3.14, we have that $\mathcal{M}(\text{GL}(2m, \mathbb{C}))_0^{\iota_{2m}}$ is isomorphic to $(\text{Sym}^{2m} T^*X)^{i_{2m}}$. We denote by $\xi_{n,0}^{x_0, i_{2m}}$ the restriction of $\xi_{n,0}^{x_0}$ to the fixed point locus. The following result gives an explicit description of the morphism of moduli spaces $\tilde{\delta}_m$.

Lemma 5.2.6. *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{M}(\text{Sp}(2m, \mathbb{C})) & \xrightarrow{\tilde{\delta}_m} & \mathcal{M}(\text{GL}(2m, \mathbb{C}))_0^{\iota_{2m}} \\ \tilde{\xi}_m^{x_0} \downarrow \cong & & \cong \downarrow \xi_{n,0}^{x_0, i_{2m}} \\ \text{Sym}^m(T^*X/\mathbb{Z}_2) & \xrightarrow{\tilde{d}_m} & (\text{Sym}^{2m} T^*X)^{i_{2m}}. \end{array}$$

Proof. Starting with the polystable $\text{Sp}(2m, \mathbb{C})$ -Higgs bundle

$$(E, \Omega, \Phi) = \bigoplus_{i=1}^m (L_i \oplus L_i^*, \Omega_i, \phi_i \oplus (-\phi_i)),$$

we have that $\tilde{d}_m \circ \tilde{\xi}_m^{x_0} ([E, \Omega, \Phi]_S)$ is equal to

$$[\xi_{1,0}^{x_0}(L_1, \phi_1), -\xi_{1,0}^{x_0}(L_1, \phi_1), \dots, \xi_{1,0}^{x_0}(L_m, \phi_m), -\xi_{1,0}^{x_0}(L_m, \phi_m)]_{\mathfrak{S}_{2m}}$$

and $\xi_{2m,0}^{x_0} \circ \tilde{\delta}_m ([E, \Omega, \Phi]_S)$ to

$$[\xi_{1,0}^{x_0}(L_1, \phi_1), \xi_{1,0}^{x_0}(L_1^*, -\phi_1), \dots, \xi_{1,0}^{x_0}(L_m, \phi_m), \xi_{1,0}^{x_0}(L_m^*, -\phi_m)]_{\mathfrak{S}_{2m}}.$$

The lemma follows from the observation that $\xi_{1,0}^{x_0}(L_i^*, -\phi_i) = -\xi_{1,0}^{x_0}(L_i, \phi_i)$. \square

Remark 5.2.7. Since \tilde{d}_m is injective, it follows from Lemma 5.2.6, that $\tilde{\delta}_m$ is an injection.

We study now the relation between $\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C}))$ and $M(\mathrm{Sp}(2m, \mathbb{C}))$. Note that $Z_m^0 = (X \times \dots \times X)$ embeds into $Z_m = (T^*X \times \dots \times T^*X)$. Restricting $\tilde{\mathcal{E}}_m$ to $X \times Z_m^0$ gives a family of polystable $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundles with zero Higgs field. Let us denote by $\tilde{\mathcal{E}}_m^0 \rightarrow X \times Z_m^0$ the underlying family of $\mathrm{Sp}(2m, \mathbb{C})$ -bundles. This family induces the following morphism from Z_m^0 to the moduli space of $\mathrm{Sp}(2m, \mathbb{C})$ -bundles

$$\nu_{\tilde{\mathcal{E}}_m^0} : X \times \dots \times X \longrightarrow M(\mathrm{Sp}(2m, \mathbb{C})).$$

By Remark 5.2.4 we see that $\nu_{\tilde{\mathcal{E}}_m^0}$ factors through

$$\nu'_{\tilde{\mathcal{E}}_m^0} : \mathrm{Sym}^m(X/\mathbb{Z}_2) \longrightarrow M(\mathrm{Sp}(2m, \mathbb{C}))$$

which is bijective.

Remark 5.2.8. Since $\nu'_{\tilde{\mathcal{E}}_m^0}$ is a bijective morphism and $M(\mathrm{Sp}(2m, \mathbb{C}))$ is a normal algebraic variety, it is an isomorphism. Taking the inverse gives the following isomorphism

$$\tilde{\zeta}_m^{x_0} : M(\mathrm{Sp}(2m, \mathbb{C})) \xrightarrow{\cong} \mathrm{Sym}^m(X/\mathbb{Z}_2) \quad (5.8)$$

$$[(E, \Omega)]_S \longmapsto [[s_{1,0}^{x_0}(L_1)]_{\mathbb{Z}_2}, \dots, [s_{1,0}^{x_0}(L_m)]_{\mathbb{Z}_2}]_{\mathfrak{S}_m}$$

We have that $\mathrm{Sym}^m(X/\mathbb{Z}_2) \cong (X \times \dots \times X)/\Gamma_m$. This quotient corresponds with the description of the moduli space $M(\mathrm{Sp}(2m, \mathbb{C})) \cong (X \otimes_{\mathbb{Z}} \Lambda)/W$ given in [FM1] where Λ and W are respectively the coroot lattice and the Weyl group of $\mathfrak{sp}(2m, \mathbb{C})$.

Recalling (4.30) we obtain

$$M(\mathrm{Sp}(2m, \mathbb{C})) \cong \mathrm{Sym}^m \mathbb{P}^1 \cong \mathbb{P}^m.$$

This agrees with [FM1] where it is stated that

$$M(\mathrm{Sp}(2m, \mathbb{C})) \cong \mathbb{WP}(1, \dots, 1) \cong \mathbb{P}^m.$$

We define the map

$$\tilde{p}_m : \mathrm{Sym}^m(T^*X/\mathbb{Z}_2) \longrightarrow \mathrm{Sym}^m(X/\mathbb{Z}_2)$$

$$[(x_1, \lambda_1), \dots, (x_m, \lambda_m)]_{\Gamma_m} \longmapsto [x_1, \dots, x_m]_{\Gamma_m}$$

Proposition 5.2.9. *There is a surjective morphism*

$$\tilde{a}_n : \mathcal{M}(\mathrm{Sp}(2m, \mathbb{C})) \longrightarrow M(\mathrm{Sp}(2m, \mathbb{C}))$$

$$[(E, \Omega, \Phi)]_S \longmapsto [(E, \Omega)]_S.$$

Furthermore the diagram

$$\begin{array}{ccc}
\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C})) & \xrightarrow{\tilde{a}_m} & M(\mathrm{Sp}(2m, \mathbb{C})) \\
\tilde{\xi}_m^{x_0} \downarrow \cong & & \cong \downarrow \tilde{\xi}_m^{x_0} \\
\mathrm{Sym}^m(T^*X/\mathbb{Z}_2) & \xrightarrow{\tilde{p}_m} & \mathrm{Sym}^m(X/\mathbb{Z}_2)
\end{array} \tag{5.9}$$

commutes.

Proof. By Proposition 5.1.2, we know that the $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundle (E, Ω, Φ) is semistable if and only if (E, Ω) is semistable.

Take a $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundle (E', Ω', Φ') S-equivalent to (E, Ω, Φ) . Then we have that (E', Ω') belongs to the S-equivalence class of (E, Ω) ; to see this, note that $\mathrm{gr}(E', \Omega', \Phi')$ and $\mathrm{gr}(E, \Omega, \Phi)$ are isomorphic, and therefore, $\mathrm{gr}(E', \Omega') \cong \mathrm{gr}(E, \Omega)$. This proves that the map \tilde{a}_m is well defined.

The diagram commutes by Remark 5.2.8. \square

We have defined an action of Γ_m on $T^*X \times \mathbb{C}^m$. Analogously, we can define the action of Γ_m on \mathbb{C}^m , and the quotient is $\mathrm{Sym}^m(\mathbb{C}/\mathbb{Z}_2)$. Let us consider the following projection

$$\tilde{\pi}_m : \mathrm{Sym}^m(T^*X/\mathbb{Z}_2) \longrightarrow \mathrm{Sym}^m(\mathbb{C}/\mathbb{Z}_2)$$

$$[(x_1, \lambda_1), \dots, (x_m, \lambda_m)]_{\Gamma_m} \longmapsto [\lambda_1, \dots, \lambda_m]_{\Gamma_m}.$$

We know that $q_{2m,2}, q_{2m,4}, \dots, q_{2m,2m}$ form a basis for the invariant polynomials of the adjoint representation of $\mathrm{Sp}(2m, \mathbb{C})$ on $\mathfrak{sp}(2m, \mathbb{C})$. Following [Hi2], we define the Hitchin map evaluating these invariant polynomials on the Higgs field

$$\tilde{b}_m : \mathcal{M}(\mathrm{Sp}(2m, \mathbb{C})) \longrightarrow \tilde{B}_m (= \bigoplus_{i=1}^m H^0(X, \mathcal{O}))$$

$$[(E, \Omega, \Phi)]_S \longmapsto (q_{2m,2}(\Phi), \dots, q_{2m,2m}(\Phi)).$$

Using $q_{2m,2}, q_{2m,4}, \dots, q_{2m,2m}$ we can construct the following isomorphism

$$\tilde{q}_m : \mathrm{Sym}^m(\mathbb{C}/\mathbb{Z}_2) \longrightarrow \mathbb{C}^m$$

$$[[\lambda_1]_{\mathbb{Z}_2}, \dots, [\lambda_m]_{\mathbb{Z}_2}]_{\mathfrak{S}_m} \longmapsto (q_{2m,2}(D_{\lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m}), \dots, q_{2m,2m}(D_{\lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m})), \tag{5.10}$$

where $D_{\lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m}$ is the diagonal matrix with eigenvalues $\lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m$.

Since $H^0(X, \mathcal{O}) \cong \mathbb{C}$ we have that $\tilde{B}_m \cong \mathbb{C}^m$ and composing with \tilde{q}_m we obtain the following isomorphism

$$\tilde{\beta}_m : \mathrm{Sym}^m(\mathbb{C}/\mathbb{Z}_2) \longrightarrow \tilde{B}_m.$$

Lemma 5.2.10. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{Sp}(2m, \mathbb{C})) & \xrightarrow{\tilde{b}_m} & \tilde{B}_m \\ \tilde{\xi}_m^{x_0} \downarrow \cong & & \cong \downarrow \tilde{\beta}_m^{-1} \\ \mathrm{Sym}^m(T^*X/\mathbb{Z}_2) & \xrightarrow{\tilde{\pi}_m} & \mathrm{Sym}^m(\mathbb{C}/\mathbb{Z}_2). \end{array} \quad (5.11)$$

Proof. If (E, Ω, Φ) is a polystable $\mathrm{Sp}(2m, \mathbb{C})$ -Higgs bundle, by Proposition 5.1.6 it decomposes as follows

$$(E, \Omega, \Phi) \cong \bigoplus_{i=1}^m \left(L_i \oplus L_i^*, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda_i & \\ & -\lambda_i \end{pmatrix} \right).$$

Then, the eigenvalues of Φ are $\lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m$ and, $\tilde{b}_m(\Phi)$ composed with $\tilde{\beta}_m$, correspond to the point $[[\lambda_1]_{\mathbb{Z}_2}, \dots, [\lambda_m]_{\mathbb{Z}_2}]_{\mathfrak{S}_m}$ of $\mathrm{Sym}^m(\mathbb{C}/\mathbb{Z}_2)$. \square

The generic element of \tilde{B}_m comes from the following element of $\mathbb{C} \times \dots \times \mathbb{C}$,

$$\bar{\lambda}_{gen} = (\lambda_1, \dots, \lambda_m),$$

where $\lambda_i \neq \pm\lambda_j$ if $i \neq j$ and for every i we have $\lambda_i \neq 0$.

Lemma 5.2.11.

$$\tilde{\pi}_m^{-1}([\bar{\lambda}_{gen}]_{\Gamma_m}) \cong X \times \dots \times X.$$

Proof. Since $\lambda_i \neq -\lambda_i$ and $\lambda_i \neq \pm\lambda_j$ for every i, j such that $i \neq j$, the stabilizer in Γ_m of $\bar{\lambda}_{gen}$ is trivial and then the stabilizer of every tuple of the form

$$((x_1, \lambda_1), \dots, (x_m, \lambda_m))$$

is trivial too. This implies that every such tuple is uniquely determined by the choice of (x_1, \dots, x_m) , and then $\tilde{\pi}_m^{-1}([\bar{\lambda}_{gen}]_{\Gamma_m})$ is isomorphic to $X \times \dots \times X$. \square

We treat now the case of an arbitrary point of \tilde{B}_m . We have that an arbitrary element of $\mathrm{Sym}^m(\mathbb{C}/\mathbb{Z}_2)$ is given by the Γ_m -orbit of the following tuple

$$\bar{\lambda}_{arb} = (0, \overset{m_0}{\cdot}, 0, \lambda_1, \overset{m_1}{\cdot}, \lambda_1, \dots, \lambda_\ell, \overset{m_\ell}{\cdot}, \lambda_\ell),$$

where $\lambda_i \neq 0$, $\lambda_i \neq \pm\lambda_j$ if $i \neq j$ and $m = m_0 + m_1 + \dots + m_\ell$.

Lemma 5.2.12.

$$\tilde{\pi}_m^{-1}([\bar{\lambda}_{arb}]_{\Gamma_m}) \cong \mathbb{P}^{m_0} \times P_{(m_1, -1)} \times \dots \times P_{(m_\ell, -1)}.$$

Proof. The stabilizer in Γ_m of $\bar{\lambda}_{arb}$ is

$$Z_{\Gamma_m}(\bar{\lambda}_{arb}) = \Gamma_{m_0} \times \mathfrak{S}_{m_1} \times \dots \times \mathfrak{S}_{m_\ell}.$$

We have a surjective morphism

$$X \times \dots \times X \longrightarrow \tilde{\pi}_m^{-1}([\bar{\lambda}_{arb}]_{\Gamma_m})$$

$$(x_1, \dots, x_m) \longmapsto [(x_1, 0), \dots, (x_{m_0}, 0), (x_{1+m_0}, \lambda_1), \dots, (x_{m_0+m_1}, \lambda_1), \dots]_{\Gamma_m}.$$

Under this morphism, two tuples give the same element if and only if they are related by the action of $Z_{\Gamma_m}(\bar{\lambda}_{arb})$. Then

$$\begin{aligned} \tilde{\pi}_m^{-1}([\bar{\lambda}_{arb}]_{\Gamma_m}) &\cong (X \times \dots \times X) / Z_{\Gamma_m}(\bar{\lambda}_{arb}) \\ &\cong \text{Sym}^{m_0}(X/\mathbb{Z}_2) \times \text{Sym}^{m_1} X \times \dots \times \text{Sym}^{m_\ell} X. \end{aligned}$$

We recall that $\text{Sym}^{m_0}(X/\mathbb{Z}_2) \cong \text{Sym}^{m_0} \mathbb{P}^1 \cong \mathbb{P}^{m_0}$ and $\text{Sym}^{m_i} X$ is the projective bundle $P_{(m_i, -1)}$. \square

Corollary 5.2.13. *The generic fibre of the Hitchin map for $\text{Sp}(2m, \mathbb{C})$ -Higgs bundles is the abelian variety $X \times \dots \times X$, the Hitchin fibre of an arbitrary element of \tilde{B}_m is a holomorphic fibration over $X \times \dots \times X$ with fibre $\mathbb{P}^{m_0} \times \mathbb{P}^{m_1-1} \times \dots \times \mathbb{P}^{m_\ell-1}$.*

We finish the section studying $\mathfrak{M}(\text{Sp}(2m, \mathbb{C}))$, the moduli space associated to the moduli functor $\text{Mod}(\tilde{\mathcal{A}}_m, \tilde{P}_m, S)$.

Proposition 5.2.14. *We have a bijective morphism $\mathcal{M}(\text{Sp}(n, \mathbb{C})) \rightarrow \mathfrak{M}(\text{Sp}(n, \mathbb{C}))$, hence $\mathcal{M}(\text{Sp}(2m, \mathbb{C}))$ is the normalization of $\mathfrak{M}(\text{Sp}(2m, \mathbb{C}))$.*

Proof. The family $\tilde{\mathcal{E}}_m$ induces a morphism

$$\nu_{\tilde{\mathcal{E}}_m} : T^* X \times \dots \times T^* X \longrightarrow \mathfrak{M}(\text{Sp}(2m, \mathbb{C})),$$

and by Remark 5.2.4 it factors through

$$\nu'_{\tilde{\mathcal{E}}_m} : \text{Sym}^m(T^* X/\mathbb{Z}_2) \longrightarrow \mathcal{M}(\text{Sp}(2m, \mathbb{C})).$$

Let us denote by $\overline{\mathfrak{M}(\text{Sp}(2m, \mathbb{C}))}$ the normalization of $\mathfrak{M}(\text{Sp}(2m, \mathbb{C}))$. We have that $\text{Sym}^m(T^* X/\mathbb{Z}_2)$ is normal, and then, by the universal property of the normalization, we know that $\nu'_{\tilde{\mathcal{E}}_m}$ factors through

$$\nu''_{\tilde{\mathcal{E}}_m} : \text{Sym}^m(T^* X/\mathbb{Z}_2) \longrightarrow \overline{\mathfrak{M}(\text{Sp}(2m, \mathbb{C}))}.$$

This map is an isomorphism since it is a bijection and $\overline{\mathfrak{M}(\text{Sp}(2m, \mathbb{C}))}$ is normal. Then $\mathcal{M}(\text{Sp}(2m, \mathbb{C}))$ is the normalization of $\mathfrak{M}(\text{Sp}(2m, \mathbb{C}))$. \square

Remark 5.2.15. Both moduli spaces would be isomorphic if $\mathfrak{M}(\text{Sp}(2m, \mathbb{C}))$ is normal, but normality in this case is an open question.

5.3 Moduli spaces of $O(n, \mathbb{C})$ -Higgs bundles

Theorem 5.3.1. *Let $\{p_{k,a}\}$, denote isolated points. Suppose $n > 4$. We have*

$$\begin{aligned}\mathfrak{M}^{st}(O(1, \mathbb{C})) &= \{p_{1,0}\} \cup \{p_{1,1}\} \cup \{p_{1,2}\} \cup \{p_{1,3}\}, \\ \mathfrak{M}^{st}(O(2, \mathbb{C})) &= \{p_{2,0}\} \cup \{p_{2,1}\} \cup \{p_{2,2}\} \cup \{p_{2,3}\} \cup \{p_{2,4}\} \cup \{p_{2,5}\}, \\ \mathfrak{M}^{st}(O(3, \mathbb{C})) &= \{p_{3,0}\} \cup \{p_{3,1}\} \cup \{p_{3,2}\} \cup \{p_{3,3}\}, \\ \mathfrak{M}^{st}(O(4, \mathbb{C})) &= \{p_{4,0}\}, \\ \mathfrak{M}^{st}(O(n, \mathbb{C})) &= \emptyset.\end{aligned}$$

Proof. We set

$$\begin{aligned}\{p_{1,a}\} &= \{[(J_a, 1, 0)]_{\cong}\}, \\ \{p_{2,a'}\} &= \{[(J_{b_i}, 1, 0) \oplus (J_{b_j}, 1, 0)]_{\cong}\}, \\ \{p_{3,a}\} &= \{[(J_{b_1}, 1, 0) \oplus (J_{b_2}, 1, 0) \oplus (J_{b_3}, 1, 0)]_{\cong}\},\end{aligned}$$

where $b_i \neq a$ and $b_i \neq b_j$ if $i \neq j$, and

$$\{p_{4,0}\} = \{[(J_0, 1, 0) \oplus (J_1, 1, 0) \oplus (J_2, 1, 0) \oplus (J_3, 1, 0)]_{\cong}\}.$$

After these identifications the proof follows from Proposition 5.1.3. \square

Let $\mathcal{E}_{(1,0)} = (\mathcal{V}_{(1,0)}, \Phi_{(1,0)})$ be the universal family of Higgs bundles with rank 1 and zero degree. We take the non vanishing section of $\Lambda^2(\mathcal{V}_{(1,0)} \oplus \mathcal{V}_{(1,0)}^*)$

$$\mathcal{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have that \mathcal{Q} anticommutes with $\Phi_{(1,0)} \oplus (-\Phi_{(1,0)})$, so

$$\mathring{\mathcal{E}}_{2,0,0} = (\mathcal{V}_{(1,0)} \oplus \mathcal{V}_{(1,0)}^*, \mathcal{Q}, \Phi_{(1,0)} \oplus (-\Phi_{(1,0)}))$$

is a family of $O(2, \mathbb{C})$ -Higgs bundles parametrized by T^*X . By Proposition 5.1.7 we know that $(\mathring{\mathcal{E}}_{2,0,0})_{z_1} \sim_S (\mathring{\mathcal{E}}_{2,0,0})_{z_2}$ if and only if $z_1 = \pm z_2$.

We define $\mathring{\mathcal{E}}'_{2m,0,0}$ as the family of $(O(2, \mathbb{C}) \times \overset{m}{\cdot} \times O(2, \mathbb{C}))$ -Higgs bundles $\mathring{\mathcal{E}}_{2,0,0} \times_X \overset{m}{\cdot}$ parametrized by $T^*X \times \overset{m}{\cdot} \times T^*X$. With the natural inclusion $i : O(2, \mathbb{C}) \times \overset{m}{\cdot} \times O(2, \mathbb{C}) \hookrightarrow O(2m, \mathbb{C})$ we define $\mathring{\mathcal{E}}_{2m,0,0}$ as the extension of structure group $i_* \mathring{\mathcal{E}}'_{2m,0,0}$.

Let $(E'_{k,a}, Q'_k, 0)$ be the stable $O(k, \mathbb{C})$ -bundle associated to $\{p_{k,a}\}$. If $n - k$ is even we define $\mathring{\mathcal{E}}_{n,k,a}$ as the direct product $(E'_{k,a}, Q'_{k,a}, 0) \oplus \mathring{\mathcal{E}}_{(n-k),0,0}$. Note that $\mathring{\mathcal{E}}_{n,k,a}$ is parametrized by $Z_{(n-k)/2} = T^*X \times \overset{(n-k)/2}{\cdot} \times T^*X$.

Remark 5.3.2. By Proposition 5.1.6 we know that every polystable $O(n, \mathbb{C})$ -Higgs bundle is contained in the family $\mathring{\mathcal{E}}_{n,k,a}$ for some value of k and a .

Let $\mathcal{F} \rightarrow X \times T$ be a family of semistable $O(2, \mathbb{C})$ -Higgs bundles given by

$$\mathcal{F} = \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & \vartheta^t \\ \vartheta & 0 \end{pmatrix}, \Phi \oplus \Phi' \right),$$

where \mathcal{V} is a family of line bundles of degree 0, $\vartheta : \mathcal{V} \rightarrow \mathcal{V}$ is an isomorphism, and Φ and Φ' are endomorphism of \mathcal{V} and \mathcal{V} satisfying $\vartheta\Phi = -\Phi'\vartheta^t$. Since $\vartheta \in H^0(X \times T, \mathcal{O}_{X \times T})$ we have that $\vartheta^{-1/2} : \mathcal{V} \rightarrow \mathcal{V}$ gives an isomorphism between

$$\left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & \vartheta^t \\ \vartheta & 0 \end{pmatrix}, \Phi \oplus \Phi' \right) \cong \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Phi \oplus (-\Phi) \right).$$

If $\mathcal{F} \rightarrow X \times T$ is a family of stable $O(k, \mathbb{C})$ -Higgs bundles and T is connected, then by Theorem 5.3.1 for every $t \in T$ we have

$$\mathcal{F}_t \cong (E'_{k,a}, Q'_k, 0)$$

for some stable $O(k, \mathbb{C})$ -Higgs bundle $(E'_{k,a}, Q'_k, 0)$.

Remark 5.3.3. Every locally graded family $\mathcal{E} \rightarrow X \times T$ of semistable $O(n, \mathbb{C})$ -Higgs bundles is such that for every $t \in T$ there exists an open subset $U \subset T$ containing t a set of families $(\mathcal{V}_1, \Phi_1), \dots, (\mathcal{V}_m, \Phi_m)$ of Higgs bundles of rank 1 and degree 0 and a stable $O(k, \mathbb{C})$ -Higgs bundles $(E'_{k,a}, Q'_k, 0)$ such that

$$\mathcal{E}|_{X \times U} \sim_S (E'_{k,a}, Q'_k, 0) \oplus \bigoplus_{j=1}^m \left(\mathcal{V}_j \oplus \mathcal{V}_j^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Phi_j \oplus (-\Phi_j) \right). \quad (5.12)$$

Proposition 5.3.4. *The connected components of $\mathcal{M}(O(2m+1, \mathbb{C}))$ are indexed by $k = 1, 3$ and $a = 0, \dots, n_k - 1$ where $n_1 = 4$, and $n_3 = 4$.*

The connected components of $\mathcal{M}(O(2, \mathbb{C}))$ are indexed by $k = 0, 2$ and $a = 0, \dots, n_k - 1$ where $n_0 = 1$ and $n_2 = 6$.

If $m > 1$, the connected components of $\mathcal{M}(O(2m, \mathbb{C}))$ are indexed by $k = 0, 2, 4$ and $a = 0, \dots, n_k - 1$ where $n_0 = 1$, $n_2 = 6$ and $n_4 = 1$.

Proof. Since the families $\mathring{\mathcal{E}}_{n,k,a}$ are locally graded and they are parametrized by the connected variety $Z_{(n-k)/2}$, all the S-equivalence classes of semistable $O(n, \mathbb{C})$ -Higgs bundles parametrized by $\mathring{\mathcal{E}}_{n,k,a}$ lie in the same connected component of $\mathcal{M}(O(n, \mathbb{C}))$.

On the other hand if $\{p_{k,a}\} \neq \{p_{k',a'}\}$, we see by Remark 5.3.3 that there is no locally graded family of semistable $O(n, \mathbb{C})$ -Higgs bundles connecting an S-equivalence class parametrized by $\mathring{\mathcal{E}}_{n,k,a}$ and an S-equivalence class parametrized by $\mathring{\mathcal{E}}_{n,k',a'}$. \square

Proposition 5.3.5. *The family $\mathring{\mathcal{E}}_{n,k,a}$ has the local universal property among locally graded families of semistable $O(n, \mathbb{C})$ -Higgs bundles parametrized by $\mathcal{M}(O(n, \mathbb{C}))_{k,a}$.*

Proof. Take any locally graded family $\mathcal{E} \rightarrow X \times T$ of semistable $O(n, \mathbb{C})$ -Higgs bundles. By Remark 5.3.3 for every $t \in T$ there exists an open subset $U \subset T$ containing t , families $(\mathcal{V}_1, \Phi_1), \dots, (\mathcal{V}_m, \Phi_m)$ of Higgs bundles of rank 1 and degree 0 and a stable $O(k, \mathbb{C})$ -Higgs bundle $(E'_{k,a}, Q'_k, 0)$ satisfying (5.12).

Recall that $\mathcal{E}_{(1,0)}$ is a universal family, therefore, for every (\mathcal{V}_i, Φ_i) there exists $f_i : U \rightarrow T^*X$ such that $(\mathcal{V}_i, \Phi_i) \sim_S f_i^* \mathcal{E}_{(1,0)}$. Setting $f = (f_1, \dots, f_m)$ we observe that

$$\mathcal{E}|_{X \times U} \sim_S f^* \mathring{\mathcal{E}}_{n,k,a}.$$

\square

We recall that Γ_m defined in (5.4) acts on $Z_m = T^*X \times \dots \times T^*X$.

Remark 5.3.6. Let $m = (n - k)/2$. The family $\mathring{\mathcal{E}}_{n,k,a}$ is parametrized by $Z_m = T^*X \times \dots \times T^*X$. We know that Γ_m acts on Z_m and we can check that for any two points $z_1, z_2 \in Z_m$ we have $(\mathring{\mathcal{E}}_{n,k,a})_{z_1} \simeq (\mathring{\mathcal{E}}_{n,k,a})_{z_2}$ if and only if $z_1 = \gamma \cdot z_2$ for some $\gamma \in \Gamma_m$ with the action described in (5.5).

Theorem 5.3.7. *There exists a coarse moduli space $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a}$ for the moduli functor $\mathrm{Mod}(\mathring{A}_n, \mathring{Q}_n, S)$ for every n, k such that $n - k$ is even.*

Let us set $m = (n - k)/2$, we have an isomorphism

$$\mathring{\xi}_{n,k,a}^{x_0} : \mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a} \xrightarrow{\cong} \mathrm{Sym}^m(T^*X/\mathbb{Z}_2)$$

$$[(E, Q, \Phi)]_S \longmapsto [[\xi_{1,0}^{x_0}(L_1, \lambda_1)]_{\mathbb{Z}_2}, \dots, [\xi_{1,0}^{x_0}(L_m, \lambda_m)]_{\mathbb{Z}_2}]_{\mathfrak{S}_m},$$

where $\mathrm{gr}(E, Q, \Phi) \cong (E'_{k,a}, Q'_k, 0) \oplus \bigoplus \left(L_j \oplus L_j^*, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} \lambda_j & \\ & -\lambda_j \end{pmatrix} \right)$ and $(E'_{k,a}, Q'_k, 0)$ is the stable bundle associated with $\{p_{k,a}\}$.

Proof. Since Z_m/Γ_m is an orbit space, the theorem is a consequence of Proposition 5.3.5 and Remark 5.3.6. \square

Recall the involution ι_n defined in (4.22). Every semistable $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle gives a fixed point of this involution since the quadratic form Q gives an isomorphism between (E, Φ) and $(E^*, -\Phi^t)$. Taking the underlying Higgs bundle, we define the following map

$$\mathring{\delta}_n : \mathcal{M}(\mathrm{O}(n, \mathbb{C})) \longrightarrow \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0^{\iota_n} \quad (5.13)$$

$$[(E, Q, \Phi)]_S \longmapsto [(E, \Phi)]_S.$$

We denote by $\mathring{\delta}_{n,k,a}$ the restriction of $\mathring{\delta}_n$ to the connected component $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a} \subset \mathcal{M}(\mathrm{O}(n, \mathbb{C}))$.

By Lemma 4.3.14, the target space is isomorphic to $(\mathrm{Sym}^n T^*X)^{i_n}$. Recall the points $\{p_{k,a}\}$ defined in Theorem 5.3.1 associated to the stable $\mathrm{O}(k, \mathbb{C})$ -Higgs bundle $(E'_{k,a}, Q'_k, 0)$ and let $[(x_{b(k,a)_1}, 0), \dots, (x_{b(k,a)_k}, 0)]_{\mathfrak{S}_k}$ be the tuple of $\mathrm{Sym}^k T^*X$ given by $\xi_{k,0}^{x_0}(E'_{k,a})$. Setting $m = (n - k)/2$, we define the following morphism

$$\mathring{d}_{n,k,a} : \mathrm{Sym}^m(T^*X/\mathbb{Z}_2) \longrightarrow (\mathrm{Sym}^n T^*X)^{i_n},$$

sending $[(x_1, \lambda_1)]_{\mathbb{Z}_2}, \dots, [(x_m, \lambda_m)]_{\mathbb{Z}_2}]_{\mathfrak{S}_m}$ in $\mathrm{Sym}^m(T^*X/\mathbb{Z}_2)$ to the following point of $(\mathrm{Sym}^n T^*X)^{i_n}$

$$[(x_{b(k,a)_1}, 0), \dots, (x_{b(k,a)_k}, 0), (x_1, \lambda_1), (-x_1, -\lambda_1), \dots, (x_m, \lambda_m), (-x_m, -\lambda_m)]_{\mathfrak{S}_n}.$$

Note that $\mathring{d}_{2m,0,0}$ is equal to \tilde{d}_m , defined in (5.7). This morphism provides an explicit description of $\mathring{\delta}_n$.

Lemma 5.3.8. *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a} & \xrightarrow{\mathring{\delta}_{n,k,a}} & \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0^{2n} \\ \xi_{n,k,a}^{x_0} \downarrow \cong & & \cong \downarrow \xi_{n,0}^{x_0, i_n} \\ \mathrm{Sym}^{(n-k)/2}(T^*X/\mathbb{Z}_2) & \xrightarrow{\mathring{d}_{n,k,a}} & (\mathrm{Sym}^n T^*X)^{i_n} \end{array}$$

Proof. The polystable representative of every S-equivalence class of $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a}$ is of the form

$$(E, Q, \Phi) \cong (E'_{k,a}, Q'_k, 0) \oplus \bigoplus \left(L_j \oplus L_j^*, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} \lambda_j & \\ & -\lambda_j \end{pmatrix} \right)$$

where $(E'_{k,a}, Q'_k, 0)$ is the stable bundle associated with $\{p_{k,a}\}$ and $\xi_{k,0}^{x_0}((E'_{k,a}, 0))$ is the tuple $[(x_{b(k,a)_1}, 0), \dots, (x_{b(k,a)_k}, 0)]_{\mathfrak{S}_k}$. We see that $\mathring{d}_{n,k,a} \circ \xi_{n,k,a}^{x_0}([(E, Q, \Phi)]_S)$ is

$$[(x_{b(k,a)_1}, 0), \dots, (x_{b(k,a)_k}, 0), \xi_{1,0}^{x_0}(L_1, \phi_1), -\xi_{1,0}^{x_0}(L_1, \phi_1), \dots, \xi_{1,0}^{x_0}(L_m, \phi_m), -\xi_{1,0}^{x_0}(L_m, \phi_m)]_{\mathfrak{S}_n}$$

while $\xi_{n,k,a}^{x_0, i_n} \circ \mathring{\delta}_{n,k,a}([(E, Q, \Phi)]_S)$ is equal to

$$[(x_{b(k,a)_1}, 0), \dots, (x_{b(k,a)_k}, 0), \xi_{1,0}^{x_0}(L_1, \phi_1), \xi_{1,0}^{x_0}(L_1^*, -\phi_1), \dots, \xi_{1,0}^{x_0}(L_m, \phi_m), \xi_{1,0}^{x_0}(L_m^*, -\phi_m)]_{\mathfrak{S}_n}.$$

We conclude the proof recalling that $\xi_{1,0}^{x_0}(L_i^*, -\phi_i) = -\xi_{1,0}^{x_0}(L_i, \phi_i)$. \square

Remark 5.3.9. Since the maps $\mathring{d}_{n,k,a}$ are injective, thanks to Lemma 5.3.8, we have that the $\mathring{\delta}_{n,k,a}$ are injective as well. We can check that $(\mathrm{Sym}^n T^*X)^{i_n}$ is the disjoint union of the images of $\mathring{d}_{n,k,a}$ and therefore, $\mathring{\delta}_n$ is a bijection.

Recall that $Z_m^0 = (X \times \cdot^m \times X)$ injects naturally in $Z_m = (T^*X \times \cdot^m \times T^*X)$. If $m = (n-k)/2$, taking the restriction of $\mathring{\mathcal{E}}_{n,k,a}$ to $X \times Z_m^0$ we obtain a family of polystable $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles with zero Higgs field. We denote by $\mathring{\mathcal{E}}_{n,k,d}^0 \rightarrow X \times Z_m^0$ the underlying family of $\mathrm{O}(n, \mathbb{C})$ -bundles. This induces a morphism from its parametrizing space to the moduli space of $\mathrm{O}(n, \mathbb{C})$ -bundles

$$\nu_{\mathring{\mathcal{E}}_{n,k,d}^0} : X \times \cdot^m \times X \longrightarrow M(\mathrm{O}(n, \mathbb{C}))_{n,k,a}.$$

Thanks to Remark 5.3.6, we see that $\nu_{\mathring{\mathcal{E}}_{n,k,d}^0}$ induces a bijective morphism

$$\nu'_{\mathring{\mathcal{E}}_{n,k,d}^0} : \mathrm{Sym}^m(X/\mathbb{Z}_2) \longrightarrow M(\mathrm{O}(n, \mathbb{C}))_{n,k,a}.$$

Remark 5.3.10. The moduli space $M(\mathrm{O}(n, \mathbb{C}))_{k,a}$ is a normal algebraic variety and therefore $\nu'_{\mathring{\mathcal{E}}_{n,k,d}^0}$ is an isomorphism. Its inverse goes as follows

$$\xi_{n,k,a}^{x_0} : M(\mathrm{O}(n, \mathbb{C}))_{k,a} \xrightarrow{\cong} \mathrm{Sym}^m(X/\mathbb{Z}_2) \tag{5.14}$$

$$[(E, Q)]_S \longmapsto [[\xi_{1,0}^{x_0}(L_1)]_{\mathbb{Z}_2}, \dots, [\xi_{1,0}^{x_0}(L_m)]_{\mathbb{Z}_2}]_{\mathfrak{S}_m},$$

and then

$$M(\mathrm{O}(n, \mathbb{C}))_{k,a} \cong \mathrm{Sym}^{(n-k)/2} \mathbb{P}^1 \cong \mathbb{P}^{(n-k)/2}.$$

Proposition 5.3.11. *There is a surjective morphism*

$$\begin{aligned} \mathring{a}_{n,k,a} : \mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a} &\longrightarrow M(\mathrm{O}(n, \mathbb{C}))_{k,a} \\ [(E, Q, \Phi)]_S &\longmapsto [(E, Q)]_S. \end{aligned}$$

Furthermore the diagram

$$\begin{array}{ccc} \mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a} & \xrightarrow{\mathring{a}_{n,k,a}} & M(\mathrm{O}(n, \mathbb{C}))_{k,a} \\ \xi_{n,k,a}^{x_0} \downarrow \cong & & \cong \downarrow \xi_{n,k,a}^{x_0} \\ \mathrm{Sym}^{(n-k)/2}(T^*X/\mathbb{Z}_2) & \xrightarrow{\tilde{P}^{(n-k)/2}} & \mathrm{Sym}^{(n-k)/2}(X/\mathbb{Z}_2) \end{array} \quad (5.15)$$

commutes.

Proof. Proposition 5.1.2 says that (E, Q, Φ) is semistable if and only if (E, Q) is semistable. We can check too that the projection preserves S-equivalence. Thus the map $\mathring{a}_{n,k,a}$ is well defined. Remark 5.3.10 implies that the diagram commutes. \square

When we restrict to $\mathrm{Sym}^n X$, the involution i_n on $\mathrm{Sym}^n T^*X$ induces the following involution

$$i_n^0 : \mathrm{Sym}^n X \longrightarrow \mathrm{Sym}^n X$$

$$[x_1, \dots, x_n]_{\mathfrak{S}_n} \longmapsto [-x_1, \dots, -x_n]_{\mathfrak{S}_n}.$$

Since an involution can be seen as the action of the finite group \mathbb{Z}_2 , the fixed point subvariety $(\mathrm{Sym}^n X)^{i_n^0}$ is smooth by the following result.

Lemma 5.3.12. *Let Γ be a finite group with a holomorphic action on the smooth holomorphic variety Y . Then, the fixed point set Y^Γ is a smooth holomorphic subvariety.*

Proof. Let h be a Γ -invariant holomorphic metric on Y . To see that such a metric exists take any holomorphic metric h' on Y and define

$$h = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma^* h'.$$

For any $y \in Y^\Gamma$, the exponential map associated to y and h gives a Γ -equivariant isomorphism $\alpha : U \rightarrow V$, where $U \subset T_y Y$ is a neighborhood of 0 and $V \subset Y$ a neighborhood y , both neighborhoods V and U can be taken to be Γ -invariant.

Since Γ acts holomorphically and α is Γ -equivariant, we obtain, by restriction, a local isomorphism $\alpha|_{U^\Gamma} : U^\Gamma \rightarrow V^\Gamma$ between the fixed point sets.

The action of Γ on $T_y Y$ is linear, and then the fixed point set $(T_y Y)^\Gamma$ is a linear subspace. Restricting to the Γ -invariant neighborhood U , the fixed point set is $U^\Gamma = U \cap (T_y Y)^\Gamma$, which is smooth. This implies that V^Γ is also smooth. \square

Remark 5.3.13. Combining (5.15), Lemma 5.3.8 and Remark 5.3.9 we obtain a bijection from $M(\mathrm{O}(n, \mathbb{C}))$ to $(\mathrm{Sym}^n X)_n^{i_0}$ that we denote by $\zeta_n^{x_0}$. Since $(\mathrm{Sym}^n X)_n^{i_0}$ is smooth by Lemma 5.3.12, $\zeta_n^{x_0}$ is an isomorphism.

The polynomials $q_{2m,2}, \dots, q_{2m,2m}$ give a basis for the invariant polynomials associated to the adjoint representation of $\mathrm{O}(2m, \mathbb{C})$ on the Lie algebra $\mathfrak{so}(2m, \mathbb{C})$. Similarly, we see that $q_{2m+1,2}, \dots, q_{2m+1,2m}$ form a basis for the invariant polynomials associated to the adjoint representation of $\mathrm{O}(2m+1, \mathbb{C})$ on the Lie algebra $\mathfrak{so}(2m+1, \mathbb{C})$. Following [Hi2] we define the Hitchin map for $\mathrm{O}(2m, \mathbb{C})$ -Higgs bundles

$$\mathring{b}_{2m} : \mathcal{M}(\mathrm{O}(2m, \mathbb{C})) \longrightarrow \mathring{B}_{2m} (= \bigoplus_{i=1}^m H^0(X, \mathcal{O}))$$

$$[(E, Q, \Phi)]_S \longmapsto (q_{2m,2}(\Phi), \dots, q_{2m,2m}(\Phi)),$$

and the Hitchin map for $\mathrm{O}(2m+1, \mathbb{C})$ -Higgs bundles

$$\mathring{b}_{2m+1} : \mathcal{M}(\mathrm{O}(2m+1, \mathbb{C})) \longrightarrow \mathring{B}_{2m+1} (= \bigoplus_{i=1}^m H^0(X, \mathcal{O}))$$

$$[(E, Q, \Phi)]_S \longmapsto (q_{2m+1,2}(\Phi), \dots, q_{2m+1,2m}(\Phi)).$$

By Proposition 5.1.6, the polystable $\mathrm{O}(n, \mathbb{C})$ -Higgs bundle (E, Q, Φ) decomposes as follows

$$(E, Q, \Phi) \cong (E_{k,a}, Q_{k,a}, 0) \oplus \bigoplus_{i=1}^{(n-k)/2} \left(L_i \oplus L_i^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda_i & \\ & -\lambda_i \end{pmatrix} \right),$$

where the stable $\mathrm{O}(k, \mathbb{C})$ -Higgs bundle $(E_{k,a}, Q_{k,a}, 0)$ is zero if $k = 0$. We see that the eigenvalues of Φ are $0, \dots, 0, \lambda_1, -\lambda_1, \dots, \lambda_{(n-k)/2}, -\lambda_{(n-k)/2}$ and therefore, $q_{n,j}(\Phi) = 0$ for every j such that $(n-k) < j \leq n$. If we denote by $\mathring{B}_{n,k,a} \subset \mathring{B}$ the image of $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a}$ under \mathring{b}_n , the previous discussion implies that $\mathring{B}_{n,k,a} = \bigoplus_{i=1}^{(n-k)/2} H^0(X, \mathcal{O})$. Since $H^0(X, \mathcal{O}) \cong \mathbb{C}$, we have that $\tilde{q}_{(n-k)/2}$ defined in (5.10) gives the following isomorphism

$$\mathring{\beta}_{n,k,a} : \mathrm{Sym}^{(n-k)/2}(\mathbb{C}/\mathbb{Z}_2) \longrightarrow \mathring{B}_{n,k,a}.$$

Lemma 5.3.14. *If $\mathring{b}_{n,k,a}$ is the restriction of \mathring{b}_n to $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a} \subset \mathcal{M}(\mathrm{O}(n, \mathbb{C}))$, we have the commutative diagram*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a} & \xrightarrow{\mathring{b}_{n,k,a}} & \mathring{B}_{n,k,a} \\ \xi_{n,k,a}^{x_0} \downarrow \cong & & \cong \downarrow \mathring{\beta}_{n,k,a}^{-1} \\ \mathrm{Sym}^{(n-k)/2}(T^*X/\mathbb{Z}_2) & \xrightarrow{\tilde{\pi}_{(n-k)/2}} & \mathrm{Sym}^{(n-k)/2}(\mathbb{C}/\mathbb{Z}_2). \end{array} \quad (5.16)$$

Proof. If (E, Q, Φ) is a bundle of $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{n,k,a}$, we recall that the eigenvalues of Φ are $0, \dots, 0, \lambda_1, -\lambda_1, \dots, \lambda_{(n-k)/2}, -\lambda_{(n-k)/2}$. Then, it is clear that the composition of $\mathring{b}_{n,k,a}^{x_0}(\Phi)$ with $\mathring{\beta}_{n,k,a}^{-1}$ gives the point $[[\lambda_1]_{\mathbb{Z}_2}, \dots, [\lambda_{(n-k)/2}]_{\mathbb{Z}_2}]_{\mathfrak{S}_{(n-k)/2}}$ of $\mathrm{Sym}^{(n-k)/2}(\mathbb{C}/\mathbb{Z}_2)$. \square

Thanks to Lemma 5.2.11 and Lemma 5.2.12, we have the following.

Corollary 5.3.15. *The generic fibre of the Hitchin map $\mathring{b}_{n,k,a}$ is the abelian variety $X \times \dots \times X$ and the Hitchin fibre of an arbitrary element of $\mathring{B}_{n,k,a}$ is a holomorphic fibration over $X \times \dots \times X$ with fibre $\mathbb{P}^{m'_0} \times \mathbb{P}^{m'_1-1} \times \dots \times \mathbb{P}^{m'_\ell}$.*

We finish the section studying $\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))_{k,a}$, the moduli space associated to the moduli functor $\mathrm{Mod}(\mathring{A}_n, \mathring{P}_n, S)$.

Proposition 5.3.16. *We have a bijective morphism $\mathcal{M}(\mathrm{O}(n, \mathbb{C})) \rightarrow \mathfrak{M}(\mathrm{O}(n, \mathbb{C}))$, hence $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a}$ is the normalization of $\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))_{k,a}$.*

Proof. The family $\mathring{\mathcal{E}}_{n,k,a}$ induces a morphism

$$\nu_{\mathring{\mathcal{E}}_{n,k,a}} : T^*X \times \dots \times T^*X \longrightarrow \mathfrak{M}(\mathrm{O}(n, \mathbb{C}))_{k,a},$$

and by Remark 5.3.6 it factors through

$$\nu'_{\mathring{\mathcal{E}}_{n,k,a}} : \mathrm{Sym}^{(n-k)/2}(T^*X/\mathbb{Z}_2) \longrightarrow \mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a}.$$

Let us denote by $\overline{\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))}$ the normalization of $\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))$. Let us remark that $\mathrm{Sym}^{(n-k)/2}(T^*X)$ is normal. Then, by the universal property of the normalization, $\overline{\nu}_{\mathring{\mathcal{E}}_{n,k,a}}$ factors through

$$\nu''_{\mathring{\mathcal{E}}_{n,k,a}} : \mathrm{Sym}^{(n-k)/2}(T^*X/\mathbb{Z}_2) \longrightarrow \overline{\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))}_{k,a}.$$

This map is an isomorphism since it is a bijection and $\overline{\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))}$ is normal. This implies that $\mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,a}$ is the normalization of $\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))_{k,a}$. \square

Remark 5.3.17. Both moduli spaces would be isomorphic if $\mathfrak{M}(\mathrm{O}(n, \mathbb{C}))_{k,a}$ is normal, but normality in this case is an open question.

