UNIVERSIDAD AUTÓNOMA DE MADRID
FACULTAD DE CIENCIAS EXACTAS Y NATURALES
INSTITUTO DE MATEMÁTICAS

Hybrid Spaces on Deligne-Mumford stacks and degeneration of measures



Universidad Autónoma
de Madrid

Por

Bilson Castro López

Dirigido por

José Ignacio Burgos Gil

Abstract

In this thesis, we present the construction of a hybrid space associated with a compact complex manifold X together with a simple normal divisor D. The notion of hybrid arises from the fact that this space combines aspects of complex geometry and polyhedral geometry. Once this topological space is established, we study its fundamental topological properties, which lead us to formulate and prove a continuity theorem for measures on the hybrid space.

After developing the theory in the case of complex varieties, the next step is to extend the construction to the setting of toroidal Deligne–Mumford stacks over $\mathbb{C}$.

The construction of these spaces and the resulting theorems are broadly motivated by the connection between amplitudes in quantum field theory and string theory. In particular, the hybrid space associated with toroidal Deligne–Mumford stacks, such as the moduli stack of stable curves of genus g with n marked points, offers a natural geometric setting in which the computation of amplitudes in these physical theories can be understood as integrals over a hybrid space associated with the moduli stack.

Resumen

En esta tesis presentamos la construcción de un espacio híbrido asociado a una variedad compleja compacta X junto con un divisor de intersección normal simple D. El carácter híbrido de este espacio proviene de la combinación entre la geometría compleja y la geometría poliedral. Una vez establecido este espacio topológico, se estudian sus propiedades topológicas fundamentales, lo que permite enunciar y demostrar un teorema de continuidad de medidas en dicho espacio.

Después de desarrollar la teoría en el caso de variedades complejas, el siguiente paso consiste en extender la construcción a los stacks toroidales de Deligne–Mumford sobre $\mathbb{C}$.

La motivación principal de esta construcción y de los resultados obtenidos proviene de la relación entre las amplitudes en teoría cuántica de campos y las amplitudes en teoría de cuerdas. En particular, el espacio híbrido asociado a stacks toroidales de Deligne–Mumford, como el stack móduli de curvas estables de género g con n puntos marcados, ofrece un marco geométrico natural en el que el cálculo de dichas amplitudes en estas teorías físicas puede interpretarse como integrales sobre el espacio híbrido asociado al stack móduli de curvas estables.

Agradecimientos

Primero, quiero agradecer a mis padres, a mis hermanos y a mi sobrino por su apoyo incondicional.

Deseo expresar mi más sincero agradecimiento a José Ignacio Burgos, pues este trabajo habría sido inconcebible sin el apoyo, la guía y la confianza que me ha brindado durante todo este proceso.

A lo largo de estos años de doctorado he tenido la oportunidad de aprender muchos e interesantes tópicos en matemáticas, distintas culturas y nuevos puntos de vista. También he tenido la fortuna, y el immenso placer de coincidir y compartir con grandes personas. En este último aspecto, la lista es enorme, y si a ello sumo las valiosas amistades con las que ya contaba antes de iniciar este camino, se hace aún más extensa.

Agradezco de corazón a todos los amigos que he hecho durante estos años y a quienes me han acompañado en esta etapa. Por mencionar solo algunos, y sin desmerecer a quienes por olvido no menciono, quiero hacer mención especial de: Alberto, Wilmar, Juan Carlos, Chanu, Eliezer, María Fernanda, Vanessa, Niklas, Oriana, Gabriela, Paul, Luis Fernando, Elisa, Yenni, Guillermo, Roberto, Pedro, Robinson, Yeison...

Y, por último, quiero agradecerle a Anggie por animarme, apoyarme y acompañarme en la distancia. Infinitas gracias.

Este trabajo fue realizado gracias al apoyo económico del contrato predoctoral PRE2019-091117 que forma parte del proyecto Severo Ochoa SEV-2015-0554-19-1 Además, parte de los contenidos de este trabajo fueron financiados por el Ministerio de Ciencia, Innovación y Universidades a través de las becas PID2019-108936GB-C21, MJIBG2 PID2022-142024NB-100 y CEX2023-001347-S.

A mi abuelo Felipe y a mi abuela Zoila...

Contents

Abstract **i**

Introduction **1**

1 Preeliminary topics **14**

1.1 Combinatorial structures of topological spaces 14

1.2 Simple normal crossing divisors and stratifications 16

1.3 Groupoids and orbifolds 21

1.4 Deligne-Mumford stack 25

1.5 Fundamental group of an stack 33

1.6 Skeletons associated with toroidal embedding of DM stacks 35

1.7 Tropicalization of the moduli of curves 37

2 Green functions and hybrid spaces **42**

2.1 Polyhedral spaces and Green functions 42

2.2 Construction of the hybrid space 45

2.3 Continuity of measures on the hybrid space 52

3 Hybrid spaces for toroidal DM stacks **69**

3.1 Local structure of analytic spaces and coverings 69

3.2 Stratifications on X^{an} 78

3.3 Local Green functions and change of charts 94

3.4 Local equivariant functions on X^{an} 102

3.5 Construction of the hybrid space 105

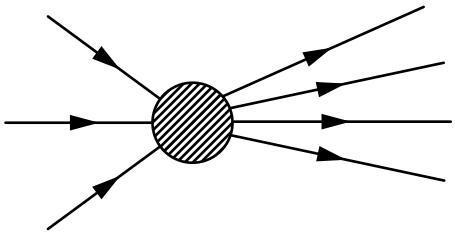
3.6 Measures on the hybrid space 113

Conclusion **122**

Introduction

In classical mechanics, if the initial state of a system is known and the governing laws are specified, the final state can be determined with certainty. In contrast, quantum mechanics is not deterministic: even with the same initial state, there exists a range of possible final states, each occurring with a certain probability. Thus, the nature of quantum mechanics is fundamentally probabilistic.

Therefore, in high-energy physics, we would like to predict the probability of an experiment involving an initial set of k elementary particles with initial momenta $p_{1,i}, \ldots, p_{k,i}$ which interact to produce a final state characterized by momenta $p_{k+1,f}, \ldots, p_{n,f}$.



The key tool for determining this probability are the scattering amplitudes, which encode the likelihood of scattering events and depends on the initial and final momenta, as well as the nature of the particles involved.

In closed string theory, scattering amplitudes arise from summing the contributions of integrals over the moduli spaces $M_{g,n}$ of (super)-Riemann surfaces of genus g with n marked points, n corresponds to the number of external momenta involved in the scattering process. Specifically,

$$\mathcal{A}_{\alpha'}^{string}(\mathbf{P}) = \sum_{g \geq 0} \int_{M_{g,n}} \exp(-i\alpha' \mathcal{F}_g) d\mu_{g,n}$$

where $\mathbf{p} = (p_{1,i}, \ldots, p_{k,i}, p_{k+1,f}, \ldots, p_{n,f})$ denotes the set of external momenta; α' is a positive real constant, equal to the square of the string length; $\mathcal{F}_g : M_{g,n} \to \mathbb{R}$ is a continuous function, which depend of the momenta $\mathbf{p}$; and $d\mu_{g,n}$ is a volume form on $M_{g,n}$ which is independent of the momenta.

If we fix values g and n, we consider the integral

$$\int_{M_{g,n}} \exp(-i\alpha' \mathcal{F}_g) d\mu_{g,n}$$

we obtain an asymptotic expansion in α' of the form

$$\int_{M_{g,n}} \exp(-i\alpha'\mathcal{F}_g)d\mu_{g,n} = A_{k_0}(\alpha')^{k_0} + A_{k_0+1}(\alpha')^{k_0+1} + \dots \tag{1}$$

where $k_0 \in \mathbb{Z}$.

In the setting of quantum field theory, scattering amplitudes are computed via Feynman diagram techniques and can be described in terms of integrals over the tropical moduli spaces:

$$\mathcal{A}^{qft}(\mathbf{p}) = \sum_{g\geq 0} \int_{M_{g,n}^{trop}} \exp(-i\mathcal{G}_g)d\nu_{g,n}$$

where $M_{g,n}^{trop}$ is the moduli space of metrized graphs of genus g and n leg. Each leg corresponds to an edge of infinite length attached to a vertex, and therefore has no terminal vertex. Graphs of this kind are known as tropical curves. The function $\mathcal{G}_g : M_{g,n}^{trop} \to \mathbb{R}$ is a continuous function which depends of the momenta $\mathbf{p}$; and $d\nu_{g,n}$ is a volumen measure on $M_{g,n}^{trop}$

Under suitable assumptions and formalism, physics suggests that for fixed values g and n, the limit of the leading term in the asymptotic expansion of the string amplitude $\mathcal{A}_{\alpha'}^{string}(\mathbf{P})$ as $\alpha' \to 0$ coincides with the corresponding quantum field theory amplitude $\mathcal{A}^{qft}(\mathbf{p})$. That is, after normalizing by $(\alpha')^{-k_0}$, we can express this limit as:

$$\lim_{\alpha'\to 0} \int_{M_{g,n}} \exp(-i\alpha'\mathcal{F}_g) \left[\frac{1}{(\alpha')^{k_0}} d\mu_{g,n}\right] = \int_{M_{g,n}^{trop}} \exp(-i\mathcal{G}_g)d\nu_{g,n} \tag{2}$$

The mathematical formalization of this convergence was introduced by Piotr Tourkine, who described this interpretation in [Tou17].

In recent years, certain mathematical objects known as hybrid spaces have gained significant importance. Roughly speaking, in the setting that concerns us, suppose that we have an open complex manifold embedded in a compact complex manifold such that the complement is a simple normal crossing divisor. To this divisor, one can associate a skeleton of a non-Archimedean analytic space. Then the resulting hybrid space is a topological space structured as a family over the interval $[0, 1]$, where each fiber for $0 < t \leq 1$ corresponds to the open complex manifold, and the fiber over $t = 0$ is given by the associated non-Archimedean space.

Regarding the moduli space of curves, Sam Payne, Dan Abramovich and Lucia Caporaso [ACP12] showed

that the tropicalization of the moduli space of curves $M_{g,n}$ is naturally identified with the tropical moduli space $M_{g,n}^{trop}$. This conclusion is based on the observation that, by considering the Deligne–Mumford compactification $\overline{M}_{g,n}$ of $\mathscr{M}_{g,n}$, the boundary of the compactification is a normal crossing divisor. In this setting, one can associate a non-Archimedean analytic space, analogous to the case of complex manifolds. This associated non-Archimedean space turns out to be precisely $M_{g,n}^{trop}$.

This point of view motivates the consideration of the hybrid space

$$M_{g,n} \times (0,1] \coprod M_{g,n}^{trop} \times \{0\}$$

analogous to the hybrid spaces constructed in the complex setting. In this context, two natural questions arise from the convergence in (2):

1. Is the function

$$\mathcal{F}_g^{hyb} : M_{g,n} \times (0,1] \coprod M_{g,n}^{trop} \times \{0\} \to \mathbb{R}, \quad (z, \alpha') \mapsto \begin{cases} \alpha' \cdot \mathcal{F}_g(z) & \text{if } \alpha' \neq 0, \\ \mathcal{G}_g & \text{if } \alpha' = 0, \end{cases} \tag{3}$$

   continuous with respect to the hybrid topology?

2. What can be said about the convergence of the measure $d\mu_{g,n}$ to the measure $d\nu_{g,n}$ as $\alpha' \to 0$?

Regarding the first question, in [ABGF16], it is shown that the functions $\alpha'\mathcal{F}_g$ converge to $\mathcal{G}_g$ as α' tends to zero, along suitable admissible paths. However, their analysis does not involve hybrid spaces; instead, they study the behavior of $\alpha'\mathcal{F}_g$ along paths in $M_{g,n}$ whose limit points lie in the boundary of the Deligne–Mumford compactification.

Based on the framework developed above, the problems addressed in this thesis are the following:

1. To extend the construction of hybrid spaces to the setting of Deligne–Mumford stacks.

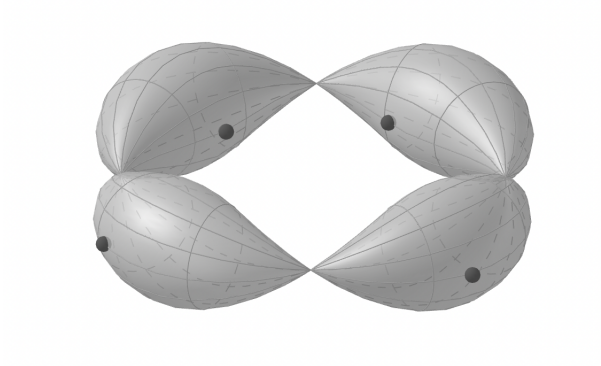
2. In the case of hybrid spaces in the complex setting, we define the notion of $\mathcal{P}$-residues of a logarithmic (n,n)-form, and we prove a continuity theorem for measures on the hybrid space. This result provides a foundation for understanding the convergence of measures in the context of the limit

$$\lim_{\alpha' \to 0} \int_{M_{g,n}} \exp(-i\alpha'\mathcal{F}_g) \left[(\alpha')^{k_0} d\mu_{g,n} \right] = \int_{M_{g,n}^{trop}} \exp(-i\mathcal{G}_g) d\nu_{g,n}$$

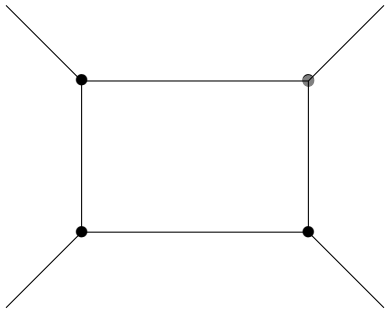
as described in (2).

A key step toward establishing a convergence theorem for measures on hybrid spaces is to understand the behavior of the measure $d\mu_{g,n}$ near the boundary of the moduli space $M_{g,n}$, and to analyze the types of singularities that arise in this setting. To illustrate this, we consider the fermionic theory of four particles at one-loop level. In this context, we examine the following illustrative example:

Let C the sigular curve of genus 1 with four marked points, represented by the complex curve:



To this curve we associate its dual graph, shown below. For each irreducible component of the curve, we assign a vertex; for each intersection point between two components, we assign an edge; and for each marked point on the curve, we attach a leg incident to the vertex corresponding to the component containing that marked point.



This curve can be represented by a point in the Faltings–Chai compactification of the moduli space of abelian varieties of genus 1 with three distinguished points. The coordinates on this moduli space are given by (τ, z_2, z_3, z_4) where τ is the modular parameter of the fundamental domain (encoding the complex structure), and z_2, z_3, z_4 are elliptic variables representing the positions of the marked points on the torus.

In these coordinates, and following [GRV08], the measure $d\mu_{1,4}$ takes the form:

$$d\mu_{1,4} = \frac{d\tau d\overline{\tau} dz_2 d\overline{z}_2 dz_3 d\overline{z}_3 dz_4 d\overline{z}_4}{\mathfrak{Im}(\tau)^5}$$

Using the theory of degeneration of abelian varieties developed by Faltings and Chai (see [FC90]), we introduce local coordinates around the degenerate curve C by setting

$$e^{2\pi i\tau} = u_1u_2u_3u_4 \quad \text{and} \quad e^{2\pi iz_j} = u_j \quad \text{for} \quad j = 2, 3, 4$$

where each u_i lies in a small complex disc. In terms of these coordinates, the measure becomes:

$$d\mu_{1,4} = \frac{du_1 d\overline{u}_1 \dots du_4 d\overline{u}_4}{u_1\overline{u}_1 \dots u_4\overline{u}_4(-\log|u_1| - \dots - \log|u_4|)^5}$$

For this concrete case, the degree of the leading term of the string amplitude $\mathcal{A}_{1,4}^{string}(\mathbf{p})$ in the asymptotic expansion is $k_0 = 1$. Adding this factor, we obtain:

$$\frac{du_1 d\overline{u}_1 \dots du_4 d\overline{u}_4}{\alpha' u_1\overline{u}_1 \dots u_4\overline{u}_4(-\log|u_1| - \dots - \log|u_4|)^5}. \tag{4}$$

Once we consider the sustitucion $x_i = -\alpha' \log|u_i|$ for $i = 1, \dots, 4$, the measure becomes:

$$\frac{dx_1 dx_2 dx_3 dx_4}{(x_1 + x_2 + x_3 + x_4)^5} \tag{5}$$

This measure is precisely $d\nu_{1,4}$.

The factor $(x_1 + \dots + x_4)$ in this measure plays a key role in the tropical moduli space $M_{1,4}^{trop}$, as it is related to the dual graph of the curve C. Indeed, it corresponds to the first Symanzik polynomial of the associated graph. If we assign to each internal edge of the graph a leght, say x_1, x_2, x_3 and x_4, then the first Symanzik polynomial is defined as the sum, over all spanning trees of the graph, of the product of the edge lengths not contained in each spanning tree.

In our case, the graph has four generating trees, obtained by deleting one internal edge each time, and the resulting polynomial is:

$$x_1 + x_2 + x_3 + x_4$$

Such Symanzik polynomials are always homogeneous.

This example illustrates the type of singularities we aim to study in order to formulate a convergence theorem for measures in the hybrid space.

Having established the motivation and origin of the problems addressed in this thesis, and having outlined the specific questions explored throughout this work, we now proceed to present in detail the main results obtained.

In the paper *Tropical and Non-Archimedean Limits of Degenerating Families of Volume Forms*, Boucksom and Jonsson [BJ17] develop a framework to study the asymptotic behavior of volume forms in degenerating families of complex varieties. In this analysis, they develop the construction of a *hybrid space*, which serves as a bridge between complex analytic geometry and polyhedral(tropical) geometry.

The construction of the hybrid space proceeds as follows. Given a complex manifold X and an effective divisor $D \subseteq X$ with simple normal crossing, we consider the disjoint union

$$X^{\mathrm{hyb}} := (X \smallsetminus D) \coprod \Delta(D)$$

where $\Delta(D)$ is the *dual complex* associated with the irreducible components of the divisor D. That is, for each irreducible component of D, one assigns a vertex; to each connected component of a non-empty intersection of irreducible components, one associates the simplex spanned by the vertices corresponding to those components. The face relations of $\Delta(D)$ are determined by the stratification induced by the pair (X, D). To endow X^{hyb} with a natural topology, they proceed as follows. For each point of D, one can choose coordinates $(U, z = (z_1, \ldots, z_n))$ such that the pullback of D to U is given by the vanishing of a monomial:

$$f_U = \prod_{j \in J} z_j^{m_j},$$

where $J \subset \{1, \ldots, n\}$ and each $z_j = 0$ defines a component of D appearing in the chart.

Having fixed it, they define:

$$\log_U : U \smallsetminus |D| \to \sigma_{Y_U}, \quad x \mapsto \left(-\frac{\log |z_j(x)|}{\log |f_U|} \right)_{j \in J}, \tag{6}$$

where σ_{Y_U} is the standard simplex corresponding to the stratum defined by the intersection of the divisors $z_j = 0$ for $j \in J$.

Based on this construction of local functions, a key result [BJ17, Proposition 2.1] states that there exists an

open neighborhood $V \subset X$ of D and a continuous map $\mathrm{Log}_V : V \smallsetminus D \to \Delta(D)$ such that for each adapted coordinate chart (U, z) with $U \subset V$, we have $\mathrm{Log}_V(U \smallsetminus D) \subset \sigma_U$, and $\mathrm{Log}_V = \log_U + O\left((\log |f_{U,z}|)^{-1}\right)$ uniformly on compact subsets of U.

Therefore, the hybrid topology defined in [BJ17] is the coarsest topology such that Log_V is continuous, and the inclusion $X \smallsetminus D \hookrightarrow X^{\mathrm{hyb}}$ is an open embbeding.

Motivated by this construction, in Chapter 2 we develop an analogous notion of a hybrid space, though with some notable differences. For instance, in our setting we work with cone complexes rather than compact polyhedral complexes, as will be explained below, and the resulting hybrid space differs slightly from the one defined in [BJ17].

A key element in our construction, analogous to the approach in [BJ17], is the use of*Green function* associated with Cartier divisors. Given a Cartier divisor D, a *Green function (of continuous type)* for D, is a continuous function $\mathfrak{g} : X \smallsetminus |D| \to \mathbb{R}$ such that for every $x \in X$ there is an open neighborhood U_x of x, and a local equation $z \in \mathcal{O}_{X,x}$ defining the Cartier divisor on U_x, such that the map $\log |z| + \mathfrak{g}$ extends to a continuous function on U_x.

One of the main motivations for taking Green functions as the starting point in the construction of the hybrid space arises from the fact that, in order to ensure the continuity of the limit in (2), the functions $\mathcal{F}_g$ $\mathcal{G}_n$, corresponding respectively to the string and the analytic settings, must satisfy a basic compatibility condition: as $\alpha' \to 0$, the function $\mathcal{F}_g$ must converge pointwise to $\mathcal{G}_g$. Since $\mathcal{F}_g$ depends, in an essential way, on Green functions, it is therefore natural, and indeed necessary, to use Green functions to define the topology of the hybrid space. This provides a coherent framework to guarantee the desired convergence.

Analogously to the local functions defined in (6), we fix a complex manifold X, with a reduced simple normal crossing divisor D. For each irreducible component D_i of D, we fix a positive Green function $\mathfrak{g}_i$. Then, for an open chart U as above, we define

$$F_U : U \smallsetminus D \to \sigma_U, \quad x \mapsto (\mathfrak{g}_j)_{j \in J}\,, \tag{7}$$

where σ_U denotes the cone generated by the rays corresponding to the irreducible components D_j of D that intersect U non-trivially. Note that in the j-th coordinated, we only evaluate the Green function $\mathfrak{g}_j$ if the corresponding component D_j meets U.

These local functions serve as the "local models" in our hybrid space and play a role analogous to the

maps defined in (6). With this analogy in mind, we introduce the cone complex $\Sigma(X, D)$, where, instead of associating points to the irreducible components of the divisor, we associate rays, and instead of simplices, we consider cones. The face relations among these cones are determined by the natural stratification induced by the pair (X, D) (see Section 2.1). With this setup, we can now state the following result adapted to our setting:

Proposition A. [cf. Proposition 2.2.3] There exists a continuous map $F : X \smallsetminus D \to \Sigma(X, D)$ with the following property: For each stratum $Y \subseteq X$, each point $x \in Y$, every adapted open U containing x with Y as its minimal stratum, and every function F_U defined by a collection $\{\mathfrak{g}_i'\}_{i\in I}$, there exists an open V of x contained in U such that on $V \smallsetminus D$ we have

$$F = F_U + \phi$$

Where ϕ is a bounded function defined in V.

The existence of the function in the above proposition is the main ingredient in defining the hybrid space. Fixing such a function F that satisfies the statement of Proposition A, we consider the set:

$$X_D^{hyb} := (X \smallsetminus D) \times (0, 1] \coprod \Sigma(X, D) \times \{0\}$$

Then, on X_D^{hyb} define the following map

$$G : X_D^{hyb} \to \Sigma(X, D), \quad (z, t) \mapsto \begin{cases} t \cdot F(z) & \text{if } t \neq 0, \\ z & \text{if } t = 0, \end{cases} \tag{8}$$

based in this construction, we define the hybrid topology

Definition 0.0.1. The *hybrid topology* is the coarsest topology on X_D^{hyb} such that:

1. $(X \smallsetminus D) \times (0, 1] \hookrightarrow X_D^{hyb}$ is an open inmmersion.
2. The projection $\pi : X_D^{hyb} \to [0, 1]$ is continuous.
3. The function G defined in (8) is continuous.

This construction highlights a key difference from the hybrid spaces defined in [BJ17], where the divisor is removed and replaced by the dual complex. In contrast, we regard $\Sigma(X, D)$ as a "degeneration" of

$X \smallsetminus D$ as the parameter t tends to zero. Moreover, in [BJ17] the associated polyhedral space is compact, whereas in our case the corresponding polyhedral space is open, that is, a cone complex.

As we show in Proposition 2.2.7, this hybrid topology will be independent of the choice of F if the complex manifold X is compact. Since we do not want a dependece of F, we will focus in this case.

Once we establish the construction of the hybrid space, we focus on proving a degeneration theorem for measures. At this stage, we return to the example of genus one with four marked points in order to identify the type of singularities that we are interested in.

Note that the measure $d\mu_{1,4}$ exhibits simple singularities together with a homogeneous polynomial in logarithmic variables. These singularities in this example motivate the introduction of $\mathcal{P}$-residues of logarithmic (n,n)-forms in the complex setting.

Specifically, suppose that X is a compact complex manifold of dimension n, and D a simple normal crossing divisor. Assume that all minimal closed strata of D have the same codimension, say k. By closed strata we mean the connected components of nonempty intersections of the irreducible components of D. Denote by Σ_k the set of such closed strata.

Definition 0.0.2. Let $\mathcal{P} := \{P_\alpha\}_{\alpha\in\Sigma_k}$ be a collection of homogeneous polynomials, each of degree r, and let ω be a continuous $(n,n)-$form defined on $X \smallsetminus D$. We say that ω is a *$\mathcal{P}$-logarithmic form* if, for every adapted chart (U,z) such that the associated stratum Y_α is a minimal stratum, ω can be written as

$$\omega_{|U} = \frac{\eta_U}{P_\alpha((-\log|z_{m_j}|)_{j=1}^k)} \frac{dz_{m_1}}{z_{m_1}} \wedge \frac{d\overline{z}_{m_1}}{\overline{z}_{m_1}} \wedge \ldots \wedge \frac{dz_{m_k}}{z_{m_k}} \wedge \frac{d\overline{z}_{m_k}}{\overline{z}_{m_k}} \tag{9}$$

Where η_U is a continuous $(n-k,n-k)$-form defined on U. If all the polynomials P_α are constant and the expression (9) holds for every adapted chart; that is, not only around the minimal strata, we refer to ω simply as a *logarithmic (n,n)-form.*

The local function η_U plays a fundamental role, since its restriction to the associated stratum is independent of the choice of adapted chart. This observation naturally motivates the definition of residues, which we will call $\mathcal{P}$-residues.

Definition 0.0.3. Let $\mathcal{P}$ be a collection of homogeneous polynomials associated to the minimal strata of X. Let Y be a minimal stratum, and let ω be a $\mathcal{P}$-logarithmic form. The *$\mathcal{P}$-residue* of ω along the stratum Y is defined to be the continuous form $\mathrm{Res}_{\mathcal{P},Y}(\omega)$ on Y, which satisfies the property that for any adapted chart (U,z) with Y as its associated stratum, $\eta_{U\,|Y} = \mathrm{Res}_{\mathcal{P}}(\omega)_{|U\cap Y}$.

If ω is a logarithmic (n,n)-form, i.e., all the polynomials P_α are constant and the local description of ω extends to every adapted chart-we simply denote the residue by $\mathrm{Res}_Y(\omega)$

Now, following the construction of the hybrid space in the compact case, we consider a family of measures $\{\mu_t\}_{0\leq t\leq 1}$ on the hybrid space X_D^{hyb}, where μ_t is supported on the fiber over t, it for each t. This family is defined as follows:

$$\mu_t = \begin{cases} \frac{t^{k-r}}{(4\pi i)^k}\omega & t \neq 0 \\ \\ \sum_{Y_\alpha \in \Sigma_k} \left(\int_{Y_\alpha} \mathrm{Res}_{\mathcal{P},Y_\alpha}(\omega) \right) \frac{d\lambda_\alpha}{P_\alpha} & t = 0 \end{cases} \tag{10}$$

Where ω is a $\mathcal{P}$-logarithmic (n,n)-form defined in $X \smallsetminus D$. Note that the factor t^{k-r} corresponds to the coefficient $(\alpha')^{k_0}$ appearing in the convergence described in (2).

Let $\Sigma(X,D)^{<k}$ denote the union of cones in $\Sigma(X,D)$ of dimension less than or equal to $k-1$. We define the open subset

$$X_D^0 := X_D^{hyb} \smallsetminus (\Sigma(X,D)^{<k} \times \{0\})$$

In this setup, we obtain the following result:

Theorem A (cf. Theorem 2.3.10)**.** Let $\{\mu_t\}_{0\leq t\leq 1}$ be the family of measures defined in (10). Then, for every continuous function $f \in C_c(X_D^0, \mathbb{R})$ we have:

$$\lim_{t\to 0} \int_{X_D^0} f d\mu_t = \int_{X_D^0} f d\mu_0$$

One of the reasons to consider continuity on the open set X_D^0 is the configuration of the homogeneous polynomials: in the maximal cones, the measure is singular along the proper faces of these cones. However, if all the polynomials are constant, i.e., if ω is a (n,n)-logarithmic form, these singularities are absent, allowing for a more refined treatment. In this case, we obtain the following degeneration theorem for measures along the entire hybrid space X_D^{hyb}:

Theorem B (cf. Theorem 2.3.12)**.** Let $\{\mu_t\}_{0\leq t\leq 1}$ be the family of measures defined in (10). Then, for every continuous function $f \in C_c(X_D^{hyb}, \mathbb{R})$, we have:

$$\lim_{t\to 0} \int_{X_D^{hyb}} f d\mu_t = \int_{X_D^{hyb}} f d\mu_0$$

Once these results are established in the complex setting, we take a further step toward extending them to the analytic coarse moduli space of toroidal Deligne–Mumford stacks over $\mathbb{C}$. In this broader context, we begin by identifying the appropriate analogue of complex cone complexes.

Let $(\mathscr{X}, \mathscr{U})$ be a toroidal Deligne–Mumford stack over $\mathbb{C}$, and let X, U denote the coarse moduli spaces of $\mathscr{X}$ and $\mathscr{U}$ respectively. Then, by the result of Payne, Abramovich, and Caporaso [ACP12], we can associate to this pair the open skeleton $\Sigma(\mathscr{X}, \mathscr{U})$, which generalizes the cone complex in the case of complex manifolds.

Now, consider X^{an} and U^{an} the complex analytification of X and U respectively. Note that, in general, X^{an} is not a complex manifold, and $X^{an} \smallsetminus U^{an}$ is not a simple normal crossing divisor, but rather a normal crossing one.

In this setting, we construct local models

$$\mathtt{F}_{\mathtt{V}} : \mathtt{V} \to \Sigma(\mathscr{X}, \mathscr{U})$$

analogous to the local models defined in (7) in the complex manifold case. Here, $\mathtt{V} \subseteq X^{an}$ denotes an open set playing the role of an adapted chart. From this construction, we obtain the following analogue of Theorem A:

Theorem C (cf. Theorem 3.5.2)**.** There exists a map $\mathtt{F} : U^{an} \to \Sigma(\mathscr{X}, \mathscr{U})$ with the following property: For each stratum M_r, each point $\mathtt{x} \in M_r$, every orbifold chart (V, G, ϕ) containing $\mathtt{x}$, with M_r as its associated stratum, and each λ_V-equivariant function F_V defined as in (3.19), there exists an open neighborhood $\mathtt{W} \subseteq \mathtt{V}$ of $\mathtt{x}$, an orbifold chart (V', G', ϕ') with associated stratum $M_{r'} \subseteq \overline{M_r}$, and a $\lambda_{V'}$-equivariant map $F' : V' \setminus D' \to \sigma_{V'}$ such that

1. The following diagram commutes:

$$\begin{array}{ccc} (\phi')^{-1}(\mathtt{W}) & \xrightarrow{F'} & \sigma_{V'} \\ \downarrow & & \downarrow \\ \mathtt{W} & \xrightarrow{\mathtt{F}_{|\mathtt{V}}} & \Sigma(\mathscr{X}, \mathscr{U}) \end{array}$$

2. If $(F_V)' : (\phi')^{-1}(\mathtt{W}) \to \sigma_{V'}$ denotes the function induced by F_V, then $(F_V)' = F' + \phi'$, where ϕ' is a bounded continuous function defined on $(\phi')^{-1}(\mathtt{W})$.

This result serves as the starting point, as it allows us to define a function $\mathtt{G}$ analogous to the function G defined in (8), We then define:

$$\mathscr{X}^{hyb} := U^{an} \times (0,1] \coprod \Sigma(\mathscr{X}, \mathscr{U})$$

And define the topology as the coarsets topology such that $\mathtt{G}$ is continuous, $U^{an} \times (0,1] \hookrightarrow \mathscr{X}^{hyb}$ is an open embedding, and the projection $\pi : \mathscr{X}^{hyb} \to [0,1]$ is continuous. This object will be our hybrid space associated to a Deligne-Mumford stacks.

In particular, in the case of the moduli of curves, we have hybrid space

$$\mathscr{M}_{g,n}^{hyb} := M_{g,n}^{an} \times (0,1] \coprod M_{g,n}^{trop} \times \{0\}$$

where $M_{g,n}$ denotes the coarse moduli space of smooth genus g and n- moarked points, and $M_{g,n}^{trop}$ is the moduli space of tropical curves of genus g and n-marked points.

Within this framework, if the function defined in equation (3) satisfies the conservation law of momenta, then it is continuous on $\mathscr{M}_{g,n}^{hyb}$. This continuity is a particular consequence of the results developed by Omid Amini, José Burgos, Javier Fresán, and Spencer Bloch in [ABGF16].

Once the construction of the hybrid space associated to a Deligne–Mumford stack has been established, we restrict our attention to those stacks for which the minimal strata of X^{an} are points. In this setting, we obtain a generalization of Theorem 2.3.10, namely, a convergence theorem for the family of measures defined on an open subset $\mathscr{X}_0^{hyb}$ of $\mathscr{X}^{hyb}$. This open set is constructed in direct analogy with the open subset X_D^0 of X_D^{hyb}. The corresponding family of measures is defined in a manner similar to the complex analytic case, and the detailed construction will be developed in Section 3.6.

Theorem D (Cf. Theorem 3.6.17)**.** Let $\{\mu_t\}_{0\leq t\leq 1}$ be a family of measures as in (3.41). Then for each continuous function $f \in C_c(\mathscr{X}_0^{hyb}, \mathbb{R})$ we have

$$\lim_{t\to 0} \int_{\mathscr{X}_0^{hyb}} f d\mu_t = \int_{\mathscr{X}_0^{hyb}} f d\mu_0$$

We conclude by highlighting several possible directions for future research arising from this work.

First, as discussed in Theorem A, the construction cannot be extended to the entire hybrid space X_D^{hyb} because of the presence of the homogeneous polynomials. Note that if we drop the factor t^{n-k} and consider

the asymptotic expansion of the integral

$$\int_{\pi^{-1}(t)} f_{|\pi^{-1}(t)} d\omega,$$

then the coefficient associated with the power t^{n-k} is precisely the conclusion of Theorem A. We may therefore expect that the subsequent terms $t^{n-k+1}, t^{n-k+2}, \ldots$ correspond, in some sense, to the contributions arising from the integrands supported on the proper faces of the cone complex $\Sigma(X, D)$.

A second line of investigation is to analyze these higher-order terms in the setting of the hybrid space $\mathscr{X}^{hyb}$, in a manner analogous to the previous discussion, in the case where the Deligne–Mumford stack has minimal strata consisting of points.

A third avenue for future work is to extend Theorem D to general Deligne–Mumford stacks, without assuming that the minimal strata are zero-dimensional.

Finally, we plan to study the moduli space of hybrid spaces in the complex-analytic setting and to compare our construction with the frameworks developed by Omid Amini and Noema Nicolussi in [AN20, AN22].

Outline

The structure of this thesis is as follows. Chapter 1 introduces the language of stratifications, polyhedral complexes, orbifolds, and Deligne–Mumford stacks. If the reader is already familiar with these topics, this chapter may be skipped. Chapter 2 presents the construction of the hybrid space in the setting of complex manifolds and outlines the convergence of measures in this context. Chapter 3 develops the construction of the hybrid space associated to a Deligne–Mumford stack and studies its properties.

Chapter 1

Preeliminary topics

1.1 Combinatorial structures of topological spaces

In this section, we give a general overview of the definitions of simplicial complexes, cone complexes, and generalized cone complexes. The first notion will provide the framework for developing triangulations adapted to stratifications, while the latter two will be related to polyhedral structures induced by stratifications arising from Cartier divisors. These notions will be treated in detail in Section 1.2.

Definition 1.1.1. Let V be a nonempty set. An *abstract simplicial complex* on V is a collection $K \subseteq \mathcal{P}(V)$ of nonempty finite subsets of V, called *simplices*, such that:

1. If $\sigma \in K$ and $\emptyset \neq \tau \subseteq \sigma$, then $\tau \in K$.
2. Every singleton $\{v\}$, with $v \in V$, is a simplex. Each such element is called a *vertex*.

We say that τ is a face of σ. Each element $\sigma = \{v_0, \ldots, v_k\} \in K$ is called a k-simplex, and its *dimension* is defined as $\dim \sigma := k$. The elements $v_0, \ldots, v_k$ are the *vertices* of σ.

Definition 1.1.2. The *geometric realization* $|K|$ of a simplicial complex K with vertex set V is the topological space

$$|K| := \bigcup_{\sigma \in K} |\sigma|$$

where each $|\sigma|$ is the standard geometric k-simplex associated to $\sigma = \{v_0, \ldots, v_k\}$, defined as:

$$|\sigma| := \left\{ \sum_{i=0}^{k} t_i v_i \in \mathbb{R}^V : \ t_i \geq 0, \ \sum_i t_i = 1 \right\}$$

with $\{e_v\}_{v \in V}$ denoting the standard basis of $\mathbb{R}^V$. Each simplex $|\sigma|$ lies naturally in a finite-dimensional subspace of $\mathbb{R}^V$, and we equip it with the subspace topology inherited from the Euclidean topology on $\mathbb{R}^V$.

Having introduced simplicial complexes, we now broaden the framework by introducing cone complexes, extended cone complexes, and generalized cone complexes.

Definition 1.1.3. Let Σ be a second-countable topological space. A *rational cone complex with integral structure* on Σ is a pair $(\{\sigma_i\}_{i\in I}, \{M_i\}_{i\in I})$, consisting of a finite collection closed subsets σ_i that cover Σ, together with finitely generated $\mathbb{R}$-vector spaces M_i of continuous $\mathbb{R}$-valued functions on σ_i, satisfying the following conditions:

1. For each index $i \in I$, the subset σ_i is homeomorphic, via the evaluation map, to a strictly convex, full-dimensional, rational polyhedral cone in the vector space $\mathrm{Hom}(M_i, \mathbb{R})$.
2. Under the identification above, any face τ of σ_i also appears as one of the subsets σ_j in the covering.
3. For any $i, j \in I$, the intersection $\sigma_i \cap \sigma_j$ is a union of faces of both σ_i and σ_j.

In this work, we will refer to rational cone complex with integral structure just as *cone complex*.

Let Σ and Σ' be cone complexes. A *morphism of cone complexes* $f : \Sigma \to \Sigma'$ is a continuous map of topological spaces such that for every cone $\sigma \subseteq \Sigma$, there exists a cone $\sigma' \subseteq \Sigma'$ and an integral linear map of cones $f_\sigma : \sigma \to \sigma'$ such that $f|_\sigma = f_\sigma$.

Let $\sigma \subseteq \mathrm{Hom}(M, \mathbb{R})$ be a cone, and denote by $S_\sigma \subseteq M$ the additive monoid of integral linear functions that are non negative on σ, then σ can be identified with $\mathrm{Hom}(S_\sigma, \mathbb{R}_{\geq 0})$. The associated *extended cone* is defined as

$$\overline{\sigma} := \mathrm{Hom}_{\mathrm{Mon}}(S_\sigma, \mathbb{R}_{\geq 0} \cup \{\infty\}),$$

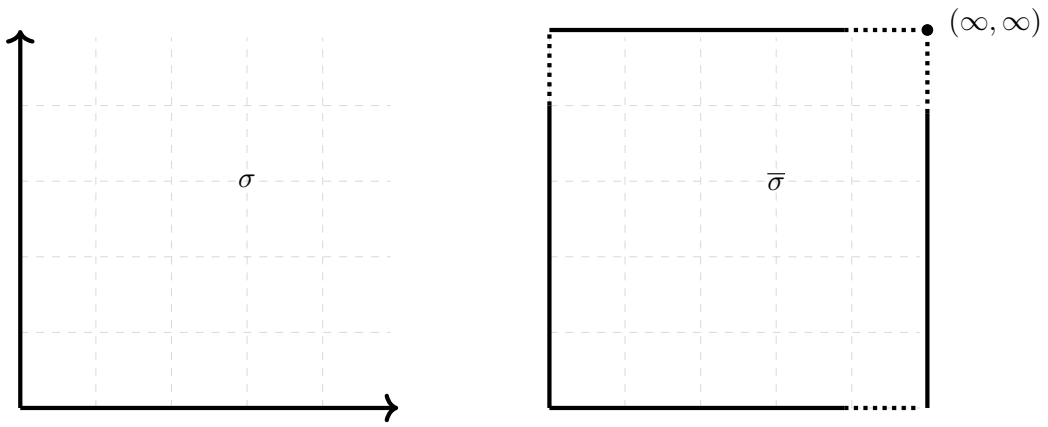
The topology on $\overline{\sigma}$ is that of pointwise convergence. The cone σ is an open dense subset of $\overline{\sigma}$, and $\overline{\sigma}$ is compact by Tychonoff's theorem. If τ is a face of σ, then the closure $\overline{\tau}$ in $\overline{\sigma}$ is canonically identified with the extended cone of τ. Such closures are referred to as *extended faces*.

It is well known that this extended cone can be written as a disjoint union

$$\overline{\sigma} = \coprod_{\tau \preceq \sigma} \tau$$

where the cones in $\overline{\sigma} \smallsetminus \sigma$ are known as the *faces at infinity*. For a more detailed treatment of this construction, see [ACP12].

Example 1.1.4. On the left-hand side of the figure, we depict a cone in $\mathbb{R}^2$,which has three faces: the two generating rays and the origin. On the right-hand side, we show its associated extended cone, where the additional faces at infinity correspond to the rays $\mathbb{R}_{\geq 0} \times \{\infty\}$ and $\{\infty\} \times \mathbb{R}_{\geq 0}$, and the point (∞, ∞).



Morphism of extended cones is a continuous map of topological spaces $f : \overline{\sigma} \to \overline{\sigma}'$ whose restriction to the interior cone $f|_\sigma : \sigma \to \tau'$ is a cone morphism, where τ' is either a face of σ', or a face of a face at infinity of σ'.

Definition 1.1.5. An *extended cone complex* is a topological space $\overline{\Sigma}$ together with a finite collection of closed subspaces $\{\overline{\sigma}_i\}_{i\in I}$ and identifications of each $\overline{\sigma}_i$ with an extended cone such that the intersection of any two $\overline{\sigma}_i \cap \overline{\sigma}_j$ is a union of extended faces of both $\overline{\sigma}_i$ and $\overline{\sigma}_j$.

A *morphism of extended cone complexes* is a continuous map of topological spaces $f : \Sigma \to \Sigma'$ such that, for each extended cone $\overline{\sigma} \subseteq \Sigma$, there exists an extended cone $\overline{\sigma}' \subseteq \Sigma'$ for which the restriction $f|_{\overline{\sigma}} : \overline{\sigma} \to \overline{\sigma}'$ is a morphism of extended cones.

We conclude this section with the following definition, adapted from [ACP12], which plays an important role in Sections 1.6 and 1.7.

Definition 1.1.6.

1. A *face morphism* of (extended) cones $\sigma \to \sigma'$ is an isomorphism onto a (extended) face of σ'. We emphasize that the image of a face morphism is not required to be a proper face of the target, so any isomorphism of cones is a face morphism.
2. A *generalized (extended) cone complex* is a topological space with a presentation as the colimit of an arbitrary finite diagram of (extended) cones with face morphisms; we emphasize that the diagram need not be a poset.

1.2 Simple normal crossing divisors and stratifications

This section explains how a simple normal crossing divisor (see Section 1.2.1) on a complex manifold or an algebraic variety defines a natural stratification of the ambient space, and how such stratifications give rise to conic structures.

Throughout this section, X will denote a complex manifold of dimension n, unless otherwise stated, and $\mathscr{M}_X$ its sheaf of meromorphic functions. We define $\mathrm{Div}(X)$ to be

$$\mathrm{Div}(X) := H^0(X, \mathscr{M}_X^\times/\mathcal{O}_X^\times)$$

Whose elements are called *Cartier divisor*. An $\mathbb{R}$-*Cartier divisor*, is any element of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. A Cartier divisor is said to be *effective* if every local equation can be taken as a holomorphic function. An $\mathbb{R}$-divisor is called $\mathbb{R}$-*efective* if there are effective Cartier divisors $D_1, \ldots, D_k$ and $a_1, \ldots, a_k \in \mathbb{R}_{>0}$ such that the $\mathbb{R}$-divisor can be written as $a_1D + \ldots + a_kD_k$.

Let D be a Cartier divisor on X. For each $x \in X$, let $f_x \in \mathscr{M}_{X,x}$ be a local equation for D. We define the support $|D|$ as

$$|D| := \{x \in X : f_x \notin \mathcal{O}_{X,x}^\times\}$$

This definition does not depend on the choice of f_x because for any other local equation g_x of D at the point x, $f_x/g_x \in \mathcal{O}_{X,x}^\times$.

If D is a Cartier divisor then there exist effective Cartier divisors $\{D_i\}_{i\in I}$, and a collection of integers $\{a_i\}_{i\in I}$, such that the support of every D_i is an irreducible (connected) component of $|D|$, and $D = \sum_{i\in I} a_iD_i$. We call $D_{i's}$ the irreducible components of D.

Definition 1.2.1. An effective Cartier divisor D on X has *normal crossing* at a point x, if there exists a complex coordinate chart $(U, z = (z_1, \ldots, z_n))$ such that D is defined in U by the equation $z_{i_1} \cdot \ldots \cdot z_{i_m} = 0$, for some $0 \leq i_1 < \ldots < i_m \leq n$. We say that D is a *normal crosing divisor* if it has normal crossing at any point of X. A normal crossing divisor D is called *simple normal crossing divisor*, if every irreducible component of $|D|$ is without self-intersection.

Remark 1.2.2.

1. If D is a simple normal crossing divisor, then $D = \sum_{i \in I} D_i$ where $D_{i's}$ are the irreducible components of D. By definition, for each $J \subseteq I$, the intersection
$$\bigcap_{j \in j} D_j$$
is either empty or has codimension $|J|$.

2. If E is an $\mathbb{R}$-divisor, and D is a simple normal crossing divisor, with $|E| \subseteq |D|$, then there exist $\{a_i \in \mathbb{R} : i \in I\}$, such that $E = \sum_{i \in I} a_i D_i$.

Following [Sta18, Tag 0BI9], we will also define simple normal crossing divisor in the case of schemes. Here, the definition of Cartier divisor is completely analogous to the one given above, where the sheaf of meromorphic functions is replaced by the sheaf of rational functions.

Definition 1.2.3. Let X be a locally Noetherian scheme. A simple normal crossing divisor on X is an effective Cartier divisor $D \subseteq X$ such that for every $x \in X$, the local ring $\mathcal{O}_{X,x}$ is regular and there exists a regular system of parameters $f_1, \ldots, f_d \in \mathfrak{m}_x$, and $1 \leq r \leq d$ such that D is cut out by $f_1 \cdot \ldots \cdot f_r$ in $\mathcal{O}_{X,x}$.

In the case of X being a variety over $\mathbb{C}$, we have that for every complex point $x \in X(\mathbb{C})$, the parameters $f_1, \ldots, f_d$ determine local equations for X^{an} around the point x. If X is regular, which implies that X^{an} is a smooth manifold, the pair $(U, (f_1, \ldots, f_d))$, where U is a small open neighborhood of x, forms a complex coordinate chart. Note that in this case, the Cartier divisor defined by $f_1, \ldots, f_r$ in X^{an} is a simple normal crossing divisor.

In conclusion, we see that the analytification of a simple normal crossing divisor in a regular complex variety gives us a simple normal crossing divisor in X^{an}.

Definition 1.2.4. Let X be a variety over an algebraically closed field.

1. A strict toroidal embedding is a variety X with an open dense subset U such that for every $x \in X$ there exists an open neighborhood $V \subseteq X$ and an étale morphism to a toric variety, that is, $\varphi : (V, V \cap U) \to (X_\sigma, T)$ with $\varphi^{-1} T = U \cap V$.

2. A toroidal embedding is a variety X together with an open dense U such that for every $x \in X$ there exists an open neigborhood $V \subseteq X$, and an étale-surjective morphism $\phi : W \to V$, with $(W, \phi^{-1}(V \cap U))$ being a strict toroidal embedding.

Remark 1.2.5.

1. From the definition above, Since locally we have étale morphisms, in particular flat, it follows that the reduced closed subscheme $D := X \setminus U$ is a divisor. And for the case of strict toroidal embeddings, the irreducible components of D are normal (see [KKMSD06, Pag 195]).

2. Following [Wł22, Lem 3.8.2], we have that for every $x \in D$ and étale morphism $\varphi : (V, V \cap U) \to (X_\sigma, T)$, with V containing x, there is a correspondence between irreducible components of D passing through x and irreducible components of $X_\sigma \setminus T$ passing through $\varphi(x)$. Moreover, this extends to a correspondence between effective Cartier divisors in $\mathrm{Spec}(\mathcal{O}_{X,x})$ and effective Cartier divisors on $\mathrm{Spec}(\mathcal{O}_{X_\sigma,\varphi(x)})$.

3. Let X be a strict toroidal embedding, and assume that X is a regular variety. Considering an étale description $\varphi : V \to X_\sigma$, we have that X_σ is regular. Therefore, for every $x \in X$, and a regular system of parameters $f_1, \ldots, f_d \in \mathcal{O}_{X,x}$ there exist $f_{i_1}, \ldots, f_{i_r}$, with $1 \leq i_1 < \ldots < i_r \leq d$ such that $\mathrm{div}(f_{i_1}), \ldots, \mathrm{div}(f_{i_r})$ are irreducible components passing through x. This implies that D defines a Cartier divisor on X, such that its support is precisely $|D|$ and it is a simple normal crossing divisor. The converse is clear; that is, a simple normal crossing divisor defines a toroidal embedding. In conclusion, in the case of a regular variety, having strict toroidal embeddings is the same as having simple normal crossing divisors.

Definition 1.2.6. Let X be a topological space. A *stratification* of X, is a collection $\{Y_\alpha\}_{\alpha\in\Lambda}$ of locally closed subspaces of X, pairwise disjoint, such that:

1. $X = \coprod_{\alpha\in\Lambda} Y_\alpha$.
2. For every pair Y_α, Y_β with $Y_\alpha \cap \overline{Y}_\beta \neq \emptyset$, we have $Y_\alpha \subseteq \overline{Y}_\beta$.

A *stratified space* is a space X together with a stratification. Every element of the stratification will be called *stratum*. We will say that the stratification is *locally finite* if each point $x \in X$ has a neighborhood that intersects only finitely many strata.

As a consequence of the above definition, for every $\alpha \in \Lambda$ there exists $\Lambda_\alpha \subseteq \Lambda$ such that $\overline{Y_\alpha} = \coprod_{\sigma\in\Lambda_\alpha} Y_\sigma$. Note that we can establish a partial order $\leq$ on the strata, saying that $Y_\alpha \leq Y_\beta$ if only if $Y_\alpha \subseteq \overline{Y}_\beta$.

Remark 1.2.7. 1. The strata which are the minimal elements under the partial order are closed.

2. If there exists a unique maximal stratum under this order, then it is open and dense in X.

3. The closure of every stratum Y_α can be written in terms of the partial order; that is

$$\overline{Y_\alpha} = \coprod_{Y_\beta \leq Y_\alpha} Y_\beta$$

   Sometimes the closure of the strata will be referred to as closed strata.

4. Note that the second condition in 1.2.6 is a local condition. Specifically, it is equivalent to requiring that for each point $x \in Y_\alpha \cap \overline{Y}_\beta$ there exists an open neighborhood $U \subseteq Y_\alpha$ of x such that $\overline{Y}_\beta$.

Example 1.2.8. Let X be a connected complex manifold, and let $D = \sum_{i\in I} D_i$ be a simple normal crossing divisor, where each D_i is an irreducible component of D. Consider the collection of connected components of nonempty expression $\bigcap_{j\in J} D_j \setminus \bigcup_{j\notin J} D_j$, where J runs over subsets of I. By the definition of a simple normal crossing divisor and item (3) of Remark 1.2.7, this collection defines a stratification of X. Moreover, this stratification is locally finite. Some authors refer to such kind of stratifications as divisorial stratifications.

Definition 1.2.9. Let X be a C^1 differentiable manifold of dimension n, and let Y, Z be two connected C^1 submanifolds such that $Z \subseteq \overline{Y} \smallsetminus Y$, with dimension of Y is equal to m. We say that the pair (Y, Z) satisfies the *Whitney condition* at a point $z \in Z$ if the following hold: For every pair of sequences $\{y_n\}_{n\in\mathbb{N}} \subseteq Y$ and $\{z_n\}_{n\in\mathbb{N}} \subseteq Z$, both converging to z, such that in a local chart at z

- The sequence of tangent spaces $\{T_{y_n}Y\}_{n\in\mathbb{N}}$ in the Grassmannian G_m^n converges to a subspace $\tau \in G_m^n$.
- The secant lines $\overline{y_n z_n}$ converge to a line l in $T_z X$

Then we have $l \subseteq \tau$.

Example 1.2.10. Let X be a complex manifold of dimension n, and let D be a simple normal crossing divisor. Then every pair of strata (Y_α, Y_β), with $Y_\alpha \subseteq \overline{Y}_\beta$ satisfies the Whitney condition at any point of Y_α. Indeed, consider the underlying smooth manifold structure of X. Fix a point $y \in Y_\alpha$, and choose a coordinate chart (U, z) around y. Assume that a Riemannian metric is given on U. Suppose that $Y_\alpha \cap U$ and $\overline{Y}_\beta \cap U$ are defined by the common vanishing of the coordinate functions $z_1, \ldots, z_s$ and $z_1, \ldots, z_t$, respectively, with $1 \leq t < s \leq n$.

Let $\{y_i\}_{i\in\mathbb{N}} \subseteq Y_\beta \cap U$ and $\{w_i\}_{i\in\mathbb{N}} \subseteq Y_\alpha \cap U$ be sequences converging to y. Then in these coordinates we can identify every tangent space $T_{y_i}Y_\beta$ with $\{0\}^{2t} \times \mathbb{R}^{2n-2t}$, and the first t coordinates of $y_i - w_i$ are zero.

By the definition of Whitney's conditions, we have that the limit

$$\lim_{n\to\infty} \frac{w_i - y_i}{||w_i - y_i||}$$

exists. Moreover, this limit lies in the limit of the tangent spaces $T_{y_i}Y_\beta$, which, under the identification above, is precisely $\{0\}^{2t} \times \mathbb{R}^{2n-2t}$. Hence, the pair (Y_α, Y_β) satisfies the Whitney conditions at every point of Y_α.

Definition 1.2.11. Let X be a differentiable manifold of class C^1. If $\Sigma = \coprod_{\alpha\in\Lambda} Y_\alpha$ is a locally finite stratification of X such that the Whitney conditions are satisfied for every pair of strata (Y_α, Y_β) with $Y_\alpha \subseteq \overline{Y}_\beta$, then we say that Σ is a *Whitney stratification.*

Whitney stratifications have good geometric and topological properties. In particular, they admit triangulations that are adapted to the stratification (See [Gor78]). This concept is made precise in the following definition, which is taken precisely from [Gor78].

Definition 1.2.12.
1. *A triangulation* of a topological space Y is a simplicial complex K together with a homeomorphism $t : |K| \to Y$.
2. A triangulation is said to be *adapted to a stratification* Σ of Y if, for each simplex $\sigma \in K$, the image $t(\mathring{\sigma})$ (i.e., the interior of σ) is contained in a unique stratum of Σ. Equivalently, for each stratum $Y \in \Sigma$, the preimage $t^{-1}(\overline{Y})$ is a subcomplex of K.

From Example 1.2.10, we saw that the stratification induced by the pair (X, D) is a Whitney stratification. Therefore, by [Gor78], the underlying smooth manifold of X admits a triangulation adapted to this stratification.

Fix a triangulation $\mathcal{T}$ adapted to the stratification induced by the pair (X, D), and let $\sigma \in \mathcal{T}$ be a simplex. Define

$$\mathrm{Star}(\sigma) := \bigcup_{\tau \succeq \sigma} \mathring{\tau}.$$

Since the collection of the relative interiors of the simplices gives a partition of X, it follows that for any pair of simplices $\sigma_1, \sigma_2 \in \mathcal{T}$ such that $\mathrm{Star}(\sigma_1) \cap \mathrm{Star}(\sigma_2) \neq \emptyset$, there exists a simplex $\tau_3 \in \mathcal{T}$ such that $\mathrm{Star}(\sigma_1) \cap \mathrm{Star}(\sigma_2) = \mathrm{Star}(\sigma_3)$. Moreover, this simplex is a minimal simplex between those simplices containing both σ_1 and σ_2 as faces. Note that each $\mathrm{Star}(\sigma)$ is open in the underlying smooth manifold of X. Therefore, $\mathrm{Star}(\sigma)$ inherits the structure of a complex submanifold of X.

