Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas



TESIS DOCTORAL

Flexibility and rigidity for maximally non-integrable distributions.

Flexibilidad y rigidez para distribuciones maximalmente no integrables.

Memoria para optar al grado de doctor presentada por

Wei Zhou

Directores:

Eduardo Fernández Fuertes
Álvaro del Pino Gómez
Francisco Presas Mata

Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas

Programa de Doctorado en Investigación Matemática

Instituto de Ciencias Matemáticas



TESIS DOCTORAL

Flexibility and rigidity for maximally non-integrable distributions.

Flexibilidad y rigidez para distribuciones maximalmente no integrables.

Memoria para optar al grado de doctor presentada por

Wei Zhou

Directores:

Eduardo Fernández Fuertes

Álvaro del Pino Gómez

Francisco Presas Mata

UNIVERSIDAD COMPLUTENSE MADRID

DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD DE LA TESIS PRESENTADA PARA OBTENER EL TÍTULO DE DOCTOR

D./Dña. Wei Zhou,
estudiante en el Programa de Doctorado Doctorado en Investigación Matemática,
de la Facultad de Ciencias Matemáticas de la Universidad Complutense de Madrid, como autor/a de la tesis presentada para la obtención del título de Doctor y titulada:

Flexibility and rigidity for maximally non-integbrable distributions

Flexibilidad y rigidez para distribuciones maximalmente no integrables

y dirigida por: Eduardo Fernández Fuertes, Francisco Presas Mata y Álvaro del Pino Gómez

DECLARO QUE:

La tesis es una obra original que no infringe los derechos de propiedad intelectual ni los derechos de propiedad industrial u otros, de acuerdo con el ordenamiento jurídico vigente, en particular, la Ley de Propiedad Intelectual (R.D. legislativo 1/1996, de 12 de abril, por el que se aprueba el texto refundido de la Ley de Propiedad Intelectual, modificado por la Ley 2/2019, de 1 de marzo, regularizando, aclarando y armonizando las disposiciones legales vigentes sobre la materia), en particular, las disposiciones referidas al derecho de cita.

Del mismo modo, asumo frente a la Universidad cualquier responsabilidad que pudiera derivarse de la autoría o falta de originalidad del contenido de la tesis presentada de conformidad con el ordenamiento jurídico vigente.

En Madrid, a 29 de septiembre de 2025

Fdo.: ZHOU WEI - X8685443E

Firmado digitalmente por ZHOU WEI - X8685443E
Fecha: 2025.09.28 15:51:49 +02'00'

Esta DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD debe ser insertada en la primera página de la tesis presentada para la obtención del título de Doctor.

A mis padres.

Agradecimientos

Quiero agradecer, en primer lugar, a mis directores Eduardo Fernández, Álvaro del Pino y Francisco (Fran) Presas por el enorme apoyo que me han brindado a lo largo de esta tesis. Me han tratado como a un amigo: con ellos he podido discutir de matemáticas y también hablar de la vida.

En orden cronológico, deseo agradecer de nuevo a Fran, quien también fue mi director de máster y me introdujo en el área de geometría simpléctica y de contacto. Aquel año nos enfrentamos juntos a un problema matemático realmente difícil; fue la primera vez que sentí cómo se hace investigación en matemáticas. Aunque fue un periodo duro, también fue el año en el que más aprendí.

Posteriormente, Eduardo y Álvaro han sido siempre cercanos y generosos: me ayudaron a resolver dudas y compartimos largas discusiones que orientaron este trabajo. Agradezco asimismo a Álvaro y Eduardo por invitarme a visitarles, y a los departamentos que me acogieron durante esas estancias por su hospitalidad y por los fructíferos intercambios científicos.

Quiero dar las gracias a mis compañeros de la universidad: Alex, Dayahna, Diego, Enrique, Guillermo, Luis, Óscar y Samuel, por tantas charlas, seminarios y problemas de matemáticas compartidos. Quiero agradecer también a mi tutor, Luis Giraldo, por su orientación académica y su apoyo continuo durante estos años.

Durante estos años de doctorado he estado financiado por la Beca de Formación de Personal Investigador (FPI) en el ICMAT, por lo que expreso también mi agradecimiento. Igualmente, doy las gracias a José Ignacio y Daniel Macías (IP del proyecto del grupo en el ICMAT) por su ayuda y por la financiación complementaria que permitió realizar mis estancias en el extranjero.

Finalmente, agradezco a las y los evaluadores de esta tesis: Georgios Dimitroglou Rizell, Fabio Gironella, Peter Lambert-Cole, Marina Logares, Javier Martínez-Aguinaga y Lisa Traynor. (En su caso, extiendo también mi agradecimiento a las/los suplentes del tribunal).

En el plano personal, quiero agradecer a mi familia y a mi pareja por su apoyo incondicional.

Contents

Agradecimientos i

Contents v

1 Introduction 1

1.1 Motivation and Overview . . . 1

1.1.1 Contact geometry as the model case . . . 2

1.1.2 Beyond contact structures . . . 2

1.2 Contents of the thesis . . . 3

1.2.1 Fat distributions and prelegendrians . . . 3

1.2.2 Engel structure as complex tangencies . . . 5

1.3 Guide to the content of thesis . . . 5

I The h-principle fails for prelegendrians in corank 2 fat distributions 9

2 Overview and main results of Part I 11

2.1 Motivation and background . . . 11

2.2 Main results . . . 14

2.3 Guide to the rest of Part I . . . 18

3 Fat manifolds and spaces of complex contact elements 21

3.1 Fat manifolds . . . 21

3.1.1 Fat Distributions . . . 21

3.1.2 Contactisation and symplectisation of a fat manifold . . . 24

3.1.3 Preisotropic and prelegendrian submanifolds . . . 26

3.1.4 Formal preisotropics and prelegendrians 29
3.2 Co-real submanifolds in almost complex manifolds 30
3.2.1 Complex linear algebra 30
3.2.2 Co-real submanifolds 32
3.3 Examples of corank 2 Fat Distributions 34
3.3.1 An alternative characterisation of fatness for corank 2 34
3.3.2 Holomorphic contact manifolds 36
3.3.3 The space of complex contact elements 37
3.3.4 Standard fat structure on $\mathbb{C}^{2n+1}$ 39
3.3.5 Affine charts in the Grassmannian 41

4 Prelegendrian Fronts **43**
4.1 Legendrian Fronts 43
4.1.1 Legendrian fibrations 43
4.1.2 Front singularities and simple fronts 44
4.2 Legendrian stabilization and loose Legendrians 45
4.2.1 N-Pushing 46
4.2.2 Legendrian Stabilization and Loose Legendrians 48
4.3 Legendrian spinning 49
4.3.1 Front spinning 50
4.3.2 A specific spinning 51
4.4 Prelegendrian fronts 53
4.4.1 Proof of Theorem 4.4.1 54
4.4.2 Simple prelegendrian Reidemeister moves 57
4.4.3 Prelegendrian spinning 58
4.5 Prelegendrian stabilization 59
4.5.1 Prelegendrian N-pushing 60
4.5.2 Prelegendrian stabilization 62

5 Existence of prelegendrian stabilisation and exotic prelegendrians **63**
5.1 Existence of compact prelegendrians and prelegendrian stabilizations 63
5.1.1 Co-real submanifolds in $\mathbb{C}^{n+1}$ graphical over $\mathbb{R}^{2n+1}$ 63
5.1.2 Existence of prelegendrian stabilizations 64

5.2 Exotic prelegendrians . . . 65
5.2.1 Auxiliary Lemmas in Contact Topology . . . 65
5.2.2 Exotic prelegendrian tori . . . 67
5.2.3 Proof of Corollary 2.2.6 . . . 69
5.2.4 Proof of Corollary 2.2.7 . . . 70
5.3 Constructing prelegendrians in non-standard fat distributions . . . 71
5.3.1 The nilpotentisation . . . 71
5.3.2 Prelegendrians in general fat manifolds . . . 72
5.3.3 Stability phenomena . . . 73

II Engel CR submanifolds of $\mathbb{C}^3$ 75

6 Engel CR submanifolds 77
6.1 Main results of part II . . . 77
6.2 Preliminaries . . . 79
6.2.1 Engel structures . . . 79
6.2.2 Prolonged distributions . . . 79
6.2.3 Engel prolongations . . . 80
6.2.4 A tubular neighbourhood for totally real submanifolds . . . 81
6.3 Convex curves with zero integral . . . 81
6.3.1 Saldanha's result . . . 82
6.3.2 The zero integral condition . . . 83
6.3.3 Proof of Proposition 6.3.6 via grafting . . . 84
6.3.4 Proof of Proposition 6.3.1 . . . 87
6.4 Zoom-in argument: proof of Theorem 6.1.1 . . . 89
6.5 Proof of Theorem 6.1.3 and Corollary 6.1.4 . . . 91

III Open questions and future work 93

7 Open questions and future work 95

Abstract

This thesis explores rigidity phenomena in the study of non-integrable distributions, beyond the setting of contact structures.

The first part focuses on a class of maximally non-integrable distributions known as fat distributions, which generalize contact structures to higher corank and admit natural symplectisations and contactisations.

In our study of fat distributions we introduce prelegendrian submanifolds, which are generalizations of Legendrians to this setting. Morally speaking, they are defined to be those submanifolds that admit a canonical Legendrian lift into the contactisation.

Our main results show that: (1) There are infinitely many exotic prelegendrian embeddings in the standard corank-2 fat space $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$, all formally isotopic but pairwise non-isotopic; these prelegendrians are distinguished by pseudoholomorphic-curve invariants of their Legendrian lifts. (2) There is a notion of prelegendrian stabilization that produces loose Legendrian lifts; it yields further families of exotic prelegendrians. (3) These results can be extended to non-standard fat manifolds using nilpotentisation models and stability arguments.

In the second part, we investigate the interaction between Engel geometry and complex geometry. We show that oriented $\mathbb{S}^1$-bundles over 3-manifolds admit co-real immersions into almost complex 6-manifolds, so that the complex tangencies define an Engel structure. In the trivial bundle case we obtain actual embeddings, and moreover we can realize smooth knotting among these embeddings. As a consequence, we construct broad classes of Engel structures in $\mathbb{C}^3$, including infinitely many knotted embeddings that are pairwise non-isotopic but whose tangencies are Engel-homotopic.

Resumen

Esta tesis explora fenómenos de rigidez en el estudio de distribuciones no integrables, más allá del ámbito de las estructuras de contacto.

La primera parte se centra en una clase de distribuciones máximamente no integrables conocidas como distribuciones fat, que generalizan las estructuras de contacto a corango superior y admiten simplectizaciones y contactizaciones de forma natural.

En nuestro estudio de las distribuciones fat, introducimos las subvariedades prelegendrianas, que son una generalización de las Legendrianas en este contexto. En términos intuitivos, se definen como aquellas subvariedades que admiten un levantamiento Legendriano canónico en la contactización.

Nuestros principales resultados muestran que: (1) Existen infinitos encajes prelegendrianos exóticos en el espacio fat estándar de corango 2, $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$, todos formalmente isotópicos pero no isotópicos entre sí; estas prelegendrianas se distinguen mediante invariantes de curvas pseudoholomorfas de sus levantamientos Legendrianos. (2) Existe una noción de estabilización prelegendriana que produce levantamientos Legendrianos "loose"; esto genera nuevas familias de prelegendrianas exóticas. (3) Estos resultados pueden extenderse a variedades fat no estándar utilizando modelos de nilpotentización y argumentos de estabilidad.

En la segunda parte, investigamos la interacción entre la geometría Engel y la geometría compleja. Mostramos que los $\mathbb{S}^1$-fibrados orientados sobre 3-variedades admiten inmersiones co-reales en variedades casi complejas de dimensión 6, de modo que las tangencias complejas definen una estructura Engel. En el caso del fibrado trivial obtenemos un encaje, y además podemos realizar knotting entre dichos encajes. Como consecuencia, construimos amplias clases de estructuras Engel en $\mathbb{C}^3$, incluidas infinitos encajes anudados que son no isotópicos entre sí, pero cuyas tangencias son Engel-homotópicas.

Chapter 1

Introduction

1.1 Motivation and Overview

One of the central concepts in differential topology and geometry is the notion of *distribution*, that is, a smooth rank-r subbundle $\mathcal{D}^r \subseteq TM^n$ of the tangent bundle of an n-manifold M^n. Distributions appear ubiquitously: when $\mathcal{D}$ is integrable in the sense of Frobenius, it gives rise to a foliation, a cornerstone of classical topology. By contrast, when $\mathcal{D}$ is non-integrable, its local geometry is richer and sometimes (e.g. in the contact setting) it dictates global dynamical and topological features of M.

A distribution $\mathcal{D}$ is called *bracket–generating* if iterated Lie brackets of vector fields tangent to $\mathcal{D}$ eventually span the whole tangent bundle. The Chow–Rashevskii theorem then asserts that on a connected manifold endowed with such a distribution, any two points can be joined by a horizontal path tangent to $\mathcal{D}$. This observation underpins modern sub-Riemannian geometry and geometric control [Cho39, Mon02, ABB20].

A classical problem in differential topology is to classify bracket–generating (in particular, maximally non-integrable) distributions up to homotopy. The guiding dichotomy is between *flexibility* and *rigidity*. Gromov's h-principle provides a framework for flexibility: when a geometric condition is sufficiently underdetermined, *formal* (bundle-theoretic) solutions deform into *genuine* solutions, so classification up to deformation reduces to algebraic–topological data [Gro86, CEM24]. When this deformation principle fails, rigidity arises and one needs finer geometric invariants, beyond purely homotopical ones. A major obstacle to detecting rigidity for maximally non-integrable distributions has been the scarcity of such invariants. Concretely, no such invariants were known outside of contact topology.

In this thesis we address this gap by focusing on the class of corank–2 fat distributions. The main contribution of the thesis is that, in this setting, tools from symplectic and contact topology, most notably Gromov's theory of pseudoholomorphic curves, can be leveraged to build invariants and thus detect rigidity (the first example thereof outside of the realm of contact structures).

1.1.1 Contact geometry as the model case

Within the maximally non-integrable class, the most studied case is $(r, n) = (2m, 2m+1)$, corresponding to *contact* distributions. By Darboux's theorem all contact structures are locally equivalent; globally, however, they split into two regimes with sharply different behaviour.

In dimension three, Eliashberg introduced the subclass of *overtwisted* contact structures, detected semi-locally by the existence of a special embedded disk with prescribed characteristic foliation (an *overtwisted disk*). He proved an h-principle for this subclass, that is, overtwisted structures are classified up to isotopy by the homotopy class of the underlying oriented 2-plane field [Eli89]. This flexible picture holds in all odd dimensions by the work of Borman–Eliashberg–Murphy, who defined a higher–dimensional overtwisted disk and established the corresponding h-principle [BEM15].

The complementary regime is that of *tight* contact structures (those containing no overtwisted disk). Here one encounters genuine rigidity. A first striking result is due to Bennequin, who studied submanifolds adapted to contact geometry—*Legendrian* and *transverse* knots—and established the Thurston–Bennequin inequality in the standard contact space $(\mathbb{R}^3, \xi_{\text{std}})$; this certifies that $(\mathbb{R}^3, \xi_{\text{std}})$ (and $\mathbb{S}^3$ with its standard contact structure) is tight [Ben83]. Eliashberg later generalised tightness obstructions beyond the standard case [Eli92]. Nowadays one can certify tightness via *fillability*: if a contact manifold is symplectically fillable, then it is tight; in particular, Stein (equivalently, Weinstein) fillability implies tightness [Eli90a, Nie06, Pre07, NvK07, CE12, BEM15].

Building on this, the classification of Legendrian submanifolds has become a central theme: these objects carry both formal (topological) data and genuinely geometric invariants. The first exotic examples (formally isotopic but not Legendrian isotopic) of Legendrian knots in $(\mathbb{R}^3, \xi_{\text{std}})$ were constructed by Chekanov and Eliashberg using the pseudoholomorphic–curve techniques initiated by Gromov [Che02, Eli98, Gro85]. In higher dimensions, similar exotic phenomena were established in [EES05b].

On the other hand, when the ambient contact manifold M has dimension ≥ 5, Murphy introduced the semi-local class of *loose* Legendrians and proved a full h-principle for them [Mur19]; moreover, looseness is intimately related to overtwistedness of the ambient contact structure [CMP19].

1.1.2 Beyond contact structures

Two natural testing grounds beyond contact structures are *even-contact* and *Engel* distributions. Both share with contact geometry the feature of a unique local normal form and consequently admit no local invariants [Car10, Mon93].

For $(r, n) = (2m-1, 2m)$, corresponding to even-contact distributions, McDuff established a full h-principle via Gromov's convex integration, showing this class is flexible [McD87].

For Engel structures, i.e. the case $(r, n) = (2, 4)$, substantial flexibility has been discovered in recent years [Vog09, CPdPP17, CdPP20, dPV20], as well as for various kinds of submanifolds within them [Ada07, Gei08a, CdP18, dPP19, FMAP20, Gom25, Keg24]. Engel structures have also been explored from a wide range of geometric perspectives [Mon99, PSC14, BM17, Mit18, KV18, Yam21, Pia22]. Despite this rapid progress in the direction of flexible behaviour, the existence of robust rigidity phenomena in Engel topology is completely open and is a major problem in the area.

Further instances of flexibility for other non-integrable families appear in [Ada03, MAdP21, MA23]. In particular, beyond the contact realm, the presence and nature of rigidity are not understood at all. The reason for this is simple: all maximally non-integrable distributions, other than contact structures, lack global stability (in the sense of Gray-Moser), making it highly non-trivial to define invariants that are persistent under homotopies.

Nonetheless, guided by the flexibility/rigidity dichotomy in contact topology, we ask:

> **Central Question.** Do rigidity phenomena arise in other classes of maximally non-integrable distributions, or in their adapted submanifolds?

1.2 Contents of the thesis

In this thesis we address whether rigidity phenomena arise beyond contact geometry by focusing on a class of distributions called *fat distributions*, a notion originating in Weinstein's theory of *fat bundles* [Wei80, Mon02]. These generalize contact distributions to higher corank. In the case $(4, 6)$, fat distributions are precisely the *elliptic* distributions conjectured to exhibit rigidity in [MAdP21].

1.2.1 Fat distributions and prelegendrians

A distribution $\mathcal{D}^r \subset TM^m$ is *fat* if for every local 1–form α annihilating $\mathcal{D}$, the restriction $\mathrm{d}\alpha|_{\mathcal{D}}$ is symplectic. In this case we refer to $(M, \mathcal{D})$ as a *fat manifold*. When $\mathcal{D}$ has corank 1, the fatness condition is exactly the contact condition. From the definition it follows that the rank of a fat distribution is even, say $\operatorname{rank}\mathcal{D} = 2n$.

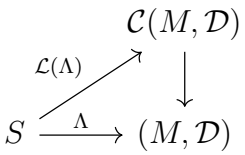
A key feature of fat distributions is the following.

Proposition. *Let $\mathcal{D} \subseteq TM$ be a fat distribution. Then there exists a canonical contact submanifold $\mathcal{C}(M, \mathcal{D}) \subseteq (\mathbb{S}T^*M, \xi_{\mathrm{can}})$, which is an isotropic fibration over $(M, \mathcal{D})$. We call $\mathcal{C}(M, \mathcal{D})$ the* contactisation *of $(M, \mathcal{D})$.*

Following the "Legendrian route" from contact topology to detect rigidity [Ben83, Eli92, CMP19], we introduce *prelegendrian* submanifolds in fat manifolds.

Definition. An $(n + k - 1)$–dimensional submanifold $\Lambda \subset (M^{2n+k}, \mathcal{D}^{2n})$ is *prelegendrian* if $\dim(T\Lambda \cap \mathcal{D}) = n$.

For every prelegendrian embedding $\Lambda : S \to (M, \mathcal{D})$, there exists a Legendrian embedding $\mathcal{L}(\Lambda) : S \to \mathcal{C}(M, \mathcal{D})$, called the *Legendrian lift* of Λ, such that the following diagram commutes:



That is, prelegendrians satisfy a Legendrian lifting property. This justifies the terminology. This construction is canonical; in particular, the Legendrian isotopy class of $\mathcal{L}(\Lambda)$ is a prelegendrian isotopy invariant, which we use to produce examples of exotic prelegendrians.

While the general theory applies in any corank, our constructions focus on a class of corank–2 fat manifolds that arise as a complex analogue of the space of contact elements. In particular, our fat manifolds are all $\mathbb{CP}^n$ or $\mathbb{C}^n$ bundles over some almost complex manifold. Among these, we single out the *standard* corank–2 fat space $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$. In this setting, our main result reads:

Theorem A (Theorem 2.2.2)**.** *There exist infinitely many prelegendrian embeddings*

$$\Lambda_s : \mathbb{T}^{2n+1} \hookrightarrow (\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}}), \qquad s \in \mathbb{Z}_{>0},$$

all formally prelegendrian isotopic. These prelegendrians are pairwise not prelegendrian isotopic, since their Legendrian lifts $\mathcal{L}(\Lambda_s)$ to the contactisation $\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$ are pairwise not Legendrian isotopic.

Since the ambient model arises from complex contact elements, there is a canonical front projection to the underlying almost complex manifold, and our construction proceeds by studying *prelegendrian fronts*. That is, fronts that represent prelegendrian submanifolds. We develop the basics of the theory of prelegendrian fronts on this thesis. After these tools are developed, our exotic prelegendrians are built by using a suitable front spinning procedure.

As a by–product of the proof, we define a *prelegendrian stabilisation* operation; this is the second main contribution of the work.

Theorem B. *Let $\Lambda \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$ be a prelegendrian, then there exist a prelegendrian $s(\Lambda)$ such that its Legendrian lift $\mathcal{L}(s(\Lambda))$ is loose.*

For a complete account, see Theorem 2.2.4, and as a consequence, we also construct exotic prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$ with topological types other than tori.

These results can be extended to fat distributions other than the model. The key observations are that: (1) Even though fat distributions themselves do not enjoy Gray stability, their contactisations do, and (2) Fat distributions do not have a unique local model, but there is a family of models that are approximate (and, in dimension 6, there is only one such model). These observations can be combined to show:

Theorem C. *Any fat manifold containing a point whose nilpotentization is standard contains compact prelegendrian submanifolds, which can also be stabilized.*

In dimension 6 the result applies to all fat manifolds (see Theorem 2.2.9). In full generality we cannot deduce that these prelegendrians are non-isotopic (since this is sensitive to the global properties of the fat distribution), but analogues of Theorem A and Theorem B hold if we restrict ourselves to fat distributions that are homotopic to the standard one via a compactly-supported deformation.

1.2.2 Engel structure as complex tangencies

Our construction of prelegendrian fronts involves the study of co-real submanifolds in almost complex manifolds. As we looked into these, we made the following observation: It is possible to give a sufficient condition for an $\mathbb{S}^1$-bundle over a 3-manifold to admit an immersion (or embedding) in $\mathbb{C}^3$ so that the complex tangencies define an Engel structure.

In doing so, we answer the following question due to Y. Eliashberg:

Question. Does there exist a compact 4-dimensional submanifold $N \subseteq \mathbb{C}^3$ whose complex tangencies $\mathcal{D}_J := TN \cap J(TN)$ form an Engel distribution?

This question was posed as well during the workshop in Engel Topology organised at AIM [Ame17b, Ame17a].

Concretely, we have proven the following:

Theorem D (Theorem 6.1.3)**.** *Let N be an oriented $\mathbb{S}^1$-bundle over a closed, oriented 3-manifold M. Then, there exists a co-real immersion $g : N \looparrowright \mathbb{C}^3$ whose complex tangencies $\mathcal{D}^g_{J_{\text{std}}}$ form an Engel distribution. If N is the trivial bundle, then g can be chosen to be an embedding.*

1.3 Guide to the content of thesis

The content of this document is based on the following two preprints:

- Eduardo Fernández, Álvaro del Pino, and Wei Zhou. The h-principle fails for prelegendrians in corank 2 fat distributions. To appear, 2025
- Eduardo Fernández, Álvaro del Pino, and Wei Zhou. Engel CR submanifolds of $\mathbb{C}^3$, 2025. https://arxiv.org/abs/2509.11488

Chapter 2. Overview and main results of Part I

In this chapter we give a preview of the first project. We introduce fat manifolds and prelegendrians and state their basic properties. We then present the main results obtained in this project and conclude with a brief guide to the remainder of Part I.

Chapter 3. Fat manifolds and spaces of complex contact elements

In this chapter we prove, in a general framework, the contactisation property of fat manifolds and the lifting property for prelegendrians. We also introduce the main object of focus in this part, the *space of complex contact elements* (the almost complex Grassmannian of complex hyperplanes), and compute its contactisation, which is canonically isomorphic to the space of (real) contact elements over the base manifold. Before doing so, we fix conventions for complex linear algebra and collect several results on co-real submanifolds that will be used in later chapters.

Chapter 4. Prelegendrian Fronts

From this point on we work in the almost complex Grassmannian. We define the *prelegendrian front* as the image of a prelegendrian under the bundle projection; since its contactisation is the space of contact elements, this image is also the front of its Legendrian lift. Accordingly, the first two sections recall basic facts about Legendrian fronts, and the third reviews the front-spinning construction, which is useful for producing high-dimensional fronts with simple singularities.

In the final two sections we introduce prelegendrian fronts and characterise a subclass of fronts that are prelegendrian. Moreover, we show that the standard operations on Legendrian fronts—such as stabilisation and front spinning—admit prelegendrian analogues in our setting.

Chapter 5. Existence of prelegendrian stabilisation and exotic prelegendrians

In this chapter we apply the theory developed in the preceding chapters to prove the existence of stabilisation (Theorem B). We then prove Theorem A, by constructing exotic prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$. Finally, using the key observation above, we prove Theorem C and derive its consequences.

Chapter 6. Engel structures as complex tangencies

This chapter begins the second project. In the first section we state the main results of this part. In the second, we review the necessary background, including: (1) the notion of prolongation, which is our main recipe for constructing Engel structures; (2) the role of convexity as a criterion for Engelness; and (3) tubular neighbourhood models around totally real submanifolds.

Next, we discuss the construction of convex bundle immersions, which provide a sufficient condition for an $\mathbb{S}^1$–bundle to admit a co-real immersion whose complex tangencies define an Engel structure. In the following section we prove that a convex

bundle immersion can always be realised as co-real with Engel complex tangencies. Finally, we combine these ingredients to prove Theorem D.

Chapter 7. Open questions and future work

In this chapter we conclude the thesis with a selection of open questions and directions for future work.

Part I

The h-principle fails for prelegendrians in corank 2 fat distributions

Chapter 2

Overview and main results of Part I

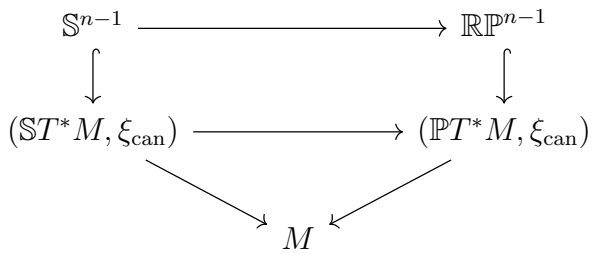
This chapter motivates and frames Part I. We first introduce all the key objects, without proofs. This includes fat manifolds and their contactisations, as well as the lifting of prelegendrian submanifolds to Legendrians in the contactisation.

We then state the main results to be proven in later chapters: (1) a characterisation of prelegendrian fronts via co-real singular loci, (2) a co-real spinning construction yielding infinitely many formally equivalent but non–prelegendrian-isotopic tori in the standard corank-2 fat space, and (3) the prelegendrian N-pushing procedure that interacts compatibly with Legendrian stabilization.

The chapter closes with a short guide to the rest of Part I.

2.1 Motivation and background

Let M^n be a smooth manifold. The space of co-oriented hyperplanes in M is $\mathbb{S}T^*M$, while the space of hyperplanes in M is $\mathbb{P}T^*M$. These manifolds carry canonical contact structures $(\mathbb{S}T^*M, \xi_{\text{can}})$ and $(\mathbb{P}T^*M, \xi_{\text{can}})$ known as the space of co-oriented contact elements, and contact elements. These contact manifolds are typical examples of *Legendrian fibrations* and fit in a commuting diagram



Here, the map $(\mathbb{S}T^*M, \xi_{\text{can}}) \to (\mathbb{P}T^*M, \xi_{\text{can}})$ is a contact double cover induced by the antipodal map on each Legendrian fibre.

Let $\mathcal{D}$ be a distribution on M. The *annihilator* of $\mathcal{D}$ is the subbundle

$$\mathcal{D}^\perp := \{\alpha \in T^*M : \alpha_{|\mathcal{D}} = 0\} \subseteq T^*M.$$

The space of co-oriented hyperplanes on M containing $\mathcal{D}$ is denoted by $\mathbb{S}\mathcal{D}^\perp \subseteq \mathbb{S}T^*M$, and the space of hyperplanes containing $\mathcal{D}$ by $\mathbb{P}\mathcal{D}^\perp \subseteq \mathbb{P}T^*M$.

In this work, we are interested in the following geometric class of maximally non-integrable distributions that generalize contact structures to higher corank:

Definition 2.1.1. Let $\mathcal{D}$ be a distribution in a smooth manifold M. We say that $\mathcal{D}$ is *fat*, and $(M, \mathcal{D})$ is a *fat manifold*, if one of the following equivalent conditions holds:

- For every local non-zero section $\alpha : U \to \mathcal{D}^\perp$, $\mathrm{d}\alpha|_{\mathcal{D}}$ is a symplectic form on $\mathcal{D}$.
- $\mathcal{C}(M, \mathcal{D}) := (\mathbb{S}\mathcal{D}^\perp, \xi_{\text{can}}) \subseteq (\mathbb{S}T^*M, \xi_{\text{can}})$ is a contact submanifold, called the *contactisation* of $(M, \mathcal{D})$.
- $\mathcal{C}_{\mathbb{P}}(M, \mathcal{D}) := (\mathbb{P}\mathcal{D}^\perp, \xi_{\text{can}}) \subseteq (\mathbb{P}T^*M, \xi_{\text{can}})$, is a contact submanifold, called the *projectivised contactisation* of $(M, \mathcal{D})$.
- $\mathcal{S}(M, \mathcal{D}) := (\mathcal{D}^\perp \setminus 0_M, \mathrm{d}\lambda_{\text{can}}) \subseteq (T^*M \setminus 0_M, \mathrm{d}\lambda_{\text{can}})$ is a symplectic submanifold, called the *symplectisation* of $(M, \mathcal{D})$. △

The equivalence between all these conditions will be proved later in this part, but is a well-known fact; see [Mon02].

From the definition it follows that the rank of $\mathcal{D}$ is $2n$ even, and that both contactisations fit in a commuting diagram of *isotropic fibrations*

$$\begin{array}{ccc} \mathbb{S}^{k-1} & \longrightarrow & \mathbb{RP}^{k-1} \\ \downarrow & & \downarrow \\ \mathcal{C}(M, \mathcal{D}) & \longrightarrow & \mathcal{C}_{\mathbb{P}}(M, \mathcal{D}) \\ & \searrow \quad \swarrow & \\ & (M, \mathcal{D}) & \end{array}$$

where k is the corank of $\mathcal{D}$. The map $\mathcal{C}(M, \mathcal{D}) \to \mathcal{C}_{\mathbb{P}}(M, D)$ is a contact double cover induced by the antipodal map on the isotropic fibres.

Remark 2.1.2. For a distribution ξ of corank 1 on a manifold M, being fat is the same as being contact. In this case, $\mathcal{C}_{\mathbb{P}}(M, \xi) = (M, \xi)$. $\mathcal{C}(M, \xi)$ is the co-orientable double cover of (M, ξ), which is disconnected if (M, ξ) is co-orientable. $\mathcal{S}(M, \xi)$ is the (complete)[1] symplectisation of (M, ξ). △

[1] In contact topology it is customary to assume that $\xi = \ker \alpha$ is co-oriented. The choice of contact form α defines a splitting $\mathcal{S}(M, \xi) \cong (M \times \mathbb{R}^*, d(t\alpha))$, and it is common to treat only the positive half $(M \times (0, +\infty), d(t\alpha))$ as the symplectisation of (M, ξ).

Remark 2.1.3. In the degenerate case $(M, \mathcal{D}) = (M, \{0\})$, we recover the space of co-oriented contact elements, resp. contact elements, as the contactisation, resp. projectivised contactisation, of $(M, \{0\})$. △

It follows that the contactisations of a fat manifold $(M, \mathcal{D})$ are compact whenever M is compact. In particular, *Gray stability* and its variants, such as the contact isotopy extension theorem, apply to them, yielding effective fat invariants:

Lemma 2.1.4. *Let $(M, \mathcal{D})$ be a compact fat manifold. The contactomorphism class of $\mathcal{C}(M, \mathcal{D})$ and $\mathcal{C}_{\mathbb{P}}(M, \mathcal{D})$ is invariant under homotopies of the underlying fat structure.*

This indicates that it is reasonable to pursue the use of symplectic and contact invariants—e.g., those arising from pseudoholomorphic curves—in order to establish *rigidity phenomena* for fat manifolds. In this thesis, we successfully follow this program to find rigidity for a certain class submanifolds of fat manifolds, called prelegendrians:

Definition 2.1.5. Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold. A $(n+k-1)$-submanifold $\Lambda^{n+k-1} \subseteq (M, \mathcal{D})$ is said to be *prelegendrian* if $\dim T\Lambda \cap \mathcal{D} = n$. △

Sometimes, when dealing with a prelegendrian submanifold Λ, we will abuse notation and treat it as an embedding by fixing some inclusion. This will be clear from context.

Remark 2.1.6. A generic Legendrian germ $L \subseteq \mathcal{C}(M, \mathcal{D})$ projects to a germ of a prelegendrian in $(M, \mathcal{D})$. This follows from the fact that a generic germ is transverse to the projection to $(M, \mathcal{D})$. △

Remark 2.1.7. Prelegendrian submanifolds of contact manifolds are precisely Legendrian submanifolds. △

The terminology *prelegendrian* is justified by the following lifting property (Proposition 3.1.13). Let $\Lambda : S \hookrightarrow (M^{2n+k}, \mathcal{D}^{2n})$ be a prelegendrian embedding. Then, there exists a unique[2] Legendrian embedding $\mathcal{L}(\Lambda) : N \hookrightarrow \mathcal{C}(M, \mathcal{D})$ making the following diagram commute:

$$\begin{array}{ccc} & & \mathcal{C}(M, \mathcal{D}) \\ & \overset{\mathcal{L}(\Lambda)}{\nearrow} & \downarrow \\ S & \xrightarrow{\Lambda} & (M, \mathcal{D}) \end{array}$$

An analogous property holds for the projective contactisation. We will refer to $\mathcal{L}(\Lambda)$ as *Legendrian lift*. The construction of the Legendrian lift is canonical. In particular, if $\mathcal{L}(\Lambda_0)$ and $\mathcal{L}(\Lambda_1)$ are not Legendrian isotopic, then Λ_0 and Λ_1 are not prelegendrian isotropic. The prelegendrian isotopy invariant that we will use to build our exotic prelegendrians is precisely the Legendrian lift.

[2]Being precise one should fix a *prelegendrian co-orientation*. We will discuss this in detail later in the next chapter. However, for all the prelegendrians treated in this thesis there will be always an obvious prelegendrian co-orientation.

Remark 2.1.8. The previously mentioned approach is somewhat reminiscent to the theory of Legendrian co-normal lifts to provide invariants of submanifolds, e.g. knot contact homology [OV00, Ng05, Ng14, She19]. In the degenerate case $(M, \{0\})$ prelegendrian submanifolds are just hypersurfaces, and its Legendrian lifts are obtained by treating them as *fronts* of Legendrians in $(\mathbb{S}T^*M, \xi_{\text{can}})$ and $(\mathbb{P}T^*M, \xi_{\text{can}})$, i.e. taking the co-normal lifts. △

2.2 Main results

Even though we will ultimately prove results for general fat manifolds of corank 2, many of our constructions deal with a concrete class thereof. Namely, the almost complex analogue of the space of (co-oriented) contact elements.

Almost complex grasssmannians

Let (X^{2n+2}, J) be an almost complex manifold, with $n \geq 1$. The space of J-complex hyperplanes on (X, J) is denoted by $\text{Gr}(X, J)$ and referred as the *complex Grassmannian* of (X, J). There is a natural structure of $\mathbb{CP}^n$-bundle $\pi_{\text{Gr}} : \text{Gr}(X, J) \to (X, J)$.

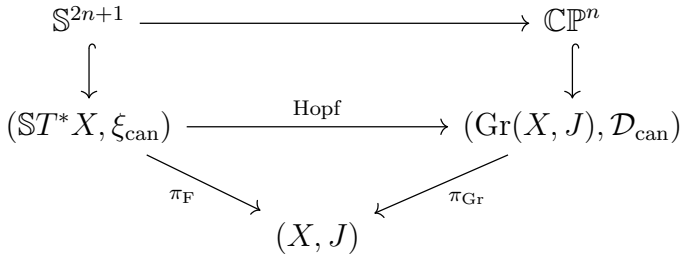
According to Proposition 3.3.5, there is a natural corank 2 fat distribution $\mathcal{D}_{\text{can}}$ on $\text{Gr}(X, J)$, defined as

$$\mathcal{D}_{\text{can}}(W) \coloneqq (\,\mathrm{d}\pi_{\text{Gr}})^{-1}(W), \qquad W \in \text{Gr}(X, J).$$

The fat manifold $(\text{Gr}(X, J), \mathcal{D}_{\text{can}})$ is referred to as the space of *J-complex contact elements* in (X, J).