5.4 Moduli spaces of $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles

Due to the isomorphism $\mathrm{SO}(2, \mathbb{C}) \cong \mathbb{C}^*$, we have that $\mathrm{SO}(2, \mathbb{C})$ -bundles are indexed by the degree of the associated line bundle.

When $n > 2$ the topological $\mathrm{SO}(n, \mathbb{C})$ -bundles are classified by the second Stiefel-Whitney class. This invariant is defined thanks to the following exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(n, \mathbb{C}) \longrightarrow \mathrm{SO}(n, \mathbb{C}) \longrightarrow 0,$$

which induces the long exact sequence

$$H^1(X, \mathrm{Spin}(n, \mathbb{C})) \longrightarrow H^1(X, \mathrm{SO}(n, \mathbb{C})) \xrightarrow{\omega_2} H^2(X, \mathbb{Z}/2\mathbb{Z}) (\cong \mathbb{Z}/2\mathbb{Z}).$$

We define the second Stiefel Whitney class as the image of a $\mathrm{SO}(n, \mathbb{C})$ -bundle under ω_2 . We note that the $\mathrm{SO}(n, \mathbb{C})$ -bundles with $\omega_2 = 0$ can be lifted to a $\mathrm{Spin}(n, \mathbb{C})$ -principal bundle, and those with $\omega_1 = 1$ can not.

We recall the description of the stable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles $(E_i^{st}, Q_i^{st}, \Phi_i^{st}, \tau_i^{st})$ given in Corollary 5.1.4, where $i = 1, 3$ and 4 .

Proposition 5.4.1. *Let $n > 2$. Let (E, Q, Φ, τ) be a polystable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle. Let ω_2 denote the second Stiefel-Whitney class of (E, Q, Φ, τ) .*

1. *if $n = 2n'$ and $\omega_2 = 0$, then $\mathrm{gr}(E, Q, \Phi, \tau)$ is a direct sum of $m_{n, \omega_2} = n'$ stable $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles of trivial degree,*
2. *if $n = 2n' + 1$ and $\omega_2 = 0$ then $\mathrm{gr}(E, Q, \Phi, \tau)$ is a direct sum of $(E_1^{st}, Q_1^{st}, \Phi_1^{st}, \tau_1^{st})$ and $m_{n, \omega_2} = n'$ stable $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles of trivial degree,*
3. *if $n = 2n' + 1$ and $\omega_2 = 1$ then $\mathrm{gr}(E, Q, \Phi, \tau)$ is a direct sum of $(E_3^{st}, Q_3^{st}, \Phi_3^{st}, \tau_3^{st})$ and $m_{n, \omega_2} = n' - 1$ stable $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles of trivial degree,*
4. *if $n = 2n'$ and $\omega_2 = 1$ then $\mathrm{gr}(E, Q, \Phi, \tau)$ is a direct sum of $(E_4^{st}, Q_4^{st}, \Phi_4^{st}, \tau_4^{st})$ and $m_{n, \omega_2} = n' - 2$ stable $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles of trivial degree.*

Proof. By Proposition 5.1.6 and Corollary 5.1.4 a semistable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundle is of the form of one of the four cases indexed in the statement. It only remains to prove that the $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles of type 1 and 2 have trivial Stiefel-Whitney class and the $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles of type 3 and 4 have non-trivial Stiefel-Whitney class.

By [FM1, Proposition 7.7 and Theorem 7.8], if (E, Q, τ) is a semistable $\mathrm{SO}(2n', \mathbb{C})$ -bundle that lifts to a $\mathrm{Spin}(2n', \mathbb{C})$ -bundle, then $(E', Q, \tau') = \mathrm{gr}(E, Q, \tau)$ is such that

$$E' \cong \bigoplus_{i=1}^m (L_i \oplus L_i^*).$$

Conversely, if (E, Q, τ) does not lift to $\mathrm{Spin}(2n', \mathbb{C})$, we have

$$E' \cong \mathcal{O} \oplus J_1 \oplus J_2 \oplus J_3 \oplus \bigoplus_{i=1}^{m-2} (L_i \oplus L_i^*).$$

This implies that the $\mathrm{SO}(2n', \mathbb{C})$ -Higgs bundles of type 1 lift to $\mathrm{Spin}(2m, \mathbb{C})$ and therefore they have trivial Stiefel-Whitney class, while the $\mathrm{SO}(2n', \mathbb{C})$ -Higgs bundles of type 4 do not lift to $\mathrm{Spin}(2m, \mathbb{C})$ and they have non-trivial Stiefel-Whitney class.

The odd case is analogous. By [FM1, Proposition 7.7 and Theorem 7.8], a semistable $\mathrm{SO}(2n', \mathbb{C})$ -bundle (E, Q, τ) that lift to a $\mathrm{Spin}(2n' + 1, \mathbb{C})$ -bundle is such that

$$E' \cong \bigoplus_{i=1}^m (L_i \oplus L_i^*),$$

and, if (E, Q, τ) does not lift to $\mathrm{Spin}(2n' + 1, \mathbb{C})$, then

$$E' \cong \mathcal{O} \oplus J_1 \oplus J_2 \oplus J_3 \oplus \bigoplus_{i=1}^{m-2} (L_i \oplus L_i^*).$$

Then, the $\mathrm{SO}(2n' + 1, \mathbb{C})$ -Higgs bundles with trivial Stiefel-Whitney class are the ones of type 2 and the $\mathrm{SO}(2n' + 1, \mathbb{C})$ -Higgs bundles with non-trivial Stiefel-Whitney class of the form of case 3.

□

Theorem 5.4.2. *We have*

$$\begin{aligned}
\mathfrak{M}^{st}(\mathrm{SO}(1, \mathbb{C}))_0 &= \{[(E_1^{st}, Q_1^{st}, \Phi_1^{st}, \tau_1^{st})]_{\cong}\}, \\
\mathfrak{M}^{st}(\mathrm{SO}(1, \mathbb{C}))_1 &= \emptyset, \\
\mathfrak{M}^{st}(\mathrm{SO}(2, \mathbb{C}))_d &= \mathfrak{M}(\mathrm{SO}(2, \mathbb{C}))_d \cong T^*X, \\
\mathfrak{M}^{st}(\mathrm{SO}(3, \mathbb{C}))_0 &= \emptyset, \\
\mathfrak{M}^{st}(\mathrm{SO}(3, \mathbb{C}))_1 &= \{[(E_3^{st}, Q_3^{st}, \Phi_3^{st}, \tau_3^{st})]_{\cong}\}, \\
\mathfrak{M}^{st}(\mathrm{SO}(4, \mathbb{C}))_0 &= \emptyset, \\
\mathfrak{M}^{st}(\mathrm{SO}(4, \mathbb{C}))_1 &= \{[(E_4^{st}, Q_4^{st}, \Phi_4^{st}, \tau_4^{st})]_{\cong}\}.
\end{aligned}$$

Suppose $n > 4$ and let ω_2 be either 0 or 1, then

$$\mathfrak{M}^{st}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2} = \emptyset.$$

Proof. This is a straightforward consequence of Corollary 5.1.4 and Proposition 5.4.1. The description of $\mathcal{M}(\mathrm{SO}(2, \mathbb{C}))$ follows from the isomorphism of Lie groups $\mathrm{SO}(2, \mathbb{C}) \cong \mathbb{C}^*$. \square

We recall $\mathring{\mathcal{E}}_{2,0,0} = (\mathcal{V}_{(1,0)} \oplus \mathcal{V}_{(1,0)}^*, \mathcal{Q}, \Phi_{(1,0)} \oplus (-\Phi_{(1,0)}))$ where \mathcal{Q} is a section of $\mathcal{O}_{x \times T^*X} \subset \mathrm{Sym}^2(\mathcal{V}_{(1,0)} \oplus \mathcal{V}_{(1,0)}^*)$ given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We see that $(\det \mathcal{Q})^{-1}$ is the section -1 of $\mathcal{O}_{X \times T^*X}$. Then the section τ of $\det(\mathcal{V}_{(1,0)} \otimes \mathcal{V}_{(1,0)}^*)$ can be taken to be the imaginary number $\sqrt{-1}$ or $-\sqrt{-1}$. We fix $\tau = \sqrt{-1}$ and we construct the following family of $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles of zero degree

$$\bar{\mathcal{E}}_2 = (\mathcal{V}_{(1,0)} \oplus \mathcal{V}_{(1,0)}^*, \mathcal{Q}, \Phi_{(1,0)} \oplus (-\Phi_{(1,0)}), \sqrt{-1}).$$

Denoting by $\bar{\mathcal{E}}'_{2m}$ the family of $(\mathrm{SO}(2, \mathbb{C}) \times \dots \times \mathrm{SO}(2, \mathbb{C}))$ -Higgs bundles $\bar{\mathcal{E}}_2 \times_X \dots \times_X \bar{\mathcal{E}}_2$, we see that this family is universal since $\bar{\mathcal{E}}_2$ is. Note that $\bar{\mathcal{E}}'_{2m}$ is parametrized by $Z_m = T^*X \times \dots \times T^*X$. Write j for the injection of $\mathrm{SO}(2, \mathbb{C}) \times \dots \times \mathrm{SO}(2, \mathbb{C})$ in $\mathrm{SO}(2m, \mathbb{C})$ and define $\bar{\mathcal{E}}_{2m,1}$ to be the extension of structure group $j_* \bar{\mathcal{E}}'_{2m}$.

We fix (n, ω_2) where $n > 2$ and set $m = m_{n, \omega_2}$. Let us take the stable $\mathrm{SO}(k, \mathbb{C})$ -Higgs bundle $(E_i^{st}, Q_i^{st}, \Phi_i^{st}, \tau_i^{st})$ associated to (n, ω_2) (it is possibly zero) and define

$$\bar{\mathcal{E}}_{n, \omega_2} = (E_i^{st}, Q_i^{st}, \Phi_i^{st}, \tau_i^{st}) \oplus \bar{\mathcal{E}}_{2m}.$$

We see that $\bar{\mathcal{E}}_{n, \omega_2}$ is a family of $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles which is parametrized by Z_m .

Remark 5.4.3. By construction, the family of $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles underlying $\bar{\mathcal{E}}_{n, \omega}$ is $\mathring{\mathcal{E}}_{n, k, 0}$ where $\{p_{k, 0}\}$ corresponds to the stable factor of $\bar{\mathcal{E}}_{n, \omega_2}$.

Suppose we have a family of stable $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles of trivial degree $\mathcal{F} \rightarrow X \times T$ of the form

$$\mathcal{F} = \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & \vartheta^t \\ \vartheta & 0 \end{pmatrix}, \Phi \oplus \Phi', \mathcal{T} \right),$$

where \mathcal{V} is a family of line bundles of degree 0, $\vartheta : \mathcal{V} \rightarrow \mathcal{V}$ is an isomorphism, Φ and Φ' are endomorphism of \mathcal{V} and \mathcal{V} (i.e. a section of $\mathcal{O}_{X \times T}$) satisfying $\vartheta \Phi = -(\Phi')^t \vartheta$ and \mathcal{T}

is a section of $\mathcal{O}_{X \times T}$ such that $\mathcal{T}^2 = -(\det \vartheta)^2$. We have that $\vartheta^{-1/2} : \mathcal{V} \rightarrow \mathcal{V}$ gives the isomorphism

$$\left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & \vartheta^t \\ \vartheta & 0 \end{pmatrix}, \Phi \oplus \Phi', \mathcal{T} \right) \cong \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Phi \oplus (-\Phi), \sqrt{-1} \right), \quad (5.17)$$

or

$$\left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & \vartheta^t \\ \vartheta & 0 \end{pmatrix}, \Phi \oplus \Phi', \mathcal{T} \right) \cong \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Phi \oplus (-\Phi), -\sqrt{-1} \right). \quad (5.18)$$

In case we obtain the isomorphism (5.18) setting $\mathcal{W} = \mathcal{V}^*$ we have

$$\left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Phi \oplus (-\Phi), -\sqrt{-1} \right) \cong \left(\mathcal{W} \oplus \mathcal{W}^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (-\Phi) \oplus \Phi, \sqrt{-1} \right),$$

and we see that every such family is isomorphic to one of the form of the second side of (5.17).

If $\mathcal{F} \rightarrow X \times T$ is a family of stable $\mathrm{SO}(k, \mathbb{C})$ -Higgs bundles and T is connected, we have by Theorem 5.4.2 that for every $t \in T$

$$\mathcal{F}_t \cong (E'_{k,a}, Q'_k, 0, 1)$$

where $(E'_{k,a}, Q'_k, 0, 1)$ is a stable $\mathrm{SO}(k, \mathbb{C})$ -Higgs bundle.

Remark 5.4.4. Every locally graded family $\mathcal{E} \rightarrow X \times T$ of semistable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles is such that for every $t \in T$ there exists an open subset $U \subset T$ containing t a set of families $(\mathcal{V}_1, \Phi_1), \dots, (\mathcal{V}_m, \Phi_m)$ of Higgs bundles of rank 1 and degree 0 and a stable $\mathrm{SO}(k, \mathbb{C})$ -Higgs bundle $(E'_{k,a}, Q'_k, 0, 1)$ such that

$$\mathcal{E}|_{X \times U} \sim_S (E'_{k,a}, Q'_k, 0, 1) \oplus \bigoplus_{j=1}^m \left(\mathcal{V}_j \oplus \mathcal{V}_j^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Phi_j \oplus (-\Phi_j), \sqrt{-1} \right). \quad (5.19)$$

Proposition 5.4.5. *The family $\overline{\mathcal{E}}_{n, \omega_2}$ has the local universal property among locally graded families of semistable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles.*

Proof. The proof is similar to the proof of Proposition 5.3.5. By Remark 5.3.3, any locally graded family $\mathcal{E} \rightarrow X \times T$ of semistable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles is such that for every $t \in T$ there exists an open subset $U \subset T$ containing t , families $(\mathcal{V}_1, \Phi_1), \dots, (\mathcal{V}_m, \Phi_m)$ of Higgs bundles of rank 1 and degree 0 and a stable $\mathrm{SO}(k, \mathbb{C})$ -Higgs bundles of the form $(E'_{k,a}, Q'_k, 0, 1)$ satisfying (5.19).

For $(n, \omega_2) = (2m, 0)$ we have that $(E'_{k,a}, Q'_k, 0) = 0$, if $(n, \omega_2) = (2m, 1)$ we have $(E'_{k,a}, Q'_k, 0) = (E_3^{st}, Q_3^{st}, 0, 1)$, for $(n, \omega_2) = (2m + 1, 0)$ we have $(E'_{k,a}, Q'_k, 0) = (E_1^{st}, Q_1^{st}, 0, 1)$ and $(E'_{k,a}, Q'_k, 0) = (E_4^{st}, Q_4^{st}, 0, 1)$ for $(n, \omega_2) = (2m + 1, 1)$. Recall that $\mathcal{E}_{(1,0)}$ is the universal family of rank 1 and degree 0 Higgs bundles, so, for every (\mathcal{V}_i, Φ_i) , there exists $f_i : U \rightarrow T^*X$ such that $(\mathcal{V}_i, \Phi_i) \sim_S f_i^* \mathcal{E}_{(1,0)}$.

Setting $f = (f_1, \dots, f_m)$ we observe that

$$\mathcal{E}|_{X \times U} \sim_S f^* \overline{\mathcal{E}}_{n, \omega_2}.$$

□

We recall the definition of Γ_m given in (5.4).

Remark 5.4.6. Let $n > 2$ and let (n, ω_2) be $(2n', 1)$, $(2n' + 1, 0)$ or $(2n' + 1, 1)$. From Proposition 5.1.8 we know that for every points $z_1, z_2 \in Z_m$ we have that $(\overline{\mathcal{E}}_{n, \omega_2})_{z_1} \cong (\overline{\mathcal{E}}_{n, \omega_2})_{z_2}$ if and only if z_1 and z_2 are related by the action of Γ_m given in (5.5).

Theorem 5.4.7. For (n, ω_2) equal to $(2m, 1)$, $(2m + 1, 0)$ and $(2m + 1, 1)$ there exists a moduli space $\mathcal{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2}$ associated to the moduli functor $\mathrm{Mod}(\overline{\mathcal{A}}_n, \overline{\mathcal{Q}}_n, S)$.

We have the following isomorphisms

$$\overline{\xi}_{2m+1,0}^{x_0} : \mathcal{M}(\mathrm{SO}(2m + 1, \mathbb{C}))_0 \xrightarrow{\cong} \mathrm{Sym}^m(T^*X/\mathbb{Z}_2),$$

$$\overline{\xi}_{2m+1,1}^{x_0} : \mathcal{M}(\mathrm{SO}(2m + 1, \mathbb{C}))_1 \xrightarrow{\cong} \mathrm{Sym}^{m-1}(T^*X/\mathbb{Z}_2),$$

and

$$\overline{\xi}_{2m,1}^{x_0} : \mathcal{M}(\mathrm{SO}(2m, \mathbb{C}))_1 \xrightarrow{\cong} \mathrm{Sym}^{m-2}(T^*X/\mathbb{Z}_2).$$

Proof. The theorem follows from Proposition 5.4.5 and Remark 5.4.6. \square

Remark 5.4.8. When (n, ω_2) is equal to $(2m, 1)$, $(2m + 1, 0)$ and $(2m + 1, 1)$ we can see that, for the appropriate value of k ,

$$\overline{\delta}_{n, \omega_2} : \mathcal{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2} \longrightarrow \mathcal{M}(\mathrm{O}(n, \mathbb{C}))_{k,0}$$

$$[(E, Q, \Phi, \tau)]_S \longmapsto [(E, Q, \Phi)]_S$$

is an isomorphism.

Remark 5.4.9. Restricting $\overline{\mathcal{E}}_{n, \omega_2}$ to $X \times Z_m^0$ gives us a family of polystable $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles with zero Higgs field. Let us denote by $\overline{\mathcal{E}}_{n, \omega_2}^0 \rightarrow X \times Z_m^0$ the underlying family of $\mathrm{O}(n, \mathbb{C})$ -bundles. As in Remarks 5.2.8 and 5.3.10, this family induces a morphism from Z_m^0 to the moduli space of $\mathrm{SO}(n, \mathbb{C})$ -bundles that, for the topological invariants considered in Remark 5.4.6, factors through $\mathrm{Sym}^{m'}(X/\mathbb{Z}_2)$ giving a bijective morphism. Since $M(\mathrm{SO}(n, \mathbb{C}))$ is normal it gives us the following isomorphisms

$$\overline{\varsigma}_{2m+1,0}^{x_0} : M(\mathrm{SO}(2m + 1, \mathbb{C}))_0 \xrightarrow{\cong} \mathrm{Sym}^m(X/\mathbb{Z}_2) \cong \mathbb{P}^m,$$

$$\overline{\varsigma}_{2m+1,1}^{x_0} : M(\mathrm{SO}(2m + 1, \mathbb{C}))_1 \xrightarrow{\cong} \mathrm{Sym}^{m-1}(X/\mathbb{Z}_2) \cong \mathbb{P}^{m-1},$$

and

$$\overline{\varsigma}_{2m,1}^{x_0} : M(\mathrm{SO}(2m, \mathbb{C}))_1 \xrightarrow{\cong} \mathrm{Sym}^{m-2}(X/\mathbb{Z}_2) \cong \mathbb{P}^{m-2}.$$

Recalling (5.8) we can check the following relations stated in [FM1],

$$M(\mathrm{SO}(2m + 1, \mathbb{C}))_1 \cong M(\mathrm{Sp}(2m - 2, \mathbb{C}))$$

and

$$M(\mathrm{SO}(2m, \mathbb{C}))_1 \cong M(\mathrm{Sp}(2m - 4, \mathbb{C})).$$

We have that $\{q_{2m+1,2}, \dots, q_{2m+1,2m}\}$ is a basis for the invariant polynomials associated to the adjoint representation of $\mathrm{SO}(2m+1, \mathbb{C})$ on the Lie algebra $\mathfrak{so}(2m+1, \mathbb{C})$. In [Hi2] the Hitchin map for $\mathrm{SO}(2m+1, \mathbb{C})$ -Higgs bundles is defined as follows

$$\bar{b}_{2m+1} : \mathcal{M}(\mathrm{SO}(2m+1, \mathbb{C})) \longrightarrow \bar{B}_{2m+1} (= \bigoplus_{i=1}^m H^0(X, \mathcal{O}))$$

$$[(E, Q, \Phi, \tau)]_S \longmapsto (q_{2m+1,2}(\Phi), \dots, q_{2m+1,2m}(\Phi)).$$

We can see that $\bar{B}_{2m+1} = \mathring{B}_{2m+1}$ and that the Hitchin map factors through $\bar{\delta}_{n,\omega_2}$, i.e. $\bar{b}_{2m+1} = \bar{\delta}_{n,\omega_2} \circ \mathring{b}_{2m+1}$. We define \bar{B}_{2m+1,ω_2} to be $\mathring{B}_{2m+1,1,0}$, if $\omega_2 = 0$, or $\mathring{B}_{2m+1,3,0}$, when $\omega_2 = 1$.

Lemma 5.4.10. *If we denote by \bar{b}_{2m+1,ω_2} the restriction of \bar{b}_{2m+1} to $\mathcal{M}(\mathrm{SO}(2m+1, \mathbb{C}))_{\omega_2} \subset \mathcal{M}(\mathrm{SO}(2m+1, \mathbb{C}))$, we have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{SO}(2m+1, \mathbb{C}))_{\omega_2} & \xrightarrow{\bar{b}_{2m+1,\omega_2}} & \bar{B}_{2m+1,\omega_2} \\ \bar{\xi}_{2m+1,\omega_2}^x \downarrow \cong & & \cong \downarrow \mathring{\beta}_{2m+1,k,0}^{-1} \\ \mathrm{Sym}^{m-\ell}(T^*X/\mathbb{Z}_2) & \xrightarrow{\tilde{\pi}_{m-\ell}} & \mathrm{Sym}^{m-\ell}(\mathbb{C}/\mathbb{Z}_2), \end{array} \quad (5.20)$$

where $\ell = 0$ and $k = 1$ if $\omega_2 = 0$, and $\ell = 1$ and $k = 3$ if $\omega_2 = 1$.

Proof. This follows from Lemma 5.3.14. \square

Corollary 5.4.11. *The generic fibres of the Hitchin fibration $\mathcal{M}(\mathrm{SO}(2m+1, \mathbb{C}))_{\omega_2} \rightarrow \bar{B}_{2m+1,\omega_2}$ are isomorphic to $X \times \dots \times X$.*

The arbitrary fibres of the Hitchin fibration $\mathcal{M}(\mathrm{SO}(2m+1, \mathbb{C}))_{\omega_2} \rightarrow \bar{B}_{2m+1,\omega_2}$ are isomorphic to a holomorphic fibration of projective spaces $\mathbb{P}^{m_0-1} \times \dots \times \mathbb{P}^{m_\ell-1}$ over $X \times \dots \times X$.

Remark 5.4.12. The generic fibre of the Hitchin fibration $\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C})) \rightarrow \tilde{B}_m$ and its corresponding fibre of the Hitchin fibration $\mathcal{M}(\mathrm{SO}(2m+1, \mathbb{C}))_0 \rightarrow \bar{B}_{2m+1,0}$ are isomorphic to $X \times \dots \times X$, this abelian variety is self-dual.

The arbitrary fibre of the Hitchin fibration $\mathcal{M}(\mathrm{Sp}(2m, \mathbb{C})) \rightarrow \tilde{B}_m$ and the corresponding fibre of the fibration $\mathcal{M}(\mathrm{SO}(2m+1, \mathbb{C}))_0 \rightarrow \bar{B}_{2m+1,0}$ are both isomorphic to a holomorphic fibration of projective spaces $\mathbb{P}^{m_0-1} \times \dots \times \mathbb{P}^{m_\ell-1}$ over $X \times \dots \times X$, the base variety is self-dual.

Let us consider the following morphism of groups

$$s_m : \mathbb{Z}_2 \times \dots \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \quad (5.21)$$

$$\bar{c} = (c_1, \dots, c_m) \longmapsto c_1 \cdots c_m.$$

The kernel of s_m is given by the tuples $\bar{c} = (c_i, \dots, c_m)$ such that only an even number of c_i are equal to -1 .

We recall that Γ_m is the semidirect product of $(\mathbb{Z}_2 \times \cdot^m \times \mathbb{Z}_2)$ and \mathfrak{S}_m . We define $\Delta_m \subset \Gamma_m$ as the subgroup given by the elements $\sigma\bar{c} \in \Gamma_m$ such that \bar{c} is contained in $\ker s_m$. The action of Γ_m on $T^*X \times \cdot^m \times T^*X$ described in (5.5) induces an action of Δ_m .

Lemma 5.4.13. *Let $z_1, z_2 \in Z_m$. Then $(\bar{\mathcal{E}}_{2m,1})_{z_1}$ is isomorphic (S-equivalent) to $(\bar{\mathcal{E}}_{2m,1})_{z_2}$ if and only if there exists $\gamma \in \Delta_m$ such that $\gamma \cdot z_1 = z_2$.*

Proof. We recall again that an isomorphism of $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles is an isomorphism of $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles that preserves the trivialization τ . Take two polystable $\mathrm{SO}(2m, \mathbb{C})$ -Higgs bundles $(E_1, Q_1, \Phi_1, \tau_1)$ and $(E_2, Q_2, \Phi_2, \tau_2)$, with

$$(E_i, Q_i, \Phi_i, \tau_i) \cong \bigoplus_{j=1}^m (E_{i,j}, Q_{i,j}, \Phi_{i,j}, \tau_{i,j})$$

where the $(E_{i,j}, Q_{i,j}, \Phi_{i,j}, \tau_{i,j})$ are zero degree $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles.

We can suppose that $(E_{1,j}, Q_{1,j}, \Phi_{1,j})$ and $(E_{2,j}, Q_{2,j}, \Phi_{2,j})$ are isomorphic $\mathrm{O}(2, \mathbb{C})$ -Higgs bundles. Then, there exist an ordering of the factors such that

$$(E_{1,j}, Q_{1,j}, \Phi_{1,j}, \tau_{1,j}) \cong (E_{2,j}, Q_{2,j}, \Phi_{2,j}, \tau_{2,j})$$

or

$$(E_{1,j}, Q_{1,j}, \Phi_{1,j}, \tau_{1,j}) \cong (E_{2,j}, Q_{2,j}, \Phi_{2,j}, -\tau_{2,j}).$$

In the second situation $(E_{1,j}, Q_{1,j}, \Phi_{1,j}, \tau_{1,j})$ and $(E_{2,j}, Q_{2,j}, \Phi_{2,j}, \tau_{2,j})$ are not isomorphic $\mathrm{SO}(2, \mathbb{C})$ -Higgs bundles (unless $L_j \cong L_j^*$ and $\lambda_j = 0$).

If we have an isomorphism of $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles that inverts an even number of τ_j then the product of all of them remains unchanged and then the $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles are isomorphic.

If our isomorphism of $\mathrm{O}(n, \mathbb{C})$ -Higgs bundles inverts an odd number of τ_j , then the product of all of them changes its sign and then the $\mathrm{SO}(n, \mathbb{C})$ -Higgs bundles can not be isomorphic.

We easily see that the elements of Γ_m that invert an even number of τ_j are those of the subgroup Δ_m . \square

Theorem 5.4.14. *There exists for $(n, \omega_2) = (2m, 0)$ a coarse moduli space $\mathcal{M}(\mathrm{SO}(n, \mathbb{C}))_0$ for the moduli functor $\mathrm{Mod}(\bar{\mathcal{A}}_{2m}, \bar{\mathcal{P}}_{2m}, S)$. The following map is an isomorphism*

$$\bar{\xi}_{m,0}^{x_0} : \mathcal{M}(\mathrm{SO}(2m, \mathbb{C}))_0 \xrightarrow{\cong} (T^*X \times \cdot^m \times T^*X) / \Delta_m \quad (5.22)$$

$$[(E, Q, \Phi)]_S \longmapsto [\xi_{1,0}^{x_0}(L_1, \lambda_1), \dots, \xi_{1,0}^{x_0}(L_m, \lambda_m)]_{\Delta_m},$$

is an isomorphism. where $\mathrm{gr}(E, Q, \Phi) \cong \bigoplus_{j=1}^m \left(L_j \oplus L_j^*, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} \lambda_j & \\ & -\lambda_j \end{pmatrix}, \sqrt{-1} \right)$.

Proof. This follows easily from Proposition 5.4.5 and Lemma 5.4.13. \square

Recall that $M(\mathrm{SO}(2m, \mathbb{C}))_0$ is the subvariety of $\mathcal{M}(\mathrm{SO}(2m, \mathbb{C}))_0$ given by the S-equivalence classes such that the associated graded object has zero Higgs field.

Remark 5.4.15. Recall the family $\bar{\mathcal{E}}_{2m,0}$ defined in 5.4.15. This family gives a morphism from Z_m^0 to $M(\mathrm{SO}(2m, \mathbb{C}))_0$ that by Lemma 5.4.13 induces a bijective morphism $Z_m^0/\Delta_m \rightarrow M(\mathrm{SO}(2m, \mathbb{C}))_0$. Since $M(\mathrm{SO}(2m, \mathbb{C}))_0$ is a normal variety, by Zariski's Main Theorem we have the following isomorphism

$$\bar{\xi}_{2m,0}^{x_0} : M(\mathrm{SO}(2m, \mathbb{C}))_0 \xrightarrow{\cong} (X \times \dots \times X) / \Delta_m. \quad (5.23)$$

Note that this agrees with the description given in [La].

Proposition 5.4.16. *There is a surjective morphism*

$$\begin{aligned} \bar{a}_{2m,0} : \mathcal{M}(\mathrm{SO}(2m, \mathbb{C}))_0 &\longrightarrow M(\mathrm{SO}(2m, \mathbb{C}))_0 \\ [(E, Q, \Phi, \tau)]_S &\longmapsto [(E, Q, \tau)]_S. \end{aligned}$$

Furthermore the diagram

$$\begin{array}{ccc} \mathcal{M}(\mathrm{SO}(2m, \mathbb{C}))_0 & \xrightarrow{\bar{a}_{2m,0}} & M(\mathrm{SO}(2m, \mathbb{C}))_0 \\ \bar{\xi}_{2m,0}^{x_0} \downarrow \cong & & \cong \downarrow \bar{\xi}_{2m,1}^{x_0} \\ (T^*X \times \dots \times T^*X) / \Delta_m & \xrightarrow{\tilde{p}_m} & (X \times \dots \times X) / \Delta_m \end{array}$$

commutes.

Proof. This follows from Proposition 5.3.11. \square

A basis for the invariant polynomials of the adjoint representation of $\mathrm{SO}(2m, \mathbb{C})$ on the Lie algebra $\mathfrak{so}(2m, \mathbb{C})$ is $\{q_{2m,2}, \dots, q_{2m,2m-2}, q_{2m,pf}\}$, where $q_{2m,pf}$ is the Pfaffian polynomial. Recall that, if D is the diagonal matrix with eigenvalues $\lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m$, evaluating the Pfaffian polynomial on D gives

$$q_{2m,pf}(D) = \lambda_1, \dots, \lambda_m.$$

Following [Hi2], we define the Hitchin map for $\mathrm{SO}(2m, \mathbb{C})$ -Higgs bundles as follows

$$\bar{b}_{2m} : \mathcal{M}(\mathrm{SO}(2m, \mathbb{C})) \longrightarrow \bar{B}_{2m} (= \bigoplus_{i=1}^m H^0(X, \mathcal{O}))$$

$$[(E, Q, \Phi, \tau)]_S \longmapsto (q_{2m+1,2}(\Phi), \dots, q_{2m,2m-2}(\Phi), q_{2m,pf}(\Phi)).$$

We note that $H^0(X, \mathcal{O}) \cong \mathbb{C}$ and therefore \bar{B}_{2m} is isomorphic to \mathbb{C}^m .

We define $\bar{b}_{2m,1}$ to be the restriction to $\mathcal{M}(\mathrm{SO}(2m, \mathbb{C}))_1$ of the Hitchin map \bar{b}_{2m} and we denote by $\bar{B}_{2m,1} \subset \bar{B}_{2m}$ the image of \bar{b}_{2m} . By Proposition 5.4.1 a polystable $\mathrm{SO}(2m, \mathbb{C})$ -Higgs bundle (E, Q, Φ, τ) with $\omega_2 = 1$ is of the form

$$(E, Q, \Phi, \tau) \cong \bigoplus_{a=0}^3 (J_a, 1, 0) \oplus \bigoplus_{j=1}^{m-2} \left(L_j \oplus L_j^*, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} \lambda_j & \\ & -\lambda_j \end{pmatrix}, \sqrt{-1} \right).$$

Then, the Higgs field of the $\mathrm{SO}(2m, \mathbb{C})$ -Higgs bundles with $\omega_2 = 1$ has, at least, four eigenvalues equal to 0. This implies that $q_{2m,pf}(\Phi)$ and $q_{2m,2m-2}(\Phi)$ are identically 0 on $\mathcal{M}(\mathrm{SO}(2m, \mathbb{C}))_1$. Then we see that $\overline{B}_{2m,1}$ is equal to $\overset{\circ}{B}_{2m,4,0}$ and isomorphic to \mathbb{C}^{m-2} . Furthermore, after Remark 5.4.8, the Hitchin map $\overline{b}_{2m,1}$ factors through $\mathcal{M}(\mathrm{O}(2m, \mathbb{C}))_{4,0}$, i.e. $\overline{b}_{2m,1} = \overset{\circ}{b}_{2m,4,0} \circ \overline{\delta}_{2m,1}$.

Lemma 5.4.17. *We have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{SO}(2m, \mathbb{C}))_1 & \xrightarrow{\overline{b}_{2m,1}} & \overline{B}_{2m,1} \\ \overline{\xi}_{2m,1}^{x_0} \downarrow \cong & & \cong \downarrow \overline{\beta}_{2m,4,0}^{-1} \\ \mathrm{Sym}^{m-2}(T^*X/\mathbb{Z}_2) & \xrightarrow{\tilde{\pi}_{m-2}} & \mathrm{Sym}^{m-2}(\mathbb{C}/\mathbb{Z}_2). \end{array} \quad (5.24)$$

Proof. This follows from Remark 5.4.8 and Lemma 5.3.14. \square

After Lemma 5.4.17 is clear that the fibres of the Hitchin map $\overline{b}_{2m,1}$ are described explicitly in Lemmas 5.2.11 and 5.2.12.

We define the Hitchin map for topologically trivial $\mathrm{SO}(2m, \mathbb{C})$ -Higgs bundles $\overline{b}_{2m,0}$ as the restriction of \overline{b}_{2m} to $\mathcal{M}(\mathrm{SO}(2m, \mathbb{C}))_0$; we can easily check that $\overline{B}_{2m,0}$, the image of $\overline{b}_{2m,0}$, is the whole \overline{B}_{2m} . Using the invariant polynomials $q_{2m,2}, \dots, q_{2m,2m-2}, q_{2m,pf}$ we define the following morphism

$$q'_m : \mathbb{C}^m \longrightarrow \mathbb{C}^m (\cong \overline{B}_{2m,0})$$

$$(\lambda_1, \dots, \lambda_m) \longmapsto (q_{2m,2}(D_{\lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m}), \dots, q_{2m,pf}(D_{\lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m})).$$

If we set $(\lambda'_1, \dots, \lambda'_m)$ to be $\gamma \cdot (\lambda_1, \dots, \lambda_m)$ for every $\gamma \in \Gamma_m$, we have that

$$q_{2m,2i}(D_{\lambda'_1, -\lambda'_1, \dots, \lambda'_m, -\lambda'_m}) = q_{2m,2i}(D_{\lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m})$$

for $i < m$ and

$$q_{2m,pf}(D_{\lambda'_1, -\lambda'_1, \dots, \lambda'_m, -\lambda'_m}) = (-1)^\ell q_{2m,pf}(D_{\lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m}),$$

where ℓ is even if $\gamma = \sigma \cdot \overline{c}$ with $\overline{c} \in \ker s_m$ (recall (5.21)) (i.e. $\gamma \in \Delta_m$), and ℓ is odd if $\gamma = \sigma \cdot \overline{c}$ with $\overline{c} \notin \ker s_m$ (i.e. $\gamma \notin \Delta_m$). We see that q'_m factors through the following bijective morphism

$$\overline{\beta}_{2m,0} : (\mathbb{C} \times \overset{.m}{\times} \mathbb{C}) / \Delta_m \longrightarrow \mathbb{C}^m (\cong \overline{B}_{2m,0}).$$

Since \mathbb{C}^m is smooth, $\overline{\beta}_{2m,0}$ is an isomorphism.

Consider the projection

$$\overline{\pi}_m : (T^*X \times \overset{.m}{\times} T^*X) / \Delta_m \longrightarrow (\mathbb{C} \times \overset{.m}{\times} \mathbb{C}) / \Delta_m$$

$$[(x_1, \lambda_1), \dots, (x_m, \lambda_m)]_{\Delta_m} \longmapsto [\lambda_1, \dots, \lambda_m]_{\Delta_m}.$$

Lemma 5.4.18. *The following diagram commutes*

$$\begin{array}{ccc}
\mathcal{M}(\mathrm{SO}(2m, \mathbb{C}))_0 & \xrightarrow{\bar{b}_{2m,0}} & \bar{B}_{2m,0} \\
\bar{\xi}_{m,0} \downarrow \cong & & \cong \downarrow \bar{\beta}_{2m,0}^{-1} \\
(T^*X \times \cdot^m \times T^*X) / \Delta_m & \xrightarrow{\bar{\pi}_m} & (\mathbb{C} \times \cdot^m \times \mathbb{C}) / \Delta_m.
\end{array} \tag{5.25}$$

Proof. The $\mathrm{SO}(2m, \mathbb{C})$ -Higgs bundle given by $[(x_1, \lambda_1), \dots, (x_m, \lambda_m)]_{\Delta_m}$ is

$$(E, Q, \Phi, \tau) = \bigoplus_{i=1}^m \left(L_i \oplus L_i^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda_i & \\ & -\lambda_i \end{pmatrix}, \sqrt{-1} \right).$$

The composition of $\bar{b}_{2m,0}(\Phi)$ with $\bar{\beta}_{2m,0}^{-1}$ gives $[(\lambda_1, \dots, \lambda_m)]_{\Delta_m}$. □

The generic element of $\bar{B}_{2m,0}$ is the Δ_m -orbit of an element of \mathbb{C}^m of the form

$$\bar{\lambda}_{gen} = (\lambda_1, \dots, \lambda_m)$$

where $\lambda_i \neq 0$ and $\lambda_i \neq \pm \lambda_j$.

Lemma 5.4.19.

$$\bar{\pi}_m^{-1}([\bar{\lambda}_{gen}]_{\Delta_m}) = X \times \cdot^m \times X.$$

Proof. We can see that the stabilizer of $\bar{\lambda}_{gen}$ is trivial and then the stabilizer of every tuple of the form

$$((x_1, \lambda_1), \dots, (x_m, \lambda_m))$$

is also trivial. So the set of Δ_m -orbits which project to $[\lambda_1, \dots, \lambda_m]_{\Delta_m}$ is isomorphic to $X \times \cdot^m \times X$. □

There is a special set of points of $\bar{B}_{2m,0}$ that come from tuples of the following form

$$\bar{\lambda}_{spc} = (\lambda_1, \cdot^{m_1}, \lambda_1, \dots, \lambda_{\ell-1}, \cdot^{m_{\ell-1}}, \lambda_{\ell-1}, \lambda_{\ell}, \cdot^{m_{\ell}-1}, \lambda_{\ell}, -\lambda_{\ell}),$$

where for every i we have $\lambda_i \neq 0$ and m_i even.

Consider the map

$$r : X \times \cdot^m \times X \longrightarrow X \times \cdot^m \times X$$

$$(x_1, \dots, x_{m-1}, x_m) \longmapsto (x_1, \dots, x_{m-1}, -x_m).$$

With this map we consider the r -action of \mathfrak{S}_m on $X \times \cdot^m \times X$. Suppose that for $\sigma \in \mathfrak{S}_m$ we denote by f_σ the permutation of $X \times \cdot^m \times X$ associated to σ . We define the r -action of σ to be the morphism $r \circ f_\sigma \circ r$. Let us call the quotient of this action $\mathrm{Sym}_r^m X$. We can construct an isomorphism between $\mathrm{Sym}_r^m X$ and $\mathrm{Sym}^m X$ and hence with $P_{(m,-1)}$.

Lemma 5.4.20.

$$\bar{\pi}_m^{-1}([\bar{\lambda}_{spc}]_{\Delta_m}) = P_{(m_1,-1)} \times \cdot^{\ell-1} \times P_{(m_{\ell-1},-1)} \times P_{(m_{\ell},-1)}.$$

Proof. The stabilizer of $\bar{\lambda}_{spc}$ is

$$Z_{\Delta_m}(\bar{\lambda}_{spc}) = \mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_{\ell-1}} \times Z_{\Delta_{m_\ell}}((\lambda_\ell, \overset{m_\ell-1}{\lambda_\ell}, -\lambda_\ell)).$$

We can check that $Z_{\Delta_{m_\ell}}((\lambda_\ell, \overset{m_\ell-1}{\lambda_\ell}, -\lambda_\ell))$ is given by the elements $\bar{c}\sigma$ of Δ_{m_i} such that σ sends the last entry of $(\lambda_\ell, \overset{m_\ell-1}{\lambda_\ell}, -\lambda_\ell)$ to the i -th entry and \bar{c} inverts the last and the i -th entry. As we see, the action of $Z_{\Delta_{m_\ell}}((\lambda_\ell, \overset{m_\ell-1}{\lambda_\ell}, -\lambda_\ell))$ is equal to the r -action of \mathfrak{S}_{m_ℓ} .

We have

$$\begin{aligned} \bar{\pi}_m^{-1}([\bar{\lambda}_{spc}]_{\Delta_m}) &\cong (X \times \overset{m}{\times} X) / Z_{\Delta_m}(\bar{\lambda}_{spc}) \\ &\cong (X \times \overset{m}{\times} X) / (\mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_{\ell-1}} \times Z_{\Delta_{m_\ell}}((\lambda_\ell, \overset{m_\ell-1}{\lambda_\ell}, -\lambda_\ell))) \\ &\cong \text{Sym}^{m_1} X \times \cdots \times \text{Sym}^{m_{\ell-1}} X \times \text{Sym}_r^{m_\ell} X. \end{aligned}$$

□

If our point of $\bar{B}_{2m,1}$ is given by a tuple different from $\bar{\lambda}_{gen}$ and from $\bar{\lambda}_{spc}$ we can always find a representant of the Δ_m -orbit with the following form

$$\bar{\lambda}_{arb} = (0, \overset{m_0}{\lambda_0}, 0, \lambda_1, \overset{m_1}{\lambda_1}, \lambda_1, \dots, \lambda_\ell, \overset{m_\ell}{\lambda_\ell}, \lambda_\ell),$$

where at least one m_i is even.

Lemma 5.4.21.

$$\bar{\pi}_m^{-1}([\bar{\lambda}_{arb}]_{\Delta_m}) = \left((X \times \cdots \times X) / \Delta_{m_0} \right) \times P_{(m_1, -1)} \times \overset{\ell}{\times} P_{(m_\ell, -1)}.$$

Proof. The stabilizer of $\bar{\lambda}_{arb}$ is

$$Z_{\Delta_m}(\bar{\lambda}_{arb}) = \Delta_{m_0} \times \mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_\ell}.$$

We have

$$\begin{aligned} \bar{\pi}_m^{-1}([\bar{\lambda}_{arb}]_{\Delta_m}) &\cong (X \times \overset{m}{\times} X) / Z_{\Delta_m}(\bar{\lambda}_{arb}) \\ &\cong (X \times \overset{m}{\times} X) / (\Delta_{m_0} \times \mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_\ell}) \\ &\cong \left((X \times \cdots \times X) / \Delta_{m_0} \right) \times \text{Sym}^{m_1} X \times \overset{\ell}{\times} \text{Sym}^{m_\ell} X. \end{aligned}$$

□

Corollary 5.4.22. *The generic fibre of the Hitchin fibration $\mathcal{M}(\text{SO}(2m, \mathbb{C}))_0 \rightarrow \bar{B}_{2m,0}$ is $X \times \overset{m}{\times} X$. The arbitrary fibre is a fibration over $X \times \overset{\ell}{\times} X$ where the fibre is $\mathbb{P}^{m_1} \times \overset{\ell}{\times} \mathbb{P}^{m_\ell} \times (X \times \overset{m_0}{\times} X) / \Delta_{m_0}$.*

We finish the section studying $\mathfrak{M}(\text{SO}(n, \mathbb{C}))_{\omega_2}$, the moduli space associated to the moduli functor $\text{Mod}(\bar{\mathcal{A}}_n, \bar{P}_n, S)$.

Proposition 5.4.23. *We have a bijective morphism $\mathcal{M}(\text{SO}(n, \mathbb{C})) \rightarrow \mathfrak{M}(\text{SO}(n, \mathbb{C}))$, hence $\mathcal{M}(\text{SO}(n, \mathbb{C}))_{\omega_2}$ is the normalization of $\mathfrak{M}(\text{SO}(n, \mathbb{C}))_{\omega_2}$.*

Proof. The family $\overline{\mathcal{E}}_{n,\omega_2}$ induces a morphism

$$\nu_{\overline{\mathcal{E}}_{n,\omega_2}} : T^*X \times^{\ell_{n,\omega_2}} T^*X \longrightarrow \mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2},$$

where $\ell_{2m+1,1} = m$, $\ell_{2m+1,-1} = m - 1$, $\ell_{2m,1} = m$ and $\ell_{2m,-1} = m - 2$. By Remark 5.4.6 and Lemma 5.4.13 it factors through

$$\nu'_{\overline{\mathcal{E}}_{n,\omega_2}} : T^*X \times^{\ell_{n,\omega_2}} T^*X / F \longrightarrow \mathcal{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2},$$

where $F = \Gamma_m$ if $(n, \omega_2) = (2m + 1, 0)$, $F = \Gamma_{m-1}$ if $(n, \omega_2) = (2m + 1, 1)$, $F = \Gamma_{m-2}$ if $(n, \omega_2) = (2m, 1)$ and $F = \Delta_m$ if $(n, \omega_2) = (2m, 0)$.

Let us denote by $\overline{\mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2}}$ the normalization of $\mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2}$. Since the quotient $T^*X \times^{\ell_{n,\omega_2}} T^*X / F$ is normal, by the universal property of the normalization, $\nu'_{\overline{\mathcal{E}}_{n,\omega_2}}$ factors through

$$\nu''_{\overline{\mathcal{E}}_{n,\omega_2}} : T^*X \times^{\ell_{n,\omega_2}} T^*X / F \longrightarrow \overline{\mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2}}.$$

This map is an isomorphism since it is a bijection and $\overline{\mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2}}$ is normal. Then $\mathcal{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2}$ is the normalization of $\mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2}$. \square

Remark 5.4.24. Both moduli spaces would be isomorphic if $\mathfrak{M}(\mathrm{SO}(n, \mathbb{C}))_{\omega_2}$ is normal, but normality in this case is an open question.