Proposition 1.2.13. Let X be a complex manifold and D a simple normal crossing divisor. Then there exists a covering $\{U_\gamma\}_{\gamma \in \mathcal{A}}$ of X with the following conditions:

1. The covering is locally finite.
2. For each U_γ, the collection of strata that have nonempty intersection with U_γ contains a unique minimal stratum. We denote this minimal stratum by $Y_{\alpha(\gamma)}$, and refer to it as the stratum associated to U_γ.
3. If $U_\gamma \cap U_{\gamma'} \neq \emptyset$, and $Y_{\alpha(\gamma)}, Y_{\alpha(\gamma')}$ are the associated strata of U_γ and $U_{\gamma'}$, respectively. Then the strata $Y_{\alpha(\gamma)}$ and $Y_{\alpha(\gamma')}$ are comparable; that is, one is contained in the closure of the other.
4. If $\emptyset \neq \mathcal{B} \subseteq \mathcal{A}$ is a finite subset, then $\bigcap_{\gamma \in \mathcal{B}} U_\gamma$ is again an element of the covering.

Proof. Let $\mathcal{T}$ be a triangulation of X. Consider the open cover $\{\mathrm{Star}(\sigma)\}_{\sigma \in \mathcal{T}}$. Since the triangulation is adapted to the stratification, it follows that for each $\sigma \in \mathcal{T}$, the set of strata that have nonempty intersection with $\mathrm{Star}(\sigma)$ has a unique minimal element. This ensures that the second condition is satisfied. Moreover, if Y_σ denotes this minimal stratum, then

$$\mathring{\sigma} \subseteq \mathrm{Star}(\sigma_x) \cap Y_\sigma.$$

We now verify the third condition. If the original cover does not satisfy this condition, we refine the triangulation by taking the barycentric subdivision of each simplex in $\mathcal{T}$. Let $\mathcal{T}'$ denote the resulting triangulation. Clearly, $\mathcal{T}'$ remains adapted to the stratification. Consider the open cover consisting of the open stars associated with $\mathcal{T}'$. As before, this refined cover satisfies the second condition. We now claim that it also satisfies the third condition:

Let τ_1 and τ_2 be simplices of $\mathcal{T}'$ such that $\mathrm{star}(\tau_1) \cap \mathrm{star}(\tau_2) \neq \emptyset$. Then there exists a minimal simplex $\sigma \in \mathcal{T}$ that contains both τ_1 and τ_2. This follows from the fact that the intersection of two open stars determined by $\mathcal{T}'$ is again the star of a simplex in $\mathcal{T}'$, and every simplex in $\mathcal{T}'$ is subordinate to simplex in $\mathcal{T}$. Now, We analyze two possible cases:

1. If the relative interiors of τ_1 and τ_2 are contained in the relative interior of σ: Since $\mathring{\sigma}$ contains points of a unique stratum, it follows that interiors of both τ_1 and τ_2 have points of the same stratum. This implies that the minimal strata of $\mathrm{St}(\tau_1)$ and $\mathrm{St}(\tau_2)$ are the same.
2. If the relative interior of τ_1 is contained in $\mathring{\sigma}$, and the relative interior of τ_2 lies at the boundary of σ: Since $\mathring{\tau_2}$ lies in the boundary of σ, the closure of the stratum associated with $\mathrm{Star}(\tau_2)$ contains the stratum associated with $\mathrm{Star}(\sigma)$. This latter stratum is precisely the one associated with $\mathrm{Star}(\tau_1)$. Therefore, the strata corresponding to $\mathrm{Star}(\tau_1)$ and $\mathrm{Star}(\tau_2)$ are comparable.

This shows that the refined open cover satisfies the condition (3).

Finally, Since X is paracompact, we may extract a locally finite subcover from the collection $\{\mathrm{Star}(\sigma)\}_{\sigma\in\mathcal{T}}$. By also including all finite intersections of elements of this subcover, we obtain an open cover that is locally finite and satisfies the first and last conditions. Hence, this refined cover satisfies all the required conditions of the statement. □

The Proposition above motivates, in a certain sense, the following definition.

Definition 1.2.14. Let X be a complex manifold, and let D a simple normal crossing divisor. We say that an open subset $U \subseteq X$ is *adapted to the stratification*, or simply an *adapted open*, if the collection of strata that have nonempty intersection with U has a unique minimal stratum.

1.3 Groupoids and orbifolds

In this section, we introduce the definitions of proper Lie groupoids and orbifolds, and establish the relationship between these two concepts.

Definition 1.3.1.

1. Let X be a paracompact Hausdorff space. A real (resp. complex) *orbifold chart* on X is a tuple (U, G, φ), where U is an open connected set in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) for some n, and G is a finite group that acts smoothly (resp. holomorphically) on U by C^∞-diffeomorphisms (resp. biholomorphisms). The map $\varphi : U \to X$ is a G-invariant continuous map such that the induced map $\overline{\varphi} : U/G \to X$ is an open embedding. We denote the image of this embedding by $\mathtt{U}$. Note that, by G-invariance, $\varphi(U) = \mathtt{U}$.

2. An *embedding* $(V, H, \phi) \hookrightarrow (U, G, \varphi)$ between two charts is a smooth (resp. homolomorphic) embedding $\iota : V \hookrightarrow U$, together with an injective group homomorphism $\rho_\iota : H \to G$, such that ι is a ρ_ι-equivariant and satifies $\varphi \circ \iota = \phi$.

3. A real (resp. complex) *orbifold atlas* on X is a family $\{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ of orbifold charts that cover X, and are compatible in the following sense: For any pair of charts $(V_\alpha, G_\alpha.\varphi_\alpha)$ and $(V_{\alpha'}, G_{\alpha'}, \varphi_{\alpha'})$, and any point $\mathtt{x} \in \mathtt{V}_\alpha \cap \mathtt{V}_{\alpha'}$ (where $\mathtt{V}_\alpha := \varphi_\alpha(V_\alpha)$, $\mathtt{V}_{\alpha'} := \varphi_{\alpha'}(V_{\alpha'})$) there exists a chart $(V_{\alpha''}, G_{\alpha''}, \varphi_{\alpha''})$ such that $\mathtt{x} \in \mathtt{V}_{\alpha''} \subseteq \mathtt{V}_\alpha \cap \mathtt{V}_{\alpha'}$, and there are smooth (resp. holomorphic) embeddings from $(V_{\alpha''}, G_{\alpha''}, \varphi_{\alpha''})$ to $(V_\alpha, G_\alpha, \varphi_\alpha)$ and $(V_{\alpha'}, G_{\alpha'}, \varphi_{\alpha'})$, respectively.

4. Two orbifold atlases are said to be *equivalent* if they admit a common refinement. An *orbifold* (of dimension n) is such a paracompact Hausdorff space X, together with an equivalence class of orbifold atlases.

Unless otherwise specified, we will omit the mention of the atlas and refer simply to the orbifold X. From now on, we will focus on real orbifolds, since every complex orbifold is a real orbifold with an extra estructure.

Given a chart $(U_\alpha, G_\alpha, \varphi_\alpha)$, note that since G_α acts smoothly on U_α, we have a homomorphism $\rho_\alpha : G_\alpha \to \mathrm{Aut}(U_\alpha)$ from the group G_α to the group of smooth automorphism of U_α. TThis observation motivates the following definition.

Definition 1.3.2. An orbifold X is called *effective* if it has an atlas where all its homomorphism of groups $\rho_\alpha : G_\alpha \to \mathrm{Aut}(U_\alpha)$ are injectives.

In [Sat57], a characterization of effective orbifolds is presented, showing that every orbifold of this kind can be realized as a quotient of the form M/G, where M is a smooth manifold and G is a Lie group acting smoothly on M

Definition 1.3.1 gives the classical definition of orbifolds in terms of local charts and compatibility conditions. Motivated by the characterization of effective orbifolds, several works in the theory of orbifolds have developed alternative approaches by describing orbifolds in terms of specific types of groupoids. In particular, [Moe02] demonstrates that groupoids offer a natural and robust language for establishing the foundations of orbifold theory.

In what follows, we adopt this perspective by working in the language of groupoids, which, as we will see in Section 1.4, is closely related to the language of stacks. The definitions and statements presented below are based on the exposition in [Moe02].

Definition 1.3.3. 1. A *groupoid* is a category in which every morphism is an isomorphism.

2. If $\mathcal{C}$ is a category with fiber products, a *groupoid object* G in $\mathcal{C}$ consists of a tuple $(G_0, G_1, s, t, m, u, i)$, where G_0 and G_1 are objects of $\mathcal{C}$, and $s, t : G_1 \to G_0, m : G_1 \times_{s,G_0,t} G_1 \to G_1, u : G_0 \to G_1$, and $i : G_1 \to G_1$ are morphisms. The morphisms s and t are called the source and target maps, respectively. The morphism m is the composition map. The morphism u, called the identity morphism, satisfies $s \circ u = t \circ u = \mathrm{id}_{G_0}$. The morphism i, referred to as the inverse map, satisfies $i^2 = \mathrm{id}_{G_1}$, together with the identities $s \circ i = t$ and $t \circ i = s$. For simplicity, we omit the notation $G_1 \times_{s,G_0,t} G_1$, and instead write $G_1 \times_{G_0} G_1$. These maps satisfy the following axioms:

   (a) The multiplication m commutes with s and t in the following sense:

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 & \xrightarrow{m} & G_1 \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle s} \\ G_1 & \xrightarrow[s]{} & G_0 \end{array} \qquad \begin{array}{ccc} G_1 \times_{G_0} G_1 & \xrightarrow{m} & G_1 \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle t} \\ G_1 & \xrightarrow[t]{} & G_0 \end{array}$$

   (b) Associativity: The following diagram commutes

$$\begin{array}{ccc} G_1 \times_{G_0} G_1 \times_{G_0} G_1 & \xrightarrow{(m,\mathrm{id})} & G_1 \times_{G_0} G_1 \\ {\scriptstyle (\mathrm{id},m)}\downarrow & & \downarrow{\scriptstyle m} \\ G_1 \times_{G_0} G_1 & \xrightarrow[m]{} & G_1 \end{array}$$

   (c) Identities: The following two diagrams are commutative

$$\begin{array}{ccc} G_0 \times_{G_0} G_1 & \xrightarrow{u\times\mathrm{id}} & G_1 \times_{G_0} G_1 \\ & \searrow^{\cong} & \downarrow{\scriptstyle m} \\ & & G_1 \end{array} \qquad \begin{array}{ccc} G_1 \times_{G_0} G_0 & \xrightarrow{\mathrm{id}\times u} & G_1 \times_{G_0} G_1 \\ & \searrow^{\cong} & \downarrow{\scriptstyle m} \\ & & G_1 \end{array}$$

(d) Inverses: The two diagrams below commutes

$$\begin{array}{ccc} G_1 & \xrightarrow{(i,\mathrm{id})} & G_1 \times_{G_0} G_1 \\ & \searrow^{u\circ t} & \downarrow m \\ & & G_1 \end{array} \qquad \begin{array}{ccc} G_1 & \xrightarrow{(\mathrm{id},i)} & G_1 \times_{G_0} G_1 \\ & \searrow^{u\circ s} & \downarrow m \\ & & G_1 \end{array}$$

Example 1.3.4. 1. Let G be a group. Then consider the category, which has a single object, denoted by $*$, whose morphisms from $*$ to itself are given by the elements of the group G, with composition defined by the group operation. This category is clearly a groupoid, and is known as the classifying groupoid of G, denoted as BG.

2. Let G be a finite algebraic group over a field k, and let X be a scheme over k. A (left) action of G on X is given by a morphism of schemes $\mu : G \times_k X \to X$ such that we have the commutativity of the following diagrams:

$$\begin{array}{ccc} G \times_k G \times_k X & \xrightarrow{m\times \mathrm{id}_X} & G \times_k X \\ \downarrow{\scriptstyle \mathrm{id}_G\times\mu} & & \downarrow{\scriptstyle \mu} \\ G \times_k X & \xrightarrow{\mu} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{(e,\mathrm{id}_X)} & G \times_k X \\ & \searrow^{\mathrm{id}_X} & \downarrow \mu \\ & & X \end{array}$$

where $m : G \times G \to G$ is the multiplication of groups, and $e : \mathrm{Spec}(k) \to G$ is the identity section of the group scheme. This action induces a groupoid object in the category of schemes $G \times X \rightrightarrows X$, where the maps s and t are given by the action map and the projection.

Based on the definition above, we now consider groupoids in the category of Hausdorff smooth manifolds. Since this category does not necessarily admit fiber products, we require that the source map s and the tarjet t be submersions. This condition ensures that the fiber product $G_1 \times_{G_0} G_1$ is a Hausdorff smooth manifold, as pullbacks along submersions always exist in the category of smooth manifolds.

In concrete terms, we have the following definition.

Definition 1.3.5. A Lie groupoid is a groupoid G in which both the object space G_0 and the morphism space G_1 are Hausdorff smooth manifolds, and the structure maps s, t, m, u and i are all smooth. In addition, the source and target maps s and t are required to be submersions.

Definition 1.3.6. A Lie groupoid is called *proper* if the map $(s, t) : G_1 \to G_0 \times G_0$ is a proper map, that is, the preimage of any compact set is compact.

A Lie groupoid G is called *étale* if the source and target maps s and t are local diffeomorphisms.

Given a point $x \in G_0$, the *isotropy group (or stabilizer)* at x, denoted G_x, is defined as $\{g \in G_1 : \ s(g) = t(g) = x\}$. That is, G_x consists of all arrows in G_1 whose source and target are both equal to x.

Let G be an étale groupoid, and let $z \in G_1$. Then there exist open neighborhoods $V \subseteq G_1$ of z and $U,\ W \subseteq G_0$ of $s(z), t(z)$, respectively. Such that the restrictions

$$s_{|V} : V \to U, \quad t_{|V} : V \to W$$

are diffeomorphisms. This implies that the map $(t\circ s^{-1})_{|U} : U \to W$ is a smooth diffeomorphism. Different choices of such neighborhoods V, yield the same germ of diffeomorphisms from $s(z)$ to $t(z)$.

Moreover, for each $x \in G_0$ and each element $y \in G_x$, the element y defines a germ of a diffeomorphism at x given by $s_{|V} : V \to U$ and $t_{|V} : V \to U$, where U is a suitable neighborhood for x. This construction induces a group homomorphism from the isotropy group G_x to $\mathrm{Diff}_x G_0$, the group of germs of diffeomorphisms of G_0 at the point x.

Definition 1.3.7. We say that the étale groupoid G is *effective* if, for each $x \in G_0$, the homomorphism from the isotropy group G_x to the group of germs of diffeomorphisms at x is injective.

Definition 1.3.8. Let G and H be Lie groupoids. A *homomorphism of Lie groupoids* (or simply a *morphism*) from G to H is a pair of smooth maps

$$\phi : G_1 \to H_1, \quad \psi : G_0 \to H_0$$

that are compatible with the groupoid structure; that is, they commute with all the structure maps (source, target, identity, composition, and inversion).

Remark 1.3.9. If G is a proper and étale Lie groupoid, then for each point $x \in G_0$, the stabilizer group G_x is finite. Moreover, since G is étale, there is a natural homomorphism from G_x to the group of germs of local diffeomorphisms at x. As a consequence, there exists an open neighborhood $U \subseteq G_0$ of x on which the finite group G_x acts smoothly. Furthermore, by étaleness, the source and target fibers over U coincide, i.e., $s^{-1}(U) = t^{-1}(U)$, and the restriction $s^{-1}(U) \rightrightarrows U$ defines a Lie subgroupoid of G. This restricted groupoid is isomorphic to the action groupoid $G_x \times U \rightrightarrows U$. This local construction shows that a proper étale Lie groupoid defines an orbifold structure on the quotient it presents.

Before introducing groupoids, we observe that effective orbifolds can be described by groupoids, and by Remark 1.3.9, every proper and étale groupoid defines an orbifold, which is not necessarily effective. This naturally raises several questions. If we start with an effective groupoid, construct the associated orbifold, and then recover a groupoid from this orbifold, do we obtain the original groupoid? Conversely, if we begin with an effective orbifold and consider its associated groupoid, is this groupoid effective? Moreover, when we take the orbifold induced by this groupoid, do we recover the original orbifold? To address these questions, we first need to clarify what it means for two such objects to be considered "equal."

Definition 1.3.10. Let G and H be Lie groupoids. A homomorphism of Lie groupoids $F = (\phi, \psi) : G \to H$ is called an *equivalence of Lie groupoids* if the following two conditions hold:

1. The map $t \circ \mathrm{pr}_1 : H_1 \times_{t,H_0,\psi} G_0 \to H_0, \quad (h, x) \mapsto t(h)$, is a surjective submersion. This condition ensures that every object of H is locally isomorphic to an object of G.

2. The following diagram is a pullback of smooth manifolds:

$$\begin{array}{ccc} G_1 & \xrightarrow{\phi} & H_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow[\psi\times\psi]{} & H_0 \times H_0 \end{array}$$

   That is, the map $\phi : G_1 \to H_1$ induces a diffeomorphism:

$$G_1 \cong G_0 \times_{H_0} H_1 \times_{H_0} G_0.$$

   This ensures that (ϕ, ψ) induces bijections between the morphism spaces in a smooth way.

Two Lie groupoids G and G' are said to be *Morita equivalent* if there exists a third groupoid H and equivalences

$$G \xleftarrow{\sim} H \xrightarrow{\sim} G'$$

Having established a series of definitions, we will enounce the Theorem that is established in [Moe02] and relate orbifolds and Lie groupoids.

Theorem 1.3.11. *Let G be a Lie groupoid. The following conditions are equivalent.*

1. *G is Morita equivalent to an effective proper étale groupoid.*
2. *G is Morita equivalent to a groupoid $G \times X \rightrightarrows X$, where G is a compact Lie group acting on a smooth manifold X, such that each stabilizer is finite and acts effectively on the normal bundle.*

In summary, we have a one-to-one correspondence, up to equivalence, between effective orbifolds and effective orbifold groupoids.

Remark 1.3.12. Finally, if G is an arbitrary étale groupoid, assume that $\{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ is the orbifold that deterimes G. For each α, consider the kernel H_α of the map from G_α to the group of diffeomorphisms of V_α to itself. Therefore, we consider the orbifold given by the charts $\{(V_\alpha, G_\alpha/H_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$. Since the actions are not trivial, this defines an effective orbifold whose underlying space is precisely the same underlying topological space of $\{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$.

1.4 Deligne-Mumford stack

In this section, we review the definitions of algebraic and analytic stacks. Most of the material presented here is based on [Gó01], [LMB99], and [Alp24] for the case of algebraic stacks, and on [Toe99] for the analytic setting.

Given a scheme S, the *big étale site* $(\mathrm{Sch}/S)_{et}$ is defined as the category of S-schemes, denoted (Sch/S), equipped with a Grothendieck topology generated by a standard notion of covering: families of étale morphisms $\{U_\alpha \to U\}_{\alpha\in\Lambda}$ such that $\coprod_\alpha U_\alpha \to U$ is surjective.

Let S be a base scheme, and let X be an S-scheme. The *functor of points* of X, denoted by h_X, is the presheaf (in fact, a sheaf) on the big étale site $(\mathrm{Sch}/S)_{et}$ defined as follows:

$$\begin{aligned} h_X : \ (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathbf{Sets} \\ T \qquad &\longmapsto \mathrm{Hom}_S(T, X) \end{aligned}$$

For simplicity, this functor of points is often denoted by $\underline{X}$ or even by X itself when there is no ambiguity between the scheme and its functor.

Let S be a scheme, and let $F, G, H : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathbf{Sets}$ be sheaves of sets on the big étale site. Suppose that $\phi : F \to H$ and $\varphi : G \to H$ are morphisms of sheaves. Then *fiber product* $F \times_H G$ is defined as the sheafification of the presheaf given by:

$$(\mathcal{F} \times \mathcal{G})(T) := \mathcal{F}(T) \times_{H(T)} \mathcal{G}(T)$$

Definition 1.4.1. 1. Let S be a scheme. A sheaf $F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathbf{Sets}$ is called *representable* if there exists an S-scheme X such that $F \cong X$. Here, this equivalence of functors means that there exists a natural isomorphism $\phi : F \Rightarrow h_X$.

2. Let $F \to G$ be a morphism of sheaves on $(\mathrm{Sch}/S)_{\text{ét}}$. We say that the morphism is *representable by schemes* if for every S-scheme X and every morphism $X \to G$, the fiber product $F \times_G X$ is representable by an S-scheme.

3. We say that a morphism $f : F \to G$ of sheave is *étale* (respectively, *surjective*, *flat*, *open*, or *closed*) if it is representable, and for every S-scheme X and every morphism $X \to G$, the base change $F \times_G X \to X$ is an étale (respectively, surjective, flat, open, or closed) morphism of schemes.

Having established the preceding definitions, we now present the definition of an algebraic space.

Definition 1.4.2. Let S be a scheme. An *algebraic space* over S is a sheaf F of sets on the big étale site $(\mathrm{Sch}/S)_{et}$ satisfying the following conditions:

1. The diagonal morphism $\Delta : F \to F \times F$ is representable by schemes. That is, for every scheme T and morphism $T \to F \times F$, the fiber product $F \times_{F \times F} T$ is representable by a scheme.

2. There exists a scheme U and a morphism of sheaves $U \to F$, such that this morphism is representable, étale, and surjective.

Example 1.4.3. 1. A basic example of an algebraic space is the functor of points of a scheme. This shows that the category of schemes over a base scheme S embeds fully faithfully into the category of algebraic spaces over S.

2. Another example of an algebraic space results from taking quotients by group actions. Let S be a base scheme, and let X be an S-scheme. As in the Example 1.3.4, suppose that a finite group G acts on X, via a morphism $\mu : X \times G \to X, \quad (x, g) \mapsto x \cdot g$. This group action induces, for every S-scheme T, an action of the group $G(T)$ on $X(T)$. Then we can consider the quotient in the category of sets
$$(X/G)(T) := X(T)/G(T)$$
this naturally defines the following presheaf
$$\begin{aligned} X/G : \ (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathbf{Sets} \\ T \qquad &\longmapsto (X/G)(T) \end{aligned}$$
This presheaf is in fact a sheaf for the étale topology. Moreover, if G acts freely[1] on X, this sheaf is an algebraic space (see [Sta18, Tag 02Z0]). In this case the natural map $X \to X/G$ is representable, étale and surjective. As we will see below, this algebraic space is not always representable by a scheme.

We will show an easy example in which not every algebraic space of the type described in 1.4.3 is a scheme. For this, we will use the following result: A necessary and sufficient condition for the existence of a quotient of a scheme X by a finite group G, in the category of schemes, is that the orbit of every point in X is contained in an affine open subset of X (see [Gro71, Prop 1.8]).

[1] An action of G on X is said to be free if for every scheme T over S, and every $x \in X(T)$, if $x \cdot g = x$ for some $g \in G(T)$, then g is the identity element of $G(T)$

Example 1.4.4. Consider the non-separated affine line $X := \mathbb{A}^1_{\mathbb{C}} \bigsqcup_{\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{0\}} \mathbb{A}^1_{\mathbb{C}}$. Recall that this scheme is obtained by gluing two copies of the affine line along the open subset $\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{0\}$. Let $\mathbb{Z}/2\mathbb{Z}$ acting on X by multiplication by -1 at every point except the two point of the origin, and swaps the two copies of the origin. Note that this action is free since no point is fixed by the non-identity element. The quotient is not representable by a scheme, since there is no affine open subset containing both origin points. This is because every affine scheme is separated, and the two origins cannot be contained in a separated open subset of X. This means the condition in [Gro71, Prop 1.8] is not satisfied.

In the remainder of this section, we focus on stacks, with particular emphasis on Deligne–Mumford stacks, as they form the central framework of Chapter 3.

Definition 1.4.5. 1. Let S be a scheme. A *presheaf of groupoids* F on the étale site $(\mathrm{Sch}/S)_{\mathrm{et}}$ is a presheaf $F : (\mathrm{Sch}/S)^{op}_{\mathrm{et}} \to \mathbf{Groupoids}$ which assigns to each S-scheme U a groupoid $F(U)$, and to each étale morphism $f : V \to U$ a functor $F(f) : F(U) \to F(V)$

Definition 1.4.6. Let S be a scheme. A *stack* over the étale site $(\mathrm{Sch}/S)_{\mathrm{et}}$ is a presheaf of groupoids

$$\mathscr{X} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathbf{Groupoids}$$

that satisfies the following conditions for every étale covering $\{U_i \to U\}$:

1. For any two objects $x, y \in \mathscr{X}(U)$, the presheaf of sets

$$\underline{\mathrm{Isom}}(x, y) : (\mathrm{Sch}/U)^{\mathrm{op}} \to \mathbf{Sets}, \quad T \mapsto \mathrm{Isom}_{\mathscr{X}(T)}(x|_T, y|_T)$$

   is a sheaf on the site $(\mathrm{Sch}/U)_{\mathrm{et}}$.

2. For any collection $x_i \in \mathscr{X}(U_i)$, and isomorphisms $\phi_{ij} : x_j|_{U_{ij}} \xrightarrow{\sim} x_i|_{U_{ij}}$ on $U_{ij} := U_i \times_U U_j$, satisfying the cocycle condition on triple overlaps:

$$\phi_{ik} = \phi_{ij}|_{U_{ijk}} \circ \phi_{jk}|_{U_{ijk}} \quad \text{on } U_{ijk} := U_i \times_U U_j \times_U U_k.$$

   There exists an object $x \in \mathscr{X}(U)$ and isomorphisms $\psi_i : x|_{U_i} \xrightarrow{\sim} x_i$ such that

$$\phi_{ij} = \psi_i|_{U_{ij}} \circ \psi_j^{-1}|_{U_{ij}}.$$

   Moreover, such an object x is unique up to a unique isomorphism in $\mathscr{X}(U)$.

Example 1.4.7. 1. Fixing a base scheme S, the functor of points of an S-scheme X is a stack. Moreover, we have that algebraic spaces are stacks.

2. Let G be a finite group scheme over a field k, acting on a scheme X over k (see example 1.3.4). Consider the groupoid presheaf $[X/G] : (\mathrm{Sch}/k)_{\mathrm{et}} \to \mathbf{Groupoids}$ defined as follows: For a k-scheme T, the groupoid $[X/G](T)$ consist of pairs of morphisms

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ \downarrow{\scriptstyle p} & & \\ T & & \end{array}$$

where p is a principal G-bundle, and f is a G-equivariant morphism. Morphisms in this groupoid are G-equivariant isomorphisms of G-bundles over T that commute with the maps to X. This construction defines a stack in the étale topology (see [LMB99][Section 2.4]), which is commonly referred to in the literature as the quotient stack $[X/G]$.

A morphism of stacks is a natural transformation between the associated sheaves of groupoids; that is, for each object in the étale site, it gives a functor between groupoids compatible with restriction maps.

Definition 1.4.8. A morphism $\mathscr{X} \to \mathscr{Y}$ of stacks is *representable (resp. representable by schemes)* if for every morphism $U \to \mathscr{Y}$ from a scheme U, the fiber product $U \times_{\mathscr{X}} \mathscr{Y}$ is an algebraic space (resp. scheme).

In the same way as in Definition 1.4.1, we consider the following definition

Definition 1.4.9. Let $\mathcal{P}$ be a property of morphism of schemes.

1. We say that $\mathcal{P}$ is étale local on the source if for every étale surjection $Z \to X$ of schemes, a morphism $X \to Y$ satisfies $\mathcal{P}$ if and only if $Z \to X \to Y$ does. $\mathcal{P}$ is étale local on the target if for every étale surjection $Z \to Y$ of schemes, a morphism $X \to Y$ satisfies $\mathcal{P}$ if only if $X \times_Y Z \to Z$ does.
2. Let $\mathcal{P}$ be stable under composition and base change and is étale local on the source and target. A morphism of stacks $\mathscr{X} \to \mathscr{Y}$ has the property $\mathcal{P}$ if for all étale presentations $V \to \mathscr{Y}$ and $U \to \mathscr{X} \times_{\mathscr{Y}} V$, the composition $U \to V$ has $\mathcal{P}$.
3. A morphism $\mathscr{X} \to \mathscr{Y}$ of stacks representable by schemes has the property $\mathcal{P}$ if for every morphism $U \to \mathscr{Y}$ from a scheme U, the morphism $U \times_{\mathscr{X}} \mathscr{Y} \to U$ of schemes has the property $\mathcal{P}$.

The properties of flatness, smothness, surjectivity, locally of finite presentation, or locally of finite type are smooth local on the source and target.

Definition 1.4.10. A Deligne Mumford (resp. algebraic) stack, is a stack $\mathscr{X}$ over $(\mathrm{Sch}/S)_{et}$ such that there exist a scheme U and a surjective, étale (resp. smooth), and representable morphism from U to $\mathscr{X}$. The morphism $U \to \mathscr{X}$ is called an *étale (resp. smooth) presentation.*

If $\mathscr{X}$ is a stack over $(\mathrm{Sch}/S)_{et}$, and $U \in (\mathrm{Sch}/S)_{et}$, then using the Yoneda's lemma for 2-categories, in this case for stacks, we have an equivalence of categories between $\mathrm{Mor}(U, \mathscr{X})$ and the groupoid fiber category $\mathscr{X}(U)$.

Example 1.4.11.
1. Schemes and algebraic spaces are a Deligne-Mumford stack.
2. Example 1.4.7 provides an instance of a Deligne–Mumford stack, commonly referred to as a quotient stack. Moreover, if we have an étale groupoid of schemes $P_1 \rightrightarrows P_0$ such that such that the associated morphism $P_1 \to P_0 \times_S P_0$, determined by the groupoid structure maps, is separated, quasi-compact, and of finite type, then the associated groupoid defines a Deligne Mumford stack $[P_1 \rightrightarrows P_0]$; see [LMB99, Prop 4.3.1].
3. The stack $\overline{\mathscr{M}}_g$ of curves of genus $g \geq 2$ is a DM stack [DM69, prop 5.1].
4. The stack $\overline{\mathscr{M}}_{g,n}$ of curves of genus g and $n-$marked points, with $3g - 3 + n > 0$ is a DM stack [KNU83, thm 2.7]

In [Alp24, Theorem 3.2.1] it is stated that the diagonal of an algebraic space is repesentable by schemes, and the diagonal of an algebraic stack is representable. In conclusion, if $\mathscr{X} \to \mathscr{Y}$ is a representable morphism of stacks, then $\mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ is representable by schemes. Based on the observation above, we now consider the following definition.

Definition 1.4.12.
1. A representable morphism $\mathscr{X} \to \mathscr{Y}$ is *separated* if the morphism $\mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$, which is representable by schemes, is proper.
2. An algebraic stack $\mathcal{X}$ is *separated* if the diagonal $\mathscr{X} \to \mathscr{X} \times \mathscr{X}$, which is representable [LMB99, Cor 3.13], is separated.

Separability is an important concept in stacks because it allows us to have better properties on stacks, for example, if $\mathcal{X}$ is a DM stack, the separability implies that any étale presentation $U \to \mathscr{X}$ is representable by schemes.

Definition 1.4.13. A substack $\mathscr{U} \subseteq \mathscr{X}$ of a stack is called an *open stack (resp closed)* if the inclusion $\mathscr{U} \to \mathscr{X}$ is representable by schemes and an open inmersion (closed immersion).

Example 1.4.14.
1. From Example 1.4.7, let $x : \mathrm{Spec}(k) \to X$ be a geometric point and denote G_x the stabilizer of the point; that is, the subgroup of $G(\mathrm{Spec}(k))$ consisting of elements that fix the point x. Then, there exists an open neighborhood $U \subseteq X$ of x that is invariant under the action of G_x. In this case, the quotient stack $[U/G_x]$ defines an open substack of $[X/G]$.
2. Following the example 1.4.11, the substack $\mathscr{M}_{g,n}$ of smooth curves of genus g and n-marked points of $\overline{\mathscr{M}}_{g,n}$ is an open substack. See [KNU83, theorem 2.7].

In general, the following fact holds for any stack admitting a surjective morphism from an algebraic space. However, since we are primarily interested in Deligne–Mumford stacks, we will develop it in that setting.

Let $p : P \to \mathscr{X}$ be an étale presentation of a separated Deligne–Mumford stack. Then this presentation defines an étale groupoid $P \times_{\mathscr{X}} P \rightrightarrows P_0$ in the category of schemes. The stack that defines this étale groupoid is isomorphic to $\mathscr{X}$ see [LMB99, Prop 3.8]. The description above in terms of groupoids gives us a characterization for the substack of $\mathscr{X}$; that is, if $s, t : P \times_{\mathscr{X}} P \to P$ is the respective groupoid, then there exists one-to-one correspondence between open (closed, locally closed) substacks of $\mathscr{X}$, and open (closed, locally closed) subschemes W of P satisfying the condition $s^{-1}(W) = t^{-1}(W)$; see [Gil84, Pag 208]. This correspondence is independent of the choice of étale presentation.

Definition 1.4.15. The inclusion $\mathscr{U} \subseteq \mathscr{X}$ is called a *toroidal embedding* if the morphism $\mathscr{U} \to \mathscr{X}$ is representable by schemes, and such representation is a toroidal embbeding.

Let $(\mathscr{X}, \mathscr{U})$ be a toroidal embedding of DM stacks, and let $P \to \mathscr{X}$ be an étale presentation. The induced toroidal embedding on the presentation is given by $\mathscr{U} \times_{\mathscr{X}} P \to P$. For simplicity, we denote $\mathscr{U} \times_{\mathscr{X}} P$ by $P_{\mathscr{U}}$. Note that we can always find an étale presentation $P \to \mathscr{X}$ such that the pair $(P, P_{\mathscr{U}})$ is a strict toroidal embedding.

Example 1.4.16.
1. Following Example 1.4.7, let G be a finite group scheme acting on a regular variety X over a field k, and consider the quotient stack $[X/G]$. Suppose that $D \subseteq X$ is a simple normal crossing divisor that is invariant under the action of G. Then the open set $U := X \setminus D$ is also G-invariant, and we may consider the induced open substack $[U/G] \subseteq [X/G]$. Since X is regular, and D is a simple normal crossing divisor, the pair (X, U) admits the structure of a toroidal

embedding (see Remark 1.2.5). Moreover, the inclusion $U \hookrightarrow X$ is G-equivariant, so the induced morphism $[U/G] \to [X/G]$ is representable by the toroidal embedding $U \hookrightarrow X$. It follows that the pair $([X/G], [U/G])$ defines a toroidal embedding of Deligne–Mumford stacks.

2. The pairs $(\overline{\mathscr{M}}_g, \mathscr{M}_g)$ and $(\overline{\mathscr{M}}_{g,n}, \mathscr{M}_{g,n})$ are toroidal embbedings of DM stacks [KNU83, Thm 2.7].

If $\mathscr{X}$ is an algebraic stack, its *underlying topological space*, denoted $|\mathscr{X}|$, is defined as the set of equivalence classes of morphisms $x : \operatorname{Spec} k \to \mathscr{X}$, where k ranges over all fields. Two such morphisms $x_1 : \operatorname{Spec} k_1 \to \mathscr{X}, x_2 : \operatorname{Spec} k_2 \to \mathscr{X}$ are considered equivalent if there exists a field k_3 containing both k_1 and k_2, and a commutative diagram:

$$\begin{array}{ccc} \operatorname{Spec} k_3 & \longrightarrow & \operatorname{Spec} k_2 \\ \downarrow & & \downarrow \\ \operatorname{Spec} k_1 & \longrightarrow & \mathscr{X} \end{array}$$

A subset $U \subseteq |\mathscr{X}|$ is declared *open* if there exists an open substack $\mathscr{U} \subseteq \mathscr{X}$ such that $|\mathscr{U}| = U$. This defines a topology on $|\mathscr{X}|$, called the Zariski topology.

We say that an algebraic stack $\mathscr{X}$ is *quasi-compact, connected*, or *irreducible* if its underlying topological space $|\mathscr{X}|$ has the corresponding property.

We now present a series of definitions concerning morphisms of algebraic stacks that are useful to keep in mind, as they will be used at various points throughout this work.

Definition 1.4.17. 1. A morphism $\mathscr{X} \to \mathscr{Y}$ is called *quasi-compact* if for every morphism $\operatorname{Spec} A \to \mathscr{Y}$, the fiber product $\operatorname{Spec} A \times_{\mathscr{Y}} \mathscr{X}$ is quasi-compact.

2. A morphism is called of *finite type* if it is locally of finite type and quasi-compact.

3. A morphism $\mathscr{X} \to \mathscr{Y}$ is *universally closed* if for every morphism $\mathscr{Z} \to \mathscr{Y}$, the morphism $\mathscr{X} \times_{\mathscr{Y}} \mathscr{X} \to \mathscr{Z}$ induces a closed map $|\mathscr{X} \times_{\mathscr{Y}} \mathscr{Z}| \to |\mathscr{Z}|$.

4. A morphism $\mathscr{X} \to \mathscr{Y}$ is called *proper* if it is universally closed, separated, and of finite type.

5. A representable morphism $\mathscr{X} \to \mathscr{Y}$ of algebraic stacks is called *locally quasi-finite*, if is locally of finite type and for every morphism $V \to \mathscr{Y}$ from a scheme, the algebraic space $\mathscr{X} \times_{\mathscr{Y}} V$ is locally quasi-finite over T. [2]

6. A morphism $\mathscr{X} \to \mathscr{Y}$ of algebraic stacks is called *locally quasi-finite*, if is locally of finite type, the diagonal $\mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ is locally quasi-finite, and for every morphism $\operatorname{Spec} k \to \mathscr{Y}$, the space $|\mathscr{X}_k|$ is discrete.

7. A morphism is called *quasi-finite* if it is locally quasi-finite and quasi-compact.

8. A morphism of stacks is called *finite* if it is proper and quasi-finite.

9. A morphism of separated DM stacks $\mathscr{X} \to \mathscr{Y}$ is called *étale* if it is representable, and for every étale presentation $V \to \mathscr{Y}$ and étale presentation $U \to \mathscr{X} \times_{\mathscr{Y}} V$, the induced composition $U \to V$ is an étale morphism of schemes.

[2] A morphism of algebraic space is locally quasi-finite if the morphism associated in the category of schemes is.

Definition 1.4.18. Let $\mathscr{X}$ be an algebraic stack. The *inertia stack* $I_{\mathscr{X}}$ is defined as the fiber product

$$I_{\mathscr{X}} := \mathscr{X} \times_{\Delta, \mathscr{X}\times\mathscr{X}, \Delta} \mathscr{X},$$

where $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is the diagonal morphism. Intuitively, $I_{\mathscr{X}}$ encodes the automorphism structure of objects in $\mathscr{X}$. More precisely, for a scheme T, the groupoid $I_{\mathscr{X}}(T)$ consists of pairs (x, α) where

- $x \in \mathscr{X}(T)$ is an object over T,
- $\alpha : x \to x$ is an automorphism of x in the fiber category over T.

Last description, allows us to view the inertia stack as a stack that parameterizes objects of $\mathscr{X}$ together with their automorphisms. This motivates the following definition.

Definition 1.4.19. Let $\mathscr{X}$ be an algebraic stack, and $x : \operatorname{Spec} k \to \mathscr{X}$ be a geometric point. The *stabilizer group* of the point x, denoted G_x, is defined as the fiber product of the morphims $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ and $(x, x) : \operatorname{Spec} k \to \mathscr{X} \times \mathscr{X}$.

Remark 1.4.20. As the diagonal morphism $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable [LMB99, Cor 3.13], the stabilizer group is an algebraic space. Moreover, if the algebraic stack is a Deligne-Mumford stack separated, then the stabilizer group is represented by a finite and reduced scheme [Alp24, Remark 3.3.13].

Finally, following [Alp24, Section 3.1.8], the stabilizer group of a point $x : \operatorname{Spec} k \to \mathscr{X}$ of $\mathscr{X}$ can also be described by the sheaf

$$\begin{aligned} \mathbf{Isom}_{\mathscr{X}(k)}(x, x) : (\mathrm{Sch}/S)_{et}/k &\to \mathrm{Sets} \\ (T \xrightarrow{f} \operatorname{Spec} k) &\mapsto \mathrm{Mor}_{\mathscr{X}(T)}(f^*x, f^*x) \end{aligned}$$

Definition 1.4.21. A morphism $\pi : \mathscr{X} \to X$ from an algebraic stack to an algebraic space is a *coarse moduli* if the following conditions hold:

1. For every algebraically closed field k, there exists a bijection between the set of isomorphism classes of $\mathscr{X}(k)$ and $X(k)$.
2. Every morphism from $\mathscr{X}$ to an algebraic space Y factors through π.

Not every algebraic stack admits a coarse moduli space. However, in the case of Deligne–Mumford stacks, the following proposition provides conditions under which a coarse moduli space exists.

Proposition 1.4.22. Let $\mathscr{X}$ be a separated DM stack of finite type over k. Then there exists a coarse moduli space $\pi : \mathscr{X} \to X$, which satisfies the following additional properties:

1. X is of finite type.
2. X is separated on k.
3. The map π is proper and finite.

Proof. See [Ryd07, pag. 2], or [Con05, thm 1.1] □

The following proposition, proved in [LMB99, Prop. 16.6.1], will be useful in establishing certain topological properties of the coarse moduli space of separated Deligne–Mumford stacks, as we will see in Chapter 3.

Proposition 1.4.23. Let $\mathscr{X}$ a DM stack of finite type over a noetherian scheme S. There exist a surjective proper morphism $Z \to \mathscr{X}$ from a quasi-projective S-scheme Z such that the map is étale on an open dense subscheme of Z.

Finally, we conclude this section with a discussion of analytic stacks of Deligne-Mumford, and complex analytifactions. The statements and definitions in this part are taken from [Toe99].

Recall that a subset $A \subseteq \mathbb{C}^n$ is called an *analytic set* if, for every point $x \in A$ there exists an open neighborhood $U \subseteq \mathbb{C}^n$ of x and finitely many holomorphic functions $f_1, \dots f_k : U \to \mathbb{C}$ such that

$$A \cap U = \{z \in U : f_1(z) = \dots = f_k(z) = 0\}$$

In this context, a *complex analytic space* is a locally ringed space over $\mathbb{C}$ that is locally isomorphic to a closed analytic subset of an open set $V \subseteq \mathbb{C}^n$, equipped with the structure sheaf of holomorphic functions modulo a coherent ideal.

Definition 1.4.24. Let $f : Y \to X$ be a morphism of complex analytic spaces. We say that f is *étale at a point* $y \in Y$ if there exist open neighborhoods $V \subseteq Y$ of y and $U \subseteq X$ of $f(y)$, such that the restricted map $f|_V : V \to U$ is an isomorphism of complex analytic spaces.

The morphism f is said to be *étale* if it is étale at every point $y \in Y$; that is, f is a local isomorphism in the analytic topology.

The morphism f is said to be *finite* if it is proper and has finite fibers.

Definition 1.4.25. The *étale site* $(\mathrm{An}/\mathbb{C})_{\mathrm{et}}$ is defined as follows: The objects of $(\mathrm{An}/\mathbb{C})_{\mathrm{et}}$ are complex analytic spaces U. Morphisms $U \to V$ in $(\mathrm{An}/\mathbb{C})_{\mathrm{et}}$ are morphisms of analytic spaces over $\mathbb{C}$.

A family $\{U_i \to U\}$ is a *covering*, if the maps $U_i \to U$ are étale morphisms, and the union of their images covers U.

Having fixed this, the notion of presheaf of groupoids on the site $(\mathrm{An}/\mathbb{C})_{\mathrm{et}}$, as well as the notion of an *analytic stack*, is analogous to the corresponding definitions in the algebraic setting.

Definition 1.4.26. A *separated analytic stack of Deligne-Mumford*, is a stack $\mathscr{X}$ over the site $(\mathrm{An}/\mathbb{C})_{\text{ét}}$ satisfying the following conditions:

1. The diagonal morphism $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable by complex analytic spaces, separated, and finite.
2. There exists a complex analytic space U and a étale, surjective morphism $U \to \mathscr{X}$

Now, we will expose a series of propositions that can be found in [Toe99, Chap V]

Proposition 1.4.27. Let $\mathscr{X}$ be an analytic stack of Deligne-Mumford.

1. There exists a complex analytic space X, and a proper morphism

$$p : \mathscr{X} \to X$$

which is universal among morphisms from $\mathscr{X}$ to analytic spaces, and which induces a bijection between the class $\mathscr{X}((\operatorname{Spec}\mathbb{C})^{an})$ of complex points of the stack and $X((\operatorname{Spec}\mathbb{C})^{an})$.

In the analytic setting, the space X is the *moduli space* for $\mathscr{X}$.

2. Let X be the moduli space of $\mathscr{X}$. Then, there exists an étale cover $U \to X$, and a complex analytic space Y, a constant sheaf of finite groups H on Y, and an action of H on Y, such that the pullback stack

$$\mathscr{X}_U := \mathscr{X} \times_X U$$

is equivalent to the classifying stack $[Y/H]$.

The analytic stack $[Y/H]$ is defined in similar way as in Example 1.4.7.

Just as in Example 1.4.11, analytic stacks can be described via groupoids arising from their étale presentations. In this sense, the analytification of a Deligne–Mumford stack can be defined as follows:

Let $\mathscr{X}$ be a separated Deligne-Mumford stack over $\mathbb{C}$, and let $P \to \mathscr{X}$ be an étale presentation. Then the associated étale groupoid $P \times_{\mathscr{X}} P \rightrightarrows P$ induces, by analytification, an étale groupoid of analytic spaces

$$(P \times_{\mathscr{X}} P)^{an} \rightrightarrows P^{an}$$

This groupoid defines an analytic stack, which we denote by $\mathscr{X}^{an}$.

Proposition 1.4.28. Let X be a scheme and H a finite group acting on X. Then there is a natural weak equivalence between the analytification of the quotient stack and the quotient of the analytified space:

$$[X/H]^{an} \simeq [X^{an}/H].$$

Proposition 1.4.29. Let $\mathscr{X}$ be an algebraic stack, and let $p : \mathscr{X} \to X$ be the projection to its coarse moduli space. Then the induced morphism on analytifications

$$p^{an} : \mathscr{X}^{an} \to X^{an}$$

makes X^{an} the coarse moduli space of $\mathscr{X}^{an}$.

1.5 Fundamental group of an stack

The main reference for this section is [Noo04]. In what follows, we discuss the fundamental group of a connected algebraic stack.

Definition 1.5.1. Let $\mathscr{X}$ be a connected algebraic stack. A morphism $f : \mathscr{Y} \to \mathscr{X}$ is called a *covering map* if $\mathscr{Y}$ is a connected algebraic stack, and f is representable by a finite étale map.

The category of covering maps on $\mathscr{X}$ is denoted by $\operatorname{Cov}(\mathscr{X})$. In addition, if $x : \operatorname{Spec} k \to \mathscr{X}$ is a geometric point, then we associate a functor

$$F_x : \operatorname{Cov}(\mathscr{X}) \to \mathbf{sets}$$

called *fiber functor*, defined as follows: for a covering map $f : \mathscr{Y} \to \mathscr{X}$, we set

$$F_x(f : \mathscr{Y} \to \mathscr{X}) := \left\{ (y, \varphi) \;\middle|\; \begin{array}{l} y : \operatorname{Spec} k \to \mathscr{Y} \text{ is a geometric point,} \\ \varphi : x \xrightarrow{\sim} f(y) \text{ is a morphism in } \mathscr{X}(k) \end{array} \right\} / \sim$$

where two pairs (y, φ) and (y', φ') are considered equivalent if there exists an isomorphism $\beta : y \to y'$ such that $f(\beta) \circ \varphi = \varphi'$. In this context, $f(y)$ denotes the composition $f \circ y$, and $f(\beta) : f(y) \xrightarrow{\sim} f(y')$.

For simplicity, we write $F_x(\mathscr{Y})$ instead of $F_x(f : \mathscr{Y} \to \mathscr{X})$.

The set $F_x(\mathscr{Y})$ is naturally in bijection with the underlying set of the fiber product $\operatorname{Spec}(k) \times_{\mathscr{X}} \mathscr{Y}$.

According to [Noo04, theorem 4.2], the pair $(\operatorname{Cov}(\mathscr{X}), F_x)$ forms a Galois category. This motivates the following definition.

Definition 1.5.2. Let $\mathscr{X}$ be a connected algebraic stack and let $x : \operatorname{Spec} k \to \mathscr{X}$ be a geometric point. We define *the étale fundamental group* $\pi_1^{\text{ét}}(\mathscr{X}, x)$ as the group of self transformations of the functor fiber F_x

Having defined the étale fundamental group of a stack, we now construct a natural homomorphism from the group of k-points of the stabilizer $G_x(k)$, associated with a geometric point $x : \operatorname{Spec}(k) \to \mathscr{X}$, to the étale fundamental group $\pi_1^{\text{ét}}(\mathscr{X}, x)$.

According to Remark 1.4.20, the k-points of the stabilizer group scheme G_x correspond to automorphisms $\gamma \in \operatorname{Mor}_{\mathscr{X}(k)}(x, x)$. Therefore, γ induces a natural self-transformation of the fiber functor F_x as follows:

Given a covering map $f : \mathscr{Y} \to \mathscr{X}$ and a class $\overline{(y, \varphi)} \in F_x(\mathscr{Y})$, we map this class to the new class $\overline{(y, \varphi \circ \gamma^{-1})}$. To verify that this assignment is well-defined, suppose that (y', φ') is another representative of the same class as (y, φ), meaning there exists an isomorphism $\beta : y \to y'$ such that $f(\beta) \circ \varphi = \varphi'$. Then, we have that $f(\beta) \circ \varphi \circ \gamma^{-1} = \varphi' \circ \gamma^{-1}$. showing that $(y, \varphi \circ \gamma^{-1})$ and $(y', \varphi' \circ \gamma^{-1})$ lie in the same equivalence class. Hence, the assignment is well-defined.

This construction yields a natural map:

$$\omega_x : G_x(k) \to \pi_1^{et}(\mathscr{X}, x) \tag{1.1}$$

which sends each $\gamma \in G_x(k)$ to the corresponding automorphism of the fiber functor F_x. Moreover, this map is a group homomorphism, as shown in [Noo04, Section 4].

Example 1.5.3. This example is established in [Noo04, Ex 4.3]. Let X be a simply connected variety over an algebraically closed field k, and let G be an algebraic group of finite type over k, acting trivially on X. Consider the quotient stack $\mathscr{X} = [X/G]$.

Then, the stabilizer group at any geometric point $x : \operatorname{Spec} k \to \mathscr{X}$ is presicely G, since the action is trivial. In this case, Noohi shows that the étale fundamental group of the stack is given by the quotient G/G_0, where G_0 is the connected subgroup of G that contains the identity. Moreover, the homomorphism $\omega_x : G_x(k) \to \pi_1^{et}(\mathscr{X}, x)$ constructed earlier coincides with the natural quotient map $G \to G/G_0$.

1.6 Skeletons associated with toroidal embedding of DM stacks

This section is based on material from [ACP12] and [Thu07]. We describe how compact analytic domains arise from toroidal embeddings, and explain how this concept extends to the setting of toroidal Deligne–Mumford stacks.

Let k be an algebraically closed field equipped with a trivial valuation. The Berkovich analytic space associated with a variety X over k will be denoted by X^{Ber}.

Definition 1.6.1. The space $X^{\beth}$ is the *compact analytic domain* of X^{Ber}, characterized by the property that, for any valued extension $K|k$, its set of K-points consists precisely of those points admitting an extension to Spec K^0.

Remark 1.6.2.
1. Every morphism $f : X \to Y$ of k-varieties induces, in a unique way, a morphism $f^{\beth} : X^{\beth} \to Y^{\beth}$; see [Thu07, Prop. 1.3].
2. If X is proper, then every $K-$point admits a prolongation to K^0. Therefore, $X^{\beth}$ is equal to X^{Ber}; see [Thu07, Rem. 1.4].

If (U, X) is a toroidal embedding, in [Thu07, Cor 3.9], is constructed a continuous idempotent map $\mathbf{p}_X : X^{\beth} \to X^{\beth}$. This map allows us to fix the following definition.

Definition 1.6.3. Let (X, U) be a toroidal embedding. The *skeleton* is defined as the subset $\overline{\Sigma}(X) \subseteq X^{\beth}$ that is the image of the map $\mathbf{p}_X$

Let us consider an example that, at first glance, may appear unrelated to our current discussion. However, once introduced, it will serve as a useful tool for presenting simple examples of skeletons.

Example 1.6.4 ([KKMSD06][Chap. II)**.**] Let (X, U) be a strict toroidal embedding, and let $E_1, \ldots, E_n$ be the irreducible components of $X \smallsetminus U$. Then X admits a canonical stratification, where each stratum is a connected component of a locally closed set of the form

$$\left(\bigcap_{i\in I} E_i\right) \smallsetminus \left(\bigcup_{j\notin I} E_j\right),$$

for some subset $I \subseteq \{1, \ldots, n\}$. Given a stratum Y, define $\mathrm{Star}(Y) \subseteq X$ as the union of all strata whose closures contain Y. With this setup, we define the following objects associated to the stratum Y:

- $M_Y := \{\text{Cartier divisors on } \mathrm{Star}(Y) \text{ supported on } X(Y) \smallsetminus X_0\}$,
- $S_Y \subset M_Y$, the submonoid of effective Cartier divisors,
- $N_Y := \mathrm{Hom}(M_Y, \mathbb{Z})$, the dual lattice,
- $\sigma_Y := \mathrm{Hom}_{\mathrm{Mon}}(S_Y, \mathbb{R}_{\geq 0}) \subseteq N_Y \otimes_{\mathbb{Z}} \mathbb{R}$, the dual rational polyhedral cone.

Whenever a stratum Y' lies in the closure of a stratum Y, the restriction of Cartier divisors induces a morphism $N_{Y'} \hookrightarrow N_Y$, which identifies $\sigma_{Y'}$ with a face of σ_Y. Gluing the cones σ_Y along these face inclusions, we obtain the cone complex $\Sigma(X)$.

Having stated the example, we now turn to the following remark.

Remark 1.6.5. If (X, U) is a strict toroidal embedding, then the image of $U^{Ber} \cap X^{\beth}$ under $\mathbf{p}_X$ is canonically identified with the cone complex $\Sigma(X)$ associated to the toroidal embedding, as constructed in Example 1.6.4. Then $\overline{\Sigma}(X)$ is the closure of $\Sigma(X)$ in $X^{\beth}$, and is canonically identified with the extended cone complex associated to $\Sigma(X)$; see [Thu07, Prop. 3.17].

In [CT07] Brian Conrad and Michell Temkim extend the notion of compact analytic domains to algebraic spaces. This allows [ACP12] to consider the following construction:

Let $\mathscr{X}$ be a separable stack of finite type over an algebraically closed field k, and let $(\mathscr{U}, \mathscr{X})$ be a toroidal DM stack, X the coarse moduli of $\mathscr{X}$, and $P \to \mathscr{X}$ an étale presentation from a scheme P such that $(P \times_{\mathscr{X}} \mathscr{U}, P)$ is a strict toroidal embedding. Then the right-exact diagram

$$P \times_{\mathscr{X}} P \rightrightarrows P \to X$$

induce a right exact diagram

$$(P \times_{\mathscr{X}} P)^{\beth} \rightrightarrows P^{\beth} \to X^{\beth}$$

see [ACP12, lemma 6.1.3]. This construction carry us to define the skeleton for DM stacks in the following sense.

Definition 1.6.6. Let $(\mathscr{X}, \mathscr{U})$ be a toroidal DM stack with X being the coarse moduli of $\mathscr{X}$. Consider $P \to \mathscr{X}$ an étale presentation from a scheme P, such that $(P \times_{\mathscr{X}} \mathscr{U}, P)$ is a strict toroidal embedding. The *skeleton* $\overline{\Sigma}(\mathscr{X})^P \subseteq X^{\beth}$ of the toroidal DM stack associated to the étale presentation is defined as the colimit of the diagram

$$\overline{\Sigma}(P \times_{\mathscr{X}} P) \rightrightarrows \overline{\Sigma}(P) \tag{1.2}$$

Remark 1.6.7.

1. In [ACP12, Proposition 6.1.4] is shown that the skeleton is independent of the choice of étale presentation. Therefore, it is denoted by $\overline{\Sigma}(\mathscr{X})$, and referred to as the *skeleton of the toroidal DM stack.* Moreover, an idempotent retraction map $\mathrm{p}_{\mathscr{X}} : X^{\beth} \to \overline{\Sigma}(\mathscr{X})$ is defined, analogous to the construction in the case of toroidal embeddings.

2. Since we can always take an étale presentation $P \to \mathscr{X}$ such that $(P \times_{\mathscr{X}} \mathscr{U}, P)$ is a strict toroidal embedding, it follows from (1.2) and example 1.6.4 that the skeleton $\overline{\Sigma}(\mathscr{X})$ is a generalized extended cone complex.

3. As in the case of toroidal embeddings, we denote by $\Sigma(\mathscr{X})$ the image under $\mathbf{p}_{\mathscr{X}}$ of the intersection $U^{Ber} \cap X^{\beth}$, where U^{Ber} is the Berkovich analytification of the coarse moduli space U of $\mathscr{U}$.

In what follows, our goal is to describe a decomposition of the skeleton $\Sigma(\mathscr{X})$ in terms of certain quotients of cones by finite group actions. To do so, we begin by fixing some preliminary data.