The *standard fat distribution* $\mathcal{D}_{\text{std}}$ on $\mathbb{C}^{2n+1}$ is defined as follows. We consider $\text{Gr}(\mathbb{C}^{n+1}, J_{\text{std}}) = \mathbb{C}^{n+1} \times \mathbb{CP}^n$ and restrict ourselves to those elements in $\mathbb{CP}^n$ representing hyperplanes graphical over $\mathbb{C}^n \times \{0\}$. This subspace can be canonically identified with $\mathbb{C}^{n+1} \times \mathbb{C}^n$ by considering the slopes of such a hyperplane. Identically, we are considering the complement of a constant hyperplane section in $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ and thus passing to an affine chart in each fibre. The fat manifold $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$ will be called the *standard (corank 2) fat space*.

By construction, the contactisation of $(\text{Gr}(X, J), \mathcal{D}_{\text{can}})$ is the space of co-oriented contact elements $\mathcal{C}(\text{Gr}(X, J), \mathcal{D}_{\text{can}}) = (\mathbb{S}T^*X, \xi_{\text{can}})$, and there is a commuting diagram



where the *Hopf map* $(\mathbb{S}T^*X, \xi_{\text{can}}) \to (\text{Gr}(X, J), \mathcal{D}_{\text{can}})$ is defined by taking the Hopf fibration in each fibre.

Prelegendrian fronts

Let $\Lambda \subseteq (\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}})$ be a prelegendrian submanifold. Its *prelegendrian front* is the singular hypersurface $\pi_{\mathrm{Gr}}(\Lambda) \subseteq (X,J)$. It follows from the construction that the prelegendrian front is nothing but the front of the Legendrian lift:

$$\pi_{\mathrm{Gr}}(\Lambda) = \pi_{\mathrm{F}}(\mathcal{L}(\Lambda)).$$

Our first contribution is the characterisation of a class of prelegendrian fronts among all legendrian fronts:

Theorem 2.2.1. *Let $\mathcal{F} \subseteq (X,J)$ be a front with only self-intersections and cusp singularities. Then, $\mathcal{F}$ is a prelegendrian front if and only if its singularity loci are* co-real *in (X,J).*

This structural result allows for the study prelegendrians from a topological perspective, via their fronts. In particular, it allows us to prove that the class of compact prelegendrians $\Lambda \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{can}})$ is non-empty, which is something not obvious just from the definitions.

Co-real spinning

In view of Theorem 2.2.1, finding prelegendrians amounts to finding compact Legendrian fronts in $\mathcal{F} \subseteq \mathbb{C}^{n+1} = J^0\mathbb{R}^{2n+1}$ whose singularity loci are co-real. To achieve this, we will implement a (co-real) version of the front spinning construction introduced by Ekholm, Etnyre and Sullivan [EES05b] (see also [EES09, Gol14, LC13]).

Co-real spinning is surprisingly flexible and will allow us to go beyond the existence of compact prelegendrians. Indeed, we obtain the following more detailed version of Theorem A:

Theorem 2.2.2. *There exist infinitely many prelegendrian embeddings*

$$\Lambda_s : \mathbb{T}^{2n+1} \hookrightarrow (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}}), \qquad s \in \mathbb{Z}_{>0},$$

all formally prelegendrian isotopic. These prelegendrians are pairwise not prelegendrian isotopic, since their Legendrian lifts $\mathcal{L}(\Lambda_s)$ to the contactisation $\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ are pairwise not Legendrian isotopic.

The Legendrian lifts $\mathcal{L}(\Lambda_s)$ are distinguished by their Legendrian Contact Homology [Che02, EES05a, EES07, EGH00, Eli98]. That is, the h-principle fails for prelegendrian submanifolds in corank 2 fat distributions, and this failure can be detected by means of symplectic and contact invariants. To the best of our knowledge this is the first instance of rigidity in the study of maximally non-integrable distributions beyond the contact case.

Remark 2.2.3. In fact, we prove more: there exist infinitely many corank 2 fat manifolds that have infinitely many prelegendrians that are pairwise not prelegendrian isotopic but are formally prelegendrian isotopic. See Remark 5.2.5 for further details. △

Prelegendrian N-pushing

To realize the formal prelegendrian isotopy between the tori from Theorem 2.2.2, we will define an *N-pushing operation* along a co-real submanifold N of a prelegendrian front. This is an adaptation of the analogous Legendrian construction treated in [Eli90b, EES05b][3].

Recall that, given a Legendrian $L \subseteq (Y, \xi)$, with $\dim L \geq 2$, there is a notion of stabilized Legendrian $s(L)$, which is loose in the sense of Murphy, and formally equivalent to L, see [Eli90b, EES05b, Mur19]. This is obtained by performing an N-pushing front move along a submanifold with $\chi(N) = 0$ near a cusp chart.

It follows that a by-product of introducing prelegendrian N-pushing is that we can establish a stabilization procedure for prelegendrians in $\mathrm{Gr}(X, J)$. This leads to the second main result of this part, a detailed version of Theorem B:

Theorem 2.2.4. *Let $\Lambda \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ be a prelegendrian submanifold. Then, there exists a prelegendrian $s(\Lambda) \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$, such that*

- *Λ and $s(\Lambda)$ are formally prelegendrian isotopic by a C^0-small formal isotopy.*
- *Their Legendrian lifts satisfy $\mathcal{L}(s(\Lambda)) \cong s(\mathcal{L}(\Lambda))$.*

We refer to $s(\Lambda)$ as a prelegendrian stabilization *of Λ.*

Remark 2.2.5. The construction in Theorem 2.2.4 is a prelegendrian front-move and can be performed locally: one can arrange that Λ and $s(\Lambda)$ differ only inside an arbitrarily small ball $\mathcal{O}p\,(p)$ around a point $p \in \Lambda$. △

This construction suggests that there could be a class of loose prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{can}})$ that is governed by an h-principle. Investigating this question will be part of future work of the author.

On the other side, the prelegendrian stabilization gives a clear procedure to find exotic pairs of prelegendrians: build prelegendrians Λ so that $\mathcal{L}(\Lambda)$ is non-loose. This follows the same line of reasoning as in Theorem 2.2.2, since proving the non-looseness of a Legendrian is a non-trivial task that usually requires input from pseudoholomorphic curve theory (or its variants in generating functions, microlocal sheaf theory, etc).

Corollary 2.2.6. *There exists a prelegendrian embedding*

$$\Lambda : \mathbb{T}^{n+1} \times \mathbb{S}^{j_0} \times \cdots \times \mathbb{S}^{j_k} \hookrightarrow (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{can}}),$$

where $j_0 + \cdots + j_k = n$, such that $\mathcal{L}(\Lambda) \subseteq \mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ is non-loose. In particular, Λ and $s(\Lambda)$ are formally prelegendrian isotopic but not prelegendrian isotopic.

[3] In [EES05b] this operation is referred as stabilization, but we preferred to call it N-pushing and leave the word "stabilization" for the actual front move producing loose Legendrians, i.e. when is performed near a cusp chart.

Conormal lifts

We were unable to find exotic pairs of prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{can}})$ whose underlying smooth manifold is not a (non-trivial) product. However, we can achieve this in the more general case of the space of complex contact elements $(\text{Gr}(X, J), \mathcal{D}_{\text{can}})$. A natural prelegendrian submanifold is the one associated with a smooth real co-oriented hypersurface $M \subseteq (X, J)$, namely

$$\Lambda_M = (M, TM \cap JTM) \subseteq (\text{Gr}(X, J), \mathcal{D}_{\text{can}}).$$

The Legendrian lift of $\mathcal{L}(\Lambda_M) = (M, TM) \subseteq (\mathbb{S}T^*X, \xi_{\text{can}})$ is the co-normal lift of the hypersurface M associated to the given co-orientation.

Therefore, our prelegendrian stabilization from Theorem 2.2.4 implies:

Corollary 2.2.7. *Let $Y \subseteq (X, J)$ be a codimension 0 compact submanifold with connected boundary $M = \partial Y$. Then, Λ_M and $s(\Lambda_M)$ are formally prelegendrian isotopic but not prelegendrian isotopic.*

Remark 2.2.8. If we apply Corollary 2.2.7 to the case in which (X, J) is $\mathbb{C}^{n+1}$, we will produce prelegendrians in $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ that are homologically essential and thus not contained in $\mathbb{C}^{2n+1}$. The reason is that M is likely to have a complex Gauss map $M \to \mathbb{CP}^n$ that is homotopically non-trivial. Finding prelegendrians contained in the standard fat $\mathbb{C}^{2n+1}$ was precisely the motivation behind Theorem 2.2.2. △

General corank-2 fat distributions

It is tempting to think of $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$ as the *model* fat distribution of corank-2, but this is not correct. It turns out that there is no Darboux theorem in this setting. Nonetheless, $(\mathbb{C}^3, \mathcal{D}_{\text{std}})$ serves as an *approximate* model when $n = 1$. This is explained in Section 5.3, where we show:

Theorem 2.2.9. *Fix a fat distribution $(M^6, \mathcal{D}^4)$. Then:*

- *It contains compact prelegendrians.*
- *The prelegendrians it contains can be stabilised.*

This is the six–dimensional case of Theorem D. In higher dimensions we obtain a weaker statement by the same method; see Section 5.3 for details.

In a similar direction, consider the following problem: we fix a path of corank-2 fat distributions $(\mathcal{D}_s)_{s\in[0,1]}$, as well as a prelegendrian Λ for $\mathcal{D}_0$. There is no Gray stability in this setting, so there is no reason for a path of prelegendrians $(\Lambda_s)_{s\in[0,1]}$ in $\mathcal{D}_s$ to exist with $\Lambda_0 = \Lambda$. However:

Lemma 2.2.10. *If Λ is compact, a path Λ_s does exist for small s. In dimension 6 a path exists for all time if $\mathcal{D}_0 = \mathcal{D}_{\text{std}}$.*

On the side of rigidity, we moreover obtain:

Theorem 2.2.11. *Let Λ and Λ' be two distinct prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$, as produced by Theorem 2.2.2 or Corollary 2.2.6. Then there is no path $(\mathcal{D}_s, \Lambda_s)_{s\in[0,1]}$ of corank-2 fat distributions and prelegendrians such that:*

- $\mathcal{D}_0 = \mathcal{D}_1 = \mathcal{D}_{\text{std}}$.
- $\mathcal{D}_s$ *agrees with* $\mathcal{D}_{\text{std}}$ *outside of a compact.*
- $\Lambda_0 = \Lambda$ *and* $\Lambda_1 = \Lambda'$.

That is, rigidity persists even if we allow homotopies of the fat structures themselves.

2.3 Guide to the rest of Part I

Part I is organized as follows.

We begin in Section 3.1 by introducing the basic objects: fat manifolds and prelegendrians. Their fundamental properties are established, including the existence of contactisations and Legendrian lifts, and we also set up the framework of formal prelegendrian embeddings. Section 3.2 reviews co-real submanifolds in almost complex manifolds, which will play a central role in our constructions. In Section 3.3 we turn to examples, focusing in particular on corank-2 fat manifolds. Among them, the space of complex contact elements (or almost complex Grassmannians) will serve as the main class of models studied throughout this part, and we describe their contactisations explicitly.

The next three sections shift attention to the contact-geometric side: Section 4.1 recalls Legendrian fibrations and introduces the notion of simple fronts; Section 4.2 reviews stabilization and the concept of loose Legendrians; and Section 4.3 presents the front-spinning construction for producing higher-dimensional examples.

With this background in place, Section 4.4 introduces prelegendrian fronts. We establish a characterisation theorem (Theorem 2.2.1) and describe the basic Reidemeister moves. In the same section we define a prelegendrian spinning construction along co-real submanifolds, providing explicit examples in the standard model $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$. The subsequent section, Section 4.5, adapts Legendrian front operations to the prelegendrian setting, giving rise to the notion of prelegendrian stabilisation.

The existence of suitable co-real submanifolds is then proven in Section 5.1. This result yields compact prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$ and shows that prelegendrian stabilisation is a non-trivial operation (Theorem 2.2.4). These tools culminate in Section 5.2, where we construct exotic prelegendrians, including infinitely many formally isotopic but not prelegendrian isotopic tori (Theorem 2.2.2) and further corollaries.

Finally, in Section 5.3 we move beyond the standard model, extending the discussion to general corank-2 fat distributions. Using nilpotentisation and stability arguments,

we establish the existence of compact prelegendrians and prelegendrian stabilisations in this more general setting.

Chapter 3

Fat manifolds and spaces of complex contact elements

This chapter develops the general framework used throughout Part I. In Section 3.1 we introduce fat manifolds and prelegendrian submanifolds, prove the equivalence between fatness and the existence of contactisations/symplectisations (Theorem 3.1.6), and establish the (projective and co-oriented) Legendrian lift for preisotropic and prelegendrian maps (Proposition 3.1.13). We then turn to the almost complex viewpoint: Section 3.2 reviews co-real submanifolds and fixes notation in complex linear algebra, and Section 3.3 constructs corank-2 fat distributions from almost complex Grassmannians, identifies their contactisations with spaces of contact elements (Theorem 3.3.7), and extracts the standard model $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$ and its contactisation (Corollary 3.3.10). Finally, Section 3.3.5 records the affine-chart descriptions used later for fronts and local operation.

3.1 Fat manifolds

In this Section we introduce the main characters of the part I: *fat manifolds* and *prelegendrian submanifolds.* We prove the key properties about them: existence of contactisations (Theorem 3.1.6) and Legendrian lifts (Proposition 3.1.13). We conclude the section by introducing formal prelegendrian embeddings, which is necessary to set up the h-principle problem.

3.1.1 Fat Distributions

Let $\mathcal{D}^r \subseteq TM$ be a distribution of rank r on a smooth manifold M. The *Lie-bracket* of $\mathcal{D}$ with itself is the $C^\infty(M)$-module of vector fields

$$[\mathcal{D}, \mathcal{D}] \coloneqq \{[u_1, u_2] \,|\, u_1, u_2 \in \Gamma(\mathcal{D})\}.$$

Throughout this thesis we will always assume that $\mathcal{D}$ is *regular,* meaning that the module $[\mathcal{D}, \mathcal{D}]$ has constant rank, hence defines a distribution that we denote by the

same symbol.

The *curvature map* of $\mathcal{D}$ is the bundle morphism

$$\Omega : \mathcal{D} \wedge \mathcal{D} \to TM/\mathcal{D}, \quad (X, Y) \mapsto -[X, Y] \mod \mathcal{D}.$$

For each distribution $H \subseteq TM$ such that $\mathcal{D} \subseteq H$, there is also an associated *H-directional curvature*

$$\begin{aligned} \Omega_H : \mathcal{D} \times \mathcal{D} &\to TM/H \\ (X, Y) &\mapsto -[X, Y] \mod H. \end{aligned}$$

It is often convenient to dualize this picture. The annihilator of $\mathcal{D}$ is the subbundle

$$(TM/\mathcal{D})^* \simeq \mathcal{D}^\perp = \{\alpha \in T^*M : \alpha_{|\mathcal{D}} = 0\},$$

consisting of all covectors vanishing on $\mathcal{D}$. Dualizing Ω and using Cartan's formula, we obtain the *dual curvature* map

$$\Omega^* : \mathcal{D}^\perp \to \wedge^2 \mathcal{D}^*, \quad \alpha \mapsto -\alpha \circ \Omega = -\alpha \circ \Omega_{\ker \alpha} = \mathrm{d}\alpha|_{\mathcal{D}}.$$

This map encodes the same information as Ω, but from the perspective of 1-forms rather than vector fields.

The basic link between curvature and bracket-generating properties is well known:

Lemma 3.1.1. *Let $\mathcal{D}$ be a distribution on a manifold M. Then the following statements are equivalent:*

1. *$[\mathcal{D}, \mathcal{D}] = TM$, i.e. $\mathcal{D}$ is bracket-generating of step 2.*
2. *The dual curvature Ω^* is a monomorphism.*
3. *The curvature Ω is an epimorphism.*

We are interested in a special subclass of step-2 distributions whose curvature exhibits maximal non-degeneracy.

Definition 3.1.2. A distribution $\mathcal{D}$ of corank k on a manifold M is said to be *fat* at a point $p \in M$ if, for every non-vanishing 1-form $\alpha \in \mathcal{D}^\perp$ at p, the 2-form $\Omega^*(\alpha)$ defines a symplectic form on $\mathcal{D}_p$.

A distribution $\mathcal{D}$ is *fat* if it is fat at every point $p \in M$. The pair $(M, \mathcal{D})$ is called a *fat manifold*, whenever $\mathcal{D}$ is fat. △

In other words, $(M, \mathcal{D})$ is a fat manifold if and only if for every (local) non-zero section of $\alpha : U \to \mathcal{D}^\perp$, the vector bundle $(\mathcal{D}_{|U}, d\alpha)$ is symplectic. In particular, fat distributions of corank 1 are precisely contact distributions. It follows that the rank of a fat distribution $\mathcal{D}$ is always an even number $2n$. We will usually write $(M^{2n+k}, \mathcal{D}^{2n})$ to denote fat manifolds of corank k.

Remark 3.1.3. A fat distribution $\mathcal{D}^{2n}$ is necessarily bracket-generating of step 2, by Lemma 3.1.1. Moreover, fat distributions are maximally non-integrable in the following sense: The distribution $\mathcal{D}^{2n}$ is fat if and only if for every hyperplane distribution $H \subseteq TM$ with $\mathcal{D} \subseteq H$, the H-directional curvature

$$\Omega_H : \mathcal{D} \wedge \mathcal{D} \to TM/H, (X, Y) \mapsto -[X, Y] \mod H,$$

is a symplectic form with values in the line bundle TM/H. In the case $H = \ker(\alpha)$, $\Omega_H = \Omega^*(\alpha)$. In particular, fat distributions came equipped, at least locally, with a $\mathbb{S}^{k-1}$-family of conformal symplectic structures. △

Inspired by [BD23], we adopt their construction and refer to it as the *fatization*.

Example 3.1.4. Let $(W^{2n}, \lambda_1, \dots, \lambda_k)$ be a smooth manifold W^{2n} equipped with $(k-1)$-sphere of Liouville forms, i.e. a k-tuple $\lambda_1, \dots, \lambda_k \in \Omega^1 W$ of 1-forms such that

1. $\mathrm{d}\lambda_i$ is a symplectic form for all $i = 1, \dots, k$.
2. For every $(a_1, \dots, a_k) \in \mathbb{S}^{k-1}$ the combination $a_1 \,\mathrm{d}\lambda_1 + a_2 d\lambda_2 + \cdots + a_k \,\mathrm{d}\lambda_k$ is symplectic.

The *fatization of* $(W^{2n}, \lambda_1, \dots, \lambda_k)$ *is the corank* k *fat manifold*

$$(W \times \mathbb{R}^k, \mathcal{D} = \ker(\,\mathrm{d}z_1 - \lambda_1) \cap \cdots \cap \ker(\,\mathrm{d}z_k - \lambda_k)),$$

where $(z_1, \dots, z_k)$ *are the standard coordinates on* $\mathbb{R}^k$*. To produce compact examples one can moreover quotient* $\mathbb{R}^k$ *to a torus* $\mathbb{T}^k$*.*

The fatization is a fat manifold. Indeed, every non-zero form $\alpha \in \mathcal{D}^\perp$*, can be uniquely expressed as*

$$\alpha = ta_1(\,\mathrm{d}z_1 - \lambda_1) + ta_2(\,\mathrm{d}z_2 - \lambda_2) + \cdots ta_2(\,\mathrm{d}z_k - \lambda_k), (t, (a_1, \dots, a_k)) \in (0, \infty) \times \mathbb{S}^{k-1}.$$

From here it follows that $\mathrm{d}\alpha = t(a_1 \,\mathrm{d}\lambda_1 + \cdots a_k \,\mathrm{d}\lambda_k)$ *is symplectic on* $\mathcal{D} \cong TW$*.*

A concrete example is $\mathbb{R}^4$*. It admits a natural* 2*-sphere of Liouville forms coming from the complex structures* i, j *and* k *given by the unit quaternions. More generally, exact hyperkähler manifolds provide a natural source of examples.*

In the non-exact setting, one can consider a k*-tuple of symplectic forms such that any combination is symplectic. If all the forms in the tuple are of integral class, one can consider the associated* $\mathbb{S}^1$*-bundles. Taking their product produces a* $\mathbb{T}^k$*-bundle with fat connection.* △

Beyond the contact case, i.e. corank 1, fatness is rare and highly constrained. The following result provides a precise set of numerical restrictions:

Theorem 3.1.5 (Rayner [Ray68])**.** *Let* $\mathcal{D}^r$ *be a distribution of positive rank* r *and corank* k *on a manifold* M^{r+k}*. If* $\mathcal{D}^r$ *is fat, then the following numerical constraints hold:*

- *r is even and $r \geq k+1$.*
- *If $k \geq 2$, then 4 divides r.*
- *The $(r-1)$-sphere S^{r-1} admits k-many linearly independent vector fields.*

Conversely, given any pair (r,k) satisfying the above conditions, there is a germ of fat distribution of rank r and corank k.

3.1.2 Contactisation and symplectisation of a fat manifold

Given a manifold M, we denote by λ_{can} the canonical Liouville 1-form in T^*M. The manifold T^*M carries a natural $\mathbb{R}^*$-action (resp. $\mathbb{R}^+$-action), induced by the associated Liouville vector field. The quotient of $T^*M \setminus 0$ under this action is the space $\mathbb{P}T^*M$ of hyperplanes in M (resp. the space $\mathbb{S}T^*M$ of co-oriented hyperplanes in M). Since the action conformally expands the symplectic form, it defines a canonical contact structure $(\mathbb{P}T^*M, \xi_{\text{can}})$ known as *space of contact elements on M (*resp. $(\mathbb{S}T^*M, \xi_{\text{can}})$ known as *space of co-oriented contact elements*).

Consider the bundle projections $\pi_{\mathbb{P}} : \mathbb{P}T^*M \to M$ and $\pi : \mathbb{S}T^*M \to M$. Then, the canonical contact structure on $\mathbb{P}T^*M$ is

$$\xi_{\text{can}}(\alpha) = (\,\mathrm{d}_\alpha \pi_{\mathbb{P}})^{-1}(\ker \alpha), \quad \alpha \in \mathbb{P}T^*M.$$

The canonical contact structure on $\mathbb{S}T^*M$ is $\xi_{\text{can}} = \ker \lambda_{\text{can}}$, hence at a point $\alpha \in \mathbb{S}T^*M$ is also given by $\xi_{\text{can}}(\alpha) = (\,\mathrm{d}_\alpha \pi)^{-1}(\ker \alpha)$. The canonical double covering $(\mathbb{S}T^*M, \xi_{\text{can}}) \to (\mathbb{P}T^*M, \xi_{\text{can}})$ is the co-oriented double contact cover of $(\mathbb{P}T^*M, \xi_{\text{can}})$, and commutes with the bundle projections.

Fix a distribution $\mathcal{D} \subseteq TM$ on M of rank k. Its annihilator $\mathcal{D}^\perp \subseteq T^*M$ is preserved by the previous actions. Thus, we define the $\mathbb{RP}^{k-1}$-subbundle $\mathbb{P}\mathcal{D}^\perp \subseteq \mathbb{P}T^*M$ (resp. the $\mathbb{S}^{k-1}$-subbundle $\mathbb{S}\mathcal{D}^\perp \subseteq \mathbb{S}T^*M$). Notice that $\mathbb{P}\mathcal{D}^\perp$ (resp. $\mathbb{S}\mathcal{D}^\perp$) is nothing but the space of hyperplanes (resp. co-oriented hyperplanes) in M that contain $\mathcal{D}$. The bundle projections will be denoted as before.

The following characterisation of fatness is the key input we will use throughout this part:

Theorem 3.1.6. *Let $\mathcal{D} \subseteq TM$ be a distribution on M. The following are equivalent:*

1. *$\mathcal{D}$ is fat.*
2. *$\mathcal{D}^\perp \setminus 0_M \subseteq (T^*M, \mathrm{d}\lambda_{\text{can}})$ is a symplectic submanifold.*
3. *$\mathbb{P}(\mathcal{D}^\perp) \subseteq (\mathbb{P}T^*M, \xi_{\text{can}})$ is a contact submanifold.*
4. *$\mathbb{S}T^*M \subseteq (\mathbb{S}T^*M, \xi_{\text{can}})$ is a contact contact submanifold.*

Proof. The equivalence between (2), (3) and (4) is a standard fact in contact and symplectic geometry, c.f. [CEM24, Gei08b]. We prove the equivalence between (1) and (2): $\mathcal{D}^\perp \setminus 0_M \subseteq T^*M$ is a symplectic if and only if $\mathcal{D}$ is fat.

Write $m = \dim(M)$, $k = \operatorname{corank}(\mathcal{D})$ and $r = \operatorname{rank}(\mathcal{D})$. Then, $\dim(\mathcal{D}^\perp) = m + k = r + 2k$ is even if and only if r is even. From now on, we will assume that $r = 2n$ is even, hence $\dim(\mathcal{D}^\perp) = 2(n + k)$.

Let $\mathbf{q} = (q_1, \dots, q_m)$ be a local coordinate chart on an open set $U \subseteq M$. Trivialize the distribution $\mathcal{D}$ on U by a collection of 1-forms $\{\alpha_1, \dots, \alpha_k\}$ such that:

$$\mathcal{D} \overset{\text{loc.}}{=} \ker \alpha_1 \cap \cdots \cap \ker \alpha_k.$$

That is, the collection $\{\alpha_1, \dots, \alpha_k\}$ forms a local frame for $\mathcal{D}^\perp$. Denote the corresponding fibre coordinates by $(a_1, \dots, a_k)$.

Hence, $(\mathbf{q}; a_1, \dots, a_k)$ gives a local coordinates on $\mathcal{D}^\perp$. The canonical Liouville 1-form on T^*M restricts to:

$$\lambda = \sum_{i=1}^{k} a_i \alpha_i,$$

and its differential is:

$$\omega \coloneqq \mathrm{d}\lambda = \mathrm{d}\Big(\sum_{i=1}^{k} a_i \alpha_i\Big) = \sum_{i=1}^{k} \Big(\mathrm{d}a_i \wedge \alpha_i + a_i \,\mathrm{d}\alpha_i\Big)$$

Since the 1-forms α_i are independent of the fibre coordinates a_i, the maximal wedge power of ω has the form:

$$\omega^{(n+k)} = c \bigwedge_{i=1}^{k} \big(\mathrm{d}a_i \wedge \alpha_i\big) \wedge \big(\sum_{i=1}^{k} a_i \,\mathrm{d}\alpha_i\big)^n. \tag{3.1}$$

Realize that $T\mathcal{D}^\perp \cong TM \oplus \mathrm{Vert} \cong \mathcal{D} \oplus TM/\mathcal{D} \oplus \mathrm{Vert}$. The first term in (3.1) is always volume form on the subspace $TM/\mathcal{D} \oplus \mathrm{Vert}$. Therefore, ω is symplectic in $\mathcal{D}^\perp \setminus 0_M$ if and only if the second term in (3.1) is a volume form on $\mathcal{D}^{2n}$ for all $(a_1, \dots, a_k) \neq (0, \dots, 0)$, which is precisely the definition of fatness. This concludes the proof. □

Definition 3.1.7. Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold.

1. The *projectivised contactisation* of $(M, \mathcal{D})$ is $\mathcal{C}_{\mathbb{P}}(M, \mathcal{D}) \coloneqq (\mathbb{P}\mathcal{D}^\perp, \xi_{\text{can}})$,
2. The (co-oriented) *contactisation* of $(M, \mathcal{D})$ is $\mathcal{C}(M, \mathcal{D}) \coloneqq (\mathbb{S}\mathcal{D}^\perp, \xi_{\text{can}})$,
3. The *symplectisation* of $(M, \mathcal{D})$ is $\mathcal{S}(M, \mathcal{D}) \coloneqq (\mathcal{D}^\perp \setminus 0_M, \mathrm{d}\lambda_{\text{can}})$. △

3.1.3 Preisotropic and prelegendrian submanifolds

Let us introduce a class of submanifolds of fat manifolds that is specially adapted to the geometry:

Definition 3.1.8. Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold. A submanifold $\Lambda \subseteq (M^{2n+k}, \mathcal{D}^{2n})$ is said to be *preisotropic* if

$$(T\Lambda + \mathcal{D})_{|\Lambda} \subseteq TM_{|\Lambda}$$

has constant corank 1. △

Equivalently, $\Lambda \subseteq (M^{2n+k}, \mathcal{D}^{2n})$ is preisotropic if

$$\dim(T\Lambda \cap \mathcal{D}) = \dim(\Lambda) - k + 1.$$

A *preisotropic immersion* is a parametrization of a possibly immersed preisotropic submanifold, and a *preisotropic embedding* is a parametrization of an embedded preisotropic submanifold.

Remark 3.1.9. In the contact case $(M^{2n+1}, \mathcal{D}^{2n})$, preisotropic submanifolds are precisely isotropic submanifolds. △

Every preisotropic submanifold $\Lambda \subseteq (M^{2n+k}, \mathcal{D}^{2n})$ carries a hyperplane distribution

$$H_\Lambda \coloneqq (T\Lambda + \mathcal{D})_{|\Lambda},$$

that we will refer to as *preisotropic hyperplane.* A *preisotropic co-orientation* is a co-orientation of H_Λ, i.e. an orientation of the line bundle $l_\Lambda \coloneqq TM_{|\Lambda}/H_\Lambda$. In the examples of preisotropics treated in this thesis there will be always an obvious preisotropic co-orientation unless otherwise stated.

Remark 3.1.10. In the contact case $(M^{2n+1}, \mathcal{D}^{2n})$, the preisotropic hyperplane is the contact structure, $H_\Lambda = \mathcal{D}_{|\Lambda}$. In particular, in co-oriented contact manifolds every preisotropic carries a natural preisotropic co-orientation. △

Since $(M^{2n+k}, \mathcal{D}^{2n})$ is fat, the curvature of $\mathcal{D}$ along the preisotropic hyperplane H_Λ defines a symplectic form

$$\Omega_{H_\Lambda} : D_{|\Lambda} \wedge \mathcal{D}_{|\Lambda} \mapsto l_\Lambda$$

with values on the line bundle l_Λ.

Lemma 3.1.11. *Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold and $\Lambda \subseteq (M^{2n+k}, \mathcal{D}^{2n})$ a preisotropic submanifold. Then, $T\Lambda \cap \mathcal{D}_{|\Lambda} \subseteq (\mathcal{D}_{|\Lambda}, \Omega_{H_\Lambda})$ is an isotropic sub-bundle. In particular,* $\mathrm{rk}(T\Lambda \cap \mathcal{D}) \leq n$, *equivalently,* $\dim(\Lambda) \leq n + k - 1$.

Proof. The computation is local so we may assume that Λ has a fixed preisotropic co-orientation, i.e $H_\Lambda = \ker \alpha$ for some 1-form α, and identify $\Omega_{H_\Lambda} = \mathrm{d}\alpha$. Since $\iota^*\alpha = 0$, for the inclusion map $\iota : \Lambda \hookrightarrow M$, we conclude that $\iota^* \mathrm{d}\alpha = 0$ on $T\Lambda \cap \mathcal{D}$, from which the result follows. □

This motivates the following definition:

Definition 3.1.12. A preisotropic submanifold $\Lambda \subseteq (M^{2n+k}, \mathcal{D}^{2n})$ is said to be *subcritical* if $\dim(\Lambda) < n + k - 1$. A *prelegendrian submanifold* is a preisotropic submanifold Λ of the maximal dimension $\dim(\Lambda) = n + k - 1$. △

Isotropic and Legendrian lifts to the contactisation

The following lifting property justifies the terminologies *preisotropic* and *prelegendrian:*

Proposition 3.1.13. *Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold.*

- *If $u : S \to (M^{2n+k}, \mathcal{D}^{2n})$ is a preisotropic immersion, then there exists a unique isotropic immersion $\widetilde{u} : S \to \mathcal{C}_{\mathbb{P}}(M, \mathcal{D})$, that makes the following diagram commute:*

$$\begin{array}{ccc} & & \mathcal{C}_{\mathbb{P}}(M,\mathcal{D}) \\ & \overset{\hat{u}}{\nearrow} & \downarrow \\ S & \xrightarrow{u} & (M,\mathcal{D}) \end{array} \tag{3.2}$$

- *If $u : S \to (M^{2n+k}, \mathcal{D}^{2n})$ is a co-oriented preisotropic immersion, then there exists a unique isotropic immersion $\widetilde{u} : S \to \mathcal{C}(M, \mathcal{D})$, that makes the following diagram commute:*

$$\begin{array}{ccc} & & \mathcal{C}(M,\mathcal{D}) \\ & \overset{\tilde{u}}{\nearrow} & \downarrow \\ & & \mathcal{C}_{\mathbb{P}}(M,\mathcal{D}) \\ & \overset{\hat{u}}{\nearrow} & \downarrow \\ S & \xrightarrow{u} & (M,\mathcal{D}) \end{array} \tag{3.3}$$

Moreover, if u is an embedding, then $\hat{u}$ and $\tilde{u}$ are also embeddings.

Proof. Recall that $\mathcal{C}_{\mathbb{P}}(M, \mathcal{D}) = (\mathbb{P}\mathcal{D}^{\perp}, \xi_{\text{can}}) \subseteq (\mathbb{P}T^*M, \xi_{\text{can}})$ while $\mathcal{C}(M, \mathcal{D}) = (\mathbb{S}\mathcal{D}^{\perp}, \xi_{\text{can}}) \subseteq (\mathbb{S}T^*M, \xi_{\text{can}})$. We denote the projections by $\pi_{\mathbb{P}} : \mathcal{C}_{\mathbb{P}}(M, \mathcal{D}) \to (M, \mathcal{D})$ and $\pi : \mathcal{C}(M, \mathcal{D}) \to (M, \mathcal{D})$ respectively.

For each $p \in S$, since u is preisotropic there is a unique 1-form $\hat{\alpha}_p \in \mathcal{D}^{\perp} = \text{Ann}(\mathcal{D})$ defined up to $\mathbb{R}^*$-scaling, such that

$$\mathrm{d}u(T_pN) + \mathcal{D}_p = \ker \tilde{\alpha}_p.$$

In the case where u is co-oriented preisotropic, there is a unique $\tilde{\alpha}_p$, defined up to positive $\mathbb{R}^+$-scaling, that induces the given co-orientation satisfying the same property.

The lifts are defined by $\hat{u}(p) \coloneqq [\hat{\alpha}_p]$ and $\tilde{u}(p) \coloneqq [\tilde{\alpha}_p]$, respectively. Differentiating the identities $\pi_{\mathbb{P}} \circ \hat{u} = u$ and $\pi \circ \tilde{u} = u$, we obtain:

$$\mathrm{d}\pi_{\mathbb{P}}(\mathrm{Im}\ \mathrm{d}\hat{u}_p) = \mathrm{Im}\ \mathrm{d}u_p \subseteq \ker \hat{\alpha}_p$$

and

$$\mathrm{d}\pi(\mathrm{Im}\ \mathrm{d}\tilde{u}_p) = \mathrm{Im}\ \mathrm{d}u_p \subseteq \ker \tilde{\alpha}_p.$$

From here it follows that both maps are isotropic immersions. Indeed, the immersion property is obvious. For the isotropic property, realize that at the point $[\hat{\alpha}_p] \in \mathbb{P}\mathcal{D}^{\perp}$ the canonical contact structure is defined as $\xi_{\mathrm{can}}([\hat{\alpha}_p]) = d\pi_{\mathbb{P}}^{-1} \ker(\hat{\alpha}_p)$, the same argument applies in the co-oriented case. □

Definition 3.1.14. Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold and $u : S \to (M, \mathcal{D})$ a (co-oriented) preisotropic immersion. The isotropics $\mathcal{I}_{\mathbb{P}}(u) \coloneqq \hat{u} : S \to \mathcal{C}_{\mathbb{P}}(M, \mathcal{D})$ and $\mathcal{I}(u) \coloneqq \tilde{u} : S \to \mathcal{C}(M, \mathcal{D})$, from the proposition above, are called the *projective isotropic lift* and the *isotropic lift* of u, respectively. △

In the prelegendrian case, we will use the notation $\mathcal{L}_{\mathbb{P}}(u) = \mathcal{I}_{\mathbb{P}}(u)$ and $\mathcal{L}(u) = \mathcal{I}(u)$ to denote the lifts, and we will refer to them as *Legendrian lifts.* When dealing with submanifolds (not maps) we will use the same notation to denote the lifts, i.e. if $\Lambda \subseteq (M, \mathcal{D})$ is preisotropic, resp. prelegendrian, we will write $\mathcal{I}(\Lambda)$, resp. $\mathcal{L}(\Lambda)$, to denote the lifts, and similarly for the projective ones.