Part II

Higgs bundles for real forms of $GL(n, \mathbb{C})$

Chapter 6

$U(p, q)$ -Higgs bundles over an elliptic curve

6.1 $U(p, q)$ -Higgs bundles

A $U(p, q)$ -Higgs bundle of type (a, b) over an elliptic curve X is a quadruple (V, W, β, γ) , where V and W are vector bundles on X of rank p and q and degree a and b respectively and $\beta : W \rightarrow V$ and $\gamma : V \rightarrow W$ are holomorphic morphisms between them.

We define the *underlying Higgs bundle* of the $U(p, q)$ -Higgs bundle (V, W, β, γ) of the $U(p, q)$ -Higgs bundle (V, W, β, γ)

$$(E, \Phi) = \left(V \oplus W, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \right).$$

Two $U(p, q)$ -Higgs bundles, (V, W, β, γ) and $(V', W', \beta', \gamma')$, are isomorphic if there exists two isomorphisms of vector bundles $f_V : V \rightarrow V'$ and $f_W : W \rightarrow W'$ such that $\beta' = f_V \circ \beta \circ f_W^{-1}$ to $\gamma' = f_W \circ \gamma \circ f_V^{-1}$.

We take the following notions of stability, semistability and polystability of $U(p, q)$ -Higgs bundles from [BGG1].

Given the $U(p, q)$ -Higgs bundle (V, W, β, γ) , we say that the pair of subbundles $V' \subset V$ and $W' \subset W$ is (β, γ) -invariant if $\beta(W') \subset V'$ and $\gamma(V') \subset W'$. A $U(p, q)$ -Higgs bundle (V, W, β, γ) is *semistable* if every pair (V', W') of (β, γ) -invariant subbundles satisfies

$$\mu(V' \oplus W') \leq \mu(V \oplus W).$$

The $U(p, q)$ -Higgs bundle is *stable* if the above inequality is strict for every pair of proper (β, γ) -invariant subbundles and *polystable* if it is a direct sum of stable $U(p_i, q_i)$ -Higgs bundles.

The *Toledo topological invariant* of a $U(p, q)$ -Higgs bundle of type (a, b) is

$$\tau = \frac{qa - pb}{p + q}.$$

In [BGG1], a constraint for the Toledo invariant of semistable $U(p, q)$ -Higgs bundles over compact Riemann surfaces of genus $g \geq 2$ is stated:

$$-\min\{p, q\}(2g - 2) \leq \tau \leq \min\{p, q\}(2g - 2). \quad (6.1)$$

Let $(V_1, W_1, \beta_1, \gamma_1)$ and $(V_2, W_2, \beta_2, \gamma_2)$ be respectively a $U(p_1, q_1)$ and a $U(p_2, q_2)$ -Higgs bundle. We write $(V_1, W_1, \beta_1, \gamma_1) \oplus (V_2, W_2, \beta_2, \gamma_2)$ for the $U(p_1 + p_2, q_1 + q_2)$ -Higgs bundle $(V_1 \oplus V_2, W_1 \oplus W_2, \beta_1 \oplus \beta_2, \gamma_1 \oplus \gamma_2)$.

Let $(V'_i, W'_i, \beta'_i, \gamma'_i)$ be a semistable $U(p_i, q_i)$ -Higgs bundle. If it is stable we define $(V_i, W_i, \beta_i, \gamma_i)$ simply as $(V'_i, W'_i, \beta'_i, \gamma'_i)$. On the other hand, if $(V'_i, W'_i, \beta'_i, \gamma'_i)$ is strictly semistable then there exists a pair (V_{i+1}, W_{i+1}) of (β'_i, γ'_i) -invariant subbundles of (V'_i, W'_i) . We take (V_{i+1}, W_{i+1}) minimal (possibly with $V_{i+1} = 0$ or $W_{i+1} = 0$), and then, with the restriction of β'_i and γ'_i to (V_{i+1}, W_{i+1}) we obtain a stable $U(p_{i+1}, q_{i+1})$ -Higgs bundle $(V_{i+1}, W_{i+1}, \beta_{i+1}, \gamma_{i+1})$. We call the quotient bundles $V'_{i+1} = V'_i/V_{i+1}$ and $W'_{i+1} = W'_i/W_{i+1}$. Since β'_i and γ'_i preserve (V_{i+1}, W_{i+1}) they induce a morphism between quotient bundles $\beta'_{i+1} : W'_{i+1} \rightarrow V'_{i+1}$ and $\gamma'_{i+1} : V'_{i+1} \rightarrow W'_{i+1}$ and then we obtain a $U(p_i - p_{i+1}, q_i - q_{i+1})$ -Higgs bundle $(V'_{i+1}, W'_{i+1}, \beta'_{i+1}, \gamma'_{i+1})$ which is semistable since $(V'_i, W'_i, \beta'_i, \gamma'_i)$ is semistable.

For an arbitrary semistable $U(p, q)$ -Higgs bundle (V, W, β, γ) , we set $(V'_0, W'_0, \beta'_0, \gamma'_0)$ to be (V, W, β, γ) and, by the above definitions, we obtain for $i = 1, \dots, \ell$ a set of stable $U(p_i, q_i)$ -Higgs bundles $(V_i, W_i, \beta_i, \gamma_i)$. We define the *associated graded object*

$$\text{gr}(V, W, \beta, \gamma) = \bigoplus_{i=1}^{\ell} (V_i, W_i, \beta_i, \gamma_i).$$

One can prove that $\text{gr}(V, W, \beta, \gamma)$ is uniquely defined up to isomorphism.

We say that two semistable $U(p, q)$ -Higgs bundles (V, W, β, γ) and $(V', W', \beta', \gamma')$ are *S-equivalent* if $\text{gr}(V, W, \beta, \gamma)$ and $\text{gr}(V', W', \beta', \gamma')$ are isomorphic.

A *family of $U(p, q)$ -Higgs bundles* \mathcal{E} parametrized by T is a quadruple $(\mathcal{V}^\mathcal{E}, \mathcal{W}^\mathcal{E}, B^\mathcal{E}, \Gamma^\mathcal{E})$, where $\mathcal{V}^\mathcal{E}$ and $\mathcal{W}^\mathcal{E}$ are families of rank p and rank q vector bundles parametrized by T and $B^\mathcal{E} : \mathcal{W} \rightarrow \mathcal{V}$ and $\Gamma : \mathcal{W} \rightarrow \mathcal{V}$ are holomorphic morphisms between them. We denote by $\check{P}_{p,q}$ this algebraic condition.

We say that \mathcal{E} is a family of semistable $U(p, q)$ -Higgs bundles if \mathcal{E}_t is a semistable Higgs bundle for every $t \in T$. Two families of semistable $U(p, q)$ -Higgs bundles \mathcal{E}_1 and \mathcal{E}_2 parametrized by the same variety T are S-equivalent if for every $t \in T$ we have $(\mathcal{E}_1)_t \sim_S (\mathcal{E}_2)_t$ are S-equivalent, we write $\mathcal{E}_1 \sim_S \mathcal{E}_2$.

We denote by $\check{\mathcal{A}}_{p,q}$ the collection of semistable $U(p, q)$ -Higgs bundles over X of topological type (a, b) and by $\check{\mathcal{A}}_{p,q}^{st}$ the subcollection of stable ones. We consider the moduli functor defined in (3.5)

$$\text{Mod}(\check{\mathcal{A}}_{p,q}, \check{P}_{p,q}, S).$$

By [BGG1] there exists a moduli space $\mathfrak{M}(U(p, q))_{(a,b)}$ parametrizing S-equivalence classes of semistable $U(p, q)$ -Higgs bundles associated to this moduli functor. One obtains that in every S-equivalence class there is a polystable $U(p, q)$ -Higgs bundle unique up to isomorphism. Then, the points of $\mathfrak{M}(U(p, q))_{(a,b)}$ can be seen also as isomorphism classes of polystable $U(p, q)$ -Higgs bundles.

For stable $U(p, q)$ -Higgs bundles, S-equivalence implies isomorphism. Recalling that S-equivalence for families of stable Higgs bundles is the same as pointwise isomorphism we consider the functor

$$\text{Mod}(\check{\mathcal{A}}_{p,q}^{st}, \check{P}_{p,q}, \overset{pt}{\cong}).$$

We denote by $\mathfrak{M}^{st}(\mathbb{U}(p, q))_{(a, b)}$ the moduli space of isomorphism (hence S -equivalence) classes of stable $\mathbb{U}(p, q)$ -Higgs bundles of topological type (a, b) associated to it.

We say that a family of semistable $\mathbb{U}(p, q)$ -Higgs bundles $\mathcal{E} \rightarrow X \times T$ is *locally graded* if for every point t of T there exists an open subset $U \subset T$ containing t and a set of families \mathcal{F}_i parametrized by U of stable $\mathbb{U}(p_i, q_i)$ -Higgs bundles and $\mathbb{U}(p_i, q_i)$ -Higgs bundles that are direct sums of stable $\mathbb{U}(p_i, 0)$ -Higgs bundles and stable $\mathbb{U}(0, q_i)$ -Higgs bundles, such that for every $t' \in U$ we have that

$$\mathcal{E}|_{X \times \{t'\}} \sim_S \bigoplus_i \mathcal{F}_i|_{X \times \{t'\}}.$$

We say that the locally graded families of $\mathbb{U}(p, q)$ -Higgs bundles satisfy the algebraic condition $\check{Q}_{p, q}$, and we consider the moduli functor associated to this algebraic condition

$$\text{Mod}(\check{\mathcal{A}}_{p, q}, \check{Q}_{p, q}, S). \quad (6.2)$$

If this functor has a moduli space we denote it by $\mathcal{M}(\mathbb{U}(p, q))$.

6.2 Stability of $\mathbb{U}(p, q)$ -Higgs bundles

Proposition 6.2.1. *Let (V, W, β, γ) be a semistable (resp. stable) $\mathbb{U}(p, q)$ -Higgs bundle. Then $(V, \beta\gamma)$ and $(W, \gamma\beta)$ are semistable (resp. stable) Higgs bundles with*

$$\mu(V) = \mu(W).$$

Proof. With no loss of generality we consider $\mu(V) \geq \mu(V \oplus W) \geq \mu(W)$. Suppose $(V, \beta\gamma)$ is not semistable, we take V' the subbundle of the first factor of the Harder-Narasimhan filtration of $(V, \beta\gamma)$, then $\mu(V') > \mu(V)$ and $(V'\beta\gamma|_{V'})$ is semistable. By Proposition 4.2.1 V' is a semistable vector bundle. We set $W' = \gamma(V')$, since $W' \cong V'/\ker \gamma$ and V' is semistable, we have $\mu(W') \geq \mu(V')$ and then

$$\mu(W') \geq \mu(V' \oplus W') \geq \mu(V') > \mu(V) \geq \mu(V \oplus W) \geq \mu(W),$$

in particular $\mu(V' \oplus W') > \mu(V \oplus W)$. By definition of W' and since V' is preserved by $\beta\gamma$, we have $\gamma(V') \subset W'$ and $\beta(W') \subset V'$. Then (V', W') is a (β, γ) -invariant pair of subbundles that contradicts the semistability of (V, W, β, γ) . We have proved that $(V, \beta\gamma)$ is semistable and therefore, by Proposition 4.2.1, so is V .

We suppose now that $\mu(V) > \mu(V \oplus W) > \mu(W)$. We take $(V, \gamma(V))$ which is obviously (β, γ) -invariant. Since V is semistable, and $\gamma(V) \cong V/\ker \gamma$ we have that $\mu(\gamma(V)) \geq \mu(V)$. This implies that $\mu(\gamma(V)) \geq \mu(V \oplus \gamma(V)) \geq \mu(V) > \mu(V \oplus W) > \mu(W)$, in particular $\mu(V \oplus \gamma(V)) > \mu(V \oplus W)$ contradicting again the semistability of (V, W, β, γ) . We have proved that $\mu(V) = \mu(V \oplus W) = \mu(W)$ and by symmetry, we have that $(W, \gamma\beta)$ is semistable too.

Finally, we suppose that (V, W, β, γ) is stable and $(V, \beta\gamma)$ strictly semistable. There exists a $(\beta\gamma)$ -invariant subbundle $V'' \subset V$ with $\mu(V'') = \mu(V)$. Since V'' and V have the same slope, $(V'', \beta\gamma|_{V''})$ is semistable and so is V'' by Proposition 4.2.1. This implies

that $W'' = \gamma(V'')$ satisfies $\mu(W'') \geq \mu(V'')$. The pair (V'', W'') is (β, γ) -invariant with slope $\mu(V'' \oplus W'') = \mu(V \oplus W)$. This contradicts the stability of (V, W, β, γ) so $(V, \beta\gamma)$ is stable. By symmetry, so is $(W, \gamma\beta)$. \square

The first implication of Proposition 6.2.1 is the rigidity of the Toledo invariant due to the equality of the slopes $\mu(V) = \mu(W)$.

Corollary 6.2.2. *Every semistable $U(p, q)$ -Higgs bundle over an elliptic curve has*

$$\tau = 0.$$

Thanks to Corollary 6.2.2 we see that (6.1) extends to the case of genus $g = 1$.

Proposition 4.2.1 implies that a Higgs bundle is semistable if and only if the underlying vector bundle is semistable; this fact, together with Proposition 6.2.1 allows us to give a complete description of semistable $U(p, q)$ -Higgs bundles.

Corollary 6.2.3. *A $U(p, q)$ -Higgs bundle is semistable if and only if V and W are semistable vector bundles of equal slope.*

We proceed with the study of stable and polystable $U(p, q)$ -Higgs bundles.

Proposition 6.2.4. *If (V, W, β, γ) is a stable $U(p, q)$ -Higgs bundle of topological type (a, b) , then the quadruple of invariants (p, q, a, b) has the form*

$$(p, p, a, a), \quad (p, 0, a, 0) \quad \text{or} \quad (0, q, 0, b),$$

with $\gcd(p, a) = 1$ and $\gcd(q, b) = 1$.

Every stable $U(p, p)$ -Higgs bundle of topological type (a, a) with $\gcd(p, a) = 1$ is isomorphic to (V, V, β, γ) , where V is any stable vector bundle of rank p and degree a , $\beta \neq 0$ and $\gamma \neq 0$. Every $U(p, p)$ -Higgs bundle of this form is stable.

Let V and V' be two stable vector bundles of rank p and degree a . Two $U(p, p)$ -Higgs bundles (V, V, β, γ) and $(V', V', \beta', \gamma')$ are isomorphic if and only if the stable Higgs bundles $(V, \beta\gamma)$ and $(V', \beta'\gamma')$ are isomorphic.

Proof. Proposition 4.2.3 implies that a Higgs bundle is stable if and only if its underlying vector bundle is stable and we recall that the only stable vector bundles over an elliptic curve are those of coprime rank and degree. This, together with Proposition 6.2.1 proves the first part of proposition.

If (V, W, β, γ) is stable then $\beta \neq 0$ and $\gamma \neq 0$, otherwise $(V, 0)$ or $(0, W)$ would contradict stability. In that case, the only possibility is that $V \cong W$ and $\beta = \lambda_1 \text{id}$ and $\gamma = \lambda_2 \text{id}$. On the other hand, a $U(p, p)$ -Higgs bundle of the form (V, V, β, γ) with V stable, $\beta \neq 0$ and $\gamma \neq 0$ is stable since every pair of (β, γ) -invariant subbundles are proper subbundles of V and, since V is stable, the slope of the direct sum is smaller than $\mu(V)$.

Consider two $U(p, p)$ -Higgs bundles (V, V, β, γ) and $(V', V', \beta', \gamma')$. An isomorphism between them induces an isomorphism between $(V, \beta\gamma)$ and $(V', \beta'\gamma')$. Recall that $\beta = \lambda_1 \text{id}$, $\gamma = \lambda_2 \text{id}$, $\beta' = \lambda'_1 \text{id}$ and $\gamma' = \lambda'_2 \text{id}$. Suppose on the contrary that $(V, \beta\gamma) \cong (V', \beta'\gamma')$. This implies that $V \cong V'$ and $\lambda_1 \lambda_2 = \lambda'_1 \lambda'_2$. Taking $f_V = \lambda_2 \text{id}$ and $f_{W'} = \lambda'_2 \text{id}$ we obtain the isomorphism between $(V, V, \lambda_1 \text{id}, \lambda_2 \text{id})$ and $(V, V, \lambda'_1 \text{id}, \lambda'_2 \text{id})$. \square

We observe that every stable $U(p, p)$ -Higgs bundle is isomorphic to one of the form $(V, V, \lambda \cdot \text{id}, \lambda \cdot \text{id})$. Note that every $U(p, 0)$ -Higgs bundle has the form $(V, 0, 0, 0)$ and respectively, every $U(0, p)$ -Higgs bundle has the form $(0, V, 0, 0)$.

Corollary 6.2.5. *There are no polystable $U(p, q)$ -Higgs bundles of topological type (a, b) unless the quadruple of invariants (p, q, a, b) is (nr, mr, nd, md) where $n, m, r \in \mathbb{Z}^+$ and $d \in \mathbb{Z}$ satisfy $\gcd(r, d) = 1$. For every polystable $U(nr, mr)$ -Higgs bundle (V, W, β, γ) of type (nd, md) , there exists $\ell \leq \min\{n, m\}$ such that*

$$(V, W, \beta, \gamma) \cong \bigoplus_{i=1}^{\ell} (V_i, V_i, \lambda_i \cdot \text{id}, \lambda_i \cdot \text{id}) \oplus \bigoplus_{j=1}^{n-\ell} (V_j, 0, 0, 0) \oplus \bigoplus_{k=1}^{m-\ell} (0, V_k, 0, 0),$$

where V_i, V_j and V_k are stable vector bundles of rank r and degree d and $\lambda_i \in \mathbb{C}^*$.

6.3 Moduli spaces of $U(p, q)$ -Higgs bundles

Let us take $\mathcal{N}(U(p, p))_{(a, a)}$ to be the subset of $\mathfrak{M}(U(p, p))_{(a, a)}$ given by the S-equivalence classes of $U(p, p)$ -Higgs bundles of the form (V, V, β, γ) . By Proposition 6.2.1, if (V, V, β, γ) is semistable, then V is semistable. When $\gcd(p, a) = 1$, the semistable vector bundle V is stable and we can define the following morphism

$$\check{\eta}_{p, a} : \mathcal{N}(U(p, p))_{(a, a)} \longrightarrow \mathfrak{M}^{st}(\text{GL}(p, \mathbb{C}))_a$$

$$[(V, V, \beta, \gamma)]_S \longmapsto [(V, \beta\gamma)]_S.$$

The map is clearly surjective and by Proposition 6.2.4 it is injective and therefore bijective. By Zariski's Main Theorem $\eta_{p, a}$ is an isomorphism

$$\check{\eta}_{p, a} : \mathcal{N}(U(p, p))_{(a, a)} \xrightarrow{\cong} \mathfrak{M}^{st}(\text{GL}(p, \mathbb{C}))_a.$$

We recall the isomorphism $\xi_{p, a}^{x_0} : \mathfrak{M}^{st}(\text{GL}(p, \mathbb{C}))_a \xrightarrow{\cong} T^*X$ given in Theorem 4.3.3.

Theorem 6.3.1. *Let p and q be nonzero positive integers. Then*

$$\mathfrak{M}^{st}(U(p, q))_{(a, b)} = \emptyset$$

unless $p = q$, $a = b$ and $\gcd(p, a) = 1$. In that case we have the following isomorphism

$$\check{\xi}_{(p, p, a, a)}^{x_0, st} : \mathfrak{M}^{st}(U(p, p))_{(a, a)} \xrightarrow{\cong} X \times \mathbb{C}^*.$$

$$[(V, V, \beta, \gamma)]_S \longmapsto \xi_{p, a}^{x_0}([(V, \beta\gamma)]_S).$$

Proof. By Proposition 6.2.4 there are stable $U(p, q)$ -Higgs bundle only in the cases stated in the proposition.

We have that $\mathfrak{M}^{st}(U(p, p))_{(a, a)} \subset \mathcal{N}(U(p, p))_{(a, a)}$ and therefore $\eta_{p, a}$ induces an isomorphism between $\mathfrak{M}^{st}(U(p, p))_{(a, a)}$ and its image in $\mathfrak{M}^{st}(\text{GL}(p, \mathbb{C}))_a$, i.e. the set of isomorphism classes of Higgs bundles with non-zero Higgs field. Finally, the image of this subset under $\xi_{p, a}^{x_0}$ is $X \times \mathbb{C}^*$. \square

Remark 6.3.2. Since $\mathcal{N}(\mathrm{U}(p, p))_{(a, a)} \subset \mathfrak{M}(\mathrm{U}(p, p))_{(a, a)}$ is isomorphic to $X \times \mathbb{C}$ and $\mathfrak{M}^{st}(\mathrm{U}(p, p))_{(a, a)} \subset \mathcal{N}(\mathrm{U}(p, p))_{(a, a)}$ is isomorphic to $X \times \mathbb{C}^*$, we have that $\mathcal{N}(\mathrm{U}(p, p))_{(a, a)}$ is the closure of $\mathfrak{M}^{st}(\mathrm{U}(p, p))_{(a, a)}$ in $\mathfrak{M}(\mathrm{U}(p, p))_{(a, a)}$. Note that $\check{\xi}_{(r, r, d, d)}^{x_0}$ can be extended to $\mathcal{N}(\mathrm{U}(p, p))_{(a, a)}$

$$\check{\xi}_{(p, p, a, a)}^{x_0, st} : \mathcal{N}(\mathrm{U}(p, p))_{(a, a)} \xrightarrow{\cong} X \times \mathbb{C}.$$

We study now the moduli spaces of semistable $\mathrm{U}(p, q)$ -Higgs bundles.

Proposition 6.3.3. *The moduli spaces $\mathcal{M}(\mathrm{U}(p, q))_{(a, b)}$ and $\mathfrak{M}(\mathrm{U}(p, q))_{(a, b)}$ are empty unless the quadruple of invariants (p, q, a, b) is (nr, mr, nd, md) where $n, m, r \in \mathbb{Z}^+$ and $d \in \mathbb{Z}$ satisfy $\gcd(r, d) = 1$.*

Proof. By Proposition 6.2.1 any semistable $\mathrm{U}(p, q)$ -Higgs bundle (V, W, β, γ) satisfy that $\mu(V) = \mu(W)$. This implies the condition on the invariants stated in the proposition. \square

Let $\gcd(r, d) = 1$ and let $\mathcal{E}_{(r, d)}^{x_0} = (\mathcal{V}_{(r, d)}^{x_0}, \Phi_{(r, d)}^{x_0})$ be the family of stable Higgs bundles of rank r and degree d parametrized by T^*X . We take the family of polystable $\mathrm{U}(p, p)$ -Higgs bundles $\check{\mathcal{E}}_{(r, r, d, d)}^{x_0} = (\mathcal{V}_{(r, d)}^{x_0}, \mathcal{V}_{(r, d)}^{x_0}, \Phi_{(r, d)}^{x_0}, \Phi_{(r, d)}^{x_0})$ parametrized by T^*X .

Remark 6.3.4. Due to Proposition 6.2.4 $\check{\mathcal{E}}_{(r, r, d, d)}^{x_0}|_{X \times \{(x, \lambda)\}} \cong \check{\mathcal{E}}_{(r, r, d, d)}^{x_0}|_{X \times \{(x', \lambda')\}}$ if and only if $(x, \lambda) = (x', \pm \lambda')$. Furthermore, for every $(x, \lambda) \in T^*X$, if we set $(V, W, \beta, \gamma) = \check{\mathcal{E}}_{(r, r, d, d)}^{x_0}|_{X \times \{(x, \lambda)\}}$ we have

$$\check{\xi}_{1, 0}^{x_0, st}((V, W, \beta, \gamma)) = \xi_{1, 0}^{x_0}(V, \beta\gamma) = (x, \lambda^2). \quad (6.3)$$

For every algebraic variety Z take the diagonal injection map $\iota : Z \rightarrow Z \times Z$ sending z to (z, z) , we have $\iota = (\iota_1, \iota_2)$ where ι_i is the image on the i -th factor. For different values of k we consider a collection of maps $\iota_k = (\iota_{k, 1}, \iota_{k, 2})$.

Suppose we have $(p, q, a, b) = (nr, mr, nd, md)$ and $0 \leq \ell \leq \min\{n, m\}$. For $\ell = 0$ we define the family of $(\mathrm{U}(r, 0) \times \cdot^n \times \mathrm{U}(r, 0) \times \mathrm{U}(0, r) \times \cdot^m \times \mathrm{U}(0, r))$ -Higgs bundles

$$\begin{aligned} \check{\mathcal{F}}_{(p, q, a, b), \ell=0}^{x_0} &= (\mathcal{V}_{(r, d)}^{x_0}, 0, 0, 0) \times_X \cdot^n \times_X (\mathcal{V}_{(r, d)}^{x_0}, 0, 0, 0) \times_X \\ &\quad \times_X (0, \mathcal{V}_{(r, d)}^{x_0}, 0, 0) \times_X \cdot^m \times_X (0, \mathcal{V}_{(r, d)}^{x_0}, 0, 0). \end{aligned}$$

Considering the injection $j : \mathrm{U}(r, 0) \times \cdot^n \times \mathrm{U}(r, 0) \times \mathrm{U}(0, r) \times \cdot^m \times \mathrm{U}(0, r) \rightarrow \mathrm{U}(p, q)$ and using the extension of structure group associated to j we define

$$\check{\mathcal{E}}_{(p, q, a, b), \ell=0}^{x_0} = j_* \check{\mathcal{F}}_{(p, q, a, b), \ell=0}^{x_0}.$$

This is a family of polystable $\mathrm{U}(p, q)$ -Higgs bundles parametrized by $T_{n, m, \ell=0}$, the subvariety of $(T^*X)^{\times n} \times (T^*X)^{\times m}$ defined as follows

$$T_{n, m, \ell=0} = (X \times \{0\}) \times \cdot^n \times (X \times \{0\}) \times ((X \times \{0\}) \times \cdot^m \times (X \times \{0\})).$$

For $\ell = 1, i \leq n$ and $j \leq m$, we define

$$\begin{aligned} \check{\mathcal{F}}_{(p, q, a, b), \ell=1, i, j}^{x_0} &= (\mathcal{V}_{(r, d)}^{x_0}, 0, 0, 0) \times_X \cdots \times_X \iota_1(\mathcal{E}_{(r, r, d, d)}^{x_0}) \times_X \cdots \times_X (\mathcal{V}_{(r, r, d, d)}^{x_0}, 0, 0, 0) \times_X \\ &\quad \times_X (0, \mathcal{V}_{(r, d)}^{x_0}, 0, 0) \times_X \cdots \times_X \iota_2(\mathcal{E}_{(r, r, d, d)}^{x_0}) \times_X \cdots \times_X (0, \mathcal{V}_{(r, d)}^{x_0}, 0, 0) \end{aligned}$$

where $\iota_1(\mathcal{E}_{(r,r,d,d)}^{x_0})$ and $\iota_2(\mathcal{E}_{(r,r,d,d)}^{x_0})$ are placed at the i -th and the j -th positions. Taking an injection in $U(p, q)$ we define

$$\check{\mathcal{E}}_{(p,q,a,b),\ell=1,i,j}^{x_0} = \mathcal{J}_* \check{\mathcal{F}}_{(p,q,a,b),\ell=1,i,j}^{x_0}.$$

This is a family of polystable $U(p, q)$ -Higgs bundles parametrized by the following subvariety of $(X \times \mathbb{C})^{\times(n+m)}$

$$T_{n,m,\ell=1,\eta_{1,i}} = ((X \times \{0\}) \times \cdots \times \iota_1(T^*X) \times \cdots \times (X \times \{0\})) \times \\ \times ((X \times \{0\}) \times \cdots \times \iota_2(T^*X) \times \cdots \times (X \times \{0\}))$$

where $\iota_1(T^*X)$ and $\iota_2(T^*X)$ are placed at the i -th and the j -th positions.

We have that $T_{n,m,\ell=0}$ and all the $T_{n,m,\ell=1,i,j}$ are subvarieties of $(T^*X)^{\times n} \times (T^*X)^{\times m}$. We note that if t is a point of the intersection of these subvarieties, the $U(p, q)$ -Higgs bundles parametrized by the different families at t are isomorphic

$$\check{\mathcal{E}}_{(p,q,a,b),\ell=1,i,j}^{x_0}|_{X \times \{t\}} \cong \check{\mathcal{E}}_{(p,q,a,b),\ell=0}^{x_0}|_{X \times \{t\}}.$$

Let $\Sigma_{\ell,n}$ be the set of unordered tuples of ℓ elements in n positions, and let $\tilde{\Sigma}_{\ell,m}$ be the set of ordered tuples of ℓ elements in m positions, when a position contains an element, we say that it is a black position and if it does not contain an element we say that the position is white. For every $1 \leq i \leq |\Sigma_{\ell,n}|$ and every $1 \leq j \leq |\tilde{\Sigma}_{\ell,m}|$, we construct $\check{\mathcal{F}}_{(p,q,a,b),\ell,i,j}^{x_0}$ as the following fibre product over X . In the p -part, for every unordered tuple σ_i we fix an order and we set a factor of the form $(\mathcal{V}_{(r,d)}^{x_0}, 0, 0, 0)$ in a white position and we place $\iota_{k,1}(\mathcal{E}_{(r,r,d,d)}^{x_0})$ in the corresponding k -th black position corresponding to the unordered tuple σ_i with the given order. In the q -part we set a factor of the form $(0, \mathcal{V}_{(r,d)}^{x_0}, 0, 0)$ in a white position of $\tilde{\sigma}_j$ and we place $\iota_{k,2}(\mathcal{E}_{(r,r,d,d)}^{x_0})$ in the corresponding k -th black position corresponding to the ordered tuple $\tilde{\sigma}_j$. Taking \mathcal{J} to be the injection in $U(p, q)$ we define $\check{\mathcal{E}}_{(p,q,a,b),\ell,i,j}^{x_0} = \mathcal{J}_* \check{\mathcal{F}}_{(p,q,a,b),\ell,i,j}^{x_0}$. This is a family of polystable $U(p, q)$ -Higgs bundles parametrized by a subvariety of $(T^*X)^{\times n} \times (T^*X)^{\times m}$ that we call $T_{n,m,\ell,i,j}$.

When $\ell > 1$, we define $T_{n,m,\ell,i,j}$ in similar terms. The p -part, is a product where in a white position of σ_i we place $X \times \{0\}$ and we place $\iota_{k,1}(T^*X)$ in the k -th black position. The q -part is a product of $X \times \{0\}$ on every white position of $\tilde{\sigma}_j$ and $\iota_{k,2}(T^*X)$ in the k -th black position.

We define $T_{n,m}$ as the subvariety of $(T^*X)^{\times n} \times (T^*X)^{\times m}$ given by

$$T_{n,m} = \bigcup_{\ell,i,j} T_{n,m,\ell,i,j}.$$

If the subvarieties of $T_{n,m,\ell,i,j}$ and $T_{n,m,\ell',i',j'} \subset (T^*X)^{\times n} \times (T^*X)^{\times m}$ intersect at the point t we have that the $U(p, q)$ -Higgs bundles parametrized by $\check{\mathcal{E}}_{(p,q,a,b),\ell,i,j}^{x_0}$ and by $\check{\mathcal{E}}_{(p,q,a,b),\ell',i',j'}^{x_0}$ at t are isomorphic. This allows us to define the family of polystable $U(p, q)$ -Higgs bundles

$$\check{\mathcal{E}}_{(p,q,a,b)}^{x_0} \rightarrow X \times T_{n,m}$$

as the family induced by all the $\check{\mathcal{F}}_{(p,q,a,b),\ell,i,j}^{x_0}$.

Since $T_{n,m}$ is a subvariety of $(T^*X)^{\times n} \times (T^*X)^{\times m}$, the product of symmetric groups $\mathfrak{S}_n \times \mathfrak{S}_m$ acts on $T_{n,m}$ permuting the factors, although $\mathfrak{S}_n \times \mathfrak{S}_m$ does not preserve necessarily the components $T_{n,m,\ell,i,j}$.

Proposition 6.3.5. *Let $(p, q, a, b) = (nr, mr, nd, md)$ with $n, m, r \in \mathbb{Z}^+$, $d \in \mathbb{Z}$ and $\gcd(r, d) = 1$. The family $\check{\mathcal{E}}_{(p,q,a,b)}^{x_0} \rightarrow X \times T_{n,m}$ has the local universal property among locally graded families of semistable $U(p, q)$ -Higgs bundles of type (a, b) .*

Proof. If we have a locally graded family of semistable $U(p, q)$ -Higgs bundles $\mathcal{E} \rightarrow X \times T$, then for every $t \in T$ there exists an open subset $U \subset T$ containing t and a set of families \mathcal{F}_i of $U(r, 0)$, $U(0, r)$ and $U(r, r)$ -Higgs bundles parametrized by U such that for every $t' \in U$ we have that

$$\mathcal{E}|_{X \times \{t'\}} \sim_S \bigoplus_i \mathcal{F}_i|_{X \times \{t'\}}.$$

Either \mathcal{F}_i is a family of stable $U(r, 0)$ or $U(0, r)$ -Higgs bundles or \mathcal{F}_i is a family of stable $U(r, r)$ -Higgs bundles and $U(r, r)$ -Higgs bundles that are direct sums of stable $U(r, 0)$ and $U(0, r)$ -Higgs bundles.

Suppose that $U = \bigcup_j U_j$ with U_j irreducible. We define $\mathcal{F}_{i,j}$ as the restriction of \mathcal{F}_i to $X \times U_j$. If $\mathcal{F}_{i,j}$ is a family of stable $U(r, 0)$ -Higgs bundles (resp. $U(0, r)$ -Higgs bundles), since $\mathcal{V}_{r,d}^{x_0}$ is a universal family, there exists $f_{i,j}^p : U_j \rightarrow X$ (resp. $f_{i,j}^q : U_j \rightarrow X$) such that $\mathcal{F}_{i,j} \sim_S (f_{i,j}^p)_* \mathcal{V}_{r,d}^{x_0}$ (resp. $(f_{i,j}^q)_* \mathcal{V}_{r,d}^{x_0}$). If $\mathcal{F}_{i,j}$ is a family of $U(r, r)$ -Higgs bundles, either it parametrizes a stable $U(r, r)$ -Higgs bundle or not. If $\mathcal{F}_{i,j}$ does not parametrize any stable $U(r, r)$ -Higgs bundle then $\mathcal{F}_{i,j} = \mathcal{F}'_{i,j} \oplus \mathcal{F}''_{i,j}$ where $\mathcal{F}'_{i,j}$ is a family of stable $U(r, 0)$ -Higgs bundles and $\mathcal{F}''_{i,j}$ is a family of stable $U(0, r)$ -Higgs bundles; this case has already been covered. If $\mathcal{F}_{i,j}$ contains a stable $U(r, r)$ -Higgs bundle, since stability is an open condition, $\mathcal{F}_{i,j}$ induces a morphism from U_j to $\mathcal{N}(U(r, r))_{(a,a)}$, the closure of $\mathfrak{M}^{st}(U(r, r))_{(a,a)}$ in $\mathfrak{M}(U(r, r))_{(a,a)}$

$$\nu_{i,j} : U_j \rightarrow \mathcal{N}(U(r, r))_{(d,d)}$$

and setting $f_{i,j}^{pq} = \check{\xi}_{(r,r,d,d)}^{x_0} \circ \nu_{i,j}$ we have that $\mathcal{F}_{i,j} \sim_S (f'_{i,j})_* \check{\mathcal{E}}_{(r,r,d,d)}^{x_0}$ by Remark 6.3.4.

With a suitable combination of $f_{i,j}$ and $f'_{i,j}$ we can construct a morphism from $U_j \rightarrow T_{n,m}$ as follows

$$f_j = (f_{1,j}^{pq}, \dots, f_{n_j,j}^{pq}, f_{n_j+1,j}^p, \dots, f_{n'_j,j}^p) \times (f_{1,j}^{pq}, \dots, f_{n_j,j}^{pq}, f_{n'_j+1,j}^q, \dots)$$

such that $\mathcal{F} \sim_S f_j^* \check{\mathcal{E}}_{(p,q,a,b)}^{x_0}$. Therefore, defining $f : U \rightarrow T_{n,m}$ as the morphism such that $f|_{U_j} = f_j$ we have

$$\mathcal{E}|_{X \times U} \sim_S f^* \check{\mathcal{E}}_{(p,q,a,b)}^{x_0}.$$

□

We recall the group Γ_k defined in (5.4). Let us consider the action of Γ_k on $T^*X \times \cdot^k \times T^*X$ induced by the permutation action of the symmetric group and the following action of group $(\mathbb{Z}_2 \times \cdot^k \times \mathbb{Z}_2)$ on $T^*X \times \cdot^k \times T^*X$

$$(1, \dots, 1, -1, 1, \dots, 1) \cdot ((x_1, \lambda_1), \dots, (x_i, \lambda_i), \dots, (x_k, \lambda_k)) = \tag{6.4} \\ = ((x_1, \lambda_1), \dots, (x_i, -\lambda_i), \dots, (x_k, \lambda_k)).$$

We see that the quotient of $T^*X \times \dots \times T^*X$ by Γ_k under this action is

$$(T^*X \times \dots \times T^*X) / \Gamma_k = \text{Sym}^k(X \times \mathbb{C}/\pm). \quad (6.5)$$

Remark 6.3.6. After (6.5), we note that $T_{n,m}/(\Gamma_n \times \Gamma_m)$ is a subvariety of $\text{Sym}^n(X \times \mathbb{C}/\pm) \times \text{Sym}^m(X \times \mathbb{C}/\pm)$.

Lemma 6.3.7. *Two points t_1 and $t_2 \in T_{n,m}$ are such that $\check{\mathcal{E}}_{(p,q,a,b)}^{x_0}|_{X \times \{t_1\}} \cong \check{\mathcal{E}}_{(p,q,a,b)}^{x_0}|_{X \times \{t_2\}}$ if and only if there exists an element of $\Gamma_n \times \Gamma_m$ sending t_1 to t_2 under the action defined in (6.4).*

Proof. We have that for every $t \in T_{n,m}$ we have that $\check{\mathcal{E}}_{(p,q,a,b)}^{x_0}|_{X \times \{t\}}$ is a direct sum of stable $U(r, 0)$ and $U(0, r)$ -Higgs bundles and stable $U(r, r)$ -Higgs bundles of the form $(V, V, \lambda \text{id}, \lambda \text{id})$. By Proposition 6.2.4 we see that $(V, V, \lambda \text{id}, \lambda \text{id}) \cong (V', V', \lambda' \text{id}, \lambda' \text{id})$ if and only if $V' \cong V$ and $\lambda' = \pm \lambda$. Then, $\check{\mathcal{E}}_{(p,q,a,b)}^{x_0}|_{X \times \{t_1\}}$ and $\check{\mathcal{E}}_{(p,q,a,b)}^{x_0}|_{X \times \{t_2\}}$ are isomorphic if and only if the first is a direct sum of the factors of the second one with the Higgs field multiplied or not by -1 . This implies that $t_2 = \sigma \cdot (z \cdot t)$, where z is an element of $\mathbb{Z}_2^{\times n} \times \mathbb{Z}_2^{\times m}$ and $\sigma \in \mathfrak{S}_n \times \mathfrak{S}_m$ is a permutation. \square

The previous results allow us to describe the moduli space of semistable $U(p, q)$ -Higgs bundles associated to the moduli functor $\text{Mod}(\check{\mathcal{A}}_{p,q}, \check{\mathcal{Q}}_{p,q}, S)$.

Theorem 6.3.8. *Let $(p, q, a, b) = (nr, mr, nd, md)$ with $n, m, r \in \mathbb{Z}^+$ and $d \in \mathbb{Z}$ such that $\text{gcd}(r, d) = 1$. Then there exists a coarse moduli space $\mathcal{M}(U(p, q))$ associated to the moduli problem $\text{Mod}(\check{\mathcal{A}}_{p,q}, \check{\mathcal{Q}}_{p,q}, S)$. We have the following isomorphism for this moduli space*

$$\check{\xi}_{(p,q,a,b)}^{x_0} : \mathcal{M}(U(p, q))_{(a,b)} \xrightarrow{\cong} T_{n,m}/(\Gamma_n \times \Gamma_m)$$

$$[(V, W, \beta, \gamma)]_S \longmapsto [\dots \xi_{r,d}^{x_0}(V_i, \sqrt{\beta_i \gamma_i}) \dots]_{\Gamma_n} \times [\dots \xi_{r,d}^{x_0}(W_i, \sqrt{\beta_i \gamma_i}) \dots]_{\Gamma_m},$$

where

$$\text{gr}(V, W, \beta, \gamma) = \bigoplus_i (V_i, W_i, \beta_i, \gamma_i).$$

Proof. By Proposition 6.3.5 $\check{\mathcal{E}}_{(p,q,a,b)}^{x_0}$ has the local universal property for the moduli problem associated to $\text{Mod}(\check{\mathcal{A}}_{p,q}, \check{\mathcal{Q}}_{p,q}, S)$. The result follows from Proposition 3.2.1 and Lemma 6.3.7. \square

Proposition 6.3.9. *We have the following commuting diagram*

$$\begin{array}{ccc} \mathcal{M}(U(p, q)) & \longrightarrow & M(\text{GL}(p, \mathbb{C}))_a \times M(\text{GL}(q, \mathbb{C}))_b \\ \check{\xi}_{(p,q,a,b)}^{x_0} \downarrow & & \downarrow \xi_{(p,a)}^{x_0} \times \xi_{(q,b)}^{x_0} \\ T_{n,m}/(\Gamma_n \times \Gamma_m) & \longrightarrow & \text{Sym}^n X \times \text{Sym}^m X. \end{array}$$

Proof. By Remark 6.3.6 $T_{n,m}/(\Gamma_n \times \Gamma_m)$ is a subvariety of $\text{Sym}^n(X \times \mathbb{C}/\pm) \times \text{Sym}^m(X \times \mathbb{C}/\pm)$. We see that $T_{n,m}/(\Gamma_n \times \Gamma_m)$ projects naturally to $X^{\times n}/\mathfrak{S}_n \times X^m/\mathfrak{S}_m$ and by Corollary 6.2.3 $\mathcal{M}(\text{U}(p, q))_{(a,b)}$ projects to $M(\text{GL}(p, \mathbb{C}))_a \times M(\text{GL}(q, \mathbb{C}))_b$. \square

Suppose that $(p, q, a, b) = (nr, mr, nd, md)$ and let (V, W, β, γ) be a polystable $\text{U}(p, q)$ -Higgs bundle of topological degree (a, b) . By Corollary 6.2.5 we have that

$$(V, W, \beta, \gamma) \cong \bigoplus_{i=1}^{\ell} (V_i, V_i, \lambda_i \cdot \text{id}, \lambda_i \cdot \text{id}) \oplus \bigoplus_{j=1}^{n-\ell} (V_j, 0, 0, 0) \oplus \bigoplus_{k=1}^{m-\ell} (0, V_k, 0, 0),$$

where V_i, V_j and V_k are stable vector bundles of rank r and degree d and $\lambda_i \in \mathbb{C}^*$.

We recall the involution $\ell_{(p+q), (a+b)}$ defined in (4.21). It is obvious that the Higgs bundles $(V_j, 0)$ and $(V_k, 0)$ are fixed under $\ell_{r,d}$, and since

$$\left(V_i \oplus V_i, \begin{pmatrix} 0 & \lambda_i \\ \lambda_i & 0 \end{pmatrix} \right) \cong \left(V_i \oplus V_i, \begin{pmatrix} \lambda_i & \\ & -\lambda_i \end{pmatrix} \right),$$

the factors $(V_i, V_i, \lambda_i \cdot \text{id}, \lambda_i \cdot \text{id})$ give fixed points of $\ell_{2r, 2d}$.

After Proposition 6.2.1 and the previous considerations, we have that the following morphism between moduli spaces is well defined

$$\begin{aligned} \check{\delta}_{(p,q,a,b)} : \mathcal{M}(\text{U}(p, q))_{(a,b)} &\longrightarrow \mathcal{M}(\text{GL}(p+q, \mathbb{C}))_{a+b}^{\ell_{(p+q), (a+b)}} \\ [(V, W, \beta, \gamma)]_S &\longmapsto \left[\left(V \oplus W, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \right) \right]_S. \end{aligned}$$

Remark 6.3.10. We can prove that $\check{\delta}_{(p,q,a,b)}$ is not injective. Take $V \not\cong W$, then the $\text{U}(p, p)$ -Higgs bundles $(V, W, 0, 0)$ and $(W, V, 0, 0)$ are not isomorphic, but they have the same image under $\check{\delta}_{(p,p,a)}$, since $(V \oplus W, 0)$ and $(W \oplus V, 0)$ are isomorphic.

We define the Hitchin map for $\text{U}(p, q)$ -Higgs bundles using this morphism and the Hitchin map for Higgs bundles $b_{(p+q, a+b)}$

$$\check{b}_{(p,q,a,b)} = b_{(p+q, a+b)} \circ \check{\delta}_{(p,q,a,b)}.$$

We denote by $\check{B}_{(p,q,a,b)}$ the image of the Hitchin map $\check{b}_{(p,q,a,b)}$. It follows from the definition that $\check{B}_{(p,q,a,b)} \subset B_{(p+q, a+b)}$, where $B_{(p+q, a+b)}$ is the image of $b_{(p+q, a+b)}$. Let us recall that $B_{(p+q, a+b)} \cong \text{Sym}^{(n+m)} \mathbb{C}$.

From now on, we suppose with no loss of generality that $n \geq m$. Let us take the injection

$$i'_{n,m} : \text{Sym}^m(\mathbb{C}/\pm) \longrightarrow \text{Sym}^{(n+m)} \mathbb{C}$$

$$[[\lambda_1]_{\mathbb{Z}_2}, \dots, [\lambda_m]_{\mathbb{Z}_2}]_{\mathfrak{S}_m} \longmapsto [0, \overset{\cdot}{\cdot}{\cdot}{\cdot}, 0, \lambda_1, -\lambda_1, \dots, \lambda_m, -\lambda_m]_{\mathfrak{S}_{(n+m)}}.$$

After Theorem 6.3.8 we can observe that $\check{B}_{(p,q,a,b)}$ coincides with $\beta_{(p+q, a+b)}(\text{im } i'_{n,m})$.

