



Universidad Autónoma de Madrid

Rigidity results for the mapping class group

Rodrigo De Pool
Supervisor: Javier Aramayona

Instituto de Ciencias Matemáticas
Consejo Superior de Investigaciones Científicas
Departamento de Matemáticas, Universidad Autónoma de Madrid

Madrid, May 2025

Este trabajo fue realizado gracias al apoyo económico del contrato predoctoral PRE2020-092939 que forma parte del proyecto Severo Ochoa CEX2019-000904-S-20-2. Además, parte de los contenidos de este trabajo fueron financiados por el Ministerio de Ciencia, Innovación y Universidades a través de las becas PGC2018-101179-B-I00 y CEX2023-001347-S.

. . . Pintor que pintas tu tierra,
si quieres pintar tu cielo,
cuando pintes angelitos
acuérdate de tu pueblo
y al lado del ángel rubio
y junto al ángel trigueño,
aunque la Virgen sea blanca,
píntame angelitos negros.

Andres Eloy Blanco.

Agradecimientos

Primero, quiero agradecer a mis padres, a mis hermanos, a mis abuelos, a mi tía y a mi prima: por los domingos familiares, por los almuerzos que preparan con tanto cariño y, sobre todo, por su amor incondicional.

Quiero agradecer a Alex y a Pablo, que generosamente me ofrecieron su amistad desde que pisé España.

Gracias a Emilio, Lucía, Óscar y Juan, por su amistad, por todas las horas de estudio juntos y por todas las aventuras universitarias.

Esta tesis es literalmente inconcebible sin el apoyo de Javier Aramayona. Gracias por todo el trabajo, los consejos, las correcciones y, especialmente, por mantenerme siempre entusiasmado con las matemáticas.

Gracias a Juan Souto, por acogerme en Rennes, prestarme su bicicleta, enseñarme geometría y aguantarme durante tres meses.

He disfrutado los años de doctorado y esto se debe en gran parte a todas las personas que he tenido la fortuna de encontrarme por el camino. Gracias a todos los amigos que he hecho en estos años: a Henrique, Sergio, Pablo, Paloma, Jan, Jorge, Edu, Javi, Joaquín, George y Yongsheng. Y, claro, gracias también a los profesores: a Andrei, Leo, Dominik, Caterina, Alejandra y Yago.

Ya por último, quiero agradecerle a María por animarme en el hartazgo y acompañarme en la alegría. Gracias por ser un sol, una flor y una copa de vino.

Contents

Resumen **viii**

Abstract **ix**

Introduction **1**

1 Braided multitwists **11**
1.1 Preliminaries 12
1.1.1 Multitwists 13
1.1.2 Geometric intersection formulas 14
1.1.3 Algebraic intersection formulas 16
1.2 Characterizing braided multitwists 16
1.2.1 Curves in braided multitwists 18
1.2.2 Reducing braided multitwists 19
1.2.3 Action on curves and their homology classes 20
1.2.4 Orbits have size two 21
1.2.5 Reduced multitwists are trivial 30

2 Classification of homomorphisms **36**
2.1 Definitions and tools 38
2.1.1 Basic properties 39
2.1.2 Some examples of homomorphisms 40
2.1.3 More tools 42
2.1.4 Homomorphisms induced by embeddings 42
2.1.5 Homomorphisms induced by multi-embeddings 44
2.2 Twist-preserving homomorphisms 47
2.2.1 Dehn twists to Dehn twists 47
2.2.2 Reducing to the irreducible case 48
2.2.3 Reducing to homomorphisms that do not factor 50
2.2.4 Reducing to surfaces with no boundary 50
2.3 Multitwist-preserving homomorphisms 52
2.4 Classification of homomorphisms 57
2.4.1 No torsion 58
2.4.2 Counting curves 60

2.4.3 Dehn twists to multitwists 62
2.5 Lifting to the diffeomorphism group 69
2.6 Maps not induced by multi-embeddings 70

3 Maps between moduli spaces 72
3.1 Teichmüller space and the Weil-Petersson metric 75
3.1.1 Preliminaries 75
3.1.2 The Weil-Petersson metric 76
3.1.3 A strictly convex function 77
3.2 Maps to Kähler manifolds 81
3.2.1 Convexity 82
3.2.2 Energy and its first variation 84
3.2.3 Energy of holomorphic maps and the Wirtinger inequality 85
3.2.4 Non-existence of certain holomorphic maps 86
3.3 Irreducibility of maps to moduli spaces 89
3.3.1 Irreducibility 89
3.3.2 Homomorphisms between mapping class groups 91
3.3.3 Holomorphisms between moduli spaces 93

4 Hopfian big mapping class groups 95
4.1 Preliminaries 98
4.1.1 Infinite-type surfaces and curves 98
4.1.2 Mapping class groups and their topology 99
4.1.3 Multitwists along infinite multicurves 100
4.2 Generating compactly supported mapping classes 101
4.3 Irreducible epimorphisms 104
4.4 From epimorphisms to isomorphisms 114
4.4.1 Moving to the full mapping class group 117
4.5 Relaxing continuity 118

5 Finite rigidity of the non-separating curve complex 120
5.1 Preliminaries 121
5.1.1 Curves 122
5.1.2 Non-separating curve complex 122
5.2 Finite rigid sets 123
5.3 Finite rigid sets for closed surfaces 124
5.3.1 Constructing the finite rigid set 124
5.3.2 Proving the rigidity of F_R 126
5.4 Finite rigid exhaustion for closed surfaces 133
5.4.1 Enlarging the finite rigid set 134
5.4.2 Constructing the exhaustion for closed surfaces 136
5.5 Finite rigid sets for punctured surfaces 137
5.5.1 Constructing the finite rigid set 137

5.6 Finite rigid exhaustion for punctured surface 141
5.6.1 Enlarging the finite rigid set 142
5.6.2 Constructing the exhaustion for punctured surfaces . . 142

6 Matching arc complexes **144**
6.1 Preliminaries 145
6.1.1 Arcs 145
6.1.2 Matching arc complex 146
6.2 Connectivity of $\mathcal{MA}(\Gamma)$ 147
6.2.1 Graphs with few vertices 148
6.3 Gromov hyperbolicity of $\mathcal{MA}(\Gamma)$ 149
6.3.1 A family of subpaths in $\mathcal{A}(S, \Gamma)$ 150

Conclusions **154**

Conclusiones **155**

Bibliography **162**

Resumen

El objetivo de esta tesis es presentar varios resultados sobre la rigidez del grupo modular y de otros espacios asociados. Además de su interés intrínseco en el ámbito de la teoría geométrica de grupos, estos resultados están motivados por la comparación entre los grupos modulares y los retículos irreducibles en grupos de Lie de rango superior. En este contexto, una pregunta análoga a la superrigidez de Margulis es si todo homomorfismo entre grupos modulares se levanta a un mapa entre los grupos de difeomorfismos. Esta pregunta guía la discusión de los dos primeros capítulos. Primero, en el Capítulo 1 introducimos una nueva herramienta para estudiar homomorfismos, a saber, una caracterización topológica de pares de multigiros que satisfacen la ecuación de trenzas. Luego, en el Capítulo 2, demostramos que bajo ciertas hipótesis en el género de las superficies, todo homomorfismo entre grupos modulares es inducido por un conjunto de embebimientos. Como consecuencia, se comprueba para estas superficies que, en efecto, todo homomorfismo entre grupos modulares se levanta a un mapa entre los grupos de difeomorfismos. Continuamos en el tercer capítulo con un resultado de rigidez similar, donde presentamos para la misma clase de superficies una clasificación geométrica de los holomorfismos entre sus espacios de moduli. En el Capítulo 4, extendemos algunas de las técnicas utilizadas al ámbito de las superficies de tipo infinito, demostrando que los epimorfismos entre ciertos grupos modulares grandes son, en realidad, isomorfismos. En particular, concluimos que, para cierta familia de superficies, los grupos modulares grandes son topológicamente hopfianos. Los dos últimos capítulos de la tesis están dedicados a las acciones simpliciales del grupo modular. Por un lado, en el Capítulo 5, demostramos la rigidez finita del complejo de curvas no separantes. Por otro, en el Capítulo 6, introducimos el complejo de arcos coincidentes y obtenemos condiciones para que este complejo sea conexo e hiperbólico.

Abstract

The purpose of this thesis is to establish various results concerning the rigidity of the mapping class group and other associated spaces. Apart from its intrinsic interest in the context of geometric group theory, these results are motivated by the comparison between mapping class groups and irreducible lattices in higher-rank Lie groups. In this setting, a natural analogue of Margulis superrigidity is whether every homomorphism between mapping class groups lifts to a map at the level of diffeomorphism groups. This questions leads the discussion for the first two chapters. We introduce in Chapter 1 a key new tool to study homomorphisms, namely a topological characterization of pairs of multitwists that satisfy the braid equation. Then, in Chapter 2 we prove that, for a range of surfaces, every homomorphism between mapping class groups is induced by a set of embeddings. As a consequence, we show that, for these surfaces, every homomorphism between mapping class groups does indeed lift to the diffeomorphism groups. A parallel rigidity result is presented in Chapter 3, where for the same range of surfaces we prove a geometric classification of holomorphic maps between their moduli spaces. In Chapter 4, we extend some of the techniques used to the setting of infinite-type surfaces, proving that epimorphisms between certain big mapping class groups are actually isomorphisms. In particular, we conclude, for a certain class of surfaces, that big mapping class groups are topologically Hopfian. The last two chapters of the thesis are devoted to simplicial actions of the mapping class group. In Chapter 5, we prove the finite rigidity of the non-separating curve complex. Finally, in Chapter 6, we introduce the matching arc complex, a variant of the arc complex that arises in the study braided Thompson groups, and we obtain conditions for this complex to be both connected and hyperbolic.

Introduction

The central object of this work is the *mapping class group* of a surface S, namely the group of orientation-preserving homeomorphisms of S up to homotopy:

$$\mathrm{Map}(S) = \mathrm{Homeo}^+(S, \partial S)/\sim .$$

Here, the surface S is allowed to have compact boundary, and both homeomorphisms and homotopies fix the boundary pointwise. There are two important variants of this group: The *extended mapping class group* $\mathrm{Map}^{\pm}(S)$ is the super-group of $\mathrm{Map}(S)$ that allows orientation reversing homeomorphisms; and, the *pure mapping class group* $\mathrm{PMap}(S)$, which is the subgroup of elements $f \in \mathrm{Map}(S)$ that act trivially on the space of ends of the surface.

The mapping class group plays a prominent role in geometric group theory and low-dimensional topology. Part of its importance comes from the fact that it arises naturally in several branches of mathematics including algebra, 3-manifold theory and algebraic geometry, as we detail below.

The algebraic point of view is exemplified by the Dehn-Nielsen-Baer theorem, which asserts that, for closed surfaces of positive genus, there is an isomorphism

$$\mathrm{Map}^{\pm}(S) \cong \mathrm{Out}(\pi_1(S)),$$

where $\mathrm{Out}(\pi_1(S))$ denotes the group of outer automorphisms of the fundamental group of S.

From the perspective of algebraic geometry, the mapping class group $\mathrm{Map}(S)$ acts holomorphically on the Teichmüller space $\mathcal{T}(S)$, and the quotient of this action is the complex orbifold $\mathcal{M}(S)$, known as the moduli space of S. The moduli space of a surface is a fundamental object in algebraic geometry and we remark that, in the spirit of Gromov, the action $\mathrm{Map}(S) \curvearrowright \mathcal{T}(S)$ connects the algebraic properties of the group $\mathrm{Map}(S)$ to the geometric properties of $\mathcal{M}(S)$.

Finally, and although slightly tangential to this thesis, we note that the structure of the mapping class group has extensively been used to study 3-dimensional hyperbolic geometry; a notable example in this direction is Brock-Canary-Minsky's solution of Thurston's Ending Lamination Conjecture [23], which parameterize hyperbolic 3-manifolds of infinite volume.

In this light, we see that the mapping class group $\mathrm{Map}(S)$ comes equipped with a plethora of actions, which turn it into a natural object of interest in geometric group theory and related areas. In this thesis, we study several algebraic properties of mapping class groups, with the vague theme of *rigidity* as connecting thread. As will become apparent, and as is customary in the theory of mapping class groups, the interplay between algebra, geometry and topology will be a central feature in the results we obtain, as well as in the techniques we use to prove them.

We now proceed to describe and motivate the results we have obtained in this thesis.

Classification of homomorphisms

There is a solid, although incomplete, comparison between $\mathrm{PMap}(S)$ and irreducible lattices in higher-rank Lie groups. This comparison has driven much of the progress in understanding $\mathrm{PMap}(S)$; for example, Grossman [34] showed that $\mathrm{PMap}(S)$ is residually finite, and Harer [37] computed the virtual cohomological dimension of $\mathrm{PMap}(S)$ by following the Borel-Serre program for lattices. In this framework, Aramayona-Souto [12, Question 5] suggested an analogue of Margulis superrigidity in the context of mapping class groups.

Problem 0.1. *Classify all homomorphisms* $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$. *Do these homomorphisms lift to maps* $\mathrm{Homeo}_c^+(S) \to \mathrm{Homeo}_c^+(S')$*?*

Problem 0.1 has been thoroughly studied by authors. The first major result in this direction is a theorem of Ivanov [50] that shows that, if S has genus $g \geq 2$, then every automorphism of $\mathrm{PMap}(S)$ is the restriction of an inner automorphism of $\mathrm{Map}^{\pm}(S)$. Even more, Ivanov-McCarthy [53] later on proved that the same conclusions actually hold for any injective endomorphism. Since then, these results were generalized in various directions: Korkmaz [57] described endomorphisms onto finite-index subgroups; Harvey-Korkmaz [38] proved the triviality of homomorphisms between mapping class groups of reducing genus; Bell-Margalit [16] proved the co-Hopfian property for mapping class groups of certain genus zero surfaces, among other results. Remarkably, Aramayona-Souto [11] generalized most of these results by classifying every homomorphism $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$ for surfaces of genus $g \geq 6$ and $g' \leq 2g-1$.

The following theorem generalizes Aramayona-Souto's classification and provides a positive answer to Problem 0.1 under suitable conditions on the genus of the surfaces. Before stating the result, let us introduce one more definition:

Let $\mathrm{Homeo}_c^+(S, \partial S)$ be the group of homeomorphisms of S with compact support in the interior of the surface. A homomorphism $\overline{\varphi} : \mathrm{Homeo}_c^+(S, \partial S) \to \mathrm{Homeo}_c^+(S', \partial S')$ is said to be *induced by a multi-embedding* if there is a set

of embeddings with disjoint image $\mathcal{I} = \{\iota_i : S \hookrightarrow S' | \ i = 1, \dots, k\}$ such that $\overline{\varphi}(\phi) = \hat{\phi}$ and $\hat{\phi}$ is defined by

$$\hat{\phi}(x) = \begin{cases} x & x \notin \iota_i(S) \ \forall i, \\ \iota_i \circ \phi \circ \iota_i^{-1}(x) & x \in \iota_i(S). \end{cases}$$

If the homomorphism $\overline{\varphi}$ descends to a map at the level of mapping class groups $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$, we also say φ is *induced by a multi-embedding*. The next result ensures that under certain hypothesis every homomorphism is of this kind.

Theorem A. *Let S, S' be two orientable finite-type surfaces of genus $g \geq 4$ and $g' \leq 6 \cdot 2^{g-4}$. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a non-trivial homomorphism, then φ is induced by a multi-embedding.*

We emphasize that Theorem A only imposes conditions on the genus of the surfaces, and has no assumptions on the number of boundary components or punctures.

Note that if $g' \leq 2g - 1$ then Theorem A recovers the classification given by Aramayona-Souto [11], namely that every homomorphism $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is induced by an embedding. In particular, we obtain a stronger version of the Ivanov-McCarthy theorem: every non-trivial endomorphism of $\mathrm{PMap}(S)$ is an inner automorphism of $\mathrm{Map}^{\pm}(S)$. Instantiating the statement of Theorem A to the case of closed surfaces we obtain the following corollary.

Corollary 0.2. *Let S_g and $S_{g'}$ be closed orientable surfaces of genus $g \geq 4$ and $g' \leq 6 \cdot 2^{g-4}$. If $g \neq g'$, every homomorphism $\mathrm{Map}(S_g) \to \mathrm{Map}(S_{g'})$ is trivial.*

Corollary 0.2 was already known for $g' \leq 2g$ by previous work of Chen-Lanier. In contrast with Chen-Lanier's proof which relies on analyzing torsion in the mapping class group, our methods resemble the standard techniques used by Ivanov and Aramayona-Souto. Indeed, to proof Theorem A we show that the genus bounds imply φ is *multitwist preserving*, meaning that the image of every non-separating Dehn twist is a multitwist. Then, that φ is induced by a multi-embedding follows from the next result.