The construction of the lift is continuous for families of preisotropics. In particular, the isotropic lifts produce effective invariants to distinguish preisotropics. For later use, we highlight the following:

Corollary 3.1.15. *Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold, and $\Lambda_0, \Lambda_1 : S \hookrightarrow (M, \mathcal{D})$ two (co-oriented) prelegendrian embeddings. Then,*

- *If $\mathcal{L}_{\mathbb{P}}(\Lambda_0)$ and $\mathcal{L}_{\mathbb{P}}(\Lambda_1)$ are not Legendrian isotopic, then Λ_0 and Λ_1 are not prelegendrian isotopic.*
- *If $\mathcal{L}(\Lambda_0)$ and $\mathcal{L}(\Lambda_1)$ are not Legendrian isotopic, then Λ_0 and Λ_1 are not co-oriented prelegendrian isotopic.*

Remark 3.1.16. In practice, we will want to distinguish prelegendrians without the co-orientation information by their lift to the (co-oriented) contactisations. For this, one would need to distinguish the set of all possible co-oriented Legendrian lifts. When the prelegendrian is connected there are just two different lifts, and both are related by the co-orientation reversing contactomorphism of $\mathcal{C}(M, \mathcal{D})$ given the antipodal action in the $\mathbb{S}^{k-1}$-fibres. Therefore, it is enough to distinguish $\mathcal{L}(\Lambda_0)$ and $\mathcal{L}(\Lambda_1)$ by means of a Legendrian invariant preserved by this action, to conclude that Λ_0 and Λ_1 are not prelegendrian isotopic. All the Legendrian invariants treated in this work satisfy that property. △

3.1.4 Formal preisotropics and prelegendrians

The study of preisotropic immersions in fat distributions was first carried out by Gromov [Gro96], who referred to them as *partially horizontal immersions*, and was subsequently developed by Bhowmick [Bho22], who completed the h-principle analysis in certain cases via the Hamilton–Moser implicit function technique.

To state the h-principle and, at the same time, to prepare for the exotic prelegendrian examples to which we return later, we introduce the following notion, which encodes the formal (i.e. topological) data of being preisotropic.

Definition 3.1.17. Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold, and let S be a smooth manifold.

1. A *formal preisotropic immersion* is a pair (f, F), such that:
   - $f : S \to M$ is a smooth map,
   - $F : TS \to TM$ is a bundle monomorphism covering f such that
     (a) $H_p = F(T_pS) + D_{f(p)}$ has corank 1, for all $p \in S$, and
     (b) $F(T_pS) \cap \mathcal{D}_{f(p)} \subseteq (\mathcal{D}_{f(p)}, \Omega_{H_p})$ is an isotropic subspace, for all $p \in S$.
2. A *formal preisotropic embedding* is a pair (f, F_s) such that:
   - $f : S \to M$ is an embedding,
   - $F_s : TN \to TM$ with $s \in [0, 1]$ is a 1-parameter family of bundle monomorphisms covering f such that $F_0 = \mathrm{d}f$,
   - (f, F_1) is a formal preisotropic immersion.
3. A *formal prelegendrian immersion (resp. embedding)* is a formal preisotropic immersion (resp. embedding) of dimension $n + k - 1$. △

Regarding preisotropic immersions there is the following h-principle:

Theorem 3.1.18. *[Bho22, Theorem 5.3] Let $(M^{2n+k}, \mathcal{D}^{2n})$ be a fat manifold and let S be a smooth manifold. Then, for any subcritical formal preisotropic immersion (f, F) from S to $(M, \mathcal{D})$, there exists a homotopy of formal preisotropic immersions (f_t, F_t) such that:*

- *$f_0 = f$ and $F_0 = F$.*
- *f_1 is a preisotropic immersion.*
- *$F_1 = \mathrm{d}f_1$.*

In the prelegendrian case, the same statement holds provided that S is an open manifold.

Remark 3.1.19. Recall that subcritical isotropic embeddings in contact manifolds satisfy an h-principle [CEM24], via the h-principle for isocontact embeddings in codimension ≥ 4. By contrast, Bhowmick establishes the result above using microflexibility and microextension arguments, which yield immersions rather than embeddings. By analogy with the rigid/flexible dichotomy in contact topology, we pursue the failure of the h-principle for prelegendrian embeddings. The h-principle problem for subcritical preisotropic embeddings and prelegendrian immersions remains to be explored. △

3.2 Co-real submanifolds in almost complex manifolds

In this section we recall basic facts about linear complex algebra, almost-complex manifolds, and co-real submanifolds that we will use in subsequent sections. We need them to 1) construct examples of corank-2 fat manifolds, and 2) build examples of compact prelegendrians in these fat manifolds. This section also serves to fix notation.

3.2.1 Complex linear algebra

Throughout this subsection we fix a real vector space V of dimension $2n$ endowed with a *complex structure* $J : V \to V$, thereby turning V into a complex vector space. There is a dual complex structure J^* on V^* defined as $J^*(\alpha) = \alpha \circ J$.

Definition 3.2.1. Given (V, J):

- A subspace $W \subseteq V$ is called J-invariant if $JW = W$.
- A $\mathbb{C}$-valued $\mathbb{R}$-linear map $\beta : V \to \mathbb{C}$ is called J-linear if $\beta \circ J = i\,\beta$.
- For a subspace $H \subseteq V$, its complex part is the J-invariant subspace $H^J := H \cap JH$.
- A subspace $S \subseteq V$ is called *real* if $S^J = \{0\}$.
- A subspace $S \subseteq V$ called *co-real* if $S + JS = V$; equivalently, the annihilator $S^\perp \subseteq V^*$ is real.
- The complexification of a linear map $\alpha : V \to \mathbb{R}$ is the J-linear map $\alpha^J := \alpha - i\,\alpha \circ J$. △

Note that if S is real then $\dim S \leq n$, while if S is co-real then $\dim S \geq n$.

The following lemma will be used later to build co-real submanifolds.

Lemma 3.2.2. *Let $S \subseteq V$ be co-real, and $H \subseteq V$ be any subspace. Then, $S + H$ is co-real.*

Proof. Since $S + JS = V$, we have

$$(S + H) + J(S + H) = S + JS + H + JH = V,$$

so $S + H$ is co-real. □

We denote the real and complex Grassmannians of V, respectively, by:

$$\mathrm{Gr}_k(V) \coloneqq \{H \subseteq V : H \text{ is a } k\text{-dimensional subspace}\},$$
$$\mathrm{Gr}_{2k}(V, J) \coloneqq \{W \subseteq V : W \text{ is } J\text{-invariant}\} \subseteq \mathrm{Gr}_{2k}(V).$$

Since we will mostly work with (real or complex) hyperplane Grassmannians, we will write $\mathrm{Gr}(V) \coloneqq \mathrm{Gr}_{2n-1}(V)$ and $\mathrm{Gr}(V, J) = \mathrm{Gr}_{2n-2}(V, J)$, whenever there is no possible confusion. Sometimes it is more convenient to work with the dual picture, called Grassmannian duality:

Lemma 3.2.3. *There are canonical bijections:*

$$\mathrm{Gr}_k(V) \to \mathrm{Gr}_{2n-k}(V^*)$$
$$H \mapsto H^\perp,$$

and

$$\mathrm{Gr}_{2k}(V, J) \to \mathrm{Gr}_{2n-2k}(V^*, J^*)$$
$$W \mapsto W^\perp.$$

The *Hopf map* is defined as:

$$\mathrm{Hopf} : \mathrm{Gr}_{2n-1}(V) \to \mathrm{Gr}_{2n-2}(V, J)$$
$$H \mapsto H^J.$$

Equivalently, $\mathrm{Hopf}(H)$ is the maximal J-invariant subspace contained in H. This is well defined since every real hyperplane is co-real. Applying Grassmannian duality and identifying with (real/complex) projectivisation, we obtain the dual Hopf map:

$$\mathrm{Hopf}^* : \mathrm{Gr}_1(V^*) \simeq \mathbb{P}(V^*) \to \mathbb{P}_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})) \simeq \mathrm{Gr}_2(V^*, J^*)$$
$$[\alpha]_{\mathbb{R}} \mapsto [\alpha^J]_{\mathbb{C}}$$

In the case $V = \mathbb{R}^{2n} \cong \mathbb{C}^n$ (i.e. $J = J_{\mathrm{std}}$), the above dual Hopf map yields the standard Hopf fibration:

$$\pi_{\mathrm{Hopf}} : \mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1}$$
$$[p_1 : \cdots : p_{2n}]_{\mathbb{R}} \mapsto [p_1 + i\, p_2 : \cdots : p_{2n-1} + i\, p_{2n}]_{\mathbb{C}}$$

With the notation above we recall the following linear-algebra lemma, that will be used later.

Lemma 3.2.4. *Let $W \in \mathrm{Gr}_{2n-2}(V)$ be a subspace of codimension 2, and let $\alpha_1, \alpha_2 \in V^*$ be linear maps with $W = \ker \alpha_1 \cap \ker \alpha_2$. Then W is J-invariant if and only if α_1^J and α_2^J are $\mathbb{C}$-linearly dependent in $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.*

Proof. By Grassmannian duality (Lemma 3.2.3), W is J-invariant if and only if the plane $\mathrm{span}_{\mathbb{R}}\{\alpha_1, \alpha_2\} \subseteq V^*$ is J^*-invariant (i.e. a complex line), which is equivalent to α_1^J and α_2^J being $\mathbb{C}$-linearly dependent. □

3.2.2 Co-real submanifolds

Let (X, J) be an almost complex manifold of real dimension $2n$, i.e. $J : TX \to TX$ is a bundle isomorphism with $J_p : T_pX \to T_pX$ making T_pX a complex vector space for each $p \in X$.

Definition 3.2.5. An immersion $f : S \to (X, J)$ is called *co-real* if for every $q \in S$ the subspace $\mathrm{d}f_q(T_qS) \subseteq T_{f(q)}X$ is co-real (with respect to $J_{f(q)}$).

A submanifold $S \subseteq (X, J)$ is called *co-real* if $TS \subseteq TX$ is a co-real subbundle. △

The class of co-real immersions/embeddings fits into the framework of *directed* immersions/embeddings [Gro86, CEM24], and the corresponding open subset of the Grassmannian bundle is ample. Consequently, Gromov's convex integration technique yields the following h-principle:

Theorem 3.2.6 (Gromov)**.** *Co-real embeddings satisfy the C^0-dense, parametric and relative to the domain, h-principle.*

Next, we will state several results about co-real submanifolds. Some of them may seem a bit unmotivated or technical, but they will become relevant in the constructions of prelegendrian stabilizations and compact prelegendrians. The first one is a (formal) topological obstruction to the existence of co-real embeddings in $\mathbb{C}^n$:

Proposition 3.2.7 ([Wel69])**.** *If $N \subseteq \mathbb{C}^n$ is an oriented, co-real submanifold, then its Euler characteristic satisfies $\chi(N) = 0$.*

Proof. Since N is co-real in $\mathbb{C}^n$, there is a splitting

$$T\mathbb{C}^n|_N \simeq (TN)^J \oplus \eta \ \oplus \nu_{N\setminus\mathbb{C}^n},$$

with ν_N the normal bundle, $\eta \subseteq TN$ and $J(\eta) = \nu_N$. In particular,

$$TN \simeq (TN)^J \oplus \nu_N.$$

Hence $e(TN) = e((TN)^J) \cup e(\nu_N)$. Since $e(\nu_N) = 0$ (see [MS74]) it follows that:

$$\chi(N) = \langle e(TN), [N] \rangle = 0. \qquad \square$$

The following result will be fundamental in our construction of exotic prelegendrians (Theorem 2.2.2) via prelegendrian spinning along a co-real torus. In particular, it will allow us to find the required torus:

Lemma 3.2.8. *Let (X, J) be an almost complex manifold such that $\pi : X \to N$ is a vector bundle over a compact manifold N. Assume that the zero section $0_N \subseteq X$ is co-real. Then,*

- *For every connection* $\mathrm{Hor} \subseteq TX$ *of X, then there exists a sufficiently small tubular neighbourhood U of 0_S such that* $\mathrm{Hor}_{|U} \subseteq TX_{|U}$ *is a co-real subbundle.*

- *Every submanifold $W \subseteq X$ fibred over N is fibred isotopic to a co-real submanifold.*

Proof. The first part is a consequence of the compactness of N and the fact that co-reality is an open condition, since $\mathrm{Hor}_p = T_pN$ for $p \in N \equiv 0_N$.

For the second part, fix a connection Hor of X that restricts to a connection of W, and consider the dilation $\delta_t : X \to X$, $t > 0$, along the fibre directions. It follows from the previous part and Lemma 3.2.2 that $\delta_t(W)$ is co-real for $t > 0$ sufficiently small. □

Sections of the complex tangencies

The following result will have a relevant role in proving that prelegendrian stabilization does not change the formal prelegendrian isotopy class. The N appearing in the statement will later be the co-real submanifold along which we perform the stabilization and $\mathcal{F}$ will be a prelegendrian front.

Proposition 3.2.9. *Let $N \subseteq \mathcal{F} \subseteq (X, J)$, where $\mathcal{F}$ is an oriented hypersurface and N is a co-real submanifold, and suppose the normal bundle $\nu_{N\setminus\mathcal{F}}$ admits a nowhere-vanishing section. Set $M \coloneqq \overline{\mathcal{O}p_{\mathcal{F}}(N)} \subseteq \mathcal{F}$. Then every nowhere-vanishing, inward-pointing section $Z \in \Gamma(TM^J_{|\partial M})$ extends to a nowhere-vanishing section $\tilde{Z} \in \Gamma(TM^J)$.*

Proof. Consider the tubular projection $\pi : M \to N$, together with a connection $\mathrm{Hor} \subseteq TM$. Define $S \coloneqq \mathrm{Hor}$. Notice that since $N \subseteq X$ is co-real we may assume that $S \subseteq TX_{|M}$ is a co-real subbundle after possibly shrinking M. Define $F \coloneqq TM \subseteq E \coloneqq TX|_M$, so we have a flag of vector bundles $S \subseteq F \subseteq E$ over M.

Observe that

$$F = TM \cong S \oplus \mathrm{Vert} \cong S \oplus \pi^*\nu_{N\setminus\mathcal{F}}.$$

Since $\nu_{N\setminus\mathcal{F}} \to N$ has a nowhere vanishing section by hypothesis, we conclude that F/S has a nowhere vanishing section. Therefore, we are in the situation of Lemma 3.2.10 (below) and there exists a non-zero section $s : M \to F^J \cap S = TM^J \cap \mathrm{Hor}$.

Finally, we will interpolate between the "almost vertical" section Z and s near ∂M to conclude. First, extend the boundary section $Z : \partial M \to TM^J_{|\partial M}$ to a collar neighbourhood, and denote the extension by $\hat{Z} : \partial M \times [0, \varepsilon) \to TM^J_{|M\times[0,\varepsilon)}$. Since Z is inward-pointing, we may assume that $\hat{Z}$ is nowhere zero neither horizontal.

Then define

$$\tilde{Z}(p) = (1 - \beta(p))s(p) + \beta(p)\hat{Z}(p), \quad \text{for } p \in M,$$

where $\beta : M \to [0, 1]$ is a bump function supports on $\partial M \times [0, \varepsilon)$, and $\beta|_{\partial M\times[0,\varepsilon/2]} \equiv 1$. Then $\tilde{Z} \in \Gamma(TM^J)$ is the required extension of Z. Indeed, it is nowhere zero because $\hat{Z}(p)$ is not horizontal while $s(p) \in \mathrm{Hor}$. □

We now prove the claimed auxiliary result.

Lemma 3.2.10. *Let $S \subseteq F \subseteq E$ be a flag of vector bundles over a manifold M such that*

- *$E \to M$ is a complex vector bundle, with fiberwise complex structure $J : E \to E$,*
- *$F \subseteq E$ has real corank one,*
- *$S \subseteq E$ is a co-real subbundle, and*
- *The normal bundle F/S admits a nowhere zero section.*

Then, there exists a nowhere zero section of $F^J \cap S$ over M.

Proof. Since $S \subseteq E$ is a co-real subbundle there is an isomorphism of vector bundles:

$$E \simeq S^J \oplus J(E/S) \oplus E/S$$

with $J(E/S) \subseteq S$.

Moreover, $E/S \cong E/F \oplus F/S$ hence, the previous isomorphism gives

$$E \simeq S^J \oplus J(F/S) \oplus J(E/F) \oplus F/S \oplus E/F.$$

On the other hand, since $F \subseteq E$ has corank 1, it is automatically co-real, hence

$$E \simeq F^J \oplus J(E/F) \oplus E/F$$

Taking the quotient by the complex line $J(E/F) \oplus E/F$ in the both expressions, we conclude that

$$F^J \simeq S^J \oplus J(F/S) \oplus F/S,$$

with $J(F/S) \subseteq S$, from which the result follows. □

3.3 Examples of corank 2 Fat Distributions

This section presents examples of corank-2 fat distributions. The most relevant one is the *space of complex contact elements* over an almost complex manifold. We will show that the contactisations of the later are canonically contactomorphic to the spaces of contact elements of the underlying almost complex manifold.

3.3.1 An alternative characterisation of fatness for corank 2

To introduce the examples of corank 2 fat distributions, we will need an alternative characterisation of of fatness in corank 2, which can be found in [Bho24, Rut23]. We begin with an algebraic definition:

Definition 3.3.1. Let (ω_1, ω_2) be a pair of symplectic forms on a real vector space W. Then there exists a unique automorphism $A : W \to W$ such that:

$$\omega_1(w_1, w_2) = \omega_2(w_1, Aw_2), \quad w_1, w_2 \in W.$$

The automorphism $A : W \to W$ is called the *connecting isomorphism* of the pair (ω_1, ω_2). △

More precisely, the connecting isomorphism is defined by $A \coloneqq I_1^{-1} \circ I_2$, where $I_i : W \to W^*$ is the isomorphism induced via $I_i(u) = \omega_i(u, -)$.

Let $\mathcal{D}^{2n}$ be a distribution on M^{2n+2}, then the dual curvature at each point $p \in M$ is

$$\Omega_p^* : \mathcal{D}_p^\perp \to \wedge^2 \mathcal{D}_p^*$$

Choosing a basis $\{e_1, e_2\}$ of $\mathcal{D}_p^\perp$, we obtain a pair of 2-forms (ω_1, ω_2) on $\mathcal{D}_p$ given by

$$\omega_i \coloneqq \Omega_p^*(e_i), \quad i = 1, 2.$$

With the above notation, we have the following:

Lemma 3.3.2. *Let M^{2n+2} be a manifold equipped with a distribution $\mathcal{D}^{2n}$. Then the following are equivalent:*

1. *$\mathcal{D}$ is fat at the point $p \in M$.*
2. *The pair of 2-forms (ω_1, ω_2) associated with any basis of $\mathcal{D}_p^\perp$, are symplectic on $\mathcal{D}_p$, and the corresponding connecting isomorphism A has no real eigenvalues.*

Proof. ($\Longrightarrow$) Suppose that $\mathcal{D}$ is fat. Then ω_1 and ω_2 are symplectic by definition. If A has a real eigenvalue, there would exist $X \in \mathcal{D}_p$ and $\lambda \in \mathbb{R}$ with $AX = \lambda X$, hence

$$\omega_1(\cdot, X) = \omega_2(\cdot, AX) = \lambda\omega_2(\cdot, X),$$

The linear combination $\omega \coloneqq \lambda\omega_1 - \omega_2$ would then be degenerate, contradicting fatness.

($\Longleftarrow$) Conversely, suppose some non-zero linear combination $\omega \coloneqq c_1\omega_1 + c_2\omega_2$ is not symplectic. Then there exists $X \in \mathcal{D}$ such that:

$$0 = \omega(\cdot, X) = c_1\omega_1(\cdot, X) + c_2\omega_2(\cdot, X).$$

Without loss of generality, assuming $c_1 \neq 0$, we obtain

$$\omega_1(\cdot, X) = \omega_2(\cdot, \lambda X), \qquad \lambda \coloneqq -\frac{c_2}{c_1},$$

which shows that A has a real eigenvalue. □

3.3.2 Holomorphic contact manifolds

Definition 3.3.3. Let M be a complex manifold of complex dimension $2n+1$. A holomorphic vector subbundle $\mathscr{L} \subseteq TM$ of complex corank 1 is called a *holomorphic contact structure*, if for every point $p \in M$, there exists an open neighbourhood $U \subseteq M$ and such that $\mathscr{L}|_U = \ker \Theta$ is defined by a holomorphic 1-form Θ satisfying:

$$\mathrm{d}\Theta \wedge (\Theta)^n \neq 0.$$

The pair $(M, \mathscr{L})$ is called a *holomorphic contact manifold.* △

A fundamental example is $\mathbb{C}^{2n+1}$ with complex coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$, equipped with the *standard holomorphic contact form*:

$$\Theta_{\mathrm{std}} = \mathrm{d}z + \sum_{i=1}^{n} y_i \, \mathrm{d}x_i,$$

whose associated holomorphic contact structure is $\ker(\Theta_{\mathrm{std}})$.

The following classical result, which exhibits a family of fat distributions of corank 2, appears in [BD23] and [Rut23].

Lemma 3.3.4. *Let $(M, \mathscr{L})$ be a holomorphic contact manifold. We view M as a smooth manifold and $\mathscr{L}$ as a real distribution of corank 2, then $\mathscr{L}$ is a fat distribution of corank 2 on M.*

Proof. By the Holomorphic Darboux Theorem [AFL17, Theorem A.2], it suffices to prove the statement for $(\mathbb{C}^{2n+1}, \ker \Theta_{\mathrm{std}})$. Identify $\mathbb{C}^{2n+1}$ with $\mathbb{R}^{4n+2}$ and and write the associated real coordinates as

$$z = z_1 + iz_2, \quad x_j = x_{j1} + ix_{j2}, \quad y_j = y_{j1} + iy_{j2}.$$

Then, we can express the holomorphic contact form as $\Theta_{\mathrm{std}} = \lambda_1 + i\lambda_2$, where

$$\lambda_1 = \mathrm{d}z_1 + \sum_j (y_{j1} \, \mathrm{d}x_{j1} - y_{j2} \, \mathrm{d}x_{j2}), \quad \lambda_2 = \mathrm{d}z_2 + \sum_j (y_{j2} \, \mathrm{d}x_{j1} + y_{j1} \, \mathrm{d}x_{j2}).$$

It is straightforward to check that $\mathscr{L}$ is spanned by the vector fields

$$X_{j1} \coloneqq \partial_{x_{j1}} - y_{j1}\partial_{z_1} - y_{j2}\partial_{z_2}, \quad X_{j2} \coloneqq \partial_{x_{j2}} + y_{j2}\partial_{z_1} - y_{j1}\partial_{z_2}$$
$$Y_{j1} \coloneqq \partial_{y_{j1}}, \quad Y_{j2} \coloneqq \partial_{y_{j2}},$$

and we compute

$$\mathrm{d}\lambda_1 = \sum_j \big(\mathrm{d}y_{j1} \wedge \mathrm{d}x_{j1} - \mathrm{d}y_{j2} \wedge \mathrm{d}x_{j2} \big),$$
$$\mathrm{d}\lambda_2 = \sum_j \big(\mathrm{d}y_{j2} \wedge \mathrm{d}x_{j1} + \mathrm{d}y_{j1} \wedge \mathrm{d}x_{j2} \big).$$

Note that each summand of $\mathrm{d}\lambda_1$ and $\mathrm{d}\lambda_2$ depends only on the block $\{X_{j1}, X_{j2}, Y_{j1}, Y_{j2}\}$; hence a direct computation shows that

$$\mathrm{d}\lambda_1(u, v) = \mathrm{d}\lambda_2(u, Jv),$$

for all $u, v \in \mathscr{L}$, where J is denoted by the standard complex structure.

This shows that the connecting isomorphism $A : \mathscr{L} \to \mathscr{L}$ associated to $(\mathrm{d}\lambda_1, \mathrm{d}\lambda_2)$ is the restriction of J to $\mathscr{L}$. In particular, A has no real eigenvalues, and hence $\mathscr{L}$ is fat by Lemma 3.3.2. □

3.3.3 The space of complex contact elements

Another class of corank-2 fat distributions arises from the study of almost-complex grassmannians, generalising the space of contact elements to the almost-complex setting. The study of such structures in the holomorphic setting was carried out in [FL22].

Let (X, J) be an almost-complex manifold of real dimension $2n$. The *almost-complex Grassmannian* of complex codimension one is the fibre bundle

$$\pi : \mathrm{Gr}_{2n-2}(X, J) \to (X, J),$$

whose fibre over $x \in X$ is

$$\mathrm{Gr}_{2n-2}(X, J)_x = \mathrm{Gr}_{2n-2}(T_x X, J_x).$$

For brevity we write $\mathrm{Gr}(X, J)$ for $\mathrm{Gr}_{2n-2}(X, J)$ and refer to it as the *almost-complex Grassmannian* of (X, J). Analogously to the construction of the space of contact elements in contact geometry, we have:

Proposition 3.3.5. *Let (X, J) be an almost complex manifold. The distribution $\mathcal{D}_{\mathrm{can}} \subseteq T\mathrm{Gr}(X, J)$ defined by:*

$$\mathcal{D}_{\mathrm{can}}(W) \coloneqq \mathrm{d}\pi^{-1}(W), W \in \mathrm{Gr}(X, J),$$

is fat. The distribution $\mathcal{D}_{\mathrm{can}}$ is called the canonical fat distribution*, and the fat manifold $(\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$ is referred to as the* space of complex contact elements *over (X, J).*

The thesis [Rut23] treats the case $\dim X = 4$. The general case can be established with an analogous computation. Instead of pursuing this route, we just remark that this will follow from the fact that its contactisation is the space of contact elements, which will be shown below.

It is straightforward to check that the canonical fat distribution is invariant under any diffeomorphism preserving the almost-complex structure. In particular, consider (X, J) an almost complex manifold and $g : X \to X$ a diffeomorphism. Then, g induces a pushforward almost complex structure $g_* J$ on X, and the map $\mathrm{Gr}(g) : \mathrm{Gr}(X, J) \to \mathrm{Gr}(X, g_* J)$ is a diffeomorphism that preserves $\mathcal{D}_{\mathrm{can}}$.

Remark 3.3.6. Fat structures of corank at least 2 have local invariants. The ones we are considering are rather special. Namely, there is a submersion $\pi_{\mathrm{Gr}} : (\mathrm{Gr}(X,J),\mathcal{D}_{\mathrm{can}}) \to (X,J)$ whose fibres are horizontal submanifolds (i.e., tangent to $\mathcal{D}_{\mathrm{can}}$). It is at the moment unknown whether a general corank-2 fat distribution admits such a submersion locally.

A certain local invariant of corank-2 fat distributions was introduced in [BD23]: the degree. The distributions $\mathcal{D}_{\mathrm{can}}$ have degree 2. A germ of fat distribution of type $(8,10)$ and degree different from 2 was constructed in [BD23].

We expect our constructions of prelegendrians to extend to arbitrary fat distributions of corank 2, but this remains an open question. △

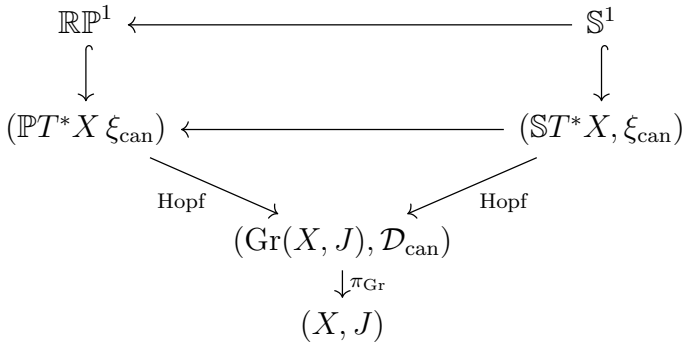
Contactisation of the space of complex contact elements

The contactisation of the space of complex contact elements $(\mathrm{Gr}(X,J),\mathcal{D}_{\mathrm{can}})$ has a particularly nice and geometric form:

Theorem 3.3.7. *Let (X,J) be an almost complex manifold. Then,*

- *The projectivised contactisation $\mathcal{C}_{\mathbb{P}}(\mathrm{Gr}(X,J),\mathcal{D}_{\mathrm{can}})$ is canonically contactomorphic to the space of contact elements $(\mathbb{P}T^*X,\xi_{\mathrm{can}})$*
- *The contactisation $\mathcal{C}(\mathrm{Gr}(X,J),\mathcal{D}_{\mathrm{can}})$ is canonically contactomorphic to the space co-oriented contact elements $(\mathbb{S}T^*X,\xi_{\mathrm{can}})$.*

Moreover, the following diagram



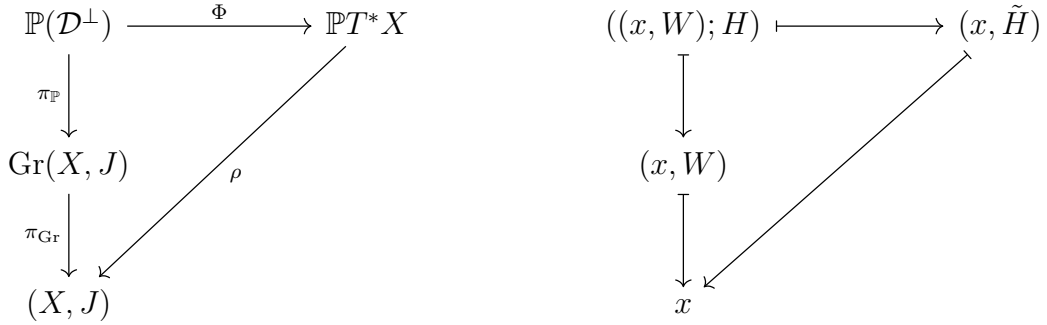
in which the diagonal maps are the fiberwise geometric Hopf maps described in the Section 3.2, commutes.

Proof. Recall that $\mathcal{C}(\mathrm{Gr}(X,J),\mathcal{D}_{\mathrm{can}}) = (\mathbb{P}(\mathcal{D}_{\mathrm{can}}^{\perp}),\xi_{\mathrm{can}})$ and denote by $\pi_{\mathrm{Gr}} : \mathrm{Gr}(X,J) \to (X,J)$ the Grassmannian bundle projection.

An element of $\mathbb{P}(\mathcal{D}_{\mathrm{can}}^{\perp})$ is a pair $((x,W);H)$ with $(x,W) \in \mathrm{Gr}(X,J)$ and $H \subseteq T_{(x,W)}\mathrm{Gr}(X,J)$ a real hyperplane. Define

$$\begin{aligned}\Phi : P(\mathcal{D}_{\mathrm{can}}^{\perp}) &\longrightarrow \mathbb{P}T^*X\\ ((x,W);H) &\mapsto \tilde{H} := (\,\mathrm{d}\pi_{\mathrm{Gr}})_{(x,W)}(H).\end{aligned}$$

Then $\tilde{H}$ is a real hyperplane in T_xX with $W \subseteq \tilde{H}$ (equivalently, $\tilde{H} \cap J\tilde{H} = W$), so Φ is a bundle map over (X, J). Choosing a connection on $\pi_{\mathrm{Gr}} : \mathrm{Gr}(X, J) \to (X, J)$ yields an inverse bundle diffeomorphism, and the resulting square commutes.



To see that Φ is a contactomorphism, differentiate the diagram. By definition of the canonical distribution ξ_{can} on $\mathbb{P}(\mathcal{D}^\perp_{\mathrm{can}})$,

$$v \in \xi_{\mathrm{can}}(H) \iff (\,\mathrm{d}\pi_{\mathbb{P}})_H(v) \in H \implies \mathrm{d}\pi_{\mathrm{Gr}}\big((\,\mathrm{d}\pi_P)_H(v)\big) \in \tilde{H}$$

where the last implication is by the definition $\tilde{H}$.

Since the diagram commutes, we have

$$\mathrm{d}\rho \circ \mathrm{d}\Psi(v) = \mathrm{d}\pi_{\mathrm{Gr}}\big((\,\mathrm{d}\pi_P)_H(v)\big) \in \tilde{H}$$

hence $\mathrm{d}\Psi(v) \in \xi_{\mathrm{can}}(\tilde{H}) = (d\rho)^{-1}(\tilde{H})$ by the definition of the tautological (Liouville) contact distribution. Thus Φ is a contactomorphism.

Finally, because Φ is bijective and $\tilde{H}$ contains the complex hyperplane W, the map

$$\pi_{\mathbb{P}} \circ \Phi^{-1} : \mathbb{P}T^*X \longrightarrow \mathrm{Gr}(X, J), \qquad \tilde{H} \longmapsto \tilde{H} \cap J\tilde{H},$$

is the fiberwise geometric Hopf map by construction. The same argument applies to the co-oriented contactisation $\mathcal{C}(\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$. □

3.3.4 Standard fat structure on $\mathbb{C}^{2n+1}$

The previous constructions can be specialised to define the standard fat distribution on $\mathbb{C}^{2n+1}$.

Equip $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ with an almost complex structure J. After choosing a complex framing, we trivialise the almost-complex Grassmannian as

$$\mathrm{Gr}(\mathbb{R}^{2n+2}, J) \simeq \mathbb{R}^{2n+2} \times \mathbb{CP}^n.$$

Fix a complex hyperplane $H \simeq \mathbb{CP}^{n-1}$ in $\mathbb{CP}^n$, and identify $\mathbb{CP}^n \setminus H \simeq \mathbb{C}^n$. Thus we obtain an open subset

$$\mathbb{R}^{2n+2} \times \mathbb{C}^n \subseteq \mathbb{R}^{2n+2} \times \mathbb{CP}^n \simeq \mathrm{Gr}(\mathbb{R}^{2n+2}, J).$$

The canonical fat structure $\mathcal{D}_{\text{can}}$ on $\mathrm{Gr}(\mathbb{R}^{2n+2}, J)$ restricts to a fat distribution on $\mathbb{R}^{2n+2} \times \mathbb{C}^n \simeq \mathbb{R}^{4n+2}$, called the *standard fat structure twisted by* J and denoted $\mathcal{D}_{\text{std},J}$.

Note that the space of hyperplanes in $\mathbb{CP}^n$ is naturally identified with the dual projective space $(\mathbb{CP}^n)^\vee \cong \mathbb{CP}^n$, which is homogeneous. Hence different choices of hyperplane and framing produce homotopic fat distributions on $\mathbb{R}^{4n+2}$, so $\mathcal{D}_{\text{std},J}$ is well defined up to homotopy.

In the case of $J \equiv J_{\text{std}}$ (the standard complex structure on $\mathbb{C}^{n+1}$), the construction yields a fat distribution on $\mathbb{C}^{2n+1}$, called the *standard fat structure* and denoted $\mathcal{D}_{\text{std}}$. Moreover

Proposition 3.3.8. *The standard fat structure* $\mathcal{D}_{\text{std}}$ *on* $\mathbb{C}^{2n+1}$ *coincides with standard holomorphic contact structure* $\ker(\Theta_{\text{std}})$.

Proof. Write the complex coordinates on $\mathbb{C}^{n+1}$ as $(x_1, \dots, x_n, z)$ as in the proof of Lemma 3.3.4. Then the canonical fat distribution $\mathcal{D}_{\text{can}}$ on the complex Grammarian $\mathrm{Gr}(\mathbb{C}^{n+1}) = \mathbb{C}^{n+1} \times \mathbb{CP}^n$ is given by

$$\mathcal{D}_{\text{can}}\big((x_1, \cdot, x_n, z); [w_1 : \cdots : w_{n+1}]\big) = \ker(w_1\,\mathrm{d}x_1 + \cdots + w_{n+1}\,\mathrm{d}z)\big).$$

Choosing the hyperplane $H \coloneqq \{w_{n+1} = 0\} \subseteq \mathbb{CP}^n$, and affine coordinates $y_j \coloneqq \frac{w_j}{w_{n+1}}$, the standard fat distribution is

$$\mathcal{D}_{\text{std}}\big((x_1, \dots, x_n, z); (y_1, \dots, y_n)\big) = \ker(y_1\,\mathrm{d}x_1 + \cdots + y_n\,\mathrm{d}x_n + \mathrm{d}z),$$

which is $\ker \Theta_{\text{std}}$, as required. □

Remark 3.3.9. The space of almost complex structures on $\mathbb{R}^{2n+2}$ is contractible, hence all standard fat distributions $\mathcal{D}_{\text{std},J}$ twisted by J, are homotopic to $\mathcal{D}_{\text{std}}$. △

Corollary 3.3.10. *The contactisation* $\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$ *is contactomorphic to* $(J^1(\mathbb{R}^{2n+1} \times \mathbb{S}^1), \xi_{\text{std}})$.