Let us recall that, using the invariant polynomials $q_{n+m,1}, \dots, q_{n+m,n+m}$, we have defined in (4.14) the isomorphism $\bar{q}_{n+m} : \text{Sym}^{(n+m)} \mathbb{C} \xrightarrow{\cong} \mathbb{C}^{(n+m)}$. We can check that $\bar{q}_{n+m}(\text{im } i'_{n,m})$ is the vanishing locus of the polynomials $q_{n+m,i}$ with $i > 2m$ or $i = 2\ell + 1$. Thus, we have $\text{im } i'_{n,m} \cong \mathbb{C}^m$ and it is smooth. As a consequence we have that $\text{im } i'_{n,m}$ is isomorphic to $\text{Sym}^m(\mathbb{C}/\pm)$, since $i'_{n,m}$ is injective. If we compose with $\beta_{(p+q,a+b)}$ we obtain the isomorphism

$$\check{\beta}_{(p,q,a,b)} : \text{Sym}^m(\mathbb{C}/\pm) \xrightarrow{\cong} \check{B}_{(p,q,a,b)}.$$

Take the natural projection map

$$\check{\pi}^k : \text{Sym}^k(X \times \mathbb{C}/\pm) \longrightarrow \text{Sym}^k(\mathbb{C}/\pm).$$

Recall from Remark 6.3.6 that $T_{n,m}/(\Gamma_n \times \Gamma_m)$ is a subvariety of $\text{Sym}^n(X \times \mathbb{C}/\pm) \times \text{Sym}^m(X \times \mathbb{C}/\pm)$ and note that the image of this subvariety under $\check{\pi}^n \times \check{\pi}^m$ is the subvariety $i_{n,m}(\text{Sym}^m(\mathbb{C}/\pm)) \times \text{Sym}^m(\mathbb{C}/\pm) \subset \text{Sym}^n(\mathbb{C}/\pm) \times \text{Sym}^m(\mathbb{C}/\pm)$.

We denote by $\check{\pi}_{n,m}$ the restriction of $\check{\pi}^n \times \check{\pi}^m$ to $T_{n,m}/(\Gamma_n \times \Gamma_m)$ composed with the projection from $i_{n,m}(\text{Sym}^m(\mathbb{C}/\pm)) \times \text{Sym}^m(\mathbb{C}/\pm)$ to $\text{Sym}^m(\mathbb{C}/\pm)$.

Lemma 6.3.11. *We have the following commuting diagram*

$$\begin{array}{ccc} \mathcal{M}(U(p, q))_{(a,b)} & \xrightarrow{\check{b}_{(p,q,a,b)}} & \check{B}_{(p,q,a,b)} \\ \check{\xi}_{(p,q,a,b)}^{x_0} \downarrow & & \cong \downarrow \beta_{(p,q,a,b)}^{-1} \\ T_{n,m}/(\Gamma_n \times \Gamma_m) & \xrightarrow{\check{\pi}_{n,m}} & \text{Sym}^m(\mathbb{C}/\pm) \end{array} \quad (6.6)$$

Proof. Let $(p, q, a, b) = (nr, mr, nd, md)$ with $\gcd(r, d) = 1$. By Corollary 6.2.5 we have that every polystable $U(p, q)$ -Higgs bundle of topological type (a, b) is of the form

$$(V, W, \beta, \gamma) \cong \bigoplus_{i=1}^{\ell} (V_i, V_i, \lambda_i \cdot \text{id}, \lambda_i \cdot \text{id}) \oplus \bigoplus_{j=1}^{n-\ell} (V_j, 0, 0, 0) \oplus \bigoplus_{k=1}^{m-\ell} (0, V_k, 0, 0),$$

where every V_ℓ is a stable vector bundle of rank r and degree d . Let us set

$$(E_i, \Phi_i) = \left(V_i \oplus V_i, \begin{pmatrix} & \lambda_i \\ \lambda_i & \end{pmatrix} \right).$$

We see that Φ_i has eigenvalues $\lambda_i, \dots, \lambda_i, -\lambda_i, \dots, -\lambda_i$.

The lemma follows from this observation, Theorem 6.3.8 and the definition of the Hitchin map $\check{b}_{(p,q,a,b)}$ as $\check{\delta}_{(p,q,a,b)} \circ b_{(p+q,a+b)}$. \square

We set

$$\bar{\lambda} = (0, \cdot^{k_0}, 0, \lambda_1, \cdot^{k_1}, \lambda_1, \dots, \lambda_\ell, \cdot^{k_\ell}, \lambda_\ell),$$

where $\lambda_i \neq 0$ for every i , $\lambda_i \neq \pm \lambda_j$ if $i \neq j$ and $m = k_0 + k_1 + \dots + k_\ell$.

Lemma 6.3.12. *We have*

$$\check{\pi}_{n,m}^{-1}([\bar{\lambda}]_{\Gamma_m}) \cong \text{Sym}^{k_0} X \times \text{Sym}^{k_1} X \times \dots \times \text{Sym}^{k_\ell} X \times \text{Sym}^{(n+k_0-m)} X.$$

Proof. We can check that

$$\begin{aligned}
\check{\pi}_{n,m}^{-1}([\bar{\lambda}]_{\Gamma_m}) &= ((\pi^{(n+m)})^{-1}(i_{n,m}([\bar{\lambda}]_{\Gamma_m})) \cap (T_{n,m}/(\Gamma_n \times \Gamma_m))) = \\
&= \left(\text{Sym}^{(n+k_0-m)} X \times \text{Sym}^{k_1} X \times \dots \times \text{Sym}^{k_\ell} X \right) \\
&\quad \times \left(\text{Sym}^{k_0} X \times \text{Sym}^{k_1} X \times \dots \times \text{Sym}^{k_\ell} X \right) \cap T_{n,m}/(\Gamma_n \times \Gamma_m) = \\
&= \left(\text{Sym}^{k_0} X \times \text{Sym}^{k_1} X \times \dots \times \text{Sym}^{k_\ell} X \times \text{Sym}^{(n+k_0-m)} X \right).
\end{aligned}$$

□

We say that the generic element of $\text{Sym}^m(\mathbb{C}/\pm)$ is given by a tuple $(\lambda_1, \dots, \lambda_m)$ where every λ_i is non-zero and if $i \neq j$ we have $\lambda_i \neq \pm\lambda_j$. We refer to the fibre of the Hitchin fibration over a generic element of $\text{Sym}^m \mathbb{C}$ as the generic Hitchin fibre.

Corollary 6.3.13. *The generic Hitchin fibre of the fibration $\mathcal{M}(\text{U}(p, p))_{(a,a)} \rightarrow \check{B}_{(p,p,a,a)}$ is the abelian variety $X \times \dots \times X$.*

The generic Hitchin fibre of the fibration $\mathcal{M}(\text{U}(p, q))_{(a,b)} \rightarrow \check{B}_{(p,q,a,b)}$ is $X \times \dots \times X \times \text{Sym}^{n-m} X$ which is not an abelian variety if $n > m + 1$.

If k_0 is the number of zeroes appearing in $\bar{\lambda}$ the dimension of the Hitchin fibre over $[\bar{\lambda}]_{\Gamma_m}$ is $n + k_0$.

We define

$$\Delta_1 = \{(x_1, x_2) \in X \times X \text{ such that } x_1 = x_2\} \subset X \times X,$$

and

$$\Delta_2 = \{(x, [(x_1, x_2)]_{\mathfrak{S}_2}) \text{ such that } x_1 = x \text{ or } x_2 = x\} \subset X \times \text{Sym}^2 X.$$

Using this notation we can give a more detailed description of $\mathcal{M}(\text{U}(p, q))_{(a,b)}$ for low values of n and m . Let $r > 0$ and d be two integers such that $\gcd(r, d) = 1$.

Remark 6.3.14. We have that

$$\mathcal{M}(\text{U}(r, r))_{(d,d)} \cong T_{1,1} = (X \times X \times \{0\}) \cup (\Delta_1 \times (\mathbb{C}/\pm)).$$

Since the intersection $(X \times X \times \{0\}) \cap (\Delta_1 \times (\mathbb{C}/\pm))$ is non-empty, this is not a normal variety.

Remark 6.3.15. We have that

$$\mathcal{M}(\text{U}(r, 2r))_{(d,2d)} \cong T_{1,2}/\Gamma_2 \cong (X \times \text{Sym}^2 X \times \{[0]_{\pm}\}) \cup (\Delta_2 \times (\mathbb{C}/\pm)).$$

Since the intersection $(X \times \text{Sym}^2 X \times \{[0]_{\pm}\}) \cap (\Delta_2 \times (\mathbb{C}/\pm))$ is non-empty, this is not a normal variety.

We study now the relation between $\mathcal{M}(\text{U}(p, q))_{(a,b)}$ and $\mathfrak{M}(\text{U}(p, q))_{(a,b)}$. The family of semistable $\text{U}(p, q)$ -Higgs bundles of type (a, b) $\check{\mathcal{E}}_{(p,q,a,b)}^{x_0} \rightarrow X \times T_{n,m}$ induces the following morphism

$$\nu_{\check{\mathcal{E}}_{(p,q,a,b)}^{x_0}} : T_{n,m} \longrightarrow \mathfrak{M}(\text{U}(p, q))_{(a,b)}.$$

By Proposition 6.3.5 $\nu_{\check{\xi}_{(p,q,a,b)}}^{\check{\xi}}$ factors through

$$\bar{\nu}_{\check{\xi}_{(p,q,a,b)}}^{\check{\xi}^{x_0}} : T_{n,m}/(\Gamma_n \times \Gamma_m) \longrightarrow \mathfrak{M}(\mathbb{U}(p,q))_{(a,b)}$$

with $\bar{\nu}_{\check{\xi}_{(p,q,a,b)}}^{\check{\xi}^{x_0}}$ is bijective. Composing with $\check{\xi}_{(p,q,a,b)}^{\check{\xi}^{x_0}}$ we obtain a bijection between our moduli spaces

$$\mathcal{M}(\mathbb{U}(p,q))_{(a,b)} \longrightarrow \mathfrak{M}(\mathbb{U}(p,q))_{(a,b)}.$$

Since $\mathcal{M}(\mathbb{U}(p,q))_{(a,b)}$ is not normal in general, $\mathfrak{M}(\mathbb{U}(p,q))_{(a,b)}$ is not normal either and we can not apply Zariski's Main Theorem. With this method we can not know if the above bijection is an isomorphism or not.

Chapter 7

$U^*(2m)$ and $GL(n, \mathbb{R})$ -Higgs bundles over an elliptic curve

7.1 $U^*(2m)$ and $GL(n, \mathbb{R})$ -Higgs bundles

A $U^*(2m)$ -Higgs bundle over the elliptic curve X is a triple (E, Ω, Φ) , where E is a rank $2m$ holomorphic vector bundle over X , $\Omega \in H^0(X, \Lambda^2 E^*)$ is a non-degenerate symplectic form on E and $\Phi \in H^0(X, \text{End } E)$ is an endomorphism of E which commutes with Ω , i.e. for every $x \in X$ we have

$$\Omega(u, \Phi(v)) = \Omega(\Phi(u), v), \quad \text{for every } u, v \in E_x.$$

Two $U^*(2m)$ -Higgs bundles, (E, Ω, Φ) and (E', Ω', Φ') are isomorphic if there exists an isomorphism $f : E' \rightarrow E$ such that $\Omega' = f^t \Omega f$ and $\Phi' = f^{-1} \Phi f$.

A $GL(n, \mathbb{R})$ -Higgs bundle over X is a triple (E, Q, Φ) , where E is a rank n holomorphic vector bundle, $Q \in H^0(X, \text{Sym}^2 E^*)$ is a non-degenerate symmetric form on E and $\Phi \in H^0(X, \text{End } E)$ is an endomorphism of E which commutes with Q , i.e. for every $x \in X$ we have

$$Q(u, \Phi(v)) = Q(\Phi(u), v), \quad \text{for every } u, v \in E_x.$$

Again, two $GL(n, \mathbb{R})$ -Higgs bundles (E, Q, Φ) and (E', Q', Φ') are isomorphic if there exists an isomorphism $f : E' \rightarrow E$ such that $Q' = f^t Q f$ and $\Phi' = f^{-1} \Phi f$.

$U^*(2m)$ and $GL(n, \mathbb{R})$ -Higgs bundles are particular cases of triples (E, Θ, Φ) , where (E, Φ) is a Higgs bundle and $\Theta : (E, \Phi) \rightarrow (E^*, \Phi^t)$ is an isomorphism of Higgs bundles that satisfies $\Theta = b\Theta^t$. If $b = 1$, we have a $GL(n, \mathbb{R})$ -Higgs bundle, and if $b = -1$ it is a $U^*(2m)$ -Higgs bundle. Since Θ is an isomorphism of Higgs bundles, we represent it as a commuting diagram

$$\begin{array}{ccc} E & \xrightarrow{\Theta} & E^* \\ \downarrow \Phi & & \downarrow \Phi^t \\ E & \xrightarrow{\Theta} & E^* \end{array}, \quad \text{where } \Theta = b\Theta^t.$$

The notions of stability, semistability and polystability of $U^*(2m)$ and $GL(n, \mathbb{R})$ -Higgs bundles presented here are the ones worked out in [GGM] (see also [GO] for the stability of $U^*(2m)$ -Higgs bundles and [BGG2] for the stability of $GL(n, \mathbb{R})$ -Higgs bundles).

Let (E, Θ, Φ) be a $U^*(2m)$ -Higgs bundle or a $GL(n, \mathbb{R})$ -Higgs bundle, since $E \cong E^*$, we have $\mu(E) = 0$. We say that (E, Θ, Φ) is *semistable* if and only if for any Φ -invariant isotropic subbundle F of E we have

$$\mu(F) \leq \mu(E) = 0$$

and it is *stable* if for any proper Φ -invariant isotropic subbundle the above inequality is strict. Furthermore, (E, Θ, Φ) is *polystable* if it is semistable and for every isotropic strict and Φ -invariant subbundle F of degree 0, there exists a Φ -invariant and coisotropic subbundle F' such that $E = F \oplus F'$.

We can construct a Jordan-Hölder filtration for semistable $U^*(2m)$ and $GL(n, \mathbb{R})$ -Higgs bundles as we did for $Sp(2m, \mathbb{C})$ and $O(n, \mathbb{C})$ -Higgs bundles in Section 3.6.

If (E, Θ, Φ) is stable we say that the Jordan-Hölder filtration is trivial $0 \subset E$, if (E, Θ, Φ) is a strictly semistable there exists an isotropic Φ -invariant subbundle E_1 with slope equal to zero, taking E_1 minimal we can assume that $(E_1, \Phi|_{E_1})$ is stable. We write E'_1 for $E_1^{\perp\Theta}$ and Φ'_1 for the induced Higgs field on the quotient E/E'_1 . Recall that Θ commutes with the Higgs field, and then it induces an isomorphism $\theta_1 : (E_1, \Phi_1) \xrightarrow{\cong} ((E/E'_1)^*, \Phi'_1)$, where Φ_1 is $\Phi|_{E_1}$.

We write \tilde{E}_1 for the quotient E_1/E_1 , and $\tilde{\Theta}_1$ and $\tilde{\Phi}_1$ for the quadratic form and the Higgs field induced by Θ and Φ on the quotient. Since (E, Θ, Φ) is semistable we have that $(\tilde{E}_1, \tilde{\Theta}_1, \tilde{\Phi}_1)$ is semistable as well. If $(\tilde{E}_1, \tilde{\Theta}_1, \tilde{\Phi}_1)$ is not stable, we can find another isotropic and Φ -invariant subbundle E_2 such that $E_1 \subset E_2 \subset E'_1$ with $(E_2/E_1, \bar{\Phi}_2)$ stable, where $\bar{\Phi}_2$ is the restriction of $\tilde{\Phi}_1$ to E_2/E_1 . Writing $E'_2 = E_2^{\perp\Theta}$ we have that this bundle is contained in E'_1 since $E_1 \subset E_2$ and then we can consider the quotient E'_1/E'_2 and $\bar{\Phi}'_2$ as the Higgs bundle induced by $\tilde{\Phi}_1$ on it. Since $\tilde{\Theta}_1$ commutes with the Higgs field induces an isomorphism $\theta_2 : (E_2/E_1, \bar{\Phi}_2) \xrightarrow{\cong} ((E'_1/E'_2)^*, \bar{\Phi}'_2)$.

We define \tilde{E}_2 as the quotient E'_2/E_2 . We have that Θ induces $\tilde{\Theta}_2$ on \tilde{E}_2 non-degenerate and so we obtain an $U^*(2m)$ or $GL(n, \mathbb{R})$ -Higgs bundle $(\tilde{E}_2, \tilde{\Theta}_2, \tilde{\Phi}_2)$. Since (E, Θ, Φ) is semistable we have that $(\tilde{E}_2, \tilde{\Theta}_2, \tilde{\Phi}_2)$ is again semistable. We repeat it until we obtain a stable $(\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k)$ or zero. This gives a *Jordan-Hölder filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset \dots \subset E'_2 \subset E'_1 \subset E'_0 = E$$

where $E_i^{\perp\Theta} = E'_i$, and for $i \neq k$ we have that $(E_i/E_{i-1}, \bar{\Phi}_i)$ and $(E'_{i-1}/E'_i, \bar{\Phi}'_i)$ are stable Higgs bundles satisfying $\theta_i : (E_i/E_{i-1}, \bar{\Phi}_i) \xrightarrow{\cong} ((E'_{i-1}/E'_i)^*, \bar{\Phi}'_i)$.

We define the *associated graded object* of every semistable $U^*(2m)$ or $GL(n, \mathbb{R})$ -Higgs bundle (E, Θ, Φ)

$$\text{gr}(E, \Theta, \Phi) = (\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k) \oplus \bigoplus_{i=1}^{k-1} \left((E_i/E_{i-1}) \oplus (E'_{i-1}/E'_i), \begin{pmatrix} 0 & b\theta'_i \\ \theta_i & 0 \end{pmatrix}, \bar{\Phi}_i \oplus \bar{\Phi}'_i \right),$$

where $b = -1$ for $U^*(2m)$ -Higgs bundles and $b = 1$ for $GL(n, \mathbb{R})$ -Higgs bundles. Recall that $(\tilde{E}_k, \tilde{\Theta}_k, \tilde{\Phi}_k)$ is stable if it exists and $\theta_i : (E_i/E_{i-1}, \bar{\Phi}_i) \xrightarrow{\cong} ((E'_{i-1}/E'_i)^*, \bar{\Phi}'_i)$.

For a given (E, Θ, Φ) , the associated graded object $\text{gr}(E, \Theta, \Phi)$ is unique up to isomorphism although the Jordan-Hölder filtration may not.

Let us write $\dot{\mathcal{A}}_m$ for the collection of all semistable $U^*(2m)$ -Higgs bundles over X . A family of $U^*(2m)$ -Higgs bundles parametrized by T is a triple $(\mathcal{V}, \Omega, \Phi)$, where (\mathcal{V}, Ω) is a family of $\text{Sp}(2m, \mathbb{C})$ -bundles parametrized by T and Φ is an endomorphism of \mathcal{F} that commutes with Ω . We call the algebraic condition defining these families \dot{P}_m .

We take $\ddot{\mathcal{A}}_n$ to be the collection of all semistable $\text{GL}(n, \mathbb{R})$ -Higgs bundles over X . A family of $\text{GL}(n, \mathbb{R})$ -Higgs bundles parametrized by T is a triple $(\mathcal{V}, \mathcal{Q}, \Phi)$, where $(\mathcal{V}, \mathcal{Q})$ is a family of $\text{O}(n, \mathbb{C})$ -bundles parametrized by T and Φ an endomorphism of \mathcal{V} commuting with \mathcal{Q} . We write \dot{P}_n for the algebraic condition defining these families.

Two semistable $U^*(2m)$ or $\text{GL}(n, \mathbb{R})$ -Higgs bundles (E, Θ, Φ) and (E', Θ', Φ') are *S-equivalent* if $\text{gr}(E, \Theta, \Phi) \cong \text{gr}(E', \Theta', \Phi')$. We define stability and S-equivalence for \dot{P}_m and \dot{P}_n -families pointwise. Recalling (3.5) we define the following moduli functors

$$\text{Mod}(\dot{\mathcal{A}}_m, \dot{P}_m, S)$$

and

$$\text{Mod}(\ddot{\mathcal{A}}_n, \dot{P}_n, S).$$

By [Sc] there exist $\mathfrak{M}(U^*(2m))$ and $\mathfrak{M}(\text{GL}(n, \mathbb{R}))$, respectively the moduli space of S-equivalence classes of semistable $U^*(2m)$ and $\text{GL}(n, \mathbb{R})$ -Higgs bundles associated to the previous moduli functors. In every S-equivalence class we always find a polystable representative which is unique up to isomorphism.

We denote by $\dot{\mathcal{A}}_m^{st}$ and $\dot{\mathcal{A}}_n^{st}$ the subcollection of $\dot{\mathcal{A}}_m$ and $\dot{\mathcal{A}}_n$ given by the stable $U^*(2m)$ and $\text{GL}(n, \mathbb{R})$ -Higgs bundles. For stable bundles S-equivalence is the same as isomorphism and then S-equivalence for families is isomorphism pointwise. We consider the moduli functors

$$\text{Mod}(\dot{\mathcal{A}}_m^{st}, \dot{P}_m, \overset{pt}{\cong})$$

and

$$\text{Mod}(\dot{\mathcal{A}}_n^{st}, \dot{P}_n, \overset{pt}{\cong}).$$

If they exists, we denote by $\mathfrak{M}^{st}(U^*(2m))$ and $\mathfrak{M}^{st}(\text{GL}(n, \mathbb{R}))$ the moduli spaces of isomorphism classes of stable $U^*(2m)$ or $\text{GL}(n, \mathbb{R})$ -Higgs bundles associated to this functor.

We say that $\mathcal{E} \rightarrow X \times T$, a family of semistable $U^*(2m)$ -Higgs bundles (resp. semistable $\text{GL}(n, \mathbb{R})$ -Higgs bundles), is *locally graded* if for every $t \in T$ there exists an open subset $U \subset T$ containing t and a set of families $\mathcal{F}'_1, \dots, \mathcal{F}'_k$ and $\mathcal{F}_1, \dots, \mathcal{F}_\ell$ parametrized by U such that the \mathcal{F}'_j are families of stable $U^*(2)$ -Higgs bundles (resp. stable $\text{GL}(1, \mathbb{R})$ -Higgs bundles) and the \mathcal{F}_i are families of the form

$$\mathcal{F}_i = \left(\mathcal{V}_i \oplus \mathcal{V}_i^*, \begin{pmatrix} 0 & b\vartheta_i^t \\ \vartheta_i & 0 \end{pmatrix}, \begin{pmatrix} \Phi_i & \\ & \Phi'_i \end{pmatrix} \right)$$

where $b = -1$ (resp. $b = 1$), \mathcal{V}_i is a family of stable vector bundles, $\vartheta_i : \mathcal{V}_i \rightarrow \mathcal{V}_i$ is an isomorphism, Φ_i and Φ'_i are, respectively, endomorphisms of \mathcal{V}_i and \mathcal{V}_i^* satisfying $\vartheta_i \Phi_i = \Phi'_i \vartheta_i^t$. The set of families is such that for every $t' \in U$ we have

$$\mathcal{E}|_{X \times \{t'\}} \sim_S \bigoplus_{j=1}^k \mathcal{F}'_j|_{X \times \{t'\}} \oplus \bigoplus_{i=1}^{\ell} \mathcal{F}_i|_{X \times \{t'\}}.$$

We say that the locally graded families of semistable $U^*(2m)$ -Higgs bundles (resp. the locally graded families of semistable $GL(n, \mathbb{R})$ -Higgs bundles) satisfy the algebraic condition \dot{Q}_m (resp. \dot{Q}_n). We consider the moduli functors for locally graded families

$$\text{Mod}(\dot{\mathcal{A}}_m, \dot{Q}_m, S) \quad \text{and} \quad \text{Mod}(\dot{\mathcal{A}}_m^{st}, \dot{Q}_m, S) \quad (7.1)$$

and

$$\text{Mod}(\ddot{\mathcal{A}}_n, \ddot{Q}_n, S) \quad \text{and} \quad \text{Mod}(\ddot{\mathcal{A}}_n^{st}, \ddot{Q}_n, S). \quad (7.2)$$

We denote by $\mathcal{M}(U^*(2m))$ and $\mathcal{M}^{st}(U^*(2m))$ and by $\mathcal{M}(GL(n, \mathbb{R}))$ and $\mathcal{M}^{st}(GL(n, \mathbb{R}))$ their associated moduli spaces whenever these moduli spaces exist.

7.2 Stability of $GL(n, \mathbb{R})$ and $U^*(2m)$ -Higgs bundles

We can observe that $U^*(2m)$ and $GL(n, \mathbb{R})$ -Higgs bundles have similarities with $Sp(2m, \mathbb{C})$ and $O(n, \mathbb{C})$ -Higgs bundles, all of them are triples (E, Θ, Φ) given by a vector bundle E , a quadratic form Θ and a Higgs field Φ . The only difference is the compatibility conditions between the Higgs field and the quadratic form in each case. The compatibility condition for $U^*(2m)$ and $GL(n, \mathbb{R})$ -Higgs bundle is $\Theta\Phi = \Phi^t\Theta$ while the compatibility condition for $Sp(2m, \mathbb{C})$ and $O(n, \mathbb{C})$ -Higgs bundles is $\Theta\Phi = -\Phi^t\Theta$. When this compatibility condition is not involved, we can extend the results obtained for $Sp(2m, \mathbb{C})$ and $O(n, \mathbb{C})$ -Higgs bundles to $U^*(2m)$ -Higgs bundles and $GL(n, \mathbb{R})$ -Higgs bundles.

Proposition 7.2.1. *Let (E, Θ, Φ) be a semistable $U^*(2m)$ or $GL(n, \mathbb{R})$ -Higgs bundle. Then (E, Φ) is a semistable Higgs bundle. If further (E, Θ, Φ) is polystable, then (E, Φ) is polystable too.*

Proof. Substituting $Sp(2m, \mathbb{C})$ and $O(n, \mathbb{C})$ by $U^*(2m)$ and $GL(n, \mathbb{R})$ in the proof of Proposition 5.1.2 leads to the result. \square

Proposition 7.2.2. *Let (E, Θ, Φ) be a semistable $U^*(2m)$ or $GL(n, \mathbb{R})$ -Higgs bundle. Then (E, Θ) is a semistable $Sp(2m, \mathbb{C})$ or $O(n, \mathbb{C})$ -bundle. If further (E, Θ, Φ) is polystable, (E, Θ) is polystable too.*

Conversely, every $U^(2m)$ or $GL(n, \mathbb{R})$ -Higgs bundle of the form (E, Θ, Φ) with (E, Θ) semistable, is semistable.*

Proof. The first part follows from the proof of Proposition 5.1.1 substituting $Sp(2m, \mathbb{C})$ and $O(n, \mathbb{C})$ by $U^*(2m)$ and $GL(n, \mathbb{R})$.

The second part is obvious since (E, Θ) semistable implies that all isotropic subbundles (including those that are Φ -invariant) have slope less than or equal to 0. \square

From the definition of the associated graded object, we know that every strictly polystable $U^*(2)$ or $GL(2, \mathbb{R})$ -Higgs bundle (E, Θ, Φ) is isomorphic to $(L \oplus E/L, \tilde{\Theta}, \Phi|_{L \oplus \tilde{\Phi}})$, where L is a stable isotropic and Φ -invariant subbundle (necessarily of rank 1 and degree 0). Since L is isotropic we have that $E/L \cong L^*$ and since Φ commutes with Θ we have

that $\tilde{\Phi} = \phi$ if $\Phi_L = \phi \in H^0(X, \mathcal{O})$. Thus, we have that (E, Θ, Φ) is isomorphic to $(L \oplus L^*, \tilde{\Theta}, \phi \otimes \text{id}_{L \oplus L^*})$, where $\tilde{\Theta}$ is off-diagonal

$$\tilde{\Theta} = \begin{pmatrix} 0 & b\theta \\ \theta & 0 \end{pmatrix}$$

with $b = -1$ in the case of $U^*(2)$ -Higgs bundles and $b = 1$ for $GL(2, \mathbb{R})$ -Higgs bundles. Composing with the isomorphism $\theta^{-1/2} \text{id}_{L \oplus L^*}$ we obtain that

$$(E, \Theta, \Phi) \cong \left(L \oplus L^*, \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \phi & \\ & \phi \end{pmatrix} \right)$$

for some $L \in \text{Pic}^0(X)$ and some $\phi \in H^0(X, \mathcal{O})$.

Proposition 7.2.3. *There are no stable $U^*(2m)$ -Higgs bundles over an elliptic curve. If (E, Ω, Φ) is a polystable $U^*(2m)$ -Higgs bundle then it is a direct sum of polystable $U^*(2)$ -Higgs bundles*

$$(E, \Omega, \Phi) \cong \bigoplus_{i=1}^m \left(L_i \oplus L_i^*, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \phi_i & \\ & \phi_i \end{pmatrix} \right)$$

where $L_i \in \text{Pic}^0(X)$ and $\phi_i \in H^0(X, \mathcal{O})$.

Proof. Suppose (E, Ω, Φ) is stable. By Lemma 7.2.1, the Higgs bundle (E, Φ) is polystable and, recalling that $\deg(E) = 0$, by Corollary 4.2.4 E decomposes as a direct sum of Higgs bundles of rank 1 and degree 0

$$(E, \Phi) = \bigoplus_i (L_i, \phi_i), \quad \phi_i = \lambda_i \otimes \text{id}_{L_i}, \quad \lambda_i \in \mathbb{C}.$$

Since Ω is antisymmetric, every L_i is isotropic. By construction L_i is also Φ -invariant and therefore there are no stable $U^*(2m)$ -Higgs bundles over an elliptic curve.

The description of polystable $U^*(2m)$ -Higgs bundles follows from the Jordan-Hölder filtration and the fact that there are no stable $U^*(2m)$ -Higgs bundles. \square

Let $J_0 (\cong \mathcal{O})$, J_1 , J_2 and J_3 be the elements of $\text{Pic}^0(X)[2]$. We note that every $GL(1, \mathbb{R})$ -Higgs bundle is stable and isomorphic to $(J_a, 1, \phi)$.

Proposition 7.2.4. *The $GL(n, \mathbb{R})$ -Higgs bundle (E, Q, Φ) is stable if and only if it is a direct sum of stable $GL(1, \mathbb{R})$ -Higgs bundles*

$$(E, Q, \Phi) \cong \bigoplus_i (J_{a_i}, 1, \phi_i)$$

where, for every $i \neq j$ we have $(J_{a_i}, 1, \phi_i) \not\cong (J_{a_j}, 1, \phi_j)$.

Proof. First we note that every $GL(n, \mathbb{R})$ -Higgs bundle of this form is stable since every Φ -invariant subbundle of degree 0 is not isotropic.

Now we suppose that (E, Q, Φ) is any stable $GL(n, \mathbb{R})$ -Higgs bundle. By Proposition 7.2.1 we have that (E, Φ) is polystable and then

$$(E, \Phi) \cong \bigoplus_i (L_i, \phi_i)$$

with (L_i, ϕ_i) a Higgs bundle of rank 1 and degree 0. Since Q commutes with Φ , Q sends (L_i, ϕ_i) to (L_i^*, ϕ_i) and if $L_i \not\cong L_i^*$ then L_i is isotropic, Φ -invariant and with $\deg L_i = 0$. Such a subbundle would contradict the stability of (E, Q, Φ) and therefore every L_i is self-dual.

Since E is a direct sum of $J_0 (\cong \mathcal{O})$, J_1 , J_2 and J_3 , we see that (E, Q, Φ) decomposes in

$$(E, Q, \Phi) \cong \bigoplus_a (E_a, Q_a, \Phi_a)$$

where $E_a \cong J_a^{\oplus n_a}$ and Q_a and Φ_a are the restrictions of Q and Φ to E_a . If (E_a, Q_a, Φ_a) has a Φ -invariant isotropic subbundle of degree 0 so does (E, Q, Φ) and this would contradict the stability of (E, Q, Φ) . Therefore (E_a, Q_a, Φ_a) is stable and has no Φ -invariant isotropic subbundle of degree 0.

Let $E_{a,i}$ be the subbundle of E_a associated to the eigenvalue ϕ_i of Φ_a . Since Q_a and Φ_a commute, Q_a preserves $E_{a,i}$ and then we obtain a decomposition of (E_a, Q_a, Φ_a)

$$(E_a, Q_a, \Phi_a) \cong \bigoplus_j (E_{a,j}, Q_{a,j}, \Phi_{a,j})$$

where $\Phi_{a,i} = \lambda_j \text{id}_{E_{a,i}}$ and $\lambda_j \neq \lambda_k$ if $j \neq k$. Every isotropic subbundle of degree 0 of $(E_{a,i}, Q_{a,i})$ is also a Φ -invariant isotropic subbundle of (E, Q, Φ) and contradicts the stability of the last. By Proposition 5.1.3, if $\text{rk } E_{a,i}$ is greater than one, there are isotropic subbundles of degree 0. Therefore $(E_{a,i}, Q_{a,i}, \Phi_{a,i}) \cong (J_a, 1, \phi_i)$. \square

Once we have described the stable $\text{GL}(n, \mathbb{R})$ -Higgs bundles, the description of polystable $\text{GL}(n, \mathbb{R})$ follows from the definition of the associated graded object.

Corollary 7.2.5. *If (E, Q, Φ) is a polystable $\text{GL}(n, \mathbb{R})$ -Higgs bundle then it is a direct sum of stable $\text{GL}(1, \mathbb{R})$ -Higgs bundles and polystable $\text{GL}(2, \mathbb{R})$ -Higgs bundles*

$$(E, Q, \Phi) \cong \bigoplus_{j=1}^{\ell} (J_{a_j}, 1, \phi_j) \oplus \bigoplus_{i=1}^m \left(L_i \oplus L_i^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \phi_i & \\ & \phi_i \end{pmatrix} \right) \quad (7.3)$$

where $n = \ell + 2m$, $J_a \in \text{Pic}^0(X)[2]$, $L_i \in \text{Pic}^0(X)$ and $\phi_j, \phi_i \in H^0(X, \mathcal{O})$.

If, for some $j \neq j'$, we have $(J_{a_j}, 1, \phi_j) \cong (J_{a_{j'}}, 1, \phi_{j'})$ in the direct sum (7.3), the $\text{GL}(n, \mathbb{R})$ -Higgs bundle (E, Q, Φ) is still polystable, since

$$(J_a, 1, \phi) \oplus (J_a, 1, \phi) \cong \left(J_a \oplus J_a^*, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} \phi & \\ & \phi \end{pmatrix} \right).$$

7.3 Moduli spaces of $U^*(2m)$ -Higgs bundles

Theorem 7.3.1. *For every $m > 0$*

$$\mathfrak{M}^{\text{st}}(U^*(2m)) = \emptyset.$$

Proof. This follows from Proposition 7.2.3. □

From Proposition 7.2.2 we know that the following morphism is well defined

$$\dot{\eta}_1 : \mathfrak{M}(U^*(2)) \longrightarrow M(\mathrm{Sp}(2)) \times H^0(X, \mathcal{O})$$

$$[(E, \Omega, \Phi)]_S \longmapsto ([(E, \Omega)]_S, \frac{1}{2} \mathrm{tr} \Phi) .$$

This map allows us to describe the moduli space associated to the functor $\mathrm{Mod}(\dot{\mathcal{A}}_1, \dot{P}_1, S)$.

Theorem 7.3.2. *We have an isomorphism*

$$\mathfrak{M}(U^*(2)) \cong \mathbb{P}^1 \times \mathbb{C} .$$

Proof. We can easily see that $\dot{\eta}_1$ is bijective. By Proposition 7.2.3 every polystable $U^*(2)$ -Higgs bundle is isomorphic to $(E, \Omega, \phi \mathrm{id}_E)$. Suppose we have two polystable $U^*(2)$ -Higgs bundles $(E, \Omega, \phi \mathrm{id}_E)$ and $(E', \Omega', \phi' \mathrm{id}_{E'})$ such that $(E, \Omega) \cong (E', \Omega')$ and $\mathrm{tr} \Phi = \mathrm{tr} \Phi'$. This last equality implies that $\phi = \phi'$, and then the isomorphism of $\mathrm{Sp}(2)$ -bundles lifts to an isomorphism of $U^*(2)$ -Higgs bundles.

Since $M(\mathrm{Sp}(2, \mathbb{C})) \times H^0(X, \mathcal{O}) \cong \mathbb{P}^1 \times \mathbb{C}$ is normal, by Zariski's Main Theorem $\dot{\eta}_1$ is an isomorphism. □

Let us take $(\mathcal{V}_{(1,0)}, \Phi_{(1,0)}) \rightarrow X \times T^*X$ to be the universal family of stable Higgs bundles of rank 1 and degree 0. We can construct the following family of polystable $U^*(2)$ -Higgs bundles

$$\dot{\mathcal{E}}_1 = \left(\mathcal{V}_{(1,0)} \oplus \mathcal{V}_{(1,0)}^*, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \Phi_{(1,0)} & 0 \\ 0 & \Phi_{(1,0)} \end{pmatrix} \right) .$$

Taking the fibre product over X we construct

$$\dot{\mathcal{E}}'_m = \dot{\mathcal{E}}_1 \times \dots \times \dot{\mathcal{E}}_1$$

and with the extension of structure group associated to the injection $j : U^*(2) \times^m U^*(2) \hookrightarrow U^*(2m)$ we define

$$\dot{\mathcal{E}}_m = j_* \dot{\mathcal{E}}'_m .$$

This is a family of polystable $U^*(2m)$ -Higgs bundles parametrized by $T^*X \times T^*X$.

Take a family $\mathcal{F} \rightarrow X \times T$ of semistable $U^*(2)$ -Higgs bundles of the form

$$\mathcal{F} = \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & -\vartheta^t \\ \vartheta & 0 \end{pmatrix}, \Phi \oplus \Phi' \right)$$

where (\mathcal{V}, Φ) and (\mathcal{V}^*, Φ') are families of rank 1 Higgs bundles of degree zero such that $\vartheta : \mathcal{V} \rightarrow \mathcal{V}$ is an isomorphism satisfying $\vartheta \Phi = -\Phi' \vartheta^t$. Taking $\vartheta^{-1/2} : \mathcal{V} \rightarrow \mathcal{V}$ we can construct an isomorphism

$$\left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & -\vartheta^t \\ \vartheta & 0 \end{pmatrix}, \Phi \oplus \Phi' \right) \cong \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \Phi \oplus \Phi' \right) .$$

Remark 7.3.3. Since by Proposition 7.2.3 there are no stable $U^*(2)$ -Higgs bundles, every locally graded family $\mathcal{E} \rightarrow X \times T$ of semistable $U^*(2m)$ -Higgs bundles is such that for every $t \in T$ there exists an open subset $U \subset T$ containing t and a set of families $(\mathcal{V}_1, \Phi_1), \dots, (\mathcal{V}_m, \Phi_m)$ of Higgs bundles of rank 1 and degree 0 such that

$$\mathcal{E}|_{X \times U} \sim_S \bigoplus_{i=1}^m \left(\mathcal{V}_i \oplus \mathcal{V}_i^*, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \Phi_i \oplus \Phi_i \right). \quad (7.4)$$

Proposition 7.3.4. $\dot{\mathcal{E}}_m$ is a family with the local universal property among locally graded families of semistable $U^*(2m)$ -Higgs bundles.

Proof. A locally graded family of semistable $U^*(2m)$ -Higgs bundles $\mathcal{F} \rightarrow X \times T$ is such that for every $t \in T$ there exists an open subset $U \subset T$ containing t and m families (\mathcal{V}_i, Φ_i) of rank 1 and degree 0 Higgs bundles parametrized by U satisfying (7.4).

Since $\mathcal{E}_{(1,0)} = (\mathcal{V}_{(1,0)}, \Phi_{(1,0)})$ is the universal family of rank 1 and degree 0 Higgs bundles we have that there exists $f_i : T \rightarrow T^*X$ such that $(\mathcal{V}_i, \Phi_i) \sim_S (f_i)_* \mathcal{E}_{(1,0)}$. Then, setting $f = (f_1, \dots, f_m)$, we have that $\mathcal{F} \sim_S f_* \dot{\mathcal{E}}_m$. \square

Let us recall the group Γ_m defined in (5.4) and consider the action of Γ_m on $T^*X \times \cdot^m \times T^*X$ induced by the permutation action of the symmetric group and the following action of group $(\mathbb{Z}_2 \times \cdot^m \times \mathbb{Z}_2)$ on $T^*X \times \cdot^m \times T^*X$

$$(1, \dots, 1, -1, 1, \dots, 1) \cdot ((x_1, \lambda_1), \dots, (x_i, \lambda_i), \dots, (x_m, \lambda_m)) = \quad (7.5) \\ = ((x_1, \lambda_1), \dots, (-x_i, \lambda_i), \dots, (x_m, \lambda_m)).$$

We see that the quotient of $T^*X \times \cdot^m \times T^*X$ by Γ_m under this action is

$$(T^*X \times \cdot^m \times T^*X) / \Gamma_m = \text{Sym}^m((X/\pm) \times \mathbb{C}) \cong \text{Sym}^m(\mathbb{P}^1 \times \mathbb{C}).$$

Remark 7.3.5. One can check that two points t_1 and t_2 of $T^*X \times \cdot^m \times T^*X$ are such that $\dot{\mathcal{E}}_m|_{X \times \{t_1\}}$ and $\dot{\mathcal{E}}_m|_{X \times \{t_2\}}$ are isomorphic if and only there is an element $\gamma \in \Gamma_m$ with $t_2 = \gamma \cdot t_1$ with the action defined in (7.5).

Theorem 7.3.6. There exists a moduli space $\mathcal{M}(U^*(2m))$ associated to the moduli functor $\text{Mod}(\dot{\mathcal{A}}_m, \dot{\mathcal{Q}}_m, S)$. We have the following isomorphism

$$\dot{\xi}_m^{x_0} : \mathcal{M}(U^*(2m)) \xrightarrow{\cong} \text{Sym}^m(\mathbb{P}^1 \times \mathbb{C})$$

$$[(E, \Omega, \Phi)]_S \longmapsto [([\zeta_{1,0}^{x_0}(L_1)]_{\pm}, \phi_1), \dots, ([\zeta_{1,0}^{x_0}(L_m)]_{\pm}, \phi_m)]_{\mathfrak{S}_m},$$

where

$$\text{gr}(E, \Omega, \Phi) \cong \bigoplus_{i=1}^m \left(L_i \oplus L_i^*, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \phi_i & \\ & \phi_i \end{pmatrix} \right)$$

Proof. This follows from Propositions 3.2.1 and 7.3.4 and Remark 7.3.5. \square

Let us study the relation between $\mathcal{M}(U^*(2m))$ and $\mathcal{M}(\text{GL}(n, \mathbb{C}))_0$. Recall the involution j_{2m} defined in (4.23). Let (E, Ω, Φ) be a semistable $U^*(2m)$ -Higgs bundle, we recall

that Ω is an antisymmetric isomorphism between (E, Φ) and (E^*, Φ^t) . Then, by Proposition 7.2.1, $[(E, \Phi)]_S$ is a fixed point of j_{2m} . We have the following morphism

$$\dot{\delta}_m : \mathcal{M}(\mathrm{U}^*(2m)) \longrightarrow (\mathcal{M}(\mathrm{GL}(2m, \mathbb{C}))_0)^{j_n}$$

$$[(E, \Omega, \Phi)]_S \longmapsto [(E, \Phi)]_S.$$

By Lemma 4.3.14, the fixed point subvariety $\mathcal{M}(\mathrm{GL}(2m, \mathbb{C}))_0^{j_n}$ is isomorphic to the variety $(\mathrm{Sym}^{2m} T^* X)^{j_{2m}}$, where the involution j_{2m} is defined in (4.26). We define

$$\dot{d}_m : \mathrm{Sym}^m((X/\pm) \times \mathbb{C}) \longrightarrow (\mathrm{Sym}^{2m} T^* X)^{j_{2m}}$$

$$[[x_1]_{\mathbb{Z}_2}, \lambda_1), \dots, ([x_m]_{\mathbb{Z}_2}, \lambda_m)]_{\mathfrak{S}_m} \longmapsto [(x_1, \lambda_1), (-x_1, \lambda_1), \dots, (x_m, \lambda_m), (-x_m, \lambda_m)]_{\mathfrak{S}_{2m}}. \quad (7.6)$$

Thanks to Lemma 4.3.14, $\mathcal{M}(\mathrm{GL}(2m, \mathbb{C}))_0^{j_{2m}}$ is isomorphic to $(\mathrm{Sym}^{2m} T^* X)^{j_{2m}}$. Let us denote by $\xi_{n,0}^{x_0, j_{2m}}$ the restriction of $\xi_{n,0}^{x_0}$ to the fixed point locus. We can give an explicit description of $\dot{\delta}_m$.

Lemma 7.3.7. *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{U}^*(2m)) & \xrightarrow{\dot{\delta}_m} & \mathcal{M}(\mathrm{GL}(2m, \mathbb{C}))_0^{j_{2m}} \\ \xi_m^{x_0} \downarrow \cong & & \cong \downarrow \xi_{n,0}^{x_0, j_{2m}} \\ \mathrm{Sym}^m((X/\pm) \times \mathbb{C}) & \xrightarrow{\dot{d}_m} & (\mathrm{Sym}^{2m} T^* X)^{j_{2m}}. \end{array}$$

Proof. By Proposition 7.2.3 every polystable $\mathrm{U}^*(2m)$ -Higgs bundle is isomorphic to

$$(E, \Omega, \Phi) = \bigoplus_{i=1}^m \left(L_i \oplus L_i^*, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} \phi_i & \\ & \phi_i \end{pmatrix} \right).$$

Let us take $(x_i, \lambda_i) = \xi_{1,0}^{x_0}(L_i, \phi_i)$, then we have that $\dot{d}_m \circ \xi_m^{x_0}([(E, \Omega, \Phi)]_S)$ is

$$[(x_1, \lambda_1), (-x_1, \lambda_1), \dots, (x_m, \lambda_m), (-x_m, \lambda_m)]_{\mathfrak{S}_{2m}}.$$

Since $\xi_{1,0}^{x_0}(L_i^*, \phi_i) = (-x_i, \lambda_i)$, this is equal to $\xi_{2m,0}^{x_0} \circ \dot{\delta}_m([(E, \Omega, \Phi)]_S)$. \square

Remark 7.3.8. By Lemma 7.3.7 and the fact that \dot{d}_m is an injection, we know that $\dot{\delta}_m$ is an injection as well.