Theorem B. *Let S, S' be two orientable finite-type surfaces and let $g \geq 3$ be the genus of S. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a multitwist-preserving map, then φ is induced by a multi-embedding.*

A key part in the proof of Theorem B involves understanding when multitwists satisfy the braid relation, this new ingredient is presented in the following section. Now, to understand *non* multitwist-preserving homomorphisms $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$ one probably requires a convenient description of roots of multitwists and their relations. These elements are quite

mysterious with non-trivial examples discovered not so long ago [65]. Thus, a complete characterization of homomorphisms $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$ remains elusive.

In any case, Theorem A implies that under suitable hypothesis every homomorphism $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ can be lifted to a homomorphism between the homeomorphism groups. Thus, for this range of surfaces we obtain a positive answer to Problem 0.1. Even more, we show that if S and S' are endowed with differentiable structures, the multi-embedding can be taken to be smooth and therefore the map can be lifted to the diffeomorphism groups.

Theorem C. *Let S, S' be two orientable finite-type smooth surfaces of genus $g \geq 4$ and $g' \leq 6 \cdot 2^{g-4}$. Any homomorphism $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ admits a lift $\overline{\varphi} : \mathrm{Diff}_c^+(S) \to \mathrm{Diff}_c^+(S')$ to the diffeomorphism groups.*

The proofs of Theorem A, Theorem B and Theorem C are presented in Chapter 2, after developing the necessary tools introduced in Chapter 1.

Braided elements

Exploiting relations between Dehn twists is a straightforward, and yet useful, approach to study quotients of the mapping class group. One spectacular example was given by Harer [36] who used the *lantern relation* to give a short proof that the first homology of $\mathrm{Map}(S)$ is trivial, a result originally proved by Mumford-Powell [71], [74]. In a similar fashion, Ivanov-McCarthy [53, Lemma 3.15] showed that two *distinct* Dehn twists d_a, d_b have non-trivial powers satisfying the braid relation

$$d_a^m d_b^m d_a^n = d_b^m d_a^n d_b^m$$

if and only if the curves intersect once and $n = m \in \{1, -1\}$. This result is a crucial ingredient in Ivanov-McCarthy's characterization of injective endomorphisms of $\mathrm{PMap}(S)$. In the general context of homomorphisms $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$, one is compelled to provide a similar description for multitwists that satisfy the braid relation. In Chapter 1 we present such description by proving the following theorem.

Theorem D. *Let S be an orientable surface. If $m_A, m_B \in \mathrm{Map}(S)$ are two multitwists that satisfy $m_A m_B m_A = m_B m_A m_B$, then*

$$m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k} m_C$$
$$m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k} m_C,$$

where $n_i \in \{1, -1\}$, the curves $a_1, \dots, a_k, b_1, \dots, b_k$ are pairwise disjoint except for $i(a_i, b_i) = 1$ and m_C is a common multitwist fixing every curve a_i and b_j.

We may rephrase Theorem D in terms of certain homomorphisms from the braid group to the mapping class group. Recall that the braid group on n-strands is given by the presentation

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} | \, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \; \forall i, \;\; [\sigma_i, \sigma_j] = 1 \; \forall |i-j| > 1 \rangle,$$

and the σ_i's are said to be the *standard braid generators*.

An embedding $\varphi : B_n \hookrightarrow \mathrm{Map}(S)$ from the braid group to the mapping class group may be defined by sending each standard generator σ_i to a Dehn twist d_{c_i}, where the curves c_i, c_j intersect once if $|i-j| = 1$ and are disjoint otherwise. Any such φ (or φ^{-1}) is known as a *geometric* embedding of the braid group into the mapping class group. If we replace Dehn twists by multitwists, then Theorem D yields a description of φ:

Corollary 0.3. *Let S be an orientable surface and let B_n be the braid group on n-strands. If $\varphi : B_n \to \mathrm{Map}(S)$ sends a standard generator to a multitwist, then φ factors as*

$$\varphi = \left(\prod_{i=1}^{k} \varphi_i \right) \circ d,$$

where $d : B_n \to \prod_{i=1}^{k} B_n$ is the diagonal homomorphism, φ_i is a geometric embedding for $1 \leq i < k$, and φ_k has cyclic image.

One could hope that every embedding $B_n \hookrightarrow \mathrm{Map}(S)$ comes from a diagonal decomposition of geometric embeddings as in Corollary 0.3; however this is not the case. For instance, Szepietowski constructed in [80] an embedding $B_n \hookrightarrow \mathrm{Map}(S)$ sending a standard generator to a non-trivial root of a Dehn twist, which in particular can have no such diagonal decomposition.

Theorem D fits into the bigger picture of results studying relations between multitwists. A closely related result is [35, Theorem 3.4] of Hamidi-Tehrani, which provides sufficient conditions for two *positive* multitwists to generate a free group of rank two. Leininger improved this result in [60, Theorem 6.1] by classifying pairs of positive multitwists generating a free group. We stress that Leininger's techniques require multitwists to be positive, therefore they cannot be applied to study general multitwists as in the statement of Theorem D. In contrast, our proof is combinatorial in nature, mainly relying on a geometric intersection formula proven by Ivanov in [52, Proposition 4.2]. Also, we note that after the classification given by Leininger and Theorem D, there is a recent result of Lamy [58] concerning the structure of certain groups generated by a finite set of (non-positive) multitwists.

As mentioned before, the original motivation for Theorem D was to further understand homomorphisms between pure mapping class groups of finite-type surfaces. However, we highlight that this result proved to be useful in the study of the Hopfian property for certain families of big mapping class groups (see Chapter 4 for details).

Holomorphic maps between moduli spaces

The *Teichmüller space* of a surface S is the space

$$\mathcal{T}(S) = \{(X, f)\}/\simeq$$

where X is a Riemann surface, $f : S \to X$ is a homeomorphism and $(X, f) \simeq (Y, g)$ if and only if $f \circ g^{-1}$ is homotopic to a bi-holomorphic map. The space $\mathcal{T}(S)$ admits a $\mathrm{PMap}(S)$ invariant structure as a complex manifold. Here $\mathrm{PMap}(S)$ acts on $\mathcal{T}(S)$ by pre-composition $[f] \cdot (X, g) = (X, g \circ f^{-1})$, and the quotient of this action yields a complex orbifold known as the *moduli space* of S

$$\mathcal{M}(S) = \mathcal{T}(S)/\,\mathrm{PMap}(S).$$

Notably, the stabilizer of every point of $\mathcal{T}(S)$ is finite and the stabilizer of infinite subgroups of $\mathrm{PMap}(S)$ is trivial. Thus, $\mathcal{T}(S)$ is a *classifying space for proper actions of* $\mathrm{PMap}(S)$. As a consequence, there is a correspondence between homotopy classes of continuous (orbifold) maps $\mathcal{M}(S) \to \mathcal{M}(S')$ and homomorphisms $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$. A more subtle question is to consider maps that preserve some structure, namely maps that preserve the complex structure of $\mathcal{M}(S)$. In a joint work with Souto, we characterize these maps between moduli space for a range of surfaces.

Theorem E. *Let S and S' be two finite-type orientable surfaces with no boundary. Assume S and S' have genus $g \geq 4$ and $g' \leq 6 \cdot 2^{g-4}$. Every non-constant holomorphic map $\mathcal{M}(S) \to \mathcal{M}(S')$ is a forgetful map. In particular, S and S' have the same genus.*

Theorem E was already known for $g' \leq 2g - 1$ by previous work of Antonakoudis-Aramayona-Souto [4]. As it was in their case, the proof of Theorem E relies on the general properties of the Weil-Petersson metric of $\mathcal{T}(S)$ and the associated Khäler structure. Together with the convexity of the distance function to the boundary $\partial\overline{\mathcal{T}(S)}$, one concludes that holomorphic maps induce irreducible homomorphisms $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$. Then, the final step is to use Theorem A to deduce that such irreducible homomorphisms are forgetful maps. A detailed account of the proof is presented in Chapter 3.

Infinite-type surfaces and the Hopfian property

A surface with non-finitely generated fundamental group is known as an *infinite-type* surface. In recent years, there has been increasing interest for mapping class groups of infinite-type surfaces, the so called *big mapping class groups*. A main objective in the area is to understand the general structure of these groups and specially their similarities and differences with mapping class groups of finite-type surfaces. In this regard, two important

examples are the Nielsen realization theorem proved in the infinite-type case by Afton-Calegari-Chen-Lyman [1]; and an Ivanov theorem for infinite-type surfaces proved by Bavard-Dowdall-Rafi [15]. Within this context, we study in Chapter 4 analogues of the Hopfian property and the forgetful map for infinite-type surfaces.

A fundamental tool in the study of mapping class groups of finite-type surfaces is the forgetful map: Given a finite-type surface S and a point $p \in S$ there is a surjective homomorphism

$$\mathcal{F} : \mathrm{PMap}(S \setminus \{p\}) \to \mathrm{PMap}(S)$$

obtained by 'forgetting the puncture p'. The following question appears as Problem 4.45 in [3]:

Question. Can you forget a non-planar end?

In joint work with Aramayona, Skipper, Tao, Vlamis and Wu, we give a negative answer to this question under a continuity assumption.

Theorem F. *Let S, S' be two orientable surfaces with no boundary, and assume S has infinite genus and no punctures. Every continuous epimorphism $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is induced by a homeomorphism $S \to S'$.*

In the statement of Theorem F we are considering $\mathrm{PMap}(S)$ as a topological group, where the topology is inherited from the compact-open topology on $\mathrm{Homeo}^+(S, \partial S)$. We remark that this topology is non-discrete if and only if S is of infinite-type.

Theorem F has natural implications for the Hopfian property of $\mathrm{PMap}(S)$. Recall that a group G is *Hopfian* if every epimorphism $G \to G$ is an automorphism. If G is a topological group, we say G is *topologically Hopfian* if every continuous epimorphism $G \to G$ is a homeomorphism.

Now, for a finite-type surface S the mapping class group is residually finite and finitely generated. In particular, this implies $\mathrm{PMap}(S)$ has the Hopfian property. On the contrary, if S is of infinite-type the group $\mathrm{PMap}(S)$ is not generally Hopfian. However, Theorem F implies that for certain infinite-type surfaces $\mathrm{PMap}(S)$ is topologically Hopfian:

Corollary 0.4. *If S is an infinite-type orientable surface with no boundary and no punctures, then $\mathrm{PMap}(S)$ is topologically Hopfian.*

In Chapter 4 we prove Theorem F in more generality. Indeed, the statement is proven for a family of subgroups of $\mathrm{Map}(S)$ that include: $\mathrm{PMap}(S)$, the closure of compactly supported mapping classes and the whole $\mathrm{Map}(S)$ if S has finitely many punctures. Also, we note that the continuity hypothesis in Theorem F can be relaxed for specific surfaces. For example, if M is a *blooming Cantor Tree*, it is a result of Vlamis [83] that $\mathrm{PMap}(M)$ has the automatic continuity property. In particular, we may upgrade Corollary 0.4 to

Corollary 0.5. *The pure mapping class group of the blooming Cantor tree surface is Hopfian.*

Simplicial actions of the mapping class group

The *curve complex* $\mathcal{C}(S)$ is the simplicial complex whose k-simplices are sets of $k+1$ (homotopy classes) of curves which are pairwise disjoint. The extended mapping class group acts naturally on the set of (homotopy classes) of curves. This action preserves disjointness of curves, and therefore extends to a simplicial action on the complex $\mathcal{C}(S)$:

$$\psi : \mathrm{Map}(S) \to \mathrm{Aut}\mathcal{C}(S).$$

Via the action ψ the curve complex works as a combinatorial model to study properties of $\mathrm{Map}^{\pm}(S)$. In fact, a celebrated theorem of Ivanov [51] ensures that for sufficiently complex surfaces the action ψ is an isomorphism, a result known as the *simplicial rigidity* of $\mathcal{C}(S)$. In turn, this result was key in establishing Ivanov-McCarthy's [53] description of injective endomorphisms of the mapping class group.

The curve complex, and its applications to the mapping class group, has motivated the study of similar complexes associated to surfaces. For example, simplicial rigidity has been established for: the arc complex [49]; the non-separating curve complex [47]; the separating curve complex [20]; and the pants graph [64] (see [68] for a survey on complexes associated to surfaces). In the following sections we present results regarding two simplicial complexes associated to a surface: the non-separating curve complex $\mathcal{N}(S)$ which is the subcomplex of $\mathcal{C}(S)$ spanned by the non-separating curves; and, a variant of the arc complex called the *matching arc complex*, that we define below.

Finite rigidity of non-separating curve complexes

A stronger notion of rigidity associated to a simplicial complex is that of *finite rigidity*: the simplicial complex $\mathcal{C}(S)$ is said to be finitely rigid if there exists a finite subcomplex X such that any locally injective simplicial map

$$\phi : X \to \mathcal{C}(S)$$

is induced by a *unique* mapping class, that is, there exists a unique $h \in \mathrm{Map}(S)$ such that the simplicial action $\psi(h) : \mathcal{C}(S) \to \mathcal{C}(S)$ satisfies $\psi(h)|_X = \phi$. Such X is called a *finite rigid set* of $\mathcal{C}(S)$ with *trivial pointwise stabilizer*.

The finite rigidity of the curve complex was proven by Aramayona and Leininger in [8], thus answering a question by Lars Louder. Furthermore, they constructed in [7] an exhaustion of $\mathcal{C}(S)$ by finite rigid sets with trivial pointwise stabilizers, from which they recovered Ivanov's result [51] on the simplicial rigidity of $\mathcal{C}(S)$.

Following the result of Aramayona and Leininger, finite rigidity has been proven for other complexes: Shinkle proved it for the arc complex [78] and the flip graph [79]; Hernández, Leininger and Maungchang proved a slightly different notion for the pants graph [42], [67]; and Huang and Tshishiku proved a weaker notion for the separating curve complex [43]. The following result establishes finite rigidity for the non-separating curve complex.

Theorem G. *Let S be an orientable finite-type surface of genus $g \geq 3$ with no boundary. There exists an exhaustion of the non-separating curve complex $\mathcal{N}(S)$ by finite rigid sets. In particular, $\mathcal{N}(S)$ is finitely rigid.*

Connectedness and hyperbolicity of matching arc complexes

Matching arc complexes, especially those associated to planar surfaces, have recently played a central role in the study of finiteness properties of braided Thompson groups [25].

We now briefly introduce this complex, see Chapter 6 for details. Given an orientable surface S with punctures and a finite simplicial graph Γ, we fix an embedding $\Gamma \hookrightarrow S$ such that the image of every vertex of Γ is a puncture of S. Say that an arc on S is compatible with Γ if there is an edge of Γ with the same endpoints. The *matching arc complex* $\mathcal{MA}(S, \Gamma)$ is the simplicial complex whose k-simplices are sets of $k+1$ (homotopy classes) of compatible arcs which are pairwise disjoint, including at their endpoints.

Zaremsky [92] asked the question of determining for which graphs Γ is the associated matching arc complex $\mathcal{MA}(S, \Gamma)$ connected and hyperbolic. In joint work with Aramayona and Fernández, we provide a partial answer to this question.

Theorem H. *Let Γ be a finite-simplicial graph with at least five non-isolated vertices. The complex $\mathcal{MA}(S, \Gamma)$ is connected if and only if the complement of every edge in Γ contains at least one edge. Furthermore, if Γ is connected and contains a triangle, then $\mathcal{MA}(S, \Gamma)$ is hyperbolic.*

The proofs for Theorem G and Theorem H are presented in chapters 5 and 6, respectively.

The results of this thesis have been included in the following articles:

1. J. Aramayona, R. de Pool, and A. Fernández. *Matching arc complexes: Connectedness and hyperbolicity*, *Expositiones Mathematicae*.

2. J. Aramayona, R. De Pool, R. Skipper, J. Tao, N. G. Vlamis and X. Wu. *Non-planar ends are continuously unforgettable*, to appear in *International Mathematics Research Notices*.

3. R. De Pool. *Braided multitwists* , to appear in *Algebraic and Geometric Topology*.

4. R. de Pool. *Finite rigid sets of the non-separating curve complex*, *Forum Mathematicum*.

5. R. De Pool. *Homomorphisms between pure mapping class groups*, preprint in arXiv: 2410.18796[math].

6. R. De Pool and J. Souto. *Holomorphic maps between moduli spaces II*, preprint in arXiv: 2412.01257[math].

Chapter 1

Braided multitwists

Let S be an orientable surface possibly with punctures or boundary. In this chapter we characterize *braided multitwists*, i.e, multitwists $m_A, m_B \in \mathrm{Map}(S)$ satisfying the braid relation $m_A m_B m_A = m_B m_A m_B$. First, let us consider some examples:

Example 1.1. *Consider a and b two curves on S intersecting once. Denote their Dehn twists by d_a and d_b. It is known that*

$$d_a d_b d_a = d_b d_a d_b,$$

and by the same equality the inverses d_a^{-1}, d_b^{-1} are also braided. It is a lemma that two Dehn twists d_a, d_b are braided if and only if the curves a and b intersect once [53, Lemma 3.15].

Now, we construct braided multitwists that are not Dehn twists.