Proof. By definition, $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}}) \subseteq \mathrm{Gr}(\mathbb{C}^{n+1}, J_{\text{std}})$ is obtained from $\mathrm{Gr}(\mathbb{C}^{n+1}, J_{\text{std}})$ by taking an affine chart and, by Theorem 3.3.7, the contactisation is

$$\mathbb{C}^{n+1} \times \mathbb{S}^{2n+1} = \mathbb{S}T^*\mathbb{C}^{n+1} \simeq \mathcal{C}(\mathrm{Gr}(\mathbb{C}^{n+1}, J_{\text{std}})) \to \mathrm{Gr}(\mathbb{C}^{n+1}, J_{\text{std}}) = \mathbb{C}^{n+1} \times \mathbb{CP}^n.$$

with the canonical contact structure. Writing the complex coordinates on $\mathbb{C}^{n+1}$ as $(x, \dots, x_n, z)$, and the induced homogeneous coordinates on $\mathbb{CP}^n$, we get

$$\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}}) = \mathbb{C}^{n+1} \times \mathrm{Hopf}^{-1}(\{z \neq 0\}) = \mathbb{C}^{n+1} \times S,$$

where $\mathrm{Hopf} : \mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1} \to \mathbb{CP}^n$ is the standard Hopf map and $S \coloneqq \mathbb{S}^{2n+1} \setminus \{z = 0\}$.

On the other hand, there is a contactomorphism

$$\begin{aligned} \mathbb{S}T^*\mathbb{R}^{2n+2} = \mathbb{R}^{2n+2} \times \mathbb{S}^{2n+1} &\to J^1(\mathbb{S}^{2n+1}) = T^*\mathbb{S}^{2n+1} \times \mathbb{R} \\ (\mathbf{p}, \mathbf{q}) &\mapsto (\mathbf{q}, \mathbf{p} - \langle \mathbf{q}, \mathbf{p}\rangle \mathbf{q}, \langle \mathbf{q}, \mathbf{p}\rangle), \end{aligned}$$

that induces a contactomorphism

$$(\mathbb{C}^{n+1} \times S, \xi_{\text{can}}) \simeq (J^1(S), \xi_{\text{std}}).$$

Since $S \simeq \mathbb{R}^{2n} \times \mathbb{S}^1$, the result follows. □

3.3.5 Affine charts in the Grassmannian

It is sometimes useful to work inside an affine chart of the abstract Grassmannian, as in the construction of $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$. In general, we cannot do this globally (there may not exist a global section of $\text{Gr}(X, J) \to X$), so we restrict to the product case $X = Q \times \mathbb{R}$.

Contact case

Let $X = Q \times \mathbb{R}$ with coordinate z on the second factor and *vertical vector field* ∂_z. Define the horizontal Grassmannian of real hyperplanes by

$$\mathbb{P}(T^*_{\text{hor}}X) \coloneqq \{H \subseteq T_pX : \partial_z \notin H\} \subseteq \mathbb{P}T^*X.$$

Being an open submanifold (a fiberwise affine chart) of $\mathbb{P}T^*X$, it inherits the canonical contact structure ξ_{can}.

Similarly, for oriented hyperplanes we write $\mathbb{S}(T^*_{\text{hor}}X) \subseteq \mathbb{S}T^*X$. We have the following standard identification:

Lemma 3.3.11. *There exists a canonical contactomorphism between* $(\mathbb{S}T^*_{\text{hor}}(Q \times \mathbb{R}), \xi_{\text{can}})$ *and* (J^1Q, ξ_{can}).

Proof. An element of $\mathbb{S}T^*_{\text{hor}}(Q \times \mathbb{R})$ is a pair $((q, z); \alpha)$ with $\alpha : T_qQ \oplus T_z\mathbb{R} \to \mathbb{R}$ and $\alpha(\partial_z) \neq 0$, hence uniquely of the form

$$\alpha = \mathrm{d}z - \lambda$$

for some $\lambda \in T^*_qQ$.

Then we define the diffeomorphism:

$$\begin{aligned} \mathbb{S}T^*_{\text{hor}}(Q \times \mathbb{R}) &\to J^1Q = T^*Q \times \mathbb{R} \\ ((q, z); \alpha) &\mapsto ((q, \lambda); z). \end{aligned}$$

This is straightforward to check that this is a contactomorphism. □

Almost complex case

Now equip $X = Q \times \mathbb{R}$ with an almost complex structure J. The *horizontal complex Grassmannian* is

$$\text{Gr}_{\text{hor}}(X, J) \coloneqq \{K \subseteq T_xX : \partial_z \notin K \text{ and } K \text{ is } J\text{-invariant}\} \subseteq \text{Gr}(X, J),$$

i.e. the fiberwise open locus of complex hyperplanes avoiding the vertical line. Equivalently, K is horizontal if and only if it does not contain the complex line $\text{span}\langle \partial_z, J\partial_z \rangle$. Thus, when $\dim_{\mathbb{R}}(Q \times \mathbb{R}) = 2n$, the fibre of $\text{Gr}_{\text{hor}}(X, J) \to X$ is

diffeomorphic to $\mathbb{C}^{n-1}$. The fiberwise Hopf map is well defined on both horizontal Grassmannians:

$$\mathrm{Hopf} : \mathbb{S}(T^*_{\mathrm{hor}}X) \longrightarrow \mathrm{Gr}_{\mathrm{hor}}(X, J).$$

In particular, there a natural inclusion

$$(\mathbb{S}(T^*_{\mathrm{hor}}X), \xi_{\mathrm{can}}) \hookrightarrow \mathcal{C}(\mathrm{Gr}_{\mathrm{hor}}(X, J), \mathcal{D}_{\mathrm{can}}) \tag{3.4}$$

Hence any prelegendrian $\Lambda \subseteq \mathrm{Gr}(X, J)$ whose Legendrian lift $\mathcal{L}(\Lambda)$ lies in $\mathbb{S}T^*X$ is automatically contained in $\mathrm{Gr}_{\mathrm{hor}}(X, J)$ whenever $\mathcal{L}(\Lambda) \subseteq \mathbb{S}T^*_{\mathrm{hor}}X$.

In the model case $Q = \mathbb{R}^{2n+1}$ and $X = \mathbb{R}^{2n+1} \times \mathbb{R}$ with J_{std}, the horizontal complex Grassmannian reproduces the standard structure:

$$\big(\mathrm{Gr}_{\mathrm{hor}}(\mathbb{C}^{n+1}, J_{\mathrm{std}}), \mathcal{D}_{\mathrm{can}}\big) \cong \big(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}}\big),$$

and we obtain the inclusion

$$\big(J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{can}}\big) \hookrightarrow \mathcal{C}\big(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}}\big). \tag{3.5}$$

Chapter 4

Prelegendrian Fronts

In this chapter we study prelegendrian submanifolds through their *fronts*.

The guiding idea is that, as in contact topology, fronts provide a powerful geometric model for representing prelegendrian submanifolds. We begin by recalling the notion of Legendrian fibrations and their associated front projections, and introduce the class of *simple* fronts that will be central in our setting.

We then review stabilization and looseness for Legendrians, and adapt these constructions to the prelegendrian case. Next, we discuss spinning techniques that allow us to construct higher-dimensional examples, and finally prove a characterisation of prelegendrian fronts in almost complex Grassmannians. This naturally leads to the definition of prelegendrian stabilization and its relation to Legendrian stabilization.

4.1 Legendrian Fronts

In view of Theorem 3.3.7, contactisations of fat complex Grassmannians are spaces of contact elements. These are examples of *Legendrian fibrations*, and admit canonical front projections, which can be used to study the Legendrians therein. In this section we provide a basic review of Legendrian fibrations and their fronts. We will define a subclass of fronts, which we call *simple*; this class will be specially relevant in our study of *prelegendrian fronts* in Section 4.4. The primary references are [AN01], [AGV12], and [Arn90].

4.1.1 Legendrian fibrations

We begin by recalling the notion of Legendrian fibrations and front projections.

Definition 4.1.1. Let (Y^{2n+1}, ξ) be a $(2n+1)$-dimensional contact manifold. A fibre bundle

$$\pi_{\mathrm{F}} : (Y^{2n+1}, \xi) \to B^{n+1}$$

over a manifold B is called a *Legendrian fibration* if all its fibres are Legendrian submanifolds of Y, i.e. if $\ker(d\pi_{\mathrm{F}}) \subseteq \xi$. The bundle map π_{F} is referred as *front projection*. △

Example 4.1.2. The following are examples of Legendrian fibrations:

- $\pi_{\mathrm{F}} : (J^1Q, \xi_{\mathrm{std}}) \to J^0Q$ with Q a smooth manifold.
- The concrete case $\pi_{\mathrm{F}} : (J^1\mathbb{R}^n, \xi_{\mathrm{std}}) \to J^0\mathbb{R}^n$ with $Q = \mathbb{R}^n$ is the *standard front projection*. Locally, every Legendrian fibration is contactomorphic to this one; see for instance [AN01, p. 72].
- The space of contact elements over a manifold X^n is a $\mathbb{RP}^{n-1}$-Legendrian fibration $\pi_{\mathbb{P}} : (\mathbb{P}T^*X, \xi_{\mathrm{can}}) \to X$. Similarly, the space of co-oriented contact elements is an $\mathbb{S}^{n-1}$-Legendrian fibration $\pi : (\mathbb{S}T^*X, \xi_{\mathrm{can}}) \to X$. △

Definition 4.1.3. A front map is a diagram consisting of an embedding of a Legendrian submanifold $\iota : L^n \hookrightarrow (Y^{2n+1}, \xi)$ and a Legendrian fibration $\pi_{\mathrm{F}} : (Y^{2n+1}, \xi) \to B^{n+1}$:

$$f : L^n \overset{\iota}{\hookrightarrow} (Y^{2n+1}, \xi) \overset{\pi_{\mathrm{F}}}{\longrightarrow} B^{n+1}.$$

By abuse of notation, we often refer to f itself as the front map. The image $\mathcal{F} \coloneqq f(L) \subseteq B^{n+1}$ is called the *wave front* (or simply the *front*) of L or f. △

Front maps are smooth maps from an n-dimensional manifold to an $(n+1)$-dimensional ambient manifold, forming a distinguished class of smooth maps with some special singularities. Roughly speaking, at each point $p \in L$, whether regular or singular, you still have a well-defined tangent map $\mathrm{d}_p f$ of full rank. Those tangent images record the original Legendrian directions and let you recover the submanifold. Locally, every front map can be described by means of the standard front projection $\pi_{\mathrm{F}} : (\mathbb{R}^{2n+1}, \xi_{\mathrm{std}}) \to \mathbb{R}^{n+1}$. Conversely, from a front $\mathcal{F} \subseteq \mathbb{R}^{n+1}$ one can recover the $\mathbf{y}$-coordinates by taking the tangencies of $T\mathcal{F}$ [1], at the very least in the smooth points of $\mathcal{F}$.

4.1.2 Front singularities and simple fronts

Let $f : L^n \to B^{n+1}$ be a front map. We now study the singularities of the map f at a point $p \in L$. By singularity, we mean an equivalence class of front map germs. Since any front map can be locally described by the standard front projection, we always present germs of front map as $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^{n+1}, 0)$. The following follows from standard transversality arguments:

Lemma 4.1.4. *Let $\mathcal{F} \subseteq B$ be a generic simple front, then the subset of singularities $\Sigma(\mathcal{F}) \subseteq B$ is a codimension 2 submanifold in B^{n+1}.*

[1] Here we identify a non-vertical oriented hyperplane $H \subseteq T\mathbb{R}^{n+1}$ with the unique 1-form $\mathrm{d}z - \mathbf{y}\,\mathrm{d}\mathbf{x}$ on $\mathbb{R}^{n+1}$ that annihilates it.

The possible singularities of a germ of a front are studied by means of generating functions or generating families of hypersurfaces [Arn90]. Although in the lower dimensional cases ($\dim L < 6$), the possible front singularities are stable and completely classified [AN01, AGV12], in full generality front singularities are intractable. In this thesis, we will focus on very *simple* singularities:

Definition 4.1.5. Let $f : L \overset{\iota}{\hookrightarrow} (Y, \xi) \overset{\pi_F}{\longrightarrow} B$ be the front map of a Legendrian submanifold L. We say that f has a *cusp point* or A_2-singularity at $p \in L$, if the germ of the front map $f|_{\mathcal{O}p(p)} : \mathcal{O}p(p) \to B^{n+1}$ is equivalent to

$$\begin{aligned} f|_{L\cap U} : \mathbb{R}^{n-1} \times \mathbb{R} &\to \mathbb{R}^n(\mathbf{x}) \times \mathbb{R}(z) \\ (q_1, \dots, q_{n-1}, t) &\mapsto \Big((q_1, \dots, q_{n-1}, -3t); 2t^3\Big) \end{aligned}$$ △

The class of (non-generic) fronts that we will work with is defined as follows:

Definition 4.1.6. Let $\mathcal{F} \subseteq B$ be a subset of B^{n+1}. We say that $\mathcal{F}$ is a *simple front* if $\mathcal{F}$ has a stratification

$$\mathcal{F} = \Sigma_{A_1}(\Sigma) \sqcup \Sigma_{A_1A_1}(\Sigma) \sqcup \Sigma_{A_2}(\Sigma)$$

such that

- for all $p \in \Sigma_{A_1}(\mathcal{F})$, p is a smooth point of $\mathcal{F}$, i.e., $\mathcal{F}$ is a smooth hypersurface near p;
- for $p \in \Sigma_{A_1A_1}(\Sigma)$, $\mathcal{F}$ has a transversal self-intersection at p;
- for $p \in \Sigma_{A_2}(\Sigma)$, p is a cusp point of $\mathcal{F}$, i.e., there exist a germ of smooth map $f : \mathbb{R}^n \to \mathbb{R}^{n+1} \simeq \mathcal{O}p(p)$ which parametrizes $\mathcal{F}$.

In this case, the sets $\Sigma_{A_1}(\mathcal{F})$, $\Sigma_{A_1A_1}(\Sigma)$ and $\Sigma_{A_2}(\Sigma)$ are called *smooth strata*, *self-intersection strata* and *cusp strata* of $\mathcal{F}$, respectively. We define the subset of singular points by

$$\Sigma(\mathcal{F}) \coloneqq \Sigma_{A_1A_1}(\Sigma) \sqcup \Sigma_{A_2}(\Sigma).$$ △

4.2 Legendrian stabilization and loose Legendrians

In this section we review the notions of Legendrian stabilization and loose Legendrians in higher-dimensional contact manifolds, as introduced in [Eli90b, EES05b, Mur19]. All the results presented here are well-known, although we give some new proofs that are better suited to the purposes of this thesis.

4.2.1 N-Pushing

We introduce an operation on fronts, called *N-pushing*, which alters the Legendrian isotopy class of a submanifold while preserving its formal Legendrian class, provided certain conditions on N are satisfied. This operation is particularly useful when showing that two Legendrians belong to the same formal class.

This construction first appeared in the works of [Eli90b, EES05b], where it was referred to as a "stabilization". However, the particular construction from [EES05b] does not, in general, produce a loose Legendrian submanifold [Mur19]. For this reason, we adopt the term *pushing*, and reserve the term of *stabilization* for the operation that produces a loose Legendrian, i.e. an N-pushing in a cusp chart.

Let Q be a manifold. Given a function $f \in C^\infty(Q)$ we denote by $j^1 f \subseteq J^1 Q = T^*Q \times \mathbb{R}$ the Legendrian submanifold of $(J^1 Q, \xi_{\text{std}})$ defined as the 1-jet of f. Given $c \in \mathbb{R}$, we will use the notation $c_Q : Q \to \mathbb{R}$ to denote the function with constant value c. Consider the disconnected Legendrian $L_0 = j^1 0_Q \sqcup j^1 1_Q$ and a compact submanifold $N \subseteq Q$ together with a smooth function $f : Q \to [0, 2]$ such that

(i) $f_{|Q \setminus \mathcal{O}p(N)} \equiv 0$,

(ii) $f(p) > 1$ for $p \in N$,

(iii) $f(p) > 0$ for $p \in \mathcal{O}p(N)$, and

(iv) The critical points of f in $\mathcal{O}p(N)$ have critical values greater than 1.

The *N-pushing* of L_0 is the disconnected Legendrian

$$L_N = j^1 f \sqcup j^1 1_Q$$

Definition 4.2.1. Let (Y^{2n+1}, ξ) be a contact manifold of dimension $2n + 1 \geq 5$, $L \subseteq Y$ a Legendrian submanifold and $U \subseteq Y$ an open codimension 0 submanifold such that

$$(U, L \cap U, \xi) \simeq (J^1 Q, L_0, \xi_{\text{std}}).$$

The N-pushing of the Legendrian L is the Legendrian $\mathrm{p}_N(L)$ obtained by replacing $(U, L \cap U, \xi)$ by $(J^1 Q, L_N, \xi_{\text{std}})$. △

The following result is well-known. However, we will include a proof since we will need to adapt it to the prelegendrian setting later:

Proposition 4.2.2 (Ekholm-Etnyre-Sullivan, Murphy)**.** *Let $L \subseteq (Y, \xi)$ be a Legendrian submanifold, and suppose that the Euler characteristic $\chi(N) = 0$ is zero. Then, the N-pushing $\mathrm{p}_N(L)$ is formally Legendrian isotopic to L in (Y, ξ).*

Proof. We will to check that the Legendrians $L_0 = j^1 0_Q \sqcup j^1 1_X$ and $L_N = j^1 f \sqcup j^1 1_Q$ are formally Legendrian isotopic by a compactly supported formal isotopy. By simplicity we will assume that $Q = \mathbb{R}^n$ since the general case is totally analogous after fixing a Riemannian metric.

To do this, it is enough to find a formal Legendrian isotopy between the Legendrian embeddings $j^1 0_{\mathbb{R}^n} : \mathbb{R}^n \to J^1\mathbb{R}^n, \mathbf{x} \mapsto (\mathbf{x}, 0, 0)$ and $j^1 f : \mathbb{R}^n \to J^1\mathbb{R}^n, \mathbf{x} \mapsto (\mathbf{x}, \frac{\partial f}{\partial \mathbf{x}}, f(\mathbf{x}))$, not intersecting the Legendrian $j^1 1_{\mathbb{R}^n}$. This isotopy should be supported in a small neighbourhood of the support of f. To construct it we will first find an isotopy of smooth embeddings, homotopic through isotopies of embeddings to the obvious Legendrian isotopy between $j^1 0_{\mathbb{R}^n}$ and $j^1 f$ induced by the linear interpolation between the functions $0_{\mathbb{R}^n}$ and f.

The required isotopy of smooth embeddings will be defined by a homotopy of non-holonomic sections of the 1-jet bundle. To define it, we will rely on the fact that a section of the 1-jet bundle can be reinterpreted as a section of the 0-jet bundle equipped with a family of non-vertical planes. The geometric idea behind the proof then goes as follows: Since the Euler characteristic $\chi(\mathcal{O}p\,(N)) = \chi(N) = 0$ vanishes, we can find a nowhere zero vector field Z tangent to $j^0 f|_{\mathcal{O}p\,(N)}$. This is enough to change the 1-jet information of $j^1 f$ by a non-holonomic deformation. Indeed, at every point of $j^0 f$ we can tilt its non-vertical tangent planes towards the vertical, by rotating Z towards ∂_z. After doing this carefully, it will be possible to interpolate linearly between $j^0 f$ and $j^0 0_{\mathbb{R}^n}$, keeping the non-holonomic 1-jet data fixed. Having tilted towards the vertical, the non-holonomic 1-jets will not intersect $j^1 1_{\mathbb{R}^n}$. The argument will conclude by tilting the 1-jet until it becomes horizontal, while fixing the 0-jet. The detailed construction of this idea is explained below.

The vector field Z is defined as follows: for $x \in f^{-1}(-\infty, 1+\epsilon)$, we define $Z = \nabla h$, where $h : j^0 f \to \mathbb{R}$ is the height function respect to the standard Euclidean metric in $\mathbb{R}^{n+1}$, where $\varepsilon > 0$ is so small that there is not critical value below $1 + \varepsilon$. In particular, $f^{-1}([1,2]) \simeq \overline{\mathcal{O}p\,(N)}$ and $f^{-1}(\{1\}) \simeq \partial\overline{\mathcal{O}p\,(N)}$. Hence, $\chi(f^{-1}([1,2])) = \chi(\mathcal{O}p\,(N)) = \chi(N) = 0$, and we can extend Z to a nowhere-zero vector field, tangent to $j^0 f$. The key property of this vector field is that the angle between Z with ∂_z is less that $\pi/2$ in the region $f^{-1}[1, 1+\epsilon]$.

We will use this vector field Z to rotate the tangent space of 0-jet $j^0 f$, which at a point $(\mathbf{x}, f(\mathbf{x}))$ is the non-vertical plane

$$T_{(\mathbf{x},f(\mathbf{x}))}(j^0 f) = \ker(\,\mathrm{d}z - \frac{\partial f}{\partial \mathbf{x}}\,\mathrm{d}\mathbf{x}) \subseteq T_{(\mathbf{x},f(\mathbf{x}))}\mathbb{R}^{n+1},$$

where $\mathbf{y} = \frac{\partial f}{\partial \mathbf{x}}$ are precisely the $\mathbf{y}$-coordinates of the Legendrian $j^1 f$.

To define the non-holonomic deformation of the planes $T_{(\mathbf{x},h(\mathbf{x}))}(j^0 f)$ we will use a bundle isomorphism

$$R_\theta : (j^0 f)^* T\mathbb{R}^{n+1} \to (j^0 f)^* T\mathbb{R}^{n+1},$$

where $\theta : \mathbb{R}^n \to [0, \frac{\pi}{2})$ is a smooth function with compact support $\mathrm{supp}(\theta) \subseteq f^{-1}[1,2]$, and $R_\theta(\mathbf{x}) : T_{(\mathbf{x},f(\mathbf{x}))}\mathbb{R}^{n+1} \to T_{(\mathbf{x},f(\mathbf{x}))}\mathbb{R}^{n+1}$ is a rotation of angle $\theta(\mathbf{x})$ about the $(n-1)$-dimensional subspace orthogonal to the oriented plane $\mathrm{span}\langle Z(\mathbf{x}), \partial_z\rangle$.

We define the function $\theta : \mathbb{R}^n \to [0, \frac{\pi}{2})$ to be

$$\theta(\mathbf{x}) = \rho(f(\mathbf{x}))\frac{\beta(\mathbf{x})}{2},$$

where

(i) $\rho : \mathbb{R} \to [0,1]$ is an increasing smooth function so that $\rho_{|\{z\leq 1\}} \equiv 0$, and $\rho_{|\{z\geq 1+\varepsilon\}} \equiv 1$.

(ii) $\beta : \mathbb{R}^n \to (0,\pi)$ satisfies that $\beta_{|\mathbb{R}^n \setminus \mathcal{O}p\,(N)} \equiv 0$, and over $\mathcal{O}p\,(N)$ is defined to be the angle

$$\beta(\mathbf{x}) \coloneqq \angle(Z(\mathbf{x}), \partial_z), \mathbf{x} \in \mathcal{O}p\,(N).$$

Now we are ready to define the required homotopy of non-holonomic sections $g_t : \mathbb{R}^n \to J^1\mathbb{R}^n$, $t \in [0,3]$, between $g_0 = j^1 f$ and $g_3 = j^1 0_{\mathbb{R}^n}$.

- For $t \in [0,1]$, define
$$g_t(\mathbf{x}) = (\mathbf{x}, \mathbf{y}_t(\mathbf{x}), f(\mathbf{x})),$$
where $\mathbf{y}_t(\mathbf{x})$ is defined by the field of non-vertical hyperplanes along $j^0 f$ given by
$$H_t(\mathbf{x}) = R_{t\theta}(T_{(\mathbf{x},f(\mathbf{x}))} j^0 f) = \ker(\,\mathrm{d}z - \mathbf{y}_t(\mathbf{x})\,\mathrm{d}\mathbf{x})$$

- For $t \in [1,2]$, define g_t just by changing the 0-jet information via a linear interpolation between the functions $f(\mathbf{x})$ and $0_{\mathbb{R}^n}$ while fixing the 1-jet information. Observe that for $\mathbf{x} \in f^{-1}[1+\varepsilon, 2]$ the angle
$$\angle(R_{\theta(\mathbf{x})}(Z(\mathbf{x})), \partial_z) = \beta(\mathbf{x}) - \frac{\beta(\mathbf{x})}{2} = \frac{\beta(\mathbf{x})}{2} < \frac{\pi}{2},$$
and for $\mathbf{x} \in f^{-1}[1, 1+\varepsilon]$,
$$\angle(R_{\theta(\mathbf{x})}(Z(\mathbf{x})), \partial_z) = \beta(\mathbf{x}) - \rho(h(\mathbf{x}))\frac{\beta(\mathbf{x})}{2} \leq \beta(\mathbf{x}) < \frac{\pi}{2}.$$
Therefore, the non-vertical hyperplanes $H_1(\mathbf{x}) = R_\theta(\mathbf{x})(T_{(\mathbf{x},f(\mathbf{x}))} j^0 f)$ are non-horizontal for $\mathbf{x} \in f^{-1}[1,2]$ and this homotopy does not intersect $j^1 1_{\mathbb{R}^n}$.

- Finally, for $t \in [2,3]$, define the homotopy by fixing the 0-jet datum while interpolating the 1-jet information $\mathbf{y}_1(\mathbf{x})$ linearly to $0_{\mathbb{R}^n}$.

This homotopy of non-holonomic sections g_t is clearly homotopic to the homotopy of holonomic sections (Legendrian embeddings) between $j^1 f$ and $j^1 0_{\mathbb{R}^n}$ induced by a linear interpolation between the functions f and $0_{\mathbb{R}^n}$. This concludes the argument. □

4.2.2 Legendrian Stabilization and Loose Legendrians

The 1-dimensional Legendrian cusp $L^0_{\text{cusp}} \subseteq J^1\mathbb{R}$ is defined by $L^0_{\text{cusp}} \cap \{x \leq 0\} = j^1 0_{\mathbb{R}} \sqcup j^1 1_{\mathbb{R}} \cap \{x \leq 0\}$, while over $\{x \geq 0\}$ the front $\pi_{\mathrm{F}}(L^0_{\text{cusp}})$ is given by a single cusp connecting $j^0 0_{\mathbb{R}}$ and $j^0 1_{\mathbb{R}}$. See Fig. 4.1

Let Q be a $(n-1)$-dimensional manifold, and define $X = \mathbb{R} \times Q$. Consider 1-jet bundle $J^1 X = J^1\mathbb{R} \times T^*Q$, and a Legendrian submanifold $L_{\text{cusp}} \subseteq J^1\mathbb{R} \times T^*Q$ defined as:

$$L_{\text{cusp}} \coloneqq L^0_{\text{cusp}} \times Q \subseteq J^1\mathbb{R} \times T^*Q$$

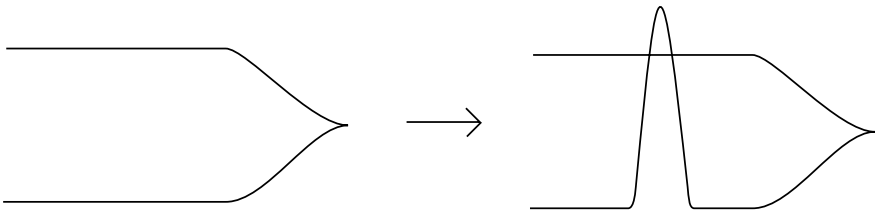


Figure 4.1: Left: the front of the cusp model L^0_{cusp}. Right: the stabilized front L_{loose}.

where $Q \subseteq T^*Q$ is the 0-section.

Notice that over $\{x \leq 0\}$ we have that

$$L_{\text{cusp}} \cap \{x \leq 0\} = (j^1 0_{\mathbb{R}} \sqcup j^1 1_{\mathbb{R}} \cap \{x \leq 0\}) \times Q = j^1 0_{\{x \leq 0\} \times Q} \sqcup j^1 1_{\{x \leq 0\} \times Q},$$

which is the situation described in the previous section. Therefore, given a closed submanifold $N \subseteq \mathbb{R} \times Q \cap \{x < 0\}$ with $\chi(N) = 0$, we define the $L_{\text{loose}} \coloneqq \mathrm{p}_N(L_{\text{cusp}})$.

Definition 4.2.3. Let (Y^{2n+1}, ξ) be a contact manifold of dimension $2n+1 \geq 5$, $L \subseteq Y$ a Legendrian submanifold and $U \subseteq Y$ an open neighbourhood such that

$$(U, L \cap U, \xi) \simeq (J^1 X, L_{\text{cusp}}, \xi_{\text{std}}),$$

for some manifold $X = \mathbb{R} \times Q$. The *stabilization* of L is the Legendrian $s(L)$ obtained by replacing $(U, L \cap U, \xi)$ by $(J^1 X, L_{\text{loose}}, \xi_{\text{std}})$. △

Definition 4.2.4. Let (Y^{2n+1}, ξ) be a contact manifold of dimension $2n+1 \geq 5$. A Legendrian submanifold $L \subseteq Y$ is *loose* if $L = s(L')$ for some Legendrian submanifold L'. △

The relevance of loose Legendrians and the justification of dropping N from the notation in the definitions relies on the following h-principle of Murphy:

Theorem 4.2.5 (Murphy [Mur19])**.** *Two loose Legendrians are Legendrian isotopic if and only if are formally Legendrian isotopic. Moreover, every formal Legendrian is formally Legendrian isotopic to a genuine Legendrian by a C^0-small isotopy.*

4.3 Legendrian spinning

In low dimensions, generic fronts are simple. In contrast, a front in high dimensions may be extremely singular and thus complicated to represent. In this section we introduce a construction of higher-dimensional fronts that uses lower-dimensional fronts as input. This construction is called *spinning*; it was introduced first in [EES05b] to construct of Legendrian submanifolds in $\mathbb{R}^{2n+1}$. It has been further studied in [EES09, Gol14, LC13].

4.3.1 Front spinning

Let us recall some basic facts about the 1-jet bundles, which is a crucial part of the spinning construction.

Lemma 4.3.1. *Let L_N and L_M be bundles over N and M, respectively, with 1-dimensional fibres. Consider a diagram of embeddings:*

$$\begin{array}{ccc} L_N & \xrightarrow{\varphi} & L_M \\ \downarrow & & \downarrow \\ N & \xrightarrow{e} & M \end{array}$$

Then, there is an induced 1-jet bundle map:

$$\hat{\varphi} : (J^1(L_N), \xi_{\text{std}}) \to (J^1(L_M), \xi_{\text{std}}),$$

which is an isocontact embedding.

In particular, $\hat{\varphi}$ sends isotropics to isotropics. In the case $\dim(N) = \dim(M)$, it defines a map between the corresponding spaces of Legendrians:

$$\hat{\varphi}_* : \mathfrak{Leg}(J^1(L_N), \xi_{\text{std}}) \to \mathfrak{Leg}(J^1(L_M), \xi_{\text{std}}).$$

Moreover, this construction works parametrically. If φ_s is a homotopy of fibred embeddings, then the induced isocontact embeddings $\hat{\varphi}_0$ and $\hat{\varphi}_1$ are contact isotopic. In particular, in the equidimensional case, the induced maps between Legendrian embedding spaces are homotopic.

Remark 4.3.2. In the case where $L_N = N \times \mathbb{R}$ and $L_M = M \times \mathbb{R}$ are trivial line bundles, the embedding $e\colon N \hookrightarrow M$ induces an isocontact embedding

$$T^*N \times \mathbb{R} \to T^*M \times \mathbb{R},$$

once a choice of Riemannian metrics has been made; note that in this standard construction the $\mathbb{R}$–coordinate is mapped identically.

However, for our purposes we need to modify the construction in order to impose an additional geometric condition on the resulting spinning, which requires us to shift the $\mathbb{R}$-coordinate. Concretely, we will only consider the case of trivial line bundles, and we will work with bundle maps of the form

$$\begin{aligned} \varphi : N \times \mathbb{R} &\to M \times \mathbb{R}, \\ (n, z) &\mapsto \big(e(n), z + h(n)\big); \end{aligned}$$

where $h \in C^\infty(N; \mathbb{R})$, and the lemma then provides the required isocontact embedding. △

The second easy observation that we use is the following:

Lemma 4.3.3. *For every smooth manifolds N^m and K, there exists a well-defined front-graph map:*

$$G_K : \mathrm{Map}(K, \mathfrak{Leg}(N, (J^1\mathbb{R}^m, \xi_{\mathrm{std}}))) \to \mathfrak{Leg}(K \times N, (J^1(K \times \mathbb{R}^m), \xi_{\mathrm{std}})).$$

Proof. Given a $L_k \in \mathrm{Map}(K, \mathfrak{Leg}(N, (J^1\mathbb{R}^m, \xi_{\mathrm{std}})))$, define the Legendrian embedding $G_K(L_k) \in \mathfrak{Leg}(K \times N, (J^1(K \times \mathbb{R}^m), \xi_{\mathrm{std}}))$ by the front map:

$$G_K(L_k) : K \times N \to J^0(K \times \mathbb{R}^m) = K \times J^0\mathbb{R}^m, \quad (k, n) \mapsto (k, \pi_{\mathrm{F}}(L_k(n))),$$

where $\pi_{\mathrm{F}} : J^1\mathbb{R}^m \to J^0\mathbb{R}^m$ is the front projection. □

Now, we can define the spinning as the following.

Definition 4.3.4. Let $e : K \times \mathbb{R}^m \to \mathbb{R}^n$ be a smooth embedding of codimension 0, covered by a fibred embedding $\varphi : J^0(K \times \mathbb{R}^m) \to J^0\mathbb{R}^n$. Let N be a smooth manifold of dimension m. The *spinning map* of N along K via φ is defined as the composition:

$$\hat{\varphi}_* \circ G_K : \mathrm{Map}(K, \mathfrak{Leg}(N, (J^1\mathbb{R}^m, \xi_{\mathrm{std}}))) \to \mathfrak{Leg}(K \times N, (J^1\mathbb{R}^n), \xi_{\mathrm{std}}).$$ △

4.3.2 A specific spinning

We will only use the spinning map in a very concrete situation, that we describe below. Let $\hat{e} : K \hookrightarrow J^0\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}_z$ be a smooth embedding graphical over $\mathbb{R}^n$, that means

$$\hat{e} = (\hat{e}_{\mathbb{R}^n}, h) : K \to \mathbb{R}^n \times \mathbb{R}_z,$$

where $\hat{e}_{\mathbb{R}^n} : K \hookrightarrow \mathbb{R}^n$ is also an embedding and $h \in C^\infty(K)$ is a smooth function. We further assume that the embedding $\hat{e}_{\mathbb{R}^n}$ has trivialisable normal bundle. We fix a framing Fr of the normal bundle of $\hat{e}_{\mathbb{R}^n}(K)$, to define an embedding:

$$e : K \times \mathbb{R}^m \to \mathbb{R}^n.$$

This codimension-0 embedding is naturally covered by the fibrewise isomorphism

$$\varphi : J^0(K \times \mathbb{R}^m) \to J^0\mathbb{R}^n, \quad (k, v, z) \mapsto (e(k, v), z + h(k)).$$

Note that the map φ is an affine isomorphism, and is invariant along the normal directions.

In this situation, given a family of Legendrian embeddings $L_k \in \mathrm{Map}(K, \mathfrak{Leg}(N, (J^1\mathbb{R}^m, \xi_{\mathrm{std}})))$, we will denote by

$$\mathrm{spin}_{\hat{e},\mathrm{Fr}}[L_k] \coloneqq \hat{\varphi}_* \circ G_K(L_k) \in \mathfrak{Leg}(N \times K, (J^1\mathbb{R}^n, \xi_{\mathrm{std}}))$$

the spinning of the family $\{L_k\}_{k \in K}$. In the cases that we will treat later the framing will be unique so we will simply write $\mathrm{spin}_{\hat{e}}[L_k]$, and if $L_k \equiv L$ is a constant family we will write $\mathrm{spin}_{\hat{e}}[L]$.

The following result follows by combining the Ekholm-Etnyre-Sullivan computation of the formal classes of a spinning [EES05a], together with Murphy classification of formal Legendrians [Mur19]. However, we provide a direct proof since it will be fundamental for our study of prelegendrians.

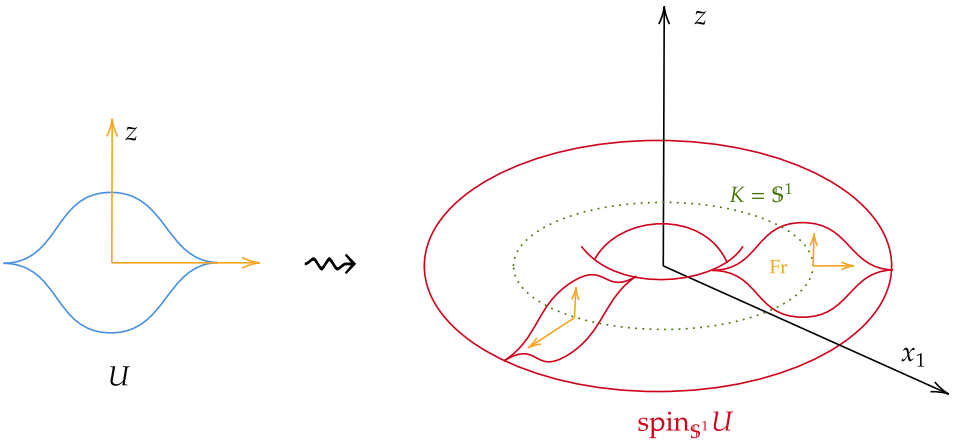


Figure 4.2: A schematic depiction of the front spinning of the Legendrian unknot $U \subseteq (J^1\mathbb{R}, \xi_{\text{std}})$ along the standard circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$.