Proposition 7.2.2 allows us to define the following morphism

$$\dot{a}_m : \mathcal{M}(\mathrm{U}^*(2m)) \longrightarrow M(\mathrm{Sp}(2m, \mathbb{C}))$$

$$[(E, \Omega, \Phi)]_S \longmapsto [(E, \Omega)]_S.$$

We can also consider the natural projection

$$\dot{p}_m : \text{Sym}^m((X/\pm) \times \mathbb{C}) \longrightarrow \text{Sym}^m(X/\pm)$$

$$[[x_1]_{\mathbb{Z}_2}, \lambda_1), \dots, ([x_m]_{\mathbb{Z}_2}, \lambda_m)]_{\mathfrak{S}_m} \longmapsto [[x_1]_{\mathbb{Z}_2}, \dots, [x_m]_{\mathbb{Z}_2}]_{\mathfrak{S}_m}.$$

Proposition 7.3.9. *We have the following commuting diagram*

$$\begin{array}{ccc} \mathcal{M}(\text{U}^*(2m)) & \xrightarrow{\dot{a}_m} & M(\text{Sp}(2m, \mathbb{C})) \\ \xi_m^{x_0} \downarrow & & \downarrow \zeta_m^{x_0} \\ \text{Sym}^m((X/\pm) \times \mathbb{C}) & \xrightarrow{\dot{p}_m} & \text{Sym}^m(X/\pm). \end{array}$$

Proof. Taking the underlying $\text{Sp}(2m, \mathbb{C})$ -bundle gives, by Proposition 7.2.2, a surjective morphism from $\mathcal{M}(\text{U}^*(2m))$ to $M(\text{Sp}(2m, \mathbb{C}))$. Also, $\text{Sym}^m((X/\pm) \times \mathbb{C})$ projects naturally to $\text{Sym}^m(X/\pm)$ and one can check that the diagram commutes. \square

Thanks to Proposition 7.2.1 we construct the following morphism between moduli spaces

$$\dot{\delta}_m : \mathcal{M}(\text{U}^*(2m)) \longrightarrow \mathcal{M}(\text{GL}(2m, \mathbb{C}))_0$$

$$[(E, \Omega, \Phi)]_S \longmapsto [(E, \Phi)]_S$$

Using $\dot{\delta}_m$ and the Hitchin map for Higgs bundles $b_{(2m,0)}$, we define the Hitchin map for $\text{U}^*(2m)$ -Higgs bundles

$$\dot{b}_m = b_{(2m,0)} \circ \dot{\delta}_m.$$

Let \dot{B}_m be the image of the Hitchin map \dot{b}_m . We recall that $\dot{B}_m \subset B_{(2m,0)}$ and $B_{(2m,0)} \cong \text{Sym}^{2m} \mathbb{C}$. Let us consider the morphism

$$i_m'' : \text{Sym}^m \mathbb{C} \longrightarrow \text{Sym}^{2m} \mathbb{C}$$

$$[\lambda_1, \dots, \lambda_m]_{\mathfrak{S}_m} \longmapsto [\lambda_1, \lambda_1, \dots, \lambda_m, \lambda_m]_{\mathfrak{S}_{2m}}.$$

We know by Theorem 7.3.6, that \dot{B}_m coincides with $\beta_{(2m,0)}(\text{im } i_m'')$.

Recalling (4.13) and (4.15), we see that $\text{im } i_m''$ is smooth and then, since i_m'' is injective, $\text{im } i_m'' \cong \text{Sym}^m \mathbb{C}$. We compose with $\beta_{(2m,0)}$ defined in (4.17), to obtain the following isomorphism

$$\dot{\beta}_m : \text{Sym}^m \mathbb{C} \xrightarrow{\cong} \dot{B}_m.$$

We consider the natural projection

$$\dot{\pi}_m : \text{Sym}^m(\mathbb{P}^1 \times \mathbb{C}) \longrightarrow \text{Sym}^m \mathbb{C}.$$

Lemma 7.3.10. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{M}(U^*(2m)) & \xrightarrow{\dot{b}_m} & \dot{B}_m \\ \xi_m^{x_0} \downarrow \cong & & \cong \downarrow \dot{\beta}_m^{-1} \\ \text{Sym}^m(\mathbb{P}^1 \times \mathbb{C}) & \xrightarrow{\dot{\pi}_m} & \text{Sym}^m \mathbb{C}. \end{array} \quad (7.7)$$

Proof. If (E, Ω, Φ) is a polystable $U^*(2m)$ -Higgs bundle, by Proposition 7.2.3 it decomposes as follows

$$(E, \Omega, \Phi) \cong \bigoplus_{i=1}^m \left(L_i \oplus L_i^*, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda_i & \\ & \lambda_i \end{pmatrix} \right).$$

Since the eigenvalues of Φ are $\lambda_1, \lambda_1, \dots, \lambda_m, \lambda_m$, we can check that the image under $\dot{\beta}_m^{-1}$ of $\dot{b}_m([[(E, \Omega, \Phi)]_S])$ is $[\lambda_1, \dots, \lambda_m]_{\mathfrak{S}_m}$. It is easy to check that $\dot{\pi}_m \circ \xi_m^{x_0}([[(E, \Omega, \Phi)]_S])$ is again $[\lambda_1, \dots, \lambda_m]_{\mathfrak{S}_m}$. \square

The generic element of \dot{B}_m comes from the following element of $\mathbb{C} \times \cdot^m \times \mathbb{C}$,

$$\bar{\lambda}_{gen} = (\lambda_1, \dots, \lambda_m),$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$ and for every i we have $\lambda_i \neq 0$.

Lemma 7.3.11.

$$\dot{\pi}_m^{-1}([\bar{\lambda}_{gen}]_{\mathfrak{S}_m}) \cong \mathbb{P}^1 \times \cdot^m \times \mathbb{P}^1.$$

Proof. Since $\lambda_i \neq \lambda_j$ for every i, j such that $i \neq j$, the stabilizer in \mathfrak{S}_m of $\bar{\lambda}_{gen}$ is trivial and then the stabilizer of every tuple of the form

$$((p_1, \lambda_1), \dots, (p_m, \lambda_m))$$

is trivial too. This implies that every such tuple is uniquely determined by the choice of (x_1, \dots, x_m) , and then $\dot{\pi}_m^{-1}([\bar{\lambda}_{gen}]_{\mathfrak{S}_m})$ is isomorphic to $\mathbb{P}^1 \times \cdot^m \times \mathbb{P}^1$. \square

We treat now the case of an arbitrary point of \dot{B}_m . The arbitrary element of $\text{Sym}^m \mathbb{C}$ is the \mathfrak{S}_m -orbit of the following tuple

$$\bar{\lambda}_{arb} = (\lambda_1, \cdot^{m_1}, \lambda_1, \dots, \lambda_\ell, \cdot^{m_\ell}, \lambda_\ell),$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$ and $m = m_1 + \dots + m_\ell$.

Lemma 7.3.12.

$$\dot{\pi}_m^{-1}([\bar{\lambda}_{arb}]_{\mathfrak{S}_m}) \cong \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_\ell}.$$

Proof. The stabilizer in \mathfrak{S}_m of $\bar{\lambda}_{arb}$ is

$$Z_{\mathfrak{S}_m}(\bar{\lambda}_{arb}) = \mathfrak{S}_{m_1} \times \dots \times \mathfrak{S}_{m_\ell}.$$

Two points of $\mathbb{P}^1 \times \cdot^m \times \mathbb{P}^1$ give the same element in $\text{Sym}^m(\mathbb{P}^1 \times \mathbb{C})$ if and only if they are related by the action of $Z_{\mathfrak{S}_m}(\bar{\lambda}_{arb})$. Then

$$\dot{\pi}_m^{-1}([\bar{\lambda}_{arb}]_{\mathfrak{S}_m}) \cong (\mathbb{P}^1 \times \cdot^m \times \mathbb{P}^1) / Z_{\mathfrak{S}_m}(\bar{\lambda}_{arb}) = \text{Sym}^{m_1} \mathbb{P}^1 \times \dots \times \text{Sym}^{m_\ell} \mathbb{P}^1.$$

We recall that $\text{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n$. \square

Corollary 7.3.13. *The generic fibre of the Hitchin map for $U^*(2m)$ -Higgs bundles is $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$, the Hitchin fibre of an arbitrary element of \dot{B}_m is $\mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_\ell}$.*

We can say something about $\mathfrak{M}(U^*(2m))$, the moduli space associated to the moduli functor $\text{Mod}(\dot{A}_m, \dot{P}_m, S)$.

Proposition 7.3.14. *We have a bijective morphism $\mathcal{M}(U^*(2m)) \rightarrow \mathfrak{M}(U^*(2m))$, hence $\mathcal{M}(U^*(2m))$ is the normalization of $\mathfrak{M}(U^*(2m))$.*

Proof. The family $\dot{\mathcal{E}}_m$ induces a morphism

$$\nu_{\dot{\mathcal{E}}_m} : T^*X \times \dots \times T^*X \longrightarrow \mathfrak{M}(U^*(2m)),$$

and by Remark 7.3.5 it factors through

$$\bar{\nu}_{\dot{\mathcal{E}}_m} : \text{Sym}^m(\mathbb{P}^1 \times \mathbb{C}) \longrightarrow \mathfrak{M}(U^*(2m)).$$

Let us denote by $\overline{\mathfrak{M}(U^*(2m))}$ the normalization of $\mathfrak{M}(U^*(2m))$. Since $\text{Sym}^m(\mathbb{P}^1 \times \mathbb{C})$ is normal, by the universal property of the normalization, $\bar{\nu}_{\dot{\mathcal{E}}_m}$ factors through

$$\tilde{\nu}_{\dot{\mathcal{E}}_m} : \text{Sym}^m(\mathbb{P}^1 \times \mathbb{C}) \longrightarrow \overline{\mathfrak{M}(U^*(2m))}.$$

This map is an isomorphism since it is a bijection and $\overline{\mathfrak{M}(U^*(2m))}$ is normal. Then $\mathcal{M}(U^*(2m))$ is the normalization of $\mathfrak{M}(U^*(2m))$. \square

Remark 7.3.15. Both moduli spaces would be isomorphic if $\mathfrak{M}(U^*(2m))$ is normal, but normality in this case is an open question.

7.4 Moduli spaces of $GL(n, \mathbb{R})$ -Higgs bundles

By Proposition 7.2.4 we have that a stable $GL(n, \mathbb{R})$ -Higgs bundle is the direct sum of n different stable $GL(1, \mathbb{R})$ -Higgs bundles. Let us define $\dot{\mathcal{E}}_{a,1}$ to be the family of $GL(1, \mathbb{R})$ -Higgs bundles parametrized by \mathbb{C} with underlying $O(1, \mathbb{C})$ -bundle $(J_1, 1)$, we set

$$\ddot{\mathcal{E}}_{1,a}|_{X \times \{\lambda\}} \cong (J_a, 1, \lambda \text{id}_{J_a}). \quad (7.8)$$

We denote by C^n the following subset of \mathbb{C}^n

$$C^n = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n \text{ such that for every } i \neq j \text{ we have } \lambda_i \neq \lambda_j\}.$$

Take the injection $\iota : GL(1, \mathbb{R}) \times \dots \times GL(1, \mathbb{R}) \hookrightarrow GL(n, \mathbb{R})$ and using the extension of structure group associated to ι let us define the following family of $GL(n, \mathbb{R})$ -Higgs bundles

$$\dot{\mathcal{E}}'_{n,a} = \iota_*(\ddot{\mathcal{E}}_{1,a} \times_X \dots \times_X \ddot{\mathcal{E}}_{1,a}).$$

This family is parametrized by \mathbb{C}^n and parametrizes stable $GL(n, \mathbb{R})$ -Higgs bundles and strictly polystable $GL(n, \mathbb{R})$ -Higgs bundles. We define $\dot{\mathcal{E}}_{n,a}^{st}$ as the restriction to $X \times C^n$ of $\dot{\mathcal{E}}'_{n,a}$. We see that $\dot{\mathcal{E}}_{n,a}^{st}$ is a family of stable $GL(n, \mathbb{R})$ -Higgs bundles.

Let us denote by Ξ_n^0 the set of points of $M(O(n, \mathbb{C}))$ obtained from the projection from $M(O(1, \mathbb{C})) \times \dots \times M(O(1, \mathbb{C}))$. We see that a point of Ξ_n^0 is completely determined by four non-negative integers n_0, n_1, n_2 and n_3 such that $n = n_0 + n_1 + n_2 + n_3$ where n_a is the number of summands of $O(1, \mathbb{C})$ -bundles of the form $(J_a, 1)$. We denote the elements of Ξ_n^0 by $\bar{n} = (n_0, n_1, n_2, n_3)$. For a given $\bar{n} \in \Xi_n^0$ we define

$$\check{\mathcal{E}}_{n, \bar{n}}^{st} = \iota_* (\check{\mathcal{E}}_{n_0, 0}^{st} \times_X \check{\mathcal{E}}_{n_1, 1}^{st} \times_X \check{\mathcal{E}}_{n_2, 2}^{st} \times \check{\mathcal{E}}_{n_3, 3}^{st})$$

where ι_* is the extension of structure group associated to the injection $\iota : \mathrm{GL}(n_0, \mathbb{R}) \times \mathrm{GL}(n_1, \mathbb{R}) \times \mathrm{GL}(n_2, \mathbb{R}) \times \mathrm{GL}(n_3, \mathbb{R}) \hookrightarrow \mathrm{GL}(n, \mathbb{R})$. We observe that $\check{\mathcal{E}}_{n, \bar{n}}^{st}$ is parametrized by $C^{n_0} \times C^{n_1} \times C^{n_2} \times C^{n_3}$.

We define Ξ_n as the following subset of $\Xi_n^0 \times \mathbb{C}^n$

$$\Xi_n = \bigcup_{\bar{n} \in \Xi_n^0} \bar{n} \times (C^{n_0} \times C^{n_1} \times C^{n_2} \times C^{n_3}).$$

We set $\check{\mathcal{E}}_n^{st}$ to be the family parametrized by Ξ_n where the restriction of $\check{\mathcal{E}}_n^{st}$ to $\bar{n} \times (C^{n_0} \times C^{n_1} \times C^{n_2} \times C^{n_3})$ for some $\bar{n} \in \Xi_n^0$ is given by $\check{\mathcal{E}}_{n, \bar{n}}^{st}$.

Remark 7.4.1. Two elements t_1 and t_2 of Ξ_n are such that $\check{\mathcal{E}}_n^{st}|_{X \times \{t_1\}} \cong \check{\mathcal{E}}_n^{st}|_{X \times \{t_2\}}$ if and only if there exists $\bar{n} \in \Xi_n^0$ and $\sigma \in \mathfrak{S}_{n_0} \times \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \times \mathfrak{S}_{n_3}$ such that $t_1, t_2 \in \bar{n} \times (C^{n_0} \times C^{n_1} \times C^{n_2} \times C^{n_3})$ and $t_2 = \sigma t_1$. The first condition is equivalent to the fact that $\check{\mathcal{E}}_n^{st}|_{X \times \{t_1\}}$ and $\check{\mathcal{E}}_n^{st}|_{X \times \{t_2\}}$ have the same underlying $O(n, \mathbb{C})$ -bundle, the second condition is equivalent to the fact that $\check{\mathcal{E}}_n^{st}|_{X \times \{t_1\}}$ and $\check{\mathcal{E}}_n^{st}|_{X \times \{t_2\}}$ are direct sums of the same $\mathrm{GL}(1, \mathbb{R})$ -Higgs bundles.

Lemma 7.4.2. *The family $\check{\mathcal{E}}_n^{st}$ has the local universal property among locally graded families of stable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles.*

Proof. Suppose that $\mathcal{F} \rightarrow X \times T$ is a family of stable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles. Suppose that $T = \cup T_i$ where each T_i is irreducible and we denote by $\mathcal{F}_i = (\mathcal{E}_i, \mathcal{Q}_i, \Phi_i)$ the restriction of \mathcal{F} to T_i . The family of underlying $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles $(\mathcal{E}_i, \mathcal{Q}_i)$ induces a morphism from T_i to Ξ_n^0 . Since \mathcal{F} is locally graded, so is \mathcal{F}_i , and by definition, there exist n families of $\mathrm{GL}(1, \mathbb{R})$ -Higgs bundles $\mathcal{F}_{i,j} = (J_{a_{i,j}}, \mathcal{Q}_{i,j}, \Phi_{i,j})$ such that

$$\mathcal{F}_i \cong \bigoplus_{j=1}^n \mathcal{F}_{i,j}.$$

Counting the number of j such that $a_{i,j} = 0, 1, 2$ or 3 we obtain $n_{0,i}, n_{1,i}, n_{2,i}$ and $n_{3,i}$ and therefore $\bar{n}_i \in \Xi_n^0$. Furthermore, the $\Phi_{i,j}$ gives us a morphism $f_{i,j} : T_i \rightarrow \mathbb{C}$, and, taking $f_i = (f_{i,1}, \dots, f_{i,n})$ we obtain a morphism from U to $\bar{n}_i \times (C^{n_{0,i}} \times C^{n_{1,i}} \times C^{n_{2,i}} \times C^{n_{3,i}}) \subset \Xi_n$. By construction, we have

$$\mathcal{F}_i \sim_S (f_i)_* \check{\mathcal{E}}_n^{st}.$$

□

Theorem 7.4.3. *There exists a moduli space of stable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles associated to $\mathrm{Mod}(\check{\mathcal{A}}_n^{\mathrm{st}}, \check{Q}_n, S)$. This moduli space is isomorphic to*

$$\mathcal{M}^{\mathrm{st}}(\mathrm{GL}(n, \mathbb{R})) \cong \bigoplus_{\bar{n} \in \Xi_n^0} ((C^{m_0}/\mathfrak{S}_{n_0}) \times (C^{m_1}/\mathfrak{S}_{n_1}) \times (C^{m_2}/\mathfrak{S}_{n_2}) \times (C^{m_3}/\mathfrak{S}_{n_3})).$$

Proof. This follows from Proposition 3.2.1, Lemma 7.4.2 and Remark 7.4.1. \square

We take the universal family $(\mathcal{F}_{(1,0)}, \Phi_{(1,0)}) \rightarrow X \times (\mathrm{Pic}^0(X) \times H^0(X, \mathcal{O}))$ of stable Higgs bundles of rank 1 and degree 0 and we construct the following family of polystable $\mathrm{GL}(2, \mathbb{R})$ -Higgs bundles

$$\check{\mathcal{E}}_2'' = \left(\mathcal{F}_{(1,0)} \oplus \mathcal{F}_{(1,0)}^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \Phi_{(1,0)} & 0 \\ 0 & \Phi_{(1,0)} \end{pmatrix} \right). \quad (7.9)$$

Recall the following involution j_n of $\mathrm{Sym}^n T^*X$ defined in (4.26). We take the projection map

$$p : T^*X \times \dots \times T^*X \longrightarrow \mathrm{Sym}^n T^*X$$

$$((x_1, \lambda_1), \dots, (x_n, \lambda_n)) \longmapsto [(x_1, \lambda_1), \dots, (x_n, \lambda_n)]_{\mathfrak{S}_n}.$$

We define \check{T}_n as the closed subvariety of $T^*X \times \dots \times T^*X$ given by the preimage of the fixed point set of j

$$\check{T}_n = p^{-1}((\mathrm{Sym}^n T^*X)^{j_n}),$$

and then

$$\check{T}_n / \mathfrak{S}_n \cong (\mathrm{Sym}^n T^*X)^{j_n}. \quad (7.10)$$

Let us call the four points of $X[2]$ y_0, y_1, y_2 and y_3 . We take the injection map $\iota : T^*X \rightarrow T^*X \times T^*X$ sending (x, λ) to $((x, \lambda), (-x, \lambda))$. We have $\iota = (\iota_1, \iota_2)$ where ι_i is the image on the i -th factor. For different values of k we consider a collection of maps $\iota_k = (\iota_{k,1}, \iota_{k,2})$.

We see that \check{T}_n is given by the union of irreducible components $\check{T}_n = \bigcup \check{T}_{n,\ell}$ where each $\check{T}_{n,\ell}$ is the set of points of $T^*X \times \dots \times T^*X$ such that for a given $\sigma_\ell \in \mathfrak{S}_n$, the subvariety $\sigma_\ell \cdot \check{T}_{n,\ell}$ has the form

$$\begin{aligned} \sigma \cdot \check{T}_{n,\ell} = & (\{y_0\} \times \mathbb{C})^{n_{0,\ell}} \times (\{y_1\} \times \mathbb{C})^{n_{1,\ell}} \times (\{y_2\} \times \mathbb{C})^{n_{2,\ell}} \times (\{y_3\} \times \mathbb{C})^{n_{3,\ell}} \times \\ & \times \iota_{1,1} T^*X \times \iota_{1,2} T^*X \times \dots \times \iota_{k_\ell,1} T^*X \times \iota_{k_\ell,2} T^*X. \end{aligned}$$

We denote by $\check{T}_{n,\ell'}$ the irreducible component $\sigma_\ell \cdot \check{T}_{n,\ell}$. Obviously $\check{T}_{n,\ell}$ and $\check{T}_{n,\ell'}$ are isomorphic, we call the isomorphism between them f_{σ_ℓ} . Recall the families $\check{\mathcal{E}}_{1,a}$ and $\check{\mathcal{E}}_2''$ defined in (7.8) and (7.9). Let us consider the family parametrized by $\check{T}_{n,\ell'}$ given by the following fibre product

$$\begin{aligned} \check{\mathcal{E}}'_{n,\ell'} = & \check{\mathcal{E}}_{1,0} \times_X \check{\mathcal{E}}_{1,0}^{n_{0,\ell'}} \times_X \check{\mathcal{E}}_{1,1} \times_X \check{\mathcal{E}}_{1,1}^{n_{1,\ell'}} \times_X \check{\mathcal{E}}_{1,1} \times_X \check{\mathcal{E}}_{1,2} \times_X \check{\mathcal{E}}_{1,2}^{n_{2,\ell'}} \times_X \check{\mathcal{E}}_{1,2} \\ & \times_X \check{\mathcal{E}}_{1,3} \times_X \check{\mathcal{E}}_{1,3}^{n_{3,\ell'}} \times_X \check{\mathcal{E}}_{1,3} \times_X \check{\mathcal{E}}_2'' \times_X \check{\mathcal{E}}_2''^{k_{\ell'}} \times_X \check{\mathcal{E}}_2''. \end{aligned}$$

Let us consider $j : \mathrm{GL}(1, \mathbb{R}) \times \cdots \times \mathrm{GL}(1, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R}) \times \cdots \times \mathrm{GL}(2, \mathbb{R})$ and using the extension of structure group associated to j we define

$$\ddot{\mathcal{E}}_{n, \ell'} = j_* \dot{\mathcal{E}}'_{n, \ell'}.$$

This is a family of polystable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles defined over $\ddot{T}_{n, \ell'}$. Using the morphism $f_{\sigma_\ell} : \ddot{T}_{n, \ell} \rightarrow \ddot{T}_{n, \ell'}$, we define the pull-back

$$\ddot{\mathcal{E}}_{n, \ell} = (f_{\sigma_\ell})^* \ddot{\mathcal{E}}_{n, \ell'}.$$

We observe that for every $t \in \ddot{T}_{n, \ell_1} \cap \ddot{T}_{n, \ell_2}$ we have that

$$\ddot{\mathcal{E}}_{n, \ell_1}|_{X \times \{t\}} \cong \ddot{\mathcal{E}}_{n, \ell_2}|_{X \times \{t\}}$$

and then, we can define $\ddot{\mathcal{E}}_n$ as the family of $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles parametrized by \ddot{T}_n such that the restriction of $\ddot{\mathcal{E}}_n$ to each $\ddot{T}_{n, \ell}$ is $\ddot{\mathcal{E}}_{n, \ell}$.

Remark 7.4.4. Take two elements t_1 and t_2 of \ddot{T}_n . We have that $\ddot{\mathcal{E}}_n|_{X \times \{t_1\}} \cong \ddot{\mathcal{E}}_n|_{X \times \{t_2\}}$ if and only if there exists $\sigma \in \mathfrak{S}_n$ such that $t_2 = \sigma \cdot t_1$.

We can check that every family $\mathcal{F} \rightarrow X \times T$ of semistable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles of the form

$$\mathcal{F} = \left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & \vartheta^t \\ \vartheta & 0 \end{pmatrix}, \Phi \oplus \Phi' \right)$$

is isomorphic to a family of the form

$$\left(\mathcal{V} \oplus \mathcal{V}^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Phi \oplus \Phi \right)$$

if (\mathcal{V}, Φ) is a family of rank 1 and degree 0 Higgs bundles.

Remark 7.4.5. Every locally graded family $\mathcal{E} \rightarrow X \times T$ of semistable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles is such that for every $t \in T$ there exists an open subset $U \subset T$ containing t , k families of $\mathrm{GL}(1, \mathbb{R})$ -Higgs bundles $\mathcal{F}_j = (J_{a_j}, 1, \Phi'_j)$ and $m = (n - k)/2$ families (\mathcal{V}_i, Φ_i) of Higgs bundles of rank 1 and degree 0 such that

$$\mathcal{E}|_{X \times U} \sim_S \bigoplus_{j=1}^k (J_{a_j}, 1, \Phi'_j) \oplus \bigoplus_{i=1}^m \left(\mathcal{V}_i \oplus \mathcal{V}_i^*, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \Phi_i \oplus \Phi_i \right). \quad (7.11)$$

Proposition 7.4.6. $\ddot{\mathcal{E}}_n$ has the local universal property among locally graded families of semistable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles.

Proof. Suppose that $\mathcal{E} \rightarrow X \times T$ is a locally graded family. Then, by Remark 7.4.5 there exists an open subset $U \subset T$ containing t , k families $\mathcal{F}_j = (J_{a_j}, 1, \Phi'_j)$ of $\mathrm{GL}(1, \mathbb{R})$ -Higgs bundles and $m = (n - k)/2$ families (\mathcal{V}_i, Φ_i) of rank 1 and degree 0 Higgs bundles parametrized by U satisfying (7.11). We set

$$\mathcal{E}_i = \left(\mathcal{V}_i \oplus \mathcal{V}_i^*, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \Phi_i \oplus \Phi_i \right).$$

We take $\ddot{T}_{n,\ell}$ to be an irreducible component of \ddot{T}_n with $k_\ell = m$. Recall that $\mathcal{E}_{(1,0)} = (\mathcal{V}_{(1,0)}, \Phi_{(1,0)})$ is the universal family of Higgs bundles of rank 1 and degree 0. For every family (\mathcal{V}_i, Φ_i) associated to \mathcal{E}_i we have $f_i : U \rightarrow T^*X$ such that $(f_i)^*(\mathcal{V}_{(1,0)}, \Phi_{(1,0)}) \sim_S (\mathcal{V}_i, \Phi_i)$. For every $\mathcal{F}_j = (J_{a_j}, 1, \Phi'_j)$, the Higgs field Φ'_j induces a morphism $f'_j : U \rightarrow \mathbb{C}$, such that $(f'_j)^*\ddot{\mathcal{E}}_{1,a_j}^{st} \sim_S \mathcal{F}'_j$. With the set of f_i and f'_j we can construct $f : U \rightarrow \ddot{T}_{n,\ell}$ such that $f^*\ddot{\mathcal{E}}_n \sim_S \bigoplus \mathcal{F}'_j \oplus \bigoplus \mathcal{E}_i$. Then

$$f^*\ddot{\mathcal{E}}_n \sim_S \mathcal{E}|_{X \times U}.$$

□

We have from (7.10) that $\ddot{T}_n / \mathfrak{S}_n$ is isomorphic to $(\text{Sym}^n T^*X)^{j_n}$.

Theorem 7.4.7. *There exists a coarse moduli space $\mathcal{M}(\text{GL}(n, \mathbb{R}))$ associated to the moduli problem $\text{Mod}(\ddot{\mathcal{A}}_n, \ddot{P}_n, S)$. We have the following isomorphism*

$$\ddot{\xi}_n^{x_0} : \mathcal{M}(\text{GL}(n, \mathbb{R})) \xrightarrow{\cong} (\text{Sym}^n T^*X)^{j_n}$$

$$[E, Q, \Phi] \longmapsto [\dots (\xi_{1,0}^{x_0}(J_{a_i}, \phi_i)) \dots (\xi_{1,0}^{x_0}(L_j, \phi_j)), (\xi_{1,0}^{x_0}(L_j^*, \phi_j)) \dots]_{\mathfrak{S}_n},$$

where

$$\text{gr}(E, Q, \Phi) \cong \bigoplus_i (J_{a_i}, 1, \phi_i) \oplus \bigoplus_j \left(L_j \oplus L_j^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \phi_j & \\ & \phi_j \end{pmatrix} \right).$$

Proof. We have by Proposition 7.4.6 that $\ddot{\mathcal{E}}_n$ has the local universal property for the moduli problem $\text{Mod}(\ddot{\mathcal{A}}_n, \ddot{P}_n, S)$. By Remark 7.4.4 we have that $\ddot{\mathcal{E}}_n$ parametrizes S-equivalent $\text{GL}(n, \mathbb{R})$ -Higgs bundles if and only if the base points are related by the action of the symmetric group \mathfrak{S}_n . We have that $\ddot{T}_n / \mathfrak{S}_n$ is a categorical quotient and then, by Proposition 3.2.1, it is isomorphic to the moduli space $\mathcal{M}(\text{GL}(n, \mathbb{R}))$. □

Take a semistable $\text{GL}(n, \mathbb{R})$ -Higgs bundle (E, Q, Φ) . We recall that Q is an isomorphism between (E, Φ) and (E^*, Φ^t) , and by Proposition 7.2.1, (E, Φ) is semistable. Then, the S-equivalence class of (E, Φ) is a point of $\mathcal{M}(\text{GL}(n, \mathbb{C}))_0$ fixed by j_n . We define

$$\ddot{\delta}_n : \mathcal{M}(\text{GL}(n, \mathbb{R})) \longrightarrow \mathcal{M}(\text{GL}(n, \mathbb{C}))_0^{j_n}$$

$$[(E, Q, \Phi)]_S \longmapsto [(E, \Phi)]_S.$$

Let us denote by $\xi_{n,0}^{x_0, j_n}$ the restriction of $\xi_{n,0}^{x_0}$ to the fixed point locus. The following diagram commutes thanks to Theorem 7.4.7

$$\begin{array}{ccc} \mathcal{M}(\text{GL}(n, \mathbb{R})) & \xrightarrow{\ddot{\delta}_n} & \mathcal{M}(\text{GL}(n, \mathbb{C}))_0^{j_n} \\ \xi_n^{x_0} \downarrow \cong & & \cong \downarrow \xi_{n,0}^{x_0, j_n} \\ (\text{Sym}^n T^*X)^{j_n} & \xlongequal{\quad} & (\text{Sym}^n T^*X)^{j_n}. \end{array}$$

Remark 7.4.8. The morphism $\ddot{\delta}_n$ is a bijection from $\mathcal{M}(\mathrm{GL}(n, \mathbb{R}))$ to $\mathcal{M}(n, \mathbb{C})_0^{j_n}$.

By Proposition 7.2.2 we can construct the following morphism between moduli spaces

$$\ddot{a}_n : \mathcal{M}(\mathrm{GL}(n, \mathbb{R})) \longrightarrow M(\mathrm{O}(n, \mathbb{C}))$$

$$[(E, Q, \Phi)]_S \longmapsto [(E, Q)]_S.$$

Recall $\xi_n^{x_0}$ from Remark 5.3.13 and $p^{(n)}$ defined in (4.11). Denote by \ddot{p}_n the restriction of $p^{(n)}$ to the subvariety of points fixed by j_n .

Proposition 7.4.9. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{GL}(n, \mathbb{R})) & \xrightarrow{\ddot{a}_n} & M(\mathrm{O}(n, \mathbb{C})) \\ \xi_n^{x_0} \downarrow & & \downarrow \xi_n^{x_0} \\ (\mathrm{Sym}^n T^* X)^{j_n} & \xrightarrow{\ddot{p}_n} & (\mathrm{Sym}^n X)^{j_n}. \end{array}$$

Proof. Taking the underlying $\mathrm{O}(n, \mathbb{C})$ -bundle of a $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundle one can construct, by Proposition 7.2.2, a surjective morphism from $\mathcal{M}(\mathrm{GL}(n, \mathbb{R}))$ to $M(\mathrm{O}(n, \mathbb{C}))$. One can check that the diagram commutes. \square

Remark 7.4.10. As a consequence of Proposition 7.4.9 the connected components of the moduli space $\mathcal{M}(\mathrm{GL}(n, \mathbb{R}))$ are indexed by the connected components of $M(\mathrm{O}(n, \mathbb{C}))$. This implies that the connected components of $\mathcal{M}(\mathrm{GL}(2m+1, \mathbb{R}))$ are indexed by $k = 1, 3$ and $a = 0, \dots, n_k - 1$ where $n_1 = 4$, and $n_3 = 4$. The connected components of $\mathcal{M}(\mathrm{GL}(2, \mathbb{R}))$ are indexed by $k = 0, 2$ and $a = 0, \dots, n_k - 1$ where $n_0 = 1$ and $n_2 = 6$. If $m > 1$, the connected components of $\mathcal{M}(\mathrm{GL}(2m, \mathbb{R}))$ are indexed by $k = 0, 2, 4$ and $a = 0, \dots, n_k - 1$ where $n_0 = 1$, $n_2 = 6$ and $n_4 = 1$.

By Proposition 7.2.1, the following morphism between moduli spaces is well defined

$$\ddot{\delta}_n : \mathcal{M}(\mathrm{GL}(n, \mathbb{R})) \longrightarrow \mathcal{M}(\mathrm{GL}(n, \mathbb{C}))_0$$

$$[(E, Q, \Phi)]_S \longmapsto [(E, \Phi)]_S$$

We can define the Hitchin map for $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles thanks to $\ddot{\delta}_n$ and $b_{(n,0)}$, the Hitchin map for Higgs bundles

$$\ddot{b}_n = b_{(n,0)} \circ \ddot{\delta}_n.$$

We can check that \ddot{b}_n is surjective, i.e. the image of \ddot{b}_n is $B_{(n,0)} = B_n$. We recall from (4.17) that $\beta_{(n,0)}$ gives an isomorphism between $B_{(n,0)}$ and $\mathrm{Sym}^n \mathbb{C}$. We set $\ddot{\pi}_n = \pi^{(n)}|_{(\mathrm{Sym}^n T^* X)^{j_n}}$.

Lemma 7.4.11. *We have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{M}(\mathrm{GL}(n, \mathbb{R})) & \xrightarrow{\ddot{b}_n} & B_{(n,0)} \\ \ddot{\xi}_n^{x_0} \downarrow & & \cong \downarrow \beta_{(n,0)}^{-1} \\ (\mathrm{Sym}^n T^*X)^{j_n} & \xrightarrow{\ddot{\pi}_n} & \mathrm{Sym}^n \mathbb{C}. \end{array}$$

Proof. This follows from the definition of \ddot{b}_n and Lemma 4.3.10. □

We take

$$\bar{\lambda} = (\lambda_1, \overset{m_1}{\cdot}, \lambda_1, \dots, \lambda_\ell, \overset{m_\ell}{\cdot}, \lambda_\ell)$$

with $n = m_1 + \dots + m_\ell$ and $\lambda_i \neq \lambda_j$ if $i \neq j$.

Lemma 7.4.12. *We have*

$$\ddot{\pi}_n^{-1}([\bar{\lambda}]_{\mathfrak{S}_n}) = (\mathrm{Sym}^{m_1} X)^{j_{m_1}} \times \dots \times (\mathrm{Sym}^{m_\ell} X)^{j_{m_\ell}}.$$

Proof. Recall that $\ddot{\pi}_n = \pi^{(n)}|_{(\mathrm{Sym}^n T^*X)^{j_n}}$, then

$$\begin{aligned} \ddot{\pi}_n^{-1}([\bar{\lambda}]_{\mathfrak{S}_n}) &= (\pi^{(n)})^{-1}([\bar{\lambda}]_{\mathfrak{S}_n}) \cap (\mathrm{Sym}^n T^*X)^{j_n} = \\ &= (\mathrm{Sym}^{m_1} X \times \dots \times \mathrm{Sym}^{m_\ell} X) \cap (\mathrm{Sym}^n T^*X)^{j_n} = \\ &= (\mathrm{Sym}^{m_1} X)^{j_{m_1}} \times \dots \times (\mathrm{Sym}^{m_\ell} X)^{j_{m_\ell}}. \end{aligned}$$

The last equality follows since j_n preserves the λ_i . □

We obtain a description of the Hitchin fibre as a consequence of Lemmas 4.3.14 and 7.4.12.

Corollary 7.4.13. *The fibre of the Hitchin fibration \ddot{b}_n is a union of products of projective spaces. The dimension of the fibre varies from 0 to the integer part of $n/2$.*

We give a more detailed description of $\mathcal{M}(\mathrm{GL}(2, \mathbb{R}))$. We denote by x_0, x_1, x_2 and x_3 the four points of $X[2]$. Let us consider the following subsets of $\mathrm{Sym}^2 T^*X$

$$B = \{[(x, \lambda), (-x, \lambda)]_{\mathfrak{S}_2} \in \mathrm{Sym}^2 T^*X\} \cong (X/\pm) \times \mathbb{C} \cong \mathbb{P}^1 \times \mathbb{C},$$

$$C_a = \{[(x_a, \lambda_1), (x_a, \lambda_2)]_{\mathfrak{S}_2} \in \mathrm{Sym}^2 T^*X\} \cong \mathrm{Sym}^2 \mathbb{C},$$

and if i denotes the i -th pair of $\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$ we set

$$C'_i = \{[(x_a, \lambda_1), (x_b, \lambda_2)]_{\mathfrak{S}_2} \in \mathrm{Sym}^2 T^*X\} \cong \mathbb{C}^2.$$

We have that C'_i does not intersect with $C'_{i'}$ if i and i' are different. Also, for every C'_i we have that $B \cap C'_i$ and $C_a \cap C'_i$ are empty. Although

$$B \cap C_a = \{[(x_a, \lambda), (x_a, \lambda)]_{\mathfrak{S}_2} \in \mathrm{Sym}^2 T^*X\} \cong \mathbb{C}.$$

Remark 7.4.14. Using this notation we have the following description

$$\mathcal{M}(\mathrm{GL}(2, \mathbb{R})) \cong \left(B \cup \bigcup_{a=0}^3 C_a \right) \cup \bigcup_{i=1}^6 C'_i.$$

Note that the intersection $B \cap C_a$ is non-empty and therefore the connected variety $(B \cup_a C_a)$ is not normal.

We finish studying the relation between $\mathcal{M}(\mathrm{GL}(n, \mathbb{R}))$ and $\mathfrak{M}(\mathrm{GL}(n, \mathbb{R}))$. The family $\check{\mathcal{E}}_n$ of semistable $\mathrm{GL}(n, \mathbb{R})$ -Higgs bundles induces the following morphism

$$\nu_{\check{\mathcal{E}}_n} : \check{T}_n \longrightarrow \mathfrak{M}(\mathrm{GL}(n, \mathbb{R})).$$

By Remark 7.4.4 it factors through

$$\bar{\nu}_{\check{\mathcal{E}}_n} : \check{T}_n / \check{\mathfrak{S}}_n \longrightarrow \mathfrak{M}(\mathrm{GL}(n, \mathbb{R}))$$

and $\bar{\nu}_{\check{\mathcal{E}}_n}$ is bijective. Composing with $\check{\xi}_n^{x_0}$ we obtain a bijection between our moduli spaces

$$\mathcal{M}(\mathrm{GL}(n, \mathbb{R})) \longrightarrow \mathfrak{M}(\mathrm{GL}(n, \mathbb{R})).$$

Since these moduli spaces are not normal in general we can not apply Zariski's Main Theorem, so we do not know if the above bijection is an isomorphism or not.

Part III

Higgs bundles for complex reductive Lie groups

Chapter 8

Representations of surface groups and G -bundles over an elliptic curve

8.1 Some results on reductive Lie groups

Let G be a connected reductive Lie group, let Z_0 be the connected component of the centre $Z_G(G)$ and denote by F the intersection $Z_0 \cap [G, G]$. We have that

$$G \cong Z_0 \times_F [G, G]. \quad (8.1)$$

We take the universal covering $D \xrightarrow{p} [G, G]$ and we define C to be $p^{-1}(F) \subset Z_D(D)$. We have a natural homomorphism $\tau : C \rightarrow Z_0$ given by $\tau(c) = p(c)$ considered as an element of Z_0 . We can rewrite (8.1) in terms of τ

$$G \cong Z_0 \times_\tau D. \quad (8.2)$$

Let us denote by \overline{G} the quotient G/F , and similarly $\overline{Z} = Z_0/F$ and $\overline{D} = D/C$ or equivalently $\overline{D} = [G, G]/F$. We have the isomorphism of groups

$$\overline{G} \cong \overline{Z} \times \overline{D}.$$

Since $q : G \rightarrow \overline{G}$ is a finite covering with group F we have an exact sequence

$$0 \longrightarrow \pi_1(G) \longrightarrow \pi_1(\overline{G}) \longrightarrow F \longrightarrow 0.$$

So, q induces an injection

$$q_G^\pi : \pi_1(G) \longrightarrow \pi_1(\overline{Z}) \times \pi_1(\overline{D}) \quad (8.3)$$

$$d \longmapsto (q_1^\pi(d), q_2^\pi(d))$$

where q_1^π and q_2^π are respectively the projections to $\pi_1(\overline{Z})$ and $\pi_1(\overline{D})$. Moreover, a pair (γ_1, γ_2) in $\pi_1(\overline{G})$ belongs to the subgroup $\pi_1(G)$ if and only if $q_1^\pi(\gamma_1)$ is the inverse in F of $q_2^\pi(\gamma_2)$. Since $\overline{D} = D/C$ and D is simply connected, we have

$$\pi_1(\overline{D}) \cong C \subset Z_D(D).$$

Since the Lie algebra \mathfrak{z} is the universal cover of \overline{Z} , the fundamental group of \overline{Z} is

$$\pi_1(\overline{Z}) \cong \exp^{-1}(F) \subset \mathfrak{z}.$$

Remark 8.1.1. We have that every $d \in \pi_1(G)$ is given by a pair $(u, c) \in \mathfrak{z} \times Z_D(D)$ such that $\exp(u) = f = p(c)$, where p is the projection of the universal covering $p : D \rightarrow [G, G]$.

Let G be a compact Lie group and let T be a maximal torus with Lie algebra \mathfrak{t} (resp. let G be a complex reductive Lie group and T be a Cartan subgroup with Lie algebra \mathfrak{t}). We define the Weyl group of G with respect to T by

$$\begin{aligned} W(G, T) &= N_G(T) / Z_G(T) \\ &= N_G(\mathfrak{t}) / Z_G(\mathfrak{t}). \end{aligned}$$

Recall from (8.2) that $G \cong Z_0 \times_{\tau} D$. This induces the decomposition of Lie algebras $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{d}$. Taking $g \in G$ of the form $[(z, d)]_{\tau}$, we see that the adjoint action of g on \mathfrak{g} leaves \mathfrak{z} invariant and is equal to the adjoint action of d on \mathfrak{d} . Let H be a maximal torus of D (resp. a Cartan subgroup of D) with Lie algebra \mathfrak{h} ; if we denote by \mathfrak{z} the Lie algebra of Z_0 we have that $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{h}$. From the previous reasoning we have

$$W(G, T) = N_D(\mathfrak{h}) / Z_D(\mathfrak{h}) = W(D, H).$$

Let us suppose for simplicity that D is a simple compact Lie group (resp. simple complex Lie group) and we take a set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$. We can consider the following subset of \mathfrak{h}

$$\Xi = \bigcup_{\alpha \in \Pi} \alpha^{-1}(\mathbb{Z}).$$

The difference $\mathfrak{h} \setminus \Xi$ is disconnected and we recall that the alcoves are the closures of the connected components of $\mathfrak{h} \setminus \Xi$. Let us take an alcove A containing the origin. Take $c \in Z_D(D)$, we know (see for instance [BtD]) that a vertex of the alcove $a_c \in A$ is such that $c = \exp(a_c)$. We see that $A - a_c$ is another alcove containing the origin. Hence there is a unique element $\omega_c \in W$ such that

$$A - a_c = \omega_c(A). \quad (8.4)$$

The following results about the action of ω_c on T are taken from [BFM] and are stated for compact semisimple Lie groups. The proofs given in [BFM] are completely algebraic and therefore the statements extend to complex semisimple Lie groups.

Lemma 8.1.2. *Let G be a compact semisimple Lie group with maximal torus T (resp. let G be a complex semisimple Lie group with Cartan subgroup T). Let $\varphi : \mathfrak{t} \rightarrow \mathfrak{t}$ be an affine linear map whose translation part is given by an element $a \in \mathfrak{t}$ which exponentiates to $c \in Z_{\tilde{G}}(\tilde{G})$ and whose linear part is an element of the Weyl group $W(G, T)$. If there is an alcove $A \subset \mathfrak{t}$ such that $\varphi(A) = A$, then φ is the action of c on the alcove given in (8.4). In particular its linear part is ω_c .*

Proof. The statement for compact Lie groups is contained in Lemma 3.1.4 of [BFM]. The proof of this lemma is purely algebraic so we can extend it to complex reductive Lie groups. \square

Proposition 8.1.3. *Let G be a compact semisimple Lie group with maximal torus T (resp. let G be a complex semisimple Lie group with Cartan subgroup T). Let $G = \prod_i G_i$ with G_i simple. Let $c \in Z_G(G)$. The following conditions are equivalent.*

1. *The fixed point set T^{ω_c} of the ω_c -action is finite.*
2. *For each i , the group G_i is $SU(n_i)$ (resp. $SL(n_i, \mathbb{C})$) and the projection c_i of c into G_i generates $Z_{G_i}(G_i)$.*

Proof. When G is compact semisimple this is Proposition 3.4.1 of [BFM]. The result extends to complex semisimple Lie groups with minor modifications since the proof is based on properties of Lie algebras. \square

Remark 8.1.4. From Remark 8.1.1 we have that every element d of $\pi_1(G)$ is determined by a pair $(u, c) \in \mathfrak{z} \times C$. We see by (8.4) that every d of $\pi_1(G)$ determines an element of the Weyl group $W(G, T)$.

Let $d \in \pi_1(G)$ with $d = (u, c)$. We denote the connected component of the fixed point set of the action of ω_c by

$$S_c = (T^{\omega_c})_0. \quad (8.5)$$

Let us take its normalizer $N_G(S_c)$ and define the quotient

$$W(G, T, c) = N_G(S_c) / Z_G(S_c) \quad (8.6)$$

to be the *Weyl group of degree c* . We abbreviate it by W_c when G and T are clear from the context. When c is the identity, one has $W_{\text{id}} = W$.

We define

$$L_c = Z_G(S_c). \quad (8.7)$$

One can easily check that $N_G(S_c) = N_G(L_c)$ and therefore

$$W(G, T, c) = N_G(L_c) / L_c.$$

Let $[\widetilde{L_c, L_c}] \xrightarrow{p_c} [L_c, L_c]$ be the universal cover; we define

$$D_c = [\widetilde{L_c, L_c}], \quad F_c = S_c \cap [L_c, L_c] \quad \text{and} \quad C_c = p_c^{-1}(F_c). \quad (8.8)$$

Finally we take $\tau_c : C_c \rightarrow F_c \subset S_c$ as the restriction of p_c to C_c .

Since L_c is the centralizer of a torus we know that it is connected and reductive. We see that S_c is the centre of L_c and by (8.2) we can decompose

$$L_c \cong S_c \times_{\tau_c} D_c. \quad (8.9)$$

The quotient $\overline{L_c} = L_c / F_c$ is isomorphic to $\overline{S_c} \times \overline{D_c}$ where

$$\overline{S_c} = S_c / F_c \quad \text{and} \quad \overline{D_c} = D_c / C_c. \quad (8.10)$$

We will give a characterization of S_c and L_c in terms of a given subset of the set of simple roots Π . Let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{d}$ with \mathfrak{z} denoting the centre and \mathfrak{d} the semisimple part of

the Lie algebra, and let \mathfrak{h} be a maximal abelian (resp. Cartan) subalgebra of \mathfrak{d} such that $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{h}$. Let us take Δ to be a root system associated to $(\mathfrak{h}, \mathfrak{d})$ and denote by $\Pi \subset \Delta$ the set of simple roots. Take a subset $B \subset \Pi$ and define $\mathfrak{t}_B = \bigcap_{\alpha \in B} \ker \alpha \subset \mathfrak{t}$; we note that $\mathfrak{z} \subset \mathfrak{t}_B$. We denote by S_B the connected subgroup of T with Lie algebra \mathfrak{t}_B and by L_B , the connected subgroup $Z_G(S_B)$.