For each étale morphism from a scheme $V \to \mathscr{X}$, define:

- M_V: the group of Cartier divisors on V supported on the boundary $V \smallsetminus (V \times_{\mathscr{X}} \mathscr{U})$,
- $S_V \subset M_V$: the submonoid of effective Cartier divisors.

Let $\mathcal{M}$ and $\mathcal{S}$ be the étale sheaves on $\mathscr{X}$ associated to the presheaves $V \mapsto M_V$ and $V \mapsto S_V$, respectively.

It is well known that the pair $(\mathscr{X}, \mathscr{U})$ induces a stratification of $\mathscr{X}$; see Lemma 3.2.1 for a detailed proof. Let $\mathscr{W} \subset \mathscr{X}$ be a stratum, and let $w \in \mathscr{W}$ be a geometric point. Consider the stalk $\mathcal{M}_w$ and $\mathcal{S}_w$. The group $\mathcal{M}_w$ is finitely generated and a free abelian group, while $\mathcal{S}_w$ is a sharp[3], saturated, finitely generated submonoid that generates $\mathcal{M}_w$ as a group [ACP12, sec 6.2]. Based on this, consider the dual cone:

$$\sigma_w := \{\ell \in \mathrm{Hom}(\mathcal{M}_w, \mathbb{R}) \mid \ell(s) \geq 0 \text{ for all } s \in \mathcal{S}_w\},$$

which is a strictly convex, full-dimensional, rational polyhedral cone. Since $\mathcal{M}$ is étale-locally constant on each strata of $\mathscr{X}$, the étale fundamental group $\pi_1^{\text{ét}}(\mathscr{W}, w)$ acts on $\mathcal{M}_w$ and preserves $\mathcal{S}_w$. See [ACP12, Sec.6.2]

Definition 1.6.8. The *monodromy group* Γ_w is the image of the étale fundamental group of the stratum W at the geometric point w in the automorphism group of $\mathcal{M}_w$:

$$\Gamma_w := \mathrm{Im}\left(\pi_1^{\text{ét}}(W, w) \to \mathrm{Aut}(\mathcal{M}_w)\right).$$

Fix a stratum $\mathscr{W}$ of $\mathscr{X}$, and w a geometric point of $\mathscr{W}$. We now recall a definition introduced in [ACP12]:

Definition 1.6.9. Let $(\mathscr{X}, \mathscr{U})$ be a toroidal embedding of DM stack, and w a geometric point. A *small toric chart* around the geometric point is an étale morphism $V \to \mathscr{X}$ satisfying

1. The toroidal embedding $(V \times_{\mathscr{X}} \mathscr{U}, V)$ is strict.
2. There exists a unique minimal stratum W of V.
3. The image of W contains the geometric point.

From the above definition, it follows that the cone complex associated with the small toric chart V is precisely a cone, which we denote by σ_V. Moreover, σ_V is canonically identified with σ_w.

Having these descriptions, one of the main results in [ACP12] is the following.

Theorem 1.6.10. *Let $\mathscr{W}_1, \ldots, \mathscr{W}_n$ be the strata of a toroidal Deligne-Mumford stack $(\mathscr{X}, \mathscr{U})$, and $\mathcal{M}$ the sheaf defined above. For $i = 1, \ldots, n$, let w_i be a geometric point of $\mathscr{W}_i$, σ_i the dual cone of $\mathcal{M}_{w_i}$, $\mathring{\sigma}_i$ the relative interior of σ_i, and Γ_i the monodromy group. Then we have a natural decomposition*

$$\Sigma(\mathscr{X}, \mathscr{U}) = \coprod_i \mathring{\sigma}_i/\Gamma_i, \quad \text{and} \quad \overline{\Sigma}(\mathscr{X}, \mathscr{U}) = \coprod_i \overset{\circ}{\overline{\sigma}}_i/\Gamma_i$$

1.7 Tropicalization of the moduli of curves

In this section we define the moduli space of tropical curves, called tropical moduli. We will show an explicit stratification of this tropical moduli in terms of non-isomorphic stable graphs of genus g and n legs. Finally, we will relate this description with the skeleton of the toroidal Deligne-Mumford stack $(\overline{\mathscr{M}}_{g,n}, \mathscr{M}_{g,n})$. For more details on this section, we refer to [AHCG11, chap. XII], [ACP12, sec. 3 and 4] and [Cap13].

[3] A monoid is said to be sharp if its only invertible element is the identity.

Definition 1.7.1. A *weighted n-marked graph* $\mathbf{G}$ is a tuple (G, g, m) where

1. G is a connected graph.
2. $g : V \to \mathbb{Z}_{\geq 0}$ is any function from the vertex of the graph to non negative integers, called the *weight function*.
3. $m : [n] \to V$ is any function called n-marking, where $[n]$ denotes the set $\{1, 2, \ldots, n\}$.

The *genus* of a weighted n-marked graph is defined as $g = |E| - |V| + 1 + \sum_v g(v)$.

Denote by L the set $\{(i, v) : m(i) = v, \quad i \in [n]\}$. For simplicity, we write v_i to refer to the pair (i, v), to refer to the pair L *legs*. Thus, we have a map $c : L \to V$. which assigns each leg to the vertex to which it is attached.

Forgetting the weight for a moment, a *morphism* $\alpha : \mathbf{G} \to \mathbf{G}'$ of marked graphs consists of a map $\alpha_L : L \to L'$ between the respective sets of legs, and a map $\alpha_{VE} : V \cup E \to V' \cup E'$, such that $\alpha_{VE}(V) \subseteq V'$ and for every edge $e \in E$, either $\alpha_{VE}(e) \in E'$ and the image of the endpoints of e are the endpoints of $\alpha_{VE}(e)$, or $\alpha_{VE}(e) \in V'$, in which case the edge is contracted to a vertex. Moreover, these maps are required to make the following diagram commute:

$$\begin{array}{ccc} L & \xrightarrow{\alpha_L} & L' \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\ V & \xrightarrow[\alpha_{VE}]{} & V' \end{array}$$

We say that a morphims α is an *isomorphism* if α induces, by restriction, three bijections:

$$\alpha_L : L \to L', \quad \alpha_V : V \to V', \quad \alpha_E : E \to E'.$$

An edge *contraction* in a graph is an operation that removes a chosen edge e connecting two distinct vertices u and v, and identifies these two vertices as a single new vertex w. After this identification, all edges previously incident to either u or v become incident to w, except for the contracted edge e, which is deleted. In this process, some edges that formerly connected u and v may become loops at w.

A *weighted graph contraction* $\pi : \mathbf{G} \to \mathbf{G}'$ is then defined as a contraction of the underlying unweighted graph obtained by a finite sequence of edge contractions, together with a canonically induced weight function on the contracted graph $\mathbf{G}'$.

The weight function g' on $\mathbf{G}'$ is determined as follows: for each vertex v' in $\mathbf{G}'$, we consider the subgraph $\pi^{-1}(v') \subseteq \mathbf{G}$ that contracts on v'. The new weight is given by

$$g'(v') := g(\pi^{-1}(v')),$$

where $\pi^{-1}(v')$ denotes the genus of the subgraph $\mathbf{G}$.

An *automorphism* of the weighted n-marked graph $\mathbf{G}$ is an isomorphim of the underlyig unweighted graph to itself such that the weight is preserve; that is, $g(v) = g(\alpha_V(v))$.

Definition 1.7.2. 1. A weighted n-marked graph $\mathbf{G}$ is called *stable* if for every vertex v, we have $2g(v) - 2 + n(v) > 0$. where $n(v)$ denotes the cardinality of the preimage of v under the marking function m.

2. A *metric graph* is a graph G together with a real leght at each edge. That is, a function $\ell : E \to \mathbb{R}_{>0}$.

3. A *tropical curve* is a stable weighted graph $\mathbf{G}$ endowed with such a metric structure, i.e., it is a stable weighted graph that is metric.

Note that contractions preserve the total genus of the graph. Moreover, any contraction of a stable weighted graph yields another stable weighted graph.

We say that two tropical curves $(\mathbf{G}, \ell)$ and $(\mathbf{G}', \ell')$, are *isomorphic* if there exists an isomorphism $\alpha : \mathbf{G} \to \mathbf{G}'$ of their underlying unweighted graphs such that

$$\ell(e) = \ell'(\alpha(e)) \quad \text{for all } e \in E, \quad \text{and} \quad g(v) = g'(\alpha(v)) \quad \text{for all } v \in V$$

Let g and n be non-negative integers satisfying $2g - 2 + n > 0$. Fix a stable weighted graph $\mathbf{G}$ of genus g and n marked points. By allowing the edge lengths to vary over all positive real numbers, we obtain all tropical curves of combinatorial type $\mathbf{G}$, where by combinatorial type we mean a tropical curve together with an identification of its underlying graph with $\mathbf{G}$. The moduli space of such tropical curves can be naturally identified with the open cone

$$\sigma^{\circ}_{\mathbf{G}} := \mathbb{R}^{|E|}_{>0}$$

If we further restrict to tropical curves of combinatorial type $\mathbf{G}$ considered up to isomorphisms, we must take the quotient by the automorphism group $\mathrm{Aut}(\mathbf{G})$. The corresponding parameter space then becomes

$$\sigma^{\circ}_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})$$

If we also allow edge lengths to be zero, interpreting this as contracting subsets of edges in $\mathbf{G}$, we obtain new tropical curves whose combinatorial types are contractions of $\mathbf{G}$. These degenerations are parametrized by the cone

$$\sigma_{\mathbf{G}} := \mathbb{R}^{|E|}_{\geq 0}.$$

If $\pi : \mathbf{G} \to \mathbf{G}'$ is a weighted contraction, there is a canonical inclusion $i_\pi : \sigma_{\mathbf{G}'} \hookrightarrow \sigma_{\mathbf{G}}$, which identifies $\sigma_{\mathbf{G}'}$ with the face of $\sigma_{\mathbf{G}'}$ where all edges contracted by π are assigned length zero.

This process leads naturally to the construction of the *moduli space of tropical curves* of genus g and n marked points.

Definition 1.7.3. As a topological space, the *coarse moduli space of tropical curves* $M^{\mathrm{trop}}_{g,n}$ is defined as the generalized cone complex

$$M^{\mathrm{trop}}_{g,n} := \varinjlim_{(\mathbf{G},\pi)} \sigma_{\mathbf{G}}$$

where the limit is taken over the category of stable weighted n-marked graphs of genus g, with morphisms given by weighted contractions and isomorphisms.

In the same way, we can consider the extended cone $\overline{\sigma}_{\mathbf{G}} := (\mathbb{R}_{\geq 0} \cup \infty)^{|E|}$, which is called the parameter space of the *extended tropical curves* with graph equal to $\mathbf{G}$.

Definition 1.7.4. The *coarse moduli space of extended tropical curves* $\overline{M}_{g,n}^{\mathrm{trop}}$ is the generalized extended cone complex

$$\overline{M}_{g,n}^{\mathrm{trop}} := \varinjlim_{(\mathbf{G},\pi)} \overline{\sigma}_{\mathbf{G}}$$

In the definition 1.7.3 (and similarly for Definition 1.7.4), oberve that every automorphism of a weighted graph $\mathbf{G}$ is, in particular, a weighted contraction. Therefore, the natural map from $\sigma_{\mathbf{G}}$ to the colimit $M_{g,n}^{\mathrm{trop}}$ factors through the quotient $\sigma_{\mathbf{G}} \to \sigma_{\mathbf{G}}/\mathrm{Aut}(\mathbf{G})$. Moreover, two points in the colimit are identified if and only if they are the images of two points in some open cone $\sigma_{\mathbf{G}}^{\circ}$ that differ by an automorphism of $\mathbf{G}$. This leads to the following proposition.

Proposition 1.7.5.

1. Let g, n integers as above. Then, the coarse moduli space of tropical curves $M_{g,n}^{\mathrm{trop}}$ decomposes as a disjoint union

$$\coprod_{\mathbf{G}} \sigma_{\mathbf{G}}^{\circ}/\mathrm{Aut}(\mathbf{G})$$

   where the disjoint union is taken over isomorphism classes of stable weighted graphs of genus g with n marked points.

2. Similarly, the extended moduli space $\overline{M}_{g,n}^{\mathrm{trop}}$ decomposes as a disjoint union

$$\coprod_{\mathbf{G}} \overline{\sigma}_{\mathbf{G}}^{\circ}/\mathrm{Aut}(\mathbf{G})$$

   where $\overline{\sigma}_{\mathbf{G}}^{\circ} := (\mathbb{R}_{>0} \cup \infty)^{|E|}$, with E being the set of edges of $\mathbf{G}$.

Due to the work in [Cap13], we have an alternative description of the (extended) moduli space of tropical curves, which we summarize here.

Let $\mathbf{G}$ be an n-marked graph, where the weight function is zero in each vertex, and let v a vertex of $\mathbf{G}$. The *valence* of v is defined as the number of incident edges plus the number of incident legs. Note that a loop is counted twice.

We say that $\mathbf{G}$ is 3-*regular* if every vertex has valence equal to 3.

Based on this, the moduli space admits the following description:

$$M_{g,n}^{\mathrm{trop}} := \left(\coprod_{\mathbf{G}\ 3\text{-regular}} \sigma_{\mathbf{G}} \right) \Big/ \cong \tag{1.3}$$

Here, the disjoint union is taken over the set of 3-regular graphs of genus g and n marked points, and with weight function equal to zero in each vertex. The equivalence relation $\cong$ identifies isomorphism classes of tropical curves.

Definition 1.7.6. Let $p \in \overline{\mathscr{M}}_{g,n}$ be a geometric point which represents a n-pointed curve $(C; p_1, \ldots, p_n)$ with at most nodes in its singularities. The *weighted dual graph* $\mathbf{G}_C$ associated to the curve is a stable weighted graph described as:

1. The vertices correspond to irreducible components of C, and it is labeled with the geometric genus of the irreducible component.

2. The set of edges corresponds to the nodes of the curve C. Here an edge is incident to two vertices if the correspondient node lies in the intersection of the two irreducible components.

3. The marked points correspond to the marked points of the curve, where a marked point is associated to a vertex if the marked point of the curve lies in the component associated with the vertex.

Having introduced the notion of skeletons for toroidal embeddings of DM stacks in Section 1.6, we now identify the skeleton of the toroidal embedding $(\overline{\mathscr{M}}_{g,n}, \mathscr{M}_{g,n})$ with the coarse moduli space of tropical curves.

Fixing a geometric point $x \in \overline{\mathscr{M}}_{g,n}$, which corresponds to a stable curve C, there exists an étale neighborhood $V \to \overline{\mathscr{M}}_{g,n}$ that parametrizes deformations of C, where a node that persists in the deformation corresponds to a smooth and irreducible divisor. Moreover, the collection of all divisors corresponding to the nodes of C is precisely the normal crossing divisor of the toroidal embedding associated to V, see [AHCG11, Chap 11- Thm 3.17]. If $\mathbf{G}_C$ is the dual graph of C, then the dual graph of any curve in the family parametrized by V is a contraction of $\mathbf{G}_C$.

Shirinking V if necessary, we can consider $V \to \overline{\mathscr{M}}_{g,n}$ a small toric chart for the geometric point x. Then the skeleton $\overline{\Sigma}(V)$ is equal to $\overline{\sigma}_{\mathbf{G}_C}$. By Theorem 1.6.10, we see that the image of $\overline{\sigma}^{\circ}_{\mathbf{G}_C}$ in $\overline{\Sigma}(\overline{\mathscr{M}}_{g,n})$ is the quotient of $\overline{\sigma}^{\circ}_{G}$ by the monodromy group H_x and by [ACP12, Proposition 7.2.1] this monodromy group is isomorphic to $\mathrm{Aut}(\mathbf{G}_C)$.

As a consequence of this description and by Theorem 1.6.10, we have

Corollary 1.7.7. The skeleton $\overline{\Sigma}(\overline{\mathscr{M}}_{g,n})$ is decomposed as $\coprod_{\mathbf{G}} \overline{\sigma}^{\circ}_{\mathbf{G}/\mathrm{Aut}(\mathbf{G})}$. Where $\mathbf{G}$ ranges over isomorphism classes of stable graphs of genus g and n marked points. TThis yields a natural bijection between $\overline{\Sigma}(\overline{\mathscr{M}}_{g,n})$ v $\overline{M}^{\mathrm{trop}}_{g,n}$. Moreover, this bijection is in fact an isomorphism of generalized extended cone complexes.

Proof. [ACP12][Cor 7.2.2] □

Chapter 2

Green functions and hybrid spaces

In this chapter, we construct a hybrid space associated with the pair (X, D), where X is a complex manifold of dimension n, and D is a simple normal crossing divisor (see 1.2.1).

To achieve this construction, we first show how to associate a cone complex with the stratification determined by the pair (X, D). Additionally, we introduce the notion of a Green function of continuous type, which plays a fundamental role in defining the hybrid space.

Finally, we focus on the case where X is compact, as this will allow us to provide a precise description of the hybrid space. Moreover, compactness will allow us to define a continuous family of suitable measures in the hybrid space.

2.1 Polyhedral spaces and Green functions

Let $D = \sum_{i\in I} D_i$ be a simple normal crossing divisor on a complex manifold X. To the pair (X, D), we associate a *cone complex*, denoted by $\Sigma(X, D)$, as follows:

For each stratum Y, which is a connected component of $\bigcap_{j\in J} D_j \smallsetminus \bigcup_{j\notin J} D_j$, for some $J \subseteq I$, we associate the cone

$$\sigma_Y := \mathbb{R}^{|J|}_{\geq 0}$$

Note, for example, that the stratum $X \smallsetminus D$ is determined by a set of indices $J = \emptyset$, so $X \smallsetminus D$ has associated the 0-dimensional cone $\{0\}$. Once the cones are defined, we establish the faces relations as follows. $\sigma_{Y'}$ will be identified as a face of σ_Y if only if $Y \subseteq \overline{Y'}$. The underlying topological space of $\Sigma(X, D)$ is defined by gluing all cones σ_Y along their common faces. We also denote this topological space by $\Sigma(X, D)$.

Example 2.1.1. Left part of the Figure 2.1 show a divisor on a surface, with two irreducible components. Here, we have 5 strata: $D_1 \smallsetminus \{Y_1, Y_2\}$ and $D_2 \smallsetminus \{Y_1, Y_2\}$, assume that these two are connected, the two points Y_1 and Y_2, and $X \smallsetminus D$. On the right, we have its associated cone complex, where the two $2-$dimesional cones glued along two rays.

Remark 2.1.2. If X is a regular variety over $\mathbb{C}$ and $U \subseteq X$ is a strict toroidal embedding (see 1.2.4) then the complex cone associated to the toroidal embedding is canonically identified with the cone complex that is associated with the simple normal crossing divisor $X^{an} \smallsetminus U^{an}$. Here X^{an} is the complex analytification

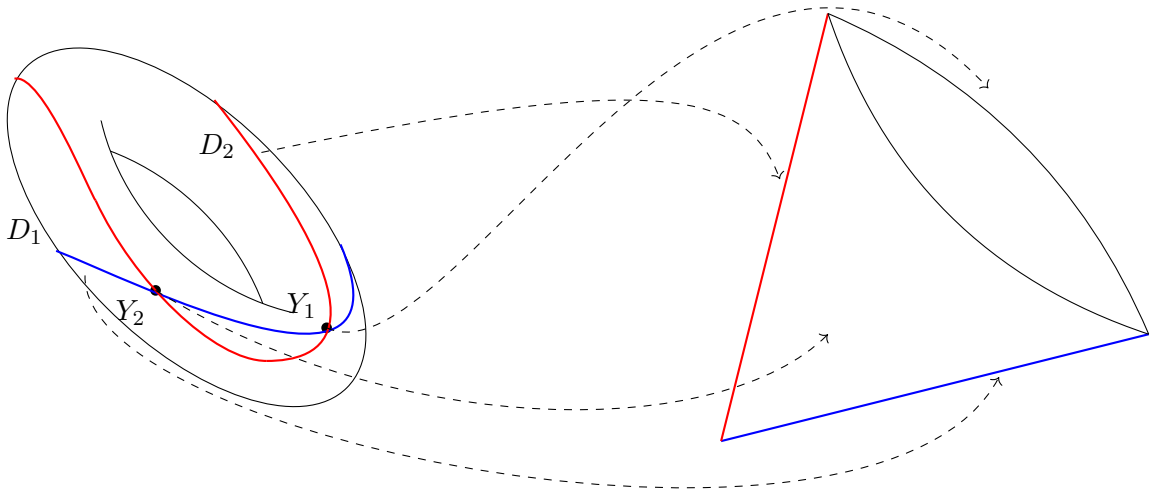


Figure 2.1: Cone complex

of X, which in this case is a smooth complex manifold. Moreover, by Remark 1.6.5, this cone complex is canonically identified with $\Sigma(X) := U^{\mathrm{Ber}} \cap \overline{\Sigma}(X)$, where $\overline{\Sigma}(X)$ is the skeleton of X; see Definition 1.6.3.

Definition 2.1.3. Let D be a Cartier divisor on a complex manifold X. A *Green function of continuous type* for D, is a continuous function $\mathfrak{g} : X \smallsetminus |D| \to \mathbb{R}$ such that for every $x \in X$ there is an open neighborhood U_x of x, and a local equation $z \in \mathcal{O}_{X,x}$ defining the Cartier divisor D on U_x, such that the map $\log|z| + \mathfrak{g}$ extends to a continuous function on U_x.

A *positive Green function* for D is a Green function $\mathfrak{g}$ of continuous type such that $\mathfrak{g}(x) \geq 0$ for all $x \in X \smallsetminus |D|$.

Example 2.1.4. If $X = \mathbb{C}$ and $D = \{0\}$, then $-\log|z|$, and $\max\{-\log|z|, \log|z|\}$ are Green functions for the divisor D. Note that only the second function is a positive Green function.

Remark 2.1.5.

- If $\mathfrak{g}, \mathfrak{g}'$ are two Green functions for D, then $\mathfrak{g} - \mathfrak{g}'$ can be extended to a continuous function on X. In particular, if X is compact, then $\mathfrak{g} - \mathfrak{g}'$ is bounded.
- If $D = \sum_{i\in I} a_i D_i$ is a divisor on X, with every $a_i \in \mathbb{Z}$. For each $i \in I$, $\mathfrak{g}_i$ is a Green function for D_i, then $\mathfrak{g}_D := \sum_{i\in I} a_i \mathfrak{g}_i$ is a Green function for D.
- If $\{\mathfrak{g}_i\}_{i=1}^m$ is a collection of Green functions for a divisor D, and $\{\chi_i\}_{i=1}^m$ a collection of continuous functions in X satisfying $\sum_{i=1}^m \chi_i = 1$, then $\sum_{i=1}^n \chi_i \mathfrak{g}_i$ is a Green function for D.

The second remark in 2.1.5 provides a way to extend the definition of a Green function of continuous type to $\mathbb{R}$-Cartier divisors.

Definition 2.1.6. Let $D = \sum_{i\in I} D_i$ be a Cartier divisor and let $E = \sum_{i\in I} a_i D_i$ be an $\mathbb{R}$-Cartier divisor, with $a_i \in \mathbb{R}$ for $i \in I$. A Green function for the $\mathbb{R}$-divisor E is a function of the form

$$\sum_{i\in I} a_i \mathfrak{g}_i$$

defined on $X \smallsetminus |E|$, where $\{\mathfrak{g}_i\}_{i\in I}$ is a family of Green functions, one for each Cartier divisor D_i.

We will say that $\sum_{i\in I} a_i \mathfrak{g}_i$ is a positive Green function for the $\mathbb{R}$-divisor E if it takes positive values.

Remark 2.1.7. From the above definition, each summand $a_i\mathfrak{g}_i$ in the positive Green function $\sum_{i\in I} a_i\mathfrak{g}_i$, which is associates to the $\mathbb{R}$-Cartier divisor E, is itself positive. Indeed, for each irreducible component D_i of D, we can choose a point $x \in D_i \smallsetminus \bigcup_{j\neq i} D_j$ and an open neighborhood V of x such that $V \cap D_j = \emptyset$ for each $j \neq i$. In this neighborhood, the functions $\mathfrak{g}_j$ for $j \neq i$ are defined and bounded in V, while $\mathfrak{g}_i$ remains unbounded. Since the total function $\sum_{i\in I} a_i\mathfrak{g}_i$ takes positive values, it follows that $a_i\mathfrak{g}_i$ must also be positive in V.

Definition 2.1.8. Let X be a complex manifold, $D = \sum_{i\in I} D_i$ a simple normal crossing divisor, and let $(U, z = (z_1, \ldots, z_n))$ be an open connected chart. We say that (U, z) is an *adapted chart* if the following conditions hold:

1. If $J = \{j \in I | D_j \cap U \neq \emptyset\}$, then $U \cap D_J$ is nonempty and equal to the intersection of U with some stratum Y of $D_J := \bigcap_{j\in J} D_j \smallsetminus \bigcup_{j\notin J} D_j$.
2. For every $j \in J$ there exists $i(j) \in \{1, \ldots, n\}$ such that $z_{i(j)}$ is a local equation of $U \cap D_j$.
3. $|z_{i(j)}| < 1$ for every $j \in J$.

Remark 2.1.9. Note that in Definition 1.2.14, we introduced the notion of an open set adapted to the stratification, or simply an adapted open, meaning that the set of strata intersecting the open set nontrivially has a unique minimal stratum. Every adapted chart is, in particular, an adapted open. However, the converse does not necessarily hold. Even if the adapted open U is a coordinate chart, this does not guarantee that it is an adapted chart. This is because, although U is equipped with local coordinates z_i, these coordinates do not necessarily define the irreducible components of the divisor on U.

Green functions are determined by local conditions. Since we are working on a complex manifold, we can construct local Green functions around the support of a simple normal crossing divisor and then patch them together appropriately to obtain a global Green function. This process will be illustrated in the following example.

Example 2.1.10. Let D_i be an irreducible component of a simple normal crossing divisor D, and let $\{(U_\alpha, z_\alpha)\}_{\alpha\in\Lambda}$ be a locally finite covering of X by adapted charts. For each chart (U_α, z_α) such that $U \cap D_i \neq \emptyset$, denote by z_{α_i} the coordinate whose zero locus coincides with $U \cap D_i$. Define a collection of local functions $\{f_\alpha\}$ as follows: If $U_\alpha \cap D_i \neq \emptyset$, set $f_\alpha = -\log|z_{\alpha_i}|$. Otherwise, set $f_\alpha = 0$. Consider $\{\chi_\alpha\}_{\alpha\in\Lambda}$ a partition of unity subordinate to the covering $\{U_\alpha\}_{\alpha\in\Lambda}$, and define the function

$$\log_{D_i} := \sum_{\alpha\in\Lambda} \chi_\alpha f_\alpha$$

Then $\log_{D_i}$ is a positive Green function associated to D_i.

2.2 Construction of the hybrid space

Given D a simple normal crossing divisor on a complex manifold X, the aim of this section is to construct a hybrid space associated with this pair. As mentioned earlier, this construction is inspired by the hybrid space introduced by Boucksom and Jonsson in [BJ17], which, in turn, was influenced by the work of Morgan and Shalen [MS84].

Let $D = \sum_{i\in I} D_i$ be a simple normal crossing divisor, $\{\mathfrak{g}_i\}_{i\in I}$ be a family of positive Green functions, where each $\mathfrak{g}_i$ is a Green function for the divisor D_i. For an adapted open set U (see Definition 1.2.14), denote by Y_U the stratum of X associated with this open set. Since strata are in correspondence with the cones of $\Sigma(X, D)$, denote by σ_U the cone associated with Y_U; we will refer to σ_U as the cone associated to the adapted open set U. With these conventions, for each adapted open set U, we associate the following function

$$F_U : U \smallsetminus D \to \sigma_U, \quad x \mapsto (\mathfrak{g}_j(x))_{j\in J} \tag{2.1}$$

Where $J \subseteq I$ is the subset of indices of the irreducible components that have nonempty intersection with the adapted open set U. Observe that in the case where $U \cap D = \emptyset$, the function F_U is identically zero, since σ_U is a point.

As each cone σ in $\Sigma(X, D)$ is identified with a subset of $\mathbb{R}^n$, for some $n \in \mathbb{N}$, we say that a function $\phi : Y \to \sigma$, from any set Y, is a *bounded function* if there exits $r > 0$ such that $\varphi(Y) \subseteq \sigma \cap B(0, r)$, where $B(0, r)$ is the open ball of radious r centered at the origin in $\mathbb{R}^n$.

Lemma 2.2.1. Let U, U' be two adapted open sets with $U' \subseteq U$, and let $F_U, F_{U'}$ be defined by the family of positive Green functions $\{\mathfrak{g}_i\}_{i\in I}$. Then for every point $x \in U'$ there exists an open neighborhood V of x contained in U' such that at $V \smallsetminus D$

$$F_U = F_{U'} + \phi$$

with ϕ being a bounded function defined on V.

Proof. Note that the stratum Y_U is contained in the closure of the stratum $Y_{U'}$. Therefore, we can identify $\sigma_{U'}$ as a face of σ_U. Following Definition 2.1.8, let J and J' be the subset of indices associated with the charts U and U', respectively. Then we have $J' \subseteq J$.

For every $x \in U'$ there exists an open neighborhood $V \subseteq U'$ of x, such that $\mathfrak{g}_j$ is defined and bounded on V for all $j \in J \smallsetminus J'$. This condition ensure that

$$F_U = F_{U'} + \phi$$

where $\phi : V \to \sigma_U$ is the bounded function whose j-th component is equal to $\mathfrak{g}_j$ for $j \in J \smallsetminus J'$, and zero for all other components. □

Lemma 2.2.2. Let U be an adapted open set, and let F_U, F'_U be defined by the family of Green functions $\{\mathfrak{g}_i\}_{i\in I}$ and $\{\mathfrak{g}'_i\}_{i\in I}$, respectively. Then for every point $x \in U$ there exists an open neighborhood V of x contained in U such that at $V \smallsetminus D$

$$F_U = F'_U + \phi$$

with ϕ a bounded function defined in V

Proof. Let $x \in U$, and let Y_x denote the stratum that contains the point x. Define J_x as the set of indices

corresponding to the irreducible components of D that contain the point x, and let J be the set of indices corresponding to the irreducible components of D that have a nonempty intersection with U. Then:

- For each $j \in J_x$ there exists an open neighborhood U_j of x such that $\mathfrak{g}_j - \mathfrak{g}'_j$ is defined and bounded in U_j.
- For each $j \in J \setminus J_x$ there exists an open neighborhood U_j of x such that $\mathfrak{g}_j$ and $\mathfrak{g}'_j$ are defined and bounded in U_j.

Taking V an open set that contains the point x, and is contained in $\bigcap_{j\in J} U_j$ we have at $V \setminus D$ that

$$F_U = F'_U + \phi$$

with ϕ a continuous bounded map defined in V. □

Proposition 2.2.3. Given a complex manifold X, together with a simple normal crossing divisor D, we associate its cone complex $\Sigma(X, D)$. Then there exists a continuous map $F : X \setminus D \to \Sigma(X, D)$ with the following property: For each stratum $Y \subseteq X$, each point $x \in Y$, every adapted open set U containing x with Y as its associated stratum, and every function F_U defined by a collection $\{\mathfrak{g}'_i\}_{i\in I}$, there exists an open set V of x contained in U such that on $V \setminus D$ we have

$$F = F_U + \phi$$

Where ϕ is a bounded function defined in V.

Proof. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a locally finite covering of X by adapted open set satisfying the conditions of Proposition 1.2.13. Let $\{\chi_\alpha\}_{\alpha\in\Lambda}$ be a partition unity subordinate to this covering, and let $\{\mathfrak{g}_i\}_{i\in I}$ be a collection of positive Green functions, with $\mathfrak{g}_i$ being a Green function of the irreducible component D_i. For every $x \in X$, define the sets

$$A_x = \{\alpha \in \Lambda : x \in U_\alpha\},\ B_x = \{\beta \in \Lambda : x \notin \operatorname{Supp}(\chi_\beta)\}$$

and consider the open set

$$W_x := \left(\bigcap_{\alpha\in A_x} U_\alpha\right) \setminus \bigcup_{\beta\in B_x} \operatorname{Supp}(\chi_\beta) \tag{2.2}$$

Since the covering satisfies the conditions of Proposition 1.2.13, the minimal strata associated to the elements of A_x are pairwise comparable. Therefore, there exists a unique minimal stratum. We denote the cone associated to this stratum by σ_{W_x}. Note that $W_x \cap \operatorname{Supp}(\chi_\alpha) \neq \emptyset$ if only if $x \in \operatorname{Supp}(\chi_\alpha)$. This would imply that σ_{U_α} is a face of σ_{W_x}. Therefore, the function

$$\sum_{\alpha\in\Lambda} \chi_\alpha F_{U_\alpha}$$

is well defined in $W_x \setminus D$, takes values in σ_{W_x}, and is continuous.

Now, if $y \in X$ is another point such that $W_y \cap W_x \neq \emptyset$. Then on the intersection $W_x \cap W_x$ this construction assigns values in both σ_{W_x} and σ_{W_y}. However, since this intersection $W_x \cap W_y$ is nonempty, it follows that the corresponding adapted open sets U_α, with $\alpha \in A_x \cup A_y$ have pairwise nonempty intersections. By the definition of adapted open sets, the associated strata are pairwise comparable, and hence the cones σ_{W_x}

and σ_{W_y} are comparable in the sense of one is a face of the other. As a consequence of this compatibility, the local functions glue together to define a global continuous function

$$F : X \smallsetminus D \to \Sigma(X, D)$$

Now, let us to see that F satisifes the conditions of the statement. Indeed, let $Y \subseteq X$ be a stratum, $x \in Y$, let U be an adapted open set that contains x, with Y as its associated stratum, and $\{\mathfrak{g}'_i\}_{i\in I}$ a collection of positive Green functions, which in turn defines the function F'_U. Let W_x be the open set defined in (2.2). Observe that for each U_α containing x, we have Y_{U_α} is contained in the closure of Y. Since the covering is locally finite, applying the lemmas 2.2.1 and 2.2.2, it follows that for each $\alpha \in A_x$ there exists V_α containing the point x, and contained in $W_x \cap U$ such that on $V_\alpha \smallsetminus D$ we have

$$F_{U_\alpha} = F'_U + \phi_\alpha$$

with ϕ_α being a bounded function defined in V_α. Finally, taking $V = \bigcap_{\alpha\in A_x} V_\alpha$, we have that at V

$$\sum_{\alpha\in\Lambda} \chi_\alpha F_{U_\alpha} = \sum_{\alpha\in\Lambda} \chi_\alpha (F_U + \phi_\alpha) = F_U + \phi$$

where $\phi := \sum_{\alpha\in\Lambda} \chi_\alpha\phi_\alpha$ is a bounded function defined on V. Therefore, F satifies the conditions of the statement. □

Given a pair (X, D), denote by X^{hyb}_D the set

$$(X \smallsetminus D) \times (0, 1] \coprod \Sigma(X, D) \times \{0\}$$

Let F be a function that satisfies the conditions of Proposition 2.2.3, then consider the function $G : X^{hyb}_D \to \Sigma(X, D)$ defined as:

$$G(z, t) = \begin{cases} tF(z) & \text{if} \quad t \neq 0 \\ \\ z & \text{if} \quad t = 0 \end{cases} \tag{2.3}$$

Definition 2.2.4. The *hybrid topology* is the coarsest topology on X^{hyb}_D such that:

1. $(X \smallsetminus D) \times (0, 1] \hookrightarrow X^{hyb}_D$ is an open inmmersion.
2. The projection $\pi : X^{hyb}_D \to [0, 1]$ is continuous.
3. The function G defined in (2.3) is continuous.

Note that this topology depends on the choice of the function G, and more specifically, on the function F. Different choices of G may give us different topologies, as we will see in the following example.

Example 2.2.5. Let $X = \mathbb{C}$, and let $D = 0$. Consider the following two positive Green functions $\mathfrak{g}_1$ and $\mathfrak{g}_2$ associated with D. The function $\mathfrak{g}_1(x)$ is given by $\max\{-\log|z|, 0\}$, while the function $\mathfrak{g}_2$ is given by $\max\{-\log|z|, \log|z|\}$. Denote by τ_1 and τ_2 the topologies determined by $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. Consider the sequence $\{(e^n, \frac{1}{n})\}_{n\in\mathbb{N}}$. Denote by G_1 and G_2 the functions determined by $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as in (2.3) respectively. The image of the elements of the sequence under G_1 are zero. This implies that the sequence converges to zero with respect to the topology induced by G_1.

Note that the image of the same sequence under G_2 is constantly equal to 1. Hence, the sequence converges to 1 in the topology induced by G_2. This show that the topologies defined by G_1 and G_2 are distinct, since the same sequence has different limits.

Note that in the example above, we used sequences without first verifying that the space is sequential. However, this is justified by the fact that X_D^{hyb} is second countable. In fact, a countable base for a point in the zero fiber $\pi^{-1}(\{0\})$ can be obtained by taking inverse images of a countable collection of open sets in $\Sigma(X, D)$, were clearly this set is second countable. On the other hand, the space $(X \setminus D) \times (0, 1]$ is clearly second countable. Therefore, the space X_D^{hyb} is second countable and, in particular, sequential.

Remark 2.2.6. Since X_D^{hyb} is a sequential space, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_D^{hyb} converges to a point x if only if, for some (and therefore all) G, the sequence $\{G(x_n)\}_{n\in\mathbb{N}}$ converges to $G(x)$, and the sequence $\{\pi(x_n)\}_{n\in\mathbb{N}}$ converges to $\pi(x)$.

We now state a proposition that provides sufficient conditions to ensure that the topology does not depend on the choice of function G.

Proposition 2.2.7. Let X be a compact complex manifold. Then the topology defined by G and G' coincides.

Proof. Let τ, τ' be the topologies determined by G and G' respectively. We will prove that a sequence converges in one topology if and only if it converges in the other topology, and that the limits coincide. It suffices to verify one direction of the equivalence, and to consider the case of a sequence $\{(x_n, t_n)\}_{n\in\mathbb{N}}$, with $t_n > 0$ for all n, and $\lim_{n\to\infty} t_n = 0$, as the remaining cases are clear. Assume that this sequence converges to the point $(x, 0)$ with respect to the topology τ. From the sequence above, consider the sequence $\{x_n\}_{n\in\mathbb{N}}$. Note that each element of this sequence is in $X \setminus D$. Since X is compact, it follows that there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ that converges to an element $x_0 \in X$. Let Y_{x_0} be the stratum containing the point x_0, U an adapted open set that contains the point x_0 and with Y_{x_0} as its associated stratum. Let $F_U : U \setminus D \to \sigma_U$ be defined as in (2.1). As the functions G and G' are determined by the functions F and F' that satisfy the conditions of the Proposition 2.2.3. Then by Lemmas 2.2.1 and 2.2.2 we have that there exists an open neighborhood V of x_0 contained in U such that at $V \setminus D$

$$F = F_U + \phi \quad F' = F_U + \phi'$$

where ϕ and ϕ' are bounded functions defined in V. Since there exists $k_0 \in \mathbb{N}$ such that $x_{n_k} \in V$ for each $k \geq k_0$, it follows that

$$\begin{aligned} x &= \lim_{k\to\infty} t_{n_k} F(x_{n_k}) \\ &= \lim_{k\to\infty} t_{n_k} (F(x_{n_k}) + \phi(x_{n_k})) \\ &= \lim_{k\to\infty} t_{n_k} F_U(x_{n_k}) \\ &= \lim_{k\to\infty} t_{n_k} F'(x_{n_k}) \end{aligned}$$

Since this limit exits, it follows from Remark 2.2.6 that the subsequence $\{(x_{n_k}, t_{n_k})\}_{n\in\mathbb{N}}$ converges to $(x, 0)$ with respect the topology τ'.

What we have proved is that for every convergent sequence in the topology τ, there exists a subsequence that converges in the topolgy τ', and with the same limit. Therefore, we can conclude that the sequence $\{(x_n, t_n)\}_{n\in\mathbb{N}}$ converges to $(x, 0)$ in the topology τ'. □

We now present an example to illustrate the hybrid topology. In particular, we will compare it with the product topology to analyze how sequences behave and how convergence differs in the two settings.

Example 2.2.8. Let $X = \mathbb{P}^1_{\mathbb{C}}$, and $D = \{0\}$. By Proposition 2.2.7, we can say that the hybrid topology on $\left(\mathbb{P}^1_{\mathbb{C}} \setminus \{0\}\right) \times (0,1] \coprod \mathbb{R}_{\geq 0} \times \{0\}$ is determined by the function $G : X_D^{hyb} \to \mathbb{R}_{\geq 0}$ which is defined as:

$$G(z,t) = \begin{cases} -t \log |z| & \text{if} \quad |z| < 1, \text{ and } t \neq 0 \\ 0 & \text{if} \quad |z| \geq 1, \text{ and } t \neq 0 \\ z & \text{if} \quad t = 0 \end{cases} \tag{2.4}$$

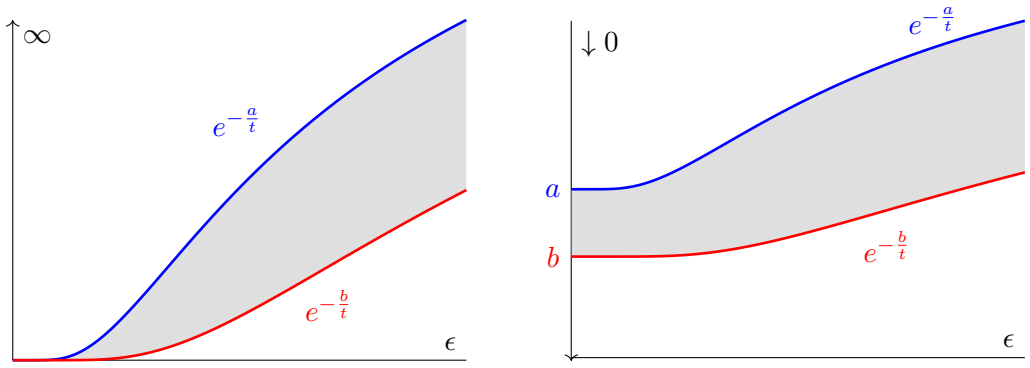
Let us to see the behavoiur of some basic open set in $\mathbb{R}_{\geq 0}$. If (a,b) is an open interval of $\mathbb{R}_{\geq 0}$, the intersection of $G^{-1}((a,b))$ with the fiber $\pi^{-1}(\{t_0\})$, for some $0 < t_0 \leq 1$ is of the form

$$\{z : e^{-\frac{b}{t_0}} < |z| < e^{-\frac{a}{t_0}}\}$$

If instead the interval is of the form $[0,b)$, then the corresponding intersection is:

$$\{z : e^{-\frac{b}{t_0}} < |z| < \infty\}$$

Figure below show the behaviour of the fiber as they tend to zero for the first case. For simplicity, the picture only show the norm of the elements of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0\}$. That is, every point (r,t), with $t > 0$, correspond to the closed set $\{z \in (\mathbb{P}^1_{\mathbb{C}} \setminus \{0\} : |z| = r\}$.



On the left hand of this picture, we have the product topology $(\mathbb{P}^1_{\mathbb{C}} \setminus \{0\}) \times [0,1]$. In this topology, a path $\gamma : (0,t] \to (\mathbb{P}^1_{\mathbb{C}} \setminus \{0\}) \times (0,1]$ with $|\gamma(t)| = e^{-\frac{a}{t}}$ converge to zero as $t \to 0$. On the right hand, we have the hybrid topology. where the same path converges to a, as t tends to zero. From these two topologies, note that the hybrid topology can be interpreted as having applied a blow-up to the origin to the picture of left side.

In the remainder of the section, we will detail some properties of the function G defined in (2.3), as well as the connection between toroidal divisors and piecewise linear functions defined in $\Sigma(X,D)$.

The following definition is taken from [ABGF16].

Definition 2.2.9. A path $\gamma : (0, r] \to X \setminus D$ is called an *admisible path* if for any adapted chart (U, z) the limit

$$\lim_{t\to 0} - t \log |z_i(\gamma(t))|$$

always exist for $i = 1, \ldots, n$, and the value of these limits lies in the interval $[0, 1)$. Here, z_i is the i-th coordinate of z.

If γ is an admissible path, denote by y_i the value of the limit $\lim_{t\to 0} - t \log |z_i(\gamma(t))|$. Then consider the function $\overline{\gamma} : [0, r] \to X_D^{hyb}$ defined as:

$$\overline{\gamma}(t) = \begin{cases} (\gamma(t), t) & \text{if} \quad t \neq 0 \\ \\ (y_j)_{j\in J} & \text{if} \quad t = 0 \end{cases} \tag{2.5}$$

where $J = \{j \in \{1, \ldots, n\} : y_j \neq 0\}$.

Proposition 2.2.10. If X is compact and γ is an admissible path, then $\overline{\gamma}$ is a continuous path.

Proof. To prove the continuity of $\overline{\gamma}$, it suffices to verify continuity at zero. Let $\{t_n\}_{n\in\mathbb{N}}$ a sequence of points in $(0, 1]$ that converges to zero. Consider the sequence $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ in $X \setminus D$. Since X is compact, there exists a point $z_0 \in X$ and a subsequence $\{\gamma(t_{n_k})\}_{k\in\mathbb{N}}$ of $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ that converges to z_0.

Let Y be the stratum that contains the point z_0, and let (U, z) be an adapted chart that contains the point z_0, and such that its associated stratum is Y. Consider the function $F_U : U \setminus D \to \sigma_U$ as in (2.1). Fix a function F that satisfies the statement of Proposition 2.2.3. After possibly shrinking U, there exists a bounded continuous function $\phi : U \to \Sigma(X, D)$ such that, on $U \setminus D$, we have

$$F = F_U + \phi$$

On the other hand, we have that there exists $k_0 \in \mathbb{N}$ such that $\gamma(t_{n_k}) \in U$ for $k \geq k_0$. This implies that for $k \geq k_0$

$$G(\gamma(t_{n_k}), t_{n_k}) = t_{n_k} F(\gamma(t_{n_k})) = t_{n_k} F_U(\gamma(t_{n_k})) + t_{n_k} \phi(\gamma(t_{n_k}))$$

Since ϕ is bounded, we have that $t_{n_k}\phi(\gamma(t_{n_k}))$ tends to zero, as t_{n_k} tends to zero. Therefore,

$$\lim_{k\to\infty} G(\gamma(t_{n_k}), t_{n_k}) = \lim_{k\to\infty} t_{n_k} F_U(\gamma(t_{n_k})) = (y_j)_{j\in J}$$

By Remark 2.2.6, this implies that $\{(\gamma(t_{n_k}), t_{n_k})\}_{k\in\mathbb{N}}$ converges to $((y_j)_{j\in J}, 0)$. Moreover, this limit is independent of the chosen subsequence or the point z_0, since γ is an admisible path. Therefore, the entire sequence $\{(\gamma(t_{n_k}), t_{n_k})\}_{n\in\mathbb{N}}$ converges to $((y_j)_{j\in J}, 0)$, and we conclude that $\overline{\gamma}$ is continuous at 0. This completes the proof. □

Proposition 2.2.11. If X is compact, then the function G is proper.

Proof. We have to show that for every compact $K \subseteq \Sigma(X, D)$, the set $G^{-1}(K)$ is compact. In fact, let $\{U_i\}_{i\in I}$ be a covering by open sets of $G^{-1}(K)$. Since the fiber at zero of $G^{-1}(K)$ is equal to K, consider the set $I_0 = \{i \in I : K \cap U_i \neq \emptyset\}$. This implies that $\{U_i \cap \pi^{-1}(0)\}_{i\in I_0}$ is a cover by open sets for K.

Hence, we can take a finite subset $I_1 \subseteq I_0$ such that $\{U_i \cap \pi^{-1}(0)\}_{i\in I_1}$ is a covering for K. For each $x \in K$ there exists an open neighborhood $V_x \subseteq \Sigma(X, D)$ of x, and $\epsilon_x > 0$ such that

$$G^{-1}(V) \cap \pi^{-1}([0, \epsilon_x)) \subseteq U_i$$

for some $i \in I_1$. Since K is compact, it can be covered by finitely many of open sets V_x. Therefore, there exists a $\epsilon > 0$ such that $\{U_i\}_{i\in I_1}$ forms a cover of $G^{-1}(K) \cap \pi^{-1}([0, \epsilon))$. It remains to prove that $G^{-1}(K) \cap \pi^{-1}([\frac{\epsilon}{2}, 1])$ can be covered by finite elements of the cover. It will suffice to show that $G^{-1}(K) \cap \pi^{-1}([\frac{\epsilon}{2}, 1])$ is a closed set of $X \times [\frac{\epsilon}{2}, 1]$. This would imply that $G^{-1}(K) \cap \pi^{-1}([\frac{\epsilon}{2}, 1])$ is compact, since $X \times [\frac{\epsilon}{2}, 1]$ is compact.

Let (z, t) be an element in the closure of $G^{-1}(K) \cap \pi^{-1}([\frac{\epsilon}{2}, 1])$, and consider $\{(z_n, t_n)\}_{n\in\mathbb{N}}$ a sequence of elements in $G^{-1}(K) \cap \pi^{-1}([\frac{\epsilon}{2}, 1])$ that converge to (z, t). If $z \in D$, and since $t_n \geq \frac{\epsilon}{2}$ for $n \in \mathbb{N}$, it follows that $\{t_n G(z_n)\}$ is unbounded in $\Sigma(X, D)$, which is a contradiction due to the compactness of K. This implies that z cannot lie in D. Hence, we have

$$G^{-1}(K) \cap \pi^{-1}([\frac{\epsilon}{2}, 1]) \cap (D \times [\epsilon/2, 1]) = \emptyset$$

Therefore, $G_D^{-1}(K) \cap \pi^{-1}([\frac{\epsilon}{2}, 1])$ is closed $X \times [\epsilon/2, 1]$, which is what we wanted to show. □

Let $E = \sum_{i\in I} a_i D_i$ be an $\mathbb{R}$-Cartier divisor supported on the divisor D; see Section 1.2. Then we can associate a piecewise linear function:

$$\phi_E : \Sigma(X, D) \to \mathbb{R}$$

Defined as follows: For each ray τ_i of $\Sigma(X, D)$, generated by a vector v_{τ_i} and associated to the irreducible divisor D_i, define $\phi(v_{\tau_i}) = a_i$. This assignment extends linearly on each cone of $\Sigma(X, D)$ yielding a piecewise linear function.

If $g_E : X \setminus |E| \to \mathbb{R}$ is a positive Green function associated to the divisor E; see 2.1.6. Then we can define a function $G_E : X_D^{hyb} \to \mathbb{R}$ as follows:

$$G_E(z, t) = \begin{cases} t \cdot g_E(z) & t \neq 0 \\ \\ \phi_E(z) & t = 0 \end{cases} \tag{2.6}$$

Proposition 2.2.12. If X is compact, then G_E is a continuous function in the hybrid topology.

Proof. Let $E = \sum_{i\in I} a_i D_i$ and $g_E = \sum_{i\in I} a_i \mathfrak{g}_i$, with $\{\mathfrak{g}_i\}_{i\in I}$ being a collection of Green functions, where each $\mathfrak{g}_i$ is associated with the irreducible component D_i. By remark 2.1.7, each summand $a_i\mathfrak{g}_i$ takes positive values. Therefore, we may assume, without loss of generality, that each $\mathfrak{g}_i$ is positive.

Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of X by adapted open sets satisfying the conditions of the Proposition 1.2.13, and let $\{\chi_\alpha\}_{\alpha\in\Lambda}$ be a partition of unity subordinated to this covering. By Proposition 2.2.3, this data induces a function F, and consequently a function $G : X_D^{hyb} \to \Sigma(X, D)$ constructed as in (2.3).

Let $\{\tau_i\}_{i\in I}$ be the rays of $\Sigma(X, D)$ associated to the irreducible divisors $D_{i's}$. Denote by $\rho_i : \Sigma(X, D) \to \tau_i$ the i-th projection, which is clearly continuous. From Remark 2.1.5, we see that $\rho_i \circ F$ is a Green function for the divisor D_i. Moreover, for every $x \in D_i$ there exists an open neighborhood V of x such that at $V \setminus D$

$$(\rho_i \circ F) = \mathfrak{g}_i \tag{2.7}$$

Therefore, the function $G_{D_i} : X_D^{hyb} \to \mathbb{R}$ defined as

$$G_{D_i}(z,t) = \begin{cases} t \cdot \mathfrak{g}_i(z) & t \neq 0 \\ \\ z & t = 0 \end{cases} \tag{2.8}$$

is a continuous function. Since $G_E = \sum_{i \in I} a_i G_{D_i}$, it follows that G_E is continuous. □

2.3 Continuity of measures on the hybrid space

In this section, unless otherwise stated, X will denote a compact manifold of dimensión n, and $D = \Sigma_{i \in I} D_i$ a simple normal crossing divisor.

Following [KKMSD06, Chap II], let Y be a stratum, and consider the following open submanifold

$$\mathrm{Star}(Y) := \bigcup_{\overline{Y'} \supseteq Y} Y'$$

Recall from Example 1.6.4 that M_Y denotes the free abelian group of Cartier divisors on $\mathrm{Star}(Y)$ supported in $D \cap \mathrm{Star}(Y)$, and that $S_Y \subseteq M_Y$ is the submonoid of effective Cartier divisors. The generators of the group M_Y correspond to the irreducible components of D that contain the stratum Y.

We now consider the dual lattice $N_Y = \mathrm{Hom}(M_Y, \mathbb{Z})$, and the associated real vector spaces $N_Y^{\mathbb{R}} := \mathrm{Hom}(M, \mathbb{R})$ and $M_Y^{\mathbb{R}} := M_Y \otimes_{\mathbb{Z}} \mathbb{R}$, which are naturally dual to each other. Recall also that the cone $\sigma_Y := \{v \in N_Y^{\mathbb{R}} : v_{|S_Y} \geq 0\}$ was previously defined. We now define $S_Y^{\mathbb{R}} := \{u \in M_Y^{\mathbb{R}} : u_{|\sigma_Y} \geq 0\}$. Then σ_Y and $S_Y^{\mathbb{R}}$ are dual cones with respect to the natural pairing between $M_Y^{\mathbb{R}}$ and $N_Y^{\mathbb{R}}$.

Now fix $r \in \mathbb{N}_{>0}$ and let

$$P \in \mathrm{Sym}^r(S_Y^{\mathbb{R}})$$

that is, P is an element of the symmetric r-th power of $S_Y^{\mathbb{R}}$. Explicitly, P can be written as a finite sum $\sum_{i=1}^{l} s_{i_1} \otimes \ldots \otimes s_{i_r}$, invariant under the action of the symmetric group $\mathfrak{S}_r$, where each $s_{i_j} \in S_Y^{\mathbb{R}}$.

Given elements $v_1, \ldots, v_r$ in $N_Y^{\mathbb{R}}$, we can evaluate P in the tuple $(v_1, \ldots, v_r)$ by

$$P(v_1, \ldots, v_n) := \sum s_{i_1}(v_1) \cdot \ldots \cdot \ldots \cdot s_{i_r}(v_r).$$

Moreover, if all $v_j \in \sigma_Y$ then each term in the sum is non-negative.

Now if we fix generators of M_Y consisting of elements in S_Y corresponding to the irreducible components of D that contain Y, we obtain coordinate systems on both $M_Y^{\mathbb{R}}$ and $N_Y^{\mathbb{R}}$, and hence we identify $M_Y^{\mathbb{R}}$ with $\mathbb{R}^k$, where k is the codimension of Y. Under this identification, the cone $S_Y^{\mathbb{R}}$ corresponds to the positive orthant $\mathbb{R}^k_{\geq 0}$, and similarly σ_Y is identified with $\mathbb{R}^k_{\geq 0}$. Therefore, P can be naturally viewed as a homogeneous polynomial of degree r with non-negative coefficients.

Let (U, z) be an adapted chart such that its associated stratum is Y. Suppose that the coordinates $z_{m_1}, \ldots, z_{m_k}$, with $1 \leq m_1 < \ldots < m_k \leq n$, are such that the zero locus of $z_{m_1} \cdot \ldots \cdot z_{m_k}$ coincides with $U \cap D$. Then the collection

$$-\log|z_{m_1}|, \ldots, -\log|z_{m_k}|$$

are local positive Green functions. Therefore, these functions define a map

$$F_U : U \smallsetminus D \to \sigma_U$$

Note the j-th coordinates of $F_U : U \smallsetminus D \to \sigma_U$ are associated to the irreducible component of D that have nonempty intersection with U, or in other word, that contains the stratum Y. This allows us to evaluate the image of F_U in the polynomial P. In other words, the polynomial P induces a real-valued function on $U \smallsetminus D$ given by

$$P(-\log|z_{m_1}|, \ldots, \log|z_{m_k}|)$$

We are abusing notation by evaluating the expressions $-\log|z_{m_j}|$ in a fixed order; this is done merely to impose a convenient indexing. Recall that each term $-\log|z_{m_j}|$ is evaluated on the coordinate corresponding to the irreducible component of the divisor D which, in the chart U, is locally defined by the equation $z_{m_j} = 0$.