Lemma 4.3.5. *Let $\hat{e} : K^{n-1} \to J^0\mathbb{R}^n$ be a smooth embedding, graphical over $\mathbb{R}^n$, with $\chi(K) = 0$. Fix moreover a framing* Fr.

If $L_0, L_1 \in \mathfrak{Leg}(J^1\mathbb{R}^1, \xi_{\text{std}})$ are two Legendrian knots with $\text{Rot}(L_0) = \text{Rot}(L_1)$, then $\text{spin}_{\hat{e},\text{Fr}}[L_0]$ and $\text{spin}_{\hat{e},\text{Fr}}[L_1]$ are formally Legendrian isotopic. Moreover, the formal Legendrian isotopy is induced by a sequence of Reidemeister moves of the 1-dimensional Legendrians L_0 and L_1, and K-pushings along parallel copies of $\hat{e}(K)$.

Proof. Following the notations above we write $e : K \times \mathbb{R}^2 \to J^0\mathbb{R}^n$, to denote the tubular neighbourhood embedding of $\hat{e}$ induced by Fr, in which we identify $\{k\} \times \mathbb{R}^2 = J^0\mathbb{R}$.

By the h-principle for Legendrian immersions in $\mathbb{R}^3$, there is a generic homotopy of Legendrian immersions L_t, $t \in [0, 1]$, between the given Legendrian knots. There is a finite number of times $0 < t_1 < \ldots < t_n < 1$ in which a unique transverse self-intersection of L_{t_i} appears at a point p_i. The spinning of L_t, $t \in [t_i - \varepsilon, t_i + \varepsilon]$, describes precisely a K-pushing along $e(K \times \{\pi_{\text{F}}(p_i)\})$. The result follows from Proposition 4.2.2. □

The original definition of spinning was given in [EES05b] for $K = \mathbb{S}^1$ and later generalized by Golovko in [Gol14] to $K = \mathbb{S}^k$. They work with the standard inclusion; specifically, they consider

$$\hat{e} : \mathbb{S}^k \hookrightarrow \mathbb{R}^{k+1} \times \{0\} \times \mathbb{R}_z \subseteq \mathbb{R}^{k+1} \times \mathbb{R}^{m-1} \times \mathbb{R}_z,$$

equipped with the canonical normal framing of $\mathbb{S}^k \subseteq \mathbb{R}^{k+1} \times \mathbb{R}^{m-1}$. In this case, we denote the spinning construction simply by $\text{spin}_{\mathbb{S}^k}[L]$ and refer to it as the *standard $\mathbb{S}^k$-spinning*. This yields a map

$$\text{spin}_{\mathbb{S}^k} : \mathfrak{Leg}(N^m, (J^1\mathbb{R}^m)) \to \mathfrak{Leg}(\mathbb{S}^k \times N^m, J^1\mathbb{R}^{m+k}).$$

Torus spinning

For our purposes (constructing prelegendrians via spinning along a general embedding of the torus) we need to verify that torus spinning produces the same Legendrian isotopy class as iterated standard $\mathbb{S}^1$-spinning. The reason is that we understand the effect that $\mathbb{S}^1$-spinning has on Legendrian Contact Homology; see [EES05b, Example 4.20].

The *standard k-torus* $\mathbb{T}^k \subseteq \mathbb{R}^{k+1}$ is defined inductively as follows:

- $\mathbb{T}^1 = \mathbb{S}^1 \subseteq \mathbb{R}^2$ is the standard inclusion.
- Assume that $\mathbb{T}^{k-1} \subseteq \mathbb{R}^k$ is defined. Consider

$$\mathbb{S}^1 \subseteq \mathbb{R}^2(x_1, x_2) \times \{0\} \subseteq \mathbb{R}^2(x_1, x_2) \times \mathbb{R}^{k-1}(x_3, \dots, x_{k+1})$$

  endowed with the canonical normal framing $\nu \simeq \operatorname{span}\langle \partial_r, \partial_{x_3}, \dots, \partial_{x_{k+1}} \rangle$. Identifying $\nu \cong \mathbb{R}^k$ and using the standard inclusion $\mathbb{T}^{k-1} \subseteq \mathbb{R}^k \simeq \nu$, we obtain the standard embedding $\mathbb{T}^k \subseteq \mathbb{R}^{k+1}$.

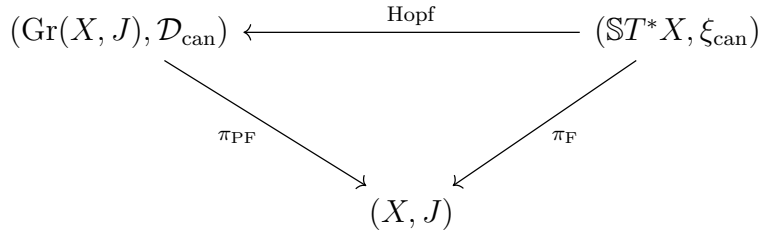
For the standard k-torus $\mathbb{T}^k \subseteq \mathbb{R}^{k+1}$ there is also a canonical normal framing; therefore we denote the corresponding spinning simply by $\operatorname{spin}_{\mathbb{T}^k}[L]$ for $L \in \mathfrak{Leg}(J^1\mathbb{R}^m, \xi_{\text{std}})$, in complete analogy with the k-sphere case. The following lemma is immediate from the definition of the standard k-torus.

Lemma 4.3.6. *Let $L \subseteq J^1\mathbb{R}^m$ be a Legendrian submanifold. Consider the standard torus $\mathbb{T}^k \subseteq \mathbb{R}^{k+1}$, then*

$$\operatorname{spin}_{\mathbb{T}^k}[L] \simeq (\operatorname{spin}_{\mathbb{S}^1})^k[L].$$

4.4 Prelegendrian fronts

Consider an almost complex manifold (X, J), and the associated fat manifold $(\operatorname{Gr}(X, J), \mathcal{D}_{\text{can}})$. By Theorem 3.3.7, the contactisation $\mathcal{C}(\operatorname{Gr}(X, J), \mathcal{D}_{\text{can}})$ is contactomorphic to the unit cotangent bundle $\mathbb{S}T^*X$, this yields the commutative diagram:



For simplicity, we assume throughout that all prelegendrians are *co-oriented* (see Section 3.1.3).

Given an embedded prelegendrian $\Lambda \subseteq \mathrm{Gr}(X, J)$, we will refer to $\pi_{\mathrm{PF}}(\Lambda) \subseteq (X, J)$ as a *prelegendrian front*.

Every generic Legendrian submanifold of $\mathcal{L}(\Lambda) \subseteq \mathbb{S}T^*X$ can be represented by its front $\mathcal{F} = \pi_{\mathrm{F}}(\mathcal{L}(\Lambda))$ in X. However, not every front $\mathcal{F}$ in X is a prelegendrian front. Strong geometric constraints must be satisfied along the singularities of the front. The main result of this section, is a complete characterization of which *simple* fronts are prelegendrian. It is a rephrasing of Theorem 2.2.1, from the Introduction:

Theorem 4.4.1 (Simple Prelegendrian Fronts)**.** *A simple front $\mathcal{F} \subseteq (X, J)$ is a prelegendrian front if and only if its singular loci $\Sigma = \Sigma(\mathcal{F})$ are co-real. Moreover, the prelegendrian $\Lambda \subseteq (\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$ determined by $\mathcal{F}$ is unique.*

Remark 4.4.2. We point out that a simple front $\mathcal{F} \subseteq (X, J)$ whose cusp locus is co-real but whose self-intersection locus is not, also defines a unique *immersed* prelegendrian $\Lambda \subseteq (\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$. The self-intersection points of Λ are precisely the preimage by π_{PF} of the complex points of the self-intersection locus. △

4.4.1 Proof of Theorem 4.4.1

Since the property of being a prelegendrian front is local, it suffices to work with the germ of $\mathcal{F}$. We therefore argue case by case according to the singularity type.

Let $p \in \mathcal{F} \subseteq (X, J)$. Since $\mathcal{F}$ is a simple front, there exists a diffeomorphism from an open neighbourhood of p in X onto an open neighbourhood of 0 in $\mathbb{R}^{2n+2}$:

$$\psi : (\mathcal{O}p\,(p), p) \to (\mathbb{R}^{2n+2}, 0),$$

such that, via ψ, the germ of the front $\mathcal{F}$ at $(\mathcal{O}p\,(p), p)$ is in the normal form of Definition 4.1.6, namely:

(i) In the case of A_1-singularity, $\mathcal{F} = \mathbb{R}^{2n+1} \times \{0\} \subseteq \mathbb{R}^{2n+2}$ is the horizontal hyperplane. These are smooth points of the front.

(ii) In the case of A_1A_1-singularity, the front $\mathcal{F}$ is the union of two hyperplanes intersecting transversely at 0. This corresponds to a transverse self–intersection of the front.

(iii) In the case of an A_2-singularity, $\mathcal{F}$ is parametrized by the front map $f : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+2}$ given by

$$\begin{aligned} f : (\mathbb{R}^{2n+1}, 0) &\to (\mathbb{R}^{2n+2}, 0) \\ (q_1, \ldots, q_{2n}, t) &\mapsto (q_1, \cdots, q_{2n}, -3t^2, 2t^3) \end{aligned} \tag{4.1}$$

This corresponds to cusp points of the front.

For simplicity, we continue to denote by J the pushforward almost–complex structure $\psi_* J$ on $\mathbb{R}^{2n+2}$. Since the almost–complex Grassmannian construction is functorial, we may reduce to the case $X = \mathbb{R}^{2n+2}$, with $\mathcal{F}$ one of the three types above.

We treat each case separately. We begin with the smooth stratum (i). In this case, the prelegendrian is determined by the complex tangencies of the front (in the contact setting, by the real tangencies). This is the content of the following:

Proposition 4.4.3 (Smooth points)**.** *The (oriented) horizontal hyperplane* $\mathcal{F} = \mathbb{R}^{2n+1} \times \{0\} \subseteq \mathbb{R}^{2n+2}$ *admits a prelegendrian lift:*

$$\begin{aligned} g : \mathbb{R}^{2n+1} &\to \mathrm{Gr}(\mathbb{R}^{2n+2}, J) = (\mathbb{R}^{2n+2} \times \mathbb{CP}^n, \mathcal{D}_{\mathrm{std},J}); \\ \mathbf{q} &\mapsto \big((\mathbf{q}, 0); H \cap J_x H\big), \end{aligned}$$

such that the following diagram commutes:

$$\begin{array}{ccc} & & \mathbb{S}T^*\mathbb{R}^{2n+2} = \mathbb{R}^{2n+2} \times \mathbb{S}^{2n+1} \\ & \overset{\mathcal{L}(g)}{\nearrow} & \big\downarrow \mathrm{Hopf} \\ \mathbb{R}^{2n+1} & \xrightarrow{\quad g \quad} & \mathrm{Gr}(\mathbb{R}^{2n+2}, J) = \mathbb{R}^{2n+2} \times \mathbb{CP}^n \end{array} \tag{4.2}$$

where $\mathcal{L}(g)(\mathbf{q}) = \big((\mathbf{q}, 0); H\big)$ *is the Legendrian lift of* g*.*

Proof. The fibrewise Hopf map $\mathrm{Hopf} : \mathbb{S}T^*\mathbb{R}^{2n+2} \to \mathrm{Gr}(\mathbb{R}^{2n+2}, J)$ (see Section 3.2) sends an oriented hyperplane to its complex part. The Legendrian determined by the hyperplane front $\mathcal{F}$ is parametrised by

$$(q_1, \dots, q_{2n+1}) \to (q_1, \dots, q_{2n+1}, 0; H) \in \mathbb{R}^{2n+2} \times \mathbb{S}^{2n+1}$$

Thus the prelegendrian map is the composition

$$\begin{aligned} g : \mathbb{R}^{2n+1} &\to \mathbb{R}^{2n+2} \times \mathbb{S}^{2n+1} \to \mathbb{R}^{2n+2} \times \mathbb{CP}^n, \\ \mathbf{q} &\mapsto (\mathbf{q}, 0; H_0) \mapsto (\mathbf{q}, 0 : H_0 \cap JH_0). \end{aligned}$$

Since the first projection parametrises $\mathcal{F}$, g is an embedding; and it is prelegendrian by the definition of $\mathcal{D}_{\mathrm{can}}$. □

Now, we deal with the self-intersection points (ii). In this case, the prelegendrian lift of each smooth branch is given by the complex tangencies of the front. The lift of both branches will be embedded if and only if the complex tangencies of each branch are transverse. Since they intersect in a $2n$-dimensional space, it is enough to require this intersection to not be complex:

Proposition 4.4.4 (Self-intersections)**.** *Let* $\mathcal{F} = H_1 \cup H_2 \subseteq \mathbb{R}^{2n+2}$ *be the front given by the union of two transversal hyperplanes. Then, it lifts to an embedded prelegendrian if and only if* $\Sigma = H_1 \cap H_2$ *is co-real.*

Proof. We have that

$$\Sigma^{2n} \cap J(\Sigma^{2n}) = (H_1 \cap JH_1)^{2n} \cap (H_2 \cap JH_2)^{2n} \subseteq \Sigma^{2n}.$$

By dimensional reasons, $H_1 \cap JH_1 = H_2 \cap JH_2$ if and only if Σ is J-complex. □

Finally, we consider the cusp stratum (iii). By the previous case, there is a genuine prelegendrian lift along the smooth stratum, which extends uniquely to the cusp points. We must check that this lift is an immersion. In the contact case this is true because the tangent space of the front rotates continuously when one crosses the cusp locus transversely. It follows that we must check that the complex tangencies also rotate across the cusp locus. The following geometric condition ensures this:

Proposition 4.4.5 (Cusps points)**.** *Let $f : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+2}$ be the front map given in (4.1), and $\mathcal{F} = f(\mathbb{R}^{2n+1})$. Then f lifts to a prelegendrian embedding if and only if $\Sigma = \Sigma(\mathcal{F})$ is co-real.*

Proof. First, write the coordinates of $\mathbb{R}^{2n+2}$ as $(x_1, \dots, x_{2n+2})$. Define the family of 1-forms $\alpha_t \coloneqq t\,\mathrm{d}x_{2n+1} + \mathrm{d}x_{2n+2}$. A simple computation shows that the tangent space of the front $\mathcal{F}$ is

$$T_{f(\mathbf{q},t)}\mathcal{F} = \ker \alpha_t.$$

The singular set of $\mathcal{F}$ is

$$\Sigma = f(\mathbb{R}^{2n} \times \{0\}) = \{x_{2n+1} = x_{2n+2} = 0\} \subseteq \mathcal{F},$$

with tangent space

$$T_{f(\mathbf{q},0)}\Sigma = \operatorname{span}(\partial_{x_1}, \dots, \partial_{x_{2n}}) = \ker(\mathrm{d}x_{2n+1}) \cap \ker(\mathrm{d}x_{2n+2}),$$

Therefore, the Legendrian embedding associated to f is

$$\begin{aligned} \tilde{g} : \mathbb{R}^{2n+1} &\to \mathbb{R}^{2n+2} \times \mathbb{S}^{2n+1}; \\ (\mathbf{q}, t) &\mapsto (f(\mathbf{q}, t), \alpha_t), \end{aligned}$$

with dual identification. According to Proposition 4.4.3, the associated prelegendrian map must be

$$\begin{aligned} g : \mathbb{R}^{2n+1} &\to \mathbb{R}^{2n+2} \times \mathbb{CP}^n; \\ (\mathbf{q}, t) &\mapsto (f(\mathbf{q}, t); (\alpha_t)^J), \end{aligned}$$

where

$$(\alpha_t)^J = t(\mathrm{d}x_{2n+1})^J + (\mathrm{d}x_{2n+2})^J$$

is the complexification of α_t.

Observe that g is injective, and, moreover, is an embedding away from $t = 0$. At the points of $\{t = 0\}$, the vectors $\{Df_{(\mathbf{q},0)}(\partial_{q_i}), i = 1, \dots, 2n\} \subseteq T_{f(\mathbf{q},t)}\mathbb{R}^{2n+2}$ span a $2n$-dimensional subspace, while $Df_{\mathbf{q},0}(\partial_t) = 0$. Therefore, g will be an immersion at a point $(\mathbf{q}, 0)$ if and only if $Dg_{(\mathbf{q},0)}(\partial_t) = (0, D_{(\mathbf{q},0)}(\alpha_t)^J(\partial_t)) \neq 0$.

Since $T_{f(\mathbf{q},0)}\Sigma = \ker(\mathrm{d}x_{2n+1}) \cap \ker(\mathrm{d}x_{2n+2})$, Lemma 3.2.4 applies, i.e. $T_{f(\mathbf{q},0)}\Sigma$ is J-invariant if and only if the complexifications $(\mathrm{d}x_{2n+1})^J$ and $(\mathrm{d}x_{2n+2})^J$ are $\mathbb{C}$–linearly dependent in $\operatorname{Hom}_{\mathbb{C}}(T_{f(\mathbf{q},0)}\mathbb{R}^{2n+2}, \mathbb{C})$.

Assume first that Σ is co-real. Then, $\{(\mathrm{d}x_{2n+1})^J, (\mathrm{d}x_{2n+2})^J\}$ are $\mathbb{C}$-linearly independent along Σ. Since being linearly independent is an open coition, we may further assume that they are linearly independent on $\mathbb{R}^{2n+2}$. Therefore, $\{(\mathrm{d}x_{2n+1})^J, (\mathrm{d}x_{2n+2})^J\}$

can be completed to a $\mathbb{C}$-basis of $\mathrm{Hom}_{\mathbb{C}}(T\mathbb{R}^{2n+2}, \mathbb{C})$. In that basis, the value of g in the $\mathbb{CP}^n$ component is of the form $[0 : \cdots : 0 : t : 1]$. In particular, the derivative with respect to t is non-zero, hence g is an immersion.

Conversely, if Σ is not co-real, then $T_{f(\mathbf{q}_0,0)}\Sigma$ is J-invariant for some $\mathbf{q}_0 \in \mathbb{R}^{n+1}$, hence $(\,\mathrm{d}x_{2n+1})^J$, $(\,\mathrm{d}x_{2n+2})^J$ are $\mathbb{C}$-linearly dependent at the point $f(\mathbf{q}_0, 0)$. As in the previous case, we complete $(\,\mathrm{d}x_{2n+2})^J$ to a basis $\{v_1, \ldots, v_n, (\,\mathrm{d}x_{2n+2})^J\}$ over $\mathbb{R}^{2n+2}$. We express $(\,\mathrm{d}x_{2n+1})^J$ as a linear combination of this basis along $\mathcal{F}$:

$$(\,\mathrm{d}x_{2n+1})^J = a_1(\mathbf{q}, t)v_1 + \cdots + a_n(\mathbf{q}, t)v_n + a_{n+1}(\mathbf{q}, t)(\,\mathrm{d}x_{2n+2})^J,$$

where $a_i(\mathbf{q}, t) \in \mathbb{C}$, $i = 1, \ldots, n+1$, are the coefficients. Moreover, by the assumption of $(\,\mathrm{d}x_{2n+1})^J$ and $(\,\mathrm{d}x_{2n+2})^J$ being $\mathbb{C}$-linearly dependent at the point $f(\mathbf{q}_0, 0)$ we have that:

- $a_{n+1}(\mathbf{q}_0, 0) \neq 0$
- $a_i(\mathbf{q}_0, t) = O(t)$ for $i = 1, \ldots, n$.

Therefore, the value of $g(\mathbf{q}_0, t)$ with t sufficiently small, in the $\mathbb{CP}^n$ component is of the form

$$\begin{aligned}[ta_1(\mathbf{q}_0, t) : \cdots : ta_n(\mathbf{q}_0, t) : 1 + ta_{n+1}(\mathbf{q}_0, t)] &= [O(t^2) : \cdots : O(t^2) : 1 + O(t)] \\ &= [O(t^2) : \cdots : O(t^2) : 1].\end{aligned}$$

This implies that the derivative of g along the t-direction vanishes at the point $(\mathbf{q}, 0)$, and g fails to be an immersion.

□

4.4.2 Simple prelegendrian Reidemeister moves

The study of prelegendrian simple fronts conducted above can be generalised to 1-parametric families. We will say that a one–parameter family of Legendrian fronts $(\mathcal{F}_s)_{s\in[0,1]}$ in (X, J) undergoes a Reidemeister I move if (1) there is a closed coorientable hypersurface $O \subseteq \mathcal{F}_0$ contained in the smooth locus such that (2) $\mathcal{F}_s$ is supported in $\mathcal{O}p\,(O) \cong O \times \mathbb{R}^2$ with $\mathcal{O}p\,(O) \cap \mathcal{F}_0 \cong O \times \mathbb{R}$, and (3) $\mathcal{F}_s$ is the product of O with the standard 1-dimensional move of $\mathbb{R}$ in $\mathbb{R}^2$ (Fig. 4.3). The Reidemeister moves II and III are defined similarly by considering O to be one of the components of the cusp and self-intersection loci, respectively. Then:

Lemma 4.4.6. *Suppose that $\mathcal{F}_0$ is a simple prelegendrian front. Introduce a Reidemeister move along a coreal submanifold $O \subseteq \mathcal{F}_0$. Then, the resulting family of fronts $(\mathcal{F}_s)_{s\in[0,1]}$ is prelegendrian.*

Proof. By construction, the singularities of $\mathcal{F}_{1/2}$ occur along an embedded co-real submanifold. The argument in the proof of Proposition 4.4.4 applies verbatim for the RII and RIII moves. For the RI move, the argument from Proposition 4.4.5 applies. □

The resulting prelegendrian homotopies are referred to as *prelegendrian Reidemeister* I, II and III moves.

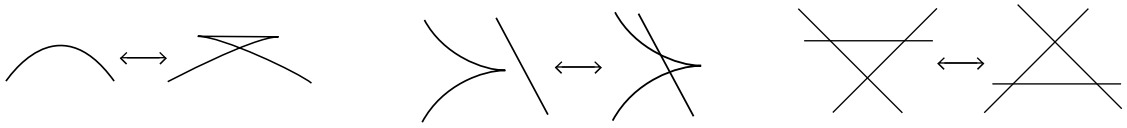


Figure 4.3: From left to right the fronts $\hat{\mathcal{F}}_s \subseteq J^0\mathbb{R}$, $s \in [0,1]$, of the 1-dimensional Reidemeister moves I, II and III. The higher dimensional moves are locally modelled on $\hat{\mathcal{F}}_s \times \mathbb{R}^n \subseteq J^0\mathbb{R} \times \mathbb{R}^n = J^0\mathbb{R}^{n+1}$, $s \in [0,1]$.

Remark 4.4.7. In plain words, Lemma 4.4.6 says that Reidemeister moves for prelegendrian fronts can be performed along co-real submanifolds. In particular, let $\mathcal{F}$ be a prelegendrian front and $N \subseteq \Sigma_{A_1}(\mathcal{F})$ a codimension 1 co-real submanifold in the smooth strata. Then, there exists a homotopy of prelegendrian fronts $\mathcal{F}_s$, $s \in [0,1]$, that describes an RI-move along N. Moreover, since being co-real is an open condition we may achieve that the region in the prelegendrian front $\mathcal{F}_1$ in which the RI appears is foliated by co-real submanifolds parallel to N. △

Remark 4.4.8. A generic 1-parametric family of Legendrians in higher-dimensions will undergo a wider class of bifurcations than those introduced above. We expect the same to be true for prelegendrians. Moreover, the three Reidemeister moves shown here are not generic and will generally appear as combinations of more elementary moves. Developing the theory further in this direction is left as an open question. △

4.4.3 Prelegendrian spinning

Theorem 4.4.1 provides a criterion ensuring that, under mild assumptions, the Legendrians in $(\mathbb{R}^{4n+3}, \xi_{\text{std}}) = (J^1\mathbb{R}^{2n+1}, \xi_{\text{std}})$ produced by the Legendrian spinning from Section 4.3.2 admit a prelegendrian front and therefore define prelegendrian embeddings in $(\mathbb{R}^{4n+2}, \mathcal{D}_{\text{std},J})$. This yields an effective method to build prelegendrian embeddings and leads us to the construction of exotic prelegendrian tori in Section 5.2.

The setup is as follows. Consider $\mathbb{R}^{2n+2} = J^0\mathbb{R}^{2n+1}$ equipped with an almost complex structure J. As explained in Section 4.3.2, given an embedding $\hat{e} = (\hat{e}_{\mathbb{R}^{2n+1}}, h) : K \hookrightarrow J^0\mathbb{R}^{2n+1}$ of codimension $m+1$ that is graphical over $\mathbb{R}^{2n+1}$, together with a framing of $\hat{e}_{\mathbb{R}^{2n+1}}$, there is a well-defined Legendrian spinning map. This operation takes as input a Legendrian $L \subseteq (J^1\mathbb{R}^m, \xi_{\text{std}})$ and produces a higher-dimensional Legendrian

$$\text{spin}_{\hat{e}}[L] \subseteq (J^1\mathbb{R}^{2n+1}, \xi_{\text{std}})$$

Our first observation concerns the singularities of the front of the Legendrian spinning:

Lemma 4.4.9. *The singular loci of the Legendrian spinning* $\pi_{\mathrm{F}}(\mathrm{spin}_{\hat{e}}[L])$ *satisfies:*

$$\Sigma\big(\pi_{\mathrm{F}}(\mathrm{spin}_{\hat{e}}[L])\big) = \varphi\big(K \times \pi_{\mathrm{F}}(L)\big),$$

In particular,

$$\Sigma\big(\pi_{\mathrm{F}}(\mathrm{spin}_{\hat{e}}[L])\big) = \bigsqcup_{p \in \Sigma(\pi_{\mathrm{F}}(L))} \varphi(K \times \{p\})$$

is foliated by submanifolds diffeomorphic to K*. Moreover, the leaves of this foliation are* C^∞*-close to* $\hat{e}(K) = \varphi(K \times \{0\})$ *whenever* $\pi_{\mathrm{F}}(L)$ *is* C^∞*-small.*

It is important to realize that every Legendrian in $(J^1\mathbb{R}^m, \xi_{\mathrm{std}})$ is Legendrian isotopic to a Legendrian whose front is C^∞-small by using the contact isotopy $\delta_t(\mathbf{x}, \mathbf{y}, z) = (t\mathbf{x}, t\mathbf{y}, t^2 z)$, for $t > 0$.

An important consequence is the following:

Corollary 4.4.10. *Assume that* $\hat{e} : K \hookrightarrow (\mathbb{R}^{2n}, J)$ *is a co-real embedding. Let* $L \subseteq (J^1\mathbb{R}^m, \xi_{\mathrm{std}})$ *be a Legendrian with sufficiently* C^∞*-small simple front. Then, the front* $\pi_{\mathrm{F}}\big(\mathrm{spin}_{\hat{e}}[L]\big)$ *is a prelegendrian front. In particular, it defines a prelegendrian embedding*

$$\mathrm{prespin}_{\hat{e}}[L] \subseteq (\mathbb{R}^{4n+2}, \mathcal{D}_{\mathrm{std},J})$$

that will be referred as prelegendrian spinning *of* L *along the (co-real embedding)* $\hat{e}$*.*

Proof. The previous lemma combined with Lemma 3.2.8 implies that the singularities of the front are co-real whenever $\pi_{\mathrm{F}}(L)$ is sufficiently C^∞-small. The result follows from Theorem 4.4.1. □

Remark 4.4.11. By definition, the Legendrian lift of the prelegendrian spinning is the Legendrian spinning:

$$\mathcal{L}(\mathrm{prespin}_{\hat{e}}[L]) = \mathrm{spin}_{\hat{e}}[L]. \qquad \triangle$$

Remark 4.4.12. We continue using the notation of the corollary, and work still under the same C^∞-small assumption on the fronts. Assume that $L_s \subseteq (J^1\mathbb{R}^m, \xi_{\mathrm{std}})$, $s \in [0,1]$, is a Legendrian isotopy whose front describes a Reidemeister I, II or III move between two simple fronts L_0 and L_1. Then, by Lemma 4.4.6, there is a prelegendrian isotopy

$$\mathrm{prespin}_{\hat{e}}[L_s] \subseteq (\mathbb{R}^{4n+2}, \mathcal{D}_{\mathrm{std},J}), s \in [0,1],$$

describing an RI, RII or RII prelegendrian move, respectively. In particular, assume that the co-real embedding $\hat{e}$ has codimension 2 and, therefore, L is a 1-dimensional Legendrian. Then, the prelegendrian spinning just depends, up to prelegendrian isotopy, on the Legendrian isotopy class of L. $\triangle$

4.5 Prelegendrian stabilization

In this section we develop the prelegendrian N-pushing operation along a co-real submanifold N, describe its effect on the formal prelegendrian class, and explain how to stabilize prelegendrians. This will lead us to the proof of Theorem 2.2.4 in the next section.

4.5.1 Prelegendrian N-pushing

In the contact case, the N-pushing construction is defined in the local model of the 1-jet bundle, where a natural front projection is available, and then transported via a Darboux chart to any contact manifold. For fat distributions of corank 2 (or higher), there is no Darboux-type theorem. For this reason, we restrict our definition to the almost complex Grassmannian, which admits a prelegendrian front projection.

Model case

First we work in a local model where $X \coloneqq J^0Q = Q \times \mathbb{R}(z)$, which allows us to consider the horizontal Grassmannians:

$$\begin{array}{ccccc} (\mathrm{Gr}_{\mathrm{hor}}(X,J), \mathcal{D}_{\mathrm{std},J}) & & \xleftarrow{\quad\text{Hopf}\quad} & & (\mathbb{S}T^*_{\mathrm{hor}}X, \xi_{\mathrm{can}}) \simeq (J^1Q, \xi_{\mathrm{std}}) \\ & \searrow^{\pi_{\mathrm{PF}}} & & \swarrow_{\pi_{\mathrm{F}}} & \\ & & (X,J) & & \end{array}$$

We use the notation of Section 4.2.1.

Proposition 4.5.1. *Let $\Lambda \subseteq \mathbb{C}^{2n+1}$ be the prelegendrian with $\pi_{\mathrm{PF}}(\Lambda) = j^0 0_Q \sqcup j^0 1_Q \subseteq J^0Q$. Suppose that $N \subseteq j^0 0$ is a co-real submanifold that admits a non-zero section in $\nu_{N\setminus Q}$. Then, there exists a prelegendrian $\mathrm{p}_N(\Lambda)$ such that*

- $\pi_{\mathrm{PF}}(\mathrm{p}_N(\Lambda)) = j^0 f \cup j^0 1_Q$,
- $\mathcal{L}(\mathrm{p}_N(\Lambda)) = \mathrm{p}_N(\mathcal{L}(\Lambda))$,
- Λ *and* $\mathrm{p}_N(\Lambda)$ *coincide outside of* $\mathrm{Gr}_{\mathrm{hor}}(\mathcal{O}p_X(N \times [0,1]), J)$,
- $\mathrm{p}_N(\Lambda)$ *is formally isotopic to* Λ *as prelegendrians.*

The prelegendrian $\mathrm{p}_N(\Lambda)$ is said to be an N-pushing of Λ.

Proof. By abuse of notation, given $A \subseteq \mathbb{R}$ we write $f^{-1}(A)$ to mean $j^0\big(f|_{f^{-1}(A)}\big) \subseteq j^0 f$.

Consider the front $\mathcal{F} \coloneqq j^0 f \cup j^0 1 \subseteq \mathbb{C}^{n+1}$. This is a simple front: its singular set consists of the transverse self-intersection $j^0 f \cap j^0 1 = f^{-1}(\{1\})$. Set $M \coloneqq f^{-1}([1,2]) \subseteq \mathbb{R}^{2n+1}$ and thus $\partial M = f^{-1}(\{1\})$. If we allow ourselves to homotope the section $j^0 1$ (relative boundary) and to shrink the model, we can assume, using the fact that co-reality is an open condition, that every shifted copy $N \times \{c\}$ is co-real and so is ∂M. Therefore, $\mathcal{F}$ is a prelegendrian front by Lemma 3.2.8 and thus defines a prelegendrian Λ'. We set $\mathrm{p}_N(\Lambda) \coloneqq \Lambda'$ and the first three claims follow.

The last claim will follow by an argument similar to the one in Proposition 4.2.2. In this case, the extra coordinates of the prelegendrian, which are an element of

$\mathbb{C}^n \subseteq \mathbb{CP}^n$, can be interpreted as the slope of the complex tangencies of the front. Suppose we now choose a non-vanishing vector field $Z \in T(j^0 f)^J$ defined over $\mathcal{O}p\,(M)$. By construction it is complex linearly independent from ∂_z and we can rotate it towards the latter. This provides a formal rotation of the complex tangencies, making them almost vertical. Once we do this, we can apply the reasoning in Proposition 4.2.2 to conclude.

It remains to produce such a $Z \in T(j^0 f)^J$. Since both ∂M and M are co-real, the same argument as in Lemma 3.2.10 shows that there is a splitting

$$TM|_{\partial M} = T(\partial M) \oplus \eta,$$

with $\eta \subseteq TM^J$ and $\eta \simeq \nu_{\partial M \setminus M}$. Choose a generator of the line bundle η that is inward-pointing along ∂M. This yields a vector field

$$V : \mathcal{O}p_M(\partial M) \longrightarrow TM^J,$$

that points transversely inward along ∂M.

Since the gradient ∇f is inward-pointing on ∂M, we can homotope it to V near the boundary and, by applying Proposition 3.2.9, obtain a vector field Z on $j^0 f$ such that

- $Z \in (T(j^0 f))^J$ on $f^{-1}([1,2])$,
- $Z = V$ on $f^{-1}([1-\varepsilon, 1+\varepsilon])$,
- $Z = \nabla f$ on $f^{-1}([0, 1-\varepsilon])$.

This concludes the proof. □

General case

We now describe the N–pushing in the general almost–complex Grassmannian setting:

Theorem 4.5.2 (Prelegendrian N-pushing)**.** *Let $\Lambda \subseteq (\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}})$ be a prelegendrian, $\mathcal{F} \coloneqq \pi_{\mathrm{PF}}(\Lambda)$, N a closed manifold, and $\Phi : N \times [0,1] \hookrightarrow X$ an embedding such that:*

- *$N_0 \coloneqq \Phi(N \times \{0\})$, $N_1 \coloneqq \Phi(N \times \{1\})$ are contained in the smooth strata $\Sigma_{A_1}(\mathcal{F})$.*
- *N_1 is a co-real submanifold of (X,J).*
- *$\Phi(N \times [0,1]) \cap \mathcal{F} = \Phi(N \times \{0,1\})$,*
- *the normal bundle $\nu_{N_1 \setminus \mathcal{F}}$ admits a non-zero section.*

Then, there exists a prelegendrian $\mathrm{p}_N(\Lambda)$ such that:

- $\Lambda \cap \mathrm{Gr}(X \backslash \mathcal{O}p\,(\Phi), J) = \mathrm{p}_N(\Lambda) \cap \mathrm{Gr}(X \backslash \mathcal{O}p\,(\Phi), J)$.
- *The Legendrian lifts* $\mathcal{L}(\Lambda)$ *and* $\mathcal{L}(\mathrm{p}_N(\Lambda))$ *in* $(\mathbb{P}T^*X, \xi_{\mathrm{std}})$ *satisfy:*

$$\mathcal{L}(\mathrm{p}_N(\Lambda)) = \mathrm{p}_N(\mathcal{L}(\Lambda))$$

- $\mathrm{p}_N(\Lambda)$ *and* Λ *are formally prelegendrian isotopic.*

The prelegendrian $p_N(\Lambda)$ *is said to be an* N*-pushing of* Λ.

Proof. Let $Q \coloneqq \mathcal{O}p_{\mathcal{F}}(N_0)$ be a neighbourhood of N_0 in $\mathcal{F}$. The embedding $\Phi : N \times [0,1] \hookrightarrow X$ extends to an embedding $\widetilde{\Phi} : Q \times [0,1] \hookrightarrow X$. Thus $\Phi(Q \times [0,1]) \subseteq (X, J)$ is identified with the model $(Q \times [0,1], \widetilde{\Phi}^* J)$ of Proposition 4.5.1, which yields the result. □

4.5.2 Prelegendrian stabilization

Following the contact geometric approach, we can use the prelegendrian N-pushing operation to define a prelegendrian stabilization.

Definition 4.5.3. Let $\Lambda \subseteq (\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$ be a prelegendrian, $p \in \mathcal{F} = \pi_{\mathrm{PF}}(\Lambda)$ be a smooth point, and $N \subseteq \mathcal{O}p\,(p) \cap \mathcal{F}$ a co-real hypersurface. We define a new prelegendrian as follows:

(i) Perform a prelegendrian RI move on Λ along N, see Remark 4.4.7. Denote the resulting prelegendrian by Λ_1.