Any $c \in Z_D(D)$ has the following form

$$c = \exp \left(\sum_{\alpha \in \Pi} m_\alpha \alpha^\vee \right).$$

We define

$$B_c = \{ \alpha \in \Pi : m_\alpha \notin \mathbb{Z} \}. \quad (8.11)$$

Lemma 8.1.5. *The derived subgroup $[L_B, L_B]$ is simply connected.*

Proof. This is contained in [B].

Although the result is stated for compact Lie groups, it extends to complex reductive Lie groups since a compact group and its complexification have the same fundamental group. \square

We denote by D_B the derived subgroup $[L_B, L_B]$; note that it is connected.

Proposition 8.1.6. *Let G be a compact (resp. complex reductive) Lie group. Take the subgroups S_c , L_c and D_c as defined in (8.5), (8.7) and (8.8). There exists $g \in G$ such that $S_{B_c} = gS_cg^{-1}$, $L_{B_c} = gL_cg^{-1}$ and $D_{B_c} = gD_cg^{-1}$.*

Proof. For G compact, this statement follows from Proposition 3.4.4 of [BFM]. Since the proof of this proposition is based in some results concerning the Lie algebras of the compact groups involved, we see that, with minor modifications, the proof extends to the case of G complex reductive. \square

We can study now a certain relation between the fundamental groups $\pi_1(G)$ and $\pi_1(L_c)$. We have that the inclusion $L_c \hookrightarrow G$ induces a homomorphism of fundamental groups

$$\mu_{G, L_c} : \pi_1(L_c) \longrightarrow \pi_1(G).$$

We recall the injection $q_G^\pi : \pi_1(G) \rightarrow \pi_1(\overline{Z}) \times \pi_1(\overline{D})$ defined in (8.3). Remark 8.1.1 says, that, under this injection, every $d \in \pi_1(G)$ is given by a pair $(u, c) \in \mathfrak{z} \times Z_D(D)$ and every ℓ of $\pi_1(L_c)$ is given by $(v, f) \in \mathfrak{s} \times Z_{D_c}(D_c)$, where (u, c) and (v, f) are such that $\exp(u) = p(c)$ and $\exp(v) = p_c(f)$. We recall that $D \xrightarrow{p} [G, G]$ and $D_c \xrightarrow{p_c} [L_c, L_c]$ are the universal covers.

Lemma 8.1.7. *Let $d \in \pi_1(G)$ such that $q_G^\pi(d) = (u, c)$. Then, there is a unique $\ell \in \pi_1(L_c)$ such that $\mu_{G, L_c}(\ell) = d$. Furthermore we have that $q_{L_c}^\pi(\ell) = (u, p(c))$.*

Proof. By Lemma 8.1.5 we have that $[L_{B_c}, L_{B_c}]$ is simply connected. By Proposition 8.1.6 so is $[L_c, L_c]$. By construction, we have that $p(c) \in [L_c, L_c]$ and $p(c) \in S_c$, thus $p(c) \in F_c \subset Z_{D_c}(D_c)$.

If $\ell \in \pi_1(L_c)$ is given by $(v, f) \in \mathfrak{s} \times Z_{D_c}(D_c)$ and we have $\mu_{G, L_c}(\ell) = d$, then $f = p(c)$ and $v = u$, since $v \in \exp^{-1}(p(c)) \subset \exp^{-1}(F) \subset \mathfrak{z}$.

Since $d = \mu_{G, L_c}(\ell)$ fixes (v, f) we have that such $\ell \in \pi_1(L_c)$ is unique. \square

We continue with characterization of the groups S_c and L_c . In the next paragraphs we will describe the structure of the Lie algebra of L_c by describing its extended Dynkin diagram.

Let $\tilde{\mathcal{D}}$ be an extended Dynkin diagram. A diagram automorphism $\sigma : \tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{D}}$ preserves the coroot integers. Suppose that Σ is a group of diagram automorphisms of $\tilde{\mathcal{D}}$. The quotient diagram $\tilde{\mathcal{D}}/\Sigma$ is defined in [BFM] as follows.

The nodes of $\tilde{\mathcal{D}}/\Sigma$ are the Σ -orbits of the nodes of $\tilde{\mathcal{D}}$. Let us denote the orbit of the node v by \bar{v} and by $n_{\bar{v}}$ the cardinality of the orbit. The description of the bonds of $\tilde{\mathcal{D}}/\Sigma$ can be obtained from the information given by the Cartan integers $n(\bar{u}, \bar{v})$. If $\bar{u} = \{u_1, u_2, \dots\}$ and $\bar{v} = \{v_1, v_2, \dots\}$ are two distinct orbits such that for every u_i, v_j we have $n(u_i, v_j) = 0$; we set $n(\bar{u}, \bar{v}) = 0$ and there is no bond connecting the \bar{u} and \bar{v} in $\tilde{\mathcal{D}}/\Sigma$. We consider now the case where there exists some $u \in \bar{u}$ and $v \in \bar{v}$ connected by a bond. First we set $\epsilon(\bar{v}) = 1$ if no two nodes of the orbit \bar{v} are connected by a bond. We set $\epsilon(\bar{v}) = 2$ if there are nodes connected by some bond (in that case the orbit consists of diagrams of type A_2 , and no two diagrams are connected by a node). Suppose that u and v are connected by a bond, then, either $Z_\Sigma(u) \subset Z_\Sigma(v)$ or $Z_\Sigma(v) \subset Z_\Sigma(u)$. In the first case we set

$$n(\bar{u}, \bar{v}) = \epsilon(\bar{v})n(u, v),$$

in the second case we define

$$n(\bar{u}, \bar{v}) = \epsilon(\bar{v}) \frac{n_{\bar{v}}}{n_{\bar{u}}} n(u, v).$$

When G is simple, we can find in [BFM] a description of S_d and the group W_d in terms of the diagram automorphisms.

Theorem 8.1.8. *Let G be a compact simple Lie group with maximal torus T (resp. let G be a complex simple Lie group with Cartan subgroup T). Let Π and \mathcal{D} denote the set of simple roots and the Dynkin diagram associated to G and let $\tilde{\Pi}$ and $\tilde{\mathcal{D}}$ denote the extended set of simple roots and the extended Dynkin diagram. Let $c \in Z_G(G)$ and denote by Σ the subgroup generated by ω_c . Let $\pi : \mathfrak{t} \rightarrow \mathfrak{t}^{\omega_c}$ be the orthogonal projection. Take \bar{S}_c and W_c as defined in (8.10) and (8.6). Then:*

1. *The restriction of π to $\tilde{\Pi}^\vee$ factors to induce an embedding of $(\tilde{\Pi}^\vee)^\Sigma$ in \mathfrak{t}^{ω_c} . This embedding identifies the set of nodes $(\tilde{\Pi}^\vee)^\Sigma$ of $\tilde{\mathcal{D}}^\vee/\Sigma$ with an extended set of simple coroots for a root system Δ_c .*
2. *$\tilde{\mathcal{D}}^\vee/\Sigma$ is the extended coroot diagram of Δ_c .*
3. *The coroot lattice of Δ_c is the fundamental group of \bar{S}_c and the group W_c is the Weyl group of Δ_c .*

Proof. When G is compact and simple this is Theorem 1.6.2 of [BFM]. Since the proof is purely algebraic, it extends, with minor modifications, to the case of G complex reductive. \square

We denote by $\tilde{H}_{G,c}$ the simply connected group given by the root system Δ_c . Applying Theorem 8.1.8, when G is a simple and simply connected Lie group one obtains the following correspondence which is worked out in [S]:

1. If $G = \mathrm{SU}(n)$ (resp. $G = \mathrm{SL}(n, \mathbb{C})$) and c is of order f dividing n , then \overline{S}_c and W_c correspond to the maximal torus and the Weyl group of $\mathrm{SU}(n/f)$ (resp. the Cartan subgroup and the Weyl group of $\mathrm{SL}(n/f, \mathbb{C})$);
2. if $G = \mathrm{Spin}(2n+1)$ (resp. $G = \mathrm{Spin}(2n+1, \mathbb{C})$) and c is of order 2, then \overline{S}_c and W_c correspond to the maximal torus and the Weyl group of $\mathrm{Sp}(2n-2)$ (resp. the Cartan subgroup and the Weyl group of $\mathrm{Sp}(2n-2, \mathbb{C})$);
3. if $G = \mathrm{Sp}(2n)$ (resp. $G = \mathrm{Sp}(2n, \mathbb{C})$), n even and c is of order 2, then \overline{S}_c and W_c correspond to the maximal torus and the Weyl group of $\mathrm{Sp}(n)$ (resp. the Cartan subgroup and the Weyl group of $\mathrm{Sp}(n, \mathbb{C})$);
4. if $G = \mathrm{Sp}(2n)$ (resp. $G = \mathrm{Sp}(2n, \mathbb{C})$), n odd and c is of order 2, then \overline{S}_c and W_c correspond to the maximal torus and the Weyl group of $\mathrm{Sp}(n-1)$ (resp. the Cartan subgroup and the Weyl group of $\mathrm{Sp}(n-1, \mathbb{C})$);
5. if $G = \mathrm{Spin}(2n)$ (resp. $G = \mathrm{Spin}(2n, \mathbb{C})$) and c is the element of order 2 such that the quotient $\mathrm{Spin}(2n)/c$ is $\mathrm{SO}(2n)$, then \overline{S}_c and W_c correspond to the maximal torus and the Weyl group of $\mathrm{Sp}(2n-4)$ (resp. the Cartan subgroup and the Weyl group of $\mathrm{Sp}(2n-4, \mathbb{C})$);
6. if $G = \mathrm{Spin}(2n)$ (resp. $G = \mathrm{Spin}(2n, \mathbb{C})$), n odd and c is of order 4, then \overline{S}_c and W_c correspond to the maximal torus and the Weyl group of $\mathrm{Sp}(n-3)$ (resp. the Cartan subgroup and the Weyl group of $\mathrm{Sp}(n-3, \mathbb{C})$);
7. if $G = \mathrm{Spin}(2n)$ (resp. $G = \mathrm{Spin}(2n, \mathbb{C})$), n even and c is one of the exotic elements of order 2, then \overline{S}_c and W_c correspond to the maximal torus and the Weyl group of $\mathrm{Spin}(n+1)$ (resp. the Cartan subgroup and the Weyl group of $\mathrm{Spin}(n+1, \mathbb{C})$);
8. if $G = E_6$ (resp. $G = E_6^{\mathbb{C}}$) and c is of order 3, then \overline{S}_c and W_c correspond to the maximal torus and the Weyl group of G_2 (resp. the Cartan subgroup and the Weyl group of $G_2^{\mathbb{C}}$);
9. if $G = E_7$ (resp. $G = E_7^{\mathbb{C}}$) and c is of order 2, then \overline{S}_c and W_c correspond to the maximal torus and the Weyl group of F_4 (resp. the Cartan subgroup and the Weyl group of $F_4^{\mathbb{C}}$).

8.2 Representations of surface groups and c -pairs

In this section we follow [BFM] where they study almost commuting pairs in compact semisimple Lie groups. We show that their results can be extended to the case of complex semisimple Lie groups.

Let G denote a compact or complex reductive Lie group. We say that two elements of a Lie group G *almost commute* if their commutator lies in the centre of the Lie group. Recall from (8.2) that $G \cong Z_0 \times_{\tau} D$ where Z_0 is the connected component of $Z_G(G)_0$, D is the universal covering of the semisimple group $[G, G]$, $F = Z_0 \cap [G, G]$, C is the preimage of F in D and τ is the natural homomorphism from C to Z_0 given by the inclusion of F in

Z_0 . Let c be an element of $C \subset Z_D(D)$. If x and y are two almost commuting elements of the form $x = [(\lambda, a)]_\tau$ and $y = [(\mu, b)]_\tau$, we say that (x, y) is a c -pair if $[a, b] = c$.

Let $R(G)_c$ denote the subset of $G \times G$ of c -pairs. The group G acts on $R(G)_c$ by conjugation; we define the moduli space of c -pairs to be the GIT quotient of $R(G)_c$ by this action

$$\mathcal{R}(G)_c = R(G)_c // G.$$

The fundamental group of an elliptic curve X is

$$\pi_1(X) = \langle \alpha, \beta : [\alpha, \beta] = \text{id} \rangle \cong \mathbb{Z} \times \mathbb{Z}. \quad (8.12)$$

Let us consider its universal central extension Γ ,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \pi_1(X) \longrightarrow 0.$$

The group Γ is generated by α, β and the central element δ , satisfying $\delta = [\alpha, \beta]$. If we identify \mathbb{Z} with the subgroup generated by δ we can define $\Gamma_{\mathbb{R}}$

$$\Gamma_{\mathbb{R}} = \mathbb{R} \times_{\mathbb{Z}} \Gamma. \quad (8.13)$$

Let G be a connected or complex reductive Lie group. A representation

$$\rho : \Gamma_{\mathbb{R}} \longrightarrow G$$

is central if $\rho(\mathbb{R})$ is contained in $Z_G(G)$, since $\rho(\mathbb{R})$ is connected and contains the unit element, it is contained in $Z_0 = Z_G(G)_0$. From a central representation $\rho : \Gamma_{\mathbb{R}} \rightarrow G$ one obtains a pair (τ, u) , where $\tau : \Gamma \rightarrow G$ is such that $\tau = \rho|_{\Gamma}$ and $u \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ is given by $u = d\rho(1)$. Conversely, suppose we have a pair (τ, u) where $\tau : \Gamma \rightarrow G$ is a representation of Γ and $u \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ satisfies $\exp(u) = \tau(\delta)$, where δ is the central element of Γ . From such (τ, u) one can construct a central representation of $\Gamma_{\mathbb{R}}$ as follows. Let us define a representation $\mu : \mathbb{R} \rightarrow Z_G(G)$ setting $\mu(\lambda) = \exp \lambda \cdot u$ and set

$$\rho : \mathbb{R} \times_{\mathbb{Z}} \Gamma \longrightarrow G$$

$$[(\lambda, \gamma)]_{\mathbb{Z}} \longmapsto \mu(\lambda) \cdot \tau(\gamma),$$

note that it is well defined since $\mu(\lambda - 1)\tau(\delta \cdot \gamma)$ is equal to $\mu(\lambda) \exp(-u)\tau(\delta)\tau(\gamma)$ and therefore to $\mu(\lambda)\tau(\gamma)$.

Suppose that the representation $\rho : \Gamma_{\mathbb{R}} \rightarrow G$ is associated with (τ, u) . Then, for every $g \in G$, the representation $g\rho g^{-1}$ is associated with $(g\tau g^{-1}, u)$. We see that $u \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ is an invariant of the conjugacy class of the representation ρ .

Every central representation $\tau : \Gamma \rightarrow G$ is completely determined by two elements of the group $a, b \in G$ that are the images of α and β under τ . Since τ is central, $\bar{c} = \tau(\delta)$ is contained in Z_0 , and since $\bar{c} = [a, b]$, we have $\bar{c} \in F$ where $F = Z_0 \cap [G, G]$. Let D be the universal covering of the semisimple group $[G, G]$, set C to be the preimage of F in D and define τ as the natural homomorphism from C to Z_0 given by the inclusion of F in Z_0 .

If G is compact or complex reductive, we have by (8.2) that $G \cong Z_0 \times_{\tau} D$. Take $a = [(z_a, d_a)]_{\tau}$ and $b = [(z_b, d_b)]_{\tau}$, and write $c = [d_a, d_b]$. We see that the pair (a, b) that completely determines the representation $\tau : \Gamma \rightarrow G$ is a c -pair.

If the representation $\tau : \Gamma \rightarrow G$ is determined by $(a, b) \in G \times G$, then, for every $g \in G$, the representation $g\tau g^{-1}$ is determined by (gag^{-1}, gbg^{-1}) . In particular $c \in C \subset Z_D(D)$ is an invariant of the conjugacy class of the representation τ .

We combine the description of ρ as (τ, u) and the description of τ by the c -pair (a, b) in the following remark.

Remark 8.2.1. Every central representation $\rho : \Gamma_{\mathbb{R}} \rightarrow G$ is determined by a triple (a, b, u) , where $(a, b) \in G \times G$ is a c -pair and $u \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ is such that $c \in Z_D(D)$ projects to $\exp(u)$.

For any $g \in G$, the representation $g\rho g^{-1}$ is determined by (gag^{-1}, gbg^{-1}, u) , where (gag^{-1}, gbg^{-1}) is a c -pair.

The pair $(u, c) \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) \times Z_D(D)$ which satisfies $\bar{c} = \exp(u) \in F$ is an invariant of the conjugacy class of ρ .

After Remark 8.2.1 we see that $\text{Hom}(\Gamma_{\mathbb{R}}, G)_{(u,c)}$ the set of central representations of $\Gamma_{\mathbb{R}}$ in G with invariants (u, c) is

$$\text{Hom}(\Gamma_{\mathbb{R}}, G)_{(u,c)} = R(G)_c,$$

where $R(G)_c$ is the set of c -pairs in G . As a consequence, the *moduli space of representations of $\Gamma_{\mathbb{R}}$ for an elliptic curve* with invariants (u, c) is equal to the moduli space of c -pairs

$$\mathcal{R}(G)_c = R(G)_c // G = \text{Hom}(\Gamma_{\mathbb{R}}, G)_{(u,c)} // G.$$

Let us study $\mathcal{R}(G)_c$ when $c = \text{id}$, that classifies commuting pairs. We need the following well known result.

Lemma 8.2.2. (Theorem 2.11 of [H]) *Let G be a compact (resp. complex reductive) Lie group. If G is simply connected, then the centralizer of any semisimple element is connected.*

Lemma 8.2.2 allows us to extend to the case of complex semisimple Lie groups the following result proved by Borel in [Bo] for compact semisimple Lie groups.

Lemma 8.2.3. *Let G be a compact semisimple (resp. complex semisimple) Lie group. Suppose G is connected and simply connected and let x, y be two semisimple elements of G such that $[x, y] = \text{id}$. Then there exists a maximal torus (resp. Cartan subgroup) $T \subset G$ with $x, y \in T$.*

Proof. We follow the proof given in [FM1]. Given $y \in G$, let T_1 be a maximal torus of G containing y and denote by G_y the centralizer $Z_G(y)$. By Lemma 8.2.2 G_y is connected. Since y is contained in T_1 we have that T_1 is contained in G_y and therefore T_1 is a maximal torus of G_y . Thus, every maximal torus of G_y is also a maximal torus of G . Since x commutes with y we have $x \in G_y$ and then it is contained in a maximal torus T of G_y . By construction y lies in the centre of G_y and then y is contained in every maximal torus of G_y . In particular it is contained in T . Thus T is a maximal torus of G that contains both x and y . \square

Borel's lemma allows us to describe the set of commuting pairs. The following result was proved in [BFM] for compact Lie groups; we see that it extends to the complex reductive case.

Proposition 8.2.4. *Let G be a compact Lie group (resp. complex reductive Lie group). Let $T \subset G$ be a maximal torus (resp. a Cartan subgroup) and let W be the Weyl group associated to it. Then*

$$\bar{\zeta}_0 : T \times T / W \longrightarrow \mathcal{R}(G)_0$$

$$[(x, y)]_W \longmapsto [(x, y)]_G$$

is a homeomorphism.

Proof. The proof follows from the arguments contained in Section 1.2 of [BFM]. Take $G \cong Z_0 \times_{\tau} D$ where D is compact semisimple (resp. complex semisimple) and simply connected and Z_0 is isomorphic to a product of $U(1)$ s (resp. a product of \mathbb{C}^* s). By Lemma 8.2.3 every commuting pair (a, b) of semisimple elements of D is conjugate by $g \in D$ to a commuting pair contained in the maximal torus \tilde{T} of D . Then the elements of G $x = [(\lambda, a)_{\tau}]$ and $y = [(\mu, b)_{\tau}]$ are conjugated by $[(\text{id}, g)]_{\tau}$ to a maximal torus T of G . Thus $\bar{\zeta}_0$ is surjective.

Since $\bar{\zeta}_0$ is continuous with the quotient topology, to see that it is a homeomorphism we only need to check that it is injective.

Suppose that two commuting pairs $(x, y) \subset T \times T$ and $(x', y') \subset T \times T$ are conjugated by some element $g \in G$. Write $gTg^{-1} = T'$ and note that T' also contains x' and y' . Consider A to be the subtorus of T generated by x' and y' . Take $Z_G(A)$ which is connected and note that T and T' are contained in $Z_G(A)$. Then, there exists an element $a \in Z_G(A)$ such that $T' = aTa^{-1}$. We have that $a^{-1}g \in N_G(T)$ and satisfies $(x', y') = (a^{-1}g)(x, y)(a^{-1}g)^{-1}$. Then (x, y) and (x', y') are related by the action of an element of the Weyl group $W = N_G(T)/T$.

We see that every commuting pair of semisimple elements is contained in $T \times T$ and every element of $T \times T$ gives a commuting pair. On the other hand two commuting pairs are conjugate in G if and only if they are related by the action of an element of the Weyl group. \square

A c -pair (x, y) has *rank zero* if $Z_G(x, y)$ is discrete, or equivalently, if $\mathfrak{z}_{\mathfrak{g}}(x, y)$ is zero. Those objects were studied in [BFM] for compact Lie groups. We can prove that their results extend to complex reductive Lie groups. To do so, we need the following preliminary lemma.

Lemma 8.2.5. *Let G be a compact (resp. complex reductive) Lie group. Let x and y be two semisimple elements of G such that $[x, y] = c$ with $c \in Z_G(G)$. If $Z_G(x, y)$ is discrete then x and y are regular.*

Proof. When G is compact the result is proved in [dS]. By Lemma 8.2.2 we have that $Z_G(x)$ is connected. If it is abelian then it is a maximal torus and we conclude the proof.

By Theorem 2.2 of [H] $Z_G(x)$ is reductive. Consider the Lie algebra $\mathfrak{z}_{\mathfrak{g}}(x)$; since it is reductive we have the decomposition $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}' \oplus \mathfrak{g}_s$ where \mathfrak{z}' is the centre and \mathfrak{g}_s is

the semisimple part. Note that since $[x, y]$ lies in the centre of G , ad_y is an automorphism of $\mathfrak{z}_{\mathfrak{g}}(x)$ and we see that it preserves \mathfrak{z}' and \mathfrak{g}_s . Since y is semisimple ad_y is semisimple. By Propositions 4.2 and 4.3 of [BM] any semisimple automorphism of a semisimple Lie algebra has non-zero fixed point set. Since $Z_G(x, y)$ is discrete, we have that $\mathfrak{z}_{\mathfrak{g}}(x, y) = 0$ and in particular $\mathfrak{g}_s^{\text{ad}_y} = 0$. This implies that \mathfrak{g}_s is 0 and then $\mathfrak{z}(x) = \mathfrak{z}'$ is abelian. \square

Proposition 8.2.6. *Let G be a connected, simply connected and compact (resp. complex reductive) Lie group. Let (x, y) be a rank zero c -pair. Then*

1. *Both x and y are regular elements of G .*
2. *The group G is a product of simple factors G_i , where each G_i is isomorphic to $\text{SU}(n_i)$ (resp. $\text{SL}(n_i, \mathbb{C})$) for some $n_i \geq 2$.*
3. *$c = c_1 \cdots c_r$, where each c_i generates the centre of G_i .*
4. *The subgroup of $G/\langle c \rangle$ generated by x and y is isomorphic to $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell$ where ℓ is the order of c .*
5. *All c -pairs in G are conjugate.*
6. *Conversely if G is as in (2) and c as in (3), then there is a rank zero c -pair in G .*

Proof. If G is a compact Lie group this statement is Proposition 4.1.1 of [BFM]. We give here the proof when G is a complex reductive Lie group which is a minor modification of the one of the compact case.

Since $xyx^{-1} = cx$ conjugation by the element y normalizes the connected group $Z_G(x)$. Recall that (x, y) has rank zero so $Z_G(x, y)$ is discrete and by Lemma 8.2.5 x and y are regular which proves 1.

Fix a Cartan subgroup T and an alcove containing the origin $A \subset \mathfrak{t}$. Up to conjugation x lies in T , and y in the normalizer of $Z_G(x)$. Since x is regular $Z_G(x) = T$, and finally, conjugating by $N_G(T)$ we can take $x = \exp a_x$ with $a_x \in A$ and $y \in N_G(T)$. Call the induced element of the Weyl group ω_y and note that a_x lies in the interior of A since x is regular. We have that $xyx^{-1} = cx$ implies that $\omega_y a_x - a_c = a_x$ for some a_c with $c^{-1} = \exp a_c$. By Lemma 8.1.2 we have that $\omega_y = \omega_c$. This implies that $\mathfrak{z}_{\mathfrak{g}}(x, y)$ contains \mathfrak{t}^{ω_c} and by hypothesis this is zero. Then Proposition 8.1.3 gives statements 2 and 3.

Statements 4, 5 and 6 follow from the study of c -pairs in $\text{SL}(n, \mathbb{C})$ that can be easily derived from the study of c -pairs in $\text{SU}(n)$ given in Section 1.3 of [FM1]. \square

Once we have a complete description of rank zero c -pairs we can describe the general case. Associated to every c -pair there is the following invariant.

Lemma 8.2.7. *Let G be a compact (resp. complex reductive) Lie group with Lie algebra $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{d}$. Denote by Π the set of simple roots of \mathfrak{d} . Let (x, y) be a c -pair in G and let S be the maximal torus (resp. Cartan subgroup) of $Z_G(x, y)$. Then there exists $B(x, y) \subset \Pi$ such that S is conjugate to $S_{B(x, y)}$.*

Proof. For the compact case this is Lemma 2.1.2 of [BFM]. Making minor modifications to the proof we can extend it to the complex reductive case. \square

Proposition 8.2.8. *Let G be a compact (resp. complex reductive) Lie group. Take S_c, L_c, D_c and B_c as in (8.5), (8.7), (8.8) and (8.11). Let (x, y) be any c -pair; then $B(x, y)$ as defined in Proposition 8.1.6 is equal to B_c . Therefore any maximal torus (resp. Cartan subgroup) is conjugate in G to S_c and (x, y) is contained in L_c after conjugation. Finally there is a unique rank zero c -pair in D_c up to conjugation.*

Proof. For the compact case this is contained in Proposition 4.2.1 of [BFM]. The proof extends to the complex reductive case since we have extended to complex reductive Lie groups all the results needed, namely 8.1.6 and 8.2.6. \square

Now we have the ingredients to describe the moduli space of c -pairs.

Theorem 8.2.9. *Let G be a compact connected Lie group with maximal torus T (resp. let G be a complex reductive connected Lie group with Cartan subgroup T). For $c \in C$ take \overline{S}_c and W_c as defined in (8.10) and (8.6). Then, we have a homeomorphism*

$$\overline{\zeta}_c : (\overline{S}_c \times \overline{S}_c) / W_c \xrightarrow{\text{homeo}} \mathcal{R}(G)_c.$$

Proof. When G is compact, this follows from Corollary 4.2.2 of [BFM]. We can extend the proof to the case in which G is complex reductive. Take S_c, L_c, D_c and C_c as in (8.5), (8.7) and (8.8). By Proposition 8.2.8 every c -pair is conjugate to a c -pair in L_c and there is a unique rank zero c -pair in D_c . By Proposition 8.2.6 D_c is a product $\text{SL}(n_1, \mathbb{C}) \cdot \dots \cdot \text{SL}(n_\ell, \mathbb{C})$ and $c = c_1 \cdot \dots \cdot c_\ell$ where c_i generates the centre of $\text{SL}(n_i, \mathbb{C})$.

Let us take (d_1, d_2) to be one representative of the unique conjugation class of the rank zero c -pair in D_c . Consider the following continuous map

$$\zeta_c : (S_c \times S_c) \longrightarrow \mathcal{R}(G)_c$$

$$(s_1, s_2) \longmapsto ([s_1, d_1]_{\tau_c}, [s_2, d_2]_{\tau_c}).$$

We take an arbitrary L_c -pair $([s'_1, d'_1]_{\tau_c}, [s'_2, d'_2]_{\tau_c})$. Since (d'_1, d'_2) is a c -pair in D_c , there exists $d \in D_c$ such that $d(d'_1, d'_2)d^{-1}$ is equal to (d_1, d_2) . Then we have that $[\text{id}, d]_{\tau_c}$ conjugates $([s'_1, d'_1]_{\tau_c}, [s'_2, d'_2]_{\tau_c})$ to a c -pair in L_c of the form $([s_1, d_1]_{\tau_c}, [s_2, d_2]_{\tau_c})$, where we recall that (s_1, s_2) lie in $S_c \times S_c$. We see that ζ_c is surjective since every c -pair in G is conjugate to a c -pair in L_c , and by the previous discussion, every c -pair in L_c lies in the image of ζ_c .

Let $d_{1,i}$ and $d_{2,i}$ be the projections of d_1 and d_2 to $\text{SL}(n_i, \mathbb{C})$. The conjugation of the c -pair $([s_1, d_1]_{\tau_c}, [s_2, d_2]_{\tau_c})$ by $[\text{id}, d_{1,i}]_{\tau_c}$ gives us $([s_1, d_1]_{\tau_c}, [s_2, c_i d_2]_{\tau_c})$ and similarly, conjugating by $[\text{id}, d_{2,i}]_{\tau_c}$ gives $([s_1, c_i^{-1} d_1]_{\tau_c}, [s_2, d_2]_{\tau_c})$. From 3 of Proposition 8.2.6 the c_i generate all $Z_{D_c}(D_c)$, so it is obvious that ζ_c factors through $\overline{S}_c \times \overline{S}_c$. Conjugating by any element of $N_G(L_c)$ we see that ζ_c factors indeed through $(\overline{S}_c \times \overline{S}_c) / W_c$. Thus, ζ_c induces the following surjective map

$$\overline{\zeta}_c : (\overline{S}_c \times \overline{S}_c) / W_c \longrightarrow \mathcal{R}(G)_c.$$

With the quotient topology it is continuous and then to prove that it is a homeomorphism we only need to prove that it is injective.

Write $Z' = Z_G([s'_1, d_1]_{\tau_c}, [s'_2, d_2]_{\tau_c})$ and suppose that there is $g \in G$ such that

$$([s_1, d_1]_{\tau_c}, [s_2, d_2]_{\tau_c}) = g([s'_1, d_1]_{\tau_c}, [s'_2, d_2]_{\tau_c})g^{-1}.$$

Then S_c and gS_cg^{-1} are maximal tori (resp. Cartan subgroups) of Z' , so there is an element $h \in Z'$ such that $hS_ch^{-1} = gS_cg^{-1}$ and then $g' = h^{-1}g$ is contained in $N_G(S_c) = N_G(L_c)$. We have that $g'([\text{id}, d_1]_{\tau_c}, [\text{id}, d_2]_{\tau_c})(g')^{-1} = ([\text{id}, d'_1]_{\tau_c}, [\text{id}, d'_2]_{\tau_c})$, where (d'_1, d'_2) is a rank zero c -pair in D_c and therefore by Proposition 8.2.6 there exists $d \in D_c$ such that $d(d'_1, d'_2)d^{-1} = (d_1, d_2)$. So, noting that $[\text{id}, d]_{\tau_c}$ commutes with S_c since $S_c = Z_{L_c}(L_c)$, we have that $g'' = [\text{id}, d]_{\tau_c} \cdot g'$ is such that

$$([s_1, d_1]_{\tau_c}, [s_2, d_2]_{\tau_c}) = ([g''s'_1(g'')^{-1}, d_1]_{\tau_c}, [g''s'_2(g'')^{-1}, d_2]_{\tau_c}) \quad (8.14)$$

with $g'' \in N_G(S_c)$. Thus (s_1, s_2) and (s'_1, s'_2) lie in the same point of $(\overline{S}_c \times \overline{S}_c)/W_c$. \square

Remark 8.2.10. When G is complex reductive $\overline{\zeta}_c$ is a morphism. In that case, if $\mathcal{R}(G)_c$ is normal, $\overline{\zeta}_c$ is an isomorphism thanks to Zariski's Main Theorem.

8.3 Review on G -bundles over elliptic curves

Let X be an elliptic curve with distinguished point x_0 . Let G be a complex connected reductive Lie group and let K be its maximal compact subgroup.

We say that a holomorphic G -bundle E over X is *stable* (resp. *semistable*) if for every reduction of structure group σ to any proper parabolic subgroup $P \subsetneq G$ giving the P -bundle E_σ and every non-trivial antidominant character $\chi : P \rightarrow \mathbb{C}^*$, we have

$$\deg \chi_* E_\sigma > 0 \quad (\text{resp. } \geq 0).$$

The G -bundle E is *polystable* if it is semistable and when there exists a parabolic $P \subsetneq G$, a reduction of structure group of E to P giving the P -bundle E_σ and χ strictly antidominant, such that

$$\deg \chi_* E_\sigma = 0,$$

there is a holomorphic reduction ς of the structure group of E_σ to the Levi subgroup $L \subset P$ giving the L -bundle E_ς .

A *family* of semistable G -bundles over X parametrized by the scheme T is a holomorphic G -bundle \mathcal{E} over $X \times T$ such that the restriction of \mathcal{E} to every slice $X \times \{t\}$ is isomorphic to a semistable G -bundle that we denote by $\mathcal{E}_t \rightarrow X$.

We say that two semistable G -bundles E and E' are *S-equivalent* if there exists a family \mathcal{E} parametrized by an irreducible scheme S and a point $s \in S$ such that for every point $t \in S$ with $t \neq s$ we have that $\mathcal{E}_t \cong E$ and $\mathcal{E}_s \cong E'$. We define *S-equivalence* to be the equivalence relation between semistable G -bundles generated by this relation. We write $E \sim_s E'$ to denote that the semistable G -bundles E and E' are S-equivalent. We say that two families of semistable G -bundles $\mathcal{E} \rightarrow X \times T$ and $\mathcal{E}' \rightarrow X \times T$ are *S-equivalent* if for every point $t \in T$ we have that $\mathcal{E}_t \sim_S \mathcal{E}'_t$.

One can see that in every S-equivalence class of semistable G -bundles there exists a polystable G -bundle unique up to isomorphism.

We denote by A_G the collection of semistable G -bundles over X of topological class $d \in \pi_1(G)$ and by A_G^{st} the subcollection of stable ones. We denote by P_G^0 the algebraic condition defined above for the definition of families of semistable G -bundles and consider the moduli functor $\text{Mod}(A_G, P_G^0, S)$ defined in (3.5). Recalling that S-equivalence for families of stable G -bundles is the same as isomorphism pointwise we also have the functor $\text{Mod}(A_G^{st}, P_G^0, \cong^{pt})$. We denote by $M(G)_d$ the moduli space of S-equivalence classes of semistable G -bundles of topological class d associated to the functor $\text{Mod}(A_G, P_G^0, S)$ and by $M^{st}(G)_d$ the moduli space of isomorphism classes of stable G -bundles of topological class d associated to the moduli functor $\text{Mod}(A_G^{st}, P_G^0, \cong^{pt})$.

The following results are valid for G -bundles over a compact Riemann surface X of arbitrary genus.

Proposition 8.3.1. (Proposition 3.2 of [Ra1]) *Let G be a complex reductive Lie group with Lie algebra \mathfrak{g} . Let $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ be the centre of \mathfrak{g} . Suppose that E is a stable G -bundle. Then $H^0(X, E(\mathfrak{g})) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$. In particular if G is semisimple $H^0(X, E(\mathfrak{g})) = 0$ and $\text{Aut } E$ is finite.*

Proposition 8.3.2. (Proposition 10.9 of [AB]) *A principal G -bundle E over X is semistable if and only if its adjoint vector bundle $E(\mathfrak{g})$ is semistable.*

Proposition 8.3.3. (Proposition 7.1 of [Ra1]) *Let G and H be reductive algebraic groups and let $\gamma : G \rightarrow H$ be a surjective homomorphism. Let E be a principal G -bundle over X and γ_*E the H -bundle constructed by the extension of structure group associated to γ . Then if γ_*E is stable (resp. semistable) E is stable (resp. semistable). If further $\ker \gamma \subset Z_G(G)$ then, conversely, if E is stable (resp. semistable) γ_*E is stable (resp. semistable).*

The Harder-Narasimhan reduction was worked out in [HN], [AB] and [RR].

Theorem 8.3.4. *Let E be an unstable principal G -bundle over X . There exists a (unique up to conjugation) parabolic subgroup P of G , a (unique up to conjugation) reduction of the structure group $\sigma \in H^0(X, E \times_G (G/P))$ giving a P -bundle E_σ and an antidominant character $\chi : P \rightarrow \mathbb{C}^*$ such that the line bundle χ^*E_σ has negative degree. Furthermore, the adjoint bundles of E and E_σ satisfy*

$$H^0(X, E(\mathfrak{g})) = H^0(X, E_\sigma(\mathfrak{p})).$$

In the particular case of an elliptic curve we have that the bundle further reduces to the Levi factor of the Harder-Narasimhan parabolic.

Theorem 8.3.5. (Proposition 2.6 of [FM1]) *Let X be an elliptic curve and let $E \rightarrow X$ be an unstable G -bundle and call its Harder-Narasimhan parabolic P . The structure group of E reduces from P to a Levi factor L of P .*

In fact, there exists a maximal parabolic subgroup P_0 of G such that the structure group of E reduces to a Levi factor L_0 of P_0 . Call this reduction of structure group σ_0 and denote by E_{σ_0} the given L_0 -bundle. We have that for every nonzero antidominant character $\chi : P_0 \rightarrow \mathbb{C}^$, the line bundle $\chi_*E_{\sigma_0}$ has negative degree.*

Recall the group $\Gamma_{\mathbb{R}}$ defined in (8.13). We study now the relation between polystable G -bundles over X and representations of $\Gamma_{\mathbb{R}}$ in K , where K is the maximal compact subgroup of the complex reductive Lie group G . We recall that the quotient of $\Gamma_{\mathbb{R}}$ by \mathbb{Z} gives a direct product

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_{\mathbb{R}} \longrightarrow \mathrm{U}(1) \times \pi_1(X) \longrightarrow 0.$$

Take the point x_0 fixed in the elliptic curve X . We take the line bundle $\mathcal{O}(x_0)$ associated with the divisor given by x_0 and let $Q_{x_0} \rightarrow X$ be the fixed $\mathrm{U}(1)$ -bundle obtained from reduction of structure group of $\mathcal{O}(x_0)$. The universal covering $\tilde{X} \rightarrow X$ is a $\pi_1(X)$ -bundle. Let us consider the fibre product $\tilde{X} \times_X Q_{x_0}$ and denote by M_{x_0} its lifting to $\Gamma_{\mathbb{R}}$.

Given a unitary representation $\rho : \Gamma_{\mathbb{R}} \rightarrow K \subset G$, we define the G -bundle E^ρ as the extension of structure group associated to ρ of M_{x_0} , i.e.

$$E^\rho = \rho_* M_{x_0}. \quad (8.15)$$

Let $\tau : \Gamma \rightarrow K$ and let $u \in \mathfrak{z}(\mathfrak{k})$ be an element of the centre of the Lie algebra of K . In [Ra1] we find a construction that, starting from a pair (τ, u) , gives a G -bundle that we denote by $E(\tau, u)$ and one can check that $E^\rho \cong E(\tau, u)$ if ρ is obtained from (τ, u) .

A Stein cover $\{U_i\}_{i \in I}$ on X is a cover such that each U_i is simply connected and each $U_i \cap U_j$ is either connected or empty for all $i \neq j$. We fix a Stein cover $\{U_i\}$ of X and a 1-cocycle $\{f_{ij}\}$ of \mathcal{O} for the cover $\{U_i\}$ such that it represents the same element as $d\bar{z}$ in the 1-dimensional space $H^1(X, \mathcal{O})$.

We write $\{h_{ij}^\rho\}$ for the normalized transition functions of E^ρ with respect to the Stein cover $\{U_i\}$. We can check that the h_{ij}^ρ are given by constant functions on $U_i \cup U_j$ taking values in the maximal compact subgroup $K \subset G$.

Let E^ρ be the polystable bundle associated to the representation $\rho : \Gamma_{\mathbb{R}} \rightarrow K$ and let $\{h_{ij}^\rho\}$ be the transition functions of E^ρ with respect to the Stein cover $\{U_i\}$. Let $y \in \mathfrak{z}_{\mathfrak{g}}(\rho)$. Denote by $E^{\rho, y}$ the G -bundle given by the transition functions

$$\{h_{ij}^\rho \exp(yf_{ij})\} \quad (8.16)$$

with respect to the covering $\{U_i\}$. We say that a G -bundle is determined by the pair (ρ, y) if it is isomorphic to $E^{\rho, y}$.

The following results of [FM1] describe semistable G -bundles over an elliptic curve X in terms of the associated representations of $\Gamma_{\mathbb{R}}$. Until the end of this section G will be a complex reductive Lie group and K its maximal compact subgroup. Denote by \mathfrak{g} and \mathfrak{k} the Lie algebras of G and K .

Proposition 8.3.6. (Theorem 3.6 of [FM1]) *Let E be a semistable principal G -bundle, let E^ρ be the polystable bundle S -equivalent to E and associated to the unitary representation $\rho : \Gamma_{\mathbb{R}} \rightarrow K$. Then for some nilpotent $y \in \mathfrak{z}_{\mathfrak{k}}(\rho)$ we have $E \cong E^{\rho, y}$.*

Proposition 8.3.7. (Theorem 4.1 of [FM1])

1. *Let E be the semistable G -bundle determined by the pair (ρ, y) , where $\rho : \Gamma_{\mathbb{R}} \rightarrow K$ and y is a nilpotent element of $\mathfrak{z}_{\mathfrak{g}}(\rho)$. Then every automorphism of E is given by a constant function g with respect to the local trivializations of E for the cover $\{U_i\}$, such that $g \in Z_G(\rho)$ and $\mathrm{Ad}(g)(y) = y$. Thus, the group of automorphisms of E is identified with the subgroup of $Z_G(\rho)$ which centralizes y .*

2. Two pairs (ρ, y) and (ρ', y') determine isomorphic holomorphic G -bundles if and only if ρ and ρ' are conjugate by an element sending y to y' . In this case, ρ and ρ' are also conjugate by an element of K .

A consequence of the first statement of Proposition 8.3.7 is the following description of the space of holomorphic sections of the adjoint bundle.

Proposition 8.3.8. (Theorem 4.6 of [FM1]) *Let E be the semistable G -bundle corresponding to the representation ρ and the nilpotent element $y \in \mathfrak{z}_{\mathfrak{g}}(\rho)$. Then*

$$H^0(X, E(\mathfrak{g})) = \ker\{\mathrm{ad} y : \mathfrak{z}_{\mathfrak{g}}(\rho) \rightarrow \mathfrak{z}_{\mathfrak{g}}(\rho)\}.$$

In Remark 8.1.4 to every element $d \in \pi_1(G)$ we associated an element of the Weyl group ω_d . As a consequence of Remark 8.2.1 and Proposition 8.2.8 we obtain the following result.

Proposition 8.3.9. (Lemma 5.17 of [FM1]) *Let G be a reductive Lie group with maximal compact subgroup K and Cartan subgroup H . Let E be a polystable G -bundle of topological type d associated to the representation $\rho : \Gamma_{\mathbb{R}} \rightarrow K$. Let $\omega_d \in W$ be the Weyl element associated to d . Then $\mathfrak{z}_{\mathfrak{g}}(\rho)$ is a reductive Lie algebra with Cartan algebra \mathfrak{h}^{ω_d} .*

The description of the moduli space of c -pairs given by [BFM] and contained in Section 8.2 and the previous description of semistable and polystable G -bundles leads [FM1] to a description of the moduli space of G -Higgs bundles.

Theorem 8.3.10. (Theorem 5.19 of [FM1]) *Suppose that G is a simply connected complex semisimple Lie group and let Λ be the coroot lattice of G . Then, there is an isomorphism of normal projective varieties*

$$M(G) \cong (X \otimes_{\mathbb{Z}} \Lambda) / W.$$

More generally, let G be a complex semisimple Lie group, K its maximal compact subgroup and T a maximal torus of K . Let c be an element of the centre of the universal cover \tilde{G} of G and suppose that G is such that $G = \tilde{G}/\langle c \rangle$. Then there exists a complex structure on $\mathcal{R}(K)_c \cong (\tilde{S}_c \times \tilde{S}_c)/W_c$ for which it is an irreducible normal projective variety isomorphic to $M(G)_c$.

Remark 8.3.11. The description of trivial G -bundles $M(G)_0$ was studied separately in [La] with a purely algebraic method.

The correspondence given by Theorem 8.1.8 at the end of Section 8.1 gives the following result which is worked out in [S] and [FM1].

Theorem 8.3.12. *Let \tilde{G} be a simple complex reductive and simply connected Lie group and let $c \in Z_{\tilde{G}}(\tilde{G})$, set $G = \tilde{G}/\langle c \rangle$. We have that $\pi_1(G) \cong \langle c \rangle$; let d be the element of $\pi_1(G)$ corresponding to c under this isomorphism. The moduli space $M(G)_d$ of holomorphic G -bundles of topological class d is isomorphic to the moduli space $M(\tilde{H}_{G,c})$ of holomorphic $\tilde{H}_{G,c}$ -bundles, where $\tilde{H}_{G,c}$ is the simple complex reductive and simply connected Lie group given by the correspondence of Theorem 8.1.8.*

Chapter 9

G -Higgs bundles over an elliptic curve

9.1 G -Higgs bundles

Let G be a connected complex reductive Lie group and let X be an elliptic curve.

A G -Higgs bundle over X is a pair (E, Φ) where E is a holomorphic G -bundle over X and Φ , called the *Higgs field*, is a holomorphic section of $E(\mathfrak{g})$.

Two G -Higgs bundles (E, Φ) and (E', Φ') are isomorphic if there exists an isomorphism of G -bundles $f : E \rightarrow E'$ such that $\Phi' = f \circ \Phi \circ f^{-1}$. We denote by $\text{Aut}(E, \Phi)$ the group of automorphisms of the G -Higgs bundle (E, Φ) and by $\text{aut}(E, \Phi)$ its Lie algebra.

Remark 9.1.1. When X is a compact Riemann surface of arbitrary genus we define a G -Higgs bundle as a pair (E, Φ) where E is a holomorphic G -bundle and Φ a section of $E(\mathfrak{g}) \otimes \Omega_X^1$, where Ω_X^1 is the canonical bundle of the Riemann surface. The previous definition holds for elliptic curves since, in that case, the canonical bundle is trivial.