Let $\{Y_\alpha\}_{\alpha\in\Sigma}$ be the stratification of (X, D), and from now on, we will assume that every minimal stratum has the same codimension, say, k. Then we denote by Σ_k the set of indices of strata of codimension m. For each $\alpha \in \Sigma_k$ consider a polynomial $P_\alpha \in \mathrm{Sym}^{r_\alpha}(S^{\mathbb{R}}_{Y_\alpha})$, for $r_\alpha \in \mathbb{N}$.

Definition 2.3.1. Let $\mathcal{P} := \{P_\alpha\}_{\alpha\in\Sigma_k}$ as above, and let ω be a continuous $(n, n)-$form defined on $X \smallsetminus D$. We say that ω is a *$\mathcal{P}$-logarithmic form* if, for every adapted chart (U, z) such that the associated stratum Y_α is a minimal stratum, ω can be written as

$$\omega_{|U} = \frac{\eta_U}{P_\alpha((-\log|z_{m_j}|)_{j=1}^k)} \frac{dz_{m_1}}{z_{m_1}} \wedge \frac{d\overline{z}_{m_1}}{\overline{z}_{m_1}} \wedge \ldots \wedge \frac{dz_{m_k}}{z_{m_k}} \wedge \frac{d\overline{z}_{m_k}}{\overline{z}_{m_k}} \tag{2.9}$$

Where η_U is a continuous $(n-k, n-k)$-form defined on U. If all the polynomials P_α are constant and the expression (2.9) holds for every adapted chart; that is, not only around the minimal strata, we refer to ω simply as a *logarithmic (n, n)-form*.

Note that the continuous form η_U of Definition 2.3.1 depends of the adapted chart. In this order, and analogously to the residue of holomorphic differentials with logarithmic poles (see [PFS08]) we define the *$\mathcal{P}$-local residue* of ω with respect to the minimal stratum Y, along an adapted chart (U, z) associated with Y, as:

$$\mathrm{Res}_{\mathcal{P}}(\omega_{|U}) := (\eta_U)_{|Y} \tag{2.10}$$

Proposition 2.3.2. Let Y be a minimal stratum, and let (U, z) and (U', z') be two adapted charts with associated stratum Y. Let ω be a $\mathcal{P}$-logarithmic form. Then on the intersection $U \cap U'$ we have

$$\mathrm{Res}_{\mathcal{P}}(\omega_{|U}) = \mathrm{Res}_{\mathcal{P}}(\omega_{|U'})$$

Proof. Without loss of generality, supose that $D_{i_1}, \ldots, D_{i_k}$ are the irreducible components which have nonempty intesection with U, and z_j (resp. z'_j) is the local equation defining the divisor D_{i_j} in U (resp. U') it for $j = 1, \ldots, k$. Assume also that P is the polynomial associated to the stratum Y.

Consider then the description of the $\mathcal{P}$-logarithmic form ω in the charts (U, z) and (U', z') respectively i.e.

$$\frac{\eta}{P((-\log|z_j|)_{j=1}^k)} \frac{dz_1}{z_1} \wedge \frac{d\overline{z}_1}{\overline{z}_1} \wedge \ldots \wedge \frac{dz_k}{z_k} \wedge \frac{d\overline{z}_k}{\overline{z}_k}, \quad \text{and} \quad \frac{\eta'}{P((-\log|z'_j|)_{j=1}^k)} \frac{dz'_1}{z'_1} \wedge \frac{d\overline{z'}_1}{\overline{z'_1}} \wedge \ldots \wedge \frac{dz'_k}{z'_k} \wedge \frac{d\overline{z'_k}}{\overline{z'_k}}$$

Since D is a simple normal crossing divisor, the transition functions between two adapted charts in the intersection $U \cap U'$ take the following form:

$$\begin{cases} z_j = z'_j u_j & \text{for} \quad j = 1, \dots, k \\ \\ z_j = w_j & \text{for} \qquad j > k \end{cases} \tag{2.11}$$

where $u_1, \dots, u_k, w_{k+1}, \dots, w_n$ are holomorphic functions depending on the coordinates $z'_1, \dots, z'_n$, and each u_j is non zero along $z'_j = 0$. Since this variable change is induced by the transition functions, which are biholomorphic, we know that the Jacobian matrix of the transformation has nonzero determinant along the intersection $U \cap U'$. Explicitly we have

$$\det \begin{pmatrix} z'_1 \frac{\partial u_1}{\partial z'_1} + u_1 & \dots & z'_1 \frac{\partial u_1}{\partial z'_k} & \dots & z'_1 \frac{\partial u_1}{\partial z'_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ z'_k \frac{\partial u_k}{\partial z'_1} & \dots & z'_k \frac{\partial u_k}{\partial z'_k} + u_k & \dots & z'_k \frac{\partial u_k}{\partial z'_n} \\ \frac{\partial w_{k+1}}{\partial z'_1} & \dots & \frac{\partial w_{k+1}}{\partial z'_k} & \dots & \frac{\partial w_{k+1}}{\partial z'_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial w_n}{\partial z'_1} & \dots & \frac{\partial w_n}{\partial z'_k} & \dots & \frac{\partial w_n}{\partial z'_n} \end{pmatrix} \neq 0$$

Considering the restriction of this determinant to the stratum Y, we have the following expression

$$\det \begin{pmatrix} u_{1|Y} & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & u_{k|Y} & \dots & 0 \\ \frac{\partial w_{k+1}}{\partial z'_1}_{|Y} & \dots & \frac{\partial w_{k+1}}{\partial z'_k}_{|Y} & \dots & \frac{\partial w_{k+1}}{\partial z'_n}_{|Y} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial w_n}{\partial z'_1}_{|Y} & \dots & \frac{\partial w_n}{\partial z'_k}_{|Y} & \dots & \frac{\partial w_n}{\partial z'_n}_{|Y} \end{pmatrix} \neq 0$$

This implies that $u_{j|Y} \neq 0$ for $j = 1, \dots, k$, and

$$\det \begin{pmatrix} \frac{\partial w_{k+1}}{\partial z'_{k+1}}_{|Y} & \dots & \frac{\partial w_{k+1}}{\partial z'_n}_{|Y} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_n}{\partial z'_{k+1}}_{|Y} & \dots & \frac{\partial w_n}{\partial z'_n}_{|Y} \end{pmatrix} \neq 0 \tag{2.12}$$

Now, taking differentials of the coordinate change described in (2.11), we obtain:

$$\begin{cases} dz_j = \left(u_j + z'_j \frac{\partial u_j}{\partial z'_j}\right) dz'_j + z'_j \sum_{l \neq j} \frac{\partial u_j}{\partial z'_l} dz'_l & \text{for} \quad j = 1, \dots, k \\ \\ dz_j = \sum_{l=1}^{n} \frac{\partial w_j}{\partial z'_l} dz'_l & \text{for} \qquad j > k \end{cases} \tag{2.13}$$

Now, for each $j = 1, \dots, k$, we divide both sides of the corresponding equation in (2.13) by z_j, and in the resulting expression, we replace z_j by $z'_j u_j$ on the right-hand side. This yields:

$$\begin{cases} \frac{dz_j}{z_j} = \left(1 + \frac{z'_j}{u_j}\frac{\partial u_j}{\partial z'_j}\right)\frac{dz'_j}{z'_j} + \frac{1}{u_j}\sum_{l\neq j}\frac{\partial u_j}{\partial z'_l}dz'_l & \text{for} \quad l = 1, \ldots, k \\ dz_j = \sum_{l=1}^{n}\frac{\partial w_j}{\partial z'_l}dz'_l & \text{for} \qquad j > k \end{cases} \tag{2.14}$$

A similar expression holds for the conjugate differentials $d\overline{z}_j$, with $\overline{u}_j$ and $\overline{w}_j$ as functions of the variables $\overline{z}'_{j's}$.

Now, observe that the wedge product $\frac{dz_1}{z_1} \wedge \frac{d\overline{z}_1}{\overline{z}_1} \wedge \ldots \wedge \frac{dz_k}{z_k} \wedge \frac{d\overline{z}_k}{\overline{z}_k}$ can be expressed in the other coordinates using the transformation in (2.14) as follows:

$$\omega_k + \left[\prod_{j=1}^{k}\left(1 + \frac{z'_j}{u_j}\frac{\partial u_j}{\partial z'_j}\right)\left(1 + \frac{\overline{z}'_j}{\overline{u}_j}\frac{\partial \overline{u}_j}{\partial \overline{z}'_j}\right)\right]\frac{dz'_1}{z'_1} \wedge \frac{d\overline{z}_1}{\overline{z}'_1} \wedge \ldots \wedge \frac{dz'_k}{z'_k} \wedge \frac{d\overline{z}_k}{\overline{z}'_k}$$

where ω_k is the resulting form whose summands do not contain the term $\frac{dz'_1}{z'_1} \wedge \frac{d\overline{z}_1}{\overline{z}'_1} \wedge \ldots \wedge \frac{dz'_k}{z'_k} \wedge \frac{d\overline{z}_k}{\overline{z}'_k}$.

For simplicity, denote by α the expression

$$\prod_{j=1}^{k}\left(1 + \frac{z'_j}{u_j}\frac{\partial u_j}{\partial z'_j}\right)\left(1 + \frac{\overline{z}'_j}{\overline{u}_j}\frac{\partial \overline{u}_j}{\partial \overline{z}'_j}\right)$$

Then, under this change of variables, the form ω_U is expressed as

$$\frac{\eta_0}{P}\omega_n + \alpha \cdot \frac{\eta_0}{P} \cdot \left[\bigwedge_{j=k+1}^{n}\left(\sum_{l=k+1}^{n}\frac{\partial w_j}{\partial z'_l}dz'_l\right) \wedge \left(\sum_{l=k+1}^{n}\frac{\partial \overline{w}_j}{\partial \overline{z}'_l}d\overline{z}'_l\right)\right] \wedge \frac{dz'_1}{z'_1} \wedge \frac{d\overline{z}_1}{\overline{z}'_1} \wedge \ldots \wedge \frac{dz'_k}{z'_k} \wedge \frac{d\overline{z}_k}{\overline{z}'_k} \tag{2.15}$$

Where ω_n is the resulting (n,n)-form arising from the wedge product of ω_k and the differentials $\sum_{l=1}^{n}\frac{\partial w_j}{\partial z'_l}dz'_l$, for $j = k+1, \ldots, n$. And η_0 is such that $\eta = \eta_0 dz_{k+1} \wedge d\overline{z}_{k+1} \wedge \ldots \wedge d\overline{z}_n$.

Expressing the equation (2.15) in terms of the wedge product $\frac{dz'_1}{z'_1} \wedge \frac{d\overline{z}_1}{\overline{z}'_1} \wedge \ldots \wedge \frac{dz'_k}{z'_k} \wedge \frac{d\overline{z}_k}{\overline{z}'_k}$ we obtain:

$$\alpha \cdot \frac{\eta_0}{P} \cdot \left[\omega_{n-k+1} + \bigwedge_{j=k+1}^{n}\left(\sum_{l=k+1}^{n}\frac{\partial w_j}{\partial z'_l}dz'_l\right) \wedge \left(\sum_{l=k+1}^{n}\frac{\partial \overline{w}_j}{\partial \overline{z}'_l}d\overline{z}'_l\right)\right] \wedge \frac{dz'_1}{z'_1} \wedge \frac{d\overline{z}_1}{\overline{z}'_1} \wedge \ldots \wedge \frac{dz'_k}{z'_k} \wedge \frac{d\overline{z}_k}{\overline{z}'_k}. \tag{2.16}$$

Note that every summand of ω_{n-k+1} contains at least one factor of the form z'_j, with $1 \leq j \leq k$. This follows from the structure of the original expression (2.15), where each summand of ω_n is not divided for any z'_j, with $j = 1, \ldots, k$.

In order to determine η', let us now express the homogeneous polynomial P from the above expression in

terms of the variables z'_j.

$$\begin{aligned} P((-\log|z_j|)_{j=1}^k) &= P((-\log|z'_j u_j|)_{j=1}^k) \\ &= P((-\log|z'_j| - |\log|u_j|)_{j=1}^k) \\ &\quad P((-\log|z'_j|)_{j=1}^k) + Q \end{aligned}$$

Where Q is a polynomial depending of $-\log|z'_1|,\ldots,-\log|z'_k|,-\log|u_1|,\ldots,-\log|u_k|$, with degree less than r with respect the logarithmic functions $-\log|z'_1|,\ldots,-\log|z'_k|$.

Now denoting

$$\beta := \bigwedge_{j=k+1}^{n}\left(\sum_{l=k+1}^{n}\frac{\partial w_j}{\partial z'_l}dz'_l\right)\wedge\left(\sum_{l=k+1}^{n}\frac{\partial \overline{w}_j}{\partial \overline{z}'_l}d\overline{z}'_l\right)$$

to simplify notation, the expression in (2.16) can be rewritten as:

$$\frac{\alpha\eta_0\left[\omega_{n-k+1}+\beta\right]}{P((-\log|z'_j|)_{j=1}^k)\left(1+\frac{Q}{P((-\log|z'_j|)_{j=1}^k)}\right)}\wedge\frac{dz'_1}{z'_1}\wedge\frac{d\overline{z}_1}{\overline{z}'_1}\wedge\ldots\wedge\frac{dz'_k}{z'_k}\wedge\frac{d\overline{z}_k}{\overline{z}'_k}$$

Observe that this expression coincides with $\omega_{|U'}$. Consequently, we identify:

$$\eta' = \frac{\alpha\eta_0(\omega_{n-k+1}+\beta)}{1+\frac{Q}{P((-\log|z'_j|)_{j=1}^k)}}$$

Now, evaluating this expression along the stratum Y, we observe the following:

- The function α is equal to 1 along the stratum Y.
- The limit of $1+\frac{Q}{P((-\log|z'_j|)_{j=1}^k)}$ as every z'_j tends to the stratum Y is equal to 1, by Remark 2.3.4.
- The form ω_{n-k+1} is zero since each sumand has at least a factor z'_j, where $1\leq j\leq k$.

Therefore, replacing β back into the expression and taking into account the observations above, we obtain:

$$\eta'_{|Y} = \eta_{0|Y}\bigwedge_{j=k+1}^{n}\left(\sum_{l=k+1}^{n}\frac{\partial w_j}{\partial z'_l}dz'_l\right)_{|Y}\wedge\left(\sum_{l=k+1}^{n}\frac{\partial \overline{w}_j}{\partial \overline{z}'_l}d\overline{z}'_l\right)_{|Y} \tag{2.17}$$

Note that the right-hand side of (2.17) corresponds exactly to the change of variables induced by restricting the transformation described in (2.11) to the stratum Y. The fact that this is a change of variables follows from (2.12). Therefore, the equality of the restrictions is proved. □

Proposition 2.3.3. Let P be a homogeneous polynomial of degree r in the variables $x_1,\ldots,x_k$, with positive coefficients. Consider $a_1,\ldots,a_k$ real numbers and denote $x=(x_1,\ldots,x_k)$. Then

$$\lim_{x\to\infty}\frac{P(x_1+a_1,\ldots,x_k+a_k)}{P(x_1,\ldots,x_k)} = 1$$

Where $x \to \infty$ means that for every $\varepsilon > 0$, there exists $M > 0$ such that $\left| \frac{P(x_1+a_1,\ldots,x_k+a_k)}{P(x_1,\ldots,x_k)} - 1 \right| < \varepsilon$ for all $x_i \geq M$, with $i = 1, \ldots, k$.

Proof. Since P is a homogeneous polynomial, it follows that for $i = 1, \ldots, k$, the polynomial $\frac{\partial P}{\partial x_i}$ is a homogeneous polynomial of degree $r-1$. Then, by Euler's homogeneous function theorem, we have

$$rP = \sum_{i=1}^{k} x_i \frac{\partial P}{\partial x_i}$$

Now, consider the Taylor expansion of the polynomial P

$$P(x_1 + a_1, \ldots, x_k + a_k) = P + \sum_i \frac{\partial P}{\partial x_i} a_i + \frac{1}{2!} \sum_{i,j} \frac{\partial^2 P}{\partial x_i \partial x_j} a_i a_j + \ldots$$

With the previous considerations, it will be sufficient to prove that for any $i = 1, \ldots, k$

$$\lim_{x \to \infty} \frac{\left(\frac{\partial P}{\partial x_i} \right)}{P} = 0 \tag{2.18}$$

Since every partial derivate of P is a homogeneous polynomial with positive coffients, we observe that for any order l

$$\lim_{x \to \infty} \frac{\left(\frac{\partial^l P}{\partial x_{i_1} \ldots \partial x_{i_l}} \right)}{P} = \lim_{x \to \infty} \frac{\left(\frac{\partial^l P}{\partial x_{i_1} \ldots \partial x_{i_l}} \right)}{\left(\frac{\partial^{l-1} P}{\partial x_{i_2} \ldots \partial x_{i_l}} \right)} \cdot \lim_{x \to \infty} \frac{\left(\frac{\partial^{l-1} P}{\partial x_{i_2} \ldots \partial x_{i_l}} \right)}{P}$$

Therefore, the desired result follows immediately by induction on l. Recall that limit in (2.18) means that for every $\varepsilon > 0$ there exists $M > 0$ such that $\frac{\frac{\partial P}{\partial x_i}(x_1,\ldots,x_k)}{P(x_1,\ldots,x_k)} < \epsilon$, whenever $x_i > M$ for every $i = 1, \ldots, k$.

From 2.3 and the hypotheses that the coefficients of the homogeneous polynomial are positive, we have for every $i = 1, \ldots, k$

$$rP \geq \frac{\partial P}{\partial x_i} x_i$$

This implies

$$\frac{\left(\frac{\partial P}{\partial x_i} \right)}{P} \leq \frac{r}{x_i}$$

Therefore, given $\varepsilon > 0$, the result follows by taking $M = \frac{r}{\epsilon}$. □

Remark 2.3.4. In Proposition 2.3.2, we consider two adapted charts (U, z) and (U', z'), both adapted to the same minimal stratum Y, which has codimension k. On the intersection $U \cap U'$, the transition functions for the first k coordinates are given by $z_j = z'_j u_j$, where each u_j is a holomorphic function in the variables $z'_1, \ldots z'_n$, and satisfies $u_j \neq 0$ along the set $\{z'_j = 0\} \cap U \cap U'$.

Shrinking the adapted charts if necessary, we may assume that there exists a constant $e_j > 0$ such that $|u_j| > e_j$ on $U \cap U'$, for each $j = 1, \ldots, k$. This implies $-\log |u_j| < -\log e_j$. Therefore, we obtain the inequality

$$|P((-\log |z_j|)_{j=1}^k)| = |P((-\log |z'_j| - |\log |u_j|)_{j=1}^k)| \leq |P((-\log |z'_j| + c_j)_{j=1}^k)|$$

where $c_j = -\log|e_j|$, for $j = 1, \ldots, k$. With this observation, and by Proposition 2.3.3, it follows that

$$\frac{P((-\log|z'_j| + c_j)_{j=1}^k)}{P((-\log|z'_j|)_{j=1}^k)} \to 1 \quad \text{as} \quad z' \to Y.$$

Consequently, we conclude that

$$\frac{P((-\log|z_j|)_{j=1}^k)}{P((-\log|z'_j|)_{j=1}^k)} \to 1 \quad \text{as} \quad z' \to Y.$$

Proposition 2.3.2 motivates the following definition.

Definition 2.3.5. Let $\mathcal{P}$ be a collection of homogeneous polynomials associated to the minimal strata of X. Let Y be a minimal stratum, and let ω be a $\mathcal{P}$-logarithmic form. The $\mathcal{P}$-*residue* of ω along the stratum Y is defined to be the continuous form $\mathrm{Res}_{\mathcal{P},Y}(\omega)$ on Y, which satisfies the property that for any adapted chart (U, z) with Y as its associated stratum, $\eta_{U|Y} = \mathrm{Res}_{\mathcal{P}}(\omega)_{|U\cap Y}$.

If ω is a logarithmic (n, n)-form-i.e., all the polynomials P_α are constant and the local description of ω extends to every adapted chart-we simply denote the residue by $\mathrm{Res}_Y(\omega)$

Recall that we are assuming that all the minimal strata are of codimension k, and that Σ_k denotes the respective collection of strata. Let $\mathcal{P}$ a collection of homogeneous polynomials, one for each minimal stratum $\sigma \in \Sigma_k$. Assume that all polynomials in $\mathcal{P}$ have the same degree, say, r. Let ω be a $\mathcal{P}$-logarithmic form.

Under these assumptions, we define a family of measures $\{\mu_t\}_{t\in[0,1]}$ on X_D^{hyb}, where each μ_t is supported on the fiber $\pi^{-1}(t)$, as follows:

$$\mu_t = \begin{cases} \frac{t^{k-r}}{(4\pi i)^k}\omega & t \neq 0 \\ \sum_{Y_\alpha\in\Sigma_k} \left(\int_{Y_\alpha} \mathrm{Res}_{\mathcal{P},Y_\alpha}(\omega)\right) \frac{d\lambda_\alpha}{P_\alpha} & t = 0 \end{cases} \tag{2.19}$$

where $d\lambda_\alpha$ denotes the Lebesgue measure on the cone σ_α associated with the minimal stratum Y_α.

Let Y be a stratum of codimension k, and let (U, z) be an adapted chart with Y as its associated stratum. Define

$$U_D := (U \smallsetminus D) \times (0, 1]$$

Assume that $z_1, \ldots, z_k$ are the coordinates such that, for each $1 \leq j \leq k$, the zero locus of z_j is equal to the intersection of an irreducible component of D with U.

As in Remark 2.1.10, consider the function from $U \smallsetminus D$ to the cone σ_U, which sends a point z to the tuple $(-\log|z_j|)_{j=1}^k$. This induces a function

$$\log_U : U_D \to \sigma_U, \quad (z, t) \mapsto (-t\log|z_j|)_{j=1}^k \tag{2.20}$$

Lemma 2.3.6. Let (U, z) be an adapted chart such that its associated stratum Y_α has codimension k. And let $\chi \in C_c(U, \mathbb{R})$. [1] Then the family of measures $\{(\log_U)_*(\chi \cdot \mu_t)\}_{0<t\leq 1}$ converges vaguely, as $t \to 0$,

[1] If X is a locally compact and Hausdorff topological space. Then $C_c(X, \mathbb{R})$ denotes the algebra of continuous functions with compact support.

to the measure $\left(\int_{Y_\alpha} \chi \cdot \mathrm{Res}_{\mathcal{P},Y_\alpha}(\omega)\right) \frac{d\lambda_\alpha}{P_\alpha}$ in the interior of the cone σ_α.

Proof. For each $\phi \in C_c(\mathring{\sigma}_\alpha, \mathbb{R})$, we have to prove that

$$\lim_{t\to 0} \int_{U \smallsetminus D} (\phi \circ \log_U) \cdot \chi d\mu_t = \left(\int_{Y_\alpha} \chi \cdot \mathrm{Res}_{\mathcal{P},Y_\alpha}(\omega)\right) \cdot \int_{\sigma_\alpha} \phi \frac{d\lambda_\alpha}{P_\alpha}$$

Note that $U \smallsetminus D$ is embedded in the $(\mathbb{C}^*)^k \times \mathbb{C}^{n-k}$, since U is an adapted chart. Then for every $0 < t \leq 1$ fixed, consider the transformation

$$\begin{aligned} T_t \,:\, & (\mathbb{R}^*)^k \times (S^1)^k \times \mathbb{C}^{n-k} \longrightarrow (\mathbb{C}^*)^k \times \mathbb{C}^{n-k} \\ & ((u_j)_{j=1}^k, (\theta_j)_{j=1}^k, z') \mapsto \left((\exp(-\frac{u_j}{t} + i\theta_j))_{j=1}^k, z'\right), \end{aligned}$$

where $z' = (z_{k+1}, \ldots, z_n)$. So, the change of variables $z_j = \exp(\frac{-u_j}{t} + i\theta_j)$ expresses the complex coordinates z_j in terms of the real variables u_j and θ_j, for $j = 1, \ldots, k$.

Differentiating this expression, we obtain the differential:

$$dz_j = \left(\frac{-\exp(\frac{-u_j}{t} + i\theta_j)}{t}\right) du_j + \left(i \exp(\frac{-u_j}{t} + i\theta_j)\right) d\theta_j$$

$$d\overline{z}_j = \left(\frac{-\exp(\frac{-u_j}{t} - i\theta_j)}{t}\right) du_j - \left(i \exp(\frac{-u_j}{t} - i\theta_j)\right) d\theta_j$$

This description implies that

$$dz_j \wedge d\overline{z}_j = \frac{2i}{t} \exp\left(\frac{-2u_j}{t}\right) du_j \wedge \theta_j$$

Moreover

$$\frac{dz_j}{z_j} \wedge \frac{d\overline{z}_j}{\overline{z}_j} = \frac{2i}{t} du_j \wedge d\theta_j$$

Therefore, we have

$$\frac{dz_1}{z_1} \wedge \frac{d\overline{z}_1}{\overline{z}_1} \wedge \ldots \wedge \frac{dz_k}{z_k} \wedge \frac{d\overline{z}_k}{\overline{z}_k} = \left(\frac{2i}{t}\right)^k du_1 \wedge d\theta_1 \wedge \ldots \wedge du_k \wedge d\theta_k \tag{2.21}$$

For simplicity, let us introduce the notations $u = (u_1, \ldots, u_k), \theta = (\theta_1, \ldots, \theta_k)$, and define the exponential map in multi-index form as $\exp(\frac{-u}{t} + i\theta) := \left(\exp(-\frac{u_j}{t} + i\theta_j)_{j=1}^k\right)$. We also denote $d\theta := d\theta_1 \ldots d\theta_k$, $du := du_1 \ldots du_k$ and $dz' = dz_{k+1} \ldots dz_n$.

With these conventions, applying a change of variables and noting that $\chi \cdot \omega = \chi \cdot \omega_{|U}$ since the support of χ is contained in U, we obtain:

$$\begin{aligned}\int_{U\smallsetminus D}(\phi\circ\log_U)\cdot\chi d\mu_t &= \int_{(\mathbb{C}^*)^k\times\mathbb{C}^{n-k}}(\phi\circ\log_U)\cdot\chi\cdot\left[\frac{t^{k-r}}{(4\pi i)^k}\omega_{|U}\right]\\ &= \int_{(\mathbb{C}^*)^k\times\mathbb{C}^{n-k}}(\phi\circ\log_U)\cdot\chi\cdot\left[\frac{t^{k-r}}{(4\pi i)^k}\eta_{|U}\frac{dz_1}{z_1}\wedge\frac{d\overline{z}_1}{\overline{z}_1}\wedge\ldots\wedge\frac{dz_k}{z_k}\wedge\frac{d\overline{z}_k}{\overline{z}_k}\right]\\ &= \frac{1}{(2\pi)^k}\int_{(\mathbb{R}_{>0})^k}\int_{\mathbb{C}^{n-k}}\int_{(S^1)^k}\phi(u)\cdot\left[\frac{\chi\cdot\eta_0}{P_\alpha(u)}\right]d\theta dz' du\end{aligned}$$

Where η_0 is such that $\eta_{|U} = \eta_0 dz_{k+1}\wedge d\overline{z}_{k+1}\wedge\ldots d\overline{z}_n$. Note that the factor t^{k-r} is canceled in two ways: On one hand, by the denominator appearing in equation (2.21), and on the other hand, by the homogeneous polynomial P_α, since when we replace $-\log|z_j|$ by $\frac{u_j}{t}$, we obtain an additional factor of $\frac{1}{t^r}$.

Observe also that $\mathbb{R}^k_{>0}$ can be naturally identified with the relative interior of the cone σ_α, and thus we have:

$$\frac{1}{(2\pi)^k}\int_{(\mathbb{R}_{>0})^k}\int_{\mathbb{C}^{n-k}}\int_{(S^1)^k}\phi(u)\cdot\left[\frac{\chi\cdot\eta_0}{P_\alpha(u)}\right]d\theta dz' du = \frac{1}{(2\pi)^k}\int_{\mathring{\sigma}_\alpha}\int_{\mathbb{C}^{n-k}}\int_{(S^1)^k}\phi(u)\cdot\left[\frac{\chi\cdot\eta_0}{P_\alpha(u)}\right]d\theta dz' du$$

This implies

$$\lim_{t\to 0}\int_{U\smallsetminus D}\phi\,(\log_U)\cdot\chi d\mu_t = \lim_{t\to 0}\frac{1}{(2\pi)^k}\int_{\mathring{\sigma}_\alpha}\int_{\mathbb{C}^{n-k}}\int_{(S^1)^k}\frac{\phi(u)}{P_\alpha(u)}[(\chi\cdot\eta_0)(e^{-\frac{u}{t}+i\theta},z')]d\theta dz' du$$

The integrand $\frac{\phi(u)}{P_\alpha(u)}[(\chi\cdot\eta_0)(e^{-\frac{u}{t}+i\theta},z')]$ defines a family of functions depending on t. For each fixed triple (u,θ,z'), this expression converges pointwise to $\frac{\phi(u)}{P_\alpha(u)}[(\chi\cdot\eta_0)(0,z')]$ as t tends to zero.

Since both ϕ and χ have compact support, it follows that for each t, the support of the function $\frac{\phi(u)}{P_\alpha(u)}[(\chi\cdot\eta_0)(e^{-\frac{u}{t}+i\theta},z')]$ is contained in the compact set $\mathrm{supp}(\phi)\times\Delta^{n-k}\times(S^1)^k$, where Δ^{n-k} is the closed polydisc $\{(z_1,\ldots,z_{n-k})\in\mathbb{C}^{n-k}:|z_i|\leq 1,\ i=1,\ldots,n-k\}$.

Moreover, for each fixed t, the function $\frac{\phi(u)}{P_\alpha(u)}[(\chi\cdot\eta_0)(e^{-\frac{u}{t}+i\theta},z')]$ is continuous and bounded on this compact set. The boundedness follows from the fact that ϕ is supported in the interior of the cone, which ensure that P_α is strictly positive on $\mathrm{Supp}(\phi)$. Hence, the family of functions parametrized by t is bounded by an integrable function. We are therefore in a position to apply the dominated convergence theorem, which yields:

$$\begin{aligned}\lim_{t\to 0}\frac{1}{(2\pi)^k}\int_{\mathring{\sigma}_\alpha}\int_{\mathbb{C}^{n-k}}\int_{(S^1)^k}\frac{\phi(u)}{P_\alpha(u)}[\chi\cdot\eta_0]d\theta dz' du &= \frac{1}{(2\pi)^k}\int_{\mathring{\sigma}_\alpha}\int_{\mathbb{C}^{n-k}}\int_{(S^1)^k}\frac{\phi(u)}{P_\alpha(u)}[(\chi\cdot\eta_0)(0,z')]d\theta dz' du\\ &= \frac{1}{(2\pi)^k}\int_{\mathring{\sigma}_\alpha}\frac{\phi(u)}{P_\alpha(u)}\left(\int_{\mathbb{C}^{n-k}}\int_{(S^1)^k}(\chi\cdot\eta_0)(0,z')d\theta dz'\right)du\\ &= \int_{\mathring{\sigma}_\alpha}\frac{\phi(u)}{P_\alpha(u)}\left(\int_{\mathbb{C}^{n-k}}(\chi\cdot\eta_0)(0,y)dz'\right)du\end{aligned}$$

Finally, observe that

$$\int_{\mathbb{C}^{n-k}}(\chi\cdot\eta_0)(0,y)dz' = \int_{Y_\alpha}\chi\cdot\mathrm{Res}_{\mathcal{P},Y_\alpha}(\omega)dz_{k+1}\wedge\ldots\wedge dz_n.$$

This completes the proof of the Lemma. □

Note that in the preceding lemma, vague convergence is established only on the interior of the cone σ_α. This restriction arises from the presence of the homogeneous polynomial P_α, which may vanish along some of the faces of the cone. Such vanishing introduces singularities in the integrand that prevent the application of the dominated convergence theorem near the boundary.

However, a situation in which vague convergence on the entire cone σ_α can be ensured occurs when each polynomial P_α is constant. In such a case, no singularities appear in the integrand of $\int_{U\smallsetminus D}(\phi\circ\log_U)\cdot\chi d\mu_t$ after the change of variables described in the proof of Lemma 2.3.6. Based on this observation, and following the same reasoning as in the proof of Lemma 2.3.6, we obtain the following result.

Lemma 2.3.7. Let (U,z) be an adapted chart such that its associated stratum Y_α has codimension k. And let $\chi\in C_c(U,\mathbb{R})$. Suppose ω is a logarithmic (n,n)-form; that is, it is given locally by

$$\omega_{|U}=\eta_U\frac{dz_{m_1}}{z_{m_1}}\wedge\frac{d\overline{z}_{m_1}}{\overline{z}_{m_1}}\wedge\ldots\wedge\frac{d\overline{z}_{m_k}}{\overline{z}_{m_k}}.$$

Then the family of measures $\{(\log_U)_*(\chi\cdot\mu_t)\}_{0<t\leq 1}$ converges vaguely, as $t\to 0$, to the measure

$$\left(\int_{Y_\alpha}\chi\cdot\mathrm{Res}_{\mathcal{P},Y_\alpha}(\omega)\right)d\lambda_\alpha$$

on the cone σ_α.

Lemma 2.3.9 provides a characterization for families of measures $\{\mu_t\}_{0\leq t\leq 1}$ on the hybrid space, where each measure is supported on a distinct fiber. This characterization tells us that if the family converges vaguely to the measure μ_0 supported on the fiber over zero, then this is equivalent to showing that the pushforward family $\{G_*(\mu_t)\}_{0<t\leq 1}$, under the map $G:X_D^{\dot{hyb}}\to\Sigma(X,D)$ defined in (2.3), converges vaguely to μ_0. Moreover, this equivalence is independent of the choice of the map G. However, before proceeding with this argument, we first need to establish some preliminary results.

Let X be a locally compact space, and consider $C_c(X,\mathbb{R})$ the space of continuous functions from X to $\mathbb{R}$ with compact support. For each compact $K\subseteq X$, denote by $C_c(X,K,\mathbb{R})$ as the subspace of $C_c(X,\mathbb{R})$ consisting of functions whose support is contained in K. This space can be naturally identified with the subspace of $C_0(K,\mathbb{R})$ consisting of continuous functions that vanish on the boundary ∂K of K.

Equipping $C_c(X,K,\mathbb{R})$ with the topology of uniform convergence via this identification, we see that it is a closed subspace of $C(K,\mathbb{R})$. Moreover, if $K_1\subseteq K_2\subseteq X$ are two compacts, then we have a canonical injection

$$C_c(X,K_1,\mathbb{R})\to C_c(X,K_2,\mathbb{R})$$

which is continuous with the topologies defined above.

Now assume that X is σ-compact. Then there exists a sequence of relatively compact open sets[2] $\{U_n\}_{n\in\mathbb{N}}$ covering X such that $\overline{U}_n\subseteq U_{n+1}$. Since the space $C_c(X,\mathbb{R})$ is expressed as a direct limit

$$\varinjlim_{n\in\mathbb{N}}C_c(X,\overline{U}_n,\mathbb{R}),$$

[2]An open set U is called a relatively compact open sets if $\overline{U}$ is a compact set

the topology we consider on $C_c(X, \mathbb{R})$ is precisely the one determined by this direct limit in the category of topological vector spaces. By [Bou04, Chap III, Sec I, Prop 1], this space is Hausdorff space. Moreover, there is a well-defined notion of boundedness in this space, and [Bou04, Chap. III, Sec. I, Prop. 2] provides a characterization of the bounded subsets: a subset $B \subseteq C_c(X, \mathbb{R})$ is bounded if and only if there exists a $n \in \mathbb{N}$ such that $B \subseteq C_c(X, \overline{U}_n, \mathbb{R})$.

According to [DJ49], we find that $C_c(X, \mathbb{R})$ is sequential, as it is the strict inductive limit of Fréchet spaces. This allows us to study the topology of $C_c(X, \mathbb{R})$ via sequences. In this setting, any convergent sequence in $C_c(X, \mathbb{R})$ necessarily forms a bounded set. Therefore, based on the characterization of bounded subsets discussed above, we obtain an explicit criterion for the convergence of sequences:

A sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements in $C_c(X, \mathbb{R})$ converges to an element $f \in C_c(X, \mathbb{R})$ if only if the following two conditions are satisfied:

1. There exists a compact K such that for each $n \in \mathbb{N}$, $\mathrm{Supp}(f_n) \subseteq K$.
2. $\lim_{n\to\infty} ||f_n - f||_{\sup} = 0$.

Now, we state a version of the Stone–Weierstrass theorem adapted to the space $C_c(X, \mathbb{R})$. The classical theorem is usually stated for the space of continuous functions vanishing at infinity; that is, continuous functions $f : X \to \mathbb{R}$ such that for each $\varepsilon > 0$, there exists a compact set $K \subseteq X$ such that $|f(x)| < \varepsilon$ for each $x \in X \setminus K$. Since we could not find a suitable reference for this specific case, we have decided to include a full proof for the sake of clarity.

Proposition 2.3.8. Let X be a locally compact and σ-compact topological space. If $\mathcal{A} \subseteq C_c(X, \mathbb{R})$ is a subalgebra with the following property: For every compact $K \subseteq X$ there exists a relatively compact open set U such that

1. The compact K is contained in U.
2. The subalgebra $\mathcal{A} \cap C_c(X, \overline{U}, \mathbb{R})$ separates points in U.
3. For each $x \in U$ there exists $f \in \mathcal{A}$ with the support contained in $\overline{U}$ such that $f(x) \neq 0$.

Then $\mathcal{A}$ is dense in $C_c(X, \mathbb{R})$.

Proof. We have to prove that for each $f \in C_c(X, \mathbb{R})$ and each $\varepsilon > 0$ there exist $g \in \mathcal{A}$ such that $||f - g||_{\sup} \leq \varepsilon$. Indeed, fix a function f, and let $K = \mathrm{Supp}(f)$, then by hypotheses there exists a relatively compact open set U such that $K \subseteq U$.

Consider $C_0(U, \mathbb{R})$, the algebra of continuous functions that vanish at infinity. Then the intersection $\mathcal{A} \cap C_c(X, \overline{U}, \mathbb{R})$ can be identified as a subalgebra of $C_0(U, \mathbb{R})$ via restriction of functions to U. Denote this subalgebra by $\mathcal{A}_U$.

By applying the Stone–Weierstrass theorem to $C_0(U, \mathbb{R})$, we find that $\mathcal{A}_U$ is dense in $C_0(U, \mathbb{R})$, since it separates points and does not vanish at the points of U by hypotheses. Therefore, as the function $f_{|U}$ is

an element of $C_0(U,\mathbb{R})$, then for $\varepsilon > 0$ there exists $h \in \mathcal{A}_U$ such that $||f_{|U} - h||_{\sup} < \varepsilon$. Hence, let the function $\overline{h} : X \to \mathbb{R}$ defined as

$$\overline{h}(x) = \begin{cases} h(x) & x \in U \\ \\ 0 & x \notin U \end{cases}$$

We have $\overline{h} \in \mathcal{A}$, and that $||f - \overline{h}||_{\sup} < \varepsilon$. Therefore, $\mathcal{A}$ is dense in $C^0(U,\mathbb{R})$, which completes the proof. □

We already know that X_D^{hyb} is locally compact, since every function $G : X_D^{hyb} \to \Sigma(X,D)$ defined in (2.3) is proper, and $\Sigma(X,D)$ is locally compact. Moreover, it is σ-compact because $\Sigma(X,D)$ is clearly σ-compact. Therefore, we can consider $C_c(X_D^{hyb},\mathbb{R})$ with the topology constructed above.

Lemma 2.3.9. Let $\{\mu_t\}_{0\leq t\leq 1}$ be the family defined in 2.19. Then for any function G constructed as in (2.3), the following equivalence holds in the sense of vague convergence of measures:

$$\lim_{t\to 0}\mu_t = \mu_0 \iff \lim_{t\to 0}(G)_*\mu_t = \mu_0$$

Proof. The direct implication follows from the continuity of G. For the converse, assume the convergence $\lim_{t\to 0}(G)_*\mu_t = \mu_0$ holds. Consider then the following subsets of the algebra $C_c(X_D^{hyb},\mathbb{R})$

1. $\mathcal{A}_1 := \{G_D^*\varphi : \varphi \in C_c(\Sigma(X,D),\mathbb{R})\}$.
2. $\mathcal{A}_2 := \{\pi^* g : g \in C^0([0,1],\mathbb{R})\}$.
3. $\mathcal{A}_3$ the set of continuous functions $f \in C_c(X_D^{hyb},\mathbb{R})$ such that $\operatorname{supp}(f) \subseteq \pi^{-1}((0,1])$, together with the constant functions on X_D^{hyb}.

Fixing these sets, consider the subalgebra $\mathcal{A} \subseteq C_c(X_D^{hyb},\mathbb{R})$ generated by functions of the form $f_1 f_2 f_3$, where $f_i \in \mathcal{A}_i$ for $i = 1,,2,3$.

Claim: The subalgebra $\mathcal{A}$ is dense in $C_c(X_D^{hyb},\mathbb{R})$ with respect to the topology described above.

Proof: To prove this, we reduce to verify the assumptions of Proposition 2.3.8. Indeed, let $K \subseteq X_D^{hyb}$ be a compact, then $G(K)$ and $\pi(K)$ are compact. Since $\Sigma(X,D)$ is not compact, then there exists $K_1 \subseteq \Sigma(X,D)$ such that $G(K) \subseteq \mathring{K}_1 \subseteq K_1$. If $\pi(K) \neq [0,1]$, then we consider a compact K_2 of $[0,1]$ such that its interior contains K, otherwise $K_2 = \pi(K)$. With above considerations, note that subalgebras

$$\{\varphi \in C_c(\Sigma(X,D),\mathbb{R}) : \operatorname{Supp}(\varphi) \subseteq \mathring{K}_1\} \quad \text{and} \quad \{g \in C^0([0,1],\mathbb{R}) : \operatorname{Supp}(g) \subseteq \mathring{K}_2\}$$

separate points and not vanishing in all points of $\mathring{K}_1$ and $\mathring{K}_2$, respectively. Consider $L = G^{-1}(K_1) \cap \pi^{-1}(K_2)$. It is clear that L is compact since G_D is proper (see 2.2.11) and $\pi^{-1}(K_2)$ is closed. Note that $K \subseteq L$, moreover, K is contained in the interior of L. In fact, if $\pi(K) \neq [0,1]$, then $K \subseteq G^{-1}(\mathring{K}_1) \cap \pi^{-1}(\mathring{K}_2) \subseteq L$. This implies that $K \subseteq \mathring{L}$, since the interior of a subset is the largest open set contained in such subset.

Now, if $\pi(K) = [0,1]$, then $L = G^{-1}(K_1)$, and it is again clear that K is contained in the interior of L.

Let $x, y \in \mathring{L}$ be two distint points, and consider the following cases:

1. If $\pi(x) = \pi(y) = 0$, then there exists $\varphi \in C_c(\Sigma(X, D), \mathbb{R})$, where its support is contained in K_1, and satisfying $\varphi(x) \neq \varphi(y)$ (Note that $G(x) = x$, and $G(y) = y$). Therefore, consdering $f_1 = G^*\varphi$, f_2 any function in $\mathcal{A}_2$ with support in $\pi^{-1}(K_2)$ and not vanishing in x, and f_3 is a nonzero constant function. We find that $f_1 f_2 f_3$ is a function in $\mathcal{A}$ where $\mathrm{Supp}(f_1 f_2 f_3) \subseteq \mathring{L}$, and is separating the point x and y.

2. If $\pi(x) = \pi(y) > 0$. In this case, can be happen that $G(x) = G(y)$, if this is the case, take $f_1 \in \mathcal{A}_1$ any function with support contained in $G^{-1}(K_1)$ not vanishing at x, $f_2 \in \mathcal{A}_2$ any function with support in $\pi^{-1}(K_2)$ and non-vanishing at x, and $f_3 \in \mathcal{A}_3$ a function separating the points x and y. Otherwise, take f_1 any with $f_1(x) \neq f_1(y)$ with support in $G^{-1}(K_2)$. f_2 any non vanishing function not vanishing in x with support in $\pi^{-1}(K_2)$, and f_3 a nonzero constant function. Therefore, the product of these three functions separates the points x and y.

3. If $\pi(x) \neq \pi(y)$ then take f_1 any function with $f_1(x) = f_1(y)$, the function f_2 is choosen in such way $f_2(x) \neq f_2(y)$, and finally, f_3 is any function with $f_3(x) = f_3(y)$. Then the product again separates such points.

This lead us to conclude that $\mathcal{A}$ separates points.

Now, it is easy to check that for every point of $\mathring{L}$ there exists an element of $\mathcal{A}$ no vanishing in such point. Hence, the conditions of the Proposition 2.3.8 have been verified. It follows that $\mathcal{A}$ is dense in $C_c(X_D^{hyb}, \mathbb{R})$, which completes the proof of the claim. ■

Having proved this claim, it now suffices to prove the assertion. $\lim_{t\to 0} \int f\mu_t = \int f\mu_0$ for every $f \in \mathcal{A}$. Moreover, by linearity, it is enough to prove this for functions that are products of three functions, as previously described. In fact,

$$\begin{aligned}\lim_{t\to 0} \int_{X\setminus D} f_1 f_2 f_3 d\mu_t &= \lim_{t\to 0} \int_{X\setminus D} G^*\varphi \cdot \pi^* g \cdot f_3 d\mu_t \\ &= \lim_{t\to 0} \int_{X\setminus D} G^*\varphi \cdot g(t) \cdot f_3 d\mu_t\end{aligned}$$

Consider the following two cases:

If $\mathrm{Supp}(f_3) \cap \pi^{-1}(\{0\}) = \emptyset$ then there exist $\varepsilon > 0$ such that $f_3(x) = 0$ for every $\pi(x) < \epsilon$, this implies that equality holds, since $\mathrm{Supp}(f_1 f_2 f_3) \subseteq \mathrm{Supp}(f_3) = 0$. Otherwise, f_3 is a constant function, which we suppose to be igual to 1. For this case we have:

$$\begin{aligned}\lim_{t\to 0} \int_{X\setminus D} G^*\varphi \cdot g(t) \cdot f_3 d\mu_t &= \lim_{t\to 0} \int_{X\setminus D} G^*\varphi \cdot g(t) d\mu_t \\ &= \lim_{t\to 0} \int_{\Sigma(X,D)} \varphi \cdot g(t) d(G_*\mu_t) \\ &= \lim_{t\to 0} g(t) \int_{\Sigma(X,D)} \varphi \cdot d(G_*\mu_t)\end{aligned}$$

Since $G_*\mu_t$ converges to μ_0, and φ is a continuous function on $\Sigma(X, D)$ of compact support, it follows that:

$$\lim_{t\to 0} \int_{\Sigma(X,D)} \varphi \cdot g(t) d(G_*\mu_t) = g(0) \int_{\Sigma(X,D)} \varphi \cdot \mu_0$$

Therefore the statement have been prooved since it is verified for the subalgebra $\mathcal{A}$. □

Let $\Sigma(X,D)^{\leq k-1} := \bigcup_{\alpha\notin\Sigma_k}\sigma_\alpha$ denote the closed set of $\Sigma(X,D)$ formed by the union of all cones in the cone complex that are not maximal. Recall that Σ_k denotes the set of indices α such that Y_α is a minimal stratum.

We then define the following open subset:

$$X_D^0 := X_D^{hyb} \smallsetminus \Sigma(X,D)^{\leq k-1}$$

Having defined this set, we now state one of the main results of this chapter, which concerns the degeneration of measures in the hybrid space associated to $\mathcal{P}$-logarithmic forms.

Theorem 2.3.10. *Let $\{\mu_t\}_{0\leq t\leq 1}$ the family of measures defined in* (2.19)*. Then for every continuous function $f\in C_c(X_D^0,\mathbb{R})$ we have*

$$\lim_{t\to 0}\int_{X_D^0} f d\mu_t = \int_{X_D^0} f d\mu_0$$

Proof. If we denote $\Sigma(X,D)^k := \Sigma(X,D)\smallsetminus\Sigma(X,D)^{\leq k-1}$, then by Lemma 2.3.9, it suffices to show that for each $\phi\in C_c(\Sigma(X,D)^k,\mathbb{R})$.

$$\lim_{t\to 0}\int_{\Sigma(X,D)^k}\phi d\left(G^*\mu_t\right) = \int_{\Sigma(X,D)^k}\phi d\mu_0$$

Since $\Sigma(X,D)^k$ is the disjoint union of the interiors of the maximal cones, the function ϕ is determined by the family $\{\phi_\alpha\}_{\alpha\in\Sigma^k}$, where each ϕ_α denotes the restriction $\phi_{|\mathring{\sigma}_\alpha}$. Therefore, the proof reduces to verifying that for each $\alpha\in\Sigma^k$,

$$\lim_{t\to 0}\int_{\sigma_\alpha}\phi_\alpha d\left(G^*\mu_t\right) = \int_{\sigma_\alpha}\phi_\alpha d\mu_0.$$

As the measure μ_0 is defined by $\sum_{Y_\alpha\in\Sigma_k}\left(\int_{Y_\alpha}\mathrm{Res}_{\mathcal{P},Y_\alpha}(\omega)\right)\frac{d\lambda_\alpha}{P_\alpha}$, it therefore suffices to prove that

$$\lim_{t\to 0}\int_{\sigma_\alpha}\phi_\alpha d\left(G^*\mu_t\right) = \left(\int_{Y_\alpha}\mathrm{Res}_{\mathcal{P},Y_\alpha}(\omega)\right)\int_{\sigma_\alpha}\phi_\alpha\frac{d\lambda_\alpha}{P_\alpha}$$

Indeed, since the equivalence in Proposition 2.3.9 is true for any G, consider a locally finite covering of X by adapted chart $\{(U_\gamma,z_\gamma)\}_{\gamma\in\Lambda}$ satisfying the conditions of Proposition 1.2.13. Moreover, since X is compact, we can take this covering to be finite.

Let $\{\chi_\gamma\}_{\gamma\in\Lambda}$ be a partition of unity subordinate to this covering.

Now consider the local functions

$$F_{U_\gamma} : U_\gamma\smallsetminus D\to\sigma_{Y_\gamma}$$

which send a point z to the tuple $(-\log|z_{\gamma_j}|)_{j\in J_\gamma}$.

Define the function

$$F = \sum\chi_\gamma F_{U_\gamma}$$

which satisfies the conditions of Proposition 2.2.3. Therefore this determines a function G, whose construction is specified in (2.3).

Now let U_β be a chart whose associated stratum is not minimal. Then $F_{|\operatorname{Supp}(\chi_\beta)}$ may take values in some maximal cones; however, it is bounded along the rays associated to those irreducible components that have empty intersection with U_β. This implies that there exists $0 < \varepsilon_\beta \leq 1$ such that

$$\operatorname{Supp}(\phi) \cap G(\operatorname{Supp}(\chi_\beta) \times \{t\}) = \emptyset$$

for each $0 < t < \varepsilon_\beta$. Since the covering is finite, there exists $\epsilon > 0$ such that

$$\operatorname{Supp}(\phi) \cap G(\operatorname{Supp}(\chi_\beta) \times \{t\}) = \emptyset$$

for each $0 < t < \varepsilon$ and for every chart U_β whose associated stratum is not minimal.

With this observation, and denoting by Λ_0 the set of indices associated to the charts such that its associated stratum is minimal, we have the following:

$$\begin{aligned}
\lim_{t\to 0} \int_{\Sigma(X,D)^k} \phi d\,(G_* \mu_t) &= \lim_{t\to 0} \int_{X_D^0} (\phi \circ G) d\mu_t \\
&= \lim_{t\to 0} \sum_{\gamma\in\Lambda} \int_{U_\gamma \smallsetminus D} (\phi \circ G)(\chi_\gamma \cdot d\mu_t) \\
&= \lim_{t\to 0} \sum_{\gamma\in\Lambda_0} \int_{U\gamma \smallsetminus D} (\phi \circ G)(\chi_\gamma \cdot d\mu_t) \\
&= \lim_{t\to 0} \sum_{\alpha\in\Sigma_k} \sum_{\gamma\in\Lambda_{0,\alpha}} \int_{U\gamma \smallsetminus D} (\phi_\alpha \circ G)(\chi_\gamma \cdot d\mu_t)
\end{aligned}$$

Where $\Lambda_{0,\alpha} \subseteq \Lambda_0$ denotes the subset of indices γ such that the chart (U_γ, z_γ) is associated precisely with the stratum Y_α.
This decomposition allows us to verify the convergence separately within each cone.

Fix $\alpha \in \Sigma_k$. For each $\gamma \in \Lambda_{0,\alpha}$, we find that $G = \log_{U_\gamma} + t \cdot \psi_\gamma$ on $(U_\gamma \smallsetminus D) \times (0, 1]$, where $\log_{U_\gamma}$ is the function defined in equation (2.20) which is precisely $t \cdot F_{U_\gamma}$, and ψ_γ is a bounded function on U_γ.

Then by the same argument as in Lemma 2.3.6, we have

$$\int_{U\gamma \smallsetminus D} (\phi \circ G)(\chi_\gamma \cdot d\mu_t) = \frac{1}{(2\pi)^k} \lim_{t\to 0} \int_{\mathring{\sigma}_\alpha} \int_{\mathbb{C}^{n-k}} \int_{(S^1)^k} \frac{\phi(u + t \cdot \psi_\gamma)}{P_\alpha(u)} (\chi_\gamma \cdot \eta_U) d\theta dz' du$$

Since $\operatorname{Supp}(\phi)$ is compact and contained in $\Sigma(X, D)^k$, it follows that $\operatorname{Supp}(\phi_\alpha)$ is compact and lies entirely within the interior of the cone σ_α. Therefore, by the same reasoning used in Lemma 2.3.6, the family of functions

$$\frac{\phi(u + t\psi_\gamma)}{P_\alpha(u)} (\chi_\gamma \cdot \eta_U)$$

converges pointwise as t tends to zero to

$$\frac{\phi(u)}{P_\alpha(u)} (\chi_\gamma \cdot \eta_U)_{|Y_\alpha}.$$

Moreover, this family is bounded by an integrable function. Hence, the Dominated Convergence Theorem

applies, and we conclude:

$$\frac{1}{(2\pi)^k}\lim_{t\to 0}\int_{\mathring{\sigma}_\alpha}\int_{\mathbb{C}^{n-k}}\int_{(S^1)^k}\frac{\phi(u+\psi_\gamma)}{P_\alpha(u+\psi_\gamma)}(\chi_\gamma\cdot\eta_U)d\theta dz' du = \left(\int_{\mathring{\sigma}_\alpha}\frac{\phi(u)}{P_\alpha(u)}du\right)\cdot\frac{1}{(2\pi)^k}\int_{\mathbb{C}^{n-k}}\int_{(S^1)^k}(\chi_\gamma\cdot\eta_U)_{|Y_\alpha}d\theta dz'$$

Finally, by proposition 2.3.6, we have

$$\frac{1}{(2\pi)^k}\int_{\mathbb{C}^{n-k}}\int_{(S^1)^k}(\chi_\gamma\cdot\eta_U)_{|Y_\alpha}d\theta dz' = \int_{\mathbb{C}^{n-k}}(\chi_\gamma\cdot\eta_U)_{|Y_\alpha}d\theta dz'$$

In consequence we conclude:

$$\begin{aligned}\lim_{t\to 0}\sum_{\gamma\in\Lambda_{0,\alpha}}\int_{U\gamma\smallsetminus D}(\phi\circ G)(\chi_\gamma\cdot d\mu_t) &= \sum_{\gamma\in\Lambda_{0,\alpha}}\left(\int_{\mathring{\sigma}_\alpha}\frac{\phi(u)}{P_\alpha(u)}du\right)\cdot\int_{\mathbb{C}^{n-k}}(\chi_\gamma\cdot\eta_{U_\gamma})_{|Y_\alpha}\\ &= \left(\int_{\mathring{\sigma}_\alpha}\frac{\phi(u)}{P_\alpha(u)}du\right)\cdot\int_{Y_\alpha}\mathrm{Res}_{\mathcal{P},Y_\alpha}(\omega)\end{aligned}$$

This completes the proof. □

The above result does not extend directly to the hybrid space X_D^{hyb}, as the measure $\frac{d\lambda_\alpha}{P_\alpha}$ is not locally finite along the faces of the maximal cone σ_α. Consequently, the Dominated Convergence Theorem cannot be applied in this broader setting. However, if we instead consider logarithmic (n,n)-forms, Theorem 2.3.10 can be extended to the hybrid space X_D^{hyb}, as we will demonstrate below.

Recall that, since ω is a logarithmic (n,n)-form, for each adapted chart (U,z), if we assume that the intersection of the locus zero $\{z_i=0\}$ for $i=1,\dots,l$ coincides with $U\cap Y$, where Y is the stratum associated with U, then locally we have

$$\omega_{|U} = \eta_U\frac{dz_1}{z_1}\wedge\frac{d\overline{z}_1}{\overline{z}_1}\wedge\ldots\wedge\frac{d\overline{z}_l}{\overline{z}_l}$$

where η_U is a continuous $(n-k,n-k)$-form defined on U. Since ω is a particular case of $\mathcal{P}$-logarithmic form, the local residue can be defined for each stratum Y, and the independence of the residue from the choice of adapted chart follows by the same argument as in Proposition 2.3.2. Therefore, we may define the residue $\mathrm{Res}_Y(\omega)$ for arbitrary strata Y, not only for the minimal ones.