(ii) Apply the N-pushing operation described in Theorem 4.5.2 to Λ_1 along a co-real $N' \cong N$ of $\mathcal{O}p\,(N) \cap \pi_{\mathrm{PF}}(\Lambda_1)$ very close to the cusp locus of the RI-move. Denote the resulting prelegendrian by $s(\Lambda)$

The prelegendrian $s(\Lambda)$ is said to be a *stabilization* of Λ. △

The following highlights the key property of the prelegendrian stabilization:

Proposition 4.5.4. *The prelegendrians* Λ *and* $s(\Lambda)$ *are formally prelegendrian isotopic. Moreover,*

$$s(\mathcal{L}(\Lambda)) \simeq \mathcal{L}(s(\Lambda)).$$

In particular, $\mathcal{L}(s(\Lambda))$ *is loose.*

Remark 4.5.5. Even though the Legendrian lift of the prelegendrian stabilization is independent of the choice of co-real submanifold (by the h-principle of Theorem 4.2.5), it is unclear whether the stabilized prelegendrian satisfies an h-principle as well. It is an open question whether prelegendrian stabilization depends on the choice of co-real submanifold N. △

Chapter 5

Existence of prelegendrian stabilisation and exotic prelegendrians

In this chapter we establish the existence of compact prelegendrians in the standard corank-2 fat space $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$, as well as the prelegendrian stabilisation operation introduced in the previous chapter. We then use these tools to construct infinite families of exotic prelegendrians, thereby proving the main rigidity results of Part I. Finally, we extend these constructions to more general fat manifolds via nilpotentisation and stability arguments.

5.1 Existence of compact prelegendrians and prelegendrian stabilizations

In this section we establish the existence of compact prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$ and prelegendrian stabilizations. To do this, we will give an explicit family of a co-real submanifolds in $\mathbb{C}^{n+1}$ which is graphical over $\mathbb{R}^{2n+1} \subseteq \mathbb{C}^{n+1} = \mathbb{R}^{2n+2} = \mathbb{R}^{2n+1} \times \mathbb{R}$. The existence of these submanifolds implies then the existence of prelegendrian stabilizations as discussed in the previous section, thus yielding the proof of Theorem 2.2.4.

5.1.1 Co-real submanifolds in $\mathbb{C}^{n+1}$ graphical over $\mathbb{R}^{2n+1}$

We begin by proving the existence of suitable co-real submanifolds to perform our prelegendrian spinning. After this, Corollary 4.4.10 trivially implies the existence of compact prelegendrians in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$.

Proposition 5.1.1. *Let $F \subseteq \mathbb{R}^n$ be a closed submanifold. Then, there exists a co-real embedding $\hat{e} : \mathbb{T}^{n+1} \times F \hookrightarrow \mathbb{C}^{n+1} = \mathbb{R}^{2n+2} = \mathbb{R}^{2n+1} \times \mathbb{R}$ which is graphical over $\mathbb{R}^{2n+1}$. In particular, there exists a co-real embedding of $\mathbb{T}^{2n}$.*

Proof of Proposition 5.1.1. We use polar coordinates (r_i, θ_i), in each $\mathbb{C}$-factor of $\mathbb{C}^{n+1} = \mathbb{C} \times \cdots \times \mathbb{C}$. Our construction takes place close to the Clifford torus $\mathbb{T}^{n+1}_{\mathrm{Cl}} \subseteq \mathbb{C}^{n+1}$, defined as the image of the embedding:

$$\begin{aligned} \mathrm{Cl} : \mathbb{T}^{n+1} = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 &\hookrightarrow \mathbb{C}^{n+1}; \\ (\theta_1, \ldots, \theta_{n+1}) &\mapsto \Big((1,\theta_1); \cdots, (1,\theta_n); (1,\theta_{n+1})\Big). \end{aligned}$$

The Clifford torus is clearly not graphical over $\mathbb{R}^{2n+1}$. However, the family of perturbations

$$\mathrm{Cl}^\delta(\theta_1, \ldots, \theta_{n+1}) = \Big((1+\delta\sin(\theta_{n+1}),\theta_1); \cdots, (1,\theta_n); (1,\theta_{n+1})\Big),$$

for $0 < \delta < 1$, is easily seen to be graphical over $\mathbb{R}^{2n+1}$.

Since being totally real is a C^1-open condition we may fix $\delta > 0$ small enough such that Cl^δ is totally real and graphical over $\mathbb{R}^{2n+1}$. Write $\mathrm{Cl}^\delta = (g, \sin(\theta_{n+1}))$, where $g : \mathbb{T}^{n+1} \hookrightarrow \mathbb{R}^{2n+1}$ is the embedding obtained by projecting Cl^δ over $\mathbb{R}^{2n+1}$.

A global trivialisation of the normal bundle of Cl^δ can be given by

$$\langle \partial_{r_1} + \partial_{r_{n+1}}, \partial_{r_2}, \ldots, \partial_{r_n} \rangle \oplus \langle \partial_z \rangle,$$

where ∂_z spans the vertical direction in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+1} \times \mathbb{R}$. Moreover, the first n-vectors project to a trivialisation of the normal bundle of g in $\mathbb{R}^{2n+1}$.

Consider the inclusion $F \hookrightarrow \mathbb{R}^n$; without loss of generality we may assume that it has image in an ε-ball, for $\varepsilon > 0$ sufficiently small. After fixing a tubular neighbourhood embedding of g, this inclusion induces an embedding $e : \mathbb{T}^{n+1} \times F \hookrightarrow \mathbb{R}^{2n+1}$.

The embedding $\hat{e} = (e, \sin(\theta_{n+1})) : \mathbb{T}^{n+1} \to \mathbb{C}^{n+1} = \mathbb{R}^{2n+1} \times \mathbb{R}$, is co-real for $\varepsilon > 0$ sufficiently small, by an application of Lemma 3.2.8. This concludes the argument. □

Note that the Clifford torus $\mathbb{T}^{n+1}_{\mathrm{Cl}} \subseteq \mathbb{C}^{n+1}$ is isotopic to the standard torus (cf. Section 4.3.2). If we take $F = \mathbb{T}^{n-1} \subseteq \mathbb{R}^n$ to be the standard torus, then the resulting embedding $e : \mathbb{T}^{2n} \hookrightarrow \mathbb{R}^{2n+1}$ is also isotopic to the standard torus. Hence, by Lemma 4.3.6, we obtain:

Lemma 5.1.2. *In the above situation of proposition, we can moreover assume that* $\hat{e} : \mathbb{T}^{2n} \to \mathbb{C}^{n+1}$ *satisfies:*

$$\mathrm{spin}_{\hat{e}}[L] \simeq (\mathrm{spin}_{\mathbb{S}^1})^{2n}[L],$$

for any Legendrian $L \subseteq J^1\mathbb{R}^m$.

5.1.2 Existence of prelegendrian stabilizations

Proof of Theorem 2.2.4. Let Λ be a prelegendrian submanifold in $(\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$. Consider a point $q \in \Lambda$ in the smooth stratum of the prelegendrian front $\pi_{\mathrm{PF}}(\Lambda)$. It

follows from Proposition 5.1.1 that, possibly after a prelegendrian front homotopy of $\mathcal{O}p\,(\pi_{\mathrm{PF}}(p)) \cap \pi_{\mathrm{PF}}(\Lambda)$, there exists a codimension 2 co-real torus $N \subseteq \mathcal{O}p\,(\pi_{\mathrm{PF}}(p)) \cap \pi_{\mathrm{PF}}(\Lambda) \subseteq \mathbb{C}^{n+1}$. This places us in the setup of Section 4.5.2. In particular, the stabilization construction defined there is not an empty operation.

The same approach works in $(\mathrm{Gr}(X, J), \mathcal{D}_{\mathrm{can}})$. Indeed, the co-real torus from Proposition 5.1.1 can be assumed to be co-real for a general J if we work in a sufficiently small ball. Indeed, the property of being co-real is preserved under C^0-perturbations of the almost complex structure, and every almost complex structure can be assumed to coincide with the standard one at a given point. □

5.2 Exotic prelegendrians

In this section we will prove our main results about the failure of the h-principle for prelegendrian embeddings: Theorem 2.2.2, Corollary 2.2.6 and Corollary 2.2.7.

5.2.1 Auxiliary Lemmas in Contact Topology

We recall two basic results in contact topology that we will need later. The first one is standard and well-known. It follows from the functoriality of jet spaces:

Lemma 5.2.1. *Let Q be a smooth manifold and $\widetilde{Q}$ its universal cover. Then, the contact universal cover of J^1Q is*

$$(\widetilde{J^1Q}, \widetilde{\xi_{\mathrm{can}}}) \simeq (J^1\widetilde{Q}, \xi_{\mathrm{can}}).$$

In particular, the contact universal cover of $(J^1(\mathbb{R}^m \times \mathbb{S}^1), \xi_{\mathrm{can}})$ is contactomorphic to $(J^1\mathbb{R}^{m+1}, \xi_{\mathrm{std}})$.

Proof. The differential of the covering map $\widetilde{Q} \to Q$ induces a symplectic covering map $(T^*\widetilde{Q}, \mathrm{d}\lambda_{\mathrm{can}}) \to (T^*Q, \mathrm{d}\lambda_{\mathrm{can}})$ which preserves the canonical 1-form. The result follows. □

The second auxiliary result that we need follows by a standard cut-off argument with contact Hamiltonians:

Lemma 5.2.2. *Let (Y^{2n+1}, ξ) be a contact manifold, such that its universal cover $(\widetilde{Y}, \tilde{\xi})$ is contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$, and $(B, \xi_{\mathrm{std}}) \subseteq (Y, \xi)$ a Darboux ball. Let $L_0, L_1 \subseteq (B, \xi_{\mathrm{std}}) \subseteq (Y, \xi)$ be two compact Legendrian submanifolds which are not Legendrian isotopic in the ball, then L_0 and L_1 are not Legendrian isotopic in (Y, ξ).*

Proof. We prove the contrapositive. Assume that there is a Legendrian isotopy $L_t \subseteq (Y, \xi)$, $t \in [0, 1]$, between L_0 and L_1. Consider a lift $(\tilde{B}, \xi_{\mathrm{std}}) \subseteq (\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$ of the Darboux ball (B, ξ_{std}), together with lifts of the initial Legendrians $\tilde{L}_0, \tilde{L}_1 \subseteq (\tilde{B}, \xi_{\mathrm{std}})$.

We claim that, since there is a Legendrian isotopy $L_t \subseteq (Y, \xi)$, $t \in [0,1]$, there exists a Legendrian isotopy $\tilde{L}_t \subseteq (\mathbb{R}^{2n+1}, \xi_{\text{std}})$, $t \in [0,1]$, in the universal cover between the Legendrians $\tilde{L}_0$ and $\tilde{L}_1$. Indeed, the Legendrian isotopy can be lifted to a Legendrian isotopy $\hat{L}_t \subseteq (\mathbb{R}^{2n+1}, \xi_{\text{std}})$, $t \in [0,1]$, starting at $\hat{L}_0 = \tilde{L}_0$ but possibly ending at Legendrian lift $\hat{L}_1$ of L_1 which is different to $\tilde{L}_1$. This lift would be in a different lift $(\hat{B}, \xi_{\text{std}})$ of the Darboux ball (B, ξ_{std}). Since the space of Darboux ball embeddings into a connected contact manifold is connected, the contact embeddings $(\tilde{B}, \xi_{\text{std}}) \hookrightarrow (\mathbb{R}^{2n+1}, \xi_{\text{std}})$ and $(\hat{B}, \xi_{\text{std}}) \hookrightarrow (\mathbb{R}^{2n+1}, \xi_{\text{std}})$ are contact isotopic. This implies that $\hat{L}_1$ and $\tilde{L}_1$ are Legendrian isotopic. The concatenation of both Legendrian isotopies is the required Legendrian isotopy $\tilde{L}_t \subseteq (\mathbb{R}^{2n+1}, \xi_{\text{std}})$, $t \in [0,1]$.

Now we will show that the Legendrian isotopy $\tilde{L}_t$ is homotopic, relative to $t \in \{0,1\}$, to a Legendrian isotopy whose image is in $(\tilde{B}, \xi_{\text{std}})$. Since the covering map defines a contactomorphism $(\tilde{B}, \xi_{\text{std}}) \to (B, \xi_{\text{std}})$ this would imply the existence of a Legendrian isotopy in (B, ξ_{std}) between L_0 and L_1, thus concluding the proof.

Fix two auxiliary Darboux balls (B_0, ξ_{std}) and (B_1, ξ_{std}) in $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$ such that

- $\tilde{L}_0, \tilde{L}_1 \subseteq B_0$
- $\tilde{L}_t \subseteq B_1$
- $B_0 \subseteq \tilde{B} \subseteq B_1$.

Since the space of Darboux balls is connected we may further assume that (B_0, ξ_{std}) is the unit ball in $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$. Consider the contact vector field

$$X = 2z\partial z + \sum_{i=1}^{n} x_i \partial x_i + y_i \partial y_i,$$

in $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$, whose contact flow is the contact dilation $\varphi_X^t(\mathbf{x}, \mathbf{y}, z) = (e^t\mathbf{x}, e^t\mathbf{y}, e^{2t}z)$, $t \in \mathbb{R}$. With respect to the standard contact form, X has contact Hamiltonian

$$H = \alpha_{\text{std}}(X) = 2z - \mathbf{y} \cdot \mathbf{x}$$

Let $\rho : \mathbb{R}^{2n+1} \to [0,1]$ be a smooth function such that

- $\rho_{|B_0} \equiv 0$ and
- $\rho_{|\mathbb{R}^{2n+1} \setminus \tilde{B}} \equiv 1$.

Denote by $\tilde{X}$ the contact vector field associated to the Hamiltonian $\tilde{H} \coloneqq \rho \cdot H$. The required Legendrian isotopy is then

$$\varphi_{\tilde{X}}^{-T}(\tilde{L}_t), t \in [0,1],$$

where $T >> 0$ is big enough. This concludes the argument. □

The previous lemma yields the following consequence, which, although not used later in the manuscript, may be of independent interest.

Corollary 5.2.3. *Let (Y^{2n+1}, ξ) be a contact manifold, such that its universal cover $(\widetilde{Y}, \tilde{\xi})$ is contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$, and $L \subseteq (B, \xi_{\text{std}})$ a Legendrian submanifold. Then, if L is non-loose in (B, ξ_{std}) then it is non-loose in (Y, ξ).*

Proof. Apply the previous result to $L_0 \coloneqq L$ and $L_1 \coloneqq s(L) \subseteq B$. Indeed, if $L_0 = L$ was loose in (Y, ξ), then it will be Legendrian isotopic to $L_1 = s(L)$ by Murphy's h-principle Theorem 4.2.5. □

Remark 5.2.4. A direct proof of the corollary can also be given, without invoking the h-principle, by using contact Hamiltonians—as in Lemma 5.2.2—to push any potential loose chart into (B, ξ). △

5.2.2 Exotic prelegendrian tori

Proof of Theorem 2.2.2. Consider the family of Legendrian Whitehead doubles of stabilized Legendrian unknots $W_s \subseteq (J^1\mathbb{R}, \xi_{\text{std}})$ with $s = 3, 5, \cdots$, depicted in Fig. 5.1. By Proposition 5.1.1, there exists a co-real embedding $\hat{e} = (\hat{e}_{\mathbb{R}^{2n+1}}, h) : \mathbb{T}^{2n} \to \mathbb{C}^{n+1} = \mathbb{R}^{2n+2} = \mathbb{R}^{2n+1} \times \mathbb{R}$, which is graphical over $\mathbb{R}^{2n+1}$.

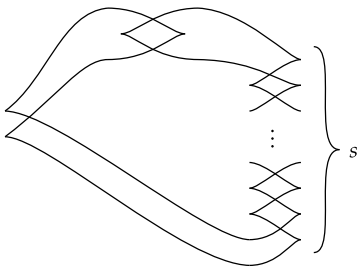


Figure 5.1: The Legendrian W_s, where s is the number of crossings in the right part.

By Corollary 4.4.10 there is well-defined infinite family of prelegendrian $(2n+1)$-tori:

$$\Lambda_s = \text{prespin}_{\hat{e}}[W_s] \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$$

obtained by prelegendrian spinning.

Observe that $\text{Rot}(W_s) = 0$ for all s. Therefore, by Lemma 4.3.5 and Remark 4.4.12 the prelegendrian fronts of Λ_s are related by a sequence of prelegendrian Reidemeister moves and N-pushings along parallel copies of $\hat{e}(\mathbb{T}^{2n})$. Theorem 4.5.2 then implies that all these prelegendrians are formally isotopic. A formal prelegendrian isotopy is depicted in Fig. 5.2.

Now we will show that the Legendrian lifts $\mathcal{L}(\Lambda_s)$ are pairwise not Legendrian isotopic, which by Corollary 3.1.15 will imply that the prelegendrians Λ_s are pairwise not prelegendrian isotopic and hence the result.

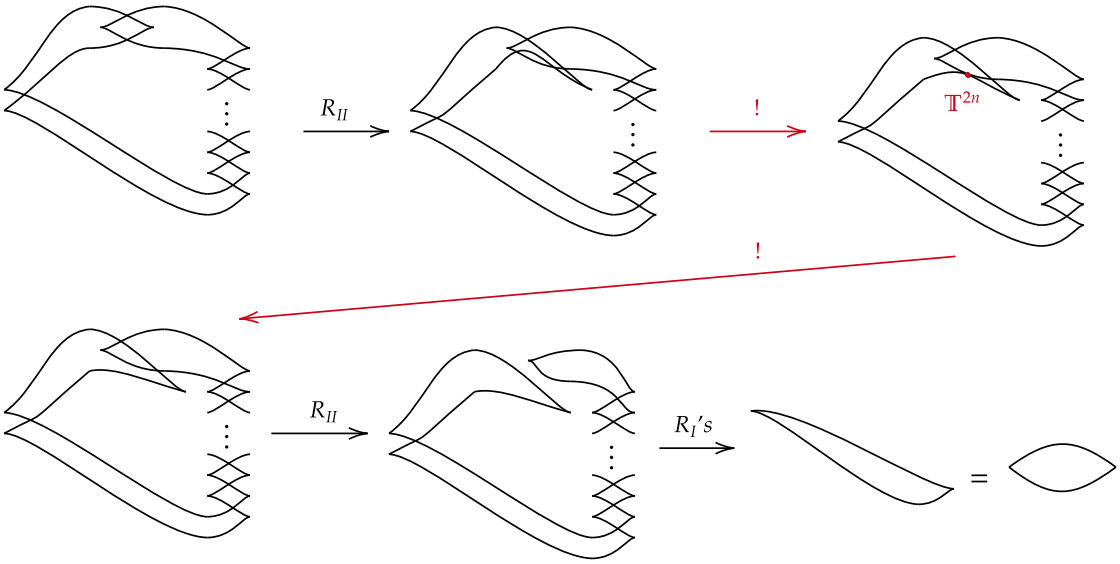


Figure 5.2: A formal prelegendrian isotopy between of $\Lambda_s = \text{prespin}_{\hat{e}}[W_s]$ and $\text{prespin}_{\hat{e}}[U]$, where $U \subseteq (J^1\mathbb{R}, \xi_{\text{std}})$ is the standard Legendrian unknot. The first and fourth arrow indicate prelegendrian RII moves, and the last one several prelegendrian RI moves along parallel copies of the co-real $\mathbb{T}^{2n}$. The second and third arrow describe a prelegendrian N-pushing along the co-real $\mathbb{T}^{2n}$.

By construction,

$$\mathcal{L}(\Lambda_s) = \text{spin}_{\hat{e}}[W_s] \subseteq (J^1\mathbb{R}^{2n+1}, \xi_{\text{std}}).$$

Moreover, as Legendrians in $(J^1\mathbb{R}^{2n+1}, \xi_{\text{std}})$, $\mathcal{L}(\Lambda_s)$ and $(\text{spin}_{\mathbb{S}^1})^{2n}[W_s]$ are Legendrian isotopic by Lemma 5.1.2. It was shown in [EES05b, Example 4.20] that the Legendrian contact homologies of $\mathcal{L}(\Lambda_s)$ are all non quasi-isomorphic, so the $\mathcal{L}(\Lambda_s)$ are pairwise not Legendrian isotopic in $(J^1\mathbb{R}^{2n+1}, \xi_{\text{std}})$. To conclude we need to tell them apart in the contactisation $\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$.

Notice that there is a natural inclusion

$$(J^1\mathbb{R}^{2n+1}, \xi_{\text{std}}) \cong (\mathbb{S}T^*_{\text{hor}}(\mathbb{R}^{2n+2}), \xi_{\text{can}}) \subseteq \mathcal{C}(\text{Gr}_{\text{hor}}(\mathbb{C}^{n+1}, J_{\text{std}}), \mathcal{D}_{\text{can}}) = \mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}}),$$

where the first contactomorphism follows from Lemma 3.3.11 and the inclusion is the one given in (3.5). Moreover, by Corollary 3.3.10 we have $\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}}) \cong (J^1(\mathbb{R}^{2n} \times \mathbb{S}^1), \xi_{\text{std}})$. Hence, Lemma 5.2.1 tell us that the contact universal cover of $\mathcal{C}(\mathbb{C}^{2n+1}, \xi_{\text{std}})$ is $(J^1\mathbb{R}^{2n+1}, \xi_{\text{std}})$. Therefore, Lemma 5.2.2 can be applied to conclude that the Legendrian $\mathcal{L}(\Lambda_s)$ are pairwise not Legendrian non isotopic in the contactisation $\mathcal{C}(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$. □

Remark 5.2.5. Let $(Q^{2n+1} \times \mathbb{R}, J)$ be an almost complex manifold, with universal cover $\mathbb{R}^{2n+1} \times \mathbb{R}$. Then, it follows from the same argument that $(\text{Gr}_{\text{hor}}(Q \times \mathbb{R}), \mathcal{D}_{\text{can}})$ has infinitely many exotic prelegendrian $(2n+1)$-tori as well. Indeed, we consider the

prelegendrian fronts appearing in the proof and we insert them in a very small ball $\mathcal{O}p\,(q,z)$, where $(q,z) \in Q \times \mathbb{R}$. It will then follow that the front singularities are coreal for J, and hence define prelegendrians in $(\mathrm{Gr}_{\mathrm{hor}}(Q \times \mathbb{R}), \mathcal{D}_{\mathrm{can}})$. Our assumption on Q then implies that the contact universal cover of $\mathcal{C}(\mathrm{Gr}_{\mathrm{hor}}(Q \times \mathbb{R}), \mathcal{D}_{\mathrm{can}})$ is contactomorphic to $(J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$, so the same proof can conclude in the same manner. This yields infinitely many examples of fat manifolds of corank 2 with infinitely many exotic prelegendrians. △

5.2.3 Proof of Corollary 2.2.6

Consider the perturbation of the Clifford torus $\hat{e} : \mathbb{T}^{n+1} \hookrightarrow \mathbb{C}^{n+1}$ from Proposition 5.1.1. The prelegendrian embedding $\Lambda : \mathbb{T}^{n+1} \times \mathbb{S}^{j_0} \times \cdots \times \mathbb{S}^{j_k} \hookrightarrow (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{can}})$ with non-loose Legendrian lift $\mathcal{L}(\Lambda)$ is going to be defined as

$$\Lambda \coloneqq \mathrm{prespin}_{\hat{e}}[L] \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}}),$$

for certain Legendrian $L \cong \mathbb{S}^{j_0} \times \cdots \times \mathbb{S}^{j_k} \subseteq (J^1\mathbb{R}^n, \xi_{\mathrm{std}})$. To conclude that $\mathcal{L}(\Lambda) = \mathrm{spin}_{\hat{e}}[L] \subseteq (J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$ is non-loose, we will show that $\mathcal{L}(\Lambda)$ is fillable [Ekh08, Mur19]. The second part of the result will then follow from Theorem 2.2.4.

By Lemma 5.1.2, the Legendrians $\mathrm{spin}_{\hat{e}}[L]$ and $(\mathrm{spin}_{\mathbb{S}^1})^{n+1}[L]$ are Legendrian isotopic. Therefore, the proof reduces to showing the existence of a Legendrian $L \cong \mathbb{S}^{j_0} \times \cdots \times \mathbb{S}^{j_k} \subseteq (J^1\mathbb{R}^n, \xi_{\mathrm{std}})$ such that

- L has a simple front, so $\Lambda = \mathrm{prespin}_{\hat{e}}[L] \subseteq (\mathbb{C}^{2n+1}, \mathcal{D}_{\mathrm{std}})$ is well-defined by Corollary 4.4.10.
- $(\mathrm{spin}_{\mathbb{S}^1})^{n+1}[L] \subseteq (J^1\mathbb{R}^{2n+1}, \xi_{\mathrm{std}})$ is Lagrangian fillable.

Let $U \cong \mathbb{S}^{j_0} \subseteq (J^1\mathbb{R}^{j_0}, \xi_{\mathrm{std}})$ be the standard Legendrian unknot. Then, the Legendrian

$$L \coloneqq \mathrm{spin}_{\mathbb{S}^{j_k}} \circ \mathrm{spin}_{\mathbb{S}^{j_{k-1}}} \circ \cdots \circ \mathrm{spin}_{\mathbb{S}^{j_1}}[U] \subseteq (J^1\mathbb{R}^n, \xi_{\mathrm{std}}),$$

satisfies the required properties: it has a simple front and, moreover, $(\mathrm{spin}_{\mathbb{S}^1})^{n+1}[L]$ is Lagrangian fillable because the $\mathbb{S}^k$-spinning of a fillable Legendrian is fillable [Gol14, Remark 2]. This concludes the argument. □

Remark 5.2.6. There is an alternative argument to conclude non-looseness in the above construction which avoids appealing to Lagrangian fillability. Dimitroglou Rizell and Golovko prove in [DRG15, Theorem 3.1] that $\mathbb{S}^m$–spinning preserves augmentations: if the Chekanov–Eliashberg DGA of a Legendrian submanifold admits an augmentation, then the DGA of its $\mathbb{S}^m$–spinning also admits one. In particular, non-acyclicity of the DGA (equivalently, nonvanishing of linearized LCH) is preserved under spinning. Thus, whenever L has nonzero Legendrian contact homology, the same holds for $\mathrm{spin}_{\mathbb{S}^m}[L]$, which already implies that the latter is non-loose. △

Remark 5.2.7. It follows from [Gol14, Theorem 1.2] that we can actually find infinitely many prelegendrians as in Corollary 2.2.6 whose Legendrian lifts are not Legendrian isotopic. We expect these prelegendrians to be formally prelegendrian isotopic, yielding new infinite families of exotic prelegendrians with topology not restricted to be tori. However, the formal computation is more involved than the case of tori, since we cannot reduce it to 1-dimensional Legendrian knot theory, as in Lemma 4.3.5. This remains an open problem. △

5.2.4 Proof of Corollary 2.2.7

The Legendrian lift of the prelegendrian $\Lambda_M = (M, TM^J) \subseteq (\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}})$ is

$$\mathcal{L}(\Lambda_M) = (M, TM) \subseteq (\mathbb{S}T^*X, \xi_{\mathrm{can}}) = \mathcal{C}(\mathrm{Gr}(X,J), \mathcal{D}_{\mathrm{can}}),$$

here the prelegendrian co-orientation is defined by the outward normal at $M = \partial Y$ defined by Y.

We claim that $\mathcal{L}(\Lambda_M)$ admits an exact Lagrangian filling in $(\mathbb{D}T^*X, \mathrm{d}\lambda_{\mathrm{can}})$, hence is non-loose [Ekh08, Mur19] and the result follows from Theorem 2.2.4. This must be well-known, however we include the proof here for completeness. Fix a Riemannian metric in M, and consider the associated metric in T^*M. Then, if ν_p, $p \in M = \partial Y$, is the outward unit normal to M, we can identify the Legendrian lift as the unit co-normal lift associated to ν, i.e. $\mathcal{L}(\Lambda_M) = \{(p, \nu_p^*) : p \in M\} \subseteq (\mathbb{S}T^*X, \xi_{\mathrm{can}})$.

Consider a tubular neighbourhood of M

$$\begin{aligned} \Phi : M \times (-\varepsilon, \varepsilon) &\to U \subseteq X \\ (p,t) &\mapsto \exp_p(t\nu_p), \end{aligned}$$

Define $\psi : U \to \mathbb{R}$ by $\psi(\Phi(p,t)) \coloneqq t$. Then, $\mathrm{d}_p\psi = \nu_p^*$ for $p \in M$, $\|\mathrm{d}\psi\| = 1$, $\psi_{|Y\cap U} < 0$ and $\psi_{|M} = 0$. Since M is separating we may extend ψ to a smooth function in X, still denoted by ψ such that $\phi_{|Y\setminus U} \leq -\varepsilon$ and $\psi_{|X\setminus Y} > 0$.

Fix a step function $\chi : (-\infty, 0] \to [-1, 0]$ such that

- $\chi(0) = 0$,
- $\chi'(0) = 1$,
- $0 \leq \chi'(t) < 1$ for all $t < 0$,
- $\mathrm{supp}(\chi') \subseteq (-\varepsilon, 0]$.

Set $f : Y \to \mathbb{R}$ by $f(p) \coloneqq \chi(\psi(p))$. Then $L \coloneqq \mathrm{graph}(\mathrm{d}f) \subseteq T^*X$ is an exact Lagrangian. By definition,

$$\mathrm{d}_p f = \chi'(\psi(p))\,\mathrm{d}_p\psi.$$

From here it follows that $\|(\mathrm{d}f)_p\| < 1$ for $p \in \mathrm{Int}(Y)$ and $\mathrm{d}_p f = \nu_p^*$ for $p \in M = \partial Y$. That is, $L \subseteq (\mathbb{D}T^*X, \mathrm{d}\lambda)$ is an exact Lagrangian filling of $\Lambda_M \subseteq (\mathbb{S}T^*X, \xi_{\mathrm{can}})$ and the result follows. □

5.3 Constructing prelegendrians in non-standard fat distributions

In this section we translate our previous results to more general fat distributions. To do this we use two main tools: (1) the nilpotentisation, to be introduced next, and (2) Gray stability for contact structures. These tools partially remedy the lack of Darboux and Gray for fat structures of higher corank.

5.3.1 The nilpotentisation

According to Cartan's study of normal forms of distributions, corank-2 fat distributions do not admit a Darboux type theorem. However, an approximate local form does exist [Ge92, Mon02], as we now explain.

Consider $(M^{4n+2}, \mathcal{D}^{4n})$ fat and fix a point $p \in M$. Then, there are local coordinates $(x_1, \dots, x_{4n}, z_1, z_2)$ and 1-forms

$$\lambda^i = \mathrm{d}z_i + \sum_{j,k} \Gamma^i_{jk} x_j \, \mathrm{d}x_k + R_i, \quad i = 1, 2,$$

such that $\mathcal{D} \overset{loc.}{=} \ker \lambda^1 \cap \ker \lambda^2$. Here R_i is a remainder/defect term

$$R_i = \sum_j f_{ij} \, \mathrm{d}z_j + \sum_j g_{ij} \, \mathrm{d}x_j,$$

with $f, g \in O(|x|^2 + |z|^2)$.

The coefficients $\{\Gamma^i_{jk}\}$ depend on p and are the structure constants of the Lie algebra structure on $\mathcal{D}^{2n}_p \oplus TM_p/\mathcal{D}_p$ induced by the curvature of $\mathcal{D}$. As p varies, this yields a weak Lie algebra bundle over M, meaning that the isomorphism type of the Lie algebra may vary from point to point. This bundle is called the *nilpotentisation* of $\mathcal{D}$. The Lie algebra at p defines a left-invariant fat distribution on the corresponding Lie group; it is given precisely by the 1-forms

$$\lambda^i = \mathrm{d}z_i + \sum_{j,k} \Gamma^i_{jk} x_j \, \mathrm{d}x_k,$$

i.e. by removing the remainder. The following standard result tells us that this provides an approximate model for $\mathcal{D}$:

Proposition 5.3.1. *Fix a fat distribution $(M^{4n+2}, \mathcal{D}^{4n})$, a point $p \in M$, and local coordinates $(x_1, \dots, x_{4n}, z_1, z_2)$, as above. Let $\mathcal{D}_{model}$ be the model fat distribution associated to the nilpotentisation at p.*

Then, there is a path of fat distributions $(\mathcal{D}_s)_{s\in[0,1]}$ on $\mathbb{R}^{4n+2}$ such that:

- *If $s > 0$, $\mathcal{D}_s$ is diffeomorphic to the restriction of $\mathcal{D}$ to the chart.*

- $\mathcal{D}_0 = \mathcal{D}_{model}$.

The proof amounts to performing a suitable weighted dilation in order to get rid of the remainder. As we take the limit as the dilation factor goes to zero, we obtain the model structure.

For the standard fat space $(\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$, this space is itself the model, and its Lie algebra is known as the *complex Heisenberg algebra*. A particularly interesting case is $n = 1$. By the classification of nilpotent 6-dimensional Lie algebras [CFS05, Mag86], there is a unique Lie algebra structure that is fat. Hence:

Corollary 5.3.2. *Fix a fat distribution $(M^6, \mathcal{D}^4)$, a point $p \in M$, and local coordinates $(x_1, \dots, x_4, z_1, z_2)$, as above. Then, there is a path of fat distributions $(\mathcal{D}_s)_{s\in[0,1]}$ on $\mathbb{C}^3$ such that:*

- *If $s > 0$, $\mathcal{D}_s$ is diffeomorphic to the restriction of $\mathcal{D}$ to the chart.*
- $\mathcal{D}_0 = \mathcal{D}_{\text{std}}$.

5.3.2 Prelegendrians in general fat manifolds

This language of approximate models allows us to prove:

Proof of Theorem 2.2.9. Use Corollary 5.3.2 to produce a path of fat distributions $\mathcal{D}_s$ starting with $\mathcal{D}_0 = \mathcal{D}_{\text{std}}$ such that $\mathcal{D}_s$ is diffeomorphic to the restriction of $\mathcal{D}$ to some small ball whose radius depends on $s > 0$. Now, consider the contactisation of the family $(B, \mathcal{D}_s)$, these define a family of (isotropic) bundle projections:

$$\text{pr}_s : (U, \xi_s) \to (B, \mathcal{D}_s),$$

where U is an open manifold diffeomorphic to $B \times \mathbb{S}^1$ and B is some open 6-ball around the origin. Consider now the prelegendrians P_0 in $(\mathbb{C}^3, \mathcal{D}_{\text{std}}) = (B, \mathcal{D}_0)$ constructed in Theorem 2.2.2, they define corresponding Legendrian lifts L_0 on $\mathcal{C}(B, \mathcal{D}_0) = (U, \xi_0)$.

These Legendrian lifts can be assumed to be contained in some compact subset $K \subseteq U$. Gray stability tells us that there is a small time $s_0 > 0$ such that a family of isocontact embeddings $\phi_s : (K, \xi_0) \to (U, \xi_s)$, $s \in [0, s_0]$, exists. Consider then the images $L_s \coloneqq \phi_s(L_0)$, these are Legendrians in (U, ξ_s). Since L_0 is transverse to the projection pr_0 and transversality is an open condition, then L_s is transverse to pr_s for sufficiently small $s > 0$, hence it projects down to a prelegendrian in $(B, \mathcal{D}_s)$, as desired. This proves the first claim.

For the second claim, we consider any prelegendrian Λ in $(M, \mathcal{D})$ and zoom in at a point $p \in \Lambda$. The zooming argument above, in a chart, allows us to produce a path of prelegendrians $\Lambda_s \subseteq (B, \mathcal{D}_s)$ such that $\Lambda_0 \simeq T_p\Lambda$ is a prelegendrian linear subspace in $(B, \mathcal{D}_0) = (\mathbb{C}^3, \mathcal{D}_{\text{std}})$. By the symmetry of the complex Heisenberg algebra, we can

assume that Λ_0 projects to a prelegendrian front; we can then stabilise and denote $\tilde{\Lambda}_0 \coloneqq s(\Lambda_0)$, and we denote its Legendrian lift by $\tilde{L}_0$.

In the contactisations $\mathrm{pr}_s \colon (U, \xi_s) \to (B, \mathcal{D}_s)$, the same construction as in the first claim produces a path of Legendrians $\tilde{L}_s \coloneqq \phi_s(\tilde{L}_0)$ contained in a compact set $K \subseteq U$, for $s \in [0, s_0]$ with $s_0 > 0$ small. Furthermore, we can assume that the loose chart of each $\tilde{L}_s$ is contained in a compact set $W \subseteq K$.

Note that $\tilde{L}_s$ is C^∞-close to the Legendrian lift $L_s = \mathcal{L}(\Lambda_s)$. Then $\tilde{L}_s$ is a graph in a Weinstein neighbourhood chart of L_s, and by cutting off with a suitable function we produce a new path of Legendrians L'_s such that

- L'_s coincides with $\tilde{L}_s$ in W, which contains a loose chart;
- L'_s coincides with L_s away from $\mathcal{O}p\,(W)$.