We say that a G -Higgs bundle (E, Φ) over X is *stable* (resp. *semistable*) if for every proper parabolic subgroup P with Lie algebra \mathfrak{p} , any non-trivial antidominant character $\chi : P \rightarrow \mathbb{C}^*$, and any reduction of structure group σ to the parabolic subgroup P giving the P -bundle E_σ such that $\Phi \in H^0(X, E_\sigma(\mathfrak{p}))$, we have

$$\deg \chi_* E_\sigma > 0 \quad (\text{resp. } \geq 0)$$

The G -Higgs bundle (E, Φ) is *polystable* if it is semistable and when there exists a parabolic subgroup $P \subsetneq G$, a strictly antidominant character $\chi : P \rightarrow \mathbb{C}^*$, and a reduction of structure group σ giving the P bundle E_σ such that

$$\Phi \in H^0(X, E_\sigma(\mathfrak{p}))$$

and

$$\deg \chi_* E_\sigma = 0,$$

there exists a holomorphic reduction ς of the structure group of E_σ to the Levi subgroup $L \subset P$ such that $\Phi \in H^0(X, E_\varsigma(\mathfrak{l}))$, where E_ς denotes the principal L -bundle obtained from the reduction of structure group ς and \mathfrak{l} is the Lie algebra of L .

A family of semistable G -Higgs bundles over X parametrized by the scheme T is a pair $\mathcal{E} = (\mathcal{F}^\mathcal{E}, \Phi^\mathcal{E})$ where $\mathcal{F}^\mathcal{E} \rightarrow X \times T$ is a family of G -bundles and $\Phi^\mathcal{E}$ is a holomorphic section of $\mathcal{F}^\mathcal{E}(\mathfrak{g})$.

Two semistable G -Higgs bundles (E, Φ) and (E', Φ') are S -equivalent if there exists a family \mathcal{E} parametrized by an irreducible scheme S and a point $s \in S$ such that for every point $t \in S$ with $t \neq s$ we have that $(\mathcal{F}_t^\mathcal{E}, \Phi_t^\mathcal{E}) \cong (E, \Phi)$ and $(\mathcal{F}_s^\mathcal{E}, \Phi_s^\mathcal{E}) \cong (E', \Phi')$ and we construct S -equivalence as the equivalence relation \sim_S obtained by the previous relation. S -equivalence for families of semistable G -Higgs bundles is defined pointwise.

We denote by \mathcal{A}_G the collection of semistable G -Higgs bundles over X of topological class d . We denote by P_G the algebraic condition defined above for the definition of families of semistable G -Higgs bundles and consider the moduli functor

$$\text{Mod}(\mathcal{A}_G, P_G, S).$$

By [Si3] there exists a moduli space $\mathfrak{M}(G)_d$ of S -equivalence classes of semistable G -Higgs bundle associated to this functor. It follows also from [Si3] that in every S -equivalence class there is always a polystable G -Higgs bundle which is unique up to isomorphism, so the points of $\mathfrak{M}(G)_d$ parametrize S -equivalence classes of semistable G -Higgs bundles or isomorphism classes of polystable G -Higgs bundles.

We denote by \mathcal{A}_G^{st} the subcollection of \mathcal{A}_G of stable G -Higgs bundles. Since stability is an open condition and two S -equivalent stable G -Higgs bundles are always isomorphic we note that for families of stable G -Higgs bundles S -equivalence corresponds to isomorphism pointwise. Then, we can consider the following moduli functor

$$\text{Mod}(\mathcal{A}_G^{st}, P_G, \overset{pt}{\cong}).$$

We denote by $\mathfrak{M}^{st}(G)_d$ the moduli space of isomorphism classes of stable G -bundles of topological class d associated to it.

Let \mathfrak{z} be the centre of \mathfrak{g} . We recall that $\text{aut}(E, \Phi)$ is the set of elements of $H^0(X, E(\mathfrak{g}))$ that commute with Φ . We denote by $\text{aut}^{ss}(E, \Phi)$ the subset of $\text{aut}(E, \Phi)$ given by sections $s \in H^0(X, E(\mathfrak{g}))$ such that for every point $x \in X$ $s(x)$ is semisimple.

Proposition 9.1.2. (Proposition 2.5 of [GGM]) *Let (E, Φ) be a polystable G -Higgs bundle. Then (E, Φ) is stable if and only if $\text{aut}^{ss}(E, \Phi) \subset H^0(X, E(\mathfrak{z}))$. Furthermore, if (E, Φ) is stable, we also have $\text{aut}(E, \Phi) \subset H^0(X, E(\mathfrak{z}))$.*

The previous result allows us to ensure for every polystable G -Higgs bundles the existence of a reduction of structure group to some Levi subgroup $L \subset G$ giving a stable L -Higgs bundle. Such a reduction is called a *Jordan-Hölder reduction* and is unique in a certain sense (see [GGM] e.g.). More precisely, one has the following.

Proposition 9.1.3. (Jordan-Hölder reduction) *Suppose (E, Φ) is a polystable G -Higgs bundle. Then there exists a reductive subgroup $L = Z_G(T)$ where T is a connected abelian subgroup of G , such that (E, Φ) admits a reduction of structure group to L giving a stable L -Higgs bundle (E_L, Φ_L) .*

If furthermore (E, Φ) admits a reduction of structure group to $L' = Z_G(T')$ where T' is another connected abelian subgroup in G , giving a stable L' -Higgs bundle $(E_{L'}, \Phi_{L'})$, then there is a $g \in G$ such that $T' = gTg^{-1}$, $E' = g \cdot E$ and $\Phi' = g\Phi g^{-1}$.

We say that a family $\mathcal{E} \rightarrow X \times T$ of semistable G -Higgs bundles is *locally graded* if for every point $t \in T$ there exists an open subset U containing t , a reductive subgroup L of G of the form $Z_G(T)$ and a family \mathcal{F} of stable L -Higgs bundles parametrized by U such that

$$\mathcal{E}|_{X \times U} \sim_S i_*(\mathcal{F})$$

where i_* is the extension of structure group associated to the injection $i : L \hookrightarrow G$. By Proposition 9.1.3, for every $t' \in U$, the L -Higgs bundle $\mathcal{F}|_{X \times \{t'\}}$ is a Jordan-Hölder reduction of $\mathcal{E}|_{X \times \{t'\}}$.

We say that locally graded families of semistable G -Higgs bundles satisfy the algebraic condition Q_G . With this new algebraic condition for families of G -Higgs bundles one can construct the moduli functor

$$\text{Mod}(\mathcal{A}_G, Q_G, S). \quad (9.1)$$

We denote by $\mathcal{M}(G)_d$ the moduli space of semistable G -Higgs bundles associated to this moduli functor if this moduli space exists.

Let Z_0 be the connected component of the centre of G . The multiplication induces a map

$$\mu : Z_0 \times G \longrightarrow G \quad (9.2)$$

$$(z, g) \longmapsto z \cdot g.$$

Given a Z_0 -Higgs bundle (L, ϕ) and a G -Higgs bundle (E, Φ) , we define

$$(L \otimes E, \Phi + \phi) = (\mu_*(L \times_X E), \Phi + \phi). \quad (9.3)$$

We need to check that this is a G -Higgs bundle. First take a covering $\{U_i\}$ of X that trivializes both L and E . Let $\{\eta_{ij}\}$ be the transition functions of the Z_0 -bundle L and $\{\varphi_{ij}\}$ the transition functions of the G -bundle E . Since the η_{ij} commute with the φ_{ij} we see that $\{\eta_{ij}\varphi_{ij}\}$ satisfy the cocycle condition and then $\mu_*(L \times_X E)$ is a well defined G -bundle. Next, we observe that the adjoint bundle $E(\mathfrak{g})$ is isomorphic to the adjoint bundle $\mu_*(L \times_X E)(\mathfrak{g})$ (recall that the centre of G acts trivially on \mathfrak{g}) and then Φ is a section of the adjoint bundle of $L \otimes E$. Finally, note that $\mathcal{O} \otimes \mathfrak{z}$ is a subbundle of $(L \otimes E)(\mathfrak{g})$ and therefore the sum $\Phi + \phi$ is a well defined Higgs field for $L \otimes E$.

The following is immediate.

Proposition 9.1.4. *Let (E, Φ) be a (semi,poly)stable G -Higgs bundle and let (L, ϕ) be a Z_0 -Higgs bundle. Then the G -Higgs bundle $(L \otimes E, \Phi + \phi)$ is (semi,poly)stable.*

9.2 Stability of G -Higgs bundles in terms of the underlying G -bundle

In this section G denotes a connected complex reductive Lie group.

Proposition 9.2.1. *Let (E, Φ) be a semistable G -Higgs bundle. Then E is a semistable G -bundle.*

Proof. Suppose E is unstable. By Theorem 8.3.4. one has that E reduces to the Harder-Narasimhan parabolic subgroup P giving E_P , and there exists a character $\chi : P \rightarrow \mathbb{C}^*$ such that $\deg \chi^* E_P < 0$. Moreover $H^0(X, E(\mathfrak{g})) = H^0(X, E_P(\mathfrak{p}))$. We thus have that $\Phi \in H^0(X, E_P(\mathfrak{p}))$ and hence the Higgs bundle (E, Φ) is unstable. \square

Proposition 9.2.2. *Let (E, Φ) be a stable G -Higgs bundle, then E is stable.*

Proof. We first note that $\Phi \in H^0(X, E(\mathfrak{g}))$ is contained in $\text{aut}(E, \Phi)$.

If (E, Φ) is stable, by Proposition 9.1.2 we have that $\text{aut}(E, \Phi) \subset H^0(X, E(\mathfrak{z}))$. By Proposition 9.1.4 $(E, 0)$ is stable too and therefore E is a stable G -bundle. \square

Recall that Proposition 9.1.3 asserts that for every polystable G -Higgs bundle (E, Φ) there exists a Jordan-Hölder reduction to a reductive Lie subgroup L , giving a stable L -Higgs bundle (E_L, Φ_L) . A direct consequence of Proposition 9.2.2 is the following.

Corollary 9.2.3. *Let (E, Φ) be a polystable G -Higgs bundle. Then E is a polystable G -bundle.*

Combining the results above with the results of Section 8.3 we are able to describe semistable, polystable and stable G -Higgs bundles. Let G be a connected complex reductive Lie group with maximal compact K . Let H be a Cartan subgroup of G and \mathfrak{h} be the corresponding Cartan subalgebra of \mathfrak{g} . Let $d \in \pi_1(G)$ be determined by $(u, c) \in \pi_1(\overline{Z}) \times \pi_1(\overline{D})$ with the notation of (8.3). Take $\omega_c \in W$ as defined in (8.4).

Proposition 9.2.4. *Every semistable G -Higgs bundle (E, Φ) is determined by a triple (ρ, y, z) where $\rho : \Gamma_{\mathbb{R}} \rightarrow K$ is a central representation, y is a nilpotent element of $\mathfrak{z}_{\mathfrak{t}}(\rho)$ and z is an element of $\mathfrak{z}_{\mathfrak{g}}(\rho)$ such that $[y, z] = 0$.*

Two triples (ρ, y, z) and (ρ', y', z') determine isomorphic semistable G -Higgs bundles if and only if there exists an element $k \in K$ such that $(\rho', y', z') = (k\rho k^{-1}, \text{ad}_k(y), \text{ad}_k(z))$.

Proof. This result follows immediately from Proposition 9.2.1 and Propositions 8.3.7 and 8.3.8. \square

Proposition 9.2.5. *Every polystable G -Higgs bundle (E, Φ) of type $d \in \pi_1(G)$ is determined by a pair (ρ, z) where $\rho : \Gamma_{\mathbb{R}} \rightarrow K$ is a central representation and z is an element of \mathfrak{h}^{ω_c} , the Cartan subalgebra of $\mathfrak{z}_{\mathfrak{g}}(\rho)$.*

Two pairs (ρ, z) and (ρ', z') determine isomorphic polystable G -Higgs bundles if and only if there exists an element $k \in K$ such that $(\rho', z') = (k\rho k^{-1}, \text{ad}_k(z))$.

Proof. The proposition follows from Corollary 4.2.4 and Proposition 9.2.4. By Proposition 8.3.9 $\mathfrak{z}_{\mathfrak{g}}(\rho)$ is a reductive subalgebra with Cartan subalgebra \mathfrak{h}^{ω_c} . Conjugating z by the reductive Lie group $\text{Aut}(E)_0 (= \exp \mathfrak{z}_{\mathfrak{g}}(\rho))$ we can always send it to the Cartan subalgebra without changing ρ . \square

Proposition 9.2.6. *A polystable G -Higgs bundle (E, Φ) determined by a pair (ρ, z) is stable if and only if $\mathfrak{z}_{\mathfrak{g}}(\rho) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$.*

Two pairs (ρ, z) and (ρ', z') determine isomorphic stable G -Higgs bundles if and only if there exists an element $k \in K$ such that $(\rho', z') = (k\rho k^{-1}, z)$.

Proof. By Corollary 9.2.3 E is polystable. By Proposition 9.1.2, E is stable if and only if $\text{aut}^{ss}(E) \subset H^0(X, E(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})))$, equivalently, if and only if $\mathfrak{z}_{\mathfrak{g}}(\rho) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$. Recall that Proposition 9.2.2 says that (E, Φ) is stable if and only if E is stable.

The second statement follows from Proposition 9.2.5. We have that z lies in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ and then, it is invariant under ad_k . \square

9.3 Stable G -Higgs bundles

We start studying stable G -Higgs bundles when the structure group G is a connected abelian complex Lie group.

Remark 9.3.1. For any abelian complex reductive Lie group G we have that every G -Higgs bundle is stable and therefore

$$\mathfrak{M}(G) \cong \mathfrak{M}^{st}(G).$$

For any G -bundle E we have, $E(\mathfrak{g}) \cong \mathcal{O} \otimes \mathfrak{g}$ and every isomorphism between G -Higgs bundles induces the identity in $H^0(X, \mathcal{O} \otimes \mathfrak{g})$, so

$$\mathfrak{M}(G) \cong H^1(X, G) \times H^0(X, \mathcal{O} \otimes \mathfrak{g}).$$

The universal cover of a connected abelian group G is its Lie algebra \mathfrak{g} and the covering map is the exponential $\exp : \mathfrak{g} \rightarrow G$. Hence the fundamental group $\pi_1(G)$ is identified with the kernel of the exponential map which is a lattice in \mathfrak{g}

$$\Lambda_G = \ker \exp_G \subset \mathfrak{g}. \tag{9.4}$$

Every element $\gamma \in \Lambda_G$ defines a cocharacter $\chi_\gamma : \mathbb{C}^* \rightarrow G$ in the following way. For any $\lambda \in \mathbb{C}^*$ we take $b \in \mathbb{C}$ such that $\lambda = \exp(2\pi i(b + \mathbb{Z}))$ and we set $\chi_\gamma(\lambda) = \exp(\gamma(b + \mathbb{Z}))$. Similarly, given a cocharacter $\chi : \mathbb{C}^* \rightarrow G$ we obtain $d\chi : \mathbb{C} \rightarrow \mathfrak{g}$ and we observe that $\gamma_\chi = d\chi(2\pi i)$ is an element of Λ_G . Furthermore, for every $\gamma_1, \gamma_2 \in \Lambda_G$ one can easily check that $\chi_{\gamma_1} \cdot \chi_{\gamma_2}$ is equal to $\chi_{\gamma_1 + \gamma_2}$.

Recall that, given two G -bundles L_1 and L_2 , we have implicitly defined $L_1 \otimes L_2$ in (9.3). Let $\mathcal{B}(\Lambda_G) = \{\gamma_1, \dots, \gamma_\ell\}$ be a basis of Λ_G . The identification between elements of the lattice $\Lambda_G \subset \mathfrak{g}$ and cocharacters $\chi : \mathbb{C}^* \rightarrow G$ gives the following isomorphism

$$\theta : \mathbb{C}^* \otimes_{\mathbb{Z}} \Lambda_G \xrightarrow{\cong} G$$

$$\sum_{\gamma_i \in \mathcal{B}(\Lambda_G)} \lambda_i \otimes_{\mathbb{Z}} \gamma_i \longmapsto \chi_{\gamma_1}(\lambda_1) \otimes \cdots \otimes \chi_{\gamma_\ell}(\lambda_\ell),$$

and

$$d\theta : \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_G \xrightarrow{\cong} \mathfrak{g}$$

$$\sum_{\gamma_i \in \mathcal{B}(\Lambda_G)} (b_i \otimes_{\mathbb{Z}} \gamma_i) \longmapsto \sum_{\gamma_i \in \mathcal{B}(\Lambda_G)} \lambda_i \cdot \gamma_i.$$

Proposition 9.3.2. *Let G be a connected complex reductive abelian group, then we have an isomorphism*

$$\xi_{G,0}^{x_0} : \mathfrak{M}(G)_0 \xrightarrow{\cong} T^*X \otimes_{\mathbb{Z}} \Lambda_G$$

$$[(L, \phi)]_S \longmapsto \sum_{\gamma_i \in \mathcal{B}(\Lambda_G)} (x_i, \lambda_i) \otimes_{\mathbb{Z}} \gamma_i$$

where $\sum_{\gamma_i \in \mathcal{B}(\Lambda_G)} (x_i, \lambda_i) \otimes_{\mathbb{Z}} \gamma_i$ is such that

$$(L, \phi) \cong \left(\bigotimes_{\gamma_i \in \mathcal{B}(\Lambda_G)} (\chi_{\gamma_i})_* L'_i, \sum_{\gamma_i \in \mathcal{B}(\Lambda_G)} d\chi_{\gamma_i} \phi'_i \right)$$

with $(L'_i, \phi'_i) = \xi_{1,0}^{x_0}(x_i, \lambda_i)$. For every $d \in \pi_1(G)$, we have

$$\mathfrak{M}(G)_d \cong T^*X \otimes_{\mathbb{Z}} \Lambda_G.$$

Proof. Since G is abelian, we have that

$$\mathfrak{M}(G)_0 \cong H^1(X, G)_0 \times H^0(X, \mathcal{O} \otimes \mathfrak{g}).$$

Since G is connected, the extension of structure group associated to θ gives an isomorphism

$$\theta_* : \text{Pic}^0(X) \otimes_{\mathbb{Z}} \Lambda_G \xrightarrow{\cong} H^1(X, G)_0$$

and $d\theta : \mathfrak{g} \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_G$ induces

$$(d\theta)_* : H^0(X, \mathcal{O}) \otimes_{\mathbb{Z}} \Lambda_G \xrightarrow{\cong} H^0(X, \mathcal{O} \otimes \mathfrak{g}).$$

Since $(\text{Pic}^0(X) \otimes_{\mathbb{Z}} \Lambda_G) \times (H^0(X, \mathcal{O}) \otimes_{\mathbb{Z}} \Lambda_G)$ is equal to $(\text{Pic}^0(X) \times H^0(X, \mathcal{O})) \otimes_{\mathbb{Z}} \Lambda_G$, we see that using the pair $(\theta_*, (d\theta)_*)$ we can construct

$$\Theta_G : ((\text{Pic}^0(X) \times H^0(X, \mathcal{O})) \otimes_{\mathbb{Z}} \Lambda_G) \xrightarrow{\cong} \mathfrak{M}(G)_0.$$

From (3.4) we have $\xi_{1,0}^{x_0} : \text{Pic}^0(X) \times H^0(X, \mathcal{O}) \xrightarrow{\cong} T^*X$ and then we can construct the isomorphism $\mu_{x_0, G} : (\text{Pic}^0(X) \times H^0(X, \mathcal{O})) \otimes_{\mathbb{Z}} \Lambda_G \xrightarrow{\cong} T^*X \otimes_{\mathbb{Z}} \Lambda_G$. The composition $\xi_{G,0}^{x_0} = \mu_{x_0, G} \circ \Theta_G^{-1}$ is an isomorphism as well.

Finally, since G is abelian and complex reductive, for every $d \in \pi_1(G)$ we have $\mathfrak{M}(G)_d \cong \mathfrak{M}(G)_0$. \square

Next, we study stable G -Higgs bundles when G is a connected complex semisimple Lie group and let K be the maximal compact subgroup of G .

Let (E, Φ) be a stable G -Higgs bundle associated to the pair (ρ, z) ; since G is semisimple ρ reduces to a representation of the fundamental group $\pi_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. A representation $\rho : \pi_1(X) \rightarrow K$ is completely determined by two elements $a, b \in K$ such that $[a, b] = \text{id}$. If we lift a and b to the universal cover of K , we obtain $\tilde{a}, \tilde{b} \in \tilde{K}$ such that $[\tilde{a}, \tilde{b}] = c$, where c is an element of the centre of \tilde{K} .

By Proposition 9.2.6 $\mathfrak{z}_{\mathfrak{g}}(\rho) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$. Since G is semisimple we have $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) = 0$, then $\mathfrak{z}_{\mathfrak{t}}(\rho) = \mathfrak{z}_{\mathfrak{t}}(a, b) = 0$ and therefore $\mathfrak{z}_{\mathfrak{t}}(\tilde{a}, \tilde{b}) = 0$.

The pair (\tilde{a}, \tilde{b}) of elements of \tilde{K} with commutator $c \in Z_{\tilde{K}}(\tilde{K})$ such that $\mathfrak{z}_{\mathfrak{t}}(\tilde{a}, \tilde{b}) = 0$ is a rank zero c -pair. As a consequence of Proposition 8.2.6 we have the following.

Proposition 9.3.3. *Let G be a connected complex semisimple Lie group. There are no stable G -Higgs bundles of topological type $d \in \pi_1(G)$ over the elliptic curve X , unless the quotient of the group by the connected component of its centre is*

$$G = \mathrm{PGL}(n_1, \mathbb{C}) \times \cdots \times \mathrm{PGL}(n_r, \mathbb{C})$$

and $d = (\tilde{d}_1, \dots, \tilde{d}_r)$, where $\tilde{d}_i = d_i \pmod{n_i}$ is such that $\mathrm{gcd}(d_i, n_i) = 1$. In that case

$$\mathfrak{M}_d^{\mathrm{st}}(G) = \{pt\}.$$

Proof. We have seen that the existence of a stable G -bundle implies the existence of a rank zero c -pair on \tilde{K} . By Proposition 8.2.6 this is only possible if \tilde{K} is a product of $\mathrm{SU}(n_i)$ with $n_i \geq 2$ and c is a product of c_i where c_i generates the centre of $\mathrm{SU}(n_i)$.

If \tilde{K} is a product of $\mathrm{SU}(n_i)$ then \tilde{G} is a product of $\mathrm{SL}(n_i, \mathbb{C})$ and then G is a product of $\mathrm{SL}(n_i, \mathbb{C})/\mathbb{Z}_{m_i}$ where \mathbb{Z}_{m_i} is a subgroup of \mathbb{Z}_{n_i} , the centre of $\mathrm{SL}(n_i, \mathbb{C})$ (m_i divides n_i).

A representation $\rho_i : \pi_1(X) \rightarrow \mathrm{SU}(n_i)/\mathbb{Z}_{m_i}$ is determined by a pair (a_i, b_i) of commuting elements of $\mathrm{SU}(n_i)/\mathbb{Z}_{m_i}$. This elements lift to $\mathrm{SU}(n_i)$ giving \tilde{a}_i and \tilde{b}_i such that $[\tilde{a}_i, \tilde{b}_i] = c_i \in \mathbb{Z}_{m_i}$ contained in \mathbb{Z}_{n_i} , the centre of $\mathrm{SU}(n_i)$. This is a rank zero c_i -pair only if c_i generates all \mathbb{Z}_{n_i} , i.e. only if $m_i = n_i$. This implies that G is a product of $\mathrm{PGL}(n_i, \mathbb{C})$.

A $\mathrm{PGL}(n_i, \mathbb{C})$ -bundle of topological type $\tilde{d}_i \in \pi_1(\mathrm{PGL}(n_i, \mathbb{C})) = \mathbb{Z}_{n_i}$ can be lifted to a principal bundle with structure group $\mathrm{SL}(n_i, \mathbb{C})/\langle \tilde{d}_i \rangle$. If the bundle comes from a representation associated to a c_i -pair with c_i generating the centre of $\mathrm{SU}(n_i)$, the bundle can not be lifted to any group different from $\mathrm{PGL}(n_i, \mathbb{C})$ and therefore it must have topological type \tilde{d}_i generating all $\pi_1(G)$. Such \tilde{d}_i has the form $\tilde{d}_i = d_i \pmod{n_i}$ with d_i and n_i coprime.

The last statement follows from Theorem 4.5.1. \square

Now we take G to be a connected complex reductive Lie group. Proposition 9.2.2 gives a restriction for the set of connected complex Lie groups such that stable G -Higgs bundles exists.

Corollary 9.3.4. *Let G be a connected complex reductive Lie group. There are no stable G -Higgs bundles over the elliptic curve X unless*

$$G/Z_0 \cong \mathrm{PGL}(n_1, \mathbb{C}) \times \cdots \times \mathrm{PGL}(n_r, \mathbb{C}),$$

and E/Z_0 has topological invariant $d = (\tilde{d}_1, \dots, \tilde{d}_r)$, where $\tilde{d}_i = d_i \pmod{n_i}$ is such that $\mathrm{gcd}(d_i, n_i) = 1$.

Proof. Denote by Z_0 the connected component of the centre of G . Consider $\gamma : G \rightarrow G/Z_0$. We obviously have that $\ker \gamma$ is contained in the centre of G and then by Proposition 8.3.3 and Propositions 9.2.2 and 9.3.3 we have the result. \square

Let us study the connected groups that satisfy $G/Z_0 = \mathrm{PGL}(n, \mathbb{C})$. Assuming that Z_0 is not trivial, then G is isomorphic to

$$G \cong Z_0 \times_{\tau} \mathrm{SL}(n, \mathbb{C})$$

where $\tau : Z_{\mathrm{SL}(n, \mathbb{C})}(\mathrm{SL}(n, \mathbb{C})) \cong \mathbb{Z}_n \rightarrow Z_0$ is a homomorphism. We can always choose an isomorphism $Z_0 \cong (\mathbb{C}^*)^s$ such that the image of τ is entirely contained in the first \mathbb{C}^* and then

$$G \cong (\mathbb{C}^*)^{s-1} \times (\mathbb{C}^* \times_{\tau} \mathrm{SL}(n, \mathbb{C})).$$

The groups $G = \mathbb{C}^* \times_{\tau} \mathrm{SL}(n, \mathbb{C})$ are determined by the representation $\tau : \mathbb{Z}_n \rightarrow \mathbb{C}^*$. The group \mathbb{Z}_n is cyclic and then its image under τ is cyclic himself, i.e. it is isomorphic to \mathbb{Z}_m with m dividing n . The representation τ is determined (up to automorphisms of \mathbb{Z}_n) by the value of m , so

$$G \cong \mathbb{C}^* \times_{\mathbb{Z}_m} \mathrm{SL}(n, \mathbb{C}).$$

Finally we have $G = \mathrm{GL}(n, \mathbb{C})/\mathbb{Z}_k$, with $k = n/m$, since

$$\begin{aligned} \mathrm{GL}(n, \mathbb{C})/\mathbb{Z}_k &\cong (\mathbb{C}^* \times_{\mathbb{Z}_n} \mathrm{SL}(n, \mathbb{C})) / \mathbb{Z}_k \\ &\cong (\mathbb{C}^*/\mathbb{Z}_k) \times_{\mathbb{Z}_{(n/k)}} \mathrm{SL}(n, \mathbb{C}) \\ &\cong \mathbb{C}^* \times_{\mathbb{Z}_{(n/k)}} \mathrm{SL}(n, \mathbb{C}). \end{aligned}$$

The fundamental group of $\mathrm{GL}(n, \mathbb{C})/\mathbb{Z}_k$ is \mathbb{Z} and the projection from $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{GL}(n, \mathbb{C})/\mathbb{Z}_k$ and further to $\mathrm{GL}(n, \mathbb{C})/\mathbb{C}^* = \mathrm{PGL}(n, \mathbb{C})$ induces a morphism between the fundamental groups

$$\pi_1(\mathrm{GL}(n, \mathbb{C})) \longrightarrow \pi_1(\mathrm{GL}(n, \mathbb{C})/\mathbb{Z}_k) \longrightarrow \pi_1(\mathrm{GL}(n, \mathbb{C})/\mathbb{C}^*) \quad (9.5)$$

$$d \longmapsto d \longmapsto d \pmod{n}$$

We can summarize all this in the following corollary.

Corollary 9.3.5. *Let G be a connected complex reductive Lie group. There are no stable G -Higgs bundles with topological invariant d over the elliptic curve X unless*

$$\begin{aligned} G &= (\mathbb{C}^*)^s \times \mathrm{PGL}(n'_1, \mathbb{C}) \times \cdots \times \mathrm{PGL}(n'_t, \mathbb{C}) \times \cdots \\ &\quad \times \mathrm{GL}(n''_1, \mathbb{C})/\mathbb{Z}_{k_1} \times \cdots \times \mathrm{GL}(n''_r, \mathbb{C})/\mathbb{Z}_{k_r} \end{aligned} \quad (9.6)$$

and $d \in \pi_1(G)$ has the form

$$d = \left(d_1, \dots, d_s, \tilde{d}_1, \dots, \tilde{d}_t, d''_1, \dots, d''_r \right), \quad (9.7)$$

where $\mathrm{gcd}(d''_j, n''_j) = 1$ and $\tilde{d}_i = d'_i \pmod{n'_i}$ with $\mathrm{gcd}(d'_i, n'_i) = 1$.

We have already studied stable Higgs bundles and stable $\mathrm{PGL}(n, \mathbb{C})$ -Higgs bundles. We complete the study of stable G -Higgs bundles over an elliptic curve with the case of stable $(\mathrm{GL}(n, \mathbb{C})/\mathbb{Z}_k)$ -Higgs bundles. Recall that the centre of $\mathrm{SL}(n, \mathbb{C})/\mathbb{Z}_k$ is \mathbb{Z}_m where $m = n/k$.

Proposition 9.3.6. *Let (E, Φ) be a stable $(\mathrm{GL}(n, \mathbb{C})/\mathbb{Z}_k)$ -Higgs bundle of degree d and let J be any element of $H^1(X, \mathbb{Z}_m)$. Then*

$$(E, \Phi) \cong (J \otimes E, \Phi).$$

Proof. By Proposition 9.2.6, the stable G -Higgs bundle (E, Φ) is determined by a pair (ρ, z) , where $\rho : \Gamma_{\mathbb{R}} \rightarrow U(n)/\mathbb{Z}_k$ and z is an element of \mathbb{C} understood as the centre of $\mathfrak{gl}(n, \mathbb{C})$.

By Remark 8.2.1 the representation ρ is determined by a triple (a, b, u) , where a and b are elements of $U(n)/\mathbb{Z}_k$ such that $c = [a, b]$ is contained in the centre of $SU(n)/\mathbb{Z}_k$, which is isomorphic to \mathbb{Z}_m . We have that c is the image $\tilde{d} = d \pmod{n}$ under the projection $\mathbb{Z}_n \rightarrow \mathbb{Z}_m$. Since E is stable we have that $\gcd(n, d) = 1$, so \tilde{d} generates all \mathbb{Z}_n and therefore c generates all \mathbb{Z}_m .

Conjugating by a we obtain $\rho' = a\rho a^{-1}$ which is determined by the triple (a, cb, u) , so $\rho' = \rho_a \cdot \rho$ where $\rho_a : \pi_1(X) \rightarrow F$ is given by (id, c) . We denote by J_a the F -bundle associated to ρ_a ; by Proposition 9.2.6 we have that $(E, \Phi) \cong (J_a \otimes E, \Phi)$.

Similarly, conjugating by b^{-1} we obtain $\rho'' = b^{-1}\rho b$ which is determined by the triple (ca, b, u) , so $\rho'' = \rho_b \cdot \rho$ where $\rho_b : \pi_1(X) \rightarrow F$ is given by (c, id) . We call J_b the F -bundle associated to ρ_b and we have $(E, \Phi) \cong (J_b \otimes E, \Phi)$.

Finally, we note that J_a and J_b generate all $H^1(X, \mathbb{Z}_m)$, so for every F -bundle J we obtain $(E, \Phi) \cong (J \otimes E, \Phi)$. \square

Recall from (8.2) that a complex reductive Lie group G is isomorphic to $Z_0 \times_{\tau} D$ where Z_0 is the connected component of the centre, D is the universal cover of $[G, G]$, F is the intersection $Z_0 \cap [G, G]$, $C \subset Z_D(D)$ is the preimage of F and $\tau : C \rightarrow Z_0$ is induced by F . One can easily generalize Proposition 9.3.6.

Corollary 9.3.7. *Let (E, Φ) be a stable G -Higgs bundle of topological class d and let J be any element of $H^1(X, F)$. Then*

$$(E, \Phi) \cong (J \otimes E, \Phi).$$

After (8.2) we have set $\overline{G} = G/F$, $\overline{Z} = Z_0/F$ and $\overline{D} = D/C$. The morphism

$$q : G \rightarrow \overline{G} \cong \overline{Z} \times \overline{D}$$

induces an extension of structure group that sends the G -bundle E to $q_*E = (q_{*,1}E, q_{*,2}E)$ where $q_{*,1}E$ is a \overline{Z} -bundle and $q_{*,2}E$ a \overline{D} -bundle. Let $q^{\pi} : \pi_1(G) \rightarrow \pi_1(\overline{G})$ be the morphism of fundamental groups induced by q .

By (9.3) one has an action of the discrete set of F -bundles, $H^1(X, F)$, on $\mathfrak{M}^{st}(G)_d$.

Proposition 9.3.8. *Let $d \in \pi_1(G)$ and denote by \overline{d} the induced element in $\pi_1(\overline{G})$. The extension of structure group associated to q induces an isomorphism*

$$q_* : \mathfrak{M}^{st}(G)_d \xrightarrow{\cong} \mathfrak{M}^{st}(\overline{G})_{\overline{d}}$$

$$[(E, \Phi)]_{\cong} \longmapsto [(q_*E, \Phi)]_{\cong}.$$

Proof. By Propositions 8.3.3 and 9.2.2 the morphism is well defined. By Proposition 8.3.1 the Higgs fields are contained in $H^0(X, \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) \otimes \mathcal{O}) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g})$ and we observe that q_* is bijective if and only if the extension of structure group q_* is bijective.

Take the short exact sequence $0 \rightarrow F \rightarrow G \rightarrow \overline{G} \rightarrow 0$ and consider the associated short exact sequence of sheaves $0 \rightarrow \underline{F} \rightarrow \underline{G} \rightarrow \underline{\overline{G}} \rightarrow 0$. By (5.7.11) of [Gr], since F is contained in the centre of G , this induces the long exact sequence

$$\dots \rightarrow H^1(X, \underline{F}) \rightarrow H^1(X, \underline{G}) \rightarrow H^1(X, \underline{\overline{G}}) \xrightarrow{\delta} H^2(X, \underline{F}).$$

Since \overline{d} is the image in $\pi_1(\overline{G})$ of $d \in \pi_1(G)$ we have that $H^1(X, \underline{\overline{G}})_{\overline{d}}$ maps to the identity under δ , therefore, we have that

$$H^1(X, \underline{G})_d \xrightarrow{q_*} H^1(X, \underline{\overline{G}})_{\overline{d}}$$

is surjective. By Corollary 9.3.7, the action of $H^1(X, \underline{F})$ on $H^1(X, \underline{G})_d$ is trivial and then, q_* is bijective.

We have that q_* is bijective. Since $\overline{G} \cong \overline{Z} \times \overline{D}$, by Proposition 9.3.2 and Proposition 9.3.3 $\mathfrak{M}^{st}(\overline{G})$ is smooth and therefore normal. By Zariski's Main Theorem q_* is an isomorphism. \square

Theorem 9.3.9. *Let G be a connected complex reductive Lie group as above and let d be an element of $\pi_1(G)$. Take \overline{Z} as above and $\Lambda_{\overline{Z}}$ as in (9.4). We have that*

$$\mathfrak{M}^{st}(G) = \emptyset,$$

unless G and d are as in (9.6) and (9.7), in which case we have an isomorphism

$$\xi_{G,d}^{x_0} : \mathfrak{M}^{st}(G) \xrightarrow{\cong} T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}}$$

$$[(E, \Phi)]_{\cong} \mapsto \xi_{\overline{Z}, q_1^{\pi}(d)}^{x_0}([(q_{*,1}E, \Phi)]_{\cong})$$

Proof. First of all, by Corollary 9.3.5, we have that the moduli space of stable G -bundles with topological class d is empty unless G and d are as in (9.6) and (9.7).

By Proposition 9.3.8 one has the following isomorphism

$$q_* : \mathfrak{M}^{st}(G)_d \xrightarrow{\cong} \mathfrak{M}^{st}(\overline{Z})_{q_1^{\pi}(d)} \times \mathfrak{M}^{st}(\overline{D})_{q_2^{\pi}(d)}$$

$$[(E, \Phi)]_{\cong} \mapsto ([(q_{*,1}E, \Phi)]_{\cong}, [(q_{*,2}E, 0)]_{\cong}).$$

Finally, from Proposition 9.3.2 and Proposition 9.3.3 we obtain that

$$\mathfrak{M}^{st}(G)_d \cong T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}} \times \{pt\}.$$

\square

We denote by $E_{(n,d)}^{x_0}$ the underlying vector bundle of the Higgs bundle that maps to $(x_0, 0)$ under $\xi_{n,d}^{x_0}$ defined in (4.9) and call the induced $\mathrm{GL}(n, \mathbb{C})/\mathbb{Z}_k$ -bundle $E_{(n,d)/k}^{x_0}$. Note that

$$\det E_{(n,d)}^{x_0} \cong \mathcal{O}(x_0)^{\otimes d}$$

and then

$$\det E_{(n,d)/k}^{x_0} \cong \mathcal{O}(x_0)^{\otimes d}.$$

We denote by $PE_{n,\tilde{d}}$ the unique (up to isomorphism) stable $\mathrm{PGL}(n, \mathbb{C})$ -bundle of degree \tilde{d} .

Take a connected complex reductive Lie group of the form (9.6) then we can define the following G -bundle

$$\begin{aligned} E_{G,d}^{x_0} = & \mathcal{O}(x_0)^{\otimes d_1} \oplus \cdots \oplus \mathcal{O}(x_0)^{\otimes d_s} \oplus PE_{n'_1, \tilde{d}_1} \oplus \cdots \oplus PE_{n'_t, \tilde{d}_t} \oplus \\ & \oplus E_{(n'_1, d'_1)/k_1}^{x_0} \oplus \cdots \oplus E_{(n'_r, d'_r)/k_r}^{x_0}. \end{aligned} \quad (9.8)$$

Let us consider the universal family of Higgs bundles of rank 1 and degree 0

$$\mathcal{E}_{(1,0)}^{x_0} = (\mathcal{F}_{(1,0)}^{x_0}, \Phi_{(1,0)}^{x_0}) \rightarrow X \times T^*X.$$

Let Z_0 be a connected complex reductive abelian Lie group and take $\Lambda_{Z_0} \subset \mathfrak{z}$ as defined in (9.4). Let $\mathcal{B}(\Lambda_{Z_0}) = \{\gamma_1, \dots, \gamma_\ell\}$ be a basis of the lattice Λ_{Z_0} . Let $\chi_\gamma : \mathbb{C}^* \rightarrow Z_0$ be the cocharacter constructed with $\gamma \in \Lambda_{Z_0}$ and $d\chi_\gamma : \mathbb{C} \rightarrow \mathfrak{z}$ its differential. We define $\mathcal{E}_{Z_0,0}^{x_0} = (\mathcal{F}_{Z_0,0}^{x_0}, \Phi_{Z_0,0}^{x_0})$ as the family of Z_0 -Higgs bundles parametrized by $X \times T^*X \otimes_{\mathbb{Z}} \Lambda_{Z_0}$ constructed as follows; take $t \in T^*X \otimes_{\mathbb{Z}} \Lambda_{Z_0}$ of the form $t = \sum_{\gamma_i \in \mathcal{B}(\Lambda_{Z_0})} (x_i, \lambda_i) \otimes_{\mathbb{Z}} \gamma_i$

$$\mathcal{E}_{Z_0,0}^{x_0}|_{X \times \{t\}} = \left(\bigoplus_{\gamma_i \in \mathcal{B}(\Lambda_{Z_0})} (\chi_{\gamma_i})_* \mathcal{F}_{(1,0)}^{x_0}|_{X \times \{x_i\}}, \sum_{\gamma_i \in \mathcal{B}(\Lambda_{Z_0})} (d\chi_{\gamma_i})_* \Phi_{(1,0)}^{x_0}|_{X \times \{\lambda_i\}} \right),$$

where $(\chi_{\gamma_i})_*$ is the extension of structure group associated to χ_{γ_i} and $(d\chi_{\gamma_i})_* : H^0(X, \mathcal{O}) \rightarrow H^0(X, \mathcal{O} \otimes \mathfrak{z})$ is induced by $d\chi_{\gamma_i}$.

Now, take G and d as in (9.6) and (9.7). The multiplication map μ defined in (9.2) and $i_* : H^0(X, \mathcal{O} \otimes \mathfrak{z}) \rightarrow H^0(X, \mathcal{O} \otimes \mathfrak{g})$ induced by $i : \mathfrak{z} \rightarrow \mathfrak{g}$ allow us to define

$$\mathcal{V}_{G,d}^{x_0} = (\mu_*(\mathcal{F}_{Z_0,0}^{x_0} \times_X E_{G,d}^{x_0}), i_*(\Phi_{Z_0,0}^{x_0})).$$

where $\mathcal{V}_{G,d}^{x_0}$ is a family of G -Higgs bundles parametrized by $T^*X \otimes_{\mathbb{Z}} \Lambda_{Z_0}$.

We denote by $\mathcal{J}_{Z_0,F}$ the image of $H^1(X, F) \times \{0\}$ under $\xi_{Z_0,0}^{x_0}$ and write $t_J \in \mathcal{J}_{Z_0,F}$ for the image of $J \in H^1(X, F)$. Let $t \in T^*X \otimes_{\mathbb{Z}} \Lambda_{Z_0}$ and let (L, ϕ) be a Z_0 -Higgs bundle such that $t = \xi_{Z_0,0}^{x_0}(L, \phi)$. Since $T^*X \otimes_{\mathbb{Z}} \Lambda_{Z_0}$ is a group we have that $\mathcal{J}_{Z_0,F}$ (and therefore $H^1(X, F)$) acts on $T^*X \otimes_{\mathbb{Z}} \Lambda_{Z_0}$ in the following way

$$\xi_{Z_0,0}^{x_0}(t_J + t) = (J \otimes L, \phi).$$

Using the notation defined in (9.3), after Corollary 9.3.7 we have that

$$\mathcal{V}_{G,d}^{x_0}|_{X \times \{t\}} \cong (L \otimes E_{G,d}^{x_0}, \phi) \cong (J \otimes L \otimes E_{G,d}^{x_0}, \phi) \cong \mathcal{V}_{G,d}^{x_0}|_{X \times \{t_J + t\}}$$

and then $\mathcal{V}_{G,d}^{x_0}$ defines a family parametrized by $(T^*X \otimes_{\mathbb{Z}} \Lambda_{Z_0})/\mathcal{J}_{Z_0,F}$.

Let us study the quotient $(T^*X \otimes_{\mathbb{Z}} \Lambda_{Z_0})/\mathcal{J}_{Z_0,F}$. Write $Z_0/F = \overline{Z}$, and consider $\Lambda_{\overline{Z}} \subset \mathfrak{z}$ as the lattice obtained from $\pi_1(\overline{Z})$. We have $\Lambda_{Z_0} \subset \Lambda_{\overline{Z}}$ and therefore we have a surjective map

$$u : T^*X \otimes_{\mathbb{Z}} \Lambda_{Z_0} \longrightarrow T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}}$$

$$\sum_{\gamma_i \in \mathcal{B}(\Lambda_{Z_0})} (x_i, \lambda_i) \otimes_{\mathbb{Z}} \gamma_i \longmapsto \sum_{\gamma_i \in \mathcal{B}(\Lambda_{Z_0})} (x_i, \lambda_i) \otimes_{\mathbb{Z}} \gamma_i,$$

where we consider $\gamma_i \in \Lambda_{Z_0}$ in the left and $\gamma_i \in \Lambda_{\overline{Z}}$ in the right. If $\{\gamma_1, \dots, \gamma_\ell\}$ is a set of generators of Λ_{Z_0} , we can choose a basis $\mathcal{B}(\Lambda_{\overline{Z}}) = \{\delta_1, \dots, \delta_\ell\}$ of $\Lambda_{\overline{Z}}$ such that $\gamma_i = m_i \delta_i$ with $m_i \in \mathbb{Z}^+$. The kernel of u is generated by $\{y_i \otimes_{\mathbb{Z}} \gamma_i : i = 1, \dots, \ell\}$ where y_i is a generator of the torsion group $X[m_i]$. On the other hand we see that the elements of $H^1(X, F)$ are generated by $(\chi_{\gamma_i})_* J_i$ where J_i is a generator of $\text{Pic}^0(X)[m_i]$. This shows that $\mathcal{J}_{Z_0,F}$ is generated by $X[m_i] \otimes_{\mathbb{Z}} \gamma_i$ and therefore $\mathcal{J}_{Z_0,F}$ is equal to $\ker u$. Then u induces an isomorphism

$$\overline{u} : (T^*X \otimes_{\mathbb{Z}} \Lambda_{Z_0})/\mathcal{J}_{Z_0,F} \xrightarrow{\cong} T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}}.$$

Taking the pull-back by $\text{id}_X \times (\overline{u})^{-1}$ of the family induced by the quotient of $\mathcal{V}_{G,d}^{x_0}$ by $H^1(X, F)$ we can define

$$\mathcal{E}_{G,d}^{x_0} \longrightarrow X \times (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}}). \quad (9.9)$$

For the moment, the family $\mathcal{E}_{G,d}^{x_0}$ is defined only for G and d of the form given in (9.6) and (9.7).

Remark 9.3.10. By construction, $\mathcal{E}_{G,d}^{x_0}$ parametrizes stable G -Higgs bundles of topological class d . One can see that the map $T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{Z}} \rightarrow \mathfrak{M}^{st}(G)_d$ induced by the family $\mathcal{E}_{G,d}^{x_0}$ is equal to $(\xi_{G,d}^{x_0})^{-1}$ which is an isomorphism. Then $\mathcal{E}_{G,d}^{x_0}$ is a universal family.

9.4 Moduli spaces of G -Higgs bundles

Let G be a connected complex reductive Lie group. Let d be an element of $\pi_1(G)$. Suppose that d is given by $(u, c) \in \pi_1(\overline{Z}) \times \pi_1(\overline{D})$ with the notation of (8.3) and take ω_c to be the element of the Weyl group defined in (8.4).

We recall the definition of S_c and L_c given in (8.5) and (8.7). The following result describes the Jordan-Hölder reduction of a polystable Higgs bundle.