With this in place, and recalling that all minimal strata have the same codimension, say k, we consider the following family of measures:

$$\mu_t = \begin{cases} \frac{t^k}{(4\pi i)^k}\omega & t\neq 0\\ \sum_{Y_\alpha\in\Sigma_k}\left(\int_{Y_\alpha}\mathrm{Res}_{Y_\alpha}(\omega)\right)d\lambda_\alpha & t=0\end{cases} \tag{2.22}$$

where, as above, Σ_k denotes the set of indices associated to the maximal cones.

With these considerations, we state the following lemma:

Lemma 2.3.11. Let $\{\mu_t\}_{0\leq t\leq 1}$ be a family of measures as in (2.22), associated with the logarithmic (n,n)-form ω. Then for any adapted chart (U,z) whose associated stratum, say Y, is not minimal, and

for every $\chi \in C_c(U, \mathbb{R})$, the family of measures $\{(\log_U)_*(\chi\mu_t)\}_{0<t\leq 1}$ converges vagely to the zero measure on the cone σ.

Proof. Let Y be a stratum of codimension $l < k$, and (U, z) an adapted chart with associated stratum equal to Y. We have to prove that for each $\phi \in C_c(\sigma, \mathbb{R})$

$$\lim_{t\to 0} \int_{U\smallsetminus D} \phi \cdot d\left(\log_{U*}(\chi\mu_t)\right) = 0$$

Indeed, we have that

$$\begin{aligned}\lim_{t\to 0} \int_{U\smallsetminus D} (\phi \circ \log_U) \cdot \chi d\mu_t &= \lim_{t\to 0} \int_{(\mathbb{C}^*)^l\times\mathbb{C}^{n-l}} (\phi \circ \log_U) \cdot \chi \cdot \left[\frac{t^k}{(4\pi i)^k}\omega_{|U}\right] \\ &= \lim_{t\to 0} \int_{(\mathbb{C}^*)^l\times\mathbb{C}^{n-l}} (\phi \circ \log_U) \cdot \chi \cdot \left[\frac{t^k\eta_U}{(4\pi i)^k \prod_{i=1}^l z_i\overline{z}_i}\right] dz_1 d\overline{z}_1 \ldots d\overline{z}_l\end{aligned}$$

Using polar coordinates as in Lemma 2.3.6, we have

$$\int_{(\mathbb{C}^*)^l\times\mathbb{C}^{n-l}} (\phi \circ \log_U) \left[\frac{t^k\chi\cdot\eta_U}{(4\pi i)^k \prod_{i=1}^l z_i\overline{z}_i}\right] dz_1 d\overline{z}_1 \ldots d\overline{z}_n = \int_{(\mathbb{R}_{>0})^l} \int_{\mathbb{C}^{n-l}} \int_{(S^1)^l} t^{k-l} \frac{\phi(u)\cdot\chi\eta_0}{(2\pi)^k(4\pi i)^{k-l}} d\theta dz' du$$

where η_0 is such that $\eta_U = \eta_0 dz_{l+1} \wedge d\overline{z}_{l+1} \wedge \ldots \wedge d\overline{z}_n$. Since $k - l > 0$, and both $\mathrm{Supp}(\chi)$ and $\mathrm{Supp}(\phi)$ are compact, it follows that the product $t^{k-l}\varphi \cdot \chi \cdot \eta_0$ is are compact, it follows that the product and bounded. Arguing as in the proof of Lemma 2.3.6, we see that this function is also bounded by an integrable function. Moreover, as $t \to 0$ the product converges pointwise to zero because $\varphi \cdot \chi \cdot \eta_0$ is in fact bounded. Therefore, by the Dominated Convergence Theorem, the corresponding integral tends to zero. □

Having assembled all the necessary ingredients, we now state the following convergence theorem in the hybrid space, in the case of logarithmic (n, n)-forms:

Theorem 2.3.12. *Let $\{\mu_t\}_{0\leq t\leq 1}$ be a family of measures as in* (2.22)*, associated with the logarithmic (n, n)-form ω. Then for each continuous function $f \in C_c(X_D^{hyb}, \mathbb{R})$ we have*

$$\lim_{t\to 0} \int_{X_D^{hyb}} f d\mu_t = \int_{X_D^{hyb}} f d\mu_0$$

Proof. By Lemma 2.3.9, it suffices to reduce to the case

$$\lim_{t\to 0} \int_{X_D^{hyb}} f d(G_*\mu_t) = \int_{X_D^{hyb}} f d\mu_0$$

and this case follows from a partition of unity argument together with Lemmas 2.3.7 and 2.3.11, as the reasoning is identical to that in the proof of Theorem 2.3.10. □

Chapter 3

Hybrid spaces for toroidal DM stacks

Throughout this chapter, $\mathscr{X}$ will denote a smooth proper Deligne-Mumford stack of finite type over $\mathbb{C}$, and $\pi : \mathscr{X} \to X$ its coarse moduli space. Recall that in this situation, X is a proper algebraic space and π is a proper map [KM97]. The fact that $\mathscr{X}$ is proper, in particular separated, implies that $\mathscr{X}$ admits an étale presentation $P \to \mathscr{X}$ from a scheme P. Since $\mathscr{X}$ is smooth, we see that P is smooth over $\mathbb{C}$, and therefore regular. We will denote by $\mathscr{X}^{an}$ the complex analytic stack associated with $\mathscr{X}$. As we mention in chapter 1, in [Toe99] it is shown that $\pi^{an} : \mathscr{X}^{an} \to X^{an}$ is the coarse moduli space in the category of analytic spaces of $\mathscr{X}^{an}$. Moreover, X^{an} is Haussdorff and compact since X is proper. Moreover, X^{an} is the coequalizer of $(P \times_{\mathscr{X}} P)^{an} \rightrightarrows P^{an}$ in the category of analytic spaces. This leads us to conclude that X^{an} is a proper étale Lie groupoid (see Definition 1.3.6). Properness follows from the fact that $\mathscr{X}$ is separated.

In this chapter, we study how the toroidal embedding $(\mathscr{X}, \mathscr{U})$ induces a natural stratification on the analytification X^{an} of the coarse moduli space of $\mathscr{X}^{an}$. We will show that this stratification can be refined into strata consisting of points with the same stabilizer; see Section 3.2.15 for further details.

Once this stratification is established, we will analyze the resulting topological properties. In particular, we show that X^{an} admits a triangulation adapted to this stratification (see 3.2.24). This result is of particular significance, as it enables the construction of a function analogous to that of Section 2.2.3, which plays a key role in defining hybrid spaces in the context of Lie groupoids.

3.1 Local structure of analytic spaces and coverings

As mentioned in the Introduction to this chapter, X^{an} is a proper étale Lie groupoid, in particular, it is an orbifold (see Remark 1.3.9). Most orbifolds arising from the analytification of coarse moduli spaces of DM stacks are ineffective. An example of it, is the analytic coarse moduli space $M_{g,n}$ of $\mathscr{M}_{g,n}$. Because in the case of $g = 2, n = 0$ there are hyperelliptic curves with non-trivial involution; see [AHCG11, Chap IX, Prop 4.11].

Since our interest lies in the underlying topological space of the analytic space X^{an}, the objective of this section is to exhibit a complex effective orbifold determined by a collection of orbifold charts, whose associated analytic space is precisely X^{an}, and such that each chart possesses properties compatible with the toroidal structure of the pair (X^{an}, U^{an}).

Before establishing this objective, we begin by showing a topological property of X^{an} that is useful to keep in mind, as we will be working with suitable coverings of X^{an}, where conditions such as metrizability and paracompactness are required. Although this fact is well known, we will include a proof for the sake of clarity.

Proposition 3.1.1. The analytic space X^{an} is metrizable and second countable

Proof. By Proposition 1.4.23, we have a surjective and proper morphism $Z \to \mathscr{X}$, from a quasi-projective $\mathbb{C}$-scheme Z. As $\pi : \mathscr{X} \to X$ is proper and surjective, then the composition $f : Z \to X$ is proper and surjective. Therefore, using GAGA arguments for the case of algebraic spaces, we find that $f^{an} : Z^{an} \to X^{an}$ is proper and surjective. Since X^{an} is locally the quotient V/G, with V being a smooth complex manifold and G a finite group acting on V; see [Gil84, Prop. 1.1] or [Vis89, Prop. 2.6], it is clear that this quotient is locally compact. Hence, we deduce that X^{an} is locally compact. As Z^{an} is contained in a projective manifold, Z^{an} is metrizable and second countable.

By the properness of f^{an}, we have $f^{an} : Z^{an} \to X^{an}$ is a closed map, with compact fibers. This implies that both, metrizability and second countability, are inherited by X^{an}; see [MH56, Theorem 2]. □

Recall that in Chapter 1, we introduced the notion of small toric charts (see Definition 1.6.9), following the framework established in [ACP12]. These charts are important because they provide information about the structure of the skeleton associated with the toroidal embedding $(\mathscr{X}, \mathscr{U})$.

As we are interested in studying X^{an}, the analytification of a small toric chart is not appropriate for local analytic considerations. To address this, we introduce a slight modification in the analytic framework. In this context, the relevant local models are analytic balls, which are particularly useful because they are contractible. Concretely, given a smooth complex manifold, one can equip it with a Riemannian metric, which in turn induces a distance function, making the notion of an open ball both precise and well-defined.

With this perspective, we now introduce the following proposition.

Proposition 3.1.2. For every $\mathtt{x} \in X^{an}$, there exists a triple (V, G, φ), where $V \subseteq \mathbb{C}^n$ is an open ball, G is a finite group acting on V, and $\varphi : V \to X^{an}$ is a G-invariant analytic map, that factorize through $\mathscr{X}^{an}$, and induces an open immersion $V/G \hookrightarrow X^{an}$ whose image contains the point $\mathtt{x}$, and satisfying the following conditions:

1. The action of G in V fixes the center of the complex ball, and this element is mapped to $\mathtt{x}$.
2. The divisor $V \smallsetminus (V \times_{\mathscr{X}^{an}} \mathscr{U}^{an})$ is a simple normal crossing divisor, invariant under the action of G
3. The stratification induced by the divisor has a unique minimal stratum, and the center of V is contained in this minimal stratum.

Proof. Let $\{V_i\}_{i\in I}$ be a collection of small toric charts for $\mathscr{X}$ such that $\coprod_{i\in I} V_i \to \mathscr{X}$ is an étale presentation. For $\mathtt{x} \in X^{an}$, consider a point $\xi \in \mathscr{X}^{an}$ that maps to $\mathtt{x}$. Denote by G the stabilizer of ξ, and let V_i be a small toric chart such that its analytification V_i^{an} contains a point u which maps to ξ. Moreover, assume that u lies in the unique minimal stratum of V_i. Recall that since $\mathscr{X}$ is smooth, each analytification of the small toric charts are complex manifolds.

Furthermore, there exists an open neighborhood $V \subseteq V_i^{an}$ of u, with G acting on V by biholomorphisms such that u is fixed under this action, and a G-invariant analytic map $\varphi : V \to X^{an}$ which induces an open immersion $\widetilde{\varphi} : V/G \to X^{an}$; see [Gil09, Prop 1.1], or Remark 1.3.9.

Taking the underlying smooth real manifold of V, we can consider a Riemannian metric on V invariant under the action of G. This allows us to consider an open real ball around the point u, which is invariant under the action of G, as this is an open set in a complex manifold, we have that it is a complex manifold. Therefore, without loss of generality, assume that V is a ball, where its center is precisely the point u.

Note that $(V, V \smallsetminus (V \times_{\mathscr{X}^{an}} \mathscr{U}^{an}))$ has a unique minimal stratum, since V contains a point of the minimal stratum of V_i^{an}. Moreover, the divisor $V \smallsetminus (V \times_{\mathscr{X}^{an}} \mathscr{U}^{an})$ is a simple normal crossing divisor, as is contained in a small toric chart. □

Remark 3.1.3. In the above proposition, note that the divisor is invariant under the action of G, since the open set $V \times_{\mathscr{X}^{an}} \mathscr{U}^{an}$ is invariant under the action of G. Therefore, each element of G permutes the irreducible components of this divisor.

We can refine the triple (V, G, φ) in Proposition 3.1.2 to obtain additional properties, as we will see in the following proposition. This proposition is based on [AHCG11, Lemma 6.2, Chapter IX].

Proposition 3.1.4. Let (V, G, φ) be a triple that satisfies the conditions of Proposition 3.1.2, with $u \in V$ being the center of this open ball. If E denotes the simple normal crossing divisor fixed by the action of G. Then there exists an adapted chart (U, z); see definition 2.1.8, that contains the point u, such that the action of G in U is linear.

Proof. It is clear that V contains an adapted chart $(U, z = (z_1, \ldots, z_n))$ that contains u, and is invariant under the action of G. Denote by $E_{j_1}, \ldots, E_{j_k}$ the irreducible components of E, then all these irreducible components have nonempty intersection with (U, z), since this chart contains the point u. Assume that zero locus of z_i is equal to $E_{j_i} \cap U$ for $i = 1, \ldots, k$. Note that each $g \in G$ induces a new system coordinates on U, given by $zg = (z_1 g, \ldots, z_n g)$, and the zero locus of $z_i g$ for $i = 1, \ldots, k$ is the set

$$\{u \in U : z_i(gu) = 0\} = \{w \in gU : z_i(w) = 0\} = g^{-1}E_i.$$

Since g acts on the divisor E, it follows that there exists a permutation $\sigma_g \in \mathcal{S}_k$ such that for $1 \leq i \leq k$, $E_{\sigma_g(i)}$ is equal to the zero locus of $z_i g$.

Now, consider the differential $dg : T_u^{1,0}U \to T_u^{1,0}U$ induced by g. Since $gE_i = E_{\sigma_{g^{-1}}(i)}$ for each $i = 1, \ldots, k$, it follows

$$dg(T_u E_i) = T_u E_{\sigma_{g^{-1}}(i)}.$$

Let $E_i' \subseteq \mathbb{C}^n$ denote the hyperplane defined by $z_i = 0$, for $i = 1, \ldots, k$. Identifying $T_u U$ with $\mathbb{C}^n$ via the coordinate chart z, we deduce that

$$dg(E_i') = E'_{\sigma_{g^{-1}}(i)}.$$

Now considering the composition $(dg)^{-1}g$, we see that the hyperplanes $E'_{i's}$ are preserved in $\mathbb{C}^n$ under this map. That is,

$$(dg)^{-1}g(E_i') = E_i' \quad \text{for} \quad i = 1 \ldots k.$$

Define the transformation

$$\alpha := \frac{1}{|G|} \sum_{g \in G} (dg)^{-1} g.$$

This defines a new coordinate chart $(U, \alpha \circ z)$, after possibly shrinking U, which preserves the coordinate hyperplanes corresponding to the irreducible components of the divisor. The fact that this is indeed a coordinate chart follows from the fact that α is a local biholomorphism around the origin. To see this, consider the differential of α

$$\begin{aligned} d\alpha &= \frac{1}{|G|}\sum_{g\in G} d(dg^{-1}\circ g) \\ &= \frac{1}{|G|}\sum_{g\in G} (d(dg^{-1})\circ g)dg \qquad (\text{ chain rule}) \\ &= \frac{1}{|G|}\sum_{g\in G} (dg^{-1}\circ g)dg \qquad (\text{since the differentials are linear maps}) \end{aligned}$$

Now, consider the differential at the origin point $z(u) = 0$ in $\mathbb{C}^n$. Since $g(u) = u$, each term $(dg^{-1}g)(0) \cdot dg(0)$ is simply the identity. It follows that $d\alpha_{|0}$ is the identity. Therefore, there exists a neigborhood around the zero in $\mathbb{C}^n$ such that α is a biholomorphism.

Moreover, for each $g \in G$ it is easy to verify that

$$\alpha g = dg\alpha$$

Hence, in these new coordinates, the action of G becomes linear and permutes the coordinate hyperplanes corresponding to the irreducible components of the divisor. □

Let (U, G, ϕ) and (V, G', ϕ') be two orbifold charts of X^{an}, as in Proposition 3.1.4. Denote by $\mathtt{U}$ and $\mathtt{V}$ the images of U and V under the maps ϕ and ϕ', respectively. Suppose that $\mathtt{V} \subseteq \mathtt{U}$. Then we ask: what is the relation between these two representations? Although this fact is well known and nicely described in the literature, we will carry out the construction, as it will be useful in the following sections.

Let $\mathcal{U} = [U/G]$ denote the quotient stack associated to the Lie groupoid $G \times U \rightrightarrows U$ in the category of analytic spaces. Recall that we are under the assumption that there is a chain of open embeddings $\mathtt{V} \hookrightarrow \mathtt{U} \hookrightarrow X^{an}$. Moreover, assume that the action of G' on V fixes a point y.

We consider the following diagram:

$$\begin{array}{ccccc} [V/G'] & \longrightarrow & \mathscr{X}^{an} & \longleftarrow & \mathcal{U} \\ \downarrow & & \downarrow & & \downarrow \\ \mathtt{V} & \hookrightarrow & X^{an} & \hookleftarrow & \mathtt{U} \end{array} \tag{3.1}$$

Since the right-hand square in (3.1) is cartesian, and since $\mathtt{U} \hookrightarrow X^{an}$ is an open embedding, it follows that $\mathcal{U} \to \mathscr{X}^{an}$ is an open embedding. Similarly, we state that $[V/G'] \to \mathscr{X}^{an}$ is an open embedding. Therefore, as $\mathtt{V} \hookrightarrow \mathtt{U}$ is an open embedding, we have an open embedding $[V/G'] \to \mathcal{U}$.

Given the étale presentation $\pi : U \to \mathcal{U}$, and identifying the quotient stack $[V/G']$ with an open substack

$\mathcal{V} \subseteq \mathcal{U}$, we can consider the following diagram:

$$\begin{array}{ccccc} V \times_{\mathcal{V}} \pi^{-1}(\mathcal{V}) & \xrightarrow{p'} & \pi^{-1}(\mathcal{V}) & \hookrightarrow & U \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi_{\pi^{-1}(\mathcal{V})}} & & \downarrow{\scriptstyle \pi} \\ V & \xrightarrow{\pi'} & \mathcal{V} & \hookrightarrow & \mathcal{U} \end{array} \tag{3.2}$$

From this diagram, we have that $V \times_{\mathcal{V}} \pi^{-1}(\mathcal{V}) \cong V \times_{\mathcal{U}} U$.

Following the definition of an analytic stack quotient; see 1.4.27, we have that the map p is a principal G-bundle and G'-equivariant at the same time. Similarly, the map p' is a principal G'-bundle, and G-equivariant. This setup induces two group actions on the fiber product $V \times_{\mathcal{V}} \pi^{-1}(\mathcal{V})$. Assuming that the actions of G on U, and G' on V are left actions, it follows that both induced actions on the fiber product are also left actions. Moreover, these actions commute with each other. The commutativity of the actions follows from the commutative diagram in (3.3), where we denote $\pi^{-1}(\mathcal{V})$ by U' to have a cleaner presentation of the diagram.

$$\begin{array}{ccccc} (G' \times V) \times_{\mathcal{V}} (G \times U') & \xrightarrow{\mu_{G'} \times Id_{G \times U'}} & V \times_{\mathcal{V}} (G \times U') & \longrightarrow & G \times U' \\ \downarrow{\scriptstyle Id_{G' \times V} \times \mu_G} & & \downarrow{\scriptstyle Id_V \times \mu_G} & & \downarrow{\scriptstyle \mu_G} \\ (G' \times V) \times_{\mathcal{V}} U' & \xrightarrow{\mu_{G'} \times Id_{U'}} & V \times_{\mathcal{V}} U' & \longrightarrow & U' \\ \downarrow & & \downarrow & & \downarrow \\ G' \times V & \xrightarrow{\mu_{G'}} & V & \longrightarrow & \mathcal{V} \end{array} \tag{3.3}$$

Recall that μ_G and $\mu_{G'}$ denote the actions of G and G' on U' and V, respectively.

Consider again the left-hand side of the diagram (3.2), let $p^{-1}(y)$ denote the fiber over the point y under p. Since p is a pincipal G-bundle, the group G acts transitively on $p^{-1}(y)$. Moreover, because V is an open ball, in particular, contractible, and p is etale, the following definition makes sense:

Definition 3.1.5. For every $z \in p^{-1}(y)$, there exists a unique holomorphic section

$$\varphi_z : V \to V \times_{\mathcal{U}} \pi^{-1}(\mathcal{V})$$

Such that $p \circ \varphi_z = \mathrm{id}_V$ and $\varphi_z(y) = z$. Furthermore, φ_z is a biholomorphism onto its image.

As a consequence of this definition, we have the decomposition

$$V \times_{\mathcal{U}} \pi^{-1}(\mathcal{V}) \;=\; \coprod_{z \in p^{-1}(y)} \varphi_z(V) \tag{3.4}$$

Since G acts transitively on $p^{-1}(y)$, we can fix a point $z_0 \in p^{-1}(y)$, such that for each $z \in p^{-1}(y)$ there is a unique $g \in G$ with $z = gz_0$. This implies that for every section φ_z we have

$$\varphi_z(y) = gz_0 = g\varphi_{z_0}(y)$$

Therefore, the section φ_z agrees with $g\varphi_{z_0}$. This allows us to give a description of (3.4) in terms of the distinguished section φ_{z_0}. That is

$$V \times_{\mathcal{U}} \pi^{-1}(\mathcal{V}) = \coprod_{g\in G} \varphi_{gz_0}(V)$$

This representation of the fiber product provides, as we will see below, an explicit description of the actions of the groups G and G', once a particular section has been fixed, in this case, the section φ_{z_0}.

1. Let $\gamma \in G$ and consider the point $\varphi_{gz_0}(w) \in V \times_{\mathcal{U}} \pi^{-1}(\mathcal{V})$. Since $\gamma\varphi_{z_0}$ defines a section, it follows that $\gamma\varphi_{gz_0}(w)$ is equal to $\varphi_{(\gamma g)z_0}(w)$.

2. Let $\gamma' \in G'$, and consider $\varphi_{gz_0}(w)$ as above. Since p is G'-equivariant, we find that $\gamma'\varphi_{gz_0}(w)$ satisfies the following
$$p(\gamma'\varphi_{gz_0}(w)) = \gamma' p(\varphi_{gz_0}(w)) = \gamma' w$$
This fact implies that, for a fixed $g \in G$, the function $\gamma'\varphi_{gz_0}\gamma'^{-1}$ defines a section. Now, if we want to see the description of this section in terms of the element z_0, we have to evaluate it at the point y. That is,
$$\begin{aligned}
\gamma'\varphi_{gz_0}\gamma'^{-1}(y) &= \gamma'\varphi_{gz_0}(\gamma'^{-1}y) \\
&= \gamma'\varphi_{gz_0}(y) \qquad (y \text{ is invariant under the action of } G') \\
&= \gamma' g z_0 \qquad (\text{Property which is satisfied by the section } \varphi_{gz_0}) \\
&= g\gamma' z_0 \quad (\text{Commutativity of actions } G \text{ and } G') \\
&= g\rho_{z_0}(\gamma')^{-1} z_0 \\
&= \varphi_{(g\rho_{z_0}(\gamma')^{-1})z_0}(y)
\end{aligned}$$
Here, $\rho_{z_0}(\gamma')^{-1}$ is the unique element in G such that $\gamma' z_0 = \rho_{z_0}(\gamma')^{-1} z_0$. The Proposition 3.1.7 below gives us a good reason to consider the element $\rho_{z_0}(\gamma')^{-1}$ in this sense. The uniqueness of $\rho_{z_0}(\gamma')^{-1}$ follows from the fact that G' acts on the fiber $p^{-1}(y)$, because y is invariant under the action of G', and G acts transitively on $p^{-1}(y)$.
Therefore, we conclude that $\gamma'\varphi_{gz_0}\gamma'^{-1} = \varphi_{g\rho_{z_0}(\gamma')^{-1}z_0}$. Consequently, the action of γ' on the element $\varphi_{gz_0}(w)$ is given by
$$\gamma'\varphi_{gz_0}(w) = \gamma'\varphi_{gz_0}(\gamma'^{-1}\gamma' w) = \gamma'\varphi_{gz_0}\gamma'^{-1}(\gamma' w) = \varphi_{g\rho_{z_0}(\gamma')^{-1}z_0}(\gamma' w) \tag{3.5}$$

Remark 3.1.6. 1. Since p' is a principal G'-bundle, it follows that $p'(\rho_{z_0}(\gamma')^{-1} z_0) = p'(\gamma' z_0) = p'(z_0)$. Now, using the fact that p' is G-invariant we also have $p'(g_{\gamma'} z_0) = g_{\gamma'} p'(z_0)$. These two facts together imply that $g_{\gamma'}$ belongs to the stabilizer group $\mathrm{Stab}_G(p'(z_0))$.
Denoting by H the above stabilizer group, we can consider the map $\rho_{z_0} : G' \to H$, where each element $\gamma' \in G'$ is sent to the unique element $\rho_{z_0}(\gamma') \in G$ which satisifies the equation
$$\gamma' z_0 = \rho_{z_0}(\gamma')^{-1} z_0 \tag{3.6}$$
That is, the element $\rho_{z_0}(\gamma')$ is exactly the inverse of $g_{\gamma'}$ defined previously.

2. If z is another element in the fiber $p^{-1}(y)$, an immediate question arises: How are ρ_{z_0} and ρ_z related?. To answer this, note that if $z, z_0 \in p^{-1}(y)$, then we have $z = gz_0$ for some element $g \in G$. Then writing the expression $\gamma' z = \rho_z(\gamma')^{-1} z$ in terms of z_0, we obtain
$$\gamma' g z_0 = \rho_z(\gamma')^{-1} g z_0.$$

Using commutativity on the left side and applying equation (3.6), we obtain the expression

$$\rho_z(\gamma') = g\rho_{z_0}(\gamma')g^{-1}$$

In this case, $\rho_z : G' \to gHg^{-1}$. This expression is expected because the value $p'(z)$ is in the orbit of $p'(z_0)$, and therefore the stabilizer of $p'(z)$ is a conjugate of the stabilizer of $p'(z_0)$.

Proposition 3.1.7. The map $\rho_{z_0} : G \to H$ is a homomorphism of groups. Moreover, it is an isomorphism.

Proof. Consider γ_1 and γ_2 two elements in G'. Then we have following sequence of equalities:

$$\begin{aligned}
\rho_{z_0}(\gamma_1\gamma_2)^{-1}z_0 &= \gamma_1\gamma_2 z_0 \quad \text{(Property of the map } \rho_{z_0} \text{ applied to } \gamma_1\gamma_2) \\
&= \gamma_1\rho_{z_0}(\gamma_2)^{-1}z_0 \quad \text{(Property of the map } \rho_{z_0} \text{ applied to } \gamma_2) \\
&= \rho_{z_0}(\gamma_2)^{-1}\gamma_1 z_0 \quad \text{(commutativity of the action } G \text{ and } G') \\
&= \rho_{z_0}(\gamma_2)^{-1}\rho_{z_0}(\gamma_1)^{-1}z_0 \quad \text{(Property of the map } \rho_{z_0} \text{ applied to } \gamma_1) \\
&= (\rho_{z_0}(\gamma_1)\rho_{z_0}(\gamma_2))^{-1}z_0
\end{aligned}$$

In order to prove that the groups are isomorphism, fix $x = p'(z_0)$. For injectivity, let $\gamma' \in G'$ with $g_{\gamma'} = e_G$. This fact implies $\gamma' z_0 = z_0$. Since p' is a principal G'-bundle, the group G' acts freely and transitively on the fiber ${p'}^{-1}(x)$. Therefore, $\gamma' = e_{G'}$.
To prove the surjectivity, let $h \in H$. Hence $p'(hz_0) = hx = x$, this implies that $hz_0 \in {p'}^{-1}(x)$. Since G' acts freely on the fiber of x, it follows that there exists a unique $\gamma' \in G'$ with $\gamma' z_0 = hz_0$. From this fact, it is clear that ${\gamma'}^{-1}$ is sent to the element h. □

Note that the section φ_{z_0} plays an important role, as it helps us to establish a function from V to U, through the composition $p' \circ \varphi_{z_0}$. Denote this function by $f_{z_0} : V \to U$. Since this function is determined by the section φ_{z_0}, which in turn depends only on the point z_0, the construction of f_{z_0} depends on the choice of an element in $p^{-1}(y)$.

Proposition 3.1.8. Let $\rho_{z_0} : G' \to H$ be the homomorphism of Proposition 3.1.7. Then the map $f_{z_0} : V \to U$ is ρ_{z_0}-equivariant. Moreover, it is a biholomorphism on its image.

Proof. Let γ' be an element of G', and w an element of V. Then we have the following sequence of equalities:

$$\begin{aligned}
f_{z_0}(\gamma' w) &= p' \circ \varphi_{z_0}(\gamma' w) \\
&= p'(\gamma' \varphi_{\rho_{z_0}(\gamma)z_0}(w)) \quad \text{(Consequence of the equation (3.5))} \\
&= p'(\varphi_{\rho_{z_0}(\gamma)z_0}(w)) \quad (p' \text{ is a principal } G'\text{-bundle}) \\
&= p'(\rho_{z_0}(\gamma)\varphi_{z_0}(w)) \quad \text{(Action of the group } G \text{ on the sections)} \\
&= \rho_{z_0}(\gamma)p'(\varphi_{z_0}(w)) \quad (p' \text{ is } G\text{-equivariant }) \\
&= \rho_{z_0}(\gamma)f_{z_0}(w).
\end{aligned}$$

It remains to show that this is a biholomorphism onto its image. Since p' is an open map, it suffices to prove that it is injective.

Let $x_1, x_2 \in V$ be two points such that $f_{z_0}(x_1) = f_{z_0}(x_2)$. This means that the points $\varphi_{z_0}(x_1)$ and $\varphi_{z_0}(x_2)$ lie in the same fiber of the map p'. Since p' is a G'-bundle, we see that there exists $\gamma' \in G'$

with $\gamma'\varphi_{z_0}(x) = \varphi_{z_0}(x_2)$. Applying p to both sides of the equation and using the G'-equivariance of the function p, we obtain:

$$x_2 = p \circ \varphi_{z_0}(x_2) = p(\gamma'\varphi_{z_0}(x_1)) = \gamma' p(\varphi_{z_0}(x_1)) = \gamma' x_1$$

Now, by equation (3.5) we have $\gamma'\varphi_{z_0}(x_1) = \varphi_{\rho_{z_0}(\gamma')^{-1}z_0}(\gamma' x_1)$. This implies $\varphi_{\rho_{z_0}(\gamma')^{-1}z_0}(x_2) = \varphi_{z_0}(x_2)$. As the images of the sections are disjoint, we conclude that

$$\rho_{z_0}(\gamma')^{-1} = e_G$$

and therefore, $\gamma' = e_{G'}$, because ρ_{z_0} is injective. It follows that $x_1 = x_2$, completing the proof of injectivity. □

The following proposition will play a key role, as it provides a way to express $\pi^{-1}(\mathcal{V})$ as a disjoint union of copies of V. This fact will be especially important, as it allows us to induce functions on $\pi^{-1}(\mathcal{V})$ from functions on V, as we will see in the section 3.5.

Proposition 3.1.9. Following the considerations of Propositions 3.1.7 and 3.1.8. We have

$$\pi^{-1}(\mathcal{V}) \cong G \times^{G'} V$$

Proof. The idea of the proof is to show that there exists a surjective map $\xi : G \times V \to \pi^{-1}(\mathcal{V})$ such that $\xi(g_1, x_1) = \xi(g_2, x_2)$ if only if $(g_1, x_1) \sim (g_2, x_2)$. Recall that $(g_1, x_1) \sim (g_2, x_2)$ if only if there exists $\gamma' \in G'$ such that $x_1 = \gamma' x_2$ and $g_1 = g_2\rho_{z_0}(\gamma')^{-1}$.

Define then the map ξ by $\xi(g, x) = g f_{z_0}(x)$. This map is clearly surjective, since p' is surjective. To prove that $\pi^{-1}(\mathcal{V})$ and $G \times^{G'} V$ are biholomorphic, suppose that $\xi(g_1, x_1) = \xi(g_2, x_2)$. Denote by g the expression $g_2^{-1}g_1$, then

$$\begin{aligned} p'(\varphi_{z_0}(x_2)) &= g_2^{-1}g_1 p'(\varphi_{z_0}(x_1)) \\ &= p'(g_2^{-1}g_1\varphi_{z_0}(x_1)) \quad (\text{Since } p' \text{ is } G\text{-equivariant}) \\ &= p'(g\varphi_{z_0}(x_1)) \end{aligned}$$

Note that $\varphi_{z_0}(x_2)$ and $g\varphi_{z_0}x_1$ are in the same fiber under the map p'. Since p' is a G'-bundle, we see that there exists $\gamma' \in G'$ such that $\gamma'\varphi_{z_0}(x_2) = g\varphi_{z_0}(x_1)$. By equation 3.5, we see that $\gamma'\varphi_{z_0}(x_2)$ is equal to $\varphi_{\rho_{z_0}(\gamma')^{-1}z_0}(\gamma' x_2)$. This implies that

$$\varphi_{\rho_{z_0}(\gamma')^{-1}z_0}(\gamma' x_2) = g\varphi_{z_0}(x_1). \tag{3.7}$$

Applying p to both sides of the equality, we obtain:

$$\gamma' x_2 = p(\varphi_{\rho_{z_0}(\gamma')^{-1}z_0}(\gamma' x_2)) = p(g\varphi_{z_0}(x_1)) = p(\varphi_{z_0}(x_1)) = x_1.$$

Using the equation (3.7) and the fact that $g\varphi_{z_0}(x_1) = \varphi_{gz_0}(x_1)$, we have:

$$\varphi_{\rho_{z_0}(\gamma')^{-1}z_0}(x_1) = \varphi_{gz_0}(x_1)$$

Since the images of the sections are disjoint, we conclude that $\rho_{z_0}(\gamma')^{-1} = g$. Therefore, $\rho(\gamma')^{-1} = g_2^{-1}g_1$. Hence $g_1 = g_2\rho(\gamma')^{-1}$.

For the converse, assume that $(g_1, x_1) \sim (g_2, x_2)$. Then there exists $\gamma' \in G'$ such that $x_1 = \gamma' x_2$ and $g_1 = g_2 \rho(\gamma')^{-1}$. Therefore, we obtain the conclusion from the following equalities:

$$\begin{aligned} g_1 f_{z_0}(x_1) &= (g_2 \rho(\gamma')^{-1})(f_{z_0}(\gamma' x_2)) \\ &= (g_2)\rho(\gamma')^{-1} f_{z_0}(\gamma' x_2) \\ &= (g_2)\rho(\gamma')^{-1}(\rho(\gamma')) f_{z_0}(x_2) \\ &= g_2 f_{z_0}(x_2) \end{aligned}$$

□

Now, we establish a short but well-known result involving three orbifold charts and the relationship between the functions f_{z_0} constructed above. We include and prove this result here, as it will be used in Section 3.4.

Consider three orbifold (U, G, ϕ), (V, G', ϕ'), and (W, G'', ϕ''), with $\mathtt{W} \subseteq \mathtt{V} \subseteq \mathtt{U}$. Let $y \in V, y' \in W$ be points invariant under the actions of G' and G'', respectively. Then, consider the corresponding commutative diagrams as in (3.2):

$$\begin{array}{ccc} W \times_{\mathcal{W}} \pi_1^{-1}(\mathcal{W}) & \xrightarrow{q'} & (\pi')^{-1}(\mathcal{W}) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \pi'_{|(\pi')^{-1}(\mathcal{W})}} \\ W & \xrightarrow{\pi''} & \mathcal{W} \end{array} \qquad \begin{array}{ccc} V \times_{\mathcal{V}} \pi^{-1}(\mathcal{V}) & \xrightarrow{p'} & \pi^{-1}(\mathcal{V}) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi_{|\pi^{-1}(\mathcal{V})}} \\ V & \xrightarrow{\pi'} & \mathcal{V} \end{array} \tag{3.8}$$

$$\begin{array}{ccc} W \times_{\mathcal{W}} \pi^{-1}(\mathcal{W}) & \xrightarrow{r'} & \pi^{-1}(\mathcal{W}) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \pi_{|\pi^{-1}(\mathcal{W})}} \\ W & \xrightarrow{\pi''} & \mathcal{W} \end{array}$$

Here, $\mathcal{W}$ is defined analogously to $\mathcal{V}$; that is, $\mathcal{W} = [W/G'']$.

Proposition 3.1.10. Let (U, G, ϕ), (V, G', ϕ'), and (W, G'', ϕ'') be three orbifold charts with $\mathtt{W} \subseteq \mathtt{V} \subseteq \mathtt{U}$. Let $f_1 : W \to \pi_1^{-1}(\mathcal{W}) \subseteq V$ and $f_2 : V \to \pi_2^{-1}(\mathcal{V}) \subseteq U$ be the functions defined by the choices of elements $z_1 \in q^{-1}(y')$ and $z_2 \in p^{-1}(y)$, respectively. Then, there exists an element $z' \in r^{-1}(y')$ such that $f_2 \circ f_1 = r' \circ \varphi_{z'}$, where $\varphi_{z'} : W \to W \times_{\mathcal{W}} \pi^{-1}(\mathcal{W})$ is the section determined by z'.

Proof. Since we have two maps from W to W and $\pi^{-1}(\mathcal{W})$, respectively, namely $f_2 \circ f_1$ and Id_W, which satisfy the commutativity condition of the fiber product, as ensured by equation (3.8), there exists a unique map $\varphi : W \to W \times_{\mathcal{W}} \pi^{-1}(\mathcal{W})$ such that $r \circ \varphi = \mathrm{Id}_W$, and $r' \circ \varphi = f_2 \circ f_1$. This implies that φ is a section, defined by some z'. □

To summarize, in this section we have shown that if (U_1, G_1, ϕ_1) and (U_2, G_2, ϕ_2) are two orbifold charts of X^{an}, such that $\phi_(U_2)$ is an open subset of $\phi_1(U_1)$, then there exists an embedding of orbifold charts

$(U_2, G_2, \phi_2) \hookrightarrow (U_1, G_1, \phi_1)$, in the sense of Definition 1.3.1. Moreover, if the charts satisfy the assumptions of Proposition 3.1.4, this construction yields a suitable cover of X^{an} by orbifold charts, as we now state:

Proposition 3.1.11. There exists an orbifold atlas $\{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ for X^{an} such that each chart $(V_\alpha, G_\alpha, \varphi_\alpha)$ satisfies the following properties:

1. $V_\alpha \subseteq \mathbb{C}^n$ is an open ball centered at the origin, and the action of G_α on V_α is linear.
2. The divisor $V_\alpha \smallsetminus V_\alpha \times_{\mathscr{X}^{an}} \mathscr{U}^{an}$ is a simple normal crossing divisor, and the stratification induced by this divisor has a unique minimal stratum.

Remark 3.1.12. Since we are interested in the underlying topological space of the analytic space X^{an}, then considering a chart covering $\{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ satisfying the properties of Proposition 3.1.11, we can take for every α, the kernel H_α of the map from G_α to the group of biholomorphisms of V_α. Therefore, we consider the orbifold given by the charts $\{(V_\alpha, G_\alpha/H_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$. Since the actions are not trivial, this defines an effective orbifold whose underlying analytic space is precisely X^{an}.

From now on, we will refer to this orbifold as the *effective orbifold* of X^{an}. This orbifold will be of great importance in the development of the following section.

3.2 Stratifications on X^{an}

The purpose of this section is to show that there exists a stratification in X^{an} that refines the stratification induced by the toroidal structure $(\mathscr{X}, \mathscr{U})$ and also refines the stratification by stabilizers described below, this result is summarized in 3.2.6. In addition, we will show that there exists a triangulation on X^{an} that is adapted to this stratification in the sense of Definition 3.2.14.

Throughout this section, we will work with the effective orbifold associated to X^{an}. Therefore, unless otherwise stated, any reference to the orbifold X^{an} will implicitly mean its associated effective orbifold.

We begin by describing the construction of a stratification on $|\mathscr{X}|$, which is induced by the toroidal embedding $(\mathscr{X}, \mathscr{U})$, and the way that this stratification induces stratifications on the coarse moduli spaces X and X^{an}.

Before proceeding, recall that to show that a given collection $\{Y_\alpha\}_{\alpha\in\Lambda}$ of locally closed, connected subsets defines a stratification of a topological space Y, we must verify the following condition:

$$Y_\alpha \cap \overline{Y}_\beta \neq \emptyset \quad \Longrightarrow \quad Y_\alpha \subseteq \overline{Y}_\beta.$$

Note that we are assuming the strata to be connected. In contrast, Definition 1.2.6 allows for strata that are not necessarily connected. Under the assumption of connected strata, the condition above is equivalent to the following local criterion:

> For every point $x \in Y_\alpha \cap \overline{Y}_\beta$ there exists an open set $U \subseteq Y_\alpha$ containing x such that $U \subseteq \overline{Y}_\beta$.

Indeed, the direct implication is clear. Conversely, if the local condition holds for every such point x, then the set $Y_\alpha \cap \overline{Y}_\beta$ is both open and closed in Y_α. Since Y_α is connected, it follows that $Y_\alpha \cap \overline{Y}_\beta = Y_\alpha$, and hence $Y_\alpha \subseteq \overline{Y}_\beta$.

Let $(\mathscr{X}, \mathscr{U})$ be a toroidal Deligne–Mumford stack. In Section 1.6, we defined an étale sheaf $\mathcal{M}$ associated with the toroidal structure. Since $\mathcal{M}$ is a locally finitely generated sheaf. One may then define the rank function

$$\mathrm{rk} : |\mathscr{X}| \to \mathbb{N}, \quad w : \mathrm{Spec}(k) \to \mathscr{X} \mapsto \dim \mathcal{M}_w \otimes_{\mathbb{Z}} \mathbb{R}.$$

The connected components of the fibers $\mathrm{rk}^{-1}(n)$ for $n \in \mathbb{N}$ defines a stratification of $|\mathscr{X}|$, as described in [ACP12, Section 6.2].

To provide an alternative and concrete construction of this stratification, let $P \to \mathscr{X}$ be an etale presentation, with P a regular and separated scheme such that $(P, P \times_{\mathscr{X}} \mathscr{U})$ is a strict toroidal embedding. Let $|P| = \coprod_{\alpha \in \lambda} Y_\alpha$ be the stratification of $|P|$ determined by the toroidal structure. We define the following equivalence relation on the strata:

$Y_\alpha \sim Y_\beta$ if and only if there exists a sequence of strata $Y_\alpha = Y_0, Y_1 \ldots, Y_k = Y_\beta$, and points $y_i \in Y_i$ and $x_i \in Y_{i+1}$ for $i = 0, \ldots, k-1$, such that $\pi(y_i) = \pi(x_i)$, where $\pi : |P| \to |\mathscr{X}|$.

Lemma 3.2.1. Let $\pi : |P| \to |\mathscr{X}|$ be as above. For each $\alpha \in \Lambda$ define:

$$Z_\alpha := \pi \left(\bigcup_{Y_\beta \sim Y_\alpha} Y_\beta \right)$$

Then the collection $\{Z_\gamma\}_{\gamma \in \Gamma}$, where $\Gamma := \Lambda_{/\sim}$, defines a stratification of $|\mathscr{X}|$.

Proof. It is immediate that $\{Z_\gamma\}_{\gamma \in \Gamma}$ forms a partition of $|\mathscr{X}|$. Moreover, each Z_γ is connected, since it is the union of connected subsets that have at least one point in common. To verify that this defines a stratification, we apply the local characterization discussed at the beginning of this section. Specifically, we need to show that if $x \in Z_\gamma \cap \overline{Z}_{\gamma'}$, then there exists an open neighborhood $U \subseteq Z_\gamma$ of x such that $U \subseteq \overline{Y}_{\gamma'}$.

Let $x \in Z_\gamma \cap \overline{Z}_{\gamma'}$, and choose $y \in |P|$ such that $\pi(y) = x$. Let Y_α be the stratum of $|P|$ that contains the point y. By Lemma 3.2.2 we have $\pi^{-1}(Z_{\gamma'}) = \bigcup_{i \in I} Y_{\beta_i}$, for some index set $I \subseteq \Lambda$. Since π is an open map, it follows that

$$\pi^{-1}(\overline{Z}_{\gamma'}) = \bigcup_{i \in I} \overline{Y}_{\beta_i}.$$

Therefore, there exists $i \in I$ such that $y \in \overline{Y}_{\beta_i}$. By the local criterion for stratifications, it follows that there exists an open subset $V \subseteq Y_\alpha$ that contains the point y, and such that $V \subseteq \overline{Y}_{\beta_i}$. Applying π, we see that $\pi(V)$ is an open subset of Z_γ that contains x, and is contained in $\overline{Z}_{\gamma'}$, as required. □

Lemma 3.2.2. Let $Z_\gamma = \pi \left(\bigcup_{Y_\beta \sim Y_\alpha} Y_\beta \right)$ be one of the strata of $|\mathscr{X}|$ defined as in Lemma 3.2.1. Then

$$\pi^{-1}(Z_\gamma) = \bigcup_{Y_{\alpha'} \sim Y_\alpha} Y_{\alpha'}.$$

Proof. If there exists a stratum Y_β such that $Y_\beta \cap \pi^{-1}(Z_\gamma) \neq \emptyset$, then there exists a point $y \in Y_\beta \cap \pi^{-1}(Z_\gamma)$. By the definition of Z_γ, there exists a point $y' \in Y_{\alpha_0}$ for some $Y_{\alpha_0} \sim Y_\alpha$ and such that $\pi(y') = \pi(y)$.

This implies $Y_\beta \sim Y_\alpha$, and consequently $Y_\beta \subseteq \bigcup_{Y_{\alpha'} \sim Y_\alpha} Y_{\alpha'}$. This shows that every stratum intersecting $\pi^{-1}(Z_\gamma)$ is contained in the union $\bigcup_{Y_{\alpha'} \sim Y_\alpha} Y_{\alpha'}$, which proves the inclusion $\pi^{-1}(Z_\gamma) \subseteq \bigcup_{Y_{\alpha'} \sim Y_\alpha} Y_{\alpha'}$.

The reverse inclusion is immediate from the definition of Z_γ, so the two sets are the same. □

Proposition 3.2.3. The stratification defined via the equivalence relation on the strata of $|P|$ (from Lemma 3.2.1) coincides with the stratification defined by the rank function rk.

Proof. Take a stratum Z_β in the stratification defined in Lemma 3.2.1. By construction, all points of Z_β come from strata Y_α in $|P|$ that are equivalent under the relation defined via the map π. Each Y_α has a constant rank, and equivalent strata have the same rank because they are connected through points mapping to the same point in $|\mathscr{X}|$. Therefore, the rank function is constant in Z_β. This means that Z_β is contained in some stratum S_γ of the rank-stratification.

Conversely, consider a stratum S_γ of the stratification defined by the rank. Its preimage under π decomposes as a disjoint union of strata in $|P|$, that is:

$$\pi^{-1}(S_\gamma) = \coprod_{\alpha \in \Lambda_\gamma} Y_\alpha$$

This equality holds because, if $Y_\alpha \cap \pi^{-1}(S_\gamma) \neq \emptyset$, then $\mathrm{rk}_{|Y_\alpha} = \mathrm{rk}_{|S_\gamma}$, and therefore $\pi(Y_\alpha) \subseteq S_\gamma$. Moreover, the strata Y_α with $\alpha \in \Lambda_\gamma$ are precisely the connected components of $\pi^{-1}(S_\gamma)$. Indeed, all points in each connected component have the same rank, since their images lie in S_γ, which implies that each such component must lie entirely within a single stratum of the stratification of $|P|$.

Considering the restriction $\pi_{|\pi^{-1}(S_\gamma)} : \pi^{-1}(S_\gamma) \to S_\gamma$, which is an open map, we see that all strata Y_α contained in $\pi^{-1}(S_\gamma)$ are equivalent to each other, since their images are open and connected subsets of the connected stratum S_γ. Therefore, $S_\gamma \subseteq Z_\beta$ for some stratum Z_β in the stratification defined via the equivalence relation.

In conclusion, every stratum of the stratification given by the equivalence is contained in a stratum of the stratification given by the rank, and vice versa. Therefore, the two stratifications coincide. □

We conclude with the following observations regarding the stratification constructed above:

Remark 3.2.4.

1. From the construction, each stratum in $|\mathscr{X}|$ correspond to a substack of $\mathscr{X}$. That is, we denote by $\{\mathscr{W}_i\}_{i\in I}$ the collection of substacks such that the underlying topological spaces $\{|\mathscr{W}_i|\}_{i\in I}$ form the stratification of $|\mathscr{X}|$. Each $\mathscr{W}_i$ is called a stratum in $\mathscr{X}$.
2. This stratification in $|\mathscr{X}|$ induces in a natural way a stratification on the coarse moduli space X of $\mathscr{X}$, by simply taking the coarse moduli space W_i of each $\mathscr{W}_i$. Consequently, passing to analytification, we obtain a stratification on X^{an}, by taking $W_i^{an} \subseteq X^{an}$ for each $i \in I$.
3. Let $V \subseteq P^{an}$ be an adapted chart with a finite group G acting on V in such way that $V/G \hookrightarrow X^{an}$ is an open immersion, then from diagram

$$\begin{array}{ccc} V & \longrightarrow & V/G \\ \downarrow & & \downarrow \\ P^{an} & \longrightarrow & X^{an} \end{array} \tag{3.9}$$

we conclude that the each strata of V is contained in a strata of X^{an}.

Having described the stratification on the coarse moduli space X, and in particular on its analytification X^{an}, induced by the toroidal structure of a Deligne–Mumford stack $(\mathscr{X}, \mathscr{U})$, we now turn to the stratification defined by stabilizers on X^{an}. Our goal is to construct a new stratification that refines both the one induced by the toroidal structure and the one defined by stabilizers.

Let $\{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ be a chart covering of the orbifold X^{an} as in Remark 3.1.12. Considering the underlying smooth real manifold of every V_α, and since G_α acts linearly on V_α, we can identify every group G_α with a finite subgroup of $GL(2n, \mathbb{R})$, where n is the complex dimension of X^{an}.

For every H subgroup of $GL(2n, \mathbb{R})$, denote by (H) the subset

$$\{K \leq GL(2n, \mathbb{R}) : gKg^{-1} = H, \quad \text{for} \quad g \in GL(2n, \mathbb{R})\}.$$

Then by [Yan63] we have a natural stratification by stabilizer class up to conjugation of X^{an}. Whose strata are precisely conected components of sets of the form

$$S_H := \{\mathtt{x} \in X^{an} : (G_{\mathtt{x}}) = (H)\}$$

Since we are assuming that X^{an} is compact, it follows that the stratification by stabilizers is finite.

Remark 3.2.5. Since we are considering the underlying smooth real manifold structure of each V_α, and the action of G_α is linear, we may equip V_α with a Riemannian metric such that G_α acts by isometries. In particular, we can choose the metric so that G_α acts by orthogonal transformations. Consequently, we may identify each G_α with a subgroup of the orthogonal group $O(2n, \mathbb{R})$

Having fixed the stratification by stabilizers, let $\{M_r\}_{r\in R}$ be the partition of X^{an} that is the result of taking the connected components of nonempty intersection of the form $T_H \cap W_i$, where T_H is a connected component of S_H, for some $H \leq GL(2n, \mathbb{R})$, and W_i is a stratum associated with the stratification induced by the toroidal structure. This new collection is clearly a finite partition, since both stratifications are finite. Moreover, we have the following statement:

Proposition 3.2.6. The partition $\{M_r\}_{r\in R}$ defined above defines a stratification of X^{an}

Proof. We have to show that $M_r \cap \overline{M}_{r'} \neq \emptyset$, implies that $M_r \subseteq \overline{M}_{r'}$. Indeed, assume that M_r and $M_{r'}$ are the connected components of intersections $T_H \cap W_i$ and $T_{H'} \cap W_j$, respectively. It is then straightforward to observe that

$$W_i \subseteq \overline{W}_j, \text{ and } H' \text{ is identified with a subgroup of } H$$

Since the elements of this partition are connected, we apply the local criterion for stratification discussed at the beginning of this section. Namely, it suffices to show that for each point $\mathtt{x} \in M_r \cap \overline{M}_{r'}$ there exists an open neighborhood $U \subseteq X^{an}$ of $\mathtt{x}$ such that $U \cap M_r \subseteq \overline{M}_{r'}$.

Let $\mathtt{x} \in M_r \cap \overline{M}_{r'}$, and consider a chart (V, H, φ) as in Proposition 3.1.4 that contains the point $\mathtt{x}$. That is, there is a point $x \in V$ such that $\varphi(x) = \mathtt{x}$, and, as we will specify more precisely below, the chart satisfies certain properties related to the stratification and the action of the group.

After shrinking V, if necessary, we may assume that its image $\varphi(V)$ is such that $\varphi(V) \cap M_r$ is equal to $\varphi(V) \cap (T_H \cap W_i)$, and W_i is the unique minimal stratum, with respect to the toroidal stratification,

between those strata that have nonempty intersection with $\varphi(V)$. This is always possible because X^{an} is metrizable; see Proposition 3.1.1.

To proceed with the proof, we now recall the properties of the chart (V, H, φ) provided by Proposition 3.1.4, taking into account the additional assumptions introduced above:

1. $V$is an open ball equipped with a system of local coordinates $(z_1, \dots, z_n)$, where the point x can be chosen as the center of V, and the action of H on V is linear.

2. Let S denote the set of points in V fixed by the action of H. Then we have $\varphi(S) = T_H \cap \varphi(V)$. Recall that S is a linear subvariety of V since the action is linear.

3. V has associated a simple normal crossing divisor

$$V \smallsetminus (V \times_{\mathscr{X}^{an}} \mathscr{U}^{an})$$

, wich is defined by the zero locus of $\prod_{i\in I} z_i$, for I a subset of $\{1, \dots, n\}$. Therefore, the unique minimal stratum W of the stratitification induced by this divisor is a linear subvariety.

4. Since the action of H permutes the irreducible components of the divisor $V \smallsetminus (V \times_{\mathscr{X}^{an}} \mathscr{U}^{an})$, it follows that W is invariant under the action of H. Consequently, by Remark 3.2.4, we have $\varphi(W) = W_i \cap \varphi(V)$.

The above facts imply that $S \cap W$ is a connected linear space with $M_r \cap \varphi(V) = \varphi(S \cap W)$.

Now let $\mathtt{y} \in M_{r'} \cap \varphi(V)$, which exists since $\mathtt{x} \in \overline{M}_{r'}$. Let y be a point in V that is mapped to $\mathtt{y}$, and let W' be the stratum of V that contains the point y. Since the stratification of V is given by coordinate hyperplanes, W' is a cone, in the sense that if $w \in W'$, and $\lambda \in \mathbb{R}_{>0}$, then $\lambda w \in W'$. Furthermore, from Remark 3.2.4, we know:

$$\varphi(W') \subseteq W_j.$$

Since the divisor of V is determined by the zero locus of $\prod_{i\in I} z_i$, then it is clear that W is given as $\{v \in V : \ z_i(v) = 0 \text{ for } i \in I\}$. As each irreducible component $E_i := \{z_i = 0\}$, for $i \in i$, of the divisor in V have real dimension $2n - 2$ then for each $J \subseteq I$ we have that $\bigcap_{j\in J} E_j \smallsetminus \bigcup_{j\notin J} E_j$ is connected. This observation implies that there exists $I' \subseteq I$ such that

$$W' = \{v \in V : \ z_i(v) = 0 \text{ for } i \in I', \text{ and } z_i(v) \neq 0 \text{ for } i \in I \smallsetminus I'\}.$$

Let $\mathtt{z} \in \varphi(V) \cap M_r$, and choose $z \in W$ mapping to $\mathtt{z}$. Consider the segment $\{z_t := (1-t)z + ty : \ 0 < t \leq 1\}$. This lies entirely in W', because the i-th component of z_t is zero if $i \in I'$, and nonzero if $i \in I \smallsetminus I'$. We claim that elements of this segment have stabilizer equal to G_y. Indeed:

- For any $g \in G_y$, we have $g(z_t) = (1 - t)g(z) + tg(y) = (1 - t)z + ty$, so $G_y \subseteq G_{z_t}$.
- Conversely, if $g \in G_{z_t}$, then $g(z_t) = z_t$, which implies $(1 - t)z + tg(y) = (1 - t)z + ty$, and hence $gy = y$. So $G_{z_t} \subseteq G_y$.

Therefore, we conclude $G_{z_t} = G_y$.

The fact above implies that $\varphi(z_t) \in W_j \cap T_{H'}$ for every $t > 0$. Since the image of this segment is connected, the entire image lies in a connected component of $W_j \cap T_{H'}$. As $\varphi(y) \in M_{r'}$, we conclude that the image of this segment is contained in $M_{r'}$.

As $\varphi(z) = \mathtt{z}$ and $\varphi(z_t) \in M_{r'}$ for $t > 0$, and $\lim_{t\to 0} z_t = z$, we conclude that $\mathtt{z} \in \overline{M}_{r'}$. This holds for every $\mathtt{z} \in M_r \cap \varphi(V)$, showing that a neighborhood of $\mathtt{x}$ in M_r lies in $\overline{M}_{r'}$. Hence we conclude that $M_r \subseteq \overline{M}_{r'}$, as desired. □

Now, we will introduce a definition from [Yan63], which will be particularly useful for our goal of constructing a triangulation adapted to the stratification described in Proposition 3.2.6.