Example 1.2. *Let $m_A = d_{a_1} d_{a_2} d_{a_3} d_{a_4}^{-1}$ and $m_B = d_{b_1} d_{b_2} d_{b_3} d_{b_4}^{-1}$ be two multitwists along the curves a_i, b_j shown in Figure 1.1. Since d_{a_i} commutes with d_{b_j} whenever $i \neq j$ and a_i, b_i intersect once, we deduce that $m_A m_B m_A = m_B m_A m_B$ from the previous example.*

Note that in Example 1.2 the multitwists m_A and m_B do not share any common Dehn twist. An easy way to create new braided multitwists is to add common components with the same power. We do so in the next example:

Example 1.3. *Let m_A, m_B be the multitwists of Example 1.2 and let c be the curve shown in Figure 1.1. Then, the multitwists $m_A d_c^n$ and $m_B d_c^n$ satisfy the braid relation.*

The goal of this chapter is to prove that braided multitwists are essentially as in Example 1.3.

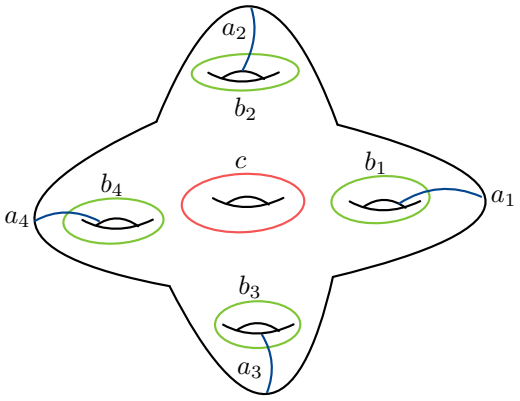


Figure 1.1: Genus five surface.

Theorem D. *Let S be an orientable surface. If $m_A, m_B \in \mathrm{Map}(S)$ are two multitwists that satisfy $m_A m_B m_A = m_B m_A m_B$, then*

$$m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k} m_C$$
$$m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k} m_C,$$

where $n_i \in \{1, -1\}$, the curves $a_1, \dots, a_k, b_1, \dots, b_k$ are pairwise disjoint except for $i(a_i, b_i) = 1$ and m_C is a common multitwist fixing every curve a_i and b_j.

As an application of Theorem D, we can describe homomorphisms from braid groups to mapping class groups that send standard generators to multitwists.

Corollary 0.3. *Let S be an orientable surface and let B_n be the braid group on n-strands. If $\varphi : B_n \to \mathrm{Map}(S)$ sends a standard generator to a multitwist, then φ factors as $\varphi = \left(\prod_{i=1}^{k} \varphi_i\right) \circ d$, where $d : B_n \to \prod_{i=1}^{k} B_n$ is the diagonal homomorphism, φ_i is a geometric embedding for $1 \leq i < k$, and φ_k has cyclic image.*

1.1 Preliminaries

Let S be a connected orientable surface. By a curve on S we mean the homotopy class of an unoriented simple closed curve that does not bound a disk or a punctured disk. Brackets around an oriented curve will be used to denote the homology class of the curve. A curve on S is *non-separating* if there exists a representative whose complement in S is connected.

For two curves a and b the *geometric intersection number* $i(a, b)$ is minimum number of intersection points between representatives of a and b.

When the curves a and b are oriented, the *algebraic intersection number* $\hat{\iota}([a], [b])$ is the sum of the signs at each intersection point. Importantly, this sum remains invariant regardless of the choice of homological representatives for $[a]$ and $[b]$.

Throughout the article, we will blur the difference between curves and their representatives. Additionally, we often consider representatives that intersect pairwise minimally. It is a theorem that such representatives always exist (see [32, Chapter 1.2]).

If a is a curve on S and A is a regular neighborhood of a given in (oriented) coordinates by $\{(h, \theta) : h \in [0, 1],\ \theta \in \mathbb{R}/(x \sim x + 2\pi)\}$, we can define the twisting map $\delta_a : S \to S$ that sends $(h, \theta) \mapsto (h, \theta + h\theta)$ on A and is the identity elsewhere. The resulting mapping class $[\delta_a] \in \mathrm{Map}(S)$ is the *Dehn twist* along the curve a and we denote it $d_a = [\delta_a]$. Note that as a matter of convention, we take our Dehn twists to be twists to the *right*. Also, we call d_a a *non-separating Dehn twist* whenever a is a non-separating curve.

A *multitwist* $m_A \in \mathrm{Map}(S)$ is a finite product of Dehn twists

$$m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k},$$

where the curves $a_1, \dots, a_k$ are distinct, pairwise disjoint and $n_i \in \mathbb{Z}$. The multitwist is *positive* if $n_i \geq 0$ for every i and it is *negative* if $n_i \leq 0$ for every i. The subindex A will be used to denote the set $A = \{a_1, \dots, a_k\}$, and we call A the set of *components* of m_A. Abusing notation, we refer to n_i as the power of a_i in m_A.

The mapping class group $\mathrm{Map}(S)$ acts on the set of curves of S and the action preserves the intersection number. We write $h \cdot a$ to denote the action of the mapping class h on the curve a. Similarly, $\mathrm{Map}(S)$ acts linearly on the homology of S and preserves the algebraic intersection number. We will write $h \cdot [a]$ for the action of the mapping class h on the homology class $[a]$.

The rest of this section is devoted to studying formulas for the geometric intersection number and algebraic intersection number, as well as some properties of multitwists. For other fundamental results and definitions on mapping class groups, we refer the reader to [32].

1.1.1 Multitwists

In this section we collect some facts about multitwists, which apply in particular to Dehn twists.

Consider two multitwists

$$m_A = d_{a_1}^{n_1} d_{a_2}^{n_2} \dots d_{a_k}^{n_k}$$
$$m_B = d_{b_1}^{m_1} d_{b_2}^{m_2} \dots d_{b_l}^{m_l},$$

where $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_l\}$ are the components of m_A and m_B, respectively. The multitwists m_A and m_B commute if only if the components are pairwise disjoint, i.e, $i(a, b) = 0$ for every $a \in A$ and $b \in B$.

A standard result [32, Lemma 3.17] ensures that the multitwists m_A and m_B satisfy $m_A = m_B$ if and only if there exists a bijection $\sigma : \{1, \dots, k\} \to \{1, \dots, l\}$ such that

$$a_i = b_{\sigma(i)} \text{ and } n_i = m_{\sigma(i)}.$$

In words, two multitwists are the same if and only if both have the same components with the same powers. As a consequence, for any $p \in \mathbb{Z} \setminus \{0\}$ we have that $m_A^p = m_B^p$ if and only if $m_A = m_B$.

The conjugate of a multitwist is also a multitwist. In fact, an element $f \in \mathrm{Map}(S)$ conjugates the multitwist m_A to

$$f \cdot m_A \cdot f^{-1} = m_{f(A)} = d_{f(a_1)}^{n_1} d_{f(a_2)}^{n_2} \dots d_{f(a_k)}^{n_k}.$$

Recall that $\mathrm{PMap}(S)$ acts transitively on the set of non-separating curves and thus any two non-separating Dehn twists are conjugate. To simplify notation, we write in the sequel $g^f = f \cdot g \cdot f^{-1}$ for any two elements $f, g \in \mathrm{Map}(S)$.

1.1.2 Geometric intersection formulas

Let a, b and c_j be curves on S and consider a positive multitwist $m_C = d_{c_1}^{n_1} \dots d_{c_k}^{n_k}$. The following is a well-known bound for the intersection number

$$i(a, b) \geq \left| i(a,\, m_C \cdot b) - \sum_{i=1}^{k} |n_i| \cdot i(a, c_i) \cdot i(b, c_i) \right|. \tag{1.1}$$

Naturally, the same bound applies for negative multitwists. For a proof of this inequality see [32, Proposition 3.4].

The formula above requires the multitwist to be either positive or negative. However, Theorem D deals with multitwists that (possibly) have mixed signs. The next bound on intersection number was proved by Ivanov for general multitwists (see [52, Proposition 4.2]).

Proposition 1.4. *Consider two curves a, b and the multitwist $m_C = d_{c_1}^{n_1} \dots d_{c_k}^{n_k}$ with $n_i \in \mathbb{Z}$. Then,*

$$i(a, b) \geq -i(a, m_C(b)) + \sum_{j=1}^{k} \widetilde{n}_j \cdot i(a, c_j) \cdot i(b, c_j),$$

where $\widetilde{n}_j = \max\{|n_j| - 2,\, 0\}$.

The proof given by Ivanov hides another formula, which will be extensively used in this article. Before introducing this formula, we need one more piece of notation.

Let a be a curve and $m_C = d_{c_1}^{n_1} \dots d_{c_k}^{n_k}$ a multitwist. The intersection points of a with curves in C define a partition $a = a_0 \cup a_1 \cup \dots \cup a_m$, where each a_i is an arc between consecutive intersection points. Since we have an arc for each intersection point, there is a total of $m + 1 = \sum_{j=1}^{k} i(a, c_j)$ arcs. Now, for each a_i set $x_i = 1$ if the subarc a_i intersects $c_k, c_l \in C$ and $n_k \cdot n_l > 0$. If $n_k \cdot n_l < 0$, set $x_i = 0$. We define the function

$$X(a, m_C) = \sum_{i=1}^{m} x_i.$$

The function $X(a, m_C)$ is non-negative and it does not depend on the choice of (minimally intersecting) representatives. This is a consequence of the next proposition.

Although Ivanov does not explicitly state the following result, it follows immediately from the first part of his proof of [52, Proposition 4.2].

Proposition 1.5 (Ivanov's hidden formula)**.** *Consider a curve a and the multitwist $m_C = d_{c_1}^{n_1} \dots d_{c_k}^{n_k}$ with $n_i \in \mathbb{Z}$. Then,*

$$i(a, m_C \cdot a) = \sum_{j=1}^{k} (|n_j| i(a, c_j) - 1) i(a, c_j) + X(a, m_C).$$

The next example illustrates how to compute $X(a,\ m_C)$ and $i(a, m_C \cdot a)$ using Proposition 1.5.

Example 1.6. *Consider the curves in Figure 1.2 and let $m_C = d_{c_1}^2 d_{c_2}^{-1} d_{c_3}$. We compute $X(a,\ m_C)$ and $i(a, m_C \cdot a)$.*

The intersection points induce the partition $a = a_0 \cup a_1 \cup a_2 \cup a_3$. Since a_0 intersects c_1, c_2 and they have powers $2 \cdot -1 < 0$, then $x_0 = 0$. Since a_1 intersects only c_1 and $2 \cdot 2 > 0$, then $x_1 = 1$. Similarly, we compute $x_2 = 1$ and $x_3 = 0$. It follows that $X(a,\ m_C) = \sum_{i=0}^{3} x_i = 2$ and we obtain $i(a, m_C \cdot a) = 8$ using Proposition 1.5.

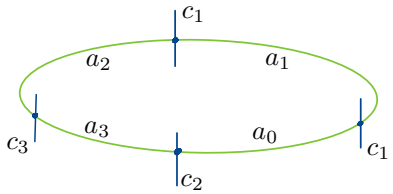


Figure 1.2: Curve $a = \bigcup_{i=0}^{3} a_i$ and the intersecting curves c_1, c_2, c_3.

1.1.3 Algebraic intersection formulas

The action of a Dehn twist on the homology of a curve is described in [32, Proposition 6.3] by the formula

$$d_c^n \cdot [a] = [a] + n \cdot \hat{\imath}([a], [c]) \cdot [c].$$

Recall that $\mathrm{Map}(S)$ acts linearly on the homology of S, thus we may iteratively use this equality to obtain a formula for the action of a multitwist. Let $m_C = d_{c_1}^{n_1} \dots d_{c_k}^{n_k}$ be a multitwist and $[a]$ the homology class of an oriented curve, it follows that

$$m_C \cdot [a] = [a] + \sum_{i=1}^{k} n_k \cdot \hat{\imath}([a], [c_i]) \cdot [c_i]. \tag{1.2}$$

From Equation 1.2 and the bilinearity of $\hat{\imath}(\cdot, \cdot)$, we derive the following result:

Lemma 1.7. *Consider a and b two oriented curves on S. Let $m_C = d_{c_1}^{n_1} \dots d_{c_k}^{n_k}$ be a multitwist in* $\mathrm{Map}(S)$*. Then,*

$$\hat{\imath}(m_C \cdot [a], [b]) = \hat{\imath}([a], [b]) + \sum_{i=1}^{k} n_k \cdot \hat{\imath}([a], [c_i]) \cdot \hat{\imath}([c_i], [b]).$$

1.2 Characterizing braided multitwists

Consider two braided multitwists

$$m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k},$$
$$m_B = d_{b_1}^{m_1} \dots d_{b_l}^{m_l}.$$

Recall the subindices A, B denote the sets $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_l\}$. The first observation is that the braid relation implies

$$(m_A m_B) m_A (m_A m_B)^{-1} = m_B.$$

In particular, for each curve a_i there exists a curve b_j such that $m_A m_B \cdot a_i = b_j$ and $n_i = m_j$. Up to re-indexing B, we may assume $m_A m_B \cdot a_i = b_i$ and $n_i = m_i$. Taking this into account, we write

$$m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k},$$
$$m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k}.$$

Remark 1.8. *We emphasize that re-indexing ensures $m_A m_B \cdot a_i = b_i$. Also, we know $m_B m_A \cdot b_i \in A$, but* a priori *$m_B m_A \cdot b_i$ could be different from a_i.*

As a first step towards Theorem D, the next lemma shows that braided multitwists decompose as a common part and two braided multitwists sharing no curve.

Lemma 1.9. *Let S be an orientable surface. If m_A, $m_B \in \mathrm{Map}(S)$ are two braided multitwists, then $m_A = m_{A'}m_C$ and $m_B = m_{B'}m_C$, where m_C is a common multitwist and $m_{A'}, m_{B'}$ are two braided multitwists sharing no curves.*

Proof. Let $C = A \cap B$ be the set of common curves of A and B. If C is empty, then we take $m_C = \mathrm{id}$ and we are done. If C is non empty:

We may assume $m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k}$, $m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k}$ such that $m_A m_B \cdot a_i = b_i$ and $C = \{a_l, \dots, a_k\}$. Note that every $a_i \in C$ is a curve in B, therefore a_i is disjoint from every curve in B and so $b_i = m_A m_B \cdot a_i = m_A \cdot a_i = a_i$. That is, we may also write $C = \{b_l, \dots, b_k\}$. Now, we define

$$m_{A'} = d_{a_1}^{n_1} \dots d_{a_{l-1}}^{n_{l-1}},$$
$$m_{B'} = d_{b_1}^{n_1} \dots d_{b_{l-1}}^{n_{l-1}}$$

and

$$m_C = d_{a_l}^{n_l} \dots d_{a_k}^{n_k}.$$

By definition $m_A = m_{A'}m_C$ and $m_B = m_{B'}m_C$, where m_C is a common multitwist and A', B' share no curves. To finish observe that $m_{A'}, m_{B'}$ are braided multitwists. Indeed, the relation $m_A m_B m_A = m_B m_A m_B$ implies

$$m_{A'}m_C m_{B'}m_C m_{A'}m_C = m_{B'}m_C m_{A'}m_C m_{B'}m_C.$$

Since m_C commutes with both $m_{A'}$ and $m_{B'}$, the equation $m_{A'}m_{B'}m_{A'} = m_{B'}m_{A'}m_{B'}$ follows immediately. □

It is worth noting that the previous lemma simplifies the proof of Theorem D to the case where the multitwists share no curves. A preliminary observation on braided multitwists sharing no curves is the following:

Lemma 1.10. *Let S be an orientable surface and consider two braided multitwists m_A, $m_B \in \mathrm{Map}(S)$. If A and B share no curves, then every $a_i \in A$ intersects at least one curve in B.*

Proof. Take $m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k}$ and $m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k}$, so that $m_A m_B \cdot a_i = b_i$.

Seeking a contradiction, suppose there is a curve $a_i \in A$ disjoint from every curve in B. Then, $b_i = m_A m_B \cdot a_i = m_A \cdot a_i = a_i$ and $a_i \in B$. But this is not possible since A and B share no curves. □

The goal of the next section is to understand how the curves in m_A and m_B may intersect each other.

1.2.1 Curves in braided multitwists

Let $m_A, m_B \in \mathrm{Map}(S)$ be two braided multitwists. In this section we describe how a curve $a \in A$ may intersect the curves in B. This description will come as a consequence of Proposition 1.5.