Proposition 9.4.1. *Every polystable G -Higgs bundle of topological type d admits a reduction of structure group to L_c giving a stable L_c -Higgs bundle of topological class d .*

Proof. Let H be the Cartan subgroup of G and \mathfrak{h} its Lie algebra. Suppose (E, Φ) is a polystable G -Higgs bundle of type d associated to the pair (ρ, z) . We have that $\text{im } \rho$ is contained in $Z_G(\mathfrak{h}^{\omega_c})$ since \mathfrak{h}^{ω_c} is contained in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}^{\omega_c})$. Also, since $z \in \mathfrak{h}^{\omega_c}$, it is straightforward that $z \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}^{\omega_c})$. Then the polystable G -Higgs bundle (E, Φ) reduces to a polystable L_c -Higgs bundle (E_L, Φ_L) . This L_c -Higgs bundle is associated to the pair (ρ_L, z_L) , where ρ_L is the restriction of ρ to L_c and z_L is simply z .

By Proposition 9.1.2 (E_L, Φ_L) is stable if $\text{aut}(E, \Phi)$ is contained in $H^0(X, E_L(\mathfrak{h}^{\omega_c}))$ where \mathfrak{h}^{ω_c} is the centre of \mathfrak{l}_c . We have that $\text{aut}(E, \Phi)$ is $\mathfrak{z}_{\mathfrak{l}_c}(\rho, z)$ by Proposition 9.2.5 and since z lies in the centre of \mathfrak{l}_c , $\text{aut}(E, \Phi)$ is isomorphic to $\mathfrak{z}_{\mathfrak{l}_c}(\rho)$. Then, (E_L, Φ_L) is stable if $\mathfrak{z}_{\mathfrak{l}_c}(\rho)$ is contained in \mathfrak{h}^{ω_c} .

We note that $\mathfrak{z}_{\mathfrak{l}_c}(\rho)$ is $\mathfrak{z}_{\mathfrak{g}}(\rho) \cap \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}^{\omega_c})$. Since \mathfrak{h}^{ω_c} is a maximal abelian subalgebra of $\mathfrak{z}_{\mathfrak{g}}(\rho)$, we have that $\mathfrak{z}_{\mathfrak{g}}(\rho) \cap \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}^{\omega_c})$ is \mathfrak{h}^{ω_c} itself.

Finally, the topological class of (E_L, Φ_L) is, by Lemma 8.1.7, the unique preimage of d under the homomorphism $\pi_1(L_c) \rightarrow \pi_1(G)$ given by the inclusion $L_c \hookrightarrow G$. For simplicity, we denote this preimage also by d . \square

Remark 9.4.2. The existence of stable L_c -Higgs bundles and Corollary 9.3.5 imply that

$$L_c \cong (\mathbb{C}^*)^{\times s} \times \text{PGL}(n'_1, \mathbb{C}) \times \cdots \times \text{PGL}(n'_t, \mathbb{C}) \times \\ \times \text{GL}(n''_1, \mathbb{C})/\mathbb{Z}_{k_1} \times \cdots \times \text{GL}(n''_r, \mathbb{C})/\mathbb{Z}_{k_r}.$$

Recall the definition of \overline{S}_c given in (8.10). We set $\Lambda_{\overline{S}_c}$ to be the lattice in $\text{Lie } \overline{S}_c = \mathfrak{h}^{\omega_c}$ given by (9.4).

Denote by i the injection $L_c \hookrightarrow G$ and by i_* the extension of structure group associated to it. Since L_c and d have the form of (9.6) and (9.7) we have defined in (9.9) the family $\mathcal{E}_{L_c, d}^{x_0}$. We set

$$\mathcal{E}_{G, d}^{x_0} = i_* \mathcal{E}_{L_c, d}^{x_0}. \quad (9.10)$$

By construction, it is a family of polystable G -Higgs bundles parametrized by $T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$.

Remark 9.4.3. By Proposition 9.4.1 every polystable G -Higgs bundle is isomorphic to $\mathcal{E}_{G, d}^{x_0}|_{X \times \{t\}}$ for some $t \in T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$.

Proposition 9.4.4. *The family $\mathcal{E}_{G, d}^{x_0}$ has the local universal property among locally graded families of semistable G -Higgs bundles.*

Proof. If the family $\mathcal{E} \rightarrow X \times T$ is locally graded, for every point $t \in T$ there exists U containing f , a subgroup $i : L \hookrightarrow G$ and a family \mathcal{F} of stable L -Higgs bundles parametrized by U such that

$$\mathcal{E}|_{X \times U} \sim_S i_*(\mathcal{F}).$$

By Proposition 9.4.1 we have that $L = L_c$ and since, by Remark 9.3.10, $\mathcal{E}_{L_c, d}^{x_0}$ is a universal family, there exists $f : U \rightarrow T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$ such that $\mathcal{F} \sim_S f^* \mathcal{E}_{L_c, d}^{x_0}$ and therefore

$$\mathcal{E} \sim_S f^* \mathcal{E}_{L_c, d}^{x_0}.$$

\square

Let us denote by K_c the maximal compact subgroup of L_c ; we note that it is a product of factors of the form of $U(1)$, $\text{PSU}(n'_i)$ and $\text{U}(n''_j)/\mathbb{Z}_{k_j}$. We recall the stable G -bundle $E_{L_c, d}^{x_0}$ defined in (9.8) and denote the unitary representation associated to $E_{L_c, d}^{x_0}$ by

$$\rho_0 : \Gamma_{\mathbb{R}} \longrightarrow K_c.$$

As in Remark 8.2.1 we denote the triple that determines ρ^0 by (a_0, b_0, u) . Note that $a_0, b_0 \in [K_c, K_c]$ since the factors of $E_{L_c, d}^{x_0}$ have determinant equal to $\mathcal{O}(x_0)^{\otimes d_i}$ for some d_i .

Since \overline{S}_c is abelian, every representation $\Gamma_{\mathbb{R}}$ factors through a representation of $U(1) \times \pi_1(X)$. A representation $\overline{\rho} : \Gamma_{\mathbb{R}} \rightarrow \overline{S}_c$ is determined by a triple $(\overline{s}_1, \overline{s}_2, \overline{u})$ with $[\overline{s}_1, \overline{s}_2] = \text{id}$ and $\exp \overline{u} = \text{id}$. If $\overline{\rho}$ is a representation with trivial topological invariant, i.e. it is associated to $(\overline{s}_1, \overline{s}_2, 0)$, then it lifts to a representation $\rho' : \Gamma_{\mathbb{R}} \rightarrow S_d$ associated to $(s_1, s_2, 0)$, where s_i projects to \overline{s}_i . If u and \overline{u} are 0, the representations ρ and $\overline{\rho}$ factors through $\pi_1(X)$. We recall that $\text{Lie}(\overline{S}_c) = \mathfrak{h}^{\omega_c}$.

Remark 9.4.5. Let $t \in T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$ and let $(\overline{L}, \overline{\phi})$ be the \overline{S}_c -Higgs bundle of topological class 0 such that $t = \xi_{\overline{S}_c, 0}^{x_0}(\overline{L}, \overline{\phi})$. Let us take (L, ϕ) to be one S_c -Higgs bundle that projects to $(\overline{L}, \overline{\phi})$. Using the notation of (9.3) we have that

$$\mathcal{E}_{G,d}^{x_0}|_{X \times \{t\}} \cong i_* (L \otimes E_{L_c, d}^{x_0}, \phi).$$

Let $\rho' : \pi_1(X) \rightarrow S_c \cap K_c$ be the unitary representation associated to L and suppose that $\phi \in H^0(X, L(\mathfrak{h}^{\omega_c}))$ is given by the element $z \in \mathfrak{h}^{\omega_c}$. We have that $\mathcal{E}_{G,d}^{x_0}|_{X \times \{t\}}$ is the polystable G -Higgs bundle associated to $(\rho' \rho_0, z)$, where we consider $\rho' \rho_0$ as a representation of $\Gamma_{\mathbb{R}}$ into G .

Recall W_c defined in (8.6). Note that W_c acts on \overline{S}_c and therefore it acts on $T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$.

Proposition 9.4.6. *Let (E_1, Φ_1) and (E_2, Φ_2) be two polystable G -Higgs bundles of topological class d parametrized by $\mathcal{E}_{G,d}^{x_0}$ at the points t_1 and $t_2 \in T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$. We have that (E_1, Φ_1) and (E_2, Φ_2) are isomorphic G -Higgs bundles if and only if there exists $\nu \in W_c$ such that $t_2 = \nu \cdot t_1$.*

Proof. Let H be a Cartan subgroup of G and \mathfrak{h} its Cartan subalgebra. Let $(\overline{L}_i, \overline{\phi}_i)$ be the \overline{S}_c -Higgs bundle that maps under $\xi_{\overline{S}_c, 0}^{x_0}$ to t_i and suppose that $(\overline{L}_i, \overline{\phi}_i)$ is associated with the pair $(\overline{\rho}_i, z_i)$ where $z_i \in \text{Lie } \overline{S}_c = \mathfrak{h}^{\omega_c}$ and $\overline{\rho}_i$ is a representation of $\pi_1(X)$ in \overline{S}_c . Let us denote by $\rho'_i : \pi_1(X) \rightarrow S_c$ the lift to S_c of $\overline{\rho}_i$.

By Remark 9.4.5 $(\rho'_1 \rho_0, z_1)$ and $(\rho'_2 \rho_0, z_2)$ are the associated pairs of (E_1, Φ_1) and (E_2, Φ_2) . Let (a_0, b_0, u) and $(a'_i, b'_i, 0)$ be the triples associated to ρ_0 and ρ'_i as described in Remark 8.2.1. Take also $(\overline{a}_i, \overline{b}_i)$ to be the projection of (a'_i, b'_i) to \overline{S}_c ; we see that $(\overline{a}_i, \overline{b}_i, 0)$ is the triple associated to $\overline{\rho}$. Recall that a_0 and b_0 belong to $[L_c, L_c]$; we note that $(a'_1 a_0, b'_1 b_0)$ and $(a'_2 a_0, b'_2 b_0)$ are c -pairs with $c = \exp u$. By Proposition 9.2.5, (E_1, Φ_1) and (E_2, Φ) are isomorphic G -Higgs bundles if and only if there exists $g \in G$ such that

$$(a'_2 a_0, b'_2 b_0) = g(a'_1 a_0, b'_1 b_0)g^{-1} \tag{9.11}$$

and

$$z_2 = \text{ad}_g z_1. \tag{9.12}$$

By (8.14) of Theorem 8.2.9 the existence of $g \in G$ satisfying (9.11) implies the existence of $g' \in N_G(S_c)$ such that

$$(a'_2, b'_2) = g'(a'_1, b'_1)(g')^{-1}$$

From the proof of Theorem 8.2.9 we know that $g' = h' h'' g$ with h' and h'' in $Z_G(S_c)$, so $\text{ad}_{g'} z_1 = \text{ad}_g z_1$ and then g' also satisfies $z_2 = \text{ad}_{g'} z_1$.

Equivalently, if we have $g' \in N_G(S_c)$ satisfying

$$(c_a a'_2, c_b b'_2) = g'(a'_1, b'_1)(g')^{-1} \quad (9.13)$$

for some c_a and c_b in C_c and

$$z_2 = \text{ad}_{g'} z_1, \quad (9.14)$$

then $(a'_0, b'_0) = (g')^{-1}(c_a^{-1}a_0, c_b^{-1}b_0)g'$ is a c -pair in $[L_c, L_c]$ and, since all the c -pairs in $[L_c, L_c]$ are conjugate we know that there exists $h \in L_c$ such that (a'_0, b'_0) is equal to $h(a_0, b_0)h^{-1}$. This implies that $(a_0, b_0) = (g')h(c_a a_0, c_b b_0)h^{-1}(g')^{-1}$. Then $g = g'h$ is such that $(g'a'_1(g')^{-1}a_0, g'b'_1(g')^{-1}b_0)$ is equal to $g(a'_1 a_0, b'_1 b_0)g^{-1}$ and so

$$(a'_2 a_0, b'_2 b_0) = g(a'_1 a_0, b'_1 b_0)g^{-1}.$$

Finally since $g = g'h$ with $h \in Z_G(S_c)$ we have that (9.14) implies $z_2 = \text{ad}_g z_1$.

We have seen that the existence of $g \in G$ satisfying conditions (9.11) and (9.12) is equivalent to the existence of $g' \in N_G(S_c)$ satisfying (9.13) and (9.12).

We denote by $\nu \in W_c$ the projection of $g' \in N_G(S_c)$ to the quotient $N_G(S_c)/Z_G(S_c)$. We note that ν induces the isomorphisms $f_\nu : \bar{S}_c \xrightarrow{\cong} \bar{S}_c$ and $df_\nu : \mathfrak{h}^{\omega_c} \xrightarrow{\cong} \mathfrak{h}^{\omega_c}$ where we recall that $\mathfrak{h}^{\omega_c} = \text{Lie}(\bar{S}_c)$. We denote by $(f_\nu)_*$ the extension of structure group associated to f_ν (note that $(f_\nu)_*$ sends an \bar{S}_c -bundle to an \bar{S}_c -bundle) and we write $(df_\nu)_* : H^0(X, \mathcal{O} \otimes \mathfrak{h}^{\omega_c}) \rightarrow H^0(X, \mathcal{O} \otimes \mathfrak{h}^{\omega_c})$ induced by df_ν .

Recalling that $(\bar{\rho}_i, z_i)$ is the pair associated to the \bar{S}_c -Higgs bundle $(\bar{L}_i, \bar{\phi}_i)$, we see that (9.13) and (9.14) is equivalent to the existence of the following isomorphism of Higgs bundles

$$(\bar{L}_2, \bar{\phi}_2) \cong ((f_\nu)_* \bar{L}_1, (df_\nu)_* \bar{\phi}_1), \quad (9.15)$$

for some $\nu \in W_c$.

Let $\mathcal{B}(\Lambda_{\bar{S}_c}) = \{\gamma_1, \dots, \gamma_\ell\}$ be a basis of the lattice $\Lambda_{\bar{S}_c}$. If $t_i = \sum_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (x_{i,j}, \lambda_{i,j}) \otimes_{\mathbb{Z}} \gamma_j$ we have by Proposition 9.3.2 that

$$(\bar{L}_i, \bar{\phi}_i) \cong \left(\bigotimes_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (\chi_{\gamma_j})_* L'_{i,j}, \sum_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (d\chi_{\gamma_j})_* \lambda_{i,j} \right)$$

where $(L'_{i,j}, \lambda_{i,j})$ is the rank 1 Higgs bundle such that $(x_{i,j}, \lambda_{i,j}) = \xi_{1,0}^{x_0}(L'_{i,j}, \lambda_{i,j})$. If we apply $(f_\nu)_*$ and $(df_\nu)_*$ we obtain

$$\begin{aligned} ((f_\nu)_* \bar{L}_i, (df_\nu)_* \bar{\phi}_i) &\cong \left(\bigotimes_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (\chi_{\nu \cdot \gamma_j})_* L'_{i,j}, \sum_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (d\chi_{\nu \cdot \gamma_j})_* \lambda_{i,j} \right) \\ &\cong \left(\bigotimes_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (\chi_{\gamma_j})_* L'_{2,j}, \sum_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (d\chi_{\gamma_j})_* \lambda_{2,j} \right) \cong \left(\bigotimes_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (\chi_{\nu \cdot \gamma_j})_* L'_{1,j}, \sum_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (d\chi_{\nu \cdot \gamma_j})_* \lambda_{1,j} \right) \end{aligned}$$

and therefore

$$\sum_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (x_{2,j}, \lambda_{2,j}) \otimes_{\mathbb{Z}} \gamma_j = \sum_{\gamma_j \in \mathcal{B}(\Lambda_{\bar{S}_c})} (x_{1,j}, \lambda_{1,j}) \otimes_{\mathbb{Z}} (\nu \cdot \gamma_j).$$

We have proved that $(E_2, \Phi_2) \cong (E_1, \Phi_1)$ if and only if $t_2 = \nu \cdot t_1$. \square

We state the main theorem of this chapter.

Theorem 9.4.7. *Let G be a connected complex reductive Lie group. Take $d \in \pi_1(G)$, where d is determined by $(u, c) \in \pi_1(\overline{Z}) \times \pi_1(\overline{D})$ with the notation of Remark 8.1.1 and let $S_c, L_c, \overline{S}_c, W_c$ and $\Lambda_{\overline{S}_c}$ be as in (8.5), (8.7), (8.10), (8.6) and (9.4). Take the natural injection $i : L_c \hookrightarrow G$ and the stable L_c -bundle $E_{L_c, d}^{x_0}$ defined in (9.8).*

There exists a coarse moduli space $\mathcal{M}(G)_d$ for the moduli problem associated to the functor $\text{Mod}(\mathcal{A}_G, Q_G, S)$. We have the isomorphism

$$\xi_{G, d}^{x_0} : \mathcal{M}(G)_d \xrightarrow{\cong} (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c$$

$$[i_*(L \otimes E_{L_c, d}^{x_0}, \phi)]_S \longmapsto \xi_{\overline{S}_c, 0}^{x_0}(\overline{L}, \overline{\phi}),$$

where (L, ϕ) is a S_c -Higgs bundle that projects to the \overline{S}_c -Higgs bundle $(\overline{L}, \overline{\phi})$.

Proof. From Proposition 3.2.1, Proposition 9.4.4 and Proposition 9.4.6 it follows that

$$\eta_{G, d}^{x_0} : (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) \longrightarrow \mathcal{M}(G)_d$$

$$t \longmapsto [(\mathcal{E}_{G, d}^{x_0})_t]_S,$$

induces an isomorphism

$$\overline{\eta}_{G, d}^{x_0} : (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c \xrightarrow{\cong} \mathcal{M}(G)_d.$$

We define $\xi_{G, d}^{x_0}$ as the inverse of this isomorphism. \square

Remark 9.4.8. Let us denote by $d = 0$ the trivial element of $\pi_1(G)$. By (8.3) d is given by the trivial elements of $\pi_1(\overline{Z}) \times \pi_1(\overline{D})$ and then it induces the trivial element of the Weyl group, $\omega_0 = \text{id}$. This implies that $H^{\omega_0} = H$ and then $L_0 = H$. We write $\Lambda_H \subset \mathfrak{h}$ for the lattice obtained from $\pi_1(H)$ and then the component of $\mathcal{M}(G)$ of topologically trivial G -Higgs bundles is

$$\mathcal{M}(G)_0 \cong (T^*X \otimes_{\mathbb{Z}} \Lambda_H) / W.$$

Since we have a natural injection of $(X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c})$ in $(T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c})$, the restriction of the family $\mathcal{E}_{G, d}^{x_0}$ to this subvariety gives a family of polystable G -Higgs bundles with zero Higgs field parametrized by $(X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c})$. If we denote by $(\mathcal{E}^0)_{G, d}^{x_0}$ the underlying family of G -bundles observe that this family give us the following morphism

$$\nu_{(\mathcal{E}^0)_{G, d}^{x_0}} : (X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) \longrightarrow \mathcal{M}(G)_d.$$

By Proposition 9.4.6 this morphism induces the following bijective morphism

$$\nu'_{(\mathcal{E}^0)_{G, d}^{x_0}} : (X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c \longrightarrow \mathcal{M}(G)_d.$$

Remark 9.4.9. Since $M(G)_d$ is a normal algebraic variety, the bijective morphism $\nu'_{(\mathcal{E}^0)_{G,d}^{x_0}}$ is an isomorphism. Its inverse is again an isomorphism

$$M(G)_d \cong (X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c. \quad (9.16)$$

This is the description of $M(G)_d$ of [La] and [FM1] which is stated in Theorem 8.3.10.

Combining Theorem 9.4.7 with Theorem 8.1.8 we obtain the following result which is analogous to Theorem 8.3.12.

Corollary 9.4.10. *Let \tilde{G} be a simple complex reductive and simply connected Lie group and let $c \in Z_{\tilde{G}}(\tilde{G})$; set $G = \tilde{G}/\langle c \rangle$. We have that $\pi_1(G) \cong \langle c \rangle$; let d be the element of $\pi_1(G)$ corresponding to c under this isomorphism. The moduli space $\mathcal{M}(G)_d$ of S -equivalence classes of semistable G -Higgs bundles of topological class d is isomorphic to the moduli space $\mathcal{M}(\tilde{H}_{G,c})$ of S -equivalence classes of semistable $\tilde{H}_{G,c}$ -Higgs bundles, where $\tilde{H}_{G,c}$ is the simple complex reductive and simply connected Lie group given by the correspondence of Theorem 8.1.8.*

We finish the section studying $\mathfrak{M}(G)_d$, the moduli space associated to the moduli functor $\text{Mod}(\mathcal{A}_G, P_G, S)$.

Proposition 9.4.11. *We have a bijective morphism $\mathcal{M}(G)_d \rightarrow \mathfrak{M}(G)_d$, hence $\mathcal{M}(G)_d$ is the normalization of $\mathfrak{M}(G)_d$.*

Proof. The family $\mathcal{E}_{G,d}^{x_0}$ induces a morphism

$$\nu_{\mathcal{E}_{G,d}^{x_0}} : (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) \longrightarrow \mathfrak{M}(G)_d,$$

and by Proposition 9.4.6 it factors through

$$\nu'_{\mathcal{E}_{G,d}^{x_0}} : (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c \longrightarrow \mathfrak{M}(G)_d.$$

Let us denote by $\overline{\mathfrak{M}(G)}_d$ the normalization of $\mathfrak{M}(G)_d$. Since $(T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c$ is normal, by the universal property of the normalization, $\nu'_{\mathcal{E}_{G,d}^{x_0}}$ factors through

$$\nu''_{\mathcal{E}_{G,d}^{x_0}} : (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c \longrightarrow \overline{\mathfrak{M}(G)}_d.$$

This map is an isomorphism since it is a bijection and $\overline{\mathfrak{M}(G)}_d$ is normal. Then $\mathcal{M}(G)_d$ is the normalization of $\mathfrak{M}(G)_d$. \square

Remark 9.4.12. Both moduli spaces would be isomorphic if $\mathfrak{M}(G)_d$ were normal, but normality in this case is an open question.

9.5 The Hitchin fibration

We describe the Hitchin map using cameral covers. A good reference for this is [DP].

Let us consider the adjoint action of the group G on the Lie algebra \mathfrak{g} and take the quotient map

$$q : \mathfrak{g} \longrightarrow \mathfrak{g} // G.$$

Let E be any holomorphic G -bundle. Since the adjoint action of G on $\mathfrak{g} // G$ is obviously trivial, we note that the fibre bundle induced by E is trivial

$$E(\mathfrak{g} // G) = \mathcal{O} \otimes (\mathfrak{g} // G).$$

The projection q induces a surjective morphism of fibre bundles

$$q_E : E(\mathfrak{g}) \longrightarrow E(\mathfrak{g} // G),$$

and q_E induces a morphism on the set of holomorphic global sections

$$(q_E)_* : H^0(X, E(\mathfrak{g})) \longrightarrow H^0(X, \mathcal{O} \otimes (\mathfrak{g} // G))$$

$$\Phi \longmapsto \Phi // G.$$

The following result is standard and implies that the map constructed above is constant along S -equivalence classes.

Lemma 9.5.1. *Let (E_1, Φ_1) and (E_2, Φ_2) be two S -equivalent semistable G -Higgs bundles. We have that*

$$(q_{E_1})_* \Phi_1 = (q_{E_2})_* \Phi_2.$$

Proof. We first prove that $(E_1, \Phi_1) \cong (E_2, \Phi_2)$ implies $(q_{E_1})_* \Phi_1 = (q_{E_2})_* \Phi_2$. If $(E_1, \Phi_1) \cong (E_2, \Phi_2)$ is equivalent to the existence of an isomorphism $f : E_1 \xrightarrow{\cong} E_2$ inducing another isomorphism $f_{\text{ad}} : E_1(\mathfrak{g}) \xrightarrow{\cong} E_2(\mathfrak{g})$ such that $\Phi_2 = f_{\text{ad}}(\Phi_1)$. We note that every isomorphism between adjoint bundles $f_{\text{ad}} : E_1(\mathfrak{g}) \longrightarrow E_2(\mathfrak{g})$ induces the trivial automorphism between the induced fibre bundles

$$\begin{array}{ccc} E_1(\mathfrak{g}) & \xrightarrow{f_{\text{ad}}} & E_2(\mathfrak{g}) \\ q_{E_1} \downarrow & & \downarrow q_{E_2} \\ \mathcal{O} \otimes (\mathfrak{g} // G) & \xrightarrow{\text{id}} & \mathcal{O} \otimes (\mathfrak{g} // G), \end{array}$$

and then $\Phi_2 = f_{\text{ad}}(\Phi_1)$ implies that $(q_{E_1})_* \Phi_1 = (q_{E_2})_* \Phi_2$.

The construction above can be done for families of G -Higgs bundles. Let $\mathcal{E} = (\mathcal{F}, \Phi)$ be a family of G -Higgs bundles parametrized by T . We have that $\mathcal{F}(\mathfrak{g} // G)$ is equal to $\mathcal{O}_{X \times T} \otimes (\mathfrak{g} // G)$ and this gives us the projection

$$q_{\mathcal{F}} : H^0(X \times T, \mathcal{F}(\mathfrak{g})) \longrightarrow H^0(X \times T, \mathcal{O}_{X \times T} \otimes (\mathfrak{g} // G)).$$

If (E, Φ) is the polystable representative of the S-equivalence class of (E_1, Φ_1) and (E_2, Φ_2) , there exists for $i = 1, 2$ a family of G -Higgs bundles $\mathcal{E}_i = (\mathcal{F}_i, \Phi_i)$ parametrized by an irreducible variety T such that for some $t_0 \in T$ we have $(E, \Phi) \cong (\mathcal{E}_i)_{t_0}$ and for every $t \in T - \{t_0\}$ we have $(E_i, \Phi_i) \cong (\mathcal{E}_i)_t$. Then

$$q_{\mathcal{F}_i}(\Phi_i)|_{X \times \{t_0\}} = q_E \Phi$$

and for $t \in T - \{t_0\}$,

$$q_{\mathcal{F}_i}(\Phi_i)|_{X \times \{t\}} = q_{E_i} \Phi_i.$$

Since $q_{\mathcal{F}_1}$ and $q_{\mathcal{F}_2}$ are a holomorphic projection, the equalities above imply that $q_{E_1} \Phi_1 = q_E \Phi = q_{E_2} \Phi_2$. \square

The Hitchin map is defined in terms of this morphism

$$b_G : \mathcal{M}(G) \longrightarrow H^0(X, \mathcal{O} \otimes (\mathfrak{g} // G)) \quad (9.17)$$

$$[(E, \Phi)]_S \longmapsto (q_E)_* \Phi;$$

by Lemma 9.5.1 we see that the Hitchin map is well defined. We know that the topological class is a discrete invariant of the moduli space of G -Higgs bundles $\mathcal{M}(G)$. When the base variety is a Riemann surface of genus greater than or equal to 2, we have that the restriction of b_G to every component $\mathcal{M}(G)_d$ (not necessarily connected) of G -Higgs bundles with topological class d is surjective. This is not the case for genus $g = 1$ and, to preserve the fact that the Hitchin map is a fibration, we set

$$B(G, d) = b_G(\mathcal{M}(G)_d),$$

and

$$b_{G,d} : \mathcal{M}(G)_d \longrightarrow B(G, d),$$

as in (9.17).

If d is the trivial element of $\pi_1(G)$, we have that $B(G, d)$ is the whole of $H^0(X, \mathcal{O} \otimes (\mathfrak{g} // G))$ since $\mathcal{M}(G)_0$ contains every S-equivalence class of the form $[(X \times G, \Phi)]_S$ where $X \times G$ is the trivial G -Higgs bundle and Φ is any element of $H^0(X, \mathcal{O} \otimes \mathfrak{g})$. If H is a Cartan subgroup with Cartan subalgebra \mathfrak{h} and Weyl group W , Chevalley's Theorem says that

$$\mathfrak{g} // G \cong \mathfrak{h} / W.$$

As a consequence we have that $H^0(X, \mathcal{O} \otimes (\mathfrak{g} // G)) \cong H^0(X, \mathcal{O} \otimes \mathfrak{h} / W)$ and since X is a compact holomorphic variety, we have that $H^0(X, \mathcal{O} \otimes \mathfrak{h} / W) \cong \mathfrak{h} / W$. Let $\Lambda_H \subset \mathfrak{h}$ be the lattice given in (9.4), so $\mathfrak{h} \cong \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H$ and furthermore $\mathfrak{h} / W \cong (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H) / W$. Composing all these isomorphisms we obtain

$$\beta_{G,0} : B(G, 0) \xrightarrow{\cong} (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H) / W.$$

Now we take $d \in \pi_1(G)$ non-trivial. Recall from Remark 8.1.1 that d is associated to $(u, c) \in \pi_1(\overline{Z}) \times \pi_1(\overline{D}) \subset \mathfrak{z} \times Z_D(D)$. By Proposition 9.2.5 we see that every polystable G -Higgs bundle of topological class d is isomorphic to (E^ρ, Φ) where E^ρ is constructed from

the representation $\rho : \Gamma_{\mathbb{R}} \rightarrow K$ and there exists a covering $\{U_i\}_{i \in I}$ of X that trivializes E^ρ such that $\Phi \in H^0(X, E^\rho(\mathfrak{g}))$ expressed in terms of $\{U_i\}_{i \in I}$ is constant and equal to $z \in \mathfrak{h}^{\omega_c} \subset \mathfrak{z}_{\mathfrak{g}}(\rho)$. Let us denote by \mathfrak{g}_c the subset of \mathfrak{g} of the elements of the form $z' = \text{ad}_d(z)$ for some $g \in G$ and some $z \in \mathfrak{h}^{\omega_c}$. By the previous description of polystable G -Higgs bundles of topological class d we have that

$$B(G, d) = H^0(X, \mathcal{O} \otimes (\mathfrak{g}_c // G)) \cong \mathfrak{g}_c // G .$$

By definition of \mathfrak{g}_c we have that $\mathfrak{g}_c // G$ is isomorphic to $\mathfrak{h}^{\omega_c} / (N_G(\mathfrak{h}^{\omega_c}) / Z_G(\mathfrak{h}^{\omega_c}))$. Take W_c , \overline{S}_c and $\Lambda_{\overline{S}_c}$ as in (8.6), (8.10) and (9.4). Recall that $\text{Lie } \overline{S}_c$ is \mathfrak{h}^{ω_c} and therefore we have that $\mathfrak{h}^{\omega_c} \cong \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}$ and furthermore $\mathfrak{h}^{\omega_c} / W_c \cong (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c$. Then, the composition of the previous isomorphisms gives us

$$\beta_{G,d} : B(G, d) \xrightarrow{\cong} (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c .$$

Let $\mathcal{B}(\Lambda_{\overline{S}_c}) = \{\gamma_1, \dots, \gamma_\ell\}$ be a basis of $\Lambda_{\overline{S}_c}$. Recalling that $T^*X \cong X \times \mathbb{C}$, we see that the projection $\pi : T^*X \rightarrow \mathbb{C}$ induces

$$\pi_{G,c} : (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c \longrightarrow (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c \quad (9.18)$$

$$\left[\sum_{\gamma_i \in \mathcal{B}(\Lambda_{\overline{S}_c})} (x_i, \lambda_i) \otimes_{\mathbb{Z}} \gamma_i \right]_{W_c} \longmapsto \left[\sum_{\gamma_i \in \mathcal{B}(\Lambda_{\overline{S}_c})} \lambda_i \otimes_{\mathbb{Z}} \gamma_i \right]_{W_c} .$$

We use this morphism to better understand the Hitchin map.

Theorem 9.5.2. *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{M}(G)_d & \xrightarrow{b_{G,d}} & B(G, d) \\ \xi_{G,d}^{x_0} \downarrow \cong & & \cong \downarrow \beta_{G,d} \\ (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c & \xrightarrow{\pi_{G,c}} & (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c . \end{array}$$

Proof. Let (E, Φ) be the polystable representative of a certain S-equivalence class in $\mathcal{M}(G)_d$. In the notation of Remark 9.4.5 we have that (E, Φ) is isomorphic to $i_*(L \otimes E_{L_c, d}^{x_0}, \phi)$ where (L, ϕ) is a S_c -Higgs bundle and $(\overline{L}, \overline{\phi})$ the induced \overline{S}_c -Higgs bundle. Note that ϕ and $\overline{\phi}$ are given by an element z of \mathfrak{h}^{ω_c} . We write

$$\xi_{\overline{S}_c, 0}^{x_0}(\overline{L}, \overline{\phi}) = \sum_{\gamma_i \in \mathcal{B}_{\Lambda_{\overline{S}_c}}} (x_i, \lambda_i) \otimes_{\mathbb{Z}} \gamma_i .$$

We note that $z = \sum_{\gamma_i \in \mathcal{B}_{\Lambda_{\overline{S}_c}}} \lambda_j \cdot \gamma_j$. On the other hand we have that $\xi_{G,d}^{x_0}([(E, \Phi)]_S)$ is equal to $\xi_{\overline{S}_c, 0}^{x_0}(\overline{L}, \overline{\phi})$ and then, we see that the diagram commutes. \square

Once we have an explicit description of the Hitchin fibration, we can describe explicitly its fibres.

Proposition 9.5.3. *Let $s \in (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c})$, then*

$$\pi_{G,c}([s]_{W_c}) \cong (X \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c}) / Z_{W_c}(s).$$

Proof. We consider the following commutative diagram

$$\begin{array}{ccc} T^*X \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c} & \xrightarrow{\tilde{q}_{G,c}} & T^*X \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c} / W_c \\ \downarrow \tilde{\pi}_{G,c} & & \downarrow \pi_{G,c} \\ \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c} & \xrightarrow{q_{G,c}} & \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c} / W_c, \end{array}$$

where $\pi_{G,c}$ is defined in (9.18) and $\tilde{\pi}_{G,c}$, $q_{G,c}$ and $\tilde{q}_{G,c}$ are the natural projections. We observe that

$$\pi_{G,c}^{-1}([s]_{W_c}) \cong \tilde{q}_{G,c} \left(\tilde{\pi}_{G,c}^{-1} \left(q_{G,c}^{-1}([s]_{W_c}) \right) \right).$$

Since $\tilde{\pi}_{G,c}^{-1}(q_{G,c}^{-1}([s]_{W_c}))$ is W_c -invariant then

$$\pi_{G,c}^{-1}([s]_{W_c}) \cong \tilde{\pi}_{G,c}^{-1} \left(q_{G,c}^{-1}([s]_{W_c}) \right) / W_c.$$

Since $Z_{W_c}(s) \subset W_c$ might not be a normal subgroup, we only know that $W_c/Z_{W_c}(s)$ is a finite set. Denote by o_ω the orbit of $\omega \in W_c$. We have

$$q_{G,c}^{-1}([s]_{W_c}) = \bigcup_{\omega \in W_c} \omega \cdot s = \bigcup_{o_\omega \in W_c/Z_{W_c}(s)} \omega \cdot s,$$

and therefore

$$\pi_{G,c}^{-1}([s]_{W_c}) \cong \left(\bigcup_{o_\omega \in W_c/Z_{W_c}(s)} \tilde{\pi}_{G,c}^{-1}(\omega \cdot s) \right) / W_c. \quad (9.19)$$

If o_{ω_1} and o_{ω_2} are different orbits we have that

$$\tilde{\pi}_{G,c}^{-1}(\omega_1 \cdot s) \cap \tilde{\pi}_{G,c}^{-1}(\omega_2 \cdot s) = \emptyset,$$

furthermore, one has

$$\tilde{\pi}_{G,c}^{-1}(\omega_2 \cdot s) = (\omega_2 \omega_1^{-1}) \cdot \tilde{\pi}_{G,c}^{-1}(\omega_1 \cdot s).$$

Then we see that (9.19) can be rewritten as follows

$$\pi_{G,c}^{-1}([s]_{W_c}) \cong \tilde{\pi}_{G,c}^{-1}(s) / Z_{W_c}(s).$$

Finally we observe that $\tilde{\pi}_{G,c}^{-1}(s) \cong X \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c}$ and the action on both sides of $Z_{W_c}(s)$ commutes. \square

Since the only element of W_c that acts trivially on $\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c}$ is the identity, we have that the subset

$$V_{G,c} = \{s \in \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c} : \text{such that there exists a non-trivial } \omega \in W_c \text{ with } s = \omega \cdot s\}$$

is a finite union of codimension at least or equal to 1 closed subsets and then $V_{G,c}$ is a closed subset of codimension greater than 1. Let us denote by $U_{G,c}$ the complement of $V_{G,c}$ in $\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c}$,

$$U_{G,c} = \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c} \setminus V_{G,c}.$$

The *generic Hitchin fibre* is the fibre over an element of the dense open subset of $B(G, c)$ given by the image of $U_{G,c}$. By construction, for any $s_a \in U_{G,c}$ we have that $Z_{W_c}(s_a) = \{\text{id}\}$ and then Proposition 9.5.3 implies the following.

Corollary 9.5.4. *Let $s_a \in U_{G,c} \subset \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_{\bar{s}_c}$; the generic Hitchin fibre is the abelian variety*

$$\pi_{G,c}^{-1}([s_a]_{W_c}) \cong X \otimes_{\mathbb{Z}} \Lambda_{\bar{s}_c}.$$

Suppose we have a pair of Langlands dual groups G and G^L and denote their Lie algebras by \mathfrak{g} and \mathfrak{g}^L respectively. Let H and H^L be Cartan subgroups of G and G^L and denote their Lie algebras by \mathfrak{h} and \mathfrak{h}^L . Let us write W and W^L for the Weyl groups of the pairs (G, H) and (G^L, H^L) . We denote by $\Lambda_H \subset \mathfrak{h}$ and by $\Lambda_{H^L} \subset \mathfrak{h}^L$ the lattices induced by $\pi_1(H)$ and $\pi_1(H^L)$ respectively. Consider a quadratic form $\langle \cdot, \cdot \rangle$ on \mathfrak{h} that projects to the Killing form on the semisimple part of \mathfrak{h} and denote by $f : \mathfrak{h} \rightarrow \mathfrak{h}^*$ the isomorphism induced by the quadratic form and Λ_H^\vee the dual lattice i.e. $\Lambda_H^\vee = \{f(z) : \text{where } z \in \mathfrak{h} \text{ is such that } \langle z, \Lambda_H \rangle \subset \mathbb{Z}\}$. The group theoretic Langlands duality says that

$$W^L = W, \quad \mathfrak{h}^L \cong \mathfrak{h}^*, \quad \text{and} \quad \Lambda_{H^L} = \Lambda_H^\vee,$$

and the action of W^L on \mathfrak{h}^L is the action of W on \mathfrak{h}^* by f . We note that in general $f(\Lambda_H) \neq \Lambda_H^\vee (= \Lambda_{H^L})$.

It is straightforward that there exists an isomorphism between the Hitchin bases of the Hitchin fibration for topologically trivial components of the moduli spaces of Higgs bundles of two Langlands dual groups

$$B(G, 0) \cong \mathfrak{h} / W \cong \mathfrak{h}^* / W \cong B(G^L, 0),$$

and then we have

$$\begin{array}{ccc} \mathcal{M}(G)_0 & & \mathcal{M}(G^L)_0 \\ & \searrow^{b_{G,0}} & \swarrow_{b_{G^L,0}} \\ & B(G, 0) \cong B(G^L, 0) & \end{array}$$

We define the isomorphism \tilde{f} making use of the following commuting diagram

$$\begin{array}{ccc} \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H & \xrightarrow{\cong} & \mathfrak{h} \\ \tilde{f} \downarrow \cong & & \cong \downarrow f \\ \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H^\vee & \xrightarrow{\cong} & \mathfrak{h}^*. \end{array}$$

Take any $s \in \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H$ and denote by z_s its image in \mathfrak{h} . By the diagram above we have that $f(z_s) \in \mathfrak{h}^*$ is the image in \mathfrak{h}^* of $\tilde{s} \in \mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H^\vee$. Since the action of W is equivariant under f we have that $Z_W(f(z_s)) = Z_W(z_s)$ and then

$$Z_W(\tilde{f}(s)) = Z_W(s). \tag{9.20}$$

Therefore the dense subset of $B(G, 0)$ given by the projection of $U_{G,0}$ is sent to the dense subset of $B(G^L, 0)$ obtained by projecting $U_{G^L,0}$. As a consequence of Proposition 9.5.3 we have the following relation between Hitchin fibres.

Corollary 9.5.5. *Let us take $s_a \in U_{G,0}$ and therefore we have that $\tilde{f}(s_a) \in U_{GL,0}$. Then*

$$\pi_{GL,0}^{-1}([\tilde{f}(s_a)]_W) \cong X \otimes_{\mathbb{Z}} \Lambda_H^{\vee}$$

is a self-dual abelian variety isomorphic to

$$\pi_{G,0}^{-1}([s_a]_W) \cong X \otimes_{\mathbb{Z}} \Lambda_H.$$

For an arbitrary $s \in (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H)$ and $\tilde{f}(s) \in (\mathbb{C} \otimes_{\mathbb{Z}} \Lambda_H^{\vee})$ we have that

$$\pi_{GL,0}^{-1}([\tilde{f}(s)]_W) \cong (X \otimes_{\mathbb{Z}} \Lambda_H^{\vee}) / Z_W(s)$$

and

$$\pi_{G,0}^{-1}([s]_W) \cong (X \otimes_{\mathbb{Z}} \Lambda_H) / Z_W(s)$$

are quotients of isomorphic self-dual abelian varieties by the action of the same finite group $Z_W(s)$ although the action of $Z_W(s)$ is different in each case.

9.6 The fibration $\mathcal{M}(G) \rightarrow M(G)$

Thanks to Proposition 9.2.1 and Corollary 9.2.3 we can construct the following morphism between moduli spaces

$$a_{G,d} : \mathcal{M}(G)_d \longrightarrow M(G)_d$$

$$[(E, \Phi)]_S \longmapsto [E]_S.$$

We see that this morphism is clearly surjective. Consider $p_{G,d}$ to be the usual projection

$$p_{G,d} : (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c \longrightarrow (X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c$$

$$[(x_1, \lambda_1) \otimes \gamma_1 + \cdots + (x_\ell, \lambda_\ell) \otimes \gamma_\ell]_{W_c} \longmapsto [x_1 \otimes \gamma_1 + \cdots + x_\ell \otimes \gamma_\ell]_{W_c}.$$

Proposition 9.6.1. *We have the following commutative diagram*

$$\begin{array}{ccc} a_{G,d} : \mathcal{M}(G)_d & \xrightarrow{a_{G,d}} & M(G)_d \\ \downarrow \xi_{G,d}^{x_0} & & \downarrow \xi_{G,d}^{x_0} \\ (T^*X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c & \xrightarrow{p_{G,d}} & (X \otimes_{\mathbb{Z}} \Lambda_{\overline{S}_c}) / W_c. \end{array}$$

Proof. This follows from the description of $\mathcal{M}(G)_d$ given in Theorem 9.4.7 and the description of $M(G)_d$ of (9.16) \square

We can give an interpretation of the projection $a_{G,d}$ in terms of a certain orbifold bundle.

Given a topological space Y , we can define an *orbifold chart* α as a triple given by a connected open subset $U_\alpha \subset \mathbb{C}^n$, a finite group Γ_α of holomorphic automorphisms of

U_α and a Γ_α -invariant map $\varphi_\alpha : U_\alpha \rightarrow \mathcal{U}_\alpha$ that induces an isomorphism between U_α/Γ_α and $\mathcal{U}_\alpha \subset Y$ open. As for the case of manifolds, we can define a notion of embedding of orbifold charts. An *orbifold atlas* \mathcal{C} for Y is a collection of such charts that cover Y and such that, for every point in the intersection of two charts, there exists a third chart that covers this point and embeds into the previous two. An *effective orbifold* \tilde{Y} is the pair (Y, \mathcal{C}) where Y is a Hausdorff topological space and \mathcal{C} is an orbifold atlas for Y . Two atlases on Y are equivalent if we can find a common refinement, an *orbifold structure* on the topological space is an equivalence class of orbifold atlases on it.

For any effective orbifold $\tilde{Y} = (Y, \mathcal{C})$ we can construct its *cotangent orbifold bundle*, denoted by $\mathcal{T}^*\tilde{Y}$. Given $(U_\alpha, \Gamma_\alpha, \varphi_\alpha)$ a chart of \mathcal{C} , we can consider the cotangent bundle T^*U_α and the induced action of Γ_α on it. The projection map $T^*U_\alpha \rightarrow U_\alpha$ is equivariant and composing with φ_α we obtain a natural projection $a_\alpha : T^*U_\alpha/\Gamma_\alpha \rightarrow \mathcal{U}_\alpha$. It can be proved (see [ALR] for details) that the bundles $a_\alpha : T^*U_\alpha/\Gamma_\alpha \rightarrow \mathcal{U}_\alpha$ can be glued together giving an orbifold structure to $\mathcal{T}^*\tilde{Y}$. Moreover the natural projection $a : \mathcal{T}^*\tilde{Y} \rightarrow \tilde{Y}$ defines a holomorphic map of orbifolds with fibres $a^{-1}(y) = T_u^*U_\alpha/\Gamma_y$, where $u \in U_\alpha$ maps to $y \in Y$.

When $Y = Z/\Gamma$ is the quotient of a holomorphic manifold Z by a finite group Γ which acts holomorphically and effectively on Z , we can define a *natural orbifold structure* on Y . For every point $z \in Z$ take Γ_z to be its stabilizer. Take any open neighbourhood V'_z of z such that for every $\gamma \in \Gamma \setminus \Gamma_z$ the intersection $\gamma(V'_z) \cap V'_z$ is empty. Write U'_z for the connected component containing z of the intersection $\bigcap_{\gamma \in \Gamma_z} \gamma(V'_z)$. Taking a holomorphic chart $g : U'_z \rightarrow U_z \subset \mathbb{C}^n$ of the manifold Z , we see that Γ_z is included in the automorphism group of U_z . Given the projection $p : Z \rightarrow Z/\Gamma$, we define $\varphi_z = p \circ g^{-1} : U_z \rightarrow \mathcal{U}_z \subset Z/\Gamma$; by construction φ_z is Γ_z invariant. The triple $(U_z, \Gamma_z, \varphi_z)$ defines an orbifold chart for every point z of Z and every sufficiently small neighbourhood V'_z . It can be proved that the set of all of these charts gives an orbifold atlas \mathcal{C} which determines an orbifold structure on $Y = Z/\Gamma$.

In (9.16) we saw that $M(G)_d$ can be described as the quotient of $X \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c}$ by the finite group W_c . Let us write $\tilde{M}(G)_d$ for the orbifold induced by this quotient. Analogously, we define $\tilde{\mathcal{M}}(G)_d$ to be the orbifold induced on the moduli space of Higgs bundles $\mathcal{M}(G)_d$ induced by the quotient $(T^*X \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c})/W_c$ given in Theorem 9.4.7.

Theorem 9.6.2. *Let G be a complex reductive Lie group. Let $\tilde{M}(G)_d$ and $\tilde{\mathcal{M}}(G)_d$ be the orbifolds defined above. We have that $\tilde{\mathcal{M}}(G)_d$ is the cotangent orbifold bundle of $\tilde{M}(G)_d$, i.e.*

$$\tilde{\mathcal{M}}(G)_d \cong \mathcal{T}^*\tilde{M}(G)_d.$$

Proof. We observe that $T^*(X \otimes_{\mathbb{Z}} \Lambda_{\bar{S}_c})$ is $T^*X \otimes \Lambda_{\bar{S}_c}$ and the action of W_c on both spaces commutes. \square

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