Definition 3.2.7. Let X be a topological manifold of dimension n, and Σ a partition. We say that (X, Σ) is a *differentiable n-space* if it satisfies the following conditions:

(a) Every element of Σ is a differentiable manifold.

(b) Σ is locally finite.

(c) For every $M_0 \in \Sigma$, with dimension equal to k and $x \in M_0$, there exists an open neighborhood U of x and a differentiable l-space (X', Σ'), with $l < n - k$ such that

   (i) There exists a homeomorphism from $\overline{U}$ to the joint $S^k * X'$, where S^k is a k-simplex. This homeomorphism induces diffeomorphism between $U \cap M_0$ and the interior of S^k, and a diffeomorphism between $U \cap M_i$ and $S^k * (X' \cap M_i) \smallsetminus (S^k \cup (X' \cap M_i))$, for each stratum M_i such that $x \in \overline{M}_i \smallsetminus M_i$.

   (ii) Once we identify X' as a subspace of X by the previous condition, we have $M' \in \Sigma'$ if only if $M' = X' \cap M_i$, for some $M_i \in \Sigma$, with $x \in \overline{M}_i \smallsetminus M_i$.

From the above definition, it is straightforward to verify that the partition Σ is, in fact, a stratification.

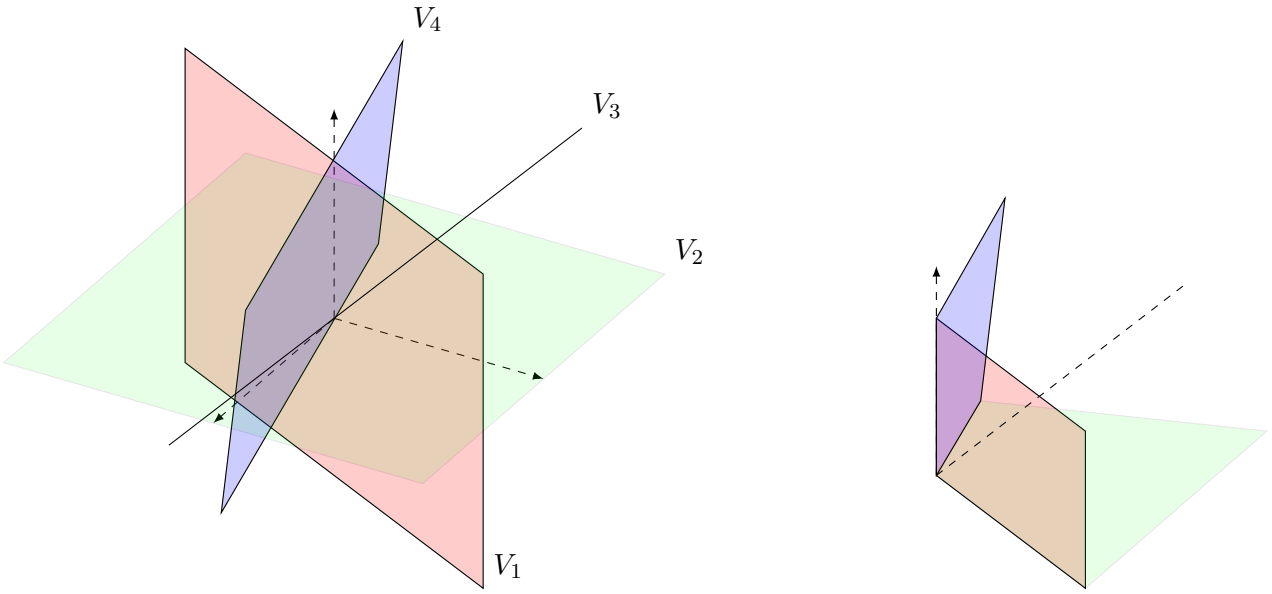
Lemma 3.2.8. Every differentiable space is triangulable, and such a triangulation can be chosen to be adapted to the stratification associated with the differentiable space (see Definition 1.2.12 in Chapter 1).

Proof. See [Yan63, Lemma 2] □

Proposition 3.2.9. Let $\{V_i\}_{i\in I}$ be a finite collection of vector subspaces of $\mathbb{R}^n$. For every $J \subseteq I$ define

$$C_J := \bigcap_{j\in J} V_j \smallsetminus \bigcup_{j\notin J} V_j$$

and consider the connected components of each such set C_J. Then the collection of all these connected components, as J ranges all subsets of I, forms a stratification of $\mathbb{R}^n$.



Before starting the proof, we refer to the illustrative picture above, which shows a stratification of $\mathbb{R}^3$ defined by four subspaces. On the right side of the image, we see a connected element represented as a polyhedral open cone with a semiline removed. This set corresponds to a connected component of the expression

$$\mathbb{R}^3 \smallsetminus \bigcup_{j=1}^{4} V_j$$

Note that removing the half-line does not affect the connectedness of that set. The picture also illustrates that this partition satisfies the conditions of a stratification.

Proof. Denote this collection by $\{M_r\}_{r\in R}$, and note that it forms a partition. Let M_r and $M_{r'}$ be elements of this collection, with $M_r \cap \overline{M}_{r'} \neq \emptyset$. We have to show that $M_r \subseteq \overline{M}_{r'}$.

Suppose that M_r and $M_{r'}$ are the connected components of C_J and $C_{J'}$, respectively. If $x \in M_r \cap \overline{M}_{r'}$, then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements in $M_{r'}$ that converge to x. Since $x_n \in V_i$ for each $n \in \mathbb{N}$ and $i \in J'$, we have that $x \in V_i$ for each $j \in J'$. Therefore, $J' \subseteq J$. This fact implies that

$$M_r \subseteq \partial C_{J'}$$

Now, let us denote by $W_{J'}$ the space $\bigcap_{i\in J'} V_i$. Since $M_r \subseteq W_{J'}$, we can consider $W_{J'}$ as the ambient space. Therefore, we can reduce to the case where $J' = \emptyset$, and $I = (J')^c$.

In this setting, $C_{\{\emptyset\}}$ is the complement of the subspaces $\{V_i\}_{i\in I}$ in $\mathbb{R}^n$. Let $K = \{i \in I : \dim V_i = n-1\}$. Then there exist hyperplane equations $\{H_k\}_{k\in K}$, with $V_k = \{x \in \mathbb{R}^n : H_k(x) = 0\}$, such that

$$M_{r'} = \{x \in \mathbb{R}^n : H_k(x) > 0, \text{ for all } k \in K\} \smallsetminus \bigcup_{k\notin K} V_k$$

Since the subspaces V_k for $k \notin K$ do not affect the connectivity of $\{x \in \mathbb{R}^n : H_k(x) \geq 0, \text{ for all } k \in K\}$, it follows that

$$\overline{M}_{r'} = \{x \in \mathbb{R}^n : H_k(x) \geq 0 \text{ for all } k \in K\}.$$

To show that $M_r \subseteq \overline{M}_{r'}$, it suffices to verify that for each $z \in M_r$, we have $H_k(z) \geq 0$ for every $k \in K$. Indeed, since M_r is a connected component of $\bigcap_{j\in J} V_j \smallsetminus \bigcup_{j\notin J} V_j$, consider a fixed $k \in K$ and observe the following cases: If $k \in J$, then $H_k(z) = 0$ for each $z \in M_r$, because $z \in V_k$. If $k \notin J$, then $V_k \cap M_r = \emptyset$, so $H_k(y) \neq 0$ for all $y \in M_r$. Now, since $M_r \cap \overline{M}_{r'} \neq \emptyset$ by assumption, there exists a point $x \in M_r \cap \overline{M}_{r'}$. Thus, for this point we have $H_k(x) > 0$. As H_k is continuous and M_r is connected, it follows (by the intermediate value theorem, or Bolzano's theorem) that $H_k(y) > 0$ for all $y \in M_r$.

In either case, we conclude that $H_k(y) \geq 0$ for all $k \in K$ and $y \in M_r$. Hence $M_r \subseteq \overline{M}_{r'}$. This concludes the proof. □

Remark 3.2.10. If $B(0,\varepsilon) \subseteq \mathbb{R}^n$ is the open ball of radius ε centered at the origin, then the stratification of $\mathbb{R}^n$ described in Proposition 3.2.9 restricts to a stratification of $B(0,\varepsilon)$. This follows from the fact that the map

$$\varphi : \mathbb{R}^n \to B(0,\varepsilon), \quad x \mapsto \frac{\varepsilon x}{1+|x|}$$

is a homeomorphism, with inverse

$$\phi : B(0,\varepsilon) \to \mathbb{R}^n, \quad y \mapsto \frac{y}{\varepsilon - |y|}.$$

Via this homeomorphism, the stratification of $\mathbb{R}^n$ induces a stratification on the ball $B(0,\varepsilon)$. Moreover, the image of a stratum M_r of $\mathbb{R}^n$ under φ is precisely $M_r \cap B(0,\varepsilon)$. This follows from the fact that if $x \in M_r$, and $\lambda \geq 0$, we see $\lambda x \in M_r$.

Therefore, since $\varphi(M_r) = M_r \cap B(0,\varepsilon)$, the stratification of $\mathbb{R}^n$ restricts to a stratification on $B(0,\varepsilon)$.

Definition 3.2.11. Let X be a topological space, and Σ a stratification. We will say that (X,Σ) is a *stratified linear space*, or that Σ is a *linear stratification*, if X is homeomorphic to an open ball $B(0,\varepsilon) \subseteq \mathbb{R}^n$, and the stratification Σ is induced by a stratification as in 3.2.10.

X is called *locally stratified linear space*, or that Σ is a *local linear stratification*, if every point of X has an open neigborhood which is a stratified linear space.

Remark 3.2.12. Since the stratification of Proposition 3.2.9 has a unique minimal stratum, it follows that every stratified linear space has a minimal stratum.

Example 3.2.13. Let V be an open ball of $\mathbb{R}^n$ centered at the origin, and G a finite group acting by linear transformations on V. Then there is a stratification Σ on V such that (V,Σ) is a stratified linear space, which induces a stratification on the quotient V/G that agrees with the stratification of V/G given by the stabilizers.

For every subgroup H of G, define

$$V_H := \{x \in V : \ H \subseteq G_x\}$$

Note that this set is a linear subspace of $\mathbb{R}^n$ intersected with the open set V, and for every $g \in G$

$$g \cdot V_H = V_{gHg^{-1}} \tag{3.10}$$

Then consider the stratification induced by the collection of subspaces $\{V_H\}_{H\in\mathcal{H}}$, where $\mathcal{H}$ is the set of subgroups of G. Let us show that this stratification induces a stratification on V/G, which is precisely that one defined by stabilizers. Indeed, if $\mathcal{I} \subseteq \mathcal{H}$ then

$$\bigcap_{H\in\mathcal{I}} V_H = V_{H_{\mathcal{I}}}$$

Where $H_{\mathcal{I}}$ is the subgruop that is generated by the union of the elements of $\mathcal{I}$. In this order, for every $H \in \mathcal{H}$

$$\bigcap_{K\in\mathcal{J}} V_K \setminus \bigcup_{K'\in\mathcal{J}^c} V_{K'} = \{x \in U : G_x = H\} \tag{3.11}$$

here, $\mathcal{J}$ denotes the set of all subgroups of H. The above argument shows that, for any given stratum, all its points have the same stabilizer.

Now, let $\varphi : V \to V/G$ denote the quotient map, and let M be a connected component of (3.11), for a fixed H. It is clear that $\varphi(M)$ is connected in V/G, and is contained in a stratum of the stratification given by stabilizers. Such a stratum must be of the form M_H, which consists of those points $\mathtt{x} \in V/G$ for which the set $(G_{\mathtt{x}})$ is equal to (H).

Now, consider $\mathtt{x}$ a point of M_H. Then there exists $x \in V$ that is mapped to $\mathtt{x}$. Since the stabilizer of x in V has the form gHg^{-1}, for some $g \in G$, we see that $g^{-1}x$ is an element with stabilizer equal to H. If M' is the connected component that contains the point $g^{-1}x$. then $\varphi(M') \subseteq M_H$. Therefore, we have the following two cases:

If $\varphi(M) \cap \varphi(M') \neq \emptyset$. Then there exist points $x_1 \in M$ and $x_2 \in M'$ such that $\varphi(x_1) = \varphi(x_2)$. This implies that there exists $g' \in G$ such that $g'x_1 = x_2$. By equation (3.10) we conclude that $g'M = M'$, and hence $\varphi(M) = \varphi(M')$.

If $\varphi(M)$ and $\varphi(M')$ are disjoint, this leads in a contradiction. Indeed, since that M_H is a connected set of V/G, and can be written as the union of finitely many subsets of the form $\varphi(M_i)$, where the M_i are some connected components of subsets as in (3.11), corresponding to subgroups H' conjugate to H. The disjointness of two such images, $\varphi(M)$ and $\varphi(M')$ implies that M_H can be obtained as a disjoint union of connected spaces, contradicting its connectedness.

With this, we have shown that $\varphi(M) = M_H$. As a consequence, the induced stratification coincides with the stratification by stabilizers.

The example above motivates us to give the following definition.

Definition 3.2.14. Let $V \subseteq \mathbb{R}^n$ be an open set, G a finite group acting on V by linear transformations, and Σ a stratification of V. We say that Σ is *compatible* with the action of G if satisfy the following properties:

1. If $M \in \Sigma$, then all the elements of M have the same stabilizer.
2. For every $M \in \Sigma$, and $g \in G$, gM is an element of Σ.

Proposition 3.2.15. Let V be an open set of $\mathbb{R}^n$, G a finite group acting on V by linear transformations, and let Σ be a linear stratification compatible with the action of G. Then Σ induces a stratification $\widetilde{\Sigma}$ on V/G. Moreover, every stratum of $\widetilde{\Sigma}$ is a differentiable manifold.

Proof. Consider the set $\overline{\Sigma} := \{M/G : M \in \Sigma\}$, and by simplicity, denote the quotient M/G by $\mathtt{M}$. Let us first see that this collection is a partition. Take elements $\mathtt{M}_1$ and $\mathtt{M}_2$ of the collection with $\mathtt{M}_1 \cap \mathtt{M}_2 \neq \emptyset$. Let $\mathtt{x}$ be an element of the intersection, and let x_1, x_2 be elements of M_1 and M_2, respectively, that are mapped to $\mathtt{x}$. Therefore, there exists $g \in G$ such that $gx_1 = x_2$. Since gM_1 is a stratum, it follows that $gM_1 = M_2$, and consequently $\mathtt{M}_1 = \mathtt{M}_2$.

Now, assume that the intersection of $\mathtt{M}_1$ with the closure of $\mathtt{M}_2$ is nonempty. We have to show that $\mathtt{M}_1$ is contained in the closure of $\mathtt{M}_2$. Indeed, since the quotient map $\pi : V \to V/G$ is an open map, it follows

$$\pi^{-1}\left(\overline{\mathtt{M}_2}\right) = \overline{\pi^{-1}(\mathtt{M}_2)} = \bigcup_{g\in G} g \cdot \overline{M_2}$$

Consider $\mathtt{x} \in \mathtt{M}_1 \cap \overline{\mathtt{M}_2}$. Then we have two points x_1, x_2 in M_1 and $\overline{M}_2$ respectiely, which are mapped to $\mathtt{x}$. This implies that there exists $g \in G$ with $gx_1 = x_2$. Consequently, we have

$$gM_1 \cap \overline{M}_2 \neq \emptyset.$$

Since Σ is a stratification, we have $gM_1 \subseteq \overline{M}_2$, and therefore $\mathtt{M}_1$ is contained in the closure of $\mathtt{M}_2$. Thus, the partition induced on the quotient is, in fact, a stratification.

As every element in the stratification Σ locally is a linear manifold, which is in fact a differentiable manifold, we see that every stratum of Σ is a differentiable manifold. From this fact, we have to show that every $\mathtt{M}$ is a differentiable manifold.

Let $\mathtt{M}$ be a stratum and $\mathtt{x} \in \mathtt{M}$. Consider a point $x \in V$ mapping to $\mathtt{x}$, and denote by M the stratum that contains the point x. Taking a slice U at the point x, we find that $M \cap U$ is a differentiable manifold and G_x acts trivially. This implies that $M \cap U \cong \mathtt{M} \cap (U/G_x)$. In conclusion, $\mathtt{M}$ is differentiable. □

The proposition above allows us to extend this local fact to the setting of effective orbifolds, as we will see in the following proposition.

Lemma 3.2.16. Let $(X, \{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda})$ be an effective orbifold, and Σ a stratification of X with the following properties:

1. Each V_α is an open ball and the action of G_α is orthogonal.
2. For every $\mathtt{M} \in \Sigma$, and $\mathtt{x} \in \mathtt{M}$. There exists a chart $(V_\alpha, G_\alpha, \varphi_\alpha)$, and a stratification Σ_α of V_α such that: $(V_\alpha, \Sigma_\alpha)$ is a stratified linear space, the stratification Σ_α is compatible with the action of G_α, and the image of the unique minimal stratum of Σ_α contains the point $\mathtt{x}$.
3. The restriction of Σ to V_α/G_α coincides with the stratification induced by Σ_α on V_α/G_α.

Then (X, Σ) is a differentiable n-space.

Proof. Conditions (2) and (3) ensure that condition (b) of Definition 3.2.7 is satisfied. Since condition (a) is a local condition, it follows directly from the Proposition 3.2.15. Therefore, it remains only to show the condition (c). The proof of this condition will be by induction on the dimension of the orbifold. The case $n = 0$ is clear, since in this case we have a discrete set of points.

Now consider X an orbifold of dimension n, and assume that the assertion holds for any dimension less than n. Let $\mathtt{M} \in \Sigma$ be a stratum, $\mathtt{x} \in \mathtt{M}$, and $(V_\alpha, G_\alpha, \varphi_\alpha)$ a chart equipped with a linear stratification Σ_α, which is associated with the point $\mathtt{x}$ as in condition (2). Denote by x the center of V_α, which is mapped to $\mathtt{x}$; that is $\varphi_\alpha(x) = \mathtt{x}$. Consider an open ball U_α that contains x, and such that the closure of U_α in $\mathbb{R}^n$ is contained in V_α. Note that G_α acts on U_α since the action is orthogonal.

Denote $F := \overline{U}_\alpha \cap \varphi_\alpha^{-1}(\mathtt{M})$, and let E be the linear closed subspace orthogonal to F. Let Y denote the boundary of E. Since G_α acts orthogonally on U_α, we have that G_α acts orthogonally on Y, and it is clear that $\dim Y/G_\alpha < n$.

By Lemmas 3.2.17 and 3.2.19 that we will prove below, the stratification Σ_α induces a stratification in Y, which makes Y a locally linear stratified space. Since G_α acts ortogonally, it follows that for every point $y \in Y$, there exists a G_y-invariant neighborhood $U \subseteq Y$ of y, such that the stratification induced on U is linear and compatible with G_y, as Σ_α is compatible with G_α. The Proposition 3.2.15 says that this linear stratification induces a stratification on U/G_y, which is the same as the partition induced by Σ to U/G_y due to the condition (2). In conclusion, we see that Y/G_α satisfies the conditions of the proposition. Hence, is a differentiable n-space.

Let C be the convex hull of $F \cup E$, and consider C/G_α. Since $\varphi_\alpha^{-1}(\mathtt{M})$ is a linear subvariety and is diffeomorphic to $\mathtt{M} \cap \varphi_\alpha(V_\alpha)$, we may take F as the simplex appearing in Definition 3.2.7. It follows that the join $[\varphi_\alpha(F)] * (Y/G_\alpha)$ coincides with the closure of C/G_α, since the join $F * Y$ consists of all line segments joining points in F to points in Y. Clearly $E \subseteq F * Y$, and hence the convex hull of $F \cup E$ is contained $F * Y$; equality holds by convexity. Moreover, as the group G_α acts linearly, we have

$$[\varphi_\alpha(F)] * (Y/G_\alpha) = C/G_\alpha$$

Consequently, X is a differentiable $n-$space. □

Lemma 3.2.17. Let V be a vector space, G be a finite group acting by linear transformations on V, and let Σ be a linear stratification adapted to G. If W is a subspace of V invariant under the action of G, then Σ induces a linear stratification on W compatible with the action of G.

Proof. If $\{V_i\}_{i\in I}$ is a finite collection of subspaces of V defining the stratification as in Proposition 3.2.9. Then $\{W \cap V_i\}_{i\in I}$ is a finite collection of subspaces that define a linear stratification in W, and this stratification coincides with the partition induced by the stratification of V to W. As the stratification in V is compatible with G, and W is invariant under the action of G, we see that the induced stratification is adapted to G. □

Remark 3.2.18. Note that if V is an open ball, invariant under the action of G, and W is a closed linear subvariety of V, also invariant under the action of G, then the statement of Lemma 3.2.17 still holds.

Lemma 3.2.19. Let W be a vector space, and Σ a linear stratification of W induced by a collection of subspaces $\{W_i\}_{i\in I}$. For every open ball $B(0,\varepsilon)$ of W, the stratification of W induces a stratification on $\partial B(0,\varepsilon)$. Moreover, this is a locally linear stratification.

Proof. Recall that for every stratum M of W, every point $x \in M$, and $\lambda > 0$, we have $\lambda x \in M$. This fact implies

$$\overline{M} \cap \partial B(0,\varepsilon) = \overline{M \cap \partial B(0,\varepsilon)} \tag{3.12}$$

Consider the partition of $\partial B(0,\epsilon)$ obtained by taking the connected components of the intersections $M \cap \partial B(0,\epsilon)$, where M ranges over the strata of W. The goal is to prove that this partition defines a stratification.

Before starting the proof, note that if the dimension of W is equal to 1, then $\partial B(0,\varepsilon)$ consists of two points. In this case, the claim is evident.

Assume then $\dim W > 1$, and consider the map $\varphi : W \smallsetminus \{0\} \to \partial B(0,\epsilon)$ where the element x is send to $\frac{\varepsilon x}{|x|}$. Note that $\varphi(M \smallsetminus \{0\}) = M \cap \partial B(0,\varepsilon)$. In this order, we have two descriptions of the elements of the partition of $\partial B(0,\epsilon)$ induced by Σ

1. If M is not a one-dimensional vector subspace of W, then $M \smallsetminus \{0\}$, and thus its image under φ, namely $M \cap \partial B(0,\epsilon)$, is also connected.
2. If M is a one-dimensional vector subspace of W, then $M \cap \partial B(0,\varepsilon)$ consists of exactly two points. In this case, the connected components of $M \cap \partial B(0,\varepsilon)$ are points.

Let N and N' be two connected components of the partition of $\partial B(0,\varepsilon)$ induced by Σ, such that $N \cap \overline{N'} \neq \emptyset$. By the description above, we may assume without loss of generality that $N = M \cap \partial B(0,\varepsilon)$ and $N' = M' \cap \partial B(0,\varepsilon)$, for some strata $M, M' \in \Sigma$. This implies that $M \cap \overline{M'} \neq \emptyset$. As Σ is a stratification, we find $M \subseteq \overline{M}'$. By (3.12), we obtain

$$N = M \cap \partial B(0,\varepsilon) \subseteq \overline{M'} \cap \partial B(0,\varepsilon) = \overline{N'}.$$

Therefore, the partition induced by the stratification Σ is indeed a stratification.

Let us now see that $\partial B(0,\varepsilon)$ together with the stratification induced by Σ is a locally stratified linear space. For $y \in \partial B(0,\varepsilon)$, consider the tangent space $T_y\partial B(0,\varepsilon)$, and let z be the antipodal point of y on the sphere $\partial B(0,\varepsilon)$. With these data, consider the stereographic projection $\xi : \partial B(0,\varepsilon) \smallsetminus \{z\} \to T_y\partial B(0,\varepsilon)$. As we said above, Σ is determined by the subspaces $\{W_i\}_{i\in I}$. Then define

$$I_y := \{i \in I : \overline{zy} \subseteq W_i\}$$

Note that for each $i \in I_y$, the image of $(\partial B(0,\varepsilon) \smallsetminus \{z\}) \cap W_i$ under ξ is a vector subspace of $T_y\partial B(0,\varepsilon)$, for simplicity, denote this vector subspace by W_i^{ξ}. Then the collection $\{W_i^{\xi}\}_{i\in I_y}$ induces a stratification on $T_y\partial B(0,\varepsilon)$.

Let U be an open neighborhood of y in $\partial B(0,\varepsilon)$, such that $\xi(U)$ is an open ball of the origin in $T_y\partial B(0,\varepsilon)$, and with the property $W_i \cap U \neq \emptyset$ if only if $i \in I_y$. In consequence, the stratification of $T_y\partial B(0,\varepsilon)$ induces a stratification in $\xi(U)$, consequently, this stratification induces a stratification in U, which agrees with the restriction of the stratification of $\partial B(0,\varepsilon)$ to U. Therefore, we have proved that the stratification in $\partial B(0,\varepsilon)$ is locally linear. □

Theorem 3.2.20. *Let (X^{an}, U^{an}) be the analytification of the moduli space associated with a toroidal embedding of Deligne–Mumford stacks $(\mathscr{X}, \mathscr{U})$, and let Σ be the stratification of X^{an} as in Proposition 3.2.6. Then there exists a triangulation of X^{an} adapted to the stratification Σ.*

Proof. Recall that we are considering the effective orbifold associated to X^{an}. In Remark 3.1.12, we constructed an atlas for this effective orbifold. For the sake of clarity in this proof, we will briefly reconstruct it here. Let us consider an orbifold atlas of X^{an}, where each orbifold chart satisfies the conditions of Proposition 3.1.11. Each orbifold chart has an associated effective chart, obtained by quotienting the acting group by its subgroup that acts trivially. We then consider the orbifold defined by these effective charts.

If $(V_\alpha, G_\alpha, \varphi_\alpha)$ is an effective chart, recall that $V_\alpha \subseteq \mathbb{C}^n$. Forgetting the complex structure, we view V_α as a subset of $\mathbb{R}^{2n}$. Since the action of G_α is linear, the underlying smooth manifold V_α can be endowed with a Riemannian metric for which the action of G_α in V_α is orthogonal.

Since, by Proposition 3.1.11, each effective orbifold chart satisfies the assumptions of Proposition 3.2.16, it follows that X^{an} is a n-differentiable space. Then, by Proposition 3.2.8, we conclude that X^{an} admits a triangulation adapted to the stratification. □

Since the stratification Σ in Proposition 3.2.6 is a refinement of the stratification by stabilizers of an orbifold, we will adapt some of the results from Section 1.2 of [IM99] to our case. This approach will lead us to the conclusion of Theorem 3.2.24, which states that one can construct a suitable atlas of orbifold charts on the analytic coarse moduli space X^{an}, in which the "intersection" of two orbifold charts is again an orbifold chart in the atlas.

Remark 3.2.21. Let $\mathcal{T}$ be a triangulation adapted to the stratified space (X^{an}, Σ), where Σ is the stratification that refines the stratification given by stabilizers, and the stratification induced by the toroidal emebeding $(\mathscr{X}, \mathscr{U})$. By replacing $\mathcal{T}$ with its barycentric subdivision, we can assume that the cover of closed simplices refines the atlas $\{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ that determines the effective orbifold associated to X^{an}. Furthermore, after a barycentric subdivision of $\mathcal{T}$, we can assume that for each simplex σ, there exists a face σ' such that the points of $\sigma \smallsetminus \sigma'$ lie in a single stratum of the stratification Σ, which refines both the stratification by stabilizers and the one induced by the toroidal embedding. The points of σ' lie in a union of strata of Σ, each of which is contained in the closure of the stratum that contains the points of $\sigma \smallsetminus \sigma'$.

From now on, we will consider triangulations that satisfy the properties described in Remark 3.2.21. Following [IM99, Lemma 1.2.2], we have the following statement.

Proposition 3.2.22. For every $\sigma \in \mathcal{T}$, which is contained in a chart $(V_\alpha, G_\alpha, \varphi_\alpha)$, we have

$$\varphi_\alpha^{-1}(\mathring{\sigma}) = \coprod_{i\in I} S_i$$

where $|I| < \infty$ and every S_i is a homeomorphic to $\mathring{\sigma}$. Moreover, each of these homeomorphisms extends to a homeomorphism between $\overline{S}_i$ and σ.

Proof. Recall that $\varphi_\alpha : V_\alpha \to V_\alpha/G_\alpha$ rectrict a covering projection on each stratum of V_α/G_α. Since the interior of σ is contained in a unique stratum of V_α/G_α, and the stratification in V_α/G_α is induced by a linear stratification in V_α compatible with G_α, it follows that the stabilizer remains constant on the interior of σ. This implies

$$\varphi_\alpha^{-1}(\mathring{\sigma}) = \coprod_{i\in I} S_i$$

where each S_i is homeomorphic to $\mathring{\sigma}$, for every $i \in I$. By continuity of φ_α, the closure of each S_i maps into σ. Moreover, it is surjective because G_α acts continuously on V_α.

We now show that the restriction $\varphi_{\alpha|\overline{S}_i} : \overline{S}_i \to \sigma$ is injective. Since φ_α is injective on S_i, assume for contradiction that there exits $x, y \in \overline{S}_i \smallsetminus S_i$ such that $\varphi_\alpha(x) = \varphi_\alpha(y) = z$. Then there exists a proper face τ of σ such that $z \in \tau$. By the same argument applied to $\mathring{\tau}$, we have

$$\varphi_\alpha^{-1}(\mathring{\tau}) = \coprod_{j\in J} R_j$$

where each R_j is homeomorphic to $\mathring{\tau}$ and the restriction $\varphi_{\alpha|R_j}$ is injective. Since $x, y \in \varphi_\alpha^{-1}(\mathring{\tau})$ and φ_α is injective in each R_j, it must be that $x \in R_{j_1}, y \in R_{j_2}$ for some $j_1 \neq j_2 \in J$.

Now, since φ_α is an open map, we can find disjoint, open, connected neighborhoods V_x and V_y of x and y, respectively. Then $\varphi_\alpha(V_x)$ and $\varphi_\alpha(V_y)$ are open neighborhoods of z. Choose an open neighborhood $W \subseteq \varphi_\alpha(V_x) \cap \varphi_\alpha(V_y)$ of z such that of $W \cap \mathring{\sigma}$ is connected.

On one hand, the preimage $(\varphi_{\alpha|\overline{S}_i})^{-1}(W \cap \mathring{\sigma}) \subseteq S_i$ is connected since $\varphi_{\alpha|S_i}$ is a homeomorphism. On the other hand, $(\varphi_{\alpha|\overline{S}_i})^{-1}(W)$ contains both x and y, so $(\varphi_{\alpha|\overline{S}_i})^{-1}(W \cap \mathring{\sigma})$ contains points from both V_x and V_y. Since V_x and V_y are disjoint, this implies that the preimage $(\varphi_{\alpha|\overline{S}_i})^{-1}(W \cap \mathring{\sigma})$ is not connected, which is a contradiction. Hence, the map must be injective.

Finally, Since φ_α is an open map, it follows $\varphi_{\alpha|\overline{S_i}} : \overline{S_i} \to \sigma$ is a homemomorphism. □

After applying a finite sequence of barycentric subdivisions to the triangulation $\mathcal{T}$ we may assume that for every simplex $\sigma \in \mathcal{T}$, the closure of its open star

$$\mathrm{St}(\sigma) := \bigcup_{\tau \prec \sigma} \mathring{\tau}$$

is contained in a chart of the atlas. That is, there exist $(V_\alpha, G_\alpha, \varphi_\alpha)$ such that

$$\overline{\mathrm{St}(\sigma)} \subseteq \varphi_\alpha(V_\alpha) \tag{3.13}$$

If X is a topological space with a triangulation $\mathcal{T}$, then for each $x \in X$, we denote by σ_x the smallest simplex in $\mathcal{T}$ that contains the point x.

Now, let $\mathtt{x} \in X^{an}$, and let $(V_\alpha, G_\alpha, \varphi_\alpha)$ be a chart such that

$$\overline{\mathrm{St}(\sigma_{\mathtt{x}})} \subseteq V_\alpha/G_\alpha$$

The idea in the following lemma is to induce a stratification in $\varphi_\alpha^{-1}(\overline{\mathrm{St}(\sigma_{\mathtt{x}})})$, following the same approach as in [IM99].

Lemma 3.2.23. The space $\varphi_\alpha^{-1}(\overline{\mathrm{St}(\sigma_{\mathtt{x}})})$ has a triangulation $\mathcal{R}$ with the following properties

1. Each connected component of $\varphi_\alpha^{-1}(\mathrm{St}(\sigma_{\mathtt{x}}))$ contains exactly a lifting x of $\mathtt{x}$.
2. The connected components of $\varphi_\alpha^{-1}(\mathrm{St}(\sigma_{\mathtt{x}}))$ are open star neighborhoods $\mathrm{St}(\sigma_x)$ in the triangulation $\mathcal{R}$. In other words
$$\varphi_\alpha^{-1}(\mathrm{St}(\sigma_{\mathtt{x}})) = \coprod_{x \in \varphi_\alpha^{-1}(\mathtt{x})} \mathrm{St}(\sigma_x)$$

Proof. By Proposition 3.2.22, for every simplex $\tau \in \overline{\mathrm{St}(\sigma_{\mathtt{x}})}$, and each connected component of $\varphi_\alpha^{-1}(\mathring{\tau})$ there exists a unique lifting $\widetilde{\tau}$ of τ containing that component. Let $\mathcal{R}$ be the collection of all such liftings, taken over all simplices in $\overline{\mathrm{St}(\sigma_{\mathtt{x}})}$. We claim that $\mathcal{R}$ defines a triangulation. It suffices to show that the intersection of any two simplices in $\mathcal{R}$ is again an element in $\mathcal{R}$. In fact, each simplex $\widetilde{\tau} \in \mathcal{R}$ is homeomorphic to its image $\tau \in \mathcal{T}$, through the restriction of φ_α.

Let $\widetilde{\rho}_1, \widetilde{\rho}_2 \in \mathcal{R}$ be two simplices with $\widetilde{\rho}_1 \cap \widetilde{\rho}_2 \neq \emptyset$. Then

$$\varphi_\alpha(\widetilde{\rho}_1 \cap \widetilde{\rho}_2) \subseteq \varphi_\alpha(\widetilde{\rho}_1) \cap \varphi_\alpha(\widetilde{\rho}_2) =: \tau$$

Where τ is a simplex in $\mathcal{T}$. Define

$$\widetilde{\tau}_i := \varphi_\alpha^{-1}(\tau) \cap \widetilde{\rho}_i, \quad \text{for} \quad i = 1, 2$$

These are two liftings of τ. They satisfy $\widetilde{\rho}_1 \cap \widetilde{\rho}_2 = \widetilde{\tau}_1 \cap \widetilde{\tau}_2$, and there exists $g \in G_\alpha$ such that $g\widetilde{\tau}_1 = \widetilde{\tau}_2$. We claim that $g \in G_x$ for every $x \in \widetilde{\rho}_1 \cap \widetilde{\rho}_2$. Suppose, by contradiction, that $gx \neq x$ for some $x \in \widetilde{\rho}_1 \cap \widetilde{\rho}_2$. Let γ be the smallest simplex that contains $\varphi_\alpha(x)$, and define

$$\widetilde{\gamma}_i := \varphi_\alpha^{-1}(\gamma) \cap \widetilde{\rho}_i, \quad i = 1, 2$$

two lifting of γ. Then $g\widetilde{\gamma}_1 = \widetilde{\gamma}_2$, since $\gamma \subseteq \tau$. As $x \in \widetilde{\gamma}_1 \cap \widetilde{\gamma}_2$, and $gx \neq x$, we have

$$\mathring{\widetilde{\gamma}}_1 \cap (g \cdot \mathring{\widetilde{\gamma}}_1) = \emptyset$$

This implies that $\varphi_\alpha(x) \in \partial\gamma$. Therefore, there exists a simplex strictly contained in γ that contains $\varphi_\alpha(x)$, contradicting the minimality of γ. Hence, we conclude that $g \in G_x$.

Now observe that $g \notin G_x$ for any $x \in \widetilde{\tau}_1 \smallsetminus (\widetilde{\tau}_1 \cap \widetilde{\tau}_2)$, since $g\widetilde{\tau}_1 = \tau_2$ and $gx = x$.

Thus, we have that $g \in G_x$ for each $x \in \widetilde{\tau}_1 \cap \widetilde{\tau}_2$ and $g \notin G_y$ for $y \in \widetilde{\tau}_1 \smallsetminus (\widetilde{\tau}_1 \cap \widetilde{\tau}_2)$. It follows that for each such pair x, y, the stabilizer G_y is a subgroup of G_x.

Since we have assumed that the triangulation $\mathcal{T}$ satisfies the properties described in Remark 3.2.21, it follows that there exists a face $\gamma \prec \tau$ such that the points of $\tau \smallsetminus \gamma$ lie entirely within a single stratum, say Y, of the stratification that refines both the stabilizer stratification and the one induced by the toroidal embedding, while the points of γ are contained in strata whose are contained in the closure of Y.

As $\widetilde{\tau}_1$ is homeomorphic to τ, and taking into account the relation between the stabilizers of points in $\widetilde{\tau}_1 \smallsetminus (\widetilde{\tau}_1 \cap \widetilde{\tau}_2)$ and those in $\widetilde{\tau}_1 \cap \widetilde{\tau}_2$, we conclude that:

$$\tau \smallsetminus \gamma \subseteq \tau \smallsetminus \varphi(\widetilde{\tau}_1 \cap \widetilde{\tau}_2),$$

If this inclusion is strict, i.e., $\varphi(\widetilde{\tau}_1 \cap \widetilde{\tau}_2) \subsetneq \gamma$, we can repeat the same argument, now replacing τ by γ, and applying Remark 3.2.21 again to the simplex γ. Since this process reduces the dimension of the simplex at each step, it must terminate after finitely many steps. Eventually, we will arrive at a face γ' of τ such that:

$$\varphi(\widetilde{\tau}_1 \cap \widetilde{\tau}_2) = \gamma'.$$

Thus, $\varphi_\alpha(\widetilde{\tau}_1 \cap \widetilde{\tau}_2)$ is a face of τ. Furthermore, since $\widetilde{\tau}_1$ is homeomorphic to τ, it follows that

$$(\varphi_{\alpha|\widetilde{\tau}_1})^{-1}(\varphi(\widetilde{\tau}_1 \cap \widetilde{\tau}_2) = \varphi_\alpha^{-1}(\varphi(\widetilde{\tau}_1 \cap \widetilde{\tau}_2)) \cap \widetilde{\tau}_1$$

is a face of $\widetilde{\tau}_1$. Hence, we conclude:

$$\widetilde{\rho}_1 \cap \widetilde{\rho}_2 = \widetilde{\tau}_1 \cap \widetilde{\tau}_2 = \varphi_\alpha^{-1}(\varphi(\widetilde{\tau}_1 \cap \widetilde{\tau}_2)) \cap \widetilde{\tau}_1$$

which shows that $\widetilde{\rho}_1 \cap \widetilde{\rho}_2$ is a simplex in $\mathcal{R}$.

Now consider two liftings of the point $\mathtt{x}$, to say x and gx. Suppose that their open stars $\mathrm{St}(\sigma_x)$ and $\mathrm{St}(\sigma_{gx})$ have nonempty intersection. Let z be a point of intersection. Then $z \in \mathring{\tau}$ for some simplex τ. By definition of the star, we have $x, gx \in \tau$. As φ_α is a homeomorphism on τ, we conclude $gx = x$. This completes the proof of the Lemma. □

The construction above allows us to establish the following theorem.

Theorem 3.2.24. *Let $(\mathscr{X}, \mathscr{U})$ be a toroidal Delgine-Mumford stack over $\mathbb{C}$, where $\mathscr{X}$ is smooth and proper on $\mathbb{C}$. Let X be the coarse moduli space of $\mathscr{X}$, and X^{an} its complex analytifaction. Let Σ be the stratification of X^{an} that refines both the stratification by stabilizers and the one induced by the toroidal embedding. Consider the effective orbifold $(X^{an}, \{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda})$, where each orbifold chart satisfies the conditions of Remark 3.1.12; that is, the divisor $V_\alpha \smallsetminus (V_\alpha \times_{\mathscr{X}^{an}} \mathscr{U}^{an})$ is a simple normal crossing divisor; this divisor induces a stratification of V_α with a unique minimal stratum, the image $\varphi_\alpha(V_\alpha)$ intersects a unique minimal stratum of Σ among those that have nonempty intersection with $\varphi_\alpha(V_\alpha)$, and the action of G_α on V_α is linear and permutes the irreducible components of the divisor.*

Then there exists an orbifold atlas $\{(W_\beta, H_\beta, \phi_\beta)\}_{\beta\in\Omega}$ satisfying the following properties:

1. *The covering $\{(W_\beta, H_\beta, \phi_\beta)\}_{\beta\in\Omega}$ refines $\{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$, and is locally finite.*
2. *Each chart $(V_\alpha, G_\alpha, \varphi_\alpha)$ is such that both W_β and its quotient W_β/H_β are contractible. The divisor $W_\beta \smallsetminus (W_\beta \times_{\mathscr{X}^{an}} \mathscr{U}^{an})$ is a simple normal crossing divisor, and the stratification induced by this divisor has a unique minimal stratum. The image $\phi_\beta(W_\beta)$ intersects a unique minimal stratum of Σ among those that have nonempty intersection with $\varphi_\beta(W_\beta)$.*
3. *The intersection of finitely many charts in the atlas is empty, or is a chart of the atlas.*
4. *For any two orbifold charts whose quotient spaces intersect nontrivially, the associated strata are comparable (i.e., one is contained in the closure of the other), and the stratum associated with the chart arising from the intersection coincides with one of the two.*

Proof. Let $\mathcal{T}$ be a triangulation of $(X^{an}, \{(V_\alpha, G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda})$ such that the open star of each simplex is contained in a chart of the orbifold atlas.

For each point $\mathtt{x} \in X^{an}$, let $\sigma_{\mathtt{x}}$ be the smallest simplex containing $\mathtt{x}$. Then the open star $\mathtt{V}_{\mathtt{x}} := \mathrm{St}(\sigma_{\mathtt{x}})$ is contained in some $\varphi_\alpha(V_\alpha)$. Choose a lifting $x \in V_\alpha$ of $\mathtt{x}$, and consider the open star $\mathrm{St}(\sigma_x)$ in V_α as in Lemma 3.2.23. Then the map

$$(\varphi_\alpha)_{|\mathrm{St}(\sigma_x)} : \mathrm{St}(\sigma_x) \to \mathrm{St}(\sigma_{\mathtt{x}})$$

is part of a chart, according to the Lemma 3.2.23, with G_x acting on $\mathrm{St}(\sigma_x)$.

From the above construction, it follows that the covering is indeed a refinement of the initial atlas. This refinement is locally finite because the triangulation partitions the space, and each point is contained in only finitely many open stars of simplices. Therefore, the cover induced by the open sets of the form $\mathrm{St}(\sigma_{\mathtt{x}})$ is locally finite. With this, the first condition is verified.

Each open set $\mathrm{St}(\sigma_{\mathtt{x}})$ has a unique minimal stratum among those strata that intersect it nontrivially. This stratum is precisely the one containing the points of $\mathring{\sigma}_{\mathtt{x}}$. Moreover, since $\mathrm{St}(\sigma_x) \subseteq V_\alpha$ for a suitable $\alpha \in \Lambda$, the divisor $\mathrm{St}(\sigma_x) \times_{\mathscr{X}^{an}} \mathscr{U}^{an}$ is a simple normal crossing divisor. The stratification induced by this divisor has a unique minimal stratum because $\mathrm{St}(\sigma_{\mathtt{x}})$ does, and the map $\varphi_{\alpha|\mathring{\sigma}_x} : \mathring{\sigma}_x \to \mathring{\sigma}_{\mathtt{x}}$ is a homeomorphism. Furthermore, this minimal stratum contains the point $\mathtt{x}$.

Finally Each open star $\mathrm{St}(\sigma_x)$ is contractible, and its quotient $\mathrm{St}(\sigma_x)/G_x$ is also contractil, since the map $\phi_x : \mathrm{St}(\sigma(x)) \to \mathrm{St}(\sigma(x))/G_x$ is open set and G_x acts continuously. These observations establish the second condition.

The third condition follows from the construction: the intersection of two open stars is again an open star.

If the covering derived from the triangulation $\mathcal{T}$ does not initially satisfy the last property, we may apply a barycentric subdivision. In that case, the result follows from Proposition 1.2.13. This completes the proof of the theorem. □

Corollary 3.2.25. Let $\{(W_\beta, H_\beta, \phi_\beta\}_{\beta\in\Omega}$ and $\{(U_\gamma, K_\gamma, \psi_\gamma\}_{\gamma\in\Gamma}$ be two chart coverings that satisfy the conditions of Theorem 3.2.24. Then there exists a chart covering that refines both coverings and satisfies the conditions of Theorem 3.2.24.

Proof. This follows by applying triangulations to the charts $\{(W_\beta, H_\beta, \phi_\beta\}_{\beta\in\Omega}$ and $\{(U_\gamma, K_\gamma, \psi_\gamma\}_{\gamma\in\Gamma}$ separately, and then taking a common refinement of these two triangulations. To this refined triangulation, we apply the construction described in Theorem 3.2.24, which produces a converging chart satisfying the required properties. □

The type of cover constructed in Theorem 3.2.24 motivates the following definition, which will be useful for the purposes of Section 3.5.

Definition 3.2.26. We say that an effective orbifold atlas $\{(V_\alpha.G_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ of X^{an} is a *good chart covering* if it satisfies the following properties

1. Each chart $(V_\alpha, G_\alpha, \varphi_\alpha)$ is such that both V_α and its quotient V_α/G_α are contractible. The divisor $V_\alpha \smallsetminus (V_\alpha \times_{\mathscr{X}^{an}} \mathscr{U}^{an})$ is a simple normal crossing divisor, and the stratification induced by this divisor has a unique minimal stratum. The image $\varphi_\alpha(V_\alpha)$ intersects a unique minimal stratum of Σ among those that have nonempty intersection with $\varphi_\alpha(V_\alpha)$.
2. The intersection of finitely many charts in the atlas is empty, or is a chart of the atlas.
3. For any two orbifold charts whose quotient spaces intersect nontrivially, the associated strata are comparable (i.e., one is contained in the closure of the other), and the stratum associated with the chart arising from the intersection coincides with one of the two.

3.3 Local Green functions and change of charts

Recall that $(\mathscr{X}, \mathscr{U})$ is a toroidal Deligne–Mumford stack over $\mathbb{C}$, with coarse moduli space X and complex analytification X^{an}. We assume that $\mathscr{X}$ is proper over $\mathbb{C}$; consequently, X^{an} is compact.

In this section, we begin by describing the local construction of Green functions on X^{an} using effective orbifold charts. These local functions will take values in quotients of the form $\sigma/\Gamma \subseteq \Sigma(\mathscr{X}, \mathscr{U})$, where Γ denotes a monodromy group associated with the corresponding chart. As we will see, this construction will be analogous to the local Green function defined in (2.1) in Section 2.2, for the case of complex manifolds.

Recall that, in the manifold case, the global Green function was obtained via a partition of unity subordinate to a suitable covering by adapted charts, allowing the local functions to be coherently glued within an appropriate cone. With this observation in mind, and since our goal here is to construct a global

Green function in the orbifold setting, the next step will be to explain how a local Green function defined on a chart $(V_\alpha, G_\alpha, \varphi_\alpha)$ can be interpreted as a function on an open subset of another chart $(V_\beta, G_\beta, \varphi_\beta)$, whenever $\varphi_\alpha(V_\alpha) \subseteq \varphi_\beta(V_\beta)$. Establishing this compatibility between local functions is essential, as it will allow us to proceed in the same way as in the complex manifold case to obtain a global function.

Since this local construction involves orbifold charts and monodromy groups arising from the toroidal structure of $(\mathscr{X}, \mathscr{U})$, we will proceed to recall some constructions from the preliminary topics in order to explain how to attach a monodromy group to a given orbifold chart.

Let $\mathscr{W}$ be a stratum of $\mathscr{X}$, determined by the toroidal structure of the pair $(\mathscr{X}, \mathscr{U})$, as especified in Remark 3.2.4. Since $\mathscr{W} \xrightarrow{i} \mathscr{X}$ is a locally closed immersion, it is represented by a locally closed immersion of schemes. Every immersion of schemes is a monomorphism, and since being a monomorphism is étale local on the target (see Definition 1.4.9), it follows that i is a monomorphism. Acording with [Sta18, Tag 06R5] we have that

$$\begin{array}{ccc} I_{\mathscr{W}} & \longrightarrow & I_{\mathscr{X}} \\ \downarrow & & \downarrow \\ \mathscr{W} & \xrightarrow{\ i\ } & \mathscr{X} \end{array} \tag{3.14}$$

Is a fiber product square. Where $I_{\mathscr{W}}$ and $I_{\mathscr{X}}$ denote the inertia stacks of $\mathscr{W}$ and $\mathscr{X}$, respectively; see 1.4.18. Since the morphism $\mathscr{W} \xrightarrow{i} \mathscr{X}$ is unramified, it follows from [Sta18, Tag 0DU9] that i induces an isomorphism between the stabilizer group scheme $\underline{G}_w$ of each geometric point $w \in |\mathscr{W}|$ and the stabilizer group scheme $\underline{G}_{i(w)}$ in $\mathscr{X}$. As a consequence, we do not need to distinguish between the stabilizer groups in $\mathscr{X}$ and those in $\mathscr{W}$.

Recall from Section 1.6, each étale map $V \to \mathscr{X}$, defines a group $\mathcal{M}_V$, which is the group of Cartier divisor of V supported in $D_V := V \smallsetminus V \times_{\mathscr{X}} \mathscr{U}$, and a submonoid $\mathcal{S}_V \subseteq \mathcal{M}_V$ consisting of effective divisors. The sheaves $\mathcal{M}$ and $\mathcal{S}$ are then defined as the sheafifications of these presheaves.

Also recall that for each geometric point $w \in \mathscr{X}$, the stalk $\mathcal{M}_w$ (resp. $\mathcal{S}_w$) is defined as

$$\mathcal{M}_w = \operatorname{Colim}_{(V,v)} M_V \quad (\text{resp. } \mathcal{S}_w)$$

Where (V, v) runs over all étale neighborhoods of w in $\mathscr{X}$. that is, commutative diagrams

$$\begin{array}{ccc} & & \operatorname{Spec} \overline{k} \\ & \swarrow^{v} & \downarrow{\scriptstyle w} \\ V & \longrightarrow & \mathscr{X} \end{array}$$

Since $\mathscr{W}$ is connected, we consider the fundamental étale group $\pi_1^{et}(\mathscr{W}, w)$ at a geometric point w of $\mathscr{W}$; see Section 1.5. Therefore, we can consider the homomorphism

$$\omega : \underline{G}_w(\mathbb{C}) \to \pi_1^{et}(\mathscr{W}, w) \tag{3.15}$$

that was defined in (1.1) in Section 1.5, where $\underline{G}_w(\mathbb{C})$ is the $\mathbb{C}$-points of the stabilizer group scheme $\underline{G}_w$. For simplicity, from now on we will denote this group by G_w.

In Section 1.6, the monodromy group Γ_w was defined as the image of the homomorphism

$$\pi_1^{et}(\mathscr{W}, w) \xrightarrow{m} \mathrm{Aut}(\mathcal{M}_w). \tag{3.16}$$

Therefore, from (3.15) and (3.16), we have the following morphism of groups

$$\lambda_w : G_w \to \Gamma_w \tag{3.17}$$

Let $V \to \mathscr{X}$ be a small toric chart, see Definition 1.6.9, such that the image of its minimal stratum contains the complex point w of $\mathscr{X}$. Then following [ACP12], we have a map

$$\sigma \to \Sigma(\mathscr{X}, \mathscr{U})$$

Where $\sigma := \mathrm{Hom}(\mathcal{S}_{\mathrm{V}}, \mathbb{R}_{\geq 0})$, and $\Sigma(\mathscr{X}, \mathscr{U})$ is the skeleton associated with the pair $(\mathscr{X}, \mathscr{U})$, constructed in [ACP12] and described in Section 1.6. Moreover, this construction induces an injective map

$$\mathring{\sigma}/\Gamma_w \to \Sigma(\mathscr{X}, \mathscr{U})$$

where $\mathring{\sigma}$ denotes the relative interior of σ. The action of Γ_w on σ coincides with the natural action of Γ_w on $\sigma_w := \mathrm{Hom}_{\mathrm{Mon}}(\mathcal{S}_w, \mathbb{R}_{\geq 0})$, via the identification between the cones σ_w and σ.

Since we have a natural inclusion $\mathring{\sigma}/\Gamma_w \hookrightarrow \sigma/\Gamma_w$, we conclude that there exists an injective map

$$\sigma/\Gamma_w \to \Sigma(\mathscr{X}, \mathscr{U}) \tag{3.18}$$

Now, consider the complex analytification X^{an} of the coarse moduli X, and let V^{an} denote the complex analytification of the small toric chart V. The natural morphism $V \to X$ induces a map $V^{an} \to X^{an}$. As w is a complex geometric point, it can be regarded as an element of X^{an}; consequently, the image of V^{an} contains this point.

By Proposition 3.1.2, we can consider $(V_\alpha, G_\alpha, \varphi_\alpha)$ an orbifold chart, such that V_α is an open complex ball contained in the analytification V^{an} of the small toric chart, with its center mapping to the complex geometric point $w \in X^{an}$, and such that G_α acts by linear transformations and permuting the irreducible components of $D_\alpha := V_\alpha \smallsetminus (V_\alpha \times_{\mathscr{X}^{an}} \mathscr{U}^{an})$. Moreover, the divisor D_α is a simple normal crossing divisor and the stratification of V_α has a unique minimal stratum since it contains the point w. By construction of this orbifold chart, we have $G_\alpha = G_w$.

Denote $D_{\alpha,1}, \ldots, D_{\alpha,n}$ the irreducible components of the divisor D, and choose $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ to be positive Green functions, where each $\mathfrak{g}_j$ is a positive Green function for the irreducible component $D_{\alpha,j}$, for $j = 1, \ldots, n$.

Since V_α has a unique minimal stratum, it follows that the intersection of the irreducible components of D_α is nonempty. Therefore, the cone complex $\Sigma(V_\alpha, D_\alpha)$, constructed as in Chapter 2, consists of a single cone, which we denote by σ_α. Moreover, this cone can be identified with the cone σ associated with the small toric chart V, since the image of this orbifold chart contains the point w.

Using the data fixed above, we now define the following function:

$$F' : V_\alpha \smallsetminus D_\alpha \to \sigma_\alpha, \quad z \mapsto (\mathfrak{g}_j(z))_{j=1}^n$$

Since σ_α is identified with σ, the group Γ_w naturally acts on σ_α. Moreover, as the group G_α acts on V_α,

we can use the homomorphism $\lambda_w : G_w \to \Gamma_w$ defined in (3.17) to construct the following function:

$$F : V_\alpha \smallsetminus D_\alpha \to \sigma_\alpha, \qquad z \mapsto \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \lambda_w(g) F'(g^{-1} z) \tag{3.19}$$

This function has the following properties, that we will establish in the next proposition.

Proposition 3.3.1. The function F is λ_w-equivariant. Moreover, for each $1 \leq j \leq n$, the j-th coordinate of F is given by the average of $|G_\alpha|$ Green functions associated with the divisor $D_{\alpha,j}$. In particular, each coordinate function of F is a Green function.

Proof. Let us first prove that this map is λ_w-equivariant. To this end, let $z \in V_\alpha \smallsetminus D_\alpha$ and $g_0 \in G_\alpha$. Then we have the following sequence of equalities:

$$\begin{aligned}
F(g_0 z) &= \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \lambda_w(g) F'(g^{-1} g_0 z) \\
&= \frac{1}{|G_\alpha|} \sum_{g_1 \in G} \lambda_w(g_0 g_1) F'(g_1^{-1} z) \quad \text{(variable change} \quad g_1^{-1} = g^{-1} g_0) \\
&= \frac{1}{|G_\alpha|} \sum_{g_1 \in G} \lambda_w(g_0) \lambda_w(g_1) F'(g_1^{-1} z) \\
&= \frac{\lambda_w(g_0)}{|G_\alpha|} \sum_{g_1 \in G_\alpha} \lambda_w(g_1) F'(g_1^{-1} z) \\
&= \lambda_w(g_0) F(z)
\end{aligned}$$

For the second part, observe that each element of the group G_α permutes the irreducible components of the divisor D_α. Consequently, the image of these elements under λ_w permutes the rays of the cone σ. Therefore, we will describe the action of these groups in terms of permutations.