As in Section 1.2, we write $m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k}$ and $m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k}$, so that $m_A m_B \cdot a_i = b_i$. Using Ivanov's formula, we deduce

$$\begin{aligned} i(a_i, b_i) &= i(a_i, m_A m_B \cdot a_i) \\ &= i(a_i, m_B \cdot a_i) \\ &= \sum_{j=1}^{k} (|n_j| i(b_j, a_i) - 1) i(b_j, a_i) + X(a_i, m_B) \end{aligned} \tag{1.3}$$

In Table 1.1 we list the possible (non-negative) values of $i(a_i, b_i)$, $X(a, m_B)$ and $|n_i|$ for which Equation (1.3) is satisfied. Notice each curve $a_i \in A$ is of a single type in Table 1.1. We collect these facts in the following proposition:

Type	$i(a_i, b_i)$	$\lvert n_i\rvert$	$X(a_i, m_B)$	**Intersecting curves**
1	0	-	0	if $i(a_i, b_j) \neq 0$ and $i \neq j$, then $i(a_i, b_j) = 1$ and $\lvert n_j\rvert = 1$.
2	1	2	0	if $i(a_i, b_j) \neq 0$ and $i \neq j$, then $i(a_i, b_j) = 1$ and $\lvert n_j\rvert = 1$.
3	1	1	1	if $i(a_i, b_j) \neq 0$, then $i(a_i, b_j) = 1$ and $\lvert n_j\rvert = 1$.
4	1	1	0	There exists b_l with $i(a_i, b_l) = 1$ and $\lvert n_l\rvert = 2$. Any other $b_j \in B \setminus \{b_l\}$ with $i(a_i, b_j) \neq 0$, satisfies $i(a_i, b_j) = 1$ and $\lvert n_j\rvert = 1$.
5	2	1	0	if $i(a_i, b_j) \neq 0$ and $i \neq j$, then $i(a_i, b_j) = 1$ and$\lvert n_j\rvert = 1$.

Table 1.1: Description of curves $a_i \in A$ in braided multitwists m_A, m_B.

Proposition 1.11. *Let S be an orientable surface and consider two braided multitwists $m_A, m_B \in \mathrm{Map}(S)$. If*

$$m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k}, \quad m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k}$$

and $m_A m_B \cdot a_i = b_i$. Then, every curve $a_i \in A$ is of Type 1, 2, 3, 4 or 5 (see Table 1.1).

Notice Proposition 1.11 already provides some restrictions on the curves $A \cup B$. For instance, we immediately know that $i(a, b) \in \{0, 1, 2\}$ for any two curves $a, b \in A \cup B$.

1.2.2 Reducing braided multitwists

Lemma 1.9 reduces the proof of Theorem D to the case where braided multitwists m_A, m_B share no curves. In this section we further simplify m_A, m_B to the case where every curve in A intersects at least two curves in B. To start, the following lemma describes curves $a \in A$ that intersect a single curve in B.

Lemma 1.12. *Let S be an orientable surface and $m_A, m_B \in \mathrm{Map}(S)$ two braided multitwists. If $a \in A$ intersects only one curve $b \in B$. Then, $b = m_A m_B \cdot a$, $i(a, b) = 1$ and $\lvert n\rvert = 1$, where n is the power of a in m_A. Moreover, b is disjoint from every curve in $A \setminus \{a\}$.*

Proof. We may assume $m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k}$ and $m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k}$, so that $m_A m_B \cdot a_i = b_i$. Further assume $a = a_1$ and $n = n_1$.

If a_1 intersects a single curve, then $X(a_1, m_B) > 0$. By Proposition 1.11 the curve a_1 is of type 3. It follows that $X(a_1, m_B) = 1$, $|n_1| = 1$ and $i(a_1, b_1) = 1$. Since a_1 intersects a single curve $b \in B$, then $b = b_1$. This establishes the first part of the lemma.

Recall that $m_A m_B m_A \cdot a_i = b_i$ and $m_A m_B m_A \cdot B = A$. Therefore, if a_1 intersects a single curve in B, then b_1 intersects a single curve in A. □

If the curve a_i is under the conditions of the previous lemma, the pair a_i, b_i can be 'deleted' from the multitwists m_A, m_B. Indeed, consider

$$m_{A'} = d_{a_1}^{n_1} \dots d_{a_{i-1}}^{n_{i-1}} \cdot d_{a_{i+1}}^{n_{i+1}} \dots d_{a_k}^{n_k},$$
$$m_{B'} = d_{b_1}^{n_1} \dots d_{b_{i-1}}^{n_{i-1}} \cdot d_{b_{i+1}}^{n_{i+1}} \dots d_{b_k}^{n_k}.$$

Then, it is easy to verify that $m_{A'} m_{B'} m_{A'} = m_{B'} m_{A'} m_{B'}$.

The previous process simplifies the braided multitwists m_A, m_B by eliminating the components that are clearly braided and do not interact with any other curves in m_A, m_B. By simplifying until there are no such pairs, we obtain *reduced* multitwists:

Definition 1.13 (Reduced multitwists)**.** *Let $m_A, m_B \in \mathrm{Map}(S)$ be two braided multitwists. We will say m_A, m_B are* reduced *if every curve $a \in A$ intersects at least two distinct curves in B.*

Remark 1.14. *If m_A, m_B are reduced multitwists, then A, B have no common curves.*

A key ingredient in the proof of Theorem D is that reduced multitwists are trivial. We record this fact in the next proposition, and dedicate most of the next sections to proving it.

Proposition 1.15. *Let S be an orientable surface. If m_A, m_B are reduced multitwists, then $m_A = m_B = 1 \in \mathrm{Map}(S)$.*

To prove Proposition 1.15 we will need to understand the action of m_A, m_B on the curves $A \cup B$.

1.2.3 Action on curves and their homology classes

Let $m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k}$ and $m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k}$ be two braided multitwists. We will consider the action of $f = m_B m_A m_B$ on the curves $A \cup B$ and the action on their homology classes.

Recall that up to re-indexing B, we may assume $f \cdot a_i = b_i$. We will write a_{i_1} for the image $f \cdot b_i = a_{i_1}$ and, in general, we write a_{i_k} for the image $f \cdot b_{i_{k-1}} = a_{i_k}$. The orbits of curves will be denoted by

$$\mathrm{Orb}_f(a_i) = \{f^k \cdot a_i |\ k \in \mathbb{Z}\}.$$

The action of f by conjugation on m_A and m_B permutes the curves $A \cup B$ and preserves the powers of the associated Dehn twists. In other words, any two curves $a, b \in A \cup B$ in the same f-orbit have associated Dehn twists with the same powers in m_A, m_B. As a consequence, two curves $a, a' \in A$ in the same f-orbit have the same type in Table 1.1.

We will also consider the action of f on the homology classes of curves in $A \cup B$. To achieve this, we need to assign orientations to the curves in $A \cup B$: for each orbit $\mathrm{Orb}_f(-)$ take a representative of the orbit $a \in \mathrm{Orb}_f(-)$ and fix an arbitrary orientation on a. Then, any other curve $c \in \mathrm{Orb}_f(a)$ in the orbit inherits an induced orientation. This orientation comes from considering the smallest $k \in \mathbb{N}$ such that $f^k \cdot a = c$. Since a is oriented it induces an orientation on $f^k \cdot a$, which we consider as the orientation of c.

The selected orientations allows us to talk about homology classes $[c]$ for curves $c \in A \cup B$. The following diagram schematizes our choice of orientations:

$$[a_i] \xrightarrow{f} [b_i] \xrightarrow{f} [a_{i_1}] \xrightarrow{f} [b_{i_1}] \xrightarrow{f} [a_{i_2}] \xrightarrow{f} \ldots \xrightarrow{f} [b_{i_m}],$$

where the orientation $[a_i]$ is arbitrary.

Remark 1.16. *Notice there could be an $m \in \mathbb{N}$ such that $f^m \cdot [a_i] = -[a_i]$.*

The next step is to show that the orbit of every curve $c \in A \cup B$ has size two, i.e, $|\mathrm{Orb}_f(c)| = 2$.

1.2.4 Orbits have size two

Proposition 1.17. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. Then, $|\mathrm{Orb}_f(c)| = 2$ for every curve $c \in A \cup B$.*

To prove Proposition 1.17 we treat each type of curve $a_i \in A$ separately (see Table 1.1). Thus, this proof is split among various lemmas and propositions below.

Lemma 1.18. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. If $a_i \in A$ is a curve of type 5, then $|\mathrm{Orb}_f(a_i)| = 2$.*

Proof. Looking for a contradiction, assume $|\mathrm{Orb}_f(a_i)| > 2$. In that case, there exists $b \in \mathrm{Orb}_f(a_i)$ such that $f \cdot b = a_i$ and $b \neq b_i$. Notice

$$i(b,\, a_i) = i(f \cdot b,\, f \cdot a_i) = i(a_i,\, b_i) = 2.$$

Thus, a_i has intersection two with two distinct curves $b, b_i \in B$. But this is not possible (see Table 1.1 for type 5 curves). □

Before continuing with other cases, we require the following lemma.

Lemma 1.19. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. For any curves $a_i \in A$ and $b \in B$, we have that*

$$\hat{\imath}([a_i],[b]) = -\hat{\imath}([a_i],\, f^2 \cdot [b])$$

.

In particular, if $\hat{\imath}([a_i],[b]) = \pm 1$ then $i(a_i,\, f^{2k} \cdot b) = 1$ for every $k \in \mathbb{Z}$.

In words, a curve $a_i \in A$ has the same algebraic intersection number (up to sign) with every curve in $\mathrm{Orb}_f(b) \cap B$.

Proof. Let $a \in A$ be the curve $a = m_B m_A \cdot b$. Note that $m_B m_A \cdot [b] = \pm[a]$ and so it follows

$$0 = \hat{\imath}([a_i],[a]) = \hat{\imath}([a_i], \pm m_B m_A \cdot [b]) = \hat{\imath}([a_i], m_B m_A \cdot [b]).$$

Now, by Equation 1.2 we have

$$\begin{aligned} 0 &= \hat{\imath}([a_i], m_B m_A \cdot [b]) \\ &= \hat{\imath}\left([a_i], m_A \cdot [b] + \sum_{b_k \in B} n_k\, \hat{\imath}(m_A \cdot [b],\, [b_k])\, [b_k] \right). \end{aligned}$$

From here, linearity yields

$$\hat{\imath}([a_i],[b]) = -\sum_{b_k \in B} n_k\, \hat{\imath}(m_A \cdot [b],\, [b_k])\, \hat{\imath}([a_i],\, [b_k]). \tag{1.4}$$

On the other hand, using the braid relation

$$\begin{aligned} \hat{\imath}([a_i],\, f^2 \cdot [b]) &= \hat{\imath}([a_i],\, (m_A m_B m_A)(m_B m_A m_B) \cdot [b]) \\ &= \hat{\imath}([a_i],\, m_B m_A m_B m_A \cdot [b]) \\ &= \hat{\imath}([a_i],\, m_B^2 m_A \cdot [b]) \\ &= \hat{\imath}\left([a_i],\, m_A \cdot [b] + 2 \sum_{b_k \in B} n_k\, \hat{\imath}(m_A \cdot [b], [b_k])\, [b_k] \right) \\ &= \hat{\imath}([a_i],\, [b]) + 2 \sum_{b_k \in B} n_k\, \hat{\imath}(m_A \cdot [b], [b_k])\, \hat{\imath}([a_i], [b_k]). \end{aligned}$$

Substituting Equation 1.4 in the previous equality, we obtain

$$\hat{\imath}([a_i],\, f^2 \cdot [b]) = -\hat{\imath}([a_i],[b]).$$

For the second part of the lemma, it is enough to observe that Proposition 1.11 implies $i(a,b) \in \{0,1,2\}$ for any two curves $a, b \in A \cup B$. Thus, $\hat{\imath}(a_i, b) = \pm 1$ implies $i(a_i, b) = 1$ and, since the algebraic intersection number only changes sign, we conclude $i(a_i,\, f^{2k} \cdot b) = 1$. □

The next lemma proves Proposition 1.17 for type 2 curves $a_i \in A$.

Lemma 1.20. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. If $a_i \in A$ is a type 2 curve, then $|\operatorname{Orb}_f(a_i)| = 2$.*

Proof. Recall we denoted $b_i = f \cdot a_i$, $a_{i_1} = f \cdot b_i$ and $b_{i_1} = f \cdot a_{i_1}$. To proceed by contradiction, we assume $|\operatorname{Orb}_f(a_i)| > 2$. In this case, $b_{i_1} \neq b_i$.

Since a_i is a type 2 curve, then $i(a_i, b_i) = 1$ and it follows from Lemma 1.19 that $i(a_i, b_{i_1}) = 1$. Moreover, a_i, b_i and b_{i_1} have exponent $|n_i| = 2$ in m_A, m_B, since a_i is of type 2 and the three curves are in the same orbit. Summarizing, $a_i \in A$ is a curve that intersects two distinct curves b_i, $b_{i_1} \in B$ both having exponent $|n_i| = 2$ in m_B. However, such a_i cannot be a curve in a braided multitwist (see Table 1.1). □

From here, the cases of Proposition 1.17 become more subtle. The next result deals with type 1 curves.

Lemma 1.21. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. If $a_i \in A$ is a type 1 curve, then $|\operatorname{Orb}_f(a_i)| = 2$.*

Proof. Consider the curve $a_{i_1} = f^2 \cdot a_i$ and the set of curves $B_i = \{b \in B |\, i(a_i, b) \neq 0\}$. Given that a_i is a type 1 curve, we know every $b \in B_i$ has exponent ± 1 in m_B and satisfies $i(a_i, b) = 1$ (see Table 1.1). Furthermore, $X(a_i, m_B) = 0$ and it follows that $|B_i|$ is even.

Let T_i be an open regular neighborhood of $a_i \cup \bigcup_{b \in B_i} b$. Note the subsurface T_i is homeomorphic to a torus with punctures. Since $X(a_i, m_B) = 0$, if two curves $\overline{b}_j, \overline{b}_{j+1} \in B_i$ bound a once punctured annulus in T_i, then their powers in m_B have opposite signs. We represent T_i as a planar torus in Figure 1.3, where we denote $\overline{b}_j$ the curves in B_i and the exponents correspond to those in m_B.

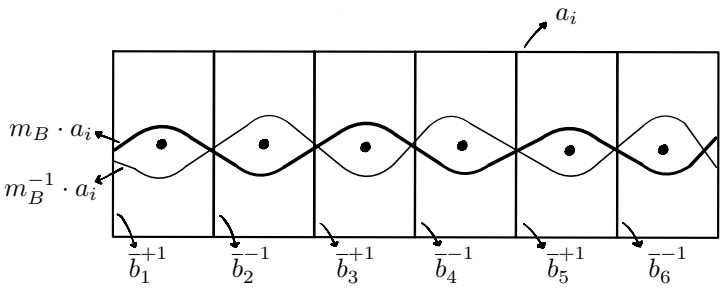


Figure 1.3: Planar torus T_i for $i(a_i, b_i) = 0$.

Given that a_i is a type 1 curve, we have that $i(a_i, b) \in \{0, 1\}$ for every $b \in B$. Since a_{i_1} is in the orbit of a_i, then a_{i_1} is also of type 1 and $i(a_{i_1}, b) \in \{0, 1\}$ for every $b \in B$. It is now a consequence of Lemma 1.19 that $i(a_i, b) =$

$i(a_{i_1}, b)$ for every $b \in B$. In particular, we know that $i(a_{i_1}, b) = i(a_i, b) = i(a_i, b_i) = 0$ for every $b \in \mathrm{Orb}_f(a_i)$. This implies

$$\begin{cases} i(a_{i_1},\, m_B \cdot a_i) = i(a_{i_1},\, b_i) = 0, \\ i(a_{i_1},\, m_B^{-1} \cdot a_i) = i(a_i,\, m_B \cdot a_{i_1}) = i(a_i, b_{i_1}) = 0. \end{cases}$$

Notice the conditions ensure that a_{i_1} is either contained in T_i or disjoint from T_i. Indeed, any curve c that intersects T_i and is not contained in T_i either satisfies $i(c,\, m_B \cdot a_i) \neq 0$ or $i(c,\, m_B^{-1} \cdot a_i) \neq 0$ (see Figure 1.3). Now, $i(a_{i_1},\, \overline{b}_1) = i(a_i, \overline{b}_1) = 1$ implies a_{i_1} intersects T_i and therefore a_{i_1} is contained in T_i.

To finish, we note that the only curve a_{i_1} in T_i satisfying $i(a_{i_1},\, m_B \cdot a_i) = i(a_{i_1},\, m_B^{-1} \cdot a_i) = 0$ is the curve $a_{i_1} = a_i$. So we conclude $|\,\mathrm{Orb}_f(a_i)| = 2$. □

Next lemma proves Proposition 1.17 for type 4 curves.

Lemma 1.22. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. If $a_i \in A$ is a type 4 curve, then $|\,\mathrm{Orb}_f(a_i)| = 2$.*

Proof. The strategy of the proof is analogous to that of Lemma 1.21. Consider $a_{i_1} = f^2 \cdot a_i$ and the set of curves $B_i = \{b \in B |\ i(b, a_i) \neq 0\}$. Since a_i is of type 4, it follows that $i(b, a_i) = 1$ for every $b \in B_i$, that $|B_i|$ is even and every curve in B_i has power ± 1 in m_B, except for a single curve which has power ± 2 (see Table 1.1).