Observe that $L'_0 = \tilde{L}_0$. By openness of transversality, we know that L'_s is transverse to pr_s for sufficiently small $s > 0$; this defines the stabilised prelegendrian of Λ.

□

The exact same reasoning shows:

Theorem 5.3.3. *Fix a fat distribution $(M^{4n+2}, \mathcal{D}^{4n})$. Suppose it contains a point whose nilpotentisation is the complex Heisenberg algebra. Then, it contains compact prelegendrians.*

5.3.3 Stability phenomena

The absence of a Darboux-type theorem for fat distributions of corank greater than 1 implies that Gray stability does not hold for these classes. Nonetheless:

Proof of Lemma 2.2.10. For the first claim we reason as in the proof of Theorem 2.2.9. We apply Gray stability in the contactisation to the Legendrian lift. This produces a family of Legendrians that, at least for small time, remain transverse to the projection to the fat manifold.

For the second claim we apply the reasoning appearing in the proof of Theorem 2.2.9 parametrically. That is: we apply the nilpotentisation parametrically on s. Then, given any prelegendrian for $\mathcal{D}_{\mathrm{std}}$ we may assume that it is contained in an arbitrary small ball, thanks to a dilation. Since $\mathcal{D}_{\mathrm{std}}$ is the model structure on the nilpotentisation, we see a copy of the prelegendrian in each $\mathcal{D}_s$, parametrically. □

Proof of Theorem 2.2.11. Suppose that such a path does exist. Since the homotopy $\mathcal{D}_s$ is compactly supported, the contactisations define a path of contact structures that is also compactly supported. As such, Gray stability applies, telling us that the Legendrian lifts are contactomorphic. Since the Legendrian lifts are distinguished by LCH, this is a contradiction. □

Part II

Engel CR submanifolds of $\mathbb{C}^3$

Chapter 6

Engel CR submanifolds

This chapter corresponds to the second part of the thesis, based on the preprint [FdPZ25a]. While Part I studied rigidity phenomena in corank-2 fat distributions through the notion of prelegendrians, here we turn to Engel geometry and its relation with complex tangencies. More precisely, we investigate co-real submanifolds of almost complex 6-manifolds whose complex tangencies carry an Engel structure. The main contribution is a sufficient condition, in terms of convex bundle immersions, that guarantees the existence of such embeddings and immersions. As a consequence, we obtain a broad class of 4-manifolds that admit co-real embeddings into $\mathbb{C}^3$ with Engel complex tangencies, together with families of knotted examples.

6.1 Main results of part II

Recall that $N^4 \subseteq \mathbb{C}^3$ is *co-real* (also called *CR-regular*) if $\mathcal{D}_J$ is a plane distribution. Remarkably, Gromov proved an h-principle for co-real embeddings [Gro86, CEM24], meaning that this condition is purely formal: an oriented 4 manifold N admits a co-real embedding into $\mathbb{C}^3$ if and only if N is parallelisable [Sla15]. This perfectly matches the fact that a closed orientable 4-manifold carries an orientable Engel structure precisely when it is parallelisable [Vog09]. As such, from an algebraic topology viewpoint, these two geometric situations are equivalent.

More generally, one can consider co-real immersions $g : N^4 \looparrowright (X^6, J)$ of a 4-manifold into an almost complex manifold of real dimension 6. Co-reality means that the complex tangencies

$$\mathcal{D}_J^g := g_*^{-1}\big(g_*TN \cap J(g_*TN)\big)$$

have rank 2.

In this chapter we focus on 4-manifolds N given as oriented $\mathbb{S}^1$-bundles over oriented 3-manifolds M^3. For these, we provide a sufficient condition to admit a co-real embedding/immersion into (X, J) so that the complex tangencies are Engel. This sufficient condition is the existence of a *convex bundle embedding/immersion* $g : N \looparrowright TM$; i.e., a bundle map that is fibrewise embedded/immersed and whose fibrewise

derivative, when normalised, traces a convex curve on the sphere (see Definition 6.2.5 for further details). Our main result reads as follows:

Theorem 6.1.1. *Let N be an oriented $\mathbb{S}^1$-bundle over a closed, oriented 3-manifold M and let $g : N \looparrowright TM$ be a convex bundle immersion. Then, for any totally real immersion $\iota : M \looparrowright (X, J)$, into an almost complex manifold of real dimension 6, there exists a co-real immersion $\tilde{g} : N \looparrowright (X, J)$ whose complex tangencies $\mathcal{D}^{\tilde{g}}_J$ form an Engel distribution. Moreover, if g and ι are embeddings, then $\tilde{g}$ is also an embedding.*

The proof of Theorem 6.1.1 is based on a zooming-in argument, which is somewhat reminiscent of Donaldson's zooming argument used in the construction of symplectic divisors [Don96] and of the corrugation phenomenon in h-principle [Sma59, The22].

Remark 6.1.2. Recall that a submanifold of an almost complex manifold is *totally real* when its tangent space contains no complex lines. Again, Gromov proved an h-principle for these, so the obstruction to its existence is purely formal [Gro86, CEM24]. In particular, Forstnerič proved that a totally real embedding $\iota : M^3 \hookrightarrow (X^6, J)$, as in Theorem 6.1.1, always exists [For86]. △

Theorem 6.1.1 yields a broad class of 4-manifolds that can be immersed, and infinitely of them embedded, into $\mathbb{C}^3$, so that the complex tangencies define an Engel structure:

Theorem 6.1.3 (Theorem D)**.** *Let N be an oriented $\mathbb{S}^1$-bundle over a closed, oriented 3-manifold M. Then, there exists a co-real immersion $g : N \looparrowright \mathbb{C}^3$ whose complex tangencies $\mathcal{D}^g_{J_{\text{std}}}$ form an Engel distribution. If N is the trivial bundle, then g can be chosen to be an embedding.*

We conjecture that g can also be chosen to be an embedding in the case that N has even Euler class. See Remark 6.5.2 for further details.

In the construction of the embedding $g : M \times \mathbb{S}^1 \hookrightarrow \mathbb{C}^3$ of Theorem 6.1.3 we can prescribe the *smooth isotopy class* of the underlying $g_{|M\times\{p\}} : M \hookrightarrow \mathbb{C}^3$. In particular, we can achieve knotting for g by knotting $g_{|M\times\{p\}}$.

Corollary 6.1.4. *Let M be a closed, oriented 3-manifold. There exist infinitely many embeddings $g_k : M \times \mathbb{S}^1 \hookrightarrow \mathbb{C}^3$, $k \in \mathbb{Z}$, that are pairwise non-isotopic and whose complex tangencies $\mathcal{D}^{g_k}_{J_{\text{std}}}$ are Engel. Moreover, all these Engel structures $\mathcal{D}^{g_k}_{J_{\text{std}}}$ are Engel homotopic.*

Remark 6.1.5. In the case where $N \subseteq \mathbb{C}^3$ is the 4-torus, regarded as the trivial $\mathbb{S}^1$-bundle over the 3-torus $M = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, we can match the Engel structure defined by the complex tangencies and the derived prolonged distribution. This can be done by choosing a base-independent (Definition 6.4.2) convex embedding $g : N \hookrightarrow TM$, and the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C} \times \mathbb{C} \times \mathbb{C} = \mathbb{C}^3$ as a totally real embedding in Theorem 6.1.1. In this case, the derived prolonged distribution and the complex tangencies match, as proven in Lemma 6.4.3. △

6.2 Preliminaries

Throughout the chapter, all manifolds, distributions, and maps are smooth. To avoid orientability issues, we assume all manifolds and bundles are equipped with fixed orientations.

6.2.1 Engel structures

A distribution $\mathcal{D} \subseteq TN$ on a manifold N is a smooth subbundle of its tangent bundle. Given two distributions $\mathcal{D}_1, \mathcal{D}_2$, their *Lie-bracket* is a module of vector fields:

$$[\mathcal{D}_1, \mathcal{D}_2] \coloneqq \{[u_1, u_2] \,|\, u_1 \in \Gamma(\mathcal{D}_1), u_2 \in \Gamma(\mathcal{D}_2)\}.$$

If $[\mathcal{D}_1, \mathcal{D}_2]$ has pointwise constant rank, it defines a distribution; we denote it by the same symbol. Throughout, we always assume that $[\mathcal{D}_1, \mathcal{D}_2]$ has constant rank.

Definition 6.2.1. Let N be a 4-manifold and $\mathcal{D} \subseteq TN$ a rank-2 distribution. We say that $\mathcal{D}$ is *Engel* if $\mathcal{D} \subsetneq [\mathcal{D}, \mathcal{D}] \subsetneq [\mathcal{D}, [\mathcal{D}, \mathcal{D}]] = TN$. △

The Engel condition is local and C^2-open, so it persists under sufficiently small C^2 perturbations.

6.2.2 Prolonged distributions

Fix a closed, oriented 3-manifold M together with an oriented $\mathbb{S}^1$-bundle $\pi : E \to M$. Let $\mathbb{S}TM$ denote the sphere bundle of oriented lines over M. We consider the following class of rank-2 distributions [Ada02, KS13, dP18]:

Definition 6.2.2. A *curve family* over M is a smooth fibred map

$$f : E \to \mathbb{S}TM.$$

At each $p \in M$, let $f_p : E_p \to \mathbb{S}(T_pM)$ denote the restriction of f to the fibres over p.

The *prolonged distribution* associated to a curve family f is the rank-2 distribution $\mathcal{D}^f_{\text{prol}} \subseteq TE$ defined by

$$\mathcal{D}^f_{\text{prol}}(q) \coloneqq (\,\mathrm{d}\pi_q)^{-1}(L_q),$$

where $L_q \subseteq T_{\pi(q)}M$ is the oriented line corresponding to the point $f_p(q) \in \mathbb{S}(T_{\pi(q)}M)$. △

One way to construct a curve family over M is to take the fibrewise derivative of a *bundle immersion* $g : E \looparrowright TM$, that is, a bundle map $g : E \to TM$ whose fibre maps $g_q : E_q \to T_qM$ are immersions. Such a map induces a curve family via

$$\begin{aligned} \mathbf{t}(g)_p \coloneqq \mathbf{t}(g_p) : E_p &\to \mathbb{S}T_pM, \\ t &\mapsto [(g_p)'(t)]; \end{aligned}$$

where $[(g_p)'(t)]$ denotes the oriented line represented by $(g_p)'(t)$. The following notion may seem a bit unmotivated at this point, but it will become relevant in our main construction:

Definition 6.2.3. If $g : E \looparrowright TM$ is a bundle immersion, its *derived prolonged distribution* is $\mathcal{D}^g_{\mathrm{der-prol}} := \mathcal{D}^{\mathbf{t}(g)}_{\mathrm{prol}}$. △

6.2.3 Engel prolongations

We now recall a sufficient condition ensuring that a prolonged distribution $\mathcal{D}^f_{\mathrm{prol}}$ is Engel.

Definition 6.2.4. A smooth immersion $\gamma : \mathbb{S}^1 \looparrowright \mathbb{S}^2 \subseteq \mathbb{R}^3$ is said to be *locally convex* (or simply *convex*) if its geodesic curvature is strictly positive everywhere. That is,

$$\det(\gamma(t), \gamma'(t), \gamma''(t)) > 0, \quad \forall\, t \in \mathbb{S}^1,$$

with respect to the standard orientation of $\mathbb{R}^3$. △

A straightforward computation shows that convexity is preserved under the action of $\mathrm{GL}^+(3)$ on $\mathbb{S}^2$. Therefore, the following is well-defined:

Definition 6.2.5. A curve family $f : E \to \mathbb{S}TM$ is *convex* if, for each $p \in M$, the curve $f_p : E_p \to \mathbb{S}T_pM$ is convex in the above sense.

A bundle immersion $g : E \looparrowright TM$ is *convex* if its fibrewise derivative $\mathbf{t}(g)$ is convex. △

The following criterion will be crucial in our construction of Engel structures. It is a particular case of [CPdPP17, Proposition 7]:

Proposition 6.2.6. *Let $f : E \to \mathbb{S}TM$ be a curve family. The following statements hold:*

(i) If f is a bundle immersion, then $\mathcal{D}^f_{\mathrm{prol}}$ is non-integrable.

(ii) If f is convex, then $\mathcal{D}^f_{\mathrm{prol}}$ is Engel.

Do note that, according to [CPdPP17, Proposition 7], the first statement in Proposition 6.2.6 is an if and only if, but the second one is not.

Corollary 6.2.7. *If $g : E \looparrowright TM$ is a convex bundle immersion, then $\mathcal{D}^g_{\mathrm{der-prol}}$ is an Engel distribution on E.*

6.2.4 A tubular neighbourhood for totally real submanifolds

Fix an oriented, closed n-dimensional manifold M. We write $M \subseteq TM$ for the zero section. Then $T(TM)$ canonically splits along M:

$$T(TM)|_M \simeq TM \oplus TM =: \mathrm{Hor} \oplus \mathrm{Vert}.$$

This allows us to consider:

Definition 6.2.8. An almost complex structure J on TM is called *splitting-adapted* if, along the zero section, $J(\mathrm{Hor}) = \mathrm{Vert}$ and $J : \mathrm{Hor} \simeq TM \to \mathrm{Vert} \simeq TM$ is the identity isomorphism. △

Which provides a model for the tubular neighbourhood of middle-dimensional totally real submanifolds:

Lemma 6.2.9. *Let $\iota : M^n \looparrowright (X^{2n}, J)$ be a totally real immersion (resp. embedding). There exists an immersion (resp. embedding) of the tangent bundle $\varphi : TM \looparrowright X$, such that:*

(i) $\varphi|_M = \iota$;

(ii) The pullback almost complex structure $\varphi^*(J)$ on TM is splitting-adapted.

Proof. For notational simplicity assume that $M \subseteq X$ is a submanifold and ι is thus the inclusion. Since M is totally real, the tangent bundle TX along M splits as $TX|_M = TM \oplus J(TM)$. Choosing an auxiliary Riemannian metric on X yields then the tubular neighbourhood embedding

$$\begin{aligned} \varphi : TM &\looparrowright (X, J), \\ q &\longmapsto \exp_p\big(J_p(q)\big), \end{aligned}$$

where $p = \mathrm{proj}_{TM}(q)$. It is straightforward to check that φ satisfies the conclusions of the lemma. □

One can then take a local chart to deduce that:

Corollary 6.2.10. *Consider (TM, J) with J splitting-adapted and fix a point $p \in M$. There are base $(x_1, \ldots, x_n)$ and fibre coordinates $(y_1, \ldots, y_n)$ around p such that $J = J_{\mathrm{std}}$ along the zero section $\mathbb{R}^n_x \subseteq T\mathbb{R}^n_x = \mathbb{R}^n_x \times \mathbb{R}^n_y = \mathbb{C}^n$. Here J_{std} denotes the standard complex structure on $\mathbb{C}^n$.*

6.3 Convex curves with zero integral

Our goal in this section is to construct Engel derived prolongations, as in Corollary 6.2.7. The statement to prove reads:

Proposition 6.3.1. *Let N be an oriented $\mathbb{S}^1$-bundle over a closed, oriented 3-manifold M. Then a convex bundle immersion $g : N \looparrowright TM$ exists. If N is the trivial bundle, the map g can be assumed to be an embedding.*

We will make use of the following auxiliary result:

Lemma 6.3.2. *Let N be an oriented $\mathbb{S}^1$-bundle over a closed, oriented 3-manifold M. Then, there exists a bundle immersion $f : N \looparrowright \mathbb{S}TM$. If the Euler class $e(N)$ is even then f can be chosen to be a bundle embedding.*

Proof. By the fibrewise Hirsch–Smale h-principle, it suffices to produce a bundle monomorphism $V(N) \to V(\mathbb{S}TM)$ between the associated vertical bundles of N and $\mathbb{S}TM$ respectively (see [CEM24]). The existence of such a monomorphism follows from the triviality of $V(N)$ and $\mathbb{S}TM \cong M \times \mathbb{S}^2 \subseteq M \times \mathbb{R}^3$, since N is oriented and M is parallelisable [BL18, MS74]. Indeed, if we denote by $\pi : N \to M$ the bundle projection and we fix a global trivialisation $V(N) \cong N \times \mathbb{R}$, then a bundle monomorphism is given by

$$\begin{aligned} F : N \times \mathbb{R} &\to V(M \times \mathbb{S}^2), \\ (p, t) &\mapsto (\pi(p), e_1, t \cdot e_2). \end{aligned}$$

For the embedding case observe that, since the Euler class $e(N)$ is even, N is isomorphic to the bundle of oriented lines of a plane distribution on M, so the result follows (see for instance [Gei08b, Proposition 4.3.2]). □

Our goal is to deform the bundle immersions f of Lemma 6.3.2 to make each f_p convex and then integrate. This will yield the claimed convex bundle immersions g of Proposition 6.3.1. Concretely, this amounts to the construction of curves in $\mathbb{R}^3$ whose (normalised) derivatives trace convex curves in $\mathbb{S}^2$. For a single curve this can be done manually: you pick a convex curve and you integrate, ensuring that the total integral is zero, so the primitive curve is also closed. However, when dealing with families, it is more handy to refer to Saldanha [Sal15].

6.3.1 Saldanha's result

In what follows, we denote $I = [0, 1]$ and think of the unit circle as $\mathbb{S}^1 = [0, 1]/_{0\sim1}$, so that the open interval $(0, 1) \subseteq \mathbb{S}^1$ is naturally embedded therein. Write $\mathfrak{Imm}$ for the space of immersions $\mathbb{S}^1 \looparrowright \mathbb{S}^2$. Following [Sal15], call a convex curve *complicated* if it is not a concatenation of embedded convex loops, and write $\mathfrak{Conv}^c \subseteq \mathfrak{Imm}$ for the subspace of such curves. The following h-principle type result holds:

Proposition 6.3.3. *[Sal15, Proposition 1.4] Let K be a compact CW-complex and $A \subseteq K$ a closed CW-subcomplex. For every continuous map $f : (K, \mathcal{O}p(A)) \to (\mathfrak{Imm}, \mathfrak{Conv}^c)$ there exists a homotopy $H_s : (K, \mathcal{O}p(A)) \to (\mathfrak{Imm}, \mathfrak{Conv}^c)$, $s \in [0, 1]$, such that:*

(i) $H_0 = f$,

(ii) $H_s(k) = f(k)$ *for all* $(s,k) \in [0,1] \times \mathcal{O}p\,(A)$,

(iii) $\mathrm{Image}(H_1) \subseteq \mathfrak{C}\mathrm{onv}^c$.

6.3.2 The zero integral condition

For our purposes we require a refinement of Proposition 6.3.3 to produce families of convex curves with vanishing integral.

A necessary condition for a smooth curve in $\mathbb{R}^3$ to have integral zero is that the convex hull of its image contains the origin $0 \in \mathbb{R}^3$. A stronger hypothesis, coming from control theory and introduced in the context of convex integration by Gromov, reads: A smooth curve $\gamma : \mathbb{S}^1 \to \mathbb{S}^2$ *strictly surrounds* the origin if

$$\mathcal{O}p\,(0) \subseteq \mathrm{Conv}(\mathrm{Img}\,\gamma).$$

Equivalently, there is no non-zero vector $u \in \mathbb{R}^3$ with $u \cdot \gamma(t) \geq 0$ for all $t \in \mathbb{S}^1$. Geometrically, this means that $\mathrm{Img}\,\gamma$ is not contained in any closed hemisphere.

The following allows us to achieve the zero integral condition:

Lemma 6.3.4 (Gromov's reparametrisation lemma [Gro86, Subsection 2.4.1])**.** *Let* K *be a compact CW-complex and* $A \subseteq K$ *a closed CW-subcomplex. Consider a* K*-parameter family of immersions* $f : K \to \mathfrak{I}\mathrm{mm}$ *such that*

(i) $f(k)$ *strictly surrounds the origin for all* $k \in K$*;*

(ii) $\displaystyle\int_0^1 f(k)(t)\,\mathrm{d}t = 0 \in \mathbb{R}^3$ *for* $k \in \mathcal{O}p\,(A)$.

Then, there exists a K*-parameter family of diffeomorphisms* $\psi : (K, \mathcal{O}p\,(A)) \to (\mathrm{Diff}(\mathbb{S}^1), \mathrm{Id})$*, such that the reparametrisations* $\tilde{f}(k) := f(k) \circ \psi(k)$ *satisfy*

$$\int_0^1 \tilde{f}(k)(t)\,\mathrm{d}t = 0, \quad \text{for all } k \in K.$$

We will show that the strictly surrounding condition can be achieved for any family of complicated curves by a suitable homotopy. For this purpose, we introduce a convenient subclass.

Definition 6.3.5. Let $\gamma : [0,1] \to \mathbb{S}^2$ be a convex curve. We say that:

- γ has a *wiggle* in a sub-interval $[a,b]$, if $\gamma(a) = \gamma(b)$ and $\gamma|_{[a,b]}$ is a smooth, closed, embedded convex curve after identifying the endpoints.
- γ has an n*-wiggle* in $[a,b]$, if there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that γ has a wiggle in $[t_i, t_{i+1}]$ for all $i = 0, \ldots, n-1$, and all arcs $\gamma|_{[t_i,t_{i+1}]}$ have the same image.

- γ *n-completely surrounds* the origin if there exist disjoint intervals $[a_1, b_1]$ and $[a_2, b_2]$ such that γ has a n-wiggle in each $[a_j, b_j]$, $j = 1, 2$, and

$$\mathcal{O}p\,(0) \subseteq \mathrm{Conv}(\mathrm{Img}\,\gamma|_{[a_1,b_1]\cup[a_2,b_2]}).$$

  In particular, γ strictly surrounds 0. △

We denote by $\mathfrak{C}\mathrm{onv}^c_n \subseteq \mathfrak{C}\mathrm{onv}^c$ the space of convex curves that n-completely surround the origin, note that if γ is not complicated (i.e. if it is a concatenation of embedded convex curves), then it never surrounds the origin. In view of Lemma 6.3.4, it is natural to deform families of complicated convex curves so that they n–completely surround the origin, for a fixed $n \geq 1$. This can be achieved by *grafting*; the precise relative statement is as follows.

Proposition 6.3.6. *Let K be a compact CW-complex and $A \subseteq K$ a closed CW-subcomplex. For every continuous map $f : (K, \mathcal{O}p\,(A)) \to (\mathfrak{C}\mathrm{onv}^c, \mathfrak{C}\mathrm{onv}^c_{n+2})$ with $n \geq 1$, there exists a homotopy $H_s : (K, \mathcal{O}p\,(A)) \to (\mathfrak{C}\mathrm{onv}^c, \mathfrak{C}\mathrm{onv}^c_{n+2})$, $s \in [0, 1]$, such that*

(i) $H_0 = f$,

(ii) $H_s(k) = f(k)$ *for all* $(s, k) \in [0, 1] \times \mathcal{O}p\,(A)$,

(iii) $\mathrm{Image}(H_1) \subseteq \mathfrak{C}\mathrm{onv}^c_n$.

The remainder of the section is devoted to its proof, which in turn implies Proposition 6.3.1.

6.3.3 Proof of Proposition 6.3.6 via grafting

A *graftable model* is a convex curve $\gamma_1 : [t_0, t_1] \to \mathbb{S}^2$ that is tangent to some latitude circles $\{z = C_i\}$ at t_i, with $C_0 > 0 > C_1$, and whose orientation matches the convex orientation of those circles. Consider then a convex curve $\gamma : \mathbb{S}^1 \to \mathbb{S}^2$ that contains a graftable model $\gamma_1 = \gamma_{|[t_0,t_1]}$. The *grafting* of γ along $[t_0, t_1]$ is the homotopy of convex curves obtained by cutting γ at t_0 and t_1, rotating the arc $\gamma_1 = \gamma_{|[t_0,t_1]}$ about the z-axis by angle $2\pi s$ for $s \in [0, 1]$, and reconnecting it along the latitude circles $z = C_i$ to the complementary arc $\gamma|_{\mathbb{S}^1\setminus[t_0,t_1]}$. For $s = 1$, the resulting curve contains two disjoint 1-wiggles whose convex hull contains the origin. In other words, the resulting convex curve[1] 1-completely surrounds the origin. See Fig. 6.2 for a schematic depiction of the grafting homotopy. We also define the *n-grafting* by repeating the grafting homotopy n times, equivalently, rotate by angle $2n\pi s$; in this case, the resulting curve n-completely surrounds the origin.

The following lemma is implicit in Saldanha's work:

[1]Technically, one needs to smooth out the homotopy constructed in this manner. This can be done in a C^1-small manner, supported in a small neighbourhood of the cutting points, while preserving convexity. This will be implicit in the remainder of the discussion.

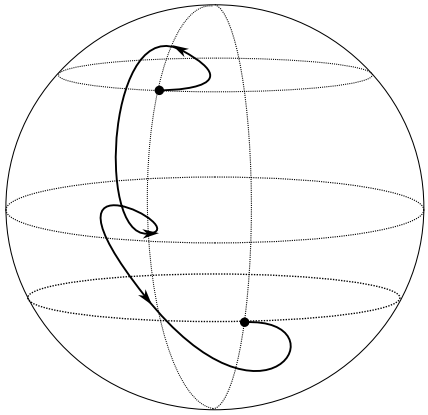


Figure 6.1: A graftable arc $\gamma_1 : [t_0, t_1] \to \mathbb{S}^2$. At its endpoints, the arc is tangent to two latitude circles.

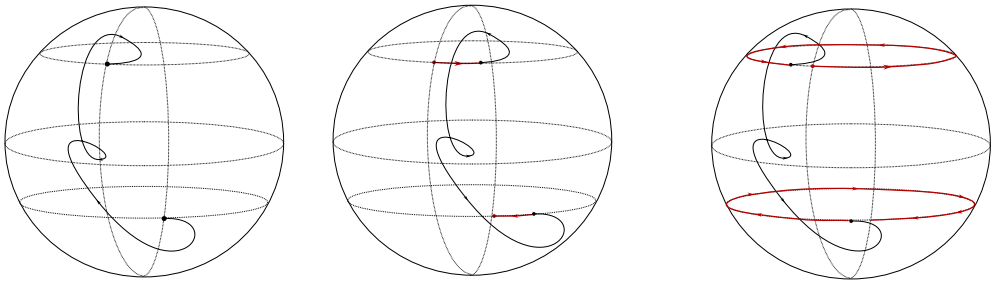


Figure 6.2: Grafting homotopy of a graftable arc. The images depict the homotopy at times $s = 0, 0.2, 0.8$, from left to right. In particular, in the last frame we see that the grafting has almost completed a full loop, which will create a wiggle in the northern hemisphere and another wiggle in the southern hemisphere (both in red).

Lemma 6.3.7. *Let $f : K \to \mathfrak{C}\mathrm{onv}^c$ be a family of complicated convex curves. Then, there are:*

- *Compact subsets $K_1, \ldots, K_m \subseteq K$ covering K.*
- *Compact subsets $K_{j,1}, K_{j,2} \subseteq K_j$ covering K_j, for each j.*
- *Continuous functions $a_j, b_j, c_j, d_j : K_j \to (0,1) \subseteq \mathbb{S}^1$, for each j, with $a_j < b_j < c_j < d_j$.*
- *Intervals $I_j(k) \coloneqq [b_j(k), c_j(k)] \subseteq \tilde{I}_j(k) \coloneqq [a_j(k), d_j(k)]$.*

Such that:

(i) For every $k \in K_i \cap K_j$ and $i \neq j$, it holds that $\tilde{I}_j(k) \cap \tilde{I}_i(k) = \emptyset$.

(ii) Over $K_{j,1}$, for each j, the arcs $f(k)\,|_{I_j(k)}$ are projectively equivalent[2] *to a graftable model.*

(iii) Over $K_{j,2}$, for each j, the arcs $f(k)|_{\tilde{I}_j(k)}$ are projectively equivalent to a graftable model.

(iv) Suppose there is $k \in K_j$ such that $f(k)$ has an n–wiggle in some $[a,b] \subseteq (0,1)$ with $n \geq 2$. Then $\tilde{I}_j(k)$ is not fully contained in $[a,b]$. Moreover, $[a,b] \cap \tilde{I}_j(k)$ contains no wiggle.

Proof. Given a convex curve $\gamma\colon [0,1] \to \mathbb{S}^2$ and $t_0 \in [0,1)$, the *next step* $\mathrm{ns}_\gamma(t_0)$ is the smallest $t_0^+ > t_0$ such that $\gamma|_{[t_0,t_0^+]}$ does not extend to an embedded convex loop. There are multiple reasons why the curve may not extend. Saldanha classifies the possibilities; see the first two paragraphs of [Sal15, Sec. 9]. A step is *bad* if $\gamma|_{[t_0,t_0^+]}$ is a wiggle. Otherwise it is *good*.

If $t_1 = \mathrm{ns}_\gamma(t_0)$ and $[t_0,t_1]$ is a good step, then for any $\varepsilon > 0$ there exist subintervals

$$[t_0,t_1] \subsetneq [\widehat{t_0},\widehat{t_1}] \subsetneq [\widetilde{t_0},\widetilde{t_1}] \subsetneq [t_0-\varepsilon, t_1+\varepsilon],$$

such that at least one of $\gamma|_{[\widehat{t_0},\widehat{t_1}]}$ or $\gamma|_{[\widetilde{t_0},\widetilde{t_1}]}$ is projectively equivalent to a graftable model. Moreover, this holds parametrically in families of good steps, see [Sal15, Lemma 8.2, Lemma 9.1].

Consider now the sequence of next steps $\{\mathrm{ns}^j_{f(k)}(0)\}$, for each j and k. Given the family $f : K \to \mathfrak{C}\mathrm{onv}^c$, we can define the subspace $K_j \subseteq K$ consisting of those $k \in K$ such that the jth-step $S_j(k) := [\mathrm{ns}^j_{f(k)}(0), \mathrm{ns}^{j+1}_{f(k)}(0)] \subseteq [0,1)$ is good. According to the definition of bad step, every complicated curve has at least one good step, so $\{K_j\}$ is a cover of K. The intervals produced in this manner, for different K_j and K_i, are not disjoint. However, being a good step is an open condition. This allows us to shift all the $S_j(k)$ to the right to make them disjoint; the definition of good step then defines the claimed $I_j(k) \subseteq \tilde{I}_j(k)$. For details, see the first three paragraphs of the proof of [Sal15, lemma 10.1].

For (iv), consider $\gamma : [0,1] \to \mathbb{S}^2$ with an n-wiggle in $[a,b]$ with partition $a = s_0 < \cdots < s_n = b$. Then, for any $s_i \leq t \leq s_{i+1}$ with $i = 0,\ldots,n-2$, it holds that $\mathrm{ns}_\gamma(t) = t^+$ satisfies $s_{i+1} \leq t^+ \leq s_{i+2}$. Moreover, $[t,t^+]$ is wiggle and thus a bad step. Hence, any good step must meet $[a,b]$ near an endpoint, i.e., inside of $[s_0,s_1)$ or $(s_{n-1},s_n]$. Since $\tilde{I}_j(k)$ is a small thickening of a good step, (iv) follows. □

Proof of Proposition 6.3.6. Our proof follows Saldanha's approach [Sal15, Lemma 10.1]. We first treat the absolute case $A = \varnothing$, since f is a family of complicated convex curves, we use the notation from Lemma 6.3.7.

Choose smaller compact subsets $W_{j,1} \Subset K_{j,1}$ and $W_{j,2} \Subset K_{j,2}$ for $j = 1,\ldots,m$, such that

$$K = \bigcup_{j=1}^{m} \mathrm{Int}(W_{j,1}) \cup \mathrm{Int}(W_{j,2}).$$

[2] That is, there exists a continuous map $k \mapsto P_k \in \mathrm{PGL}^+(3,\mathbb{R})$ such that $P_k \circ f(k)\,|_{I_j(k)}$ is a graftable model.

We construct the homotopy inductively for $j = 1, \dots, m$, modifying the family only over K_j.

Set $f_0 := f$. On $K_{1,1}$ perform an n–grafting of $f_0(k)$ along $I_1(k)$ for $k \in W_{1,1}$, obtaining $\tilde{f}_0$, and keep $\tilde{f}_0(k) = f_0(k)$ for $k \notin K_{1,1}$. On $K_{1,1} \setminus W_{1,1}$ interpolate with a bump function $\beta \colon K \to [0,1]$ satisfying $\beta \equiv 1$ on $W_{1,1}$ and $\beta \equiv 0$ on $K \setminus K_{1,1}$.

Next, on $K_{1,2}$ perform an n-grafting of $\tilde{f}_0(k)$ along $\tilde{I}_1(k)$ for $k \in W_{1,2}$, producing f_1, and set $f_1(k) = \tilde{f}_0(k)$ for $k \notin K_{1,2}$; interpolate on $K_{1,2} \setminus W_{1,2}$ with another cutoff as before. The pair of n-wiggles created in the first step lies inside $\tilde{I}_1(k)$, and the grafting homotopy (rotation about the z-axis) preserves the n-completely surrounding condition. This yields a homotopy from f_0 to f_1 supported in K_1, with $f_1(k)$ n-completely surrounding the origin for $k \in W_{1,1} \cup W_{1,2}$.

Iterating this two-step procedure for $j = 2, \dots, m$ yields maps $f_2, \dots, f_m$. Because the intervals $\tilde{I}_j(k)$ are pairwise disjoint, each stage preserves the previously created n-wiggles. Consequently, for every $k \in K$, the curve $f_m(k)$ n-completely surrounds the origin.

For the relative case, fix a compact subset $W \subseteq \mathcal{O}p\,(A)$ such that $A \subseteq \mathrm{Int}(W)$. We perform the construction above with cutoffs so that the entire homotopy is supported in $K \setminus W$. Consider $k \in \mathcal{O}p\,(A) \setminus W$ and denote by B_k and B'_k the two intervals supporting the $(n+2)$-wiggles. According to Lemma 6.3.7 (iv), each $\tilde{I}_j(k)$ can meet B_k or B'_k only along a subsegment that is itself not a wiggle. Since k may be in multiple K_j, both ends of B_k or B'_k may intersect one such thickened good step.

Consequently, since the grafting is supported inside $\bigcup_j \tilde{I}_j(k)$, it can remove at most one wiggle from each side of B_k or B'_k. It follows that at least n wiggles remain from each. We deduce that the n-completely surrounding condition holds on the collar $\mathcal{O}p\,(A) \setminus W$, and thus globally, while the homotopy remains equal to f on W. □

6.3.4 Proof of Proposition 6.3.1

Since M is a closed, oriented 3-manifold, we can fix a trivialisation $TM \simeq M \times \mathbb{R}^3$ for the rest of the proof.

Assume first that $N \cong M \times \mathbb{S}^1$ is trivial. Fix an immersed curve $\gamma : \mathbb{S}^1 \looparrowright \mathbb{S}^2$ and apply Proposition 6.3.3 to homotope it to a complicated convex curve. Apply then Proposition 6.3.6 to homotope it further so it becomes completely surrounding. Lastly, apply Gromov's reparametrisation Lemma 6.3.4 so its integral is zero. Denote the resulting curve by $\eta : \mathbb{S}^1 \looparrowright \mathbb{S}^2$. Then η has a primitive $\nu : \mathbb{S}^1 \looparrowright \mathbb{R}^3$. By transversality, after a C^3–small perturbation, we may assume that ν is embedded while $\mathbf{t}(\nu)$ remains convex. Define a convex bundle embedding by

$$g : M \times \mathbb{S}^1 \hookrightarrow M \times \mathbb{R}^3 \cong TM; \quad (p,t) \mapsto (p, \nu(t)).$$

This concludes the argument in the trivial bundle case.

For a general bundle N we proceed as follows. Fix a finite cover by compact subsets $\{K_j\}_{j=1,\dots,m}$ of M over which N trivialises, and a smaller cover by compact sets

$\{W_j\}_{j=1,\dots,m}$ with $W_j \Subset K_j$. Without loss of generality, assume that the transition maps $\sigma_{ij} : K_i \cap K_j \times \mathbb{S}^1 \to K_i \cap K_j \times \mathbb{S}^1$ are valued in $\mathrm{SO}(2)$, and that the refined cover $\{W_j\}$ has order 3, so any point of M lies in at most 4 of the W_j (see [HW15, Theorem V.1]).

By Lemma 6.3.2, there exists a bundle immersion $f : N \looparrowright \mathbb{S}TM \simeq M \times \mathbb{S}^2 \subseteq M \times \mathbb{R}^3$. This immersion can be described as a collection of continuous maps

$$f_j : K_j \longrightarrow \mathfrak{Imm}, \qquad j = 1, \dots, m,$$

which are compatible with the transition maps in the obvious sense: on $K_i \cap K_j$ one has $f_i(k) \circ \sigma_{ij}(k) = f_j(k)$.

Our goal is to deform these maps so that their values lie in the subspace of convex curves with zero total integral. Notice that the latter condition is independent of the chosen trivialisations, because the transition functions are $\mathrm{SO}(2)$-valued. After this is achieved, the desired convex bundle immersion is obtained by fibrewise integration.