For each $g \in G_\alpha$ there exists $\gamma_g \in \mathcal{S}_n$ such that for each $1 \leq j \leq n$ $g \cdot D_{\alpha,j} = D_{\alpha,\gamma_g(j)}$. Moreover, it is clear that $\gamma_{g^{-1}} = (\gamma_g)^{-1}$. This implies that $\mathfrak{g}_j \circ g$ is a Green function for the component $D_{\alpha,\gamma_{g^{-1}}(j)}$. In consequence, the j-th coordinate is described as

$$\frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \mathfrak{g}_{\gamma_g(j)} \cdot g$$

As this is average of positive Green functions, we have that this is a positive Green function. Therefore, the claim is proved. □

Since F is λ_w-equivariant, we naturally have a function

$$\mathtt{F}_\alpha : (V_\alpha \smallsetminus D_\alpha)/G_\alpha \to \sigma_V/\Gamma_w \subseteq \Sigma(\mathscr{X}, \mathscr{U}). \tag{3.20}$$

Note that $(V_\alpha \smallsetminus D_\alpha)/G_\alpha$ can be identified with $\mathtt{V}_\alpha \cap U^{an}$, where $\mathtt{V}_\alpha$ denotes the image of V_α under φ_α, and U^{an} is the complex analytification of the coarse moduli space of the open substack $\mathscr{U} \subseteq \mathscr{X}$.

The construction above serves as the starting point, since the local functions defined here will be used to build the global Green function on $U^{an} \subseteq X^{an}$. This open subset plays the role of the analogue of $X \setminus D$,

where the global Green function was defined in Chapter 2, in the case where X is a complex manifold and D a simple normal crossing divisor.

Our next task is to explain how to relate these local functions on the intersections of charts, taking into account that they are induced by orbifold charts. Therefore, in what follows, we describe how to relate functions of the form (3.19) when one orbifold chart is contained in another.

Let $(V_\alpha, G_\alpha, \phi_\alpha)$ and $(V_\beta, G_\beta, \phi_\beta)$ be two orbifold charts of X^{an} satisfying the conditions of Proposition 3.1.2, that is, V_α and V_β are open balls of $\mathbb{C}^n$ centered at the origin, and the groups G_α and G_β act linearly, preserving their corresponding divisors. Assume that there is an inclusion $\mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$; in other words, $[V_\beta/G_\beta] \hookrightarrow [V_\alpha/G_\alpha]$ an open immersion of analytic stacks.

Consider the following data:

1. Denote by $D_\beta \subseteq V_\beta$ and $D_\alpha \subseteq V_\alpha$ the respective invariant simple normal crossing divisors.
2. Let y denote the center of V_β, then it is clear that this oiint lies in the minimal stratum of V_β, and is invariant under the action of G_β.
3. Let σ_α and σ_β be the associated cones corresponding to the pairs (V_α, D_α) and (V_β, D_β), respectively.
4. Let Γ_α, and Γ_β denote the monodromy groups associated with the charts V_α and V_β respectively.
5. Denote $\lambda_\alpha : G_\alpha \to \Gamma_\alpha$ and $\lambda_\beta : G_\beta \to \Gamma_\beta$ the respective homomorphisms as in (3.17).

As a consequence of the inclusion $\mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$, and following Section 3.1, we have the commutative diagram:

$$\begin{array}{ccc} V_\beta \times_{\mathcal{V}_\beta} \pi_\alpha^{-1}(\mathcal{V}_\beta) & \xrightarrow{p_\alpha} & \pi_\alpha^{-1}(\mathcal{V}_\beta) \\ \downarrow{\scriptstyle p_\beta} & & \downarrow{\scriptstyle \pi_{\alpha|\pi_\alpha^{-1}(\mathcal{V}_\beta)}} \\ V_\beta & \xrightarrow{\pi_\beta} & \mathcal{V}_\beta \end{array} \tag{3.21}$$

where $\mathcal{V}_\beta$ denotes the analytic stack $[V_\beta/G_\beta]$, and p_β is a principal G_α-bundle.

Let $y \in V_\beta$ be the center point, and consider the fiber $p_\beta^{-1}(y)$. Since V_β is chosen to be contractible, by Definition 3.1.5 we have that for each $z \in p_\beta^{-1}(y)$ there exists a unique section $\varphi_z : V_\beta \to V_\beta \times_{\mathcal{V}_\beta} \pi_\alpha^{-1}(\mathcal{V}_\beta)$ such that $p_\beta \circ \varphi_z = \mathrm{id}_{V_\beta}$ and $\varphi_z(y) = z$. Composing φ_z with p_α, we obtain the map

$$f_z : V_\beta \to \pi_\alpha^{-1}(\mathcal{V}_\beta). \tag{3.22}$$

In Proposition 3.1.7, we showed that the element z also induces a monomorphism of groups $\rho_{z_0} : G_\beta \to G_\alpha$. In this setting, we proved tha f_z is a ρ_z-equivariant map, biholomorphic onto its image, and that it induces a biholomorphism

$$\pi_\alpha^{-1}(\mathcal{V}_\beta) \simeq G_\alpha \times^{G_\alpha} V_\beta$$

In addition, since the inverse image of the divisor on V_α under f_z coincides with the divisor on V_β, we can identify the cone σ_β as a face of σ_α. As this face morphism depends on the map f_z, we denote this identification by $\iota_z : \sigma_\beta \to \sigma_\alpha$. Hence, this embedding is not canonical.

From the above disscusion, choose $z_0 \in p_\beta^{-1}(y)$ and consider the following diagram of group morphisms.

$$\begin{array}{ccccc} H & \longrightarrow & G_\alpha & \xrightarrow{\lambda_\alpha} & \Gamma_{V_\alpha} \\ {\scriptstyle \rho_{z_0}}\uparrow & & & & \\ G_\beta & \xrightarrow{\lambda_\beta} & \Gamma_{V_\beta} & & \end{array} \tag{3.23}$$

Here, H is the subgroup of G_α which is isomorphic to G_β.

Let $F_\beta : V_\beta \setminus D_\beta \to \sigma_{V_\beta}$ be the λ_β-equivariant function defined as in (3.19) from a collection of positive Green functions associated to the irreducible components of D_α. In Section 3.1, it was shown that the spaces $\phi_\alpha^{-1}(\mathtt{V}_\beta)$ and $G_\alpha \times^{G_\beta} V_\beta$, are biholomorphic, with the biholomorphism determined by the map f_{z_0}. Under this identification, each element $v \in \phi_\alpha^{-1}(\mathtt{V}_\beta)$ is represented by an equivalence class $\widetilde{(g,x)}$, with $g \in G_\alpha$ and $x \in V_\beta$. This description allows us to define the following map:

$$F_\beta^\alpha : \phi_\alpha^{-1}(\mathtt{V}_\beta) \setminus D_\alpha \to \sigma_\alpha, \quad v \mapsto F_\beta^\alpha(v) := \lambda_\alpha(g)\left[(\iota_{z_0} \circ F_\beta)(x)\right] \tag{3.24}$$

This definition is independent of the choice of representative, and the map is λ_α-equivariant, as we will show in the following result.

Proposition 3.3.2. The map F_β^α is well defined and λ_α-equivariant.

Proof. Let (g_1, x_1) and (g_2, x_2) be two elements representing the same equivalence class. Therefore, there exists an element $\gamma' \in G_\beta$ such that $x_2 = \gamma' x_1$ and $g_2 = g_1 \rho_{z_0}(\gamma')^{-1}$. Hence, we have

$$\lambda_\alpha(g_2)\left[(\iota_{z_0} \circ F_\beta)(x_2)\right] = \lambda_\alpha(g_1\rho(\gamma')^{-1})\left[(\iota_{z_0} \circ F_\beta)(\gamma' x)\right]$$

Due to λ_β-equivariance of F_β, the expression above becomes

$$\lambda_\alpha(g_1\rho(\gamma')^{-1})\left[(\iota_{z_0} \circ F_\beta)(\gamma' x))\right] = \lambda_\alpha(g_1)\lambda_\alpha(\rho(\gamma')^{-1})\left[\iota_{z_0}(\lambda_i(\gamma') \cdot F_\beta(x))\right]$$

Note that γ' permutes the irreducible components of the divisor D_β that pass through y. Consequently, $\rho_{z_0}(\gamma')$ permutes the irreducible components of the divisor D_α that pass through the point $f_{z_0}(y)$. This observation, together with the following commutative diagram

$$\begin{array}{ccc} V_\beta & \xrightarrow{f_{z_0}} & f_{z_0}(V_\beta) \\ \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \rho_{z_0}(\gamma')} \\ V_\beta & \xrightarrow{f_{z_0}} & f_{z_0}(V_\beta) \end{array}$$

allows us to conclude the commutativity of the following diagram:

$$\begin{array}{ccc} \sigma_\alpha & \xrightarrow{\iota_{z_0}} & \sigma_\alpha \\ \downarrow{\scriptstyle \lambda_\beta(\gamma')} & & \downarrow{\scriptstyle \lambda_\alpha(\rho_{z_0}(\gamma'))} \\ \sigma_\alpha & \xrightarrow{\iota_{z_0}} & \sigma_\alpha \end{array} \tag{3.25}$$

In consequence, we have the following sequence of equalities

$$\begin{aligned}\lambda_\alpha(g_1)\lambda_\alpha(\rho(\gamma')^{-1})\left[\iota_{z_0}(\lambda_\beta(\gamma')\cdot F_\beta(x))\right] &= \lambda_\alpha(g_1)\lambda_\alpha(\rho_{z_0}(\gamma')^{-1})\lambda_\alpha(\rho_{z_0}(\gamma'))\left[(\iota_{z_0}\circ F_\beta)(x)\right]\\ &= \lambda_\alpha(g_1)\left[(\iota_{z_0}\circ F_\beta)(x)\right]\end{aligned}$$

We have shown that the value of the map is the same for any choice of representatives, and therefore F_β^α is well defined.

We now prove that the map F_β^α is λ_α-equivariant. Let $g \in G_\alpha$, and let $v = (g_0, x)$ be a representative of the class. Then we have the following sequence of equalities:

$$\begin{aligned}F_\beta^\alpha(gv) &= \lambda_\alpha(gg_0)\left[(\iota_{z_0}\circ F_\beta)(x)\right]\\ &= \lambda_\alpha(g)\lambda_\alpha(g_0)\left[(\iota_{z_0}\circ F_\beta)(x)\right]\\ &= \lambda_\alpha(g)F_\beta^\alpha(v)\end{aligned}$$

This shows that F_β^α is λ_α-equivariant, as we claimed. □

Remark 3.3.3. As F_β^α is λ_α-equivariant, it induces a map

$$\mathtt{F}_\beta^\alpha : \mathtt{V}_\beta \cap U^{an} \to \sigma_\alpha/\Gamma_\alpha$$

similarly, the map $F_\beta : V_\beta \to \sigma_\beta$ induces a function

$$\mathtt{F}_\beta : \mathtt{V}_\beta \cap U^{an} \to \sigma_\beta/\Gamma_\beta$$

Note that $\mathtt{V}_\beta \cap U^{an}$ is precisely $(V_\beta \smallsetminus D_\beta)/G_\beta$; an analogous description holds for the index α.

The values of $\mathtt{F}_\beta$ and $\mathtt{F}_\beta^\alpha$ coincide in $\Sigma(\mathscr{X}, \mathscr{U})$. Indeed, suppose that $v_1, v_2 \in \phi_\alpha^{-1}(\mathtt{V}_\beta)$ are sent to the same element in X^{an} under ϕ_α. Then there exists $g \in G_\alpha$ such that $v_2 = g \cdot v_2$. If v_1 and v_2 are represented by the classes $\widetilde{(g_1, x_1)}$ and $\widetilde{(g_2, x_2)}$ in $G_\alpha \times^{G_\beta} V_\beta$, the action of g affects only the first component. By taking $g = g_1^{-1}$, we obtain an element $v_3 \in \phi_\alpha^{\mathtt{V}_\beta}$ corresponding to the class $\widetilde{(e_{G_\alpha}, x_1)}$, whose image under ϕ_α is the same as v_1. Evaluating v_3, we obtain:

$$F_\beta^\alpha(v_3) = \lambda_\alpha(e_{G_\alpha})[\iota_{z_0}\circ F_\beta(x_1)] = \iota_{z_0}\circ F_\beta(x_1).$$

Since $F_\beta(x)$ and $\iota_{z_0}\circ F_\beta(x_1)$ have the same image in $\Sigma(\mathscr{X}, \mathscr{U})$, it follows that $\mathtt{F}_\beta$ and $\mathtt{F}_\beta^\alpha$ coincide as functions into $\Sigma(\mathscr{X}, \mathscr{U})$.

Remark 3.3.4. Note that the map F_β^α defined above depends on the choice of the point $z_0 \in p_\beta^{-1}(y)$. In this context, a natural question arises: What happens to F_β^α if we choose a different point in $p^{-1}(y)$?

If $f_z : V_\beta \to \phi_\alpha^{-1}(\mathtt{V}_\beta)$, for z in the fiber $p^{-1}(y)$ is another function, we have shown that there exists an element $g \in G_\alpha$ such that $z = gz_0$. Then we had $f_z = gf_{z_0}$, and in consequence the biholomorphism

$$\xi_z : G_\alpha \times^{G_\beta} V_\beta \to \phi_\alpha^{-1}(\mathtt{V}_\beta)$$

induced by f_z sends a point (g', x) to the value $g'f_{gz_0}(x) = g'gf_{z_0}(x)$, see Proposition 3.1.9. Therefore, if the value (g', x) is sent to a point v in $\phi_\alpha^{-1}(\mathtt{V}_\beta)$ via the map ξ_{z_0}, then the point that maps to v via ξ_z is precisely $(g'g^{-1}, x)$.

As the map $\iota_{z_0} : \sigma_\beta \to \sigma_\alpha$ is determined by the choice of z_0 and $f_z = gf_{z_0}$, it follows that

$$\iota_z = \lambda_\alpha(g) \circ f_{z_0}$$

From these observations, and denoting by F_β^α and $F_\beta^{\alpha\prime}$ the functions induced by F_β via the choices f_{z_0} and f_z, respectively, we obtain the following sequence of equalities.

$$\begin{aligned} F_\beta^{\alpha\prime}(v) &= \lambda_\alpha(g'g^{-1})\,[(\iota_{gz_0} \circ F_\beta)(x)] \\ &= \lambda_\alpha(g')\lambda_\alpha(g^{-1})\,[\lambda_\alpha(g)(\iota_{z_0} \circ F_\beta)(x)] \\ &= \lambda_\alpha(g')\,[(\iota_{z_0} \circ F_\beta)(x)] \\ &= F_\beta^\alpha(v) \end{aligned}$$

Therefore, we conclude that the construction of the map F_β^α is independent of the choice of the point z_0.

Remark 3.3.5. The above remark allows us to conclude that the function F_β^α is the unique λ_α-equivariant function from $\phi_\alpha^{-1}(\mathtt{V}_\beta \cap U^{an})$ to σ_α that makes the diagram

$$\begin{array}{ccc} \varphi_\beta^{-1}(\mathtt{V}_\beta \cap U^{an}) & \xrightarrow{\;f_z\;} & \phi_\alpha^{-1}(\mathtt{V}_\beta \cap U^{an}) \\ \Big\downarrow{\scriptstyle F_\beta} & & \Big\downarrow{\scriptstyle F_\beta^\alpha} \\ \sigma_\beta & \xrightarrow{\;\iota_z\;} & \sigma_\alpha \end{array}$$

commute, for each $z \in p_\beta^{-1}(y)$.

We recall some previously established properties. These are restated here, as they will be important for the proof of Proposition 3.3.7 below.

Remark 3.3.6.

1. Let $(V_\alpha, G_\alpha, \phi_\alpha)$ be an orbifold chart, and let D_α be a divisor invariant under the action of G_α. If $\{D_{\alpha,i}\}_{i\in I_\alpha}$ are the irreducible components of D_α, recall that G_α permutes these components. In this context, if $\mathfrak{g}_{\alpha,i}$ is a Green function for the component $D_{\alpha,i}$, and $\gamma \in G_\alpha$, then the function $\mathfrak{g}_{\alpha,i} \circ \gamma^{-1}$ is a Green function for the component $\gamma D_{\alpha,i}$.

2. If $(V_\alpha, G_\alpha, \phi_\alpha)$, $(V_\beta, G_\beta, \phi_\beta)$ are two orbifold charts with $\mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$, where $\mathtt{V}_\alpha := \phi_\alpha(V_\alpha)$, $\mathtt{V}_\beta := \phi_\beta(V_\beta)$, and if $\mathfrak{g}$ is a Green function for a divisor $D_{\beta,i}$, then $\mathfrak{g} \circ f_{z_0}^{-1}$ is a Green function for the divisor $f_{z_0}(D_{\beta,i})$. Moreover, for any $\gamma \in G_\alpha$, the function $\mathfrak{g} \circ (\gamma \circ f_{z_0})^{-1}$ is a Green function for $\gamma \circ f_{z_0}(D_{\beta,i})$. Where $D_{\beta,i}$ is an irreducible component of the divisor D_β of V_β.

The following proposition can be viewed as a generalization of Propositions 2.2.1 and 2.2.2, since every complex manifold may be regarded as a particular case of a complex orbifold.

Proposition 3.3.7. Let $(V_\alpha, G_\alpha, \phi_\alpha)$ and $(V_\beta, G_\beta, \phi_\beta)$ such that $\mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$, and let D_α and D_β be the respective invariant divisors. Let F_α and F_β functions as in (3.19), and let F_β^α be the function induced by F_β. Then for each point $\mathtt{z} \in \mathtt{V}_\beta$ there exists a chart $(V_\tau, G_\tau, \phi_\tau)$, with $\mathtt{V}_\tau \subseteq \mathtt{V}_\beta$ and an open neighborhood of $\mathtt{z}$ such that at $\phi_\alpha^{-1}(\mathtt{V}_\tau) \smallsetminus D_\alpha$

$$F_\alpha = F_\beta^\alpha + \text{bounded function.}$$

Proof. First, let us fix the following data:

1. The set $\{D_{\alpha,i}\}_{i\in I_\alpha}$ and $\{D_{\beta,j}\}_{j\in I_\beta}$ denote the irreducible components of D_α and D_β, respectively.
2. Let $\{\mathfrak{g}_{\alpha,i}\}_{i\in I_\alpha}$ and $\{\mathfrak{g}_{\beta,j}\}_{j\in I_\beta}$ be the collection of positive Green functions defining F_α and F_β, respectively.

Since $f_{z_0} : V_\beta \to \phi_\alpha^{-1}(\mathtt{V}_\beta)$ is a biholomorphism with its image, we see that for every $j \in I_\beta$ there exists a unique $i(j) \in I_\alpha$ such that

$$f_{z_0}(D_{\beta,j}) = D_{\alpha,i(j)} \cap f_{z_0}(V_\beta).$$

Hence, by Remark 3.3.6 we have $\mathfrak{g}_{\beta,j} \circ (f_{z_0|f_{z_0}(V_\beta)})^{-1}$ is a Green function for $D_{\alpha,i(j)} \cap f_{z_0}(V_\beta)$. Therefore, by Proposition 3.3.1 we see that the $i(j)$-th coordinates of the functions F_β^α and F_α are Green function of the divisor $D_{\alpha,i(j)} \cap f_{z_0}(V_\beta)$.

For simplicity, denote the i-coordinate of the functions F_β^α and F_α by $\mathfrak{h}_i$ and $\mathfrak{h}'_i$ respectively.

Let $\mathtt{z} \in \mathtt{V}_\beta$, and choose a point $z \in f_{z_0}(V_\beta) \subseteq V_\alpha$ such that $\phi_\alpha(z) = \mathtt{z}$. Define

$$I_\beta^0 := \{j \in I_\beta : z \in f_{z_0}(D_{\beta,j})\}$$

Then there exists a G_z-invariant open neighborhood W_1 of z contained in $f_{z_0}(V_\beta)$ such that for each $j \in I_\beta^0$, on $W_1 \smallsetminus D_\alpha$ we have:

$$\mathfrak{h}_{i(j)} = \mathfrak{h}'_{i(j)} + \psi_{i(j)}$$

where $\psi_{i(j)}$ is a continuous bounded function defined on W_1.

Finally, define

$$I_\alpha^0 := \{i \in I_\beta : i = i(j), \text{ for some } j \in I_\beta^0\}.$$

Then there exists a G_z-invariant open neighborhood W_2 of z such that for each $i \in I_\alpha \smallsetminus I_\alpha^0$, the Green function $\mathfrak{h}_i$ is defined and bounded in W_3, and the corresponding coordinate of F_β^α is zero or bounded.

Now, let

$$W' := W_1 \cap W_2$$

and set $W := G_\alpha \cdot W'$. By construction, these properties hold on every G_α-translate of z. Therefore, we have

$$F_\alpha = F_\beta^\alpha + \text{bounded function.}$$

In conclusion, the chart $(W', G_z, \phi_{\alpha|W'})$ satisfies the conditions stated in the proposition. □

3.4 Local equivariant functions on X^{an}

Let $(V_\alpha, G_\alpha, \phi_\alpha)$ and $(V_\beta, G_\beta, \phi_\beta)$ be two orbifold charts of X^{an}, with $\mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$. In the previous section, see Remark 3.3.4, we showed that the function F_α^β, induced by a λ_β-equivariant function F_β, does not depend on the choice of biholomorphism, and that this induced function is itself λ_α-equivariant.

Note that these facts and the specific properties of such induced functions are not exclusive to equivariant functions arising from Green functions. Therefore, in this section, we extend these results to arbitrary local equivariant functions since this generalization will be of particular interest in the next section, where we construct the global Green functions.

Since we will be working with equivariant functions, it is convenient to recall the stratification introduced in Section 3.1.

For the analytic complex space X^{an}, which is the analytification of the coarse moduli space $\mathscr{X}$ of the toroidal embedding $(\mathscr{X}, \mathscr{U})$, we have a stratification $\Sigma = \{M_r\}_{r\in R}$ that refines both the stratification by stabilizers and the toroidal stratification. By definition, the relation $M_r \leq M_{r'}$ implies that there exist strata W and W' of the toroidal stratification containing M_r and $M_{r'}$, respectively, such that $W \subseteq \overline{W'}$ and moreover, for each $x \in M_r$ and $x' \in M_{r'}$, the stabilizer of x', is isomorphic, as an abstract group, to a subgroup of the stabilizer of x.

Since X^{an} is metrizable 3.1.1, in this section we will consider orbifold chart $(V_\alpha, G_\alpha, \phi_\alpha)$ satisfying the properties of Proposition 3.1.2, such that each chart has an associated stratum M_{r_α}, which is the unique minimal stratum of Σ between those that have nonempty intersection with $\mathtt{V}_\alpha$. We also fix a distinguished point $x_\alpha \in V_\alpha$, which is invariant under the action of G_α.

Proposition 3.4.1. Let $(V_\alpha, G_\alpha, \phi_\alpha)$ and $(V_\beta, G_\beta, \phi_\beta)$ be two orbifold charts with $\mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$, and assume that $M_{r_\alpha} = M_{r_\beta}$. If $F : \phi_\alpha^{-1}(\mathtt{W}) \to \sigma_\alpha$ is a λ_α-equivariant function defined on an open $\mathtt{W}$ of $\mathtt{V}_\beta$, then there exists a unique λ_β-equivariant function $F_\beta : \phi_\beta^{-1}(\mathtt{W}) \to \sigma_\beta$ that makes the following diagram commute for each $z \in p_\beta^{-1}(y)$:

$$\begin{array}{ccc} \varphi_\beta^{-1}(\mathtt{W}) & \xrightarrow{f_z} & \phi_\alpha^{-1}(\mathtt{W}) \\ \downarrow{\scriptstyle (F)_\beta} & & \downarrow{\scriptstyle F} \\ \sigma_\beta & \xrightarrow{\iota_z} & \sigma_\alpha \end{array}$$

Here, f_z denotes the map associated to $z \in p_\beta^{-1}(x_\beta)$, and we are considering its restriction to the open subset $\varphi_\beta^{-1}(\mathtt{W}) \subseteq V_\beta$.

Proof. Recall that $p_\beta : V_\beta \times_{\mathcal{V}_\beta} \pi_\alpha^{-1}(\mathcal{V}_\beta) \to V_\beta$ is the G_α-bundle defined in (3.21) in the previous section. Fix a point $z \in p_\beta^{-1}(x_\beta)$. Since $M_{r_\alpha} = M_{r_\beta}$, the homomorphism $\rho_z : G_\beta \to G_\alpha$ is an isomorphism. Therefore, $f_z : V_\beta \to \pi_\alpha^{-1}(\mathcal{V}_\beta)$ is in fact a biholomorphism.

The equality $M_{r_\alpha} = M_{r_\beta}$ also implies that $\iota_z : \sigma_\beta \to \sigma_\alpha$ is an isomorphism of cones. This follows from the fact that the stratification Σ refines the toroidal stratification, and the cones associated with points belonging to the same stratum are isomorphic to each other, see the beginning of Section 3.3, where this construction is explained.

Using the biholomorphism f_z, which is ρ_z-equivariant, together with the isomorphism of cones ι_z, we obtain in a natural way a λ_β-equivariant function $F_\beta : \phi_\beta^{-1}(\mathtt{W}) \to \sigma_\beta$ that makes the diagram commute.

Finally, if $f_{z'}$ is another map determined by some $z' \in p_\beta^{-1}(x_\beta)$, then there exists $g \in G_\alpha$ such that $z' = gz$. It follows that $f_{z'} = gf_z$. By the equivariance of F and the isomorphism $\rho_z : G_\beta \to G_\alpha$, the construction of F_β is therefore independent of the choice of z, and hence unique. □

Proposition 3.4.2. Let $(V_\alpha, G_\alpha, \phi_\alpha)$ and $(V_\beta, G_\beta, \phi_\beta)$ be two orbifold charts with $\mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$, and $M_{r_\alpha} \leq M_{r_\beta}$. If $F : \phi_\beta^{-1}(\mathtt{W}) \to \sigma_\beta$ is a λ_β-equivariant function defined on an open $\mathtt{W}$ of $\mathtt{V}_\beta$, then there exists a unique λ_α-equivariant function $(F)^\alpha : \phi_\alpha^{-1}(\mathtt{W}) \to \sigma_\alpha$ that makes the following diagram commute for each

element $z \in p_\beta^{-1}(y)$:

$$\begin{array}{ccc} \varphi_\beta^{-1}(\mathtt{W}) & \xrightarrow{f_z} & \phi_\alpha^{-1}(\mathtt{W}) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F^\alpha} \\ \sigma_\beta & \xrightarrow{\iota_z} & \sigma_\alpha \end{array}$$

Proof. Fix a point $z \in p_\beta^{-1}(x_\beta)$. The ρ_z-equivariant holomorphism $f_z : V_\beta \to (\mathcal{V}_\beta)$ induces a biholomorphism $\pi_\alpha^{-1}(\mathcal{V}_\beta) \simeq G_\alpha \times^{G_\beta} V_\beta$, and we also have the corresponding homomorphism of cones $\iota_z : \sigma_\beta \to \sigma_\alpha$.

Therefore, as in the previous section, and noting that $\phi_\beta^{-1}(\mathtt{W}) \subseteq V_\beta$, each element $v \in \phi_\alpha^{-1}(\mathtt{W})$ is represented by an equivalence class $\widetilde{(g,x)}$, where $g \in G_\alpha$ and $x \in \phi_\beta^{-1}(\mathtt{W})$. We then define the function

$$F^\alpha : \phi_\alpha^{-1}(\mathtt{W}) \to \sigma_\alpha, \quad v \mapsto \lambda_\alpha(g)[\iota_z(F(x))],$$

in the same way as the local Green function F_α^β in (3.24) in the previous section.

It is straightforward to verify, by the same arguments used there, that this function is well defined, λ_α-equivariant, and independent of the choice of the element in $p_\beta^{-1}(x_\beta)$, since those proofs relied only on the equivariance properties. Hence, F^α is the unique function making the diagram commute. □

Remark 3.4.3. In Propositions 3.4.1 and 3.4.2, the commutativity of the square needs to be verified only for a single element $z \in p_\beta^{-1}(y)$, since the induced function in both cases is λ_β-equivariant. Indeed, for any $z_1, z_2 \in p_\beta^{-1}(y)$, there exists $g \in G_\alpha$ such that $z_2 = gz_1$, and by the construction described in Section 3.1, we have $f_{z_2} = gf_{z_1}$.

Proposition 3.4.4. Let $(V_\alpha, G_\alpha, \phi_\alpha), (V_\beta, G_\beta, \phi_\beta)$ and $(V_\gamma, G_\gamma, \phi_\gamma)$ be three orbifold charts with distinguished points $x_\gamma \in V_\gamma$ and $x_\beta \in V_\beta$, invariant under the actions of G_γ and G_β, respectively, and such that $\mathtt{V}_\gamma \subseteq \mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$. If $F : \phi_\gamma^{-1}(\mathtt{W}) \to \sigma_\gamma$ is a λ_γ-equivariant function, then $((F)^\beta)^\alpha = (F)^\alpha$.

Proof. Chosee a $f_1 : V_\gamma \to \phi_\beta^{-1}(\mathtt{V}_\gamma)$ which is defined by the choice of an element in $p_\gamma^{-1}(x_\gamma)$, where p_γ is the G_β-bundle defined in a same way as in (3.21). Then we have:

$$\begin{array}{ccc} V_\gamma & \xrightarrow{f_1} & \phi_\beta^{-1}(\mathtt{V}_\gamma) \\ \downarrow{\scriptstyle F_\gamma} & & \downarrow{\scriptstyle (F)^\beta} \\ \sigma_\gamma & \xrightarrow{\iota_1} & \sigma_\beta \end{array}$$

Extending this square to include $\phi_\alpha^{-1}(\mathtt{V}_\gamma)$, we obtain the following diagram:

$$\begin{array}{ccccc} V_\gamma & \xrightarrow{f_1} & \phi_\beta^{-1}(\mathtt{V}_\gamma) & \xrightarrow{f_2} & \phi_\alpha^{-1}(\mathtt{V}_\gamma) \\ \downarrow{\scriptstyle F_\beta} & & \downarrow{\scriptstyle (F)^\beta} & & \downarrow{\scriptstyle ((F)^\beta)^\alpha} \\ \sigma_\gamma & \xrightarrow{\iota_1} & \sigma_\beta & \xrightarrow{\iota_2} & \sigma_\alpha \end{array}$$

From Proposition 3.1.10, we know that $f_2 \circ f_1$ is defined by an element $z \in p_\gamma^{-1}(x_\gamma)$. Hence, by the uniqueness stated in Proposition 3.4.2 and Remark 3.4.3, we conclude that $(F^\beta)^\alpha = (F)^\alpha$. □

Remark 3.4.5. Following the same argument as in Proposition 3.4.4, consider three orbifold charts $(V_\alpha, G_\alpha, \phi_\alpha), (V_\beta, G_\beta, \phi_\beta)$ and $(V_\gamma, G_\gamma, \phi_\gamma)$ three orbifold charts, with $M_{r_\alpha} = M_{r_\beta} = M_{r_\gamma}$ and distinguished points $x_\gamma \in V_\gamma$ and $x_\beta \in V_\beta$ whose images under ϕ_γ and ϕ_β lie in M_{r_γ} and M_{r_β}, respectively. If moreover, $\mathtt{V}_\gamma \subseteq \mathtt{V}_\beta \subseteq \mathtt{V}_\alpha$ and $F : \phi_\alpha^{-1}(\mathtt{W}) \to \sigma_\alpha$ is a λ_α-equivariant function, then $((F)_\beta)_\gamma = (F)_\gamma$.

3.5 Construction of the hybrid space

Given a toroidal Deligne-Mumford stack $(\mathscr{X}, \mathscr{U})$ over $\mathbb{C}$, recall that we are assuming $\mathscr{X}$ to be proper and smooth. These conditions ensure that the analytification X^{an} of the coarse moduli space of $\mathscr{X}$ is compact, and that it forms the underlying topological space of a proper étale groupoid in the category of complex manifolds.

The goal of this section is to construct the hybrid space associated with $(\mathscr{X}, \mathscr{U})$. To achieve this, we first determine a global function on U^{an} such that its local locally structure is similar with the functions $\mathtt{F}_\alpha$ defined in (3.20). As we will see, this construction can be regarded as a generalization of the global functions introduced in Proposition 2.2.3.

Let $(X^{an}, \mathcal{V})$ be the effective orbifold associated with X^{an}, where $\mathcal{V} = \{(V_\alpha, G_\alpha, \phi_\alpha)\}_{\alpha\in\Lambda}$ denotes a good covering chart; see Definition 3.2.26. In order to construct a collection of local functions as in (3.19), we fix the following data for each orbifold chart $(V_\alpha, G_\alpha, \phi_\alpha)$ in $\mathcal{V}$:

1. As in the previous section, $\mathtt{V}_\alpha$ denotes the quotient V_α/G_α in X^{an}; that is, the image of V_α under the map ϕ_α.
2. Let D_α be the simple normal crossing divisor on V_α that is invariant under the action of G_α, and denote $\mathtt{D}_\alpha$ its image in X^{an}. Note that $\mathtt{D}_\alpha = \mathtt{V}_\alpha \cap (X^{an} \smallsetminus U^{an})$.
3. Denote by M_{r_α} the associated stratum of $\mathtt{V}_\alpha$. Recall that the stratification under consideration is the one introduced in Proposition 3.2.6, namely, the stratification that refines both the stratification by stabilizers of X^{an} and the stratification induced by the toroidal structure of the pair $(\mathscr{X}, \mathscr{U})$. This stratum is the unique minimal stratum among those that have nonempty intersection with $\mathtt{V}_\alpha$.
4. Let x_α be the distinguished point of V_α. This point is invariant under the action of G_α, and its image is contained in M_{r_α}. Therefore, denote by Γ_α the monodromy group associated with the image of this point in $\mathscr{X}^{an}$, and $\lambda_\alpha : G_\alpha \to \Gamma_\alpha$ the homomorphisms determined as in (3.17).
5. Denote by σ_α the associated cone of the pair (V_α, D_α).
6. Let $\{g_{\alpha,i}\}_{i\in I_\alpha}$ be a collection of positive Green functions, one for each irreducible component of D_α.

Finally, consider $\{\chi_\alpha\}_{\alpha\in\Lambda}$ a partition of unity subordinate to the covering $\{\mathtt{V}_\alpha\}_{\alpha\in\Lambda}$. The existence of such a partition is guaranteed because the analytic space X^{an} is metrizable; see Proposition 3.1.1.

Observe that, by the definition of a good covering chart (see 3.2.26), if $\mathtt{V}_\alpha \cap \mathtt{V}_\beta \neq \emptyset$, then the associated strata are comparable. Moreover, there exists a chart $(V_\gamma, G_\gamma, \phi_\gamma)$, with $\gamma \in \Lambda$, such that $\mathtt{V}_\alpha \cap \mathtt{V}_\beta = \mathtt{V}_\gamma$, and the associated stratum M_{r_γ} coincides with either M_{r_α} or M_{r_β}.

For each $\alpha \in \Lambda$, consider the λ_α-equivariant function

$$F_\alpha : V_\alpha \smallsetminus D_\alpha \to \sigma_\alpha \tag{3.26}$$

constructed as in (3.19) from the collecction of positive Green functions specified above.

For each point $\mathtt{x} \in X^{an}$, consider the following index sets:

$$A_\mathtt{x} := \{\alpha \in \Lambda : \mathtt{x} \in \mathtt{V}_\alpha\}, \quad B_\mathtt{x} := \{\beta \in \Lambda : \mathtt{x} \notin \mathrm{Supp}(\chi_\beta)\},$$

and define the open subset

$$\mathtt{W}_\mathtt{x} := \left(\bigcap_{\alpha \in A_\mathtt{x}} \mathtt{V}_\alpha\right) \smallsetminus \left(\bigcup_{\beta \in B_\mathtt{x}} \mathrm{Supp}(\chi_\beta)\right). \tag{3.27}$$

Since the covering is a good covering by charts, note that the intersection $\bigcap_{\alpha\in A_\mathtt{x}} \mathtt{V}_\alpha$ is equal to some $\mathtt{V}_{\alpha'}$, with $\alpha' \in A_\mathtt{x}$.

Let $\alpha_\mathtt{x} \in A_\mathtt{x}$ be any index such that the stratum associated with $\mathtt{V}_{\alpha_\mathtt{x}}$ is minimal among the collection

$$\{M_{r_\alpha} : \ \alpha \in A_\mathtt{x}, \ \ M_{r_\alpha} \text{ stratum associated with } \ \mathtt{V}_\alpha\}.$$

Note that the index $\alpha_\mathtt{x}$ is not necessarily equal to α'. Then for every $\beta \in A_\mathtt{x}$ there exists $\gamma(\beta) \in A_\mathtt{x}$ such that

$$\mathtt{V}_{\alpha_\mathtt{x}} \cap \mathtt{V}_\beta = \mathtt{V}_{\gamma(\beta)}.$$

Since $M_{r_{\gamma(\beta)}} = M_{r_\beta}$, it follows from Proposition 3.4.1 that for each index $\beta \in A_\mathtt{x}$, the restriction of F_β, $F_{\beta_{|\phi_\beta^{-1}(\mathtt{W}_\mathtt{x})}} : \phi_\beta^{-1}(\mathtt{W}_\mathtt{x}) \to \sigma_\beta$ induces a continuous function

$$(F_\beta)_{\gamma(\beta)} : \phi_{\gamma(\beta)}^{-1}(\mathtt{W}_\mathtt{x}) \to \sigma_{\gamma(\beta)}$$

Moreover, since $\mathtt{V}_{\gamma(\beta)} \subseteq \mathtt{V}_{\alpha_\mathtt{x}}$ and $M_{r_{\alpha_\mathtt{x}}} \leq M_{r_{\gamma(\beta)}}$, it follows from Proposition 3.4.2 that $(F_\beta)_{\gamma(\beta)}$ induces a continuous function

$$((F_\beta)_{\gamma(\beta)})^{\alpha_\mathtt{x}} : \phi_{\alpha_\mathtt{x}}^{-1}(\mathtt{W}_\mathtt{X}) \to \sigma_{\alpha_\mathtt{x}}$$

To simplify notation, we denote this function by $F^{\alpha_\mathtt{x}}_{\beta,\gamma(\beta)} := ((F_\beta)_{\gamma(\beta)})^{\alpha_\mathtt{x}}$.

Now, we fix the set of indices

$$\Lambda_\mathtt{x} := \{\alpha \in \Lambda : \mathtt{V}_\alpha \cap \mathtt{V}_{\alpha_\mathtt{x}} \neq \emptyset\}. \tag{3.28}$$

Note that $A_\mathtt{x}$ is contained in $\Lambda_\mathtt{x}$. Fixing it, we see that the collection $\{\chi_\alpha\}_{\alpha\in\Lambda_\mathtt{x}}$ induces a partition of unity

$$\{\chi_\alpha \circ \phi_{\alpha_\mathtt{x}}\}_{\alpha\in\Lambda_\mathtt{x}},$$

on $V_{\alpha_\mathtt{x}}$, which is subordinate to the covering $\{\phi_{\alpha_\mathtt{x}}^{-1}(\mathtt{V}_\alpha)\}_{\alpha\in\Lambda_\mathtt{x}}$. In this way, we define the following continuous function:

$$F_\mathtt{x} : \phi_{\alpha_\mathtt{x}}^{-1}(\mathtt{W}_\mathtt{x}) \smallsetminus D_{\alpha_\mathtt{x}} \to \sigma_{\alpha_\mathtt{x}}, \quad F_\mathtt{x} := \sum_{\beta\in A_\mathtt{x}} (\chi_\beta \circ \phi_{\alpha_\mathtt{x}}) F^{\alpha_\mathtt{x}}_{\beta,\gamma(\beta)} \tag{3.29}$$

Note that although the partition of unity is indexed by $\Lambda_\mathtt{x}$, the sum is taken over the set of indices $A_\mathtt{x}$, since it is defined on the open set $\phi_{\alpha_\mathtt{x}}^{-1}(\mathtt{W}_\mathtt{x}) \subseteq V_{\alpha_\mathtt{x}}$. It is clear that the function $F_\mathtt{x}$ is $\lambda_{\alpha_\mathtt{x}}$-equivariant, and

thus induces a continuous function on the quotient space.

$$\mathtt{F}_{\mathtt{x}} : \mathtt{W}_{\mathtt{x}} \smallsetminus \mathtt{D}_{\alpha_{\mathtt{x}}} \to \Sigma(\mathscr{X}, \mathscr{U}) \tag{3.30}$$

The next result explains how the local functions defined in (3.30) are related on the intersection of open sets of the form $\mathtt{W}_{\mathtt{x}}$. In particular, it shows what happens when a different index $\alpha_{\mathtt{x}}$ is chosen in the above construction.

Proposition 3.5.1. Let $\mathtt{x}, \mathtt{y}$ be two points of X^{an}, and define $\mathtt{W}_{\mathtt{x}}$ and $\mathtt{W}_{\mathtt{y}}$ as above. Then the functions $\mathtt{F}_{\mathtt{x}}$ and $\mathtt{F}_{\mathtt{y}}$ agree on the intersection $\mathtt{W}_{\mathtt{x}} \cap \mathtt{W}_{\mathtt{y}}$. In particular, these functions do not depend on the choice of $\alpha_{\mathtt{x}}$.

Proof. If $\mathtt{W}_{\mathtt{x}} \cap \mathtt{W}_{\mathtt{y}} \neq \emptyset$, then the open sets $\mathtt{V}_{\alpha_{\mathtt{x}}}$ and $\mathtt{V}_{\alpha_{\mathtt{y}}}$ are comparable. Without loss of generality, assume that

$$M_{r_{\alpha_{\mathtt{x}}}} \leq M_{r_{\alpha_{\mathtt{y}}}}$$

Recall that $\mathrm{Supp}(\chi_\alpha) \cap \mathtt{W}_{\mathtt{x}} \neq \emptyset$ if only if $\mathtt{x} \in \mathrm{Supp}(\chi_\alpha)$, and in particular $\mathtt{x} \in \mathtt{V}_\alpha$. The same argument holds when replacing $\mathtt{W}_{\mathtt{x}}$ by $\mathtt{W}_{\mathtt{y}}$. Let $\mathtt{z} \in \mathtt{W}_{\mathtt{x}} \cap \mathtt{W}_{\mathtt{y}}$, and define

$$C_{\mathtt{z}} = \{\alpha \in \Lambda : \mathtt{z} \in \mathrm{Supp}(\chi_\alpha)\}.$$

We claim that $C_{\mathtt{z}} \subseteq A_{\mathtt{x}} \cap A_{\mathtt{y}}$. Indeed, fix $\alpha \in C_{\mathtt{z}}$. Since $\mathtt{z} \in \mathrm{Supp}(\chi_\alpha)$ and $\mathtt{z} \in \mathtt{W}_{\mathtt{x}}$, it follows that $\mathrm{Supp}(\chi_\alpha) \cap \mathtt{W}_{\mathtt{x}} \neq \emptyset$, and thus $\mathtt{x} \in \mathrm{Supp}(\chi_\alpha) \subseteq \mathtt{V}_\alpha$ by the above recall, so $\alpha \in A_{\mathtt{x}}$. Similarly, since $\mathtt{z} \in \mathtt{W}_{\mathtt{y}}$, we also get $\mathtt{y} \in \mathrm{Supp}(\chi_\alpha)$. Hence, $\alpha \in A_{\mathtt{x}} \cap A_{\mathtt{y}}$, as desired.

Take points $z_1 \in V_{\alpha_{\mathtt{x}}}$ and $z_2 \in V_{\alpha_{\mathtt{y}}}$ mapping to $\mathtt{z}$ under the quotient maps, then we have the following expressions:

$$F_{\mathtt{x}}(z_1) = \sum_{\beta \in C_{\mathtt{z}}} \chi_\beta(\mathtt{z}) \cdot F^{\alpha_{\mathtt{x}}}_{\beta,\gamma(\beta)}(z_1), \quad F_{\mathtt{y}}(z_2) = \sum_{\beta \in C_{\mathtt{z}}} \chi_\beta(\mathtt{z}) \cdot F^{\alpha_{\mathtt{y}}}_{\beta,\delta(\beta)}(z_2)$$

where $\gamma(\beta), \delta(\beta)$ are such that

$$\mathtt{V}_{\gamma(\beta)} = \mathtt{V}_\beta \cap \mathtt{V}_{\alpha_{\mathtt{y}}}, \quad \mathtt{V}_{\delta(\beta)} = \mathtt{V}_\beta \cap \mathtt{V}_{\alpha_{\mathtt{y}}}.$$

The goal is to show that $F_{\mathtt{x}}(z_1)$ and $F_{\mathtt{y}}(z_2)$ agree once we pass to the quotient, i.e., they define the same element in $\Sigma(\mathscr{X}, \mathscr{U})$.

By the assumptions fixed at the beginning of the proof, we have

$$\mathtt{V}_\tau = \mathtt{V}_{\alpha_{\mathtt{x}}} \cap \mathtt{V}_{\alpha_{\mathtt{y}}}$$

for some $(V_\tau, G_\tau, \phi_\tau)$ of the covering, with $M_{r_\tau} = M_{r_{\alpha_{\mathtt{y}}}}$. Then by Propositions 3.4.1 and 3.4.2, $F_{\mathtt{y}}$ induces the function

$$((F_{\mathtt{y}})_\tau)^{\alpha_{\mathtt{x}}} : \phi^{-1}_{\alpha_{\mathtt{x}}}(\mathtt{W}_{\mathtt{y}} \cap \mathtt{W}_{\mathtt{x}}) \smallsetminus D_{\alpha_{\mathtt{x}}} \to \sigma_{\alpha_{\mathtt{x}}}.$$

Recall that $F_{\mathtt{y}}$ is given by

$$F_{\mathtt{y}} = \sum_{\beta \in A_{\mathtt{y}}} (\chi_\beta \circ \phi_\beta) \cdot F^{\alpha_{\mathtt{y}}}_{\beta,\delta(\beta)}.$$

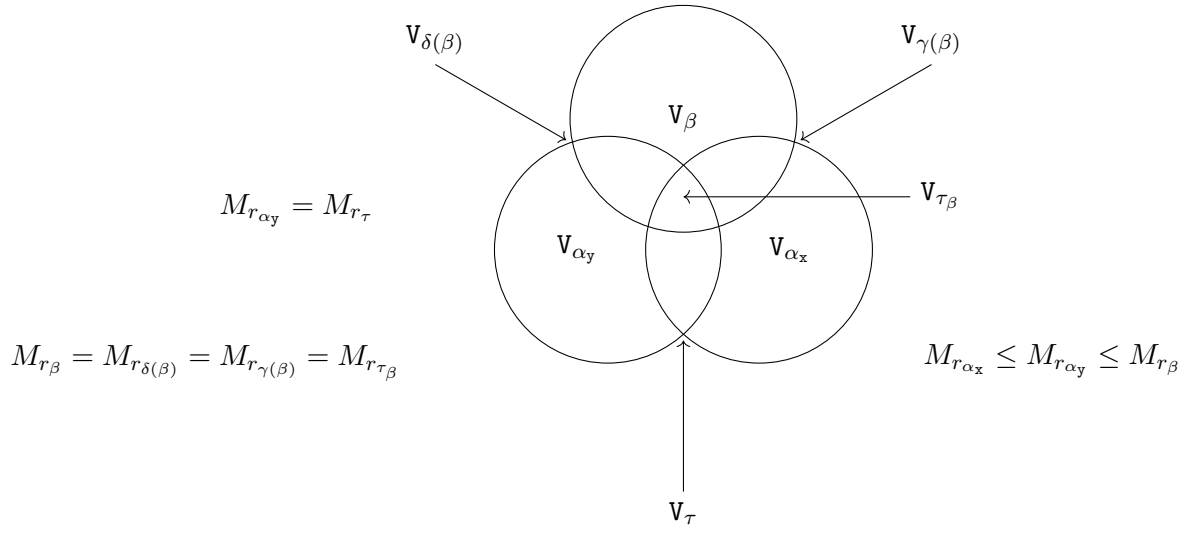
Since the partition of unity depends only on the values in X^{an}, it follows that induced function $((F_{\mathtt{y}})_\tau)^{\alpha_{\mathtt{x}}}$ is the result of applying the same process to each $F^{\alpha_{\mathtt{y}}}_{\beta,\delta(\beta)}$, for $\beta \in A_{\mathtt{y}}$. That is,

$$((F_{\mathtt{y}})_\tau)^{\alpha_{\mathtt{x}}} = \sum_{\beta \in A_{\mathtt{y}}} (\chi_\beta \circ \phi_{\alpha_{\mathtt{x}}})[((F^{\alpha_{\mathtt{y}}}_{\beta,\delta(\beta)})_\tau)^{\alpha_{\mathtt{x}}}]$$

the proof will be guaranteed if we can show that, for each $\beta \in C_{\mathtt{z}}$

$$((F^{\alpha_{\mathtt{y}}}_{\beta,\delta(\beta)})_\tau)^{\alpha_{\mathtt{x}}} = F^{\alpha_{\mathtt{x}}}_{\beta,\gamma(\beta)}$$

To verify this equality, consider the following diagram, which illustrates the corresponding intersections of the charts in X^{an}, and the comparison of the strata:



Note that by Proposition 3.4.1, we have that the function F_β induces a function $(F_\beta)_{\tau_\beta} : V_{\tau_\beta} \to \sigma_{\tau_\beta}$, which in turn induces a function $((F_\beta)_{\tau_\beta})^{\alpha_{\mathtt{y}}}$. By Proposition 3.4.4 and Remark 3.4.5, this function coincides with $(F^{\alpha_{\mathtt{x}}}_{\beta,\delta(\beta)})_{|\phi^{-1}_{\alpha_{\mathtt{y}}}(\mathtt{V}_{\tau_\beta})}$.

Now, $((F_\beta)_{\tau_\beta})^{\alpha_{\mathtt{y}}}$ induces a function on the chart $(V_\tau, G_\tau, \phi_\tau)$, which is precisely $((F_\beta)_{\tau_\beta})^\tau$, since we again apply Proposition 3.4.4 and Remark 3.4.5.

Consequently, this function induces $(((F_\beta)_{\tau_\beta})^\tau)^{\alpha_{\mathtt{x}}}$, and, by the same proposition and remark, we obtain that it is equal to $((F_\beta)_{\tau_\beta})^{\alpha_{\mathtt{x}}}$. Therefore in $\phi^{-1}_{\alpha_{\mathtt{x}}}(\mathtt{V}_{\tau_\beta})$ we have

$$F^{\alpha_{\mathtt{x}}}_{\beta,\gamma(\beta)} = ((F^{\alpha_{\mathtt{y}}}_{\beta,\delta(\beta)})_\tau)^{\alpha_{\mathtt{x}}}$$

Hence, we obtain

$$((F_{\mathtt{y}})_\tau)^{\alpha_{\mathtt{x}}} = F_{\alpha_{\mathtt{x}}}$$

on the restriction $\phi^{-1}_{\alpha_{\mathtt{x}}}(\mathtt{W}_{\mathtt{x}} \cap \mathtt{W}_{\mathtt{y}})$. Since $F_{\mathtt{y}}$ and $((F_{\mathtt{y}})_\tau)^{\alpha_{\mathtt{x}}}$ coincide once we pass to the quotient (see Remark 3.3.3), it follows that $F_{\mathtt{y}}(z_2)$ coincides with $((F_{\mathtt{y}})_\tau)^{\alpha_{\mathtt{x}}}(z_1)$ in the quotient. Therefore $F_{\mathtt{x}}(z_1)$ and $F_{\mathtt{y}}(z_2)$ define the same element in $\Sigma(\mathscr{X}, \mathscr{U})$. □

We are now in a position to state the main theorem of this section:

Theorem 3.5.2. *There exists a map $\mathtt{F} : U^{an} \to \Sigma(\mathscr{X}, \mathscr{U})$ with the following property: For each stratum M_r, each point $\mathtt{x} \in M_r$, every orbifold chart (V, G, ϕ) containing $\mathtt{x}$, with M_r as its associated stratum, and each λ_V-equivariant function F_V defined as in (3.19), there exists an open neighborhood $\mathtt{W} \subseteq \mathtt{V}$ of $\mathtt{x}$, an orbifold chart (V', G', ϕ') with associated stratum $M_{r'} \subseteq \overline{M_r}$, and a $\lambda_{V'}$-equivariant map $F' : V' \setminus D' \to \sigma_{V'}$ such that*

1. *The following diagram commutes:*

$$\begin{array}{ccc} (\phi')^{-1}(\mathtt{W}) & \xrightarrow{F'} & \sigma_{V'} \\ \downarrow & & \downarrow \\ \mathtt{W} & \xrightarrow{\mathtt{F}_{|\mathtt{V}}} & \Sigma(\mathscr{X},\mathscr{U}) \end{array}$$

2. *If $(F_V)' : (\phi')^{-1}(\mathtt{W}) \to \sigma_{V'}$ denotes the function induced by F_V, then*

$$(F_V)' = F' + \psi',$$

where ψ' is a bounded continuous function defined on $(\phi')^{-1}(\mathtt{W})$.

Proof. Let $\mathtt{F}$ be defined locally as in (3.30). Since this local construction is independent of the choice of the open set used in (3.27), the function $\mathtt{F}$ is well-defined on U^{an}. Note that $\mathtt{F}$ depends on the choice of a good covering chart and a corresponding family of Green functions, one such family for each chart in the covering. Let us now verify that $\mathtt{F}$ satisfies the property stated in the theorem.

For any point $\mathtt{x} \in X^{\mathrm{an}}$, consider the open set $\mathtt{W}_{\mathtt{x}}$ defined as in (3.27). Then, the restriction of $\mathtt{F}$ to $\mathtt{W}_{\mathtt{x}}$ is precisely the function

$$\mathtt{F}_{\mathtt{x}} : \mathtt{W}_{\mathtt{x}} \smallsetminus \mathtt{D}_{\alpha_{\mathtt{x}}} \to \Sigma(\mathscr{X},\mathscr{U})$$

constructed as in (3.30). Since for each element $\beta \in A_{\mathtt{x}}$, for $A_{\mathtt{x}}$ defined as in 3.28, the associated stratum satisfies $M_{r_\beta} \leq M_r$, and since $A_{\mathtt{x}}$ is finite, we can choose a G-invariant contractible open subset $W' \subseteq V$ that contains the point $\mathtt{x}$ in its image, and such that its quotient $\mathtt{W}' := \phi(W') \subseteq \mathtt{V}_\beta$ for each $\beta \in A_{\mathtt{x}}$.

We then define by $F_{W'}$ the restriction of F_V to W'. From Proposition 3.4.2, $F_{W'}$ induces, for each $\beta \in A_{\mathtt{x}}$, a function $(F_{W'})^\beta : \phi_\beta^{-1}(\mathtt{W}') \smallsetminus D_\beta \to \sigma_\beta$.

By Proposition 3.3.7, there exists an open subset $W_\beta \subseteq V_\beta$ where its image under ϕ_β is contained in $\mathtt{W}'$ and contains the point $\mathtt{x}$, and such that at $W_\beta \smallsetminus D_\beta$

$$F_\beta = (F_{W'})^\beta + \psi_\beta \tag{3.31}$$

where ψ_β a bounded continuous function defined on W_β. Therefore, we define

$$\mathtt{W} := \bigcap_{\beta \in A_{\mathtt{x}}} \phi_\beta(W_\beta)$$

In this order, consider this open, the orbifold chart $(V_{\alpha_{\mathtt{x}}}, G_{\alpha_{\mathtt{x}}}, \phi_{\alpha_{\mathtt{x}}})$, and the function $F_{\mathtt{x}}$ restricted to $\phi_{\alpha_{\mathtt{x}}}^{-1}(\mathtt{W})$. Therefore, these choices guarantee the first condition of the theorem by construction of $\mathtt{F}_{\mathtt{x}}$.

Now, note that in $\phi_{\alpha_{\mathtt{x}}}^{-1}(\mathtt{W}) \smallsetminus D_{\alpha_{\mathtt{x}}}$, for each $\beta \in A_{\mathtt{x}}$, we have:

$$(\chi_\beta \circ \phi_{\alpha_{\mathtt{x}}}) \cdot ((F_{W'})^\beta)^{\alpha_{\mathtt{x}}} = (\chi_\beta \circ \phi_{\alpha_{\mathtt{x}}}) \cdot (((F_\beta)_{\gamma(\beta)})^{\alpha_{\mathtt{x}}} + \phi'_\beta)$$

where ϕ'_β is the function that induces ϕ_β from (3.31).