Let T_i be an open regular neighborhood of $a_i \cup \bigcup_{b \in B_i} b$. The above paragraph implies T_i is a torus with punctures. Since $X(a_i,\, m_B) = 0$, then two curves $\overline{b}_j, \overline{b}_k \in B_i$ bounding a single punctured annulus in T_i have powers of opposite sign in m_B. In Figure 1.4 we represent T_i as a planar torus, where we label $\overline{b}_j$ the curves in B_i and the exponents correspond to those in m_B.

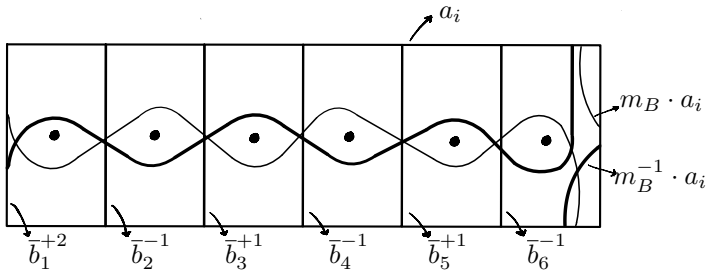


Figure 1.4: Planar torus T_i for $i(a_i, b_i) = 1$ and $X(a_i, m_B) = 0$.

Given that a_i and a_{i_1} are type 4 curves, then $i(a_i, b), i(a_{i_1}, b) \in \{0, 1\}$ for every $b \in B$. Thus, Lemma 1.19 implies $i(a_i, b) = i(a_{i_1}, b)$ for every $b \in B$. It follows a_{i_1} is a curve satisfying

$$\begin{cases} i(a_{i_1},\, \overline{b}_j) = 1, \\ i(a_{i_1},\, a_i) = 0, \\ i(a_{i_1},\, m_B \cdot a_i) = i(a_{i_1},\, b_i) = 1, \\ i(a_{i_1},\, m_B^{-1} \cdot a_i) = i(a_i,\, m_B \cdot a_{i_1}) = i(a_i, b_{i_1}) = 1. \end{cases}$$

The only curve satisfying above equations is $a_{i_1} = a_i$ (see Figure 1.4). One may check this fact by considering the point of intersection between a_{i_1} and $\overline{b}_k$, where $\overline{b}_k \in B_i$ is the only curve in B_i with exponent two. Then, by inspection, it is clear a_{i_1} is contained in T_i. Readily, we obtain that a_{i_1} and a_i are homotopic, that is, $a_{i_1} = a_i$. Thus, $|\operatorname{Orb}_f(a_i)| = 2$. □

Lastly, we consider the longer case of type 3 curves. We work through this case in the next three lemmas.

Lemma 1.23. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. If $a_i \in A$ is a type 3 curve and $|\operatorname{Orb}_f(a_i)| > 2$, then there exist $b_k \in B$ that intersects a_i and satisfies $|\operatorname{Orb}_f(b_k)| = 2$. Also, any other $b \in B \setminus \{b_k\}$ intersecting a_i satisfies $|\operatorname{Orb}_f(b)| = 4$.*

Proof. Let $B_i = \{b \in B |\, i(a_i, b) \neq 0\}$ and T_i an open regular neighborhood of the curves $a_i \cup \bigcup_{b \in B_i} b$. Given that a_i is of type 3, we deduce $|B_i|$ is odd (see Table 1.1) and T_i is a torus with $|B_i|$ punctures.

The intersection points of a_i with curves in B_i induce a cyclic order in B_i. Considering this order, we may write $\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_{|B_i|}$ to denote the curves in B_i. Since $X(a_i,\, m_B) = 1$, we can arrange the subindices in $\overline{b}_j$ so that the powers in m_B alternate sign. Even though the signs alternate, the curves $\overline{b}_1$ and $\overline{b}_{|B_i|}$ have the same sign since $|B_i|$ is odd. In Figure 1.5 we represent T_i as a planar torus.

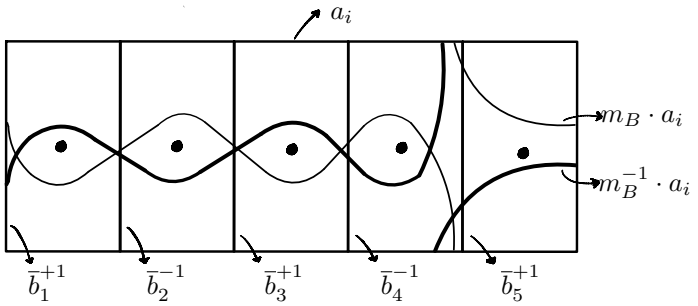


Figure 1.5: The planar torus T_i for $i(a_i, b_i) = 1$ and $X(a_i, m_B) = 1$.

Let $a_{i_1} = f^2 \cdot a_i \in A$. By hypothesis $a_{i_1} \neq a_i$. As in previous lemmas, we may use Proposition 1.11 and Lemma 1.19 to check that a_{i_1} satisfies the

equations

$$\begin{cases} i(a_{i_1}, a_i) = 0, \\ i(a_{i_1}, b) = i(a_i, b) \qquad \forall b \in B, \\ i(a_{i_1}, m_B \cdot a_i) = 1, \\ i(a_{i_1}, m_B^{-1} \cdot a_i) = 1. \end{cases}$$

From these conditions it follows that either $a_{i_1} = a_i$ or $a_{i_1} \cap T_i$ has a representative as in Figure 1.6. Namely, a_{i_1} has a representative in minimal position with curves in $A \cup B$, therefore disjoint from a_i and intersecting once every $b \in B_i$; also, $a_{i_1} \cap T_i$ is an arc with endpoints in a single puncture of T_i and $T_i \setminus (a_i \cup a_{i_1})$ is the union of an annulus and a $|B_i| - 1$ punctured annulus. To check that $a_{i_1} = a_i$ or $a_{i_1} \cap T_i$ has a representative as in Figure 1.6, we may consider the point of intersection of a_{i_1} with $\overline{b}_{|B_i|}$ (see Figure 1.5). Then, by inspection, we see that a curve a_{i_1} satisfying above equations must be as claimed.

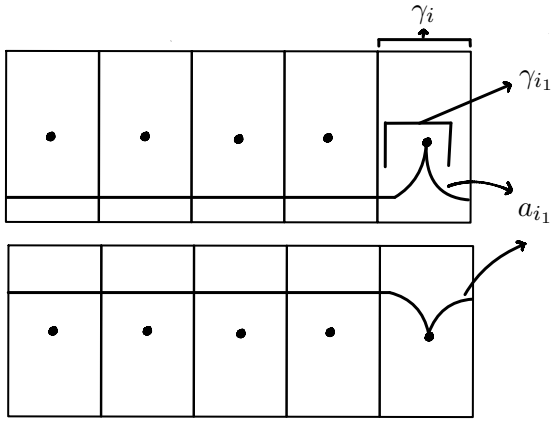


Figure 1.6: The two possible arcs for $a_{i_1} \cap T_i$.

(We warn the reader that in the next paragraphs we silently use the Alexander Method [32, Proposition 2.8]. We are taking representatives of a_i, a_{i_1}, every $b \in B_i$ and a representative of f^2, so that f^2 permutes the curves $b \in B_i$ and sends a_i to a_{i_1}.)

Now, denote γ_i the subarc of a_i that goes from $a_i \cap \overline{b}_{|B_i|}$ to $a_i \cap \overline{b}_1$ and is disjoint from the rest of $\overline{b}_j$'s. In the same style, define γ_{i_1} to be the subarc of a_{i_1} going from $a_{i_1} \cap \overline{b}_{|B_i|}$ to $a_{i_1} \cap \overline{b}_1$ and is disjoint from the rest of $\overline{b}_j$'s (see Figure 1.6). Observe that $f^2 \cdot \gamma_i$ is a subarc of a_{i_1} with interior disjoint from the curves $\overline{b}_j$'s. The endpoints of $f^2 \cdot \gamma_i$ are points of intersection of a_{i_1} with $f^2 \cdot \overline{b}_1$ and with $f^2 \cdot \overline{b}_{|B_i|}$. Since $\overline{b}_1$ and $\overline{b}_{|B_i|}$ have the same power in m_B and f^2 preserves the exponents, the curves $f^2 \cdot \overline{b}_1$ and $f^2 \cdot \overline{b}_{|B_i|}$ have the same exponent. But the only subarc of a_{i_1} as above is γ_{i_1}, that is, γ_{i_1} is the only subarc of a_{i_1} between consecutive intersection points such that the

intersecting curves have the same power. Thus, $f^2 \cdot \gamma_i = \gamma_{i_1}$ and either f^2 fixes $\overline{b}_1$ and $\overline{b}_{|B_i|}$, or it interchanges them.

If f^2 fixes both $\overline{b}_1$ and $\overline{b}_{|B_i|}$, we have that f^2 fixes every $\overline{b}_j$. To see this, consider the subarc of a_i from $\overline{b}_1$ to $\overline{b}_2$ (disjoint from every other $\overline{b}_j$). Naturally, f^2 must send this subarc to the subarc of a_{i_1} from $\overline{b}_1$ to $\overline{b}_2$. Since f^2 fixes $\overline{b}_1$, then it also fixes $\overline{b}_2$. By repeating the argument on adjacent subarcs, we conclude f^2 fixes every $\overline{b}_j$. However, this implies $|\operatorname{Orb}_f(\overline{b}_j)| = 2$ for every $\overline{b}_j \in B_i$ and, since $b_i \in B_i$, this contradicts the hypothesis $|\operatorname{Orb}_f(a_i)| > 2$. Thus, f^2 cannot fix $\overline{b}_1$ and $\overline{b}_{|B_i|}$.

If, on the contrary, f^2 interchanges $\overline{b}_1$ and $\overline{b}_{|B_i|}$, i.e, $f^2 \cdot \overline{b}_1 = \overline{b}_{|B_i|}$ and $f^2 \cdot \overline{b}_{|B_i|} = \overline{b}_1$. Again, using the previous argument on subarcs, it is an easy exercise to check that $f^2 \cdot \overline{b}_j = \overline{b}_{|B_i|+1-j}$ and $f^2 \cdot \overline{b}_{|B_i|+1-j} = \overline{b}_j$. Therefore, $|\operatorname{Orb}_f(b)| = 4$ for every $b \in B_i \setminus \{\overline{b}_{\frac{|B_i|+1}{2}}\}$ and $|\operatorname{Orb}_f(\overline{b}_{\frac{|B_i|+1}{2}})| = 2$. The b_k of the statement corresponds to $b_k = \overline{b}_{\frac{|B_i|+1}{2}}$. □

Lemma 1.24. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. If $a_i \in A$ is a type 3 curve and $|\operatorname{Orb}_f(a_i)| > 2$, then there exists a unique $b_k \in B$ that intersects a_i and such that $|\operatorname{Orb}_f(b_k)| = 2$. Furthermore, let a_k be the curve $f \cdot a_k = b_k$. Then $i(a_i, b) = i(a_k, b)$ for every $b \in B$.*

Proof. The first part of the lemma follows from Lemma 1.23. We only need to see that $i(a_i, b) = i(a_k, b)$ for every $b \in B$.

Denote $B_i = \{b \in B |\, i(b, a_i) \neq 0\}$ and $B_k = \{b \in B |\, i(b, a_k) \neq 0\}$. First, we will see $B_i \subset B_k$:

Let $b_j \in B_i$. From Table 1.1, we know that $i(a_i, b_j) = 1$. Now, using Lemma 1.19 we get $i(b_i, a_j) = 1$ and thus

$$\begin{aligned} \pm 1 &= \hat{\imath}([b_i], [a_j]) \\ &= \hat{\imath}(m_A m_B \cdot [a_i], [a_j]) \\ &= \hat{\imath}(m_B \cdot [a_i], [a_j]) \\ &= \sum_{b_l \in B} n_l \, \hat{\imath}([a_i], [b_l]) \, \hat{\imath}([b_l], [a_j]) \\ &= \sum_{b_l \in B_i \cap B_j} n_l \, \hat{\imath}([a_i], [b_l]) \, \hat{\imath}([b_l], [a_j]). \end{aligned} \tag{1.5}$$

Now, considering Lemma 1.23, we can partition the set B_i as

$$B_i = B \cap \left(\operatorname{Orb}_f(\overline{b}_2) \sqcup \operatorname{Orb}_f(\overline{b}_3) \sqcup \cdots \sqcup \operatorname{Orb}_f(\overline{b}_{\frac{|B_i|-1}{2}}) \sqcup \{b_k\} \right),$$

where each orbit has two curves in B (except for $\{b_k\}$). This induces a partition of $B_i \cap B_j$ as disjoint union of $\operatorname{Orb}_f(\overline{b}_m) \cap B_j$ and $\{b_k\} \cap B_j$. We

are looking to prove $b_k \in B_j$, so seeking a contradiction assume $b_k \notin B_j$. By means of Lemma 1.19, we have

$$\begin{aligned} n_l\, \hat{\imath}([a_i],[b_l])\, \hat{\imath}([b_l],[a_j]) &= n_l\, \hat{\imath}([a_i], f^2\cdot[b_l])\, \hat{\imath}(f^2\cdot[b_l],[a_j]) \\ &= \pm n_l\, \hat{\imath}([a_i],[f^2\cdot b_l])\, \hat{\imath}([f^2\cdot b_l],[a_j]). \end{aligned}$$

Note that for $\overline{b}_m \neq b_k$ each orbit $B_j \cap \mathrm{Orb}_f(\overline{b}_m)$ has two elements, which contribute either as $2\cdot n_l\, \hat{\imath}([a_i],[b_l])\, \hat{\imath}([b_l],[a_j])$ or as zero to the sum. Therefore, if $b_k \notin B_j$ the sum

$$\sum_{b_l \in B_i \cap B_j} n_l\, \hat{\imath}([a_i],[b_l])\, \hat{\imath}([b_l],[a_j])$$

is even. But this clearly contradicts that it is equal to ± 1 (see Equation 1.5). Hence, $b_k \in B_j$ and even more the argument yields that $n_k \hat{\imath}([a_i],\,[b_k])\, \hat{\imath}([b_k],[a_j]) = \pm\hat{\imath}([b_k],[a_j])$ is odd. Given that $i(b_k,\, a_j) \in \{0,1,2\}$, it follows that $i(b_k,\, a_j) = 1$ and by Lemma 1.19 we have $i(a_k, b_j) = 1$. We conclude $1 = i(a_i, b) = (a_k, b)$ for every $b \in B_i$ and therefore $B_i \subset B_k$.

To complete the lemma, we need to see $B_k \subset B_i$. Pursuing a contradiction, assume there exists $b_j \in B$ with $i(a_k, b_j) \neq 0$ and $i(a_i, b_j) = 0$. First, observe that $i(a_i, b_i) = i(a_i, b_k) = 1$ implies $j \neq i$ and $j \neq k$. Now, $i(a_k, b_j) = 1$ follows from $j \neq k$ (see Table 1.1). Moreover, $0 = i(a_i,\, b_j) = i(b_i,\, a_{j_1})$ and by Lemma 1.19 we have $\hat{\imath}([b_i],\,[a_j]) = 0$. After this, consider

$$\begin{aligned} 0 &= \hat{\imath}([b_i],\,[a_j]) \\ &= \hat{\imath}(m_A m_B\cdot[a_i],\,[a_j]) \\ &= \sum_{b_l \in B_i} n_l\, \hat{\imath}([a_i],[b_l])\, \hat{\imath}([b_l],[a_j]) \\ &= \sum_{b_l \in B_i \cap B_j} n_l\, \hat{\imath}([a_i],[b_l])\, \hat{\imath}([b_l],[a_j]). \end{aligned}$$

Replicating the argument above yields that the previous sum is odd. Indeed, each orbit $\mathrm{Orb}_f(\overline{b}_m) \cap B_j$ contributes an even number to the sum, plus

$$n_k\, \hat{\imath}([a_i],[b_k])\, \hat{\imath}([b_k],[a_j]) = \pm 1.$$

But the sum cannot be odd and equal to zero. Thus, no such b_j can exist. In other words, $B_k = B_i$ and it follows that $i(a_i,\, b) = i(a_k,\, b)$ for every $b \in B$. □

Finally, we prove Proposition 1.17 for type 3 curves.

Lemma 1.25. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. If $a_i \in A$ is a type 3 curve, then $|\,\mathrm{Orb}_f(a_i)| = 2$.*

Proof. Assume $|\operatorname{Orb}_f(a_i)| > 2$. By Lemma 1.24, there exist $b_k \in B_i$ with $|\operatorname{Orb}_f(a_k)| = 2$ and $i(a_k, b) = i(a_i, b)$ for every $b \in B$. Since $|B_i| = |B_k|$ is odd, we know from Table 1.1 that a_k is a type 3 curve and so $i(a_k, b_k) = 1$, $|n_k| = 1$ and $X(a_k, m_B) = 1$.

Take any curve $b_j \in B_k \backslash \{b_k\}$, from Lemma 1.23 we have that $|\operatorname{Orb}_f(b_j)| = 4$. Notice that a_j is a type 3 curve, since lemmas 1.18, 1.20, 1.21 and 1.22 rule out any other type in Table 1.1. Now, considering Lemma 1.24, we deduce $i(a_k, b) = i(a_j, b)$ for every $b \in B$. As a consequence, any two curves $b_j, b_l \in B_k = B_i$ satisfy $i(a_j, b) = i(a_l, b)$ for every $b \in B$.

We continue by considering T_k an open regular neighborhood of the curves $a_k \cup \bigcup_{b \in B_k} b$ (see Figure 1.7). For any $b_j \in B_k$ consider the curve a_j such that $f \cdot a_j = b_j$. By the previous paragraph, we know a_j satisfies the equations

$$\begin{cases} i(a_j, a_k) = 0, \\ i(a_j, b) = i(a_k, b) \qquad \forall b \in B, \\ i(a_j, m_B \cdot a_k) = 1, \\ i(a_j, m_B^{-1} \cdot a_k) = 1. \end{cases}$$

Since a_j satisfies the same conditions as a_{i_1} in the proof of Lemma 1.23, we may use the same argument to find a representative of a_j disjoint from a_k such that $a_j \cap T_k$ is an arc with endpoints in a single puncture, and $T_k \setminus (a_k \cup a_j)$ is the union of an annulus and a $|B_k| - 1$ punctured annulus. Simpler, $a_j \cap T_k$ is as in Figure 1.7.

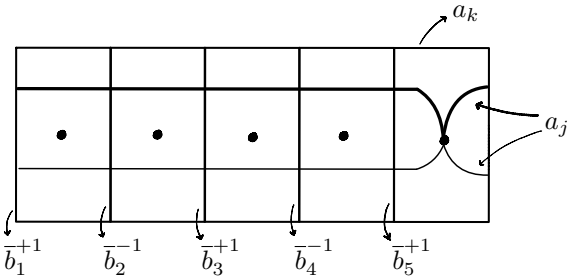


Figure 1.7: The planar torus T_k. The thick line and the thin line represent the two possible arcs for $a_j \cap T_i$.

We emphasize that $a_j \cap T_k$ is as in Figure 1.7 for any curve $a_j \in A$ with $f \cdot a_j = b_j \in B_k$. By taking appropriate representatives of each a_j, we may assume that $\overline{b}_1$ and $\overline{b}_{|B_i|}$ induce the same cyclic order on the a_j's (see Figure 1.8).

Now, consider T the regular neighborhood of $\bigcup_{b_j \in B_k} b_j \cup \bigcup_{b_j \in B_k} a_j$. The neighborhood T is homeomorphic to a torus with punctures. We represent T as a planar torus in Figure 1.9, where we denote $\overline{a}_j$ the curves that satisfy $f \cdot \overline{a}_j = \overline{b}_j$. To check that T is a torus with punctures, it is enough to consider a regular neighborhood of the curves as depicted in Figure 1.8.

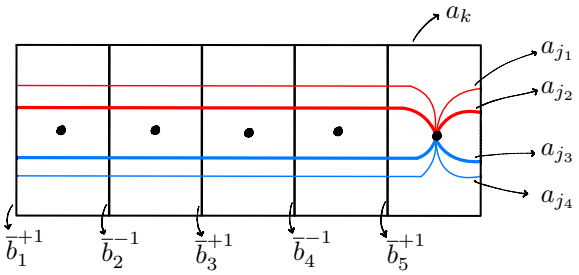


Figure 1.8: The planar torus T_k. The arcs $a_{j_l} \cap T_k$ for $a_{j_l} \in A$ are represented with different colors or thickness.

Recall that the intersection of a_k with curves $\overline{b}_j$ induces the cyclic order $\overline{b}_1, \dots, \overline{b}_{|B_i|}$. In the same fashion, the intersection of b_k with curves $\overline{a}_j$ induces the cyclic order $\overline{a}_1, \dots, \overline{a}_{|B_i|}$ (see Figure 1.9). This simply follows from the fact that $f \cdot \overline{a}_j = \overline{b}_j$ and $f \cdot a_k = b_k$.

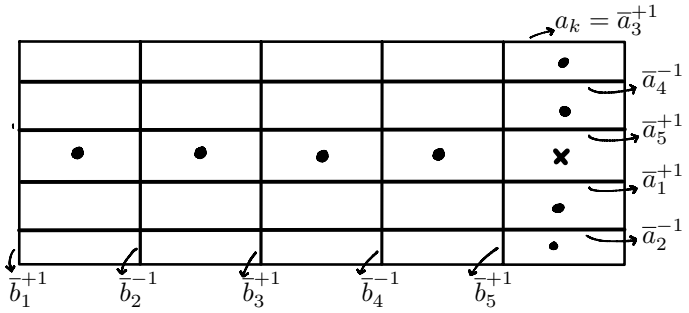


Figure 1.9: The planar torus T. The puncture marked with a cross might be a disk.

Finally, one may directly check that $i(m_B m_A \cdot \overline{b}_{|B_i|}, \overline{a}_1) \neq 0$ by taking a representative of $m_B m_A \cdot \overline{b}_{|B_i|}$ in T and using the bigon criterion. This clearly yields a contradiction, as $m_B m_A \cdot \overline{b}_{|B_i|} \in A$ and any two curves in A are disjoint. □

Collecting the previous lemmas we conclude $|\operatorname{Orb}_f(a_i)| = 2$ for any curve $a_i \in A$. This ends up the proof of Proposition 1.17.

Proof of Proposition 1.17. Consider a curve $c \in A \cup B$. To prove $|\operatorname{Orb}_f(c)| = 2$, it is enough to prove it for $c \in A$. By Proposition 1.11 the curve c is of type 1, 2, 3, 4 or 5, so it satisfies the conditions of either Lemma 1.18, 1.20, 1.21, 1.22 or 1.25. In any case, the conclusion is $|\operatorname{Orb}_f(c)| = 2$. □

1.2.5 Reduced multitwists are trivial

We are close to proving Proposition 1.15. Before doing so, we devote the next lemmas to understand three concrete cases.

Lemma 1.26. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. If $a_i \in A$ is a type 2 curve, then a_i intersects at least three distinct curves in B.*

Proof. Seeking a contradiction, assume a_i intersects at most two curves in B. Since the multitwists are reduced, a_i intersects exactly two curves in B.

Take $b_i, b_j \in B$ to be the two curves that intersect a_i. We may assume without loss of generality that $n_i = 2$ and $n_j = -1$ (see Figure 1.10). We have that

$$\begin{aligned}
\pm 1 &= \hat{\imath}(m_A m_B \cdot [a_i],\, [a_j]) \\
&= \hat{\imath}(m_B \cdot [a_i], [a_j]) \\
&= \hat{\imath}([a_i] + 2\hat{\imath}([a_i], [b_i])\,[b_i] - \hat{\imath}([a_i], [b_j])\,[b_j],\, [a_j]) \\
&= 2\hat{\imath}([a_i], [b_i])\,\hat{\imath}([b_i], [a_j]) - \hat{\imath}([a_i], [b_j])\,\hat{\imath}([b_j], [a_j]) \\
&= \pm 2 \pm \hat{\imath}([b_j], [a_j]).
\end{aligned}$$

Notice that for the previous equation to be satisfied, it is required that $i(b_j, a_j) = 1$.

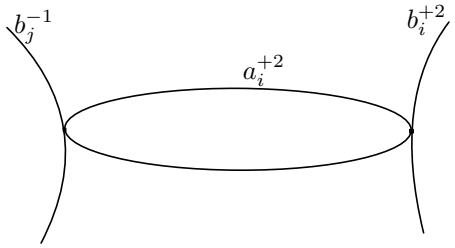


Figure 1.10: Regular neighborhood of $i(a_i, b_i) = 1$ with $|n_i| = 2$ such that a_i intersects only two curves in B.

To finish, consider T_i a regular neighborhood of $a_i \cup b_i \cup b_j$. We represent T_i as a planar torus in Figure 1.11. Now, we know a_j satisfies the equations

$$\begin{cases}
i(a_j,\, b_i) = 1, \\
i(a_j,\, a_i) = 0, \\
i(a_j,\, m_B \cdot a_i) = 1, \\
i(a_j,\, m_B^{-1} \cdot a_i) = 1.
\end{cases}$$

These conditions imply that a_j is a curve contained in T_i. But the only curve in T_i satisfying above equations is a_i, so $a_j = a_i$. However, this is impossible since we assumed $b_j \neq b_i$.

□

Lemma 1.27. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. If $a_i \in A$ is a type 1 curve, then a_i intersects at least three curves in B.*

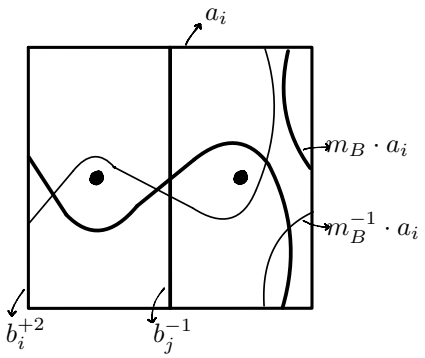


Figure 1.11: Planar torus T_i with $i(a_i, b_i) = 1$ and $|n_i| = 2$, such that a_i intersects only two curves in B.

Proof. Seeking a contradiction, assume a_i intersects at most two curves in B. Since the multitwists are reduced, a_i intersects exactly two curves in B.

Take $b_j, b_k \in B$ to be the two curves that intersect a_i. Without loss of generality, we may assume the Dehn twists in m_B have exponents $n_i = n_k = 1$ and $n_j = -1$ (see Figure 1.12).

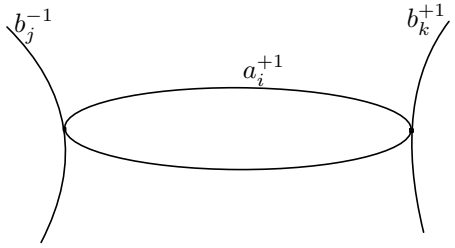


Figure 1.12: Regular neighborhood of a_i with $i(a_i, b_i) = 0$ such that a_i intersects exactly two curves in B.

Let T_i be a regular neighborhood of $a_i \cup b_j \cup b_k$. The subsurface T_i is homeomorphic to a torus with two punctures. We represent T_i as a planar torus in Figure 1.13. Note that a_j satisfies the equations

$$\begin{cases} i(a_j, m_B \cdot a_i) = 1, \\ i(a_j, m_B^{-1} \cdot a_i) = 1. \end{cases}$$

Any curve a_j satisfying the conditions above is either a_i or is a curve disjoint from T_i (see Figure 1.13). If $i(a_j, b_j) \neq 0$, then a_j intersects T_i and so $a_j = a_i$. However, this is not possible since $b_j \neq b_i$. Therefore, we can assume $i(a_j, b_j) = 0$.

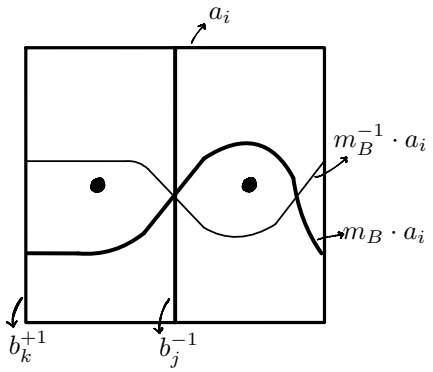


Figure 1.13: Regular neighborhood of $i(a_i, b_i) = 0$, where a_i intersects two curves in B.

Notice that

$$\begin{aligned}
\pm 1 &= \hat{\imath}(m_B \cdot [a_i],\, [a_j]) \\
&= \hat{\imath}([a_i] + \hat{\imath}([a_i],[b_k])\,[b_k] - \hat{\imath}([a_i],[b_j])\,[b_j],\, [a_j]) \\
&= \hat{\imath}([a_i],[b_k])\,\hat{\imath}([b_k],[a_j]) - \hat{\imath}([a_i],[b_j])\,\hat{\imath}([b_j],[a_j]) \\
&= \pm\hat{\imath}([b_k],[a_j]) \pm \hat{\imath}([b_j],[a_j]).
\end{aligned}$$

If $i(a_j, b_j) = 0$ then the previous equation implies $i(a_j, b_k) = 1$. Again, it follows that a_j is not disjoint from T_i and so we have the contradiction $a_i = a_j$. □

Lemma 1.28. *Consider two reduced multitwists m_A, m_B and let $f = m_A m_B m_A$. If $a_i \in A$ is a type 5 curve and a_i intersects at most three curves in B, then there exists $a_j \in A$ such that:*

- *either a_j is a type 1 curve,*
- *or $i(a_j, b_j) = 1$ (type 2, 3 or 4),*
- *or $i(a_j, b_j) = 2$ and a_j intersects at least four curves in B.*

Proof. Since m_A, m_B are reduced, a_i must intersect at least two curves. Using that $i(a_i,\, b_i) = 2$ and $X(a_i,\, m_B) = 0$ (see Table 1.1 for type 5 curves), we deduce a_i intersects exactly three curves.

Let $b_i, b_j, b_k \in B$ be the curves intersecting a_i. We may assume that

$n_j = n_k = -1$ and $n_i = 1$. Now, we have

$$\begin{aligned}
\pm 1 &= \hat{\imath}([a_j],[b_i]) \\
&= \hat{\imath}([a_j], m_B \cdot [a_i]) \\
&= \hat{\imath}([a_j],[a_i] + \hat{\imath}([a_i],[b_i])[b_i] - \hat{\imath}([a_i],[b_j])[b_j] - \hat{\imath}([a_i],[b_k])[b_k]) \\
&= \hat{\imath}([a_i],[b_i])\,\hat{\imath}([a_j],[b_i]) - \hat{\imath}([a_i],[b_j])\,\hat{\imath}([a_j],[b_j]) - \hat{\imath}([a_i],[b_k])\,\hat{\imath}([a_j],[b_k]) \\
&= \pm\hat{\imath}([a_i],[b_i]) \pm \hat{\imath}([a_j],[b_j]) \pm \hat{\imath}([a_j],[b_k]).
\end{aligned}$$

If $i(a_j, b_j) \in \{0, 1\}$, then we already have the statement. Suppose that $i(a_j, b_j) = 2$.

Using that $i(a_i, b_i) = 2$, the previous equation implies that $i(a_j, b_k) = 1$. Since $i(a_j, b_j) = 2$, $X(a_j, m_B) = 0$ and $n_k = n_j = -1$, it follows that a_j must intersect at least four curves in B as otherwise $X(a_j, m_B) \neq 0$. □

Lastly, we are ready to prove Proposition 1.15.

Proof of Proposition 1.15. Assume m_A, m_B are non-trivial and let $f = m_A m_B m_A$. For each curve $a_i \in A$ consider the subset $B_i \subset B$ of curves intersecting a_i. Since m_A is non-trivial, there exists an $a_i \in A$.

If $i(a_i,\, b_i) \leq 1$ and $|B_i| \geq 3$, consider the graph $\mathcal{G}$ isomorphic to the union of curves $a_i \cup \bigcup_{b \in B_i} b$. Recall that Proposition 1.17 implies that $f^2 \cdot c = c$ for every $c \in A \cup B$, so f^2 defines an automorphism of the graph $\mathcal{G}$ by means of the Alexander method. Even more, f^2 fixes every vertex in $\mathcal{G}$ and so it follows that f^2 preserves the orientation of a_i. In other words, $f^2 \cdot [a_i] = [a_i]$. Notice there exists $b \in B_i$ with $i(a_i, b) = 1$ and

$$\hat{\imath}([a_i],[b]) = \hat{\imath}(f^2 \cdot [a_i], f^2 \cdot [b]) = \hat{\imath}([a_i], f^2 \cdot [b]),$$

but this clearly contradicts Lemma 1.19. As a result, there cannot exist $a_i \in A$ with $i(a_i,\, b_i) \leq 1$ and $|B_i| \geq 3$.

Suppose $a_i \in A$ is a curve with $i(a_i,\, b_i) = 2$ and $|B_i| \geq 4$, then the previous argument yields the same contradiction. For $i(a_i, b_i) = 2$, we emphasize $|B_i| \geq 4$ is used to ensure f^2 fixes every vertex in $\mathcal{G}$.

To finish, we list the rest of the cases according to Proposition 1.11:

- If a_i is a type 1 curve, then Lemma 1.27 ensures $|B_i| \geq 3$ and we already proved this leads to contradiction.
- If a_i is a type 2 curve, then Lemma 1.26 implies $|B_i| \geq 3$. Again, we know this leads to contradiction.
- If a_i is a type 4 curve, then Table 1.1 guarantees the existence of another curve $a_j \in A$ of type 2. However, this puts us in the previous case, that has already been discarded.

- If a_i is a type 3 curve, then $|B_i|$ is odd. Since the multitwists are reduced, it follows that $|B_i| \geq 3$ and we already checked this case drives to contradiction.

- At last, if a_i is a type 5 curve and $|B_i| \leq 3$, then Lemma 1.28 implies the existence of an a_j of type 1, 2, 3, 4, or 5, and if a_j is of type 5 then $|B_j| \geq 4$. However, we proved in the previous cases that such a_j cannot exist. Thus, a_i of type 5 and $|B_i| \leq 3$ also leads to contradiction.

We have seen that the existence of $a_i \in A$ leads to a contradiction independently of its type. Therefore, A is empty and so is B. In other words, $m_A = m_B = 1 \in \mathrm{Map}(S)$. □

With this in hand, we may quickly prove Theorem D.

Proof of Theorem D. Consider two braided multitwists m_A, m_B. Following Section 1.2 we can write

$$m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k},$$
$$m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k}.$$

where $m_A m_B m_A \cdot a_i = b_i$. Additionally, we write $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$. Using Lemma 1.9, we may assume that A and B share no curves and by Lemma 1.10 we may assume every curve $a \in A$ intersects some curve $b \in B$.

Let $a_i \in A$ be a curve intersecting a single curve b in B. In this case, Lemma 1.12 guarantees $b = b_i = f \cdot a_i$, $i(a_i, b_i) = 1$ and $|n_i| = 1$. Moreover, we can decompose $m_A = m_{A'} \circ d_{a_i}^{n_i}$ and $m_B = m_{B'} \circ d_{b_i}^{n_i}$, so that $m_{A'}, m_{B'}$ are still braided and share no curves.

Repeating the above process until every curve $a_i \in A'$ intersects at least two curves in B', leaves us with two reduced multitwists $m_{A'}, m_{B'}$. But by Proposition 1.15, we have that $m_{A'} = m_{B'} = 1$. Thus, by Lemma 1.12 we conclude that the curves $a_1, \dots, a_k, b_1, \dots, b_k$ are pairwise disjoint except for $i(a_i, b_i) = 1$ and that $n_i \in \{-1, 1\}$, just as we claimed in Theorem D. □

Chapter 2

Classification of homomorphisms

A central objective in the study of mapping class groups is to connect the geometry of the surface S to the algebraic structure of the group $\mathrm{PMap}(S)$. In this chapter, we focus on homomorphisms $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$ between pure mapping class groups. Our first result relates these homomorphisms to maps between the diffeomorphism groups.

Let $\mathrm{Diff}_c^+(S)$ be the group of orientation preserving diffeomorphisms of S compactly supported on $S \setminus \partial S$, and let $\pi : \mathrm{Diff}_c^+(\cdot) \to \mathrm{PMap}(\cdot)$ be the map projecting each diffeomorphism to its homotopy class. It is a question of Aramayona and Souto [12, Question 5] whether every homomorphism $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ admits a lift $\overline{\varphi}$ to the diffeomorphism groups, that is, whether we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Diff}_c^+(S) & \overset{\overline{\varphi}}{\dashrightarrow} & \mathrm{Diff}_c^+(S') \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathrm{PMap}(S) & \xrightarrow{\varphi} & \mathrm{PMap}(S'). \end{array}$$

The next theorem yields a positive answer to [12, Question 5] under suitable conditions on the genus of the surfaces.

Theorem C. *Let S, S' be two orientable finite-type surfaces of genus $g \geq 4$ and $g' \leq 6 \cdot 2^{g-4}$. Any homomorphism $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ admits a lift $\overline{\varphi} : \mathrm{Diff}_c^+(S) \to \mathrm{Diff}_c^+(S')$ to the diffeomorphism groups.*

We emphasize that Theorem C only imposes conditions on the genus of the surfaces, and has no assumptions on the number of boundary components or punctures.

A theorem of Hurtado [45] states that every homomorphism from the identity component of $\mathrm{Diff}_c^+(S)$ to the identity component of $\mathrm{Diff}_c^+(S')$ is an 'extended topologically diagonal map'. Although we do not rely on

Hurtado's result, the lifts $\overline{\varphi} : \mathrm{Diff}_c^+(S) \to \mathrm{Diff}_c^+(S')$ from Theorem C admit an analogous description. Namely, we say that a homomorphism $\overline{\varphi} : \mathrm{Diff}_c^+(S) \to \mathrm{Diff}_c^+(S')$ is *induced by a multi-embedding* if there is a set of disjoint embeddings $\{\iota_i : S \hookrightarrow S' |\ i = 1, 2, \dots, n\}$ such that $\overline{\varphi}$ is given by $\phi \mapsto \hat{\phi}$, where $\hat{\phi}$ is defined as

$$\hat{\phi}(x) = \begin{cases} x & x \notin \iota_i(S)\ \forall i, \\ \iota_i \circ \phi \circ \iota_i^{-1}(x) & x \in \iota_i(S). \end{cases}$$

If $\overline{\varphi}$ is a lift of the homomorphism $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$, we also say that φ is induced by a multi-embedding.

To prove Theorem C we actually show that, within the same genus bounds, every homomorphism $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is induced by a *multi-embedding.*

Theorem A. *Let S, S' be two orientable finite-type surfaces of genus $g \geq 4$ and $g' \leq 6 \cdot 2^{g-4}$. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a non-trivial homomorphism, then φ is induced by a multi-embedding.*

Theorem A extends some already known results. For instance, when $g' = g$, we recover results of Ivanov [50] and Ivanov-McCarthy [51] on injective endomorphisms of the mapping class group. Also, if $g' \leq 2g - 1$, the statement recovers Aramayona-Souto's classification [11], which asserts that every non-trivial homomorphism $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is induced by an embedding. Another immediate consequence of Theorem A is the following generalization of [27, Theorem 1] by Chen-Lanier.

Corollary 0.2. *Let $S_g, S_{g'}$ be two closed orientable surfaces of genus $g \geq 4$ and $g' \leq 6 \cdot 2^{g-4}$. If $g \neq g'$, every homomorphism $\mathrm{Map}(S_g) \to \mathrm{Map}(S_{g'})$ is trivial.*

We stress the importance of the genus bounds in Theorem A. For instance, for some g' large enough, we may consider a finite quotient of $\mathrm{PMap}(S)$ that embeds in $\mathrm{PMap}(S')$. Such map cannot be induced by a multi-embedding. Similarly, if S has genus $g \leq 2$ and S' is any closed surface, there are homomorphisms $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$ with finite image $\mathbb{Z}/2\mathbb{Z}$. Again, such maps are not induced by multi-embeddings.

More interestingly, the statement of Theorem A requires some exponential upper bound on g'. Indeed, if $g' \geq (g-1) \cdot 2^{2g} + 1$, there are injections between mapping class groups that are not induced by multi-embeddings. In Chapter 2.6 we produce such homomorphisms following a construction of Ivanov-McCarthy [53].

The proof of Theorem A relies on the following result of Bridson [21]: If S has genus $g \geq 3$, then every homomorphism $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ sends Dehn twists to roots of multitwists. Building on this, we show that

the genus bounds imply that φ is *multitwists-preserving*, meaning that the image of every non-separating Dehn twist is a multitwist. Then, that φ is induced by a multi-embedding follows from the next result.

Theorem B. *Let S, S' be two orientable finite-type surfaces and let $g \geq 3$ be the genus of S. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a multitwist-preserving map, then φ is induced by a multi-embedding.*

Outline of the chapter

In Section 2.1 we introduce some definitions and the necessary tools. In Section 2.2 we prove a weaker version of Theorem B. In Section 2.3 we prove Theorem B. In Section 2.4 we prove Theorem A. In Section 2.5 we prove Theorem C. We finish by constructing a homomorphisms not induced by multi-embedding in Section 2.6.

2.1 Definitions and tools

In this section we introduce some basic definitions and results about mapping class groups that are used throughout the chapter. For a thorough introduction to mapping class groups we refer the reader to the book of Farb-Margalit [32].

All surfaces in this chapter are assumed to be orientable and of finite-type, that is, with finitely generated fundamental group. Alternatively, we may define an orientable finite-type surface S as a compact surface minus a finite set of points in its interior. The points removed are called *punctures* of S. The genus g, the number of boundary components b and the number of punctures p, characterize the surface S up to homeomorphism.

We denote $\mathrm{Homeo}^+(S, \partial S)$ the group of homeomorphisms of S that preserve orientation and fix the boundary pointwise. Let $\mathrm{Homeo}_0^+(S, \partial S)$ be the subgroup of elements homotopic to the identity via homotopies fixing the boundary pointwise. The mapping class group $\mathrm{Map}(S)$ of S may be defined as the quotient $\mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}_0^+(S, \partial S)$. Recall that $\mathrm{Map}(S)$ has many equivalent definitions, for instance, by upgrading homeomorphisms to diffeomorphisms. We will introduce additional definitions as needed, deferring their presentation until required.

Two curves a and b are said to be *disjoint* if $i(a, b) = 0$. A *multicurve* on S is a set of pairwise disjoint curves. Recall that if S has genus g, b boundary components and p punctures, then every multicurve η on S contains at most $3g - 3 + p + b$ curves.

Recall a multitwist $m_A \in \mathrm{Map}(S)$ is a finite product of Dehn twists

$$m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k},$$

where $A = \{a_1, \ldots, a_k\}$ is the set of components and the n_i are non-zero integers. We allow the set of components to be empty, in which case the multitwist is the identity. Notably, multitwists may have roots. We say $f \in \mathrm{Map}(S)$ is a *proper root of a multitwist* if f is not a multitwist itself, but f^k is a multitwist for some $k \in \mathbb{N}$. The *degree* of the proper root f is the smallest $k > 0$ such that f^k is a multitwist. For instance, finite order elements are proper roots of the identity. For further examples, we refer the reader to [65] and [75].

Whenever we want to refer to a mapping class $f \in \mathrm{PMap}(S)$ as the homotopy class of a homeomorphism $\phi \in \mathrm{Homeo}^+(S, \partial S)$, we write $f = [\phi]$. An element $f \in \mathrm{Map}(S)$ is said to be *reducible* if there exists a multicurve η such that f fixes η setwise, and η is said to be a *reducing multicurve* for f. If f is not reducible, we say f is *irreducible*. An element $f \in \mathrm{Map}(S)$ *fixes no curve* if $f(c) \neq c$ for every curve c. Note that fixing a curve implies reducibility, but the converse does not hold, that is, there are reducible elements that fix no curve.

Now, we extend the notions of reducibility, irreducibility and fixing no curve to homomorphisms. A map $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is *reducible* if there exists a multicurve η on S' such that η is fixed setwise by every element in the image of φ. A homomorphism φ is *irreducible* if it is not reducible. Analogously, the homomorphism φ *fixes no curve* if for every curve c on S' there exists $f \in \varphi(\mathrm{PMap}(S))$ such that $f(c) \neq c$. As before, a reducible homomorphism might fix a curve or fix no curve, but an irreducible homomorphism necessarily fixes no curve.

2.1.1 Basic properties

Let S be an orientable finite-type surface of genus $g \geq 1$. The Dehn-Lickorish Theorem [61] asserts that $\mathrm{PMap}(S)$ is finitely generated by Dehn twists. For surfaces of genus $g \geq 3$, we use the convenient generating set produced by Humphries [44], namely, the set of non-separating Dehn twists $\mathcal{H} = \{d_c | \, c \in H\}$ where H is the set of curves depicted in Figure 2.1. In general, we say a set $\mathcal{H}' \subset \mathrm{PMap}(S)$ is a *Humphries generating set* if $\mathcal{H}'$ is conjugate to $\mathcal{H}$.

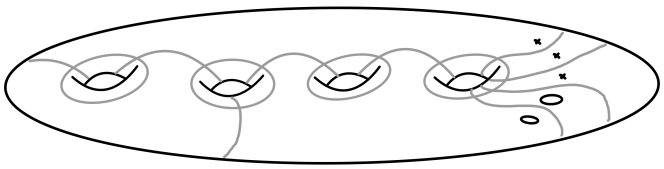


Figure 2.1: Curves in a Humphries generating set for a surface of genus four, with two boundary components and three punctures.

Mumford [71] showed that the abelianization of $\mathrm{PMap}(S)$ is finite for surfaces of genus $g \geq 2$. Powell [74] improved on this result by showing that the abelianization is actually trivial if $g \geq 3$. This result is frequently used in the proofs below, and so we state it here.

Theorem 2.1 ([71] , [74])**.** *Let S be an orientable finite-type surface of genus g. If $g = 2$, then the abelianization of $\mathrm{PMap}(S)$ is $\mathbb{Z}/10\mathbb{Z}$. If $g \geq 3$, then $\mathrm{PMap}(S)$ has trivial abelianization.*

The next result we introduce follows from a combination of the Nielsen realization theorem (proved by Kerckhoff [55]) and the Hurwitz's automorphism theorem.

Theorem 2.2 ([55])**.** *Let S be a orientable finite-type surface of genus $g \geq 2$. Every finite subgroup $H < \mathrm{PMap}(S)$ has order $|H| \leq 84(g-1)$.*

Nakajima [73] and Maclachlan [62] specialized the previous theorem to the case of finite abelian subgroups.

Theorem 2.3 ([73], [62])**.** *Let S be an orientable finite-type surface of genus $g \geq 2$. Then, every finite abelian subgroup $H < \mathrm{Map}(S)$ has order $|H| \leq 4g + 4$.*

For later use, we record the following two properties.

Remark 2.4. *Let S be an orientable finite-type surface with boundary, then $\mathrm{Map}(S)$ has no finite order elements.*

Remark 2.5. *Let S be an orientable finite-type surface with at least one puncture, then every finite subgroup of $\mathrm{PMap}(S)$ is cyclic.*

2.1.2 Some examples of homomorphisms

A natural starting point to characterize maps $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is to consider some examples. Hence, we introduce well-known constructions of homomorphisms between pure mapping class groups.

Let $S \subset S'$ be a closed subsurface of S. There exists a natural homomorphism known as the *inclusion homomorphism*

$$\hat{\imath} : \mathrm{PMap}(S) \to \mathrm{PMap}(S'). \tag{2.1}$$

The map $\hat{\imath}$ is defined as follows: For every class $[\phi] \in \mathrm{PMap}(S)$, consider the homeomorphism $\hat{\phi} : S' \to S'$ acting as ϕ on S and acting as the identity on $S' \setminus S$; then, the map $\hat{\imath}$ is given by the equation $\hat{\imath}([\phi]) = [\hat{\phi}]$. A result [32, Lemma 3.16] ensures that $\hat{\imath}$ is a well-defined homomorphism with kernel generated by Dehn twists d_a for each curve $a \in \partial S$ bounding a puncture in S' and by multitwists $d_b d_c^{-1}$, where $b, c \in \partial S$ are homotopic curves in S'.

A special case of the inclusion homomorphism is the *boundary deletion map*. Let S be a surface with boundary. Let S' be the surface obtained by gluing to S a punctured disk along a boundary curve $a \in \partial S$. The inclusion $\iota : S \hookrightarrow S'$ induces a homomorphism $\hat{\iota} : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ and the kernel of $\hat{\iota}$ is the group $\mathbb{Z} = \langle d_a \rangle$.

Let S be a surface and η a multicurve on S. The stabilizer $\mathcal{Z}(\eta)$ of η is the subgroup of $\mathrm{PMap}(S)$ fixing the multicurve η setwise. We denote $\mathcal{Z}_0(\eta) \leq \mathcal{Z}(\eta)$ the subgroup of elements that fix each curve $c \in \eta$ with orientation. There exists a natural homomorphism from $\mathcal{Z}_0(\eta)$ obtained by 'cutting' S along the curves in η. To be more precise, for each curve $c \in \eta$ take $\overline{U_c}$ to be a closed regular neighborhood of c. The components of $S \setminus \bigcup_{c\in\eta} \overline{U_c}$ are open subsurfaces $R_i \subset S$. Then, the *cutting homomorphism* is defined as the map

$$\mathrm{Cut}_\eta : \mathcal{Z}_0(\eta) \to \prod_{i=1}^{m} \mathrm{PMap}(R_i), \tag{2.2}$$

where each $f \in \mathcal{Z}_0(\eta)$ is sent to $(f_1, \ldots, f_n)$ and f_i is the restriction of f to the subsurface R_i. The kernel of Cut_η is the group $\mathbb{Z}^{|\eta|}$ generated by the Dehn twists d_a for $a \in \eta$.

Remark 2.6. *If $f \in \mathcal{Z}_0(\eta)$ is a proper root of degree k of a multitwist and the components of f^k are contained in η, then $\mathrm{Cut}_\eta(f)$ has order k.*

Let S be a surface with punctures. The *Birman exact sequence* is the following short exact sequence

$$1 \to \pi_1(\overline{S}, \{*\}) \to \mathrm{PMap}(S) \xrightarrow{\mathcal{F}} \mathrm{PMap}(\overline{S}) \to 1, \tag{2.3}$$

where $\overline{S} \setminus \{*\} = S$. In words, $\overline{S}$ is the surface obtained by 'filling' the puncture $*$ on S. The *forgetful* map $\mathcal{F}$ is defined as $\mathcal{F}([\phi]) = [\overline{\phi}]$, where $\overline{\phi}$ acts as ϕ on S and sends the new point $*$ to itself. It is a theorem [32, Chapter 4.2] that $\mathcal{F}$ is a well-defined homomorphism and the kernel is isomorphic to $\pi_1(\overline{S}, \{*\})$.

The next exact sequence is a consequence of the Birman-Hilden theory [18] and concerns centralizers of finite order elements (see also [11, Theorem 2.8]). Let $h \in \mathrm{Map}(S)$ be a finite order element and denote $\mathcal{Z}(h)$ its centralizer in $\mathrm{Map}(S)$. Let $\sigma : S \to S$ be a finite order homeomorphism representing h and consider the surface O obtained by removing the singular points of the quotient $S/\langle\sigma\rangle$. We have the following exact sequence

$$1 \to \langle h \rangle \to \mathcal{Z}(h) \to \mathrm{Map}^{\pm}(O). \tag{2.4}$$

Here $\mathrm{Map}^{\pm}(O)$ denotes the group of homeomorphisms (possibly inverting the orientation) modulo homotopy.

2.1.3 More tools

In this section we introduce two results regarding quotients of the mapping class group. The first one is a theorem of Berrick-Gebhardt-Paris [17] bounding the finite actions of $\mathrm{PMap}(S)$.

Theorem 2.7 ([17])**.** *Let S be an orientable finite-type surface of genus $g \geq 3$. If $\mathrm{PMap}(S)$ acts non-trivially on a finite set F, then $|F| \geq 2^{g-1}(2^g - 1)$.*

Remark 2.8. *Theorem 2.7 is stated in [17] only for compact surfaces with (possibly empty) boundary. The result for punctured surfaces follows from their statement. We sketch the argument.*

Let $S_{g,p}^b$ be the surface of genus g with b boundary components and p punctures. Let S_g^{p+b} be the surface of genus g with $p+b$ boundary components. The boundary deletion map induces an epimorphism

$$\mathrm{PMap}(S_g^{b+p}) \xrightarrow{\Omega} \mathrm{PMap}(S_{g,p}^b).$$

If $\mathrm{PMap}(S_{g,p}^b)$ acts non-trivially on a set F, then $\mathrm{PMap}(S_g^{p+b})$ acts on F via the map Ω. It follows from the compact case that $|F| \geq 2^{g-1}(2^g - 1)$, as we wanted.

The next result of Bridson [21] controls the image of a Dehn twist under homomorphisms.

Theorem 2.9 ([21])**.** *Let S, S' be orientable finite-type surfaces and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a homomorphism. If S has genus $g \geq 3$, the image of Dehn twist is a root of a multitwist.*

We devote the rest of the section to introduce homomorphisms induced by embeddings and homomorphisms induced by multi-embeddings.

2.1.4 Homomorphisms induced by embeddings

Let S and S' be two surfaces. An *embedding* from S to S' is a continuous injective map $\iota : S \hookrightarrow S'$. Fix an embedding ι and let $\phi \in \mathrm{Homeo}_c^+(S, \partial S)$ be a compactly supported homeomorphism. We may extend ϕ to a homeomorphism $\widehat{\phi} : S' \to S'$ by letting

$$\widehat{\phi}(x) = \begin{cases} x & x \notin \iota(S), \\ \iota \circ \phi \circ \iota^{-1}(x) & x \in \iota(S). \end{cases} \qquad (*)$$

In this way, an embedding ι naturally induces an injective homomorphism between the compactly supported homeomorphism groups

$$\widehat{\iota} : \mathrm{Homeo}_c^+(S, \partial S) \hookrightarrow \mathrm{Homeo}_c^+(S', \partial S')$$
$$\phi \mapsto \widehat{\phi}.$$

In *some* cases the map $\widehat{\iota}$ descends to a homomorphism between mapping class groups, which motivates the next definition.

Definition 2.10. *An embedding $\iota : S \to S'$* induces a map between mapping class groups *if the map $\widehat{\iota} : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ given by $\widehat{\iota}([\phi]) = [\widehat{\phi}]$ is a well-defined homomorphism, where $\widehat{\phi}$ is defined as* $(*)$.

A source of examples for Definition 2.10 comes from the inclusion homomorphism. Recall that if S is a subsurface of S' and S is closed as a subset, then there is an inclusion map $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ at the level of mapping class groups. The map φ is precisely the homomorphism induced by the natural embedding $S \to S'$.

As hinted before, not every embedding induces a map between mapping class groups. We illustrate this in the following example.

Example 2.11. *Let S' be a compact surface of genus $g \geq 2$ and let S be the* interior *of a genus $g-1$ subsurface. The embedding $\iota : S \hookrightarrow S'$ given by inclusion does* not *induce a homomorphism between the mapping class groups.*

With this example in mind, we characterize which embeddings induce homomorphisms.

Lemma 2.12. *Let S and S' be two orientable surface of finite-type. An embedding $\iota : S \hookrightarrow S'$ induces a map between mapping class groups if and only if for every puncture p of S and every simple closed curve γ homotopic to p, the curve $\iota(\gamma)$ bounds a disk or a once-punctured disk.*

If $\iota(\gamma)$ bounds a once-punctured disk, we say ι sends the puncture p to a puncture in S'. If $\iota(\gamma)$ bounds a disk, we say ι sends the puncture p to a disk.

Proof. If ι is an embedding inducing a map between mapping class groups, it follows quickly that every puncture is sent to a puncture or a disk. Thus, we only prove the converse.

Consider the map $\widehat{\iota} : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ given by $\widehat{\iota}([\phi]) = [\widehat{\phi}]$. To check $\widehat{\iota}$ is well-defined, first assume that $\iota : S \to S'$ sends every puncture to a puncture. In this case, $\iota(S)$ is a subsurface of S' which is closed as a subset. Therefore, if $\phi_1, \phi_2 \in \mathrm{Homeo}_c^+(S, \partial S)$ are two homotopic homeomorphisms of S, then $\widehat{\phi_1}$ and $\widehat{\phi_2}$ are homotopic homeomorphisms of S'. It follows that $\widehat{\iota}$ is a well-defined. To check $\widehat{\iota}$ is homomorphism it is enough to observe that $\widehat{\phi_1 \circ \phi_2} = \widehat{\phi_1} \circ \widehat{\phi_2}$.

Assume now that ι sends only one puncture p on S to a disk on S'. We consider two cases: either $\overline{\iota(S)} \setminus \iota(S)$ is a single point or it is a curve. In the first case, we may factor ι as

$$S \overset{F}{\hookrightarrow} \overline{S} \overset{\iota_1}{\to} S',$$

where $\overline{S} \setminus \{p\} = S$, and both F and ι_1 are the natural inclusions. Notice ι_1 induces a map between mapping class groups. Indeed, since ι sends only

one puncture to a disk, it follows ι_1 sends every puncture to a puncture and therefore induces a map $\mathrm{PMap}(\overline{S}) \to \mathrm{PMap}(S')$. Moreover, F induces the forgetful map $\mathcal{F} : \mathrm{PMap}(S) \to \mathrm{PMap}(\overline{S})$. Since $\widehat{\iota}$ coincides with the composition $\widehat{\iota_1} \circ \mathcal{F}$, it follows $\widehat{\iota}$ is well-defined. Now, in the case $\overline{\iota(S)} \setminus \iota(S)$ is a curve, we may again consider an appropriate $\overline{S}$, F and ι_1 so that $\widehat{\iota}$ is the composition $\widehat{\iota_1} \circ \mathcal{F}$.

To finish, if $\iota : S \to S'$ sends two or more punctures to disks, we may show $\widehat{\iota} : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is well-defined by inductively repeating the argument in the previous paragraph. □

Remark 2.13. *If S has no punctures, Lemma 2.12 implies that every embedding $S \hookrightarrow S'$ induces a map between mapping class groups.*

2.1.5 Homomorphisms induced by multi-embeddings

Let S and S' be two surfaces. A *multi-embedding* from S to S' is a collection of disjoint embeddings

$$\mathcal{I} = \{\iota_i : S \hookrightarrow S' |\, \iota_i(S) \cap \iota_j(S) = \emptyset \ \forall i \neq j\}.$$

Fix a multi-embedding $\mathcal{I} = \{\iota_i |\, i = 1, \ldots, n\}$ from S to S' and let $\phi \in \mathrm{Homeo}_c^+(S, \partial S)$ be a compactly supported homeomorphism. We may extend ϕ to a homeomorphism $\widehat{\phi} : S' \to S'$ by letting

$$\widehat{\phi}(x) = \begin{cases} x & x \notin \iota_i(S) \ \forall i, \\ \iota_i \circ \phi \circ \iota_i^{-1}(x) & x \in \iota_i(S). \end{cases} \qquad (**)$$

Again, the multi-embedding $\mathcal{I}$ induces an injective homomorphism

$$\widehat{\mathcal{I}} : \mathrm{Homeo}_c^+(S, \partial S) \hookrightarrow \mathrm{Homeo}_c^+(S', \partial S').$$
$$\phi \mapsto \widehat{\phi}.$$

In *some* cases $\widehat{\mathcal{I}}$ descends to a homomorphism $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$. The next two examples illustrate such homomorphisms.

Example 2.14. *Let S' be a closed surface of genus $2g$ and let S be a surface of genus $g - 1$ with one boundary component. Consider two embeddings*

$$\iota_i : S \to S' \text{ for } i = 1, 2$$

such that $\iota_1(S) \cap \iota_2(S) = \emptyset$. By Remark 2.13, each ι_i induces a homomorphism $\widehat{\iota_i}$ between mapping class groups. Notice that elements in the image of $\widehat{\iota_1}$ commute with elements in the image of $\widehat{\iota_2}$, since they have disjoint support. Therefore, we may consider the product homomorphism $\prod \widehat{\iota_i} : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ defined by $\prod \widehat{\iota_i}(f) = \widehat{\iota_1}(f)\widehat{\iota_2}(f)$.

We may also construct the homomorphism $\prod \widehat{\iota_i}$ *as follows. Let* $\phi \in \mathrm{Homeo}_c^+(S)$ *be a homeomorphism and consider the extension* $\widehat{\phi} : S' \to S'$ *defined as in* $(**)$*. As before,* $\widehat{\phi} \in \mathrm{Homeo}_c^+(S')$ *and we may define the map* $\widehat{\mathcal{I}}([\phi]) = [\widehat{\phi}]$*. It is now possible to check that* $\widehat{\mathcal{I}}$ *is exactly the map* $\prod \widehat{\iota_i}$*.*

In the previous example, the induced map $\widehat{\mathcal{I}} : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ coincides with the product map $\prod \widehat{\iota_i}$. In general, this might not be the case. For instance, a multi-embedding $\mathcal{I} = \{\iota_1, \iota_2\}$ might induce a homomorphism $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$, while each ι_i separately does not induce a homomorphism. We illustrate this case in Example 2.15. Before doing so, we require a couple of definitions.

Fix two oriented surfaces S, S' and let $\iota : S \to S'$ be an embedding. We say ι *preserves* orientation if ι sends the orientation of S to the orientation of S'. Otherwise, we say ι *reverses* orientation. Additionally, we say two embeddings $\iota_1, \iota_2 : S \to S'$ *induce opposite orientations* if one of them preserves and the other one reverses the orientation.

Example 2.15. *Let* S' *be a closed surface of genus* $2g$ *and let* S *be a once-punctured surface with genus* g *and no boundary. Consider two embeddings*

$$\iota_i : S \to S' \text{ for } i = 1, 2$$

such that ι_1, ι_2 *induce opposite orientations and* $\iota_1(S) \cap \iota_2(S) = \emptyset$*.*

Note that neither ι_1 *nor* ι_2 *induce a map between mapping class groups. However, the multi-embedding* $\mathcal{I} = \{\iota_1, \iota_2\}$ *induces a map*

$$\widehat{\mathcal{I}} : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$$

by letting $\widehat{\mathcal{I}}([\phi]) = [\widehat{\phi}]$*, where* $\widehat{\phi}$ *is the extension of* $\phi \in \mathrm{Homeo}_c^+(S, \partial S)$ *defined in* $(**)$*. For a proof that* $\widehat{\mathcal{I}}$ *is well-defined see Lemma 2.18 below.*

Motivated by the previous examples, we give the following definition.

Definition 2.16. *The multi-embedding* $\mathcal{I} = \{\iota_i : S \hookrightarrow S' | \ i = 1, 2, \ldots, k\}$ induces a homomorphism between mapping class groups *if the map* $\widehat{\mathcal{I}} : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ *given by* $\widehat{\mathcal{I}}([\phi]) = [\widehat{\phi}]$ *is a well-defined homomorphism, where* $\phi \in \mathrm{Homeo}_c^+(S, \partial S)$ *and*

$$\widehat{\phi}(x) = \begin{cases} x & x \notin \iota_i(S) \ \forall i, \\ \iota_i \circ \phi \circ \iota_i^{-1}(x) & x \in \iota_i(S). \end{cases}$$

Remark 2.17. *The notion of multi-embedding induced map is analogous to the notion of 'topologically diagonal map' introduced by Mann for diffeomorphism groups (see [63, Definition 1.4]).*

The next lemma characterizes multi-embeddings inducing homomorphisms between mapping class groups.

Lemma 2.18. *The multi-embedding $\mathcal{I} = \{\iota_i : S \hookrightarrow S' : i = 1, 2, \ldots, k\}$ induces a homomorphism between mapping class groups if and only if for every curve γ on S that bounds a puncture and for every embedding $\iota_i \in \mathcal{I}$ one of the following conditions hold:*

- *either $\iota_i(\gamma)$ bounds a disk,*
- *or $\iota_i(\gamma)$ bounds a once-punctured disk,*
- *or there exists an embedding $\iota_j \in I$ such that $\iota_j(\gamma)$ is homotopic to $\iota_i(\gamma)$ and both embeddings induce opposite orientations.*

In the third case, when $\iota_i(\gamma)$ is non trivial, we say ι_i sends a puncture to a non-trivial curve.

Proof. If $\mathcal{I}$ is multi-embedding inducing a map between mapping class groups, then it is an exercise to see that for every embedding and every puncture one of the three conditions above holds. We prove the converse.

Let $\{p_1, \ldots, p_n\}$ be the set of punctures such that p_i is sent to a non-trivial curve by some embedding in $\mathcal{I}$. Let U_{p_i} be an open regular neighborhood for each p_i and consider the subsurface $\overline{S} = S \setminus \bigcup_i U_{p_i}$. The surface $\overline{S}$ has one boundary component for each puncture p_i. We may embed $\overline{S}$ in S' by composing the inclusions $\overline{S} \overset{s}{\hookrightarrow} S \overset{\iota_i}{\hookrightarrow} S$, where s is the inclusion as a subset. Now, by Lemma 2.12, the embeddings $\iota_i \circ s$ induce a map between mapping class groups. Even more, if $i \neq j$ then elements in the image of $\widehat{\iota_i \circ s}$ commute with elements in the image of $\widehat{\iota_j \circ s}$. As a consequence, the set of maps $\{\iota_i \circ s\}$ induces a product map

$$\mathrm{PMap}(\overline{S}) \xrightarrow{\prod \widehat{\iota_i \circ s}} \mathrm{PMap}(S'),$$

where $\prod \widehat{\iota_i \circ s}(f) = \widehat{\iota_1 \circ s}(f) \cdot \widehat{\iota_2 \circ s}(f) \cdot \ldots \cdot \widehat{\iota_{|\mathcal{I}|} \circ s}(f)$.

On one side, the three conditions in the statement imply that

$$\prod \widehat{\iota_i \circ s}(d_a) = \mathrm{id}$$

for every curve $a \in \partial\overline{S}$ homotopic to a puncture p_i in S. On the other side, the embedding s induces the 'boundary deletion' map $\widehat{s} : \mathrm{PMap}(\overline{S}) \to \mathrm{PMap}(S)$. From these two facts, we conclude that $\prod \widehat{\iota_i \circ s}$ factors through $\widehat{s}$, that is, there exists ψ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{PMap}(\overline{S}) & \xrightarrow{\prod \widehat{\iota_i \circ s}} & \mathrm{PMap}(S') \\ {\scriptstyle \widehat{s}}\downarrow & \nearrow_{\psi} & \\ \mathrm{PMap}(S) & & \end{array}$$

It is immediate that $\psi([\phi]) = [\widehat{\phi}]$. In words, ψ is the map induced by the multi-embedding $\mathcal{I}$ and $\psi = \widehat{\mathcal{I}}$ is a well-defined homomorphism, as we wanted. □

2.2 Twist-preserving homomorphisms

In this section we prove that homomorphisms $\mathrm{PMap}(S) \to \mathrm{PMap}(S')$ that send Dehn twists to Dehn twists are induced by embeddings. The main ideas behind the proof of this result are contained in Aramayona-Souto [11, Sections 9 and 10]. In here, we collect the necessary modifications to their statements and arguments.

Definition 2.19 (Twist-preserving homomorphisms)**.** *Let S, S' be orientable finite-type surfaces and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a homomorphism. The map φ is* twist-preserving *if the image of a non-separating Dehn twist is a non-trivial power of a Dehn twist.*

Theorem 2.20. *Let S, S' be orientable finite-type surface. If S has genus $g \geq 3$ and $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a twist-preserving homomorphism, then φ is induced by an embedding.*

The following sections contain the proof Theorem 2.20.

2.2.1 Dehn twists to Dehn twists

A priori, a twist-preserving homomorphism