To achieve this, proceed inductively over the compact sets $W_j \Subset K_j$, $j = 1, \dots, m$. Set $n := 2 \cdot 4 + 1 = 9$ which will be used in Proposition 6.3.6.

For the first chart, apply Proposition 6.3.3, Proposition 6.3.6 and Lemma 6.3.4 (in the absolute case) to the map $f_1 : K_1 \to \mathfrak{Imm}$. This yields a homotopy $f_1^s : K_1 \to \mathfrak{Imm}$, $s \in [0,1]$, with $f_1^0 = f_1$ and $f_1^1 : K_1 \to \mathfrak{Conv}_n^c$ having zero integral. We cut off this deformation outside of W_1: we fix a bump function $\rho : K_1 \to [0,1]$, supported in $\mathcal{O}p\,(W_1)$, with $\rho_{|W_1} \equiv 1$, and define $\tilde{f}_1(k) := f_1^{\rho(k)}(k)$. Since the deformation is supported in $\mathcal{O}p\,(W_1)$ it defines a global bundle immersion $\tilde{f} : N \looparrowright \mathbb{S}TM$ with the required properties over W_1.

Since Proposition 6.3.3, Proposition 6.3.6 and Lemma 6.3.4 admit parametric and relative versions, the same argument can be iterated over $W_2, \dots, W_m$, keeping the previously treated region fixed at each step. Since every point of M lies in at most four of the sets W_j, the multiplicity of the wiggles may decrease by at most 2 per step, hence by at most 8 in total. It follows that, if we start from $\mathfrak{Conv}_n^c$ with $n = 9$, the final family will still lie in $\mathfrak{Conv}_{n-8}^c = \mathfrak{Conv}_1^c$, so the strictly surrounding condition persists throughout the induction. I.e., we have obtained a bundle immersion $\hat{f} : N \looparrowright \mathbb{S}TM$ with local representatives

$$\hat{f}_j : K_j \longrightarrow \mathfrak{Conv}_{n-8}^c \subseteq \mathfrak{Imm}, \qquad j = 1, \dots, m.$$

We may assume that these curves have zero total integral after an application of Lemma 6.3.4.

The final step of the proof amounts to integrating the family $\hat{f}$. Define local maps

$$g_j \colon K_j \longrightarrow \mathfrak{Imm}(\mathbb{S}^1, \mathbb{R}^3), \qquad g_j(k)(t) := \int_0^t \hat{f}_j(k)(\tau)\, d\tau, \qquad j = 1, \dots, m,$$

by fibrewise integration. Since $\int_0^1 \hat{f}_j(k)(\tau)\, d\tau = 0$, each $g_j(k)$ is a closed curve in $\mathbb{R}^3$.

On overlaps $K_i \cap K_j$ the families g_i and g_j may differ by a translation and thus not define a bundle map. Namely, if we write

$$\sigma_{ij}(k) \colon \mathbb{S}^1 \to \mathbb{S}^1, \ t \mapsto t - \delta(k),$$

for the transition function, then we get:

$$(g_i \circ \sigma_{ij})(k)(t) = \int_0^{t-\delta(k)} \hat{f}_i(k)(\tau)\, d\tau = \int_0^{t-\delta(k)} \hat{f}_j(k)(\tau + \delta(k)\, d\tau$$
$$= \int_{\delta(k)}^{t} \hat{f}_j(k)(\tau)\, d\tau = g_j(k)(t) \; - \underbrace{\int_0^{\delta(k)} \hat{f}_j(k)(\tau)\, d\tau}_{=: v_{ij}(k)}.$$

I.e., the translation vector $v_{ij} \colon K_i \cap K_j \to \mathbb{R}^3$ is independent of t, as expected.

Since $\mathbb{R}^3$ is contractible, we can now shift the $\{g_j\}$ to match them up. Concretely: choose a partition of unity $\{\phi_j\}$ subordinate to $\{K_j\}$ and, define a bundle map $g : N \to TM \simeq M \times \mathbb{R}^3$ by

$$g(q) = \sum_{j=1}^{m} \phi_j(\pi(q)) g_j(\pi(q)), \quad q \in N,$$

where $\pi : N \to M$ is the bundle projection. In the trivialisation over K_j (write $k = \pi(q)$) we have, using $(g_i \circ \sigma_{ij})(k) = g_j(k) - v_{ij}(k)$,

$$g|_{K_j}(k) = \phi_j(k) g_j(k) + \sum_{i \neq j} \phi_i(k)(g_i \circ \sigma_{ij})(k) = g_j - \sum_{i \neq j} \phi_i(k) v_{ij}(k).$$

The correction term is independent of $t \in \mathbb{S}^1$, hence the fibrewise derivative satisfies $\mathbf{t}(g)|_{K_j} = \mathbf{t}(g_j) = \hat{f}_j$, which is convex by construction. Therefore g is a well-defined bundle immersion with convex fibrewise derivative. □

6.4 Zoom-in argument: proof of Theorem 6.1.1

Let N be an oriented $\mathbb{S}^1$-bundle over the closed, oriented 3-manifold M and let $g : N \looparrowright TM$ be a convex bundle immersion. According to the local model Lemma 6.2.9, we may assume that $(X, J) = (TM, J)$, with J splitting-adapted and M embedded as the zero section.

For $\lambda > 0$, consider the fibrewise dilations

$$\delta_\lambda : TM \to TM,$$

which scale each fibre by λ, and define a one-parameter family of convex bundle immersions by

$$g^\lambda := \delta_\lambda \circ g : N \looparrowright (TM, J).$$

The result will follow if we show that $\mathcal{D}_J^{g^\lambda}$ is Engel for all sufficiently small $\lambda > 0$. To this end, we prove the following claim.

Claim 6.4.1. *For every point $p \in M$, there exists $\lambda(p) > 0$ sufficiently small and a neighbourhood $U(p)$ of E_p such that $\mathcal{D}_J^{g^\lambda}$ is Engel on $U(p)$ for all $0 < \lambda < \lambda(p)$.*

Once the claim is established, Theorem 6.1.1 follows immediately by a standard compactness argument.

Since the claim is local, choose coordinates as in Corollary 6.2.10:

$$\begin{aligned} \mathbb{R}^3_x \times \mathbb{R}^3_y &\to T\mathbb{R}^3_x \\ (x_i, y_i) &\mapsto \sum y_i \cdot \partial_{x_i}, \end{aligned}$$

with J splitting-adapted along $\mathbb{R}^3_x$ and satisfying $J = J_{\text{std}}$ along the zero section. In these coordinates, the family of convex bundle immersions has the form

$$\begin{aligned} g^\lambda : \mathbb{R}^3_x \times \mathbb{S}^1 &\looparrowright \mathbb{R}^3_x \times \mathbb{R}^3_y \\ (x,t) &\mapsto \big(x, \lambda\gamma(x,t)\big), \end{aligned}$$

for some $\gamma : \mathbb{R}^3_x \times \mathbb{S}^1 \to \mathbb{R}^3_y$.

We now introduce a special case of bundle immersion that will be important in the proof of the claim.

Definition 6.4.2. A bundle immersion $g(x,t) = (x, \gamma(x,t)) : \mathbb{R}^3_x \times \mathbb{S}^1 \looparrowright \mathbb{R}^3_x \times \mathbb{R}^3_y$ is called *base-independent* if γ does not depend on x. Equivalently, $\frac{\partial}{\partial x}\gamma \equiv 0$. △

Proof of Claim 6.4.1 and Theorem 6.1.1. We divide the proof into two steps. First, we treat the case where g is base-independent and $J \equiv J_{\text{std}}$ is the standard complex structure. Then, we reduce the general case to this one by using a *zooming-in* argument.

Step I: Base-independent case. We observe:

Lemma 6.4.3. *If g is base-independent and $J \equiv J_{\text{std}}$ on $\mathbb{C}^3 = \mathbb{R}^3_x \times \mathbb{R}^3_y$, then*

$$\mathcal{D}^g_{\text{der}-\text{prol}} = \mathcal{D}^g_{J_{\text{std}}}$$

on $\mathbb{R}^3_x \times \mathbb{S}^1$.

Proof. By hypothesis, $g(x,t) = (x, \gamma(t))$ so the differential of g is the map $g_* = \text{Id}_{T\mathbb{R}^3_x} \times (\gamma)_*$. I.e., the image of g_* at (x,t) is $T\mathbb{R}^3_x \oplus \langle \gamma'(t) \rangle \subseteq T\mathbb{R}^3_x \times T\mathbb{R}^3_y$. The element $J_{\text{std}}(\gamma'(t))$ is contained in $T\mathbb{R}^3_x$ and is identified with $-\gamma'(t)$ under the canonical identification $T\mathbb{R}^3_x \simeq T\mathbb{R}^3_y$. It follows that the complex tangency, pulled back to the domain, is $\langle \gamma'(t), \partial_t \rangle$. This is exactly how $\mathcal{D}^g_{\text{der}-\text{prol}}(x,t)$ was defined. □

In particular, since g was chosen to be a convex bundle immersion, we deduce (Proposition 6.2.6) that $\mathcal{D}^g_{J_{\text{std}}} = \mathcal{D}^g_{\text{der}-\text{prol}}$ is Engel in the base-independent, $J = J_{\text{std}}$ case.

Step II: The zooming argument. We now reduce the general claim to the previous case. At an informal level, our argument is based on the fact that rescaling the base by $\lambda \to 0$ makes the bundle immersion closer to being base-independent, whereas rescaling base and fibre simultaneously forces the pulled-back almost complex structure to converge to the standard one.

Consider the family of "zooming" coordinate changes:

$$\begin{aligned}\Delta_\mu : \mathbb{R}^3_a \times \mathbb{R}^3_b &\to \mathbb{R}^3_x \times \mathbb{R}^3_y,\\ (a,b) &\mapsto (\mu a, \mu b) :\end{aligned}$$

for $\mu > 0$.

In the coordinates given by Δ_λ (i.e., $\mu = \lambda$), the bundle immersion g^λ becomes:

$$\tilde{g}^\lambda(a,t) = (\Delta_\lambda)^{-1} g^\lambda(\lambda a, t) = (a, \gamma(\lambda a, t)).$$

while the almost complex structure J transforms as:

$$J^\lambda_{(a,b)}(V) := (\Delta_\lambda)^* J(V) = (\Delta_\lambda)^{-1}\big(J_{(\lambda a,\lambda b)}(\lambda V)\big) = J_{(\lambda a,\lambda b)}(V), \quad \forall\, V \in \mathbb{R}^3_a \times \mathbb{R}^3_b,$$

which remains splitting-adapted by construction.

Lemma 6.4.4. *The pairs $(\tilde{g}^\lambda, J^\lambda)$ converge to $(\tilde{g}^0, J_{\text{std}})$ in the weak C^∞-topology, where $\tilde{g}^0(x,t) = (x, \gamma(0,t))$ is a base-independent bundle immersion.*

Proof. We compute

$$(\tilde{g}^\lambda - \tilde{g}^0)(a,t) = (0, \gamma(\lambda a, t) - \gamma(0,t)).$$

Expanding $\gamma(\lambda a, t)$ in the first variable around 0 via Taylor's theorem yields the desired convergence. The convergence of J^λ to J_{std} follows similarly. □

By the definition of the weak Whitney C^∞ topology, $\mathcal{D}^{\tilde{g}^\lambda}_{J^\lambda}$ is C^∞-close to $\mathcal{D}^{\tilde{g}^0}_{J_{\text{std}}}$ near the origin. Since $\tilde{g}^0$ is base-independent, Step I implies that $\mathcal{D}^{\tilde{g}^0}_{J_{\text{std}}}$ is Engel. Because the Engel condition is C^2-open, $\mathcal{D}^{\tilde{g}^\lambda}_{J^\lambda}$ is also Engel for λ small enough. This proves Claim 6.4.1, and therefore Theorem 6.1.1. □

6.5 Proof of Theorem 6.1.3 and Corollary 6.1.4

The last ingredient we need is Gromov's h-principle for totally real embeddings, combined with the fact, due to Forstnerič [For86, Gro86], that any embedding of a 3-manifold into $\mathbb{C}^3$ is formally totally real.

Proposition 6.5.1. *Any embedding $\iota : M \hookrightarrow \mathbb{C}^3$ is isotopic, through embeddings, to a totally real embedding.*

Proof of Theorem 6.1.3. Let N be an oriented $\mathbb{S}^1$-bundle over a closed, oriented 3-manifold M. By Proposition 6.5.1, we can find a totally real embedding $M \hookrightarrow \mathbb{C}^3$ in any given isotopy class. According to Proposition 6.3.1 we can construct a convex bundle immersion $g : N \looparrowright TM$ (an embedding when N is trivial). Then Theorem 6.1.1 applies, yielding the desired co-real immersion whose complex tangencies are Engel. □

Remark 6.5.2. When $e(N)$ is even, Lemma 6.3.2 provides a bundle embedding $g : N \hookrightarrow TM$. We conjecture that a convex bundle embedding also exists. This would produce embeddings of more general $\mathbb{S}^1$-bundles in Theorem 6.1.3, and is left as an open question. Alternatively, one may produce a convex bundle immersion and try to achieve the fibrewise embedding condition. △

Proof of Corollary 6.1.4. The proof follows by applying the construction above to an infinite family of totally real embeddings $i_k : M^3 \hookrightarrow \mathbb{C}^3$, $k \in \mathbb{Z}$, pairwise not smoothly isotopic.

The existence of such a family relies on the works of Haefliger and Skopenkov [Hae62, Hae66, Sko08]. Indeed, in view of the h-principle from Proposition 6.5.1 it is enough to find an infinite family of smooth embeddings pairwise not smoothly isotopic to conclude. In the case $M = \mathbb{S}^3$, the existence of a family of embeddings $i_k : \mathbb{S}^3 \hookrightarrow \mathbb{C}^3$, $k \in \mathbb{Z}$, pairwise not smoothly isotopic follows from [Hae62, Hae66]. This proves the result for $M = \mathbb{S}^3$.

For the general case, consider a smooth embedding $f : M^3 \hookrightarrow \mathbb{R}^5 \subseteq \mathbb{C}^3$, which exists because of [Wal65]. We may assume that f is disjoint from all Haefliger's spheres i_k. Then, the embeddings $f \# i_k : M \hookrightarrow \mathbb{C}^3$, $k \in \mathbb{Z}$, obtained by ambient connected sum of f and i_k, are pairwise not smoothly isotopic as proved in [Sko08]. This concludes the argument. □

Part III

Open questions and future work

Chapter 7

Open questions and future work

We conclude this thesis with a collection of open problems that naturally emerge from the results presented here. These questions indicate possible directions for further development of the theory of fat distributions, prelegendrians, and Engel structures.

Fat manifolds and prelegendrians

A first basic problem concerns the topology of prelegendrians:

Question. Does there exist a prelegendrian embedding $\Lambda : \mathbb{S}^{2n+1} \hookrightarrow (\mathbb{C}^{2n+1}, \mathcal{D}_{\text{std}})$ of the sphere?

All examples constructed in this thesis arise from spinning operations, and thus have the topology of a (non-trivial) product. It could be that understanding higher order singularities of prelegendrian fronts is required to answer this question. More generally, one could ask about the construction of prelegendrian embeddings of other smooth types.

In a different direction:

Question. Does the h-principle fail for prelegendrian embeddings in corank $k > 2$ fat manifolds?

The case $k = 4$ is currently under investigation, building on the methods developed here. For higher coranks, our understanding of fat structures is still too limited to manipulate prelegendrians effectively.

Analogously to contact geometry, one may also ask about the role of subcritical preisotropics in fat manifolds. The h-principle for subcritical preisotropic immersions, and for prelegendrian immersions of open manifolds, has been established by Bhowmick, building on Gromov's work [Bho22, Gro96]. This motivates the following problem:

Question. Does the h-principle hold for subcritical preisotropic *embeddings* and for prelegendrian immersions in corank $k > 1$ fat manifolds?

For $k = 1$ (the classical contact case), the answer is affirmative by results of Gromov [CEM24, Gro86].

The stabilization procedure developed in Theorem 2.2.4 suggests a natural notion of flexibility:

Question. Does there exist a class of *loose prelegendrians* (containing our stabilized prelegendrians), that is governed by an h-principle?

If the answer to this question is negative, then it is not hard to see that the following question also has a negative answer:

Question. Is the isotopy class of the prelegendrian lift $\mathcal{L}(\Lambda)$ a complete prelegendrian invariant? Is it a complete prelegendrian invariant among the class of formally equivalent prelegendrians?

This question is a concrete incarnation of Eliashberg's "pseudoholomorphic curves or nothing" philosophy. The problem is also analogous to the fundamental question of whether the Legendrian isotopy class of the co-normal lift is a complete isotopy invariant. This is known to be true for knots in $\mathbb{R}^3$ by a celebrated result of Shende [ENS18, She19]. The study of Legendrian singularities relative to isotropic foliations [AG18] may play a role here.

Another interesting question is about the prelegendrian itself. If $\Lambda \subseteq (M, \mathcal{D})$ is prelegendrian, then $\mathcal{D}_\Lambda \coloneqq T\Lambda \cap \mathcal{D} \subseteq T\Lambda$ is a distribution of Λ. So the following questions are natural:

Question. (a) Let $\mathcal{D}' \subseteq T\Lambda$ be distribution, does there exist a prelegendrian embedding $\Lambda \hookrightarrow (M, \mathcal{D})$ such that $\mathcal{D}_\Lambda = \mathcal{D}'$?

(b) Let $\Lambda \subseteq (M, \mathcal{D})$ be a prelegendrian submanifold. Can a small perturbation of $\mathcal{D}_\Lambda$ be realized by perturbations of the embedding $\Lambda \hookrightarrow (M, \mathcal{D})$?

Finally, the most fundamental questions concern fat distributions themselves:

Question. Recall that fat distributions satisfy the h-principle on open manifolds, according to Gromov's h-principle for open and diff-invariant relations [CEM24, Gro86]. Then, focusing on closed manifolds:

(a) Does the h-principle fail for corank $k > 1$ fat distributions themselves?

(b) If the answer to (a) is positive, does there exist an *overtwisted* class of fat distributions?

(c) Is there a fat manifold with overtwisted contactisation?

We conjecture that the answer to (a) is positive at least for $k = 2$, in agreement with the conjecture in [MAdP21] on $(4, 6)$ fat distributions. For (c), one possible strategy would be to define a "pre-overtwisted" object lifting to an overtwisted disk in the contactisation. It is worth noting that all fat manifolds appearing in this thesis (e.g. complex Grassmannians) have tight contactisations, as these are spaces of contact elements and hence fillable.

Engel structures and complex tangencies

Eliashberg's question on the existence of Engel structures as CR submanifolds is very natural from the viewpoint of contact topology,: If $Y^{2n-1} \subset \mathbb{C}^n$ is a smooth real hypersurface, its complex tangencies $\xi_{J_{\text{std}}} \coloneqq TY \cap J_{\text{std}}(TY)$ define a corank-one distribution that is contact precisely when Y is strictly pseudoconvex; moreover, in this case, $(Y, \xi_{J_{\text{std}}})$ is tight.

Our construction from Theorem 6.1.3 provides the first examples of Engel structures arising naturally from complex geometry. Analogously to the reasoning above, this leads to the following problem, which is very much open:

Question. Can some form of rigidity be established in Engel Topology using this relation to Complex Geometry?

It is unclear to us how to pursue this further.

One can also ask whether the methods presented in Theorems 6.1.1 and 6.1.3 and Corollary 6.1.4 to construct Engel structures can be generalised further. Namely, our examples are arise as small deformations of derived prolonged distributions associated with convex bundle immersions. In particular, one cannot fully prescribe the Engel structure in advance (not even up to homotopy). Motivated by the facts that every parallelisable 4-manifold admits an Engel structure [Vog09] and that there is no formal obstruction to admitting a co-real embedding into $\mathbb{C}^3$ [Sla15, Gro86], we are led to the following questions:

Question. Can every oriented $\mathbb{S}^1$-bundle with even Euler class be co-real Engel embedded into $\mathbb{C}^3$? More generally, can every parallelisable 4-manifold be embedded into $\mathbb{C}^3$ in such a way that its complex tangencies define an Engel structure?

The following is closer to the idea of studying deformation theory in CR and complex geometry:

Question. Suppose $N \subseteq \mathbb{C}^3$ is co-real with Engel complex tangencies $\mathcal{D}_{J_{\text{std}}}$. Can every perturbation of $\mathcal{D}_{J_{\text{std}}}$ be induced by a perturbation of the embedding $N \hookrightarrow \mathbb{C}^3$?

Guided by the characterisation of Engelness for curve families [CdPP20], one is also naturally led to ask:

Question. Can the Cartan prolongation of a contact structure [Car24, Mon99] be realized as the complex tangencies of a co-real submanifold in $\mathbb{C}^3$?

Lastly, let us remark that the following major open question remains:

Question. Is there some form of homotopical rigidity in Engel topology?

The results of this thesis (combined with the state-of-the-art) indicate that there are two opposing takeaways:

- Engel flexibility seems to be a generalisation of contact flexibity. Whether Engel rigidity exists is unknown, but it suffers from the fact that Engel structures are not fat.
- Fat rigidity appears to be the right generalisation of contact rigidity to higher corank (and the methods of this thesis should allow us to pursue this further). Whether there is flexibility for the construction of fat structures is very much unclear and, in fact, all the examples we know of seem to be rather special/geometric.

Bibliography

[ABB20] Andrei Agrachev, Davide Barilari, and Ugo Boscain. *A Comprehensive Introduction to Sub-Riemannian Geometry*, volume 181 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020.

[Ada02] Jiro Adachi. Engel structures with trivial characteristic foliations. *Algebr. Geom. Topol.*, 2:239–255, 2002.

[Ada03] Jiro Adachi. Global stability of distributions of higher corank of derived length one. *Int. Math. Res. Not.*, (49):2621–2638, 2003.

[Ada07] Jiro Adachi. Classification of horizontal loops in the standard Engel space. *Int. Math. Res. Not. IMRN*, (3):Art. ID rnm008, 29, 2007.

[AFL17] Antonio Alarcón, Franc Forstnerič, and Francisco J. López. Holomorphic Legendrian curves. *Compositio Mathematica*, 153(9):1945–1986, 2017.

[AG18] Daniel Álvarez-Gavela. The simplification of singularities of Lagrangian and Legendrian fronts. *Invent. Math.*, 214(2):641–737, 2018.

[AGV12] V.I. Arnold, S.M. Gusein-Zade, and A.N. Varchenko. *Singularities of Differentiable Maps, Volume 1: Classification of Critical Points, Caustics and Wave Fronts*. Birkhäuser, Boston, 2012.

[Ame17a] American Institute of Mathematics. Engel structure workshop problem list. http://aimpl.org/engelstr/4/, 2017. Workshop held April 17–21, 2017.

[Ame17b] American Institute of Mathematics. Engel structure workshop summary. https://aimath.org/pastworkshops/engelstrrep.pdf, 2017. Workshop held April 17–21, 2017.

[AN01] V. I. Arnold and S. P. Novikov, editors. *Dynamical systems. IV*, volume 4 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, revised edition, 2001. Symplectic geometry and its applications, A translation of *Current problems in mathematics. Fundamental directions, Vol. 4 (Russian)*, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1985 [MR0842907 (87j:58032)], Translated by G. Wasserman.

[Arn90] V. I. Arnold. *Singularities of caustics and wave fronts*, volume 62 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990.

[BD23] Aritra Bhowmick and Mahuya Datta. Existence of horizontal immersions in fat distributions. *Internat. J. Math.*, 34(10):Paper No. 2350056, 33, 2023.

[BEM15] Matthew Strom Borman, Yakov Eliashberg, and Emmy Murphy. Existence and classification of overtwisted contact structures in all dimensions. *Acta Math.*, 215(2):281–361, 2015.

[Ben83] Daniel Bennequin. Entrelacements et équations de Pfaff. In *Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982)*, volume 107-108 of *Astérisque*, pages 87–161. Soc. Math. France, Paris, 1983.

[Bho22] Aritra Bhowmick. *Horizontal, Contact and Partially Horizontal Immersions in Fat Distributions*. PhD thesis, Indian Statistical Institute, 2022. https://digitalcommons.isical.ac.in/doctoral-theses/569/.

[Bho24] Aritra Bhowmick. On horizontal immersions of discs in fat distributions of type (4,6). *J. Topol. Anal.*, 16(1):125–153, 2024.

[BL18] Riccardo Benedetti and Paolo Lisca. Framing 3-manifolds with bare hands. *Enseign. Math.*, 64(3-4):395–413, 2018.

[BM17] I. Yu. Beschastnyi and A. V. Medvedev. Left-invariant sub-Riemannian Engel structures: Abnormal geodesics and integrability. *SIAM Journal on Control and Optimization*, 55(2):1083–1109, 2017.

[Car10] Elie Cartan. Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre. *Ann. Sci. École Norm. Sup. (3)*, 27:109–192, 1910.

[Car24] E. Cartan. Sur les variétés à connexion projective. *Bull. Soc. Math. France*, 52:205–241, 1924.

[CdP18] Roger Casals and Álvaro del Pino. Classification of Engel knots. *Math. Ann.*, 371(1-2):391–404, 2018.

[CdPP20] Roger Casals, Álvaro del Pino, and Francisco Presas. Loose Engel structures. *Compos. Math.*, 156(2):412–434, 2020.

[CE12] Kai Cieliebak and Yakov Eliashberg. *From Stein to Weinstein and Back*, volume 59 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, Providence, Rhode Island, 2012.

[CEM24] Kai Cieliebak, Yakov Eliashberg, and Nikolai Mishachev. *Introduction to the h -Principle*, volume 239. American Mathematical Society, 2024.

[CFS05] Sergio Console, Anna Fino, and Evangelia Samiou. The moduli space of six-dimensional two-step nilpotent Lie algebras. *Ann. Global Anal. Geom.*, 27(1):17–32, 2005.

[Che02] Yuri Chekanov. Differential algebra of Legendrian links. *Invent. Math.*, 150(3):441–483, 2002.

[Cho39] Wei-Liang Chow. über Systeme von linearen partiellen Differentialgleichungen erster Ordnung. *Math. Ann.*, 117:98–105, 1939.

[CMP19] Roger Casals, Emmy Murphy, and Francisco Presas. Geometric criteria for overtwistedness. *J. Amer. Math. Soc.*, 32(2):563–604, 2019.

[CPdPP17] Roger Casals, José Luis Pérez, Álvaro del Pino, and Francisco Presas. Existence h-principle for Engel structures. *Invent. Math.*, 210(2):417–451, 2017.

[Don96] S. K. Donaldson. Symplectic submanifolds and almost-complex geometry. *J. Differential Geom.*, 44(4):666–705, 1996.

[dP18] Álvaro del Pino. On the classification of prolongations up to Engel homotopy. *Proc. Amer. Math. Soc.*, 146(2):891–907, 2018.

[dPP19] Álvaro del Pino and Francisco Presas. Flexibility for tangent and transverse immersions in Engel manifolds. *Rev. Mat. Complut.*, 32(1):215–238, 2019.

[dPV20] Álvaro del Pino and Thomas Vogel. The Engel-Lutz twist and overtwisted Engel structures. *Geom. Topol.*, 24(5):2471–2546, 2020.

[DRG15] Georgios Dimitroglou Rizell and Roman Golovko. Estimating the number of Reeb chords using a linear representation of the characteristic algebra. *Algebr. Geom. Topol.*, 15(5):2887–2920, 2015.

[EES05a] Tobias Ekholm, John Etnyre, and Michael Sullivan. The contact homology of Legendrian submanifolds in $\mathbb{R}^{2n+1}$. *J. Differential Geom.*, 71(2):177–305, 2005.

[EES05b] Tobias Ekholm, John Etnyre, and Michael Sullivan. Non-isotopic Legendrian submanifolds in $\mathbb{R}^{2n+1}$. *J. Differential Geom.*, 71(1):85–128, 2005.

[EES07] Tobias Ekholm, John Etnyre, and Michael Sullivan. Legendrian contact homology in $P \times \mathbb{R}$. *Trans. Amer. Math. Soc.*, 359(7):3301–3335, 2007.

[EES09] Tobias Ekholm, John B. Etnyre, and Joshua M. Sabloff. A duality exact sequence for Legendrian contact homology. *Duke Math. J.*, 150(1):1–75, 2009.

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to symplectic field theory. pages 560–673. 2000. GAFA 2000 (Tel Aviv, 1999).

[Ekh08] Tobias Ekholm. Rational symplectic field theory over $\mathbb{Z}_2$ for exact Lagrangian cobordisms. *J. Eur. Math. Soc. (JEMS)*, 10(3):641–704, 2008.

[Eli89] Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, 98(3):623–637, 1989.

[Eli90a] Yakov Eliashberg. Filling by holomorphic discs and its applications. In *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, volume 151 of *London Math. Soc. Lecture Note Ser.*, pages 45–67. Cambridge Univ. Press, Cambridge, 1990.

[Eli90b] Yakov Eliashberg. Topological characterization of Stein manifolds of dimension > 2. *Internat. J. Math.*, 1(1):29–46, 1990.

[Eli92] Yakov Eliashberg. Contact 3-manifolds twenty years since J. Martinet's work. *Ann. Inst. Fourier (Grenoble)*, 42(1-2):165–192, 1992.

[Eli98] Yakov Eliashberg. Invariants in contact topology. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, pages 327–338, 1998.

[ENS18] Tobias Ekholm, Lenhard Ng, and Vivek Shende. A complete knot invariant from contact homology. *Invent. Math.*, 211(3):1149–1200, 2018.

[FdPZ25a] Eduardo Fernández, Álvaro del Pino, and Wei Zhou. Engel CR submanifolds of $\mathbb{C}^3$, 2025. https://arxiv.org/abs/2509.11488.

[FdPZ25b] Eduardo Fernández, Álvaro del Pino, and Wei Zhou. The h-principle fails for prelegendrians in corank 2 fat distributions. To appear, 2025.

[FL22] Franc Forstnerič and Finnur Lárusson. Holomorphic Legendrian curves in projectivised cotangent bundles. *Indiana Univ. Math. J.*, 71(1):93–124, 2022.

[FMAP20] Eduardo Fernández, Javier Martínez-Aguinaga, and Francisco Presas. Fundamental groups of formal Legendrian and horizontal embedding spaces. *Algebr. Geom. Topol.*, 20(7):3219–3312, 2020.

[For86] Franc Forstnerič. On totally real embeddings into $\mathbf{C}^n$. *Exposition. Math.*, 4(3):243–255, 1986.

[Ge92] Zhong Ge. Betti numbers, characteristic classes and sub-Riemannian geometry. *Illinois J. Math.*, 36(3):372–403, 1992.

[Gei08a] Hansjörg Geiges. Horizontal loops in Engel space. *Math. Ann.*, 342(2):291–296, 2008.

[Gei08b] Hansjörg Geiges. *An Introduction to Contact Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2008.

[Gol14] R. Golovko. A note on the front spinning construction. *Bull. Lond. Math. Soc.*, 46(2):258–268, 2014.

[Gom25] Robert E. Gompf. Transverse tori in engel manifolds, 2025. https://arxiv.org/abs/2210.04292.

[Gro85] M. Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[Gro86] Mikhael Gromov. *Partial Differential Relations*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1986.

[Gro96] Mikhael Gromov. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian Geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996.

[Hae62] André Haefliger. Knotted $(4k-1)$-spheres in $6k$-space. *Ann. of Math. (2)*, 75:452–466, 1962.

[Hae66] André Haefliger. Differential embeddings of S^n in S^{n+q} for $q > 2$. *Ann. of Math. (2)*, 83:402–436, 1966.

[HW15] Witold Hurewicz and Henry Wallman. *Dimension theory*, volume 4. Princeton university press, 2015.

[Keg24] Marc Kegel. Non-isotopic transverse tori in Engel manifolds. *Rev. Mat. Iberoam.*, 40(1):43–56, 2024.

[KS13] Mirko Klukas and Bijan Sahamie. On prolongations of contact manifolds. *Proc. Amer. Math. Soc.*, 141(9):3257–3263, 2013.

[KV18] Dieter Kotschick and Thomas Vogel. Engel structures and weakly hyperbolic flows on four-manifolds. *Comment. Math. Helv.*, 93(3):475–491, 2018.

[LC13] Peter Lambert-Cole. Legendrian products, 2013. https://arxiv.org/abs/1301.3700.

[MA23] Javier Martínez-Aguinaga. Existence and classification of maximal growth distributions, 2023. https://arxiv.org/abs/2308.10762.

[MAdP21] Javier Martínez-Aguinaga and Álvaro del Pino. Convex integration with avoidance and hyperbolic (4,6) distributions, 2021. https://arxiv.org/abs/2112.14632.

[Mag86] L. Magnin. Sur les algèbres de Lie nilpotentes de dimension ≤ 7. *J. Geom. Phys.*, 3(1):119–144, 1986.

[McD87] Dusa McDuff. Applications of convex integration to symplectic and contact geometry. *Ann. Inst. Fourier (Grenoble)*, 37(1):107–133, 1987.

[Mit18] Yoshihiko Mitsumatsu. Geometry and dynamics of engel structures, 2018.

[Mon93] Richard Montgomery. Generic distributions and Lie algebras of vector fields. *J. Differential Equations*, 103(2):387–393, 1993.

[Mon99] Richard Montgomery. Engel deformations and contact structures. In *Northern California Symplectic Geometry Seminar*, volume 196 of *Amer. Math. Soc. Transl. Ser. 2*, pages 103–117. Amer. Math. Soc., Providence, RI, 1999.

[Mon02] Richard Montgomery. *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[MS74] John Willard Milnor and James D. Stasheff. *Characteristic Classes*. Number 76. Princeton university press, 1974.

[Mur19] Emmy Murphy. Loose legendrian embeddings in high dimensional contact manifolds, 2019. https://arxiv.org/abs/1201.2245.

[Ng05] Lenhard Ng. Knot and braid invariants from contact homology. I. *Geom. Topol.*, 9:247–297, 2005.

[Ng14] Lenhard Ng. A topological introduction to knot contact homology. In *Contact and symplectic topology*, volume 26 of *Bolyai Soc. Math. Stud.*, pages 485–530. János Bolyai Math. Soc., Budapest, 2014.

[Nie06] Klaus Niederkrüger. The plastikstufe—a generalization of the overtwisted disk to higher dimensions. *Algebr. Geom. Topol.*, 6:2473–2508, 2006.

[NvK07] Klaus Niederkrüger and Otto van Koert. Every contact manifolds can be given a nonfillable contact structure. *Int. Math. Res. Not. IMRN*, (23):Art. ID rnm115, 22, 2007.

[OV00] Hirosi Ooguri and Cumrun Vafa. Knot invariants and topological strings. *Nuclear Phys. B*, 577(3):419–438, 2000.

[Pia22] Nicola Pia. Riemannian properties of Engel structures. *Int. Math. Res. Not. IMRN*, (7):4890–4930, 2022.

[Pre07] Francisco Presas. A class of non-fillable contact structures. *Geom. Topol.*, 11:2203–2225, 2007.

[PSC14] Francisco Presas and Luis E. Solá Conde. Holomorphic Engel structures. *Rev. Mat. Complut.*, 27(1):327–344, 2014.

[Ray68] C. B. Rayner. The exponential map for the Lagrange problem on differentiable manifolds. *Philos. Trans. Roy. Soc. London Ser. A*, 262:299–344, 1967/68.

[Rut23] Nina Rutten. Fat prolonged distributions of type (4,6). Master's thesis, Utrecht University, 2023. https://studenttheses.uu.nl/handle/20.500.12932/44715.

[Sal15] Nicolau C. Saldanha. The homotopy type of spaces of locally convex curves in the sphere. *Geom. Topol.*, 19(3):1155–1203, 2015.

[She19] Vivek Shende. The conormal torus is a complete knot invariant. *Forum Math. Pi*, 7:e6, 16, 2019.

[Sko08] Arkadiy Skopenkov. A classification of smooth embeddings of 3-manifolds in 6-space. *Math. Z.*, 260(3):647–672, 2008.

[Sla15] Marko Slapar. CR regular embeddings and immersions of compact orientable 4-manifolds into $\mathbb{C}^3$. *Internat. J. Math.*, 26(5):1550033, 5, 2015.

[Sma59] Stephen Smale. The classification of immersions of spheres in Euclidean spaces. *Ann. of Math. (2)*, 69:327–344, 1959.

[The22] Mélanie Theillière. Convex integration theory without integration. *Math. Z.*, 300(3):2737–2770, 2022.

[Vog09] Thomas Vogel. Existence of Engel structures. *Ann. of Math. (2)*, 169(1):79–137, 2009.

[Wal65] C. T. C. Wall. All 3-manifolds imbed in 5-space. *Bull. Amer. Math. Soc.*, 71:564–567, 1965.

[Wei80] Alan Weinstein. Fat bundles and symplectic manifolds. *Adv. in Math.*, 37(3):239–250, 1980.

[Wel69] R. O. Wells, Jr. Compact real submanifolds of a complex manifold with nondegenerate holomorphic tangent bundles. *Math. Ann.*, 179:123–129, 1969.

[Yam21] Koji Yamazaki. Developing maps and engel automorphisms, 2021.