Note that $((F_{W'})^\beta)^{\alpha_{\mathtt{x}}}$ depends on both $F_{W'}$ and the group $G_{\alpha_{\mathtt{x}}}$. Therefore, by the uniqueness of Propo-

sition 3.4.2, for any β, β' in $A_{\mathtt{x}}$, we have:

$$((F_{W'})^{\beta})^{\alpha_{\mathtt{x}}} = ((F_{W'})^{\beta'})^{\alpha_{\mathtt{x}}}$$

In particular, this implies $(F_{W'})^{\alpha_{\mathtt{x}}} = F_{\mathtt{x}} + \psi$, where ψ is a bounded function defined on $\phi_{\alpha_{\mathtt{x}}}^{-1}(\mathtt{W})$. With this, we have the second condition of the theorem. □

Given a toroidal Deligne-Mundford stack $(\mathscr{X}, \mathscr{U})$, let (X^{an}, U^{an}) denote its associated pair in the category of analytic spaces. We define the set:

$$\mathscr{X}^{hyb} := U^{an} \times (0,1] \coprod \Sigma(\mathscr{X}, \mathscr{U}) \times \{0\}$$

Let $\mathtt{F}$ be any fixed function that satisfies the conditions of Proposition 3.5.2. Using this function, define the map

$$\mathtt{G} : \mathscr{X}^{hyb} \to \Sigma(\mathscr{X}, \mathscr{U})$$

by

$$\mathtt{G}(z,t) = \begin{cases} t \cdot \mathtt{F}(z) & \text{if} \quad t \neq 0 \\ \\ z & \text{if} \quad t = 0 \end{cases} \tag{3.32}$$

In analogy with Definition 2.2.4, we introduce the following:

Definition 3.5.3. The *hybrid topology* on $\mathscr{X}^{hyb}$ is the coarsest topology such that:

1. The inclusion $U^{an} \times (0,1] \hookrightarrow \mathscr{X}^{hyb}$ is an open inmmersion.
2. The projection $\pi : \mathscr{X}^{hyb} \to [0,1]$ is continuous.
3. The function $\mathtt{G}$ defined in (3.32), is continuous.

As in Section 2.2, note that the construction of the function $\mathtt{G}$ depends on the choice of $\mathtt{F}$. A priori, different choices of $\mathtt{F}$ may yield different functions $\mathtt{G}$, and thus different topologies.

Suppose now that $\mathtt{F}'$ is another function satisfying the conditions of Theorem 3.5.2. This rise to another function $\mathtt{G}'$ defined analogously to $\mathtt{G}$.

What follows is the proof that the hybrid topology on $\mathscr{X}^{hyb}$ is independent of the particular choice of the function $\mathtt{F}$. Before proceeding with this proof, we state a lemma that will be useful in establishing the independence of the topology induced by the function $\mathtt{G}$.

Lemma 3.5.4.
1. The topological space $\Sigma(\mathscr{X}, \mathscr{U})$ is second-countable.
2. The hybrid space $\mathscr{X}^{hyb}$, with any topology induced by a choice of function $\mathtt{G}$, is second-countable; in particular, it is a sequential space.

Proof. Let $P_0 \to \mathscr{X}$ be an étale presentation such that $(P_0, P_0 \times_{\mathscr{X}} \mathscr{U})$ is a strict toroidal embedding, and consider the groupoid $P_1 \rightrightarrows P_0$ associated to this presentation. From Section 1.1, we have that the diagram of associated skeletons

$$\overline{\Sigma}(P_1, P_1 \times_{\mathscr{X}} \mathscr{U}) \rightrightarrows \overline{\Sigma}(P_0, P_0 \times_{\mathscr{X}} \mathscr{U}) \to \overline{\Sigma}(\mathscr{X}, \mathscr{U})$$

is exact in the category of topological spaces.

According to [Thu07], the skeletons $\overline{\Sigma}(P_1, P_1 \times_{\mathscr{X}} \mathscr{U})$ and $\overline{\Sigma}(P_0, P_0 \times_{\mathscr{X}} \mathscr{U})$ can each be written as a finite union of compact polyhedra of the form $(\mathbb{R}_{\geq 0} \cup \infty)^n$, for some n. It follows that these spaces are locally metrizable. They are also compact and Hausdorff, since P_0 and P_1 are separable schemes. These properties imply that both skeletons are metrizable spaces.

Now, since $\overline{\Sigma}(P_0, P_0 \times_{\mathscr{X}} \mathscr{U})$ is a compact metric space and the map

$$\overline{\Sigma}(P_0, P_0 \times_{\mathscr{X}} \mathscr{U}) \to \overline{\Sigma}(\mathscr{X}, \mathscr{U})$$

is a quotient map, and $\overline{\Sigma}(\mathscr{X}, \mathscr{U})$ is Hausdorff, it follows that $\overline{\Sigma}(\mathscr{X}, \mathscr{U})$ is metrizable[1]. In particular, $\overline{\Sigma}(\mathscr{X}, \mathscr{U})$ is second-countable, since $\overline{\Sigma}(\mathscr{X}, \mathscr{U})$ is compact. In consequence, we have that the open set $\Sigma(\mathscr{X}, \mathscr{U})$ is second-countable.

Now, let us show that $\mathscr{X}^{hyb}$ is second countable. For this, fix a function $\mathtt{G}$ defining the topology on $\mathscr{X}^{hyb}$. Since $\mathtt{G}$ is continuous and surjective, we may choose a countable basis for $\Sigma(\mathscr{X}, \mathscr{U})$, which provides a countable basis for the fiber over $t = 0$.

Since X^{an} is compact, it can be covered by finitely many charts (V, G, ϕ), where V is a complex manifold. Each V is second-countable, and since ϕ is an open map, the image $\phi(V)$ is an open set, second-countable subset of X^{an}. Therefore, X^{an} is second-countable, in particular U^{an} so is. This implies that $U^{an} \times (0, 1]$ is second-countable.

From the above facts, we conclude that $\mathscr{X}^{hyb}$ is second-countable. In particular, it is a sequential space. □

Proposition 3.5.5. Since X^{an} is compact, the topologies on $\mathscr{X}^{hyb}$ defined by any two choices of functions $\mathtt{G}$ and $\mathtt{G}'$ as in Definition 3.5.3 coincide.

Proof. Let τ, and τ' be the topologies associated to the functions $\mathtt{G}_1$ and $\mathtt{G}_2$ respectively. Since both $(\mathscr{X}^{hyb}, \tau)$ and $(\mathscr{X}^{hyb}, \tau')$ are sequential spaces, it suffices to prove the following statement:

A sequence $\{(z_n, t_n)\}_{n\in\mathbb{N}}$ in $\mathscr{X}^{hyb}$ converges to a point $(y, 0)$ with respect to the topology τ if only if it also converges to $(y, 0)$ with respect to τ'.

Indeed, suposse that $\{(z_n, t_n)\}_{n\in\mathbb{N}}$ converges to a point $(y, 0)$ in the topology τ. Consider the sequence $\{z_n\}_{n\in\mathbb{N}}$ formed by the first component of the above sequence. Since X^{an} is compact, there exists a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ and $z_0 \in X^{an}$ such that the limit of that subsequence is z_0.

Let M_r be the stratum of X^{an} that contains the point z_0, and let $\mathtt{F}_1, \mathtt{F}_2$ be the functions defining $\mathtt{G}$ and $\mathtt{G}'$, respectively.

Let $(V_\alpha, G_\alpha, \phi_\alpha)$ be any orbifold chart such that $z_0 \in \phi_\alpha(V_\alpha)$, whose associated stratum is M_r, and let

$$F_\alpha : V_\alpha \smallsetminus D_\alpha \to \sigma_\alpha$$

be any λ_α-equivariant map defined by a collection of Green functions as in (3.19). Then by Proposition 3.5.2, there exist:

[1] If $f : X \to Y$ a quotient map between topological spaces, with X a compact metrizable space and Y Hausdorff, then Y is metrizable; see [Eng89][Theorem. 4.2.13]

- An open subset $\mathtt{W} \subseteq \phi_\alpha(V_\alpha)$ that contains the point z_0.
- Orbifold charts (V_1, G_1, ϕ_1) and (V_2, G_2, ϕ_2) containing z_0, whose associated strata are contained in $\overline{M}_r$.
- λ_1 and λ_2 equivariant maps $F_1 : V_1 \smallsetminus D_1 \to \sigma_1$ and $F_2 : V_2 \smallsetminus D_2 \to \sigma_2$, respectively.

such that:

1. The following diagrams commute:

$$\begin{array}{ccc} \phi_1^{-1}(\mathtt{W}) & \xrightarrow{F_1} & \sigma_1 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle i_1} \\ \mathtt{W} & \xrightarrow{\mathtt{F}_1} & \Sigma(\mathscr{X}, \mathscr{U}) \end{array} \qquad \begin{array}{ccc} \phi_2^{-1}(\mathtt{W}) & \xrightarrow{F_2} & \sigma_2 \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle i_2} \\ \mathtt{W} & \xrightarrow{\mathtt{F}_2} & \Sigma(\mathscr{X}, \mathscr{U}) \end{array} \tag{3.33}$$

2. If $(F_\alpha)^1$ and $(F_\alpha)^2$ denote the functions induced by F_α on the charts $(V_1, , G_1, \phi_1)$ and (V_2, G_2, ϕ_2) respectively, then there exist bounded continuous functions $\phi_1 : \phi_1^{-1}(\mathtt{W}) \to \sigma_1$ and $\psi_2 : \psi_2^{-1}(\mathtt{W}) \to \sigma_2$ such that
$$(F_\alpha)^1 = F_1 + \psi_1, \quad (F_\alpha)^2 = F_2 + \psi_2 \tag{3.34}$$

Since the monodromy group acts by permuting the coordinates of the cones, it follows that, for the map $i_1 : \sigma_1 \to \Sigma(\mathscr{X}, \mathscr{U})$ (similar for i_2), and for any $t \in \mathbb{R}_{\geq 0}$ and $v \in \sigma_1$, we have that

$$i(t \cdot v) = t \cdot i(v). \tag{3.35}$$

Now, assume without loss of generality, that the elements of the sequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ lie in $\mathtt{W}$. Since the maps ϕ_1, ϕ_2 are open, and V_1, V_2 first-countable, it follows by by [Siw71][Proposition 2.4] that there exist points $w \in \phi_1^{-1}(z_0), v \in \phi_2^{-1}(z_0)$ and sequences $\{w_k\}_{k\mathbb{N}}, \{v_k\}_{k\in\mathbb{N}}$ with $w_k \in \phi_1^{-1}(z_{n_k}), v_k \in \phi_2^{-1}(z_{n_k})$ with

$$\lim_{k\to\infty} w_k = w, \quad \lim_{k\to\infty} v_k = v.$$

Then we have:

$$\begin{aligned} y &= \lim_{k\to\infty} t_{n_k} \cdot \mathtt{F}_1(z_{n_k}) \\ &= \lim_{k\to\infty} [t_{n_k} \cdot i_1(F_1(w_k))] \quad \text{(commutativity of (3.33))} \\ &= \lim_{k\to\infty} [i_1(t_{n_k} \cdot F_1(w_k))] \quad \text{(by (3.35))} \\ &= i_1(\lim_{k\to\infty} t_{n_k} \cdot F_1(w_k)) \quad \text{(continutiy of } i_1) \\ &= i_1(\lim_{k\to\infty} t_{n_k} \cdot [F_1(w_k) + \psi_1(w_k)]) \quad (\phi_1 \text{ is bounded}) \\ &= i_1(\lim_{k\to\infty} t_{n_k} \cdot (F_\alpha)^1(w_k)) \quad \text{(by 3.34)} \\ &= \lim_{k\to\infty} i_1(t_{n_k} \cdot (F_\alpha)^1(w_k)) \quad \text{(continutiy of } i_1) \\ &= \lim_{k\to\infty} t_{n_k} \cdot i_1((F_\alpha)^1(w_k)) \quad \text{(by (3.35))} \end{aligned}$$

Since $(F_\alpha)^1$ is λ_1-equiavriant, then denoting by $\mathtt{F}^1_\alpha$ the induced map in the quotient, we have

$$i_1(\cdot(F_\alpha)^1(w_k)) = \mathtt{F}^1_\alpha(z_{n_k}), \quad k \in \mathbb{N}.$$

Moreover, letting $\mathtt{F}_\alpha$ denote the function on the quotient associated to F_α, by Remark 3.3.3, the values of $\mathtt{F}_\alpha$ and $\mathtt{F}^1_\alpha$ coincide in the quotient. Therefore,

$$\lim_{k\to\infty} t_{n_k} \cdot \mathtt{F}_1(z_{n_k}) = \lim_{k\to\infty} t_{n_k} \cdot \mathtt{F}^1_\alpha(z_{n_k}) = \lim_{k\to\infty} t_{n_k} \cdot \mathtt{F}_\alpha(z_{n_k})$$

Applying the same reasoning for $\mathtt{F}_2$ and the data associated to (V_2, G_2, ϕ_2), we obtain an analogous description for $\mathtt{F}_2$, hence

$$\lim_{k\to\infty} t_{n_k} \cdot \mathtt{F}_2(z_{n_k}) = y$$

Since this convergence is independent of the subsequence, we have that

$$\lim_{n\to\infty} t_n \cdot \mathtt{F}_1(z_n) = y$$

Then we conclude that $\{(z_n, t_n)\}_{n\in\mathbb{N}}$ converges to $(y, 0)$ with respect to the topology τ'. This conclusion follows from the criterion that a sequence $\{(z_n, t_n)\}_{n\in\mathbb{N}}$ converges to a point (y, s) in $\mathscr{X}^{hyb}$ if only if

$$\lim_{n\to\infty} \pi(z_n, t_n) = \pi(y, s), \quad \text{and} \quad \lim_{n\to\infty} \mathtt{G}(z_n, t_n) = \mathtt{G}(y, s)$$

Since the reverse implication follows by the same reasoning, and both $(\mathscr{X}^{hyb}, \tau)$ and $(\mathscr{X}, \tau')$ are sequentials, it follows that $\tau = \tau'$.

Therefore, the proposition is proved. □

Proposition 3.5.6. The function $\mathtt{G}$ is proper.

Proof. The proof is analogous to that of Proposition 2.2.11, as it relies purely on topological arguments. The crucial observation is that both X^{an} and $\Sigma(\mathscr{X}, \mathscr{U})$ are sequential spaces,which is the relevant property that enables this analogy. □

3.6 Measures on the hybrid space

In this section, we focus on a toroidal Deligne–Mumford stack $(\mathscr{X}, \mathscr{U})$ that is smooth and proper over $\mathbb{C}$, whose minimal strata in the stratification of X^{an}, which is the one given by the toroidal embedding, are points. In particular, these strata are minimal strata in the stratification that refine both the stratification by stabilizers and that induced by the toroidal embedding.

In Section 2.3, we established a continuity theorem for measures on a hybrid space associated with a complex manifold and a simple normal crossing divisor.

The goal of this section is to construct continuous families of measures on the hybrid space $\mathscr{X}^{hyb}$, defined in the previous section, that generalize the specific case of Section 2.3, corresponding to complex manifolds whose minimal strata, determining the simple normal crossing divisor, are points.

We will start with a brief introduction on measures, and differential forms on étale proper Lie groupoids. This introduction is taken from [ST24] and [Beh04].

If M is a smooth manifold, then $\mathcal{A}^q$ denotes the sheaf of smooth differential q-forms on M, and $\mathcal{A}^q(M)$ denotes the $\mathcal{C}^\infty(M)$-module of global sections of $\mathcal{A}^q$.

Definition 3.6.1. Let P be an étale proper groupoid in the category of smooth manifolds. The space of differential forms on P is defined as

$$\Omega^q(P) := \ker(s^* - t^* : \Omega^q(P_0) \to \Omega^q(P_1)).$$

Definition 3.6.2.
1. A subset $C \subset P_0$ is called *clean* if the restriction $t_{|s^{-1}(C)}$ is proper. Equivalently, if the restriction $s_{|t^{-1}(C)}$ is proper.
2. A subset $C \subset P_1$ is called *clean* if it is closed and both $s(C) \subset P_0$ and $t(C) \subset P_0$ are clean.

Example 3.6.3. Let $G := \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{R}^2$ by permuting the coordinates. Consider the subset $C := \{(x, y) \in \mathbb{R}^2 : \ x = y\}$. This set is clean, since the restriction of both the projection and the action of G on C is the identity, which is clearly proper.

Remark 3.6.4. As a consequence of Definition 3.6.2, the following properties hold:

1. Let $C \subset P_1$ be a clean subset. Then the restrictions $s|_C$ and $t|_C$ are proper maps.
2. A closed subset of a clean subset $C \subset P_i$ is clean, for $i = 0, 1$.
3. A finite union of clean subsets of P_i is clean, for $i = 0, 1$.

The next lemma establishes an equivalent characterization for clean subset of a groupoid.

Lemma 3.6.5. Let $\pi : P_0 \to |P|$ be the natural map from the object space to the underlying topological space of the orbifold. A subset $C \subset P_0$ is clean if and only if the restriction $\pi_{|C} : C \to |P|$ is a proper map.

Proof. see [ST24][Lemma 2.26]. □

Remark 3.6.6. As a consequence of the Lemma 3.6.5, if the groupoid is compact; that is, if its underlying topological space is compact, then a subset is clean if and only if it is compact.

Let $\mathcal{A}^q_{\mathrm{cl}}(P_0)$, $\mathcal{A}^q_{\mathrm{cl}}(P_1)$ denote the locally convex subspaces of $\mathcal{A}^q(P_0)$, $\mathcal{A}^q(P_1)$, consisting of differential q-forms with clean support on P_0 and P_1, respectively.

Since the restrictions of the source and target maps $s, t : P_1 \to P_0$ to a clean subset of P_1 are proper, we may define the pushforward of differential forms with clean support via integration along the fibers.

Based on the observation above, we define the space of differential forms with clean support on P by

$$\mathcal{A}^q_{\mathrm{cl}}(P) := \operatorname{coker}\left(s_* - t_* : \mathcal{A}^q_{\mathrm{cl}}(P_1) \to \mathcal{A}^q_{\mathrm{cl}}(P_0)\right).$$

Concerning differential forms with clean support, it is shown in [ST24] that for each class $[\alpha] \in \mathcal{A}^q_{\mathrm{cl}}(P_0)$ the expression $s_* t^* \alpha$ lies in $\mathcal{A}^q(P_0)$, and moreover, it vanishes if α lies in the image of $s_* - t_*$. This observation allows us to define the map

$$J : \mathcal{A}^q_{\mathrm{cl}}(P) \to \mathcal{A}^q(P), \quad J([\alpha]) := t_* s^* \alpha$$

Definition 3.6.7. Let $s, t : P_1 \rightrightarrows P_0$ be a groupoid. *A partition of unity* is a smooth function $\rho : P_0 \to [0, 1]$ of clean support such that

$$t_* s^* \rho = 1$$

Example 3.6.8. If M is a smooth manifold and $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover by charts, then the groupoid defined by $P_0 := \coprod_{\alpha\in\Lambda} U_\alpha$ and $P_1 = \coprod_{\alpha,\beta} U_\alpha \cap U_\beta$ together with the natural source and target projections, forms an orbifold presenting M. A partition of unity in the sense of Definition 3.6.7 induces a classical partition of unity subordinate to the covering $\{U_\alpha\}_{\alpha\in\Lambda}$.

If the groupoid is proper and étale, then a partition of unity always exists; see [ST24, Lemma 4.12]. In this context, given a partition of unity ρ, one defines a continuous linear map

$$K : \mathcal{A}^q(P) \to \mathcal{A}^q_{\mathrm{cl}}(P), \quad K(\alpha) := [\rho\alpha]. \tag{3.36}$$

According to [ST24], the maps J and K are inverses of each other, and K does not depend on the choice of the partition ρ.

Remark 3.6.9. If we are only interested in integration theory, we may use the treatment of [Dem97, Chapter 1, Section 2.c] for locally integrable n-forms in place of smooth forms. In this setting, we can define the functions J and K can be defined in the space of locally integrable forms. Moreover, by an analogous argument, the isomorphism (3.36) continues to hold for locally integrable n-forms.

Let $F : P \to Q$ be a morphism of groupoids such that F is a proper submersion; that is, both maps $F_1 : P_1 \to Q_1$ and $F_0 : P_0 \to Q_0$ are proper submersions. It is shown in [ST24] that the map $F_0 : \mathcal{A}^q_{\mathrm{cl}}(P_0) \to \mathcal{A}^q_{\mathrm{cl}}(Q_0)$ sends elements in the image of $s_* - t_*$ to elements in the image of $r_* - u_*$, where, $r, u : Q_1 \to Q_0$ denote the source and target maps of the groupoid Q, respectively. This fact allows us to define a well-defined map

$$F_* : \mathcal{A}^q_{\mathrm{cl}}(P) \to \mathcal{A}^q_{\mathrm{cl}}(Q), \quad F_*([\alpha]) := [(F_0)_*\alpha].$$

As a consequence, we obtain a map

$$\mathcal{F}_* : \mathcal{A}^q(P) \longrightarrow \mathcal{A}^q(Q)$$

defined by the composition $\mathcal{F}_* := J \circ F_* \circ K$.

Remark 3.6.10. In the case of smooth manifolds, given a map $f : M \to *$, from an oriented manifold M to a point, one defines a pushforward operation on compactly supported differential forms. For each $\alpha \in \Omega^q(M)$ of compact support, the pushforward $f_*\alpha$ is defined as:

$$f_*\alpha = \begin{cases} \int_M \alpha & \text{if} \quad q = \dim M \\ \\ 0 & \text{if} \quad q \neq \dim M \end{cases} \tag{3.37}$$

Definition 3.6.11. An *orientation* of an étale proper groupoid P consists of orientations on P_0 and P_1 such that the étale maps $s, t : P_0 \to P_1$ are orientation preserving.

Taking into account Remark 3.6.10, let P be an orientable proper étale groupoid, and let pt denote the proper étale groupoid associated with a single point, that is, the groupoid whose space of objects and arrows are both a single point. Consider the unique smooth functor $F : P \to \mathrm{pt}$. If $\xi \in \mathcal{A}^q(P)$ is such that the image of its support under the projection $\pi : P_0 \to |P|$ is compact, then the integral of ξ over P is defined as:

$$\int_P \xi := \mathcal{F}_*\xi.$$

In order to describe $\mathcal{F}_*$, let $\rho : P_0 \to [0,1]$ be a partition of unity on P. Using the fact that $J_{\mathrm{pt}} = \mathrm{Id}$, we obtain for each $\xi \in \mathcal{A}^n(P)$, where $n = \dim P$

$$\int_P \xi = \mathcal{F}_*\xi = (F_0)_*(K\xi) = (F_0)_*(\rho\xi) = \int_{P_0} \rho\xi.$$

In this way, if $f : |P| \to \mathbb{R}$ is a continuous function we have

$$\int_{|P|} f \cdot \xi := \int_{P_0} f \circ \pi \cdot \rho\xi \tag{3.38}$$

Having defined the integration theory for orbifolds, Our aim is to develop an integration theory on the analytic stack $\mathscr{X}^{an}$, where $(\mathscr{X}, \mathscr{U})$ is a toroidal Deligne–Mumford stack whose minimal strata, in the stratification induced by the toroidal structure, consist of isolated points.

To this end, note that the preceding analysis involving real-valued differential forms extends naturally to complex differential forms $\Omega^{p,q}(\mathscr{X}^{an})$, for $p, q \in \mathbb{N}$. This construction is independent of the choice of étale presentation: any two such presentations are Morita equivalent, and hence the corresponding complexes of differential forms are canonically isomorphic [Beh04, Cor. 3].

Therefore, it remains to show that the integration itself does not depend on the choice of an étale presentation of $\mathscr{X}^{an}$. Although this fact is well known, we include a proof for completeness.

Proposition 3.6.12. Let $(\mathscr{X}, \mathscr{U})$ be a regular proper toroidal Deligne–Mumford stack, and let $\mathscr{X}^{an}$ denote the analytification of $\mathscr{X}$. Then the integral defined in (3.38) does not depend on the choice of étale presentation of $\mathscr{X}^{an}$.

Proof. Let $P_0 \to \mathscr{X}^{an}$, and $Q_0 \to \mathscr{X}^{an}$ be two étale presentations, and let $\xi \in \Omega^{n,n}(P_0)$ be a form representing an element of $\Omega^{n,n}(\mathscr{X}^{an})$, that is, $s^*\xi = t^*\xi$, for $s, t : P_1 \to P_0$, where $P_1 = P_0 \times_{\mathscr{X}^{an}} P_0$. Set similarly $Q_1 := Q_0 \times_{\mathscr{X}^{an}} Q_0$, and consider the following diagram:

$$\begin{array}{ccccc}
 & & P_0 \times_{\mathscr{X}^{an}} Q_1 & \xrightarrow{\phi'} & Q_1 \\
 & & \Big\downarrow{\scriptstyle u'} \Big\downarrow{\scriptstyle r'} & & \Big\downarrow{\scriptstyle u} \Big\downarrow{\scriptstyle r} \\
P_1 \times_{\mathscr{X}^{an}} Q_0 & \underset{t'}{\overset{s'}{\rightrightarrows}} & P_0 \times_{\mathscr{X}^{an}} Q_0 & \xrightarrow{\phi} & Q_0 \\
\Big\downarrow{\scriptstyle \varphi'} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow \\
P_1 & \underset{t}{\overset{s}{\rightrightarrows}} & P_0 & \longrightarrow & \mathscr{X}^{an}
\end{array} \tag{3.39}$$

Let $\rho_0 : P_0 \to \mathbb{R}$ be a partition of unity for the orbifold groupoid $P_1 \rightrightarrows P_0$. Then $\rho_0\xi \in \Omega^{n,n}_{cl}(P_0)$. Hence, $\varphi^*\rho_0\xi \in \Omega^{n,n}_{cl}(P_0 \times_{\mathscr{X}^{an}} Q_0)$ with respect to the groupoid

$$P_1 \times_{\mathscr{X}^{an}} Q_0 \rightrightarrows P_0 \times_{\mathscr{X}^{an}} Q_0.$$

To explain this, let $C := \mathrm{Supp}\ \rho_0\xi$. Since the map $t_{|t^{-1}(C)} : t^{-1}(C) \to C$ is proper, this follows that $\varphi^{-1}(C) = \mathrm{Supp}\ \varphi^*\rho_0\xi$, and that $t'_{|t'^{-1}(\varphi^{-1}(C))} : t'^{-1}(\varphi^{-1}(C)) \to \varphi^{-1}(C)$ is proper. This implies that $\varphi^*\rho_0\xi \in \Omega^{n,n}_{cl}(P_0 \times_{\mathscr{X}^{an}} Q_0)$.

As $\phi_{|\varphi^{-1}(C)}$ is proper, we may therefore define

$$\phi_*\varphi^*\rho_0\xi \in \Omega^{n,n}(Q_0).$$

We claim that this form descends to an element of $\Omega^{n,n}(\mathscr{X}^{an})$. In other words, we must show $u^*\phi_*\varphi^*\rho_0\xi = r^*\phi_*\varphi^*\rho_0\xi$. For this we recall the following standard fact. Consider a fiber product

$$\begin{array}{ccc} M \times_N P & \xrightarrow{p} & P \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ M & \xrightarrow{q} & N \end{array}$$

in the category of smooth manifolds. If f is a proper submersion and $\alpha \in \Omega^k(P)$, then $q^*f_*\alpha = g_*p^*\alpha$.

Applying this fact to the two squares in diagram (3.39), we obtain

$$u^*\phi_*\varphi^*\rho_0\xi = \phi'_*u'^*\varphi^*\rho_0\xi, \quad r^*\phi_*\varphi^*\rho_0\xi = \phi'_*r'^*\varphi^*\rho_0\xi.$$

But since $P_0 \times_{\mathscr{X}^{an}} Q_1 \rightrightarrows P_0 \times_{\mathscr{X}^{an}} Q_0$ is a groupoid, we have $u'^*\varphi^*\rho_0\xi = r'^*\varphi^*\rho_0\xi$. This proves the claim.

Finally, let $f : X^{an} \to \mathbb{R}$ be continuous and let $\xi \in \Omega^{n,n}(\mathscr{X}^{an})$. Using the two presentations, we obtain two expressions for the integral:

$$\int_{X^{an}} f\xi := \int_{P_0} \pi \circ f \cdot \rho_0\xi, \quad \int_{X^{an}} f\xi := \int_{Q_0} \pi' \circ f \cdot \rho_1\phi_*\varphi^*\rho_0\xi$$

where $\rho_1 : Q_0 \to \mathbb{R}$ is a partition of unity and $\pi : P_0 \to X^{an}$, $\pi' : Q_0 \to X^{an}$ are the natural projections. We now verify that these two integrals agree.

On the one hand, by the definition of pushforward we have

$$\int_{Q_0} \pi' \circ f \cdot \rho_1\phi_*\varphi^*\rho_0\xi = \int_{P_0 \times_{\mathscr{X}^{an}} Q_0} \phi \circ \pi' \circ f \cdot \phi \circ \rho_1 \cdot \varphi^*\rho_0\xi.$$

Note that $\phi \circ \rho_1 : P_0 \times_{\mathscr{X}^{an}} Q_0 \to \mathbb{R}$ is a partition of unity for the groupoid $P_1 \times_{\mathscr{X}^{an}} Q_0 \rightrightarrows P_0 \times_{\mathscr{X}^{an}} Q_0$.

Also observe that

$$\int_{P_0 \times_{\mathscr{X}^{an}} Q_0} \phi \circ \pi' \circ f \cdot \phi \circ \rho_1 \cdot \varphi^*\rho_0\xi = \int_{P_0 \times_{\mathscr{X}^{an}} Q_0} \varphi \circ \pi \circ f \cdot \phi \circ \rho_1 \cdot \varphi^*\rho_0\xi,$$

since $\phi \circ \pi' = \varphi \circ \pi$ on the fiber product.

On the other hand, since $P_1 \times_{\mathscr{X}^{an}} Q_0 \rightrightarrows P_0 \times_{\mathscr{X}^{an}} Q_0$ is an orbifold, we have

$$\int_{P_0} \pi \circ f \cdot \rho_0\xi = \int_{P_0 \times_{\mathscr{X}^{an}} Q_0} \varphi \circ \pi \circ f \cdot \phi \circ \rho_1 \cdot \varphi^*\rho_0\xi$$

Combining the previous equalities, we obtain equality of the two representations of the integral on X^{an}. This completes the proof that the integral does not depend on the choice of étale presentation.

□

Definition 3.6.13. Let $(\mathscr{X}, \mathscr{U})$ be a toroidal DM stack regular, and let n be the dimension of $\mathscr{X}^{an}$. Let $P_0 \to \mathscr{X}^{an}$ be an étale presentation such that $P_0 \times_{\mathscr{X}^{an}} \mathscr{U}^{an}$ is a simple normal crossing divisor. A $\mathcal{P}$-logarithmic (n,n)-form on $\mathscr{X}^{an}$ is a $\mathcal{P}$-logarithmic form ω on P_0 satisfying $s^*\omega = t^*\omega$.

Proposition 3.6.14. Let $(\mathscr{X}, \mathscr{U})$ be a proper and regular toroidal Deligne–Mumford stack. Then there exist $\mathcal{P}$-logarithmic forms on $\mathscr{X}^{an}$.

Proof. Assume that n is the dimension of X^{an}, and consider $P_0 \to \mathscr{X}^{an}$ an étale presentation, such that $P_0 \smallsetminus (P_0 \times_{\mathscr{X}^{an}} \mathscr{U}^{an})$ is a simple normal crossing divisor. Consider a good covering chart $\{(V_\alpha, G_\alpha, \phi_\alpha)\}_{\alpha\in\Lambda}$ of X^{an}, see Definition 3.2.26, where each V_α is contained in P_0. Since X^{an} is compact, we may assume that this covering is finite.

We then define the associated groupoid by:

$$Q_1 := \coprod_{\alpha,\beta} V_\alpha \times_{\mathscr{X}^{an}} V_\beta, \quad Q_0 := \coprod_{\alpha\in\Lambda} V_\alpha. \tag{3.40}$$

This groupoid $Q_1 \rightrightarrows Q_0$ is étale and proper, and presents the analytic stack $\mathscr{X}^{an}$.

For each $\alpha \in \Lambda$, let U_α denote the dense open subset defined by the fiber product $V_\alpha \times_{\mathscr{X}^{an}} \mathscr{U}^{an}$, and by $j_\alpha : U_\alpha \to V_\alpha$ the inclusion map. Consider also an (n,n)-form $\omega_\alpha \in \Gamma(j_{\alpha*}\Omega^{n,n}, V_\alpha)$, whose support $\mathrm{Supp}(\omega_\alpha) \subseteq V_\alpha$ is compact. Among all such forms, we focus on those satisfying the following condition in the charts that contains the minimal strata of the toroidal stratification:

Let $\alpha \in \Lambda$ be an index such that the image of V_α under ϕ_α contains a minimal stratum with respect to the toroidal stratification. Since the image of the strata of V_α under the map ϕ_α are contained in the strata of X^{an}, see Remark 3.2.4, and V_α has a unique minimal stratum, we conclude that this stratum is a point. In this situation, we are in the case of maximal degeneration: if the dimension of V_α is n, then the divisor $V_\alpha \smallsetminus V_\alpha \times_{\mathscr{X}^{an}} \mathscr{U}^{an}$ has n irreducible components, whose intersection is single point.

Therefore, for this α we require that ω_α is a $\mathcal{P}$-logarithmic form, see Definition 2.3.1). That is, suppose that $z = (z_{\alpha_1}, \ldots, z_{\alpha_n})$ are local coordinates on V_α, and let W_α denote the unique minimal stratum of V_α (which, in this case, is a point), and let $P_\alpha \in \mathrm{Sym}^r(W_\alpha)$, then ω_α can be written as

$$\omega_\alpha = \frac{f_\alpha dz_{\alpha_1} \wedge d\overline{z}_{\alpha_1} \wedge \ldots \wedge d\overline{z}_{\alpha_n}}{P_\alpha((-\log|z_{\alpha_j}|)_{j=1}^n) \prod_{j=1}^n z_{\alpha_j}\overline{z}_{\alpha_j}}$$

where f_α is a continuous function defined on V_α.

With this construction, we have that $\{\omega_\alpha\}_{\alpha\in\Lambda}$ is a $\mathcal{P}$-logarithmic form in P_0. For simplicity, we denote this form by ω. Consequently, $s_*t^*\omega \in \Omega^{n,n}(\mathscr{X}^{an})$. Moreover, this is a $\mathcal{P}$-logarithmic form, since s and t are étale maps, that is, local isomorphisms, and therefore preserve the local structure of ω. □

We now proceed to define a family of measures on the hybrid space $\mathscr{X}^{hyb}$. Fix an étale presentation $P_0 \to \mathscr{X}^{an}$ such that the divisor $D_0 := P_0 \smallsetminus P_0 \times_{\mathscr{X}^{an}} \mathscr{U}^{an}$ is a simple normal crossing divisor. Let ω a $\mathcal{P}$-logarithmic form on $\mathscr{X}^{an}$, with homogeneous polynomials of degree r, and let $q : \Sigma(P_0, D_0) \to \Sigma(\mathscr{X}, \mathscr{U})$ be the induced map between the associated polyhedral complexes.

Using these data, we define a family of measures $\{\mu_t\}_{0\le t\le 1}$ in $\mathscr{X}^{hyb}$, where each μ_t is supported on the

fiber $\mathscr{X}_t^{hyb} := \pi^{-1}(t)$, with $\pi : \mathscr{X}^{hyb} \to [0,1]$ the natural projection. The measures are defined by

$$\mu_t = \begin{cases} \frac{t^{n-r}}{(4\pi i)^n}\omega & t \neq 0 \\ \sum_{Y_i \in \Sigma_n} \left(\int_{Y_i} \mathrm{Res}_{Y_i}(\omega) \right) d\left(q_* \left(\frac{\lambda_i}{P_i} \right) \right) & t = 0 \end{cases} \tag{3.41}$$

where, Σ_n denotes the set of indices associated with the maximal cones of $\Sigma(P_0, D_0)$, λ_i the Lebesgue measure in the maxinal cone σ_i, and P_i i s the homogeneous polynomial of degree r associated with the strata Y_i. Recall that these minimal strata are points, since we are in the case of maximal degeneration.

Lemma 3.6.15. Let $\{\mu_t\}_{0 \leq t \leq 1}$ be the family defined in (3.41). Then for any function $\mathtt{G}$ constructed as in (3.32), the following equivalence holds in the sense of vague convergence of measures:

$$\lim_{t \to 0} \mu_t = \mu_0 \iff \lim_{t \to 0} \mathtt{G}_* \mu_t = \mu_0$$

Proof. Recall that $\Sigma(\mathscr{X}, \mathscr{U})$ is locally compact, as it is contained in the compact space $\overline{\Sigma}(\mathscr{X}, \mathscr{U})$. Furthermore, Proposition 3.5.4 establishes that $\Sigma(\mathscr{X}, \mathscr{U})$ is second countable. It follows that $\Sigma(\mathscr{X}, \mathscr{U})$ is σ-compact. Since the map $\mathtt{G}$ is proper, the proof can be carried out in the same way as in Lemma 2.3.9. □

Let $(V_\alpha, G_\alpha, \phi_\alpha)$ an orbifold chart, with $V_\alpha \subseteq P_0$, and let $z = (z_1, \ldots, z_n)$ be local coordinates on V_α. Define the set

$$\mathscr{V}_\alpha := (\mathtt{V}_\alpha \cap U^{an}) \times (0,1]$$

Where $\mathtt{V}_\alpha$ denotes the image of V_α under the map ϕ_α.

Assume that the unique minimal stratum of V_α has codimension k. We may choose the coordinates $z_1, \ldots, z_k$ so that the toroidal divisor D of V_α, determined by the toroidal structure, is defined by the zero locus of the product $\prod_{j=1}^k z_j$. In this case, the minimal stratum is given by the common zero locus of $z_1, \ldots, z_k$.

Consider the set $(V_\alpha \times_{\mathscr{X}^{an}} \mathscr{U}^{an}) \times (0,1]$ as introduced in Section 2.3. Define the map from this set to the cone σ_α, which sends a point (z,t) to the tuple $(-t \log |z_j|)_{j=1}^k$, which is the map denoted $\log_{V_\alpha}$ in section 2.3.

As in Section 3.3, consider the group homomorphisms $\lambda_\alpha : G_\alpha \to \Gamma_\alpha$, where Γ_α is the group action on the cone σ_α. Using this, we define a λ_α-equivariant map

$$F_\alpha : (V_\alpha \smallsetminus V_\alpha \times_{\mathscr{X}^{an}} \mathscr{U}^{an}) \times (0,1] \to \sigma, \quad (z,t) \mapsto \frac{1}{|G_\alpha|} \sum_{g \in G_\alpha} \lambda_\alpha(g) \cdot \log_{V_\alpha}(g^{-1}z, t)$$

This map naturally descends to a well-defined function

$$\log_{\mathscr{V}_\alpha} : \mathscr{V}_\alpha \to \sigma_\alpha / \Gamma_\alpha, \quad (\overline{z}, t) \mapsto \overline{F_\alpha(z,t)} \tag{3.42}$$

Here, $\overline{z}$ denotes the class of the element $z \in V_\alpha$ in $\mathtt{V}_\alpha$, and the right-hand side is the image of $F_\alpha(z,t)$ in the quotient $\sigma_\alpha / \Gamma_\alpha$.

The following Lemma is a generalization of Proposition 2.3.6.

Lemma 3.6.16. Let $(V_\alpha, G_\alpha, \phi_\alpha)$ be a chart as above, and suppose that its minimal stratum $Y_\alpha \subseteq V_\alpha$ has codimension k. Let $\chi \in C_c(\mathtt{V}_\alpha, \mathbb{R})$, where $\mathtt{V}_\alpha$ is the image of V_α under ϕ_α. Then the family of measures $\{(\log_{\mathscr{V}_\alpha})_*(\chi \cdot \frac{t^{n-r}}{(4\pi i)^n} \cdot \omega)\}_{0<t\leq 1}$ converges vaguely, as $t \to 0$, to the measure $\left(\int_{Y_\alpha} \chi \circ \phi_\alpha \cdot \mathrm{Res}_{Y_\alpha}(\omega)\right) \cdot d\left(q_*\left(\frac{\lambda_\alpha}{P_\alpha}\right)\right)$ in the interior of $\sigma_\alpha/\Gamma_\alpha$ if the image of Y_α under ϕ_α is contained in a minimal stratum of X^{an} with respect to the toroidal stratification, which in this case is a point. Otherwise, the limit measure is zero.

Proof. In analogy to Proposition 2.3.6, we aim to prove, in the case associated with the minimal strata, that for each $\varphi \in C_c(\mathring{\sigma}_\alpha/\Gamma_\alpha, \mathbb{R})$, the following holds:

$$\lim_{t\to 0} \int_{\mathtt{V}_\alpha \cap U^{an}} (\varphi \circ \log_{\mathscr{V}_\alpha}) \chi \omega = \left(\int_{Y_\alpha} \chi \circ \phi_\alpha \cdot \mathrm{Res}_{Y_\alpha}(\omega)\right) \cdot \int_{\sigma_\alpha/\Gamma_\alpha} \varphi d\left(q_*\left(\frac{\lambda_\alpha}{P_\alpha}\right)\right) \tag{3.43}$$

Indeed, the left-hand side of (3.43) can be rewritten as

$$\begin{aligned}\lim_{t\to 0} \int_{\mathtt{V}_\alpha \cap U^{an}} (\varphi \circ \log_{\mathscr{V}_\alpha}) \cdot \chi \cdot \omega &= \lim_{t\to 0} \int_{\mathtt{V}_\alpha \cap U^{an}} \frac{t^{n-r}}{(4\pi i)^n} (\varphi \circ \log_{\mathscr{V}_\alpha}) \cdot \chi \cdot \omega \\ &= \lim_{t\to 0} \int_{U_\alpha} \frac{t^{n-r}}{(4\pi i)^n} (\varphi \circ \log_{\mathscr{V}_\alpha} \circ \phi_\alpha) \cdot \chi \circ \phi_\alpha \cdot \omega\end{aligned}$$

where by simplicity, we have denoted by U_α to the set $V_\alpha \smallsetminus V_\alpha \times_{\mathscr{X}^{an}} \mathscr{U}^{an}$. Now, using the commutativity of the diagram

$$\begin{array}{ccc} U_\alpha \times (0,1] & \xrightarrow{\phi_\alpha \times Id} & (\mathtt{V}_\alpha \cap U^{an}) \times (0,1] \\ \downarrow{\scriptstyle \log_{V_\alpha}} & & \downarrow{\scriptstyle \log_{\mathscr{V}_\alpha}} \\ \sigma & \xrightarrow{\quad q \quad} & \sigma_\alpha/\Gamma_\alpha \end{array}$$

we obtain

$$\lim_{t\to 0} \int_{U_\alpha} \frac{t^{n-r}}{(4\pi i)^n} (\varphi \circ \log_{\mathscr{V}_\alpha} \circ \phi_\alpha) \cdot \chi \circ \phi_\alpha \cdot \omega = \lim_{t\to 0} \int_{U_\alpha} \frac{t^{n-r}}{(4\pi i)^n} (\varphi \circ \log_{V_\alpha} \circ q) \cdot \chi \circ \phi_\alpha \cdot \omega$$

By Proposition 2.3.6, we have

$$\lim_{t\to 0} \int_{U_\alpha} \frac{t^{n-r}}{(4\pi i)^n} (\varphi \circ q \circ \log_{V_\alpha}) \cdot \chi \circ \phi_\alpha \cdot \omega = \left(\int_{Y_\alpha} \chi \circ \phi_\alpha \cdot \mathrm{Res}_{Y_\alpha}(\omega)\right) \int_{\sigma_\alpha} (\varphi \circ q) \frac{d\lambda_\alpha}{P_\alpha} \tag{3.44}$$

Finally, applying the pushforward of the measure, we obtain

$$\left(\int_{Y_\alpha} \chi \circ \phi_\alpha \cdot \mathrm{Res}_{Y_\alpha}(\omega)\right) \int_{\sigma_\alpha} (\varphi \circ q) \frac{d\lambda_\alpha}{P_\alpha} = \left(\int_{Y_\alpha} \chi \circ \phi_\alpha \cdot \mathrm{Res}_{Y_\alpha}(\omega)\right) \int_{\sigma_\alpha/\Gamma_\alpha} \varphi d\left(q_*\left(\frac{d\lambda_\alpha}{P_\alpha}\right)\right).$$

If the stratum of associated to the chart $(V_\alpha, G_\alpha, \phi_\alpha)$ is not minimal, then performing the same steps as above yields the same representation as in (3.44). However, by Proposition 2.3.11, the right-hand side of that equation vanishes. Hence, the limit is zero in this case as well. This completes the proof. □

Let $\Sigma(\mathscr{X}, \mathscr{U})^{\leq n-1} := \bigcup_{i\in I_{n-1}} \sigma_i/\Gamma_i$ denote the subset of $\Sigma(\mathscr{X}, \mathscr{U})$ formed by the union of all those quotients that are not maximal.

We then define the following open subset:

$$\mathscr{X}_0^{hyb} := \mathscr{X}^{hyb} \smallsetminus \Sigma(\mathscr{X}, \mathscr{U})^{\leq n-1}$$

Having introduced this set, we may now state one of the main results of this chapter, describing the degeneration behavior of the family of measures on $\mathscr{X}^{hyb}$ associated to the $\mathcal{P}$-logarithmic forms defined in (3.41)

Theorem 3.6.17. *Let $\{\mu_t\}_{0\leq t\leq 1}$ be a family of measures as in* (3.41)*. Then for each continuous function $f \in C_c(\mathscr{X}_0^{hyb}, \mathbb{R})$ we have*

$$\lim_{t\to 0} \int_{\mathscr{X}_0^{hyb}} f d\mu_t = \int_{\mathscr{X}_0^{hyb}} f d\mu_0$$

Proof. By Lemma 3.6.15, it suffices to establish the convergence

$$\lim_{t\to 0} \int_{\mathscr{X}_0^{hyb}} f d(G_* \mu_t) = \int_{\mathscr{X}_0^{hyb}} f d\mu_0.$$

The proof then follows the same strategy as in Theorem 2.3.12, which corresponds to a particular case of the present statement.

Specifically, let $P_0 \to \mathscr{X}^{an}$ be an etale presentation, and choose a good covering chart $\{(V_\alpha, G_\alpha, \phi_\alpha)\}_{\alpha\in\Lambda}$, where each $V_\alpha \subseteq P_0$. Since X^{an} is compact, we can assume that the number of charts is finite. Recall that ϕ_α is simply the restriction of $P_0 \to X^{an}$ to V_α. Take a standard partition of unity $\{\chi_\alpha\}_{\alpha\in\Lambda}$ subordinate to the cover $\{\mathtt{V}_\alpha\}_{\alpha\in\Lambda}$. Consider $Q_0 = \coprod_{\alpha\in\Lambda} V_\alpha$, then $\{\chi_\alpha \circ \phi_\alpha \cdot \omega\}_{\alpha\in\Lambda}$ is a $\mathcal{P}$-logarithmic form of compact support on Q_0. Hence,

$$\int_{\mathscr{X}_0^{hyb}} f d(\mathtt{G}_* \mu_t) = \int_{Q_0} f \circ \phi \cdot \rho \cdot \frac{t^{n-r}\omega}{(4\pi i)^n},$$

where $\rho = \{\chi_\alpha \circ \phi_\alpha\}_{\alpha\in\Lambda}$ and $\phi : Q_0 \to X^{an}$ is such that $\phi_{|V_\alpha} = \phi_\alpha$.

The desired convergence then follows by applying Lemma 3.6.16 to each chart and summing over $\alpha \in \Lambda$. □

As a final remark, we note that the constructions and results of this section, and of the previous one, apply directly to the toroidal Deligne–Mumford stack $(\overline{\mathscr{M}_{g,n}}\mathscr{M}_{g,n})$. In this case, the associated skeleton is precisely the tropical moduli space $M_{g,n}^{\mathrm{trop}}$, and the resulting hybrid space takes the form

$$\overline{\mathscr{M}}_{g,n}^{hyb} = M_{g,n} \times (0,1] \coprod M_{g,n}^{\mathrm{trop}} \times \{0\}$$

Consequently, our main theorem yields a degeneration result for the family of measures associated to $\mathcal{P}$-logarithmic forms, describing their convergence on the interiors of maximal cells of $M_{g,n}^{trop}$.

Conclusions

The main goal of this thesis has been the development of a hybrid space $\mathscr{X}^{hyb}$ associated with a toroidal Deligne-Mumford stack $(\mathscr{X}, \mathscr{U})$. In the particular case of the moduli stack of curves $(\overline{\mathscr{M}_{g,n}}, \mathscr{M}_{g,n})$, this construction provides an initial framework for understanding the relationship between amplitudes in quantum field theory and amplitudes in string theory.

To achieve this goal, we first formalized the construction of a hybrid space X_D^{hyb} associated to a pair (X, D), where X is a complex manifold, and D a simple normal crossing divisor. This construction is analogous to the one considered in [BJ17], with the essential difference that we work with cone complexes rather than compact polyhedral spaces.

In the compact case, our construction leads to two degeneration theorems for measures, namely Theorems 2.3 and 2.3.12. These measures are associated to a (n, n)-forms called $\mathcal{P}$-logarithmic form, a new class of differential forms introduced in this thesis, here n is the dimesion of the complex manifold X. Although these forms define singular measures on X, they induce continuous measures on the corresponding hybrid space.

Conclusiones

El objetivo principal de esta tesis ha sido el desarrollo de un espacio híbrido $\mathscr{X}^{hyb}$ asociado a un stack de Deligne–Mumford toroidal $(\mathscr{X}, \mathscr{U})$. En el caso particular del stack moduli de curvas $(\overline{\mathscr{M}_{g,n}}, \mathscr{M}_{g,n})$, esta construcción proporciona un marco inicial para comprender la relación entre las amplitudes en la teoría cuántica de campos y las amplitudes en la teoría de cuerdas.

Para alcanzar este objetivo, lo primero que se hizo fue formalizar la construcción de un espacio híbrido X_D^{hyb} asociado a un par (X, D) donde X es una variedad compleja y D un divisor a cruces normales simple. Esta construcción es análoga a la considerada en [BJ17], con la diferencia esencial de que trabajamos con complejos cónicos en lugar de espacios poliedrales compactos.

En el caso compacto, nuestra construcción conduce a dos teoremas de degeneración de medidas, a saber, los Teoremas 2.3 y 2.3.12. Estas medidas están asociadas a una (n, n)-forma denominada forma $\mathcal{P}$-logarítmica, una nueva clase de formas diferenciales introducida en esta tesis, aquí n denota la dimensión de la variedad compleja X. Aunque dichas formas definen medidas singulares sobre X, inducen medidas continuas en el espacio híbrido correspondiente.

Bibliography

[ABGF16] Omid Amini, Spencer J. Bloch, José I. Burgos Gil, and Javier Fresán. Feynman amplitudes and limits of heights. *Izvestiya: Mathematics*, 80(5):813–848, 2016.

[ACP12] Dan Abramovich, Lucia Caporaso, and Sam Payne. The tropicalization of the moduli space of curves. *Annales Scientifiques de l'Ecole Normale Superieure*, 48, 12 2012.

[AHCG11] E. Arbarello, J.D. Harris, M. Cornalba, and P. Griffiths. *Geometry of Algebraic Curves: Volume II with a contribution by Joseph Daniel Harris.* Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2011.

[Alp24] Jarod Alper. Stacks and moduli, 2024. URL: `https://sites.math.washington.edu/~jarod/moduli.pdf`. Last version 2024/05/21.

[AN20] Omid Amini and Noema Nicolussi. Moduli of hybrid curves i: Variations of canonical measures. *arXiv preprint arXiv:2007.07130*, 2020.

[AN22] Omid Amini and Noema Nicolussi. Moduli of hybrid curves ii: Tropical and hybrid laplacians. *arXiv preprint arXiv:2203.12785*, 2022.

[Beh04] Kai Behrend. Cohomology of stacks. In *Intersection theory and moduli*, ICTP Lect. Notes, XIX, pages 249–294. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[BJ17] Sébastien Boucksom and Mattias Jonsson. Tropical and non-Archimedean limits of degenerating families of volume forms. *Journal de l'École polytechnique — Mathématiques*, 4:87–139, 2017.

[Bou04] Nicolas Bourbaki. *Integration I: Chapters 1–6.* Elements of Mathematics. Springer-Verlag, Berlin, 2004. Translated from the 2nd French edition by Sterling K. Berberian.

[Cap13] Lucia Caporaso. Algebraic and tropical curves: comparing their moduli spaces. In Gavril Farkas and Ian Morrison, editors, *Handbook of Moduli, Volume I*, volume XXIV of *Advanced Lectures in Mathematics*, pages 119–160. International Press, Somerville, MA, 2013. Originally available as arXiv:1101.4821 (Jan 2011).

[Con05] Brian Conrad. Keel–mori theorem via stacks. In *https://api.semanticscholar.org/CorpusID:12268216*, 2005.

[CT07] Brian Conrad and Michael Temkin. Non-archimedean analytification of algebraic spaces. *Journal of Algebraic Geometry*, 18, 07 2007.

[Dem97] Jean-Pierre Demailly. Complex analytic and differential geometry, 1997. Available online, version of May 2014.

[DJ49] Schwartz Laurent Dieudonne Jean. La dualite dans les espaces $\mathcal{F}$ et $\mathcal{LF}$. *Annales de l'institut Fourier*, 1:61–101, 1949.

[DM69] Pierre Deligne and David Mumford. The irreducibility of the space of curves of given genus. *Publications Mathématiques de l'IHÉS*, 36:75–109, 1969.

[Eng89] Ryszard Engelking. *General Topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989.

[FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of Abelian Varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1990.

[Gil84] Henri Gillet. Intersection theory on algebraic stacks and q-varieties. *Journal of Pure and Applied Algebra*, 34(2):193–240, 1984.

[Gil09] Henri Gillet. Arithmetic intersection theory on deligne-mumford stacks. *Fields Communications, American Mathematical Society*, 04 2009.

[Gor78] R. Mark Goresky. Triangulation of stratified objects. *Proceedings of the American Mathematical Society*, 72(1):193–200, 1978.

[Gro71] Alexander Grothendieck. Revêtements étales et groupe fondamental (sga 1), exposé v. In Alexander Grothendieck and Michèle Raynaud, editors, *Séminaire de Géométrie Algébrique du Bois Marie 1960–61 (SGA 1)*, volume 224 of *Lecture Notes in Mathematics*. Springer, Berlin, 1971. `https://publications.ias.edu/sga`.

[GRV08] Michael B. Green, Jorge G. Russo, and Pierre Vanhove. Low energy expansion of the four-particle genus-one amplitude in type ii superstring theory. *JHEP*, 2008(02):020, 2008.

[Gó01] Tomás L. Gómez. Algebraic stacks. *Proceedings Mathematical Sciences*, 111(1):1–31, 2001.

[IM99] D.A Pronk I Moerdijk. Simplicial cohomology of orbifolds. *Indagationes Mathematicae*, 10, Issue 2:269–293, 1999.

[KKMSD06] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal Embeddings 1*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2006.

[KM97] Seán Keel and Shigefumi Mori. Quotients by groupoids. *Annals of Mathematics*, 145(1):193–213, 1997.

[KNU83] FINN F. KNUDSEN. The projectivity of the moduli space of stable curves, ii: The stacks m g,n. *Mathematica Scandinavica*, 52(2):161–199, 1983.

[LMB99] G. Laumon and L. Moret-Bailly. *Champs algébriques*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 1999.

[MH56] Kiiti Morita and Sitiro Hanai. Closed mappings and metric spaces. *Proceedings of the Japan Academy*, 32(1):10 – 14, 1956.

[Moe02] Ieke Moerdijk. Orbifolds as groupoids: an introduction. In *Orbifolds in Mathematics and Physics*, volume 310 of *Contemporary Mathematics*, pages 205–222. American Mathematical Society, 2002.

[MS84] John W. Morgan and Peter B. Shalen. Valuations, trees, and degenerations of hyperbolic structures. i. *Annals of Mathematics, Second Series*, 120(3):401–476, 1984.

[Noo04] Behrang Noohi. Fundamental groups of algebraic stacks. *Journal of the Institute of Mathematics of Jussieu*, 3(1):69–103, 2004.

[PFS08] Chris Peters, Joseph Fourier, and Jozef Steenbrink. *Mixed Hodge structures*. Springer Berlin, Heidelberg, 2008.

[Ryd07] David Rydh. Existence and properties of geometric quotients. *Journal of Algebraic Geometry*, 22, 08 2007.

[Sat57] Ichiro Satake. The gauss–bonnet theorem for v-manifolds. *Journal of the Mathematical Society of Japan*, 9(4):464–492, 1957.

[Siw71] F. Siwiec. Sequence-covering and countably bi-quotient mappings. *Topology Appl.*, 1(2):143–154, 1971.

[ST24] Jake P. Solomon and Sara B. Tukachinsky. Differential forms on orbifolds with corners. *Journal of Topology and Analysis*, 16(4):561–615, 2024.

[Sta18] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2018.

[Thu07] Amaury Thuillier. Géométrie toroïdale et géométrie analytique non archimédienne. application au type d'homotopie de certains schémas formels. *manuscripta mathematica*, 123:381–451, 07 2007.

[Toe99] Bertrand Toen. K-theory and cohomology of algebraic stacks: Riemann-roch theorems, d-modules and gaga theorems, 1999.

[Tou17] Piotr Tourkine. Tropical amplitudes. *Annales Henri Poincaré*, 18(6):2199–2249, Feb 2017.

[Vis89] Angelo Vistoli. Intersection theory on algebraic stacks and on their moduli spaces. *Inventiones mathematicae*, 97(3):613–670, 1989.

[Wł22] Jarosław Włodarczyk. Functorial resolution except for toroidal locus. toroidal compactification. *Advances in Mathematics*, 407:108551, 2022.

[Yan63] C. T. Yang. The triangulability of the orbit space of a differentiable transformation group. *Bulletin of the American Mathematical Society*, 69(3):405 – 408, 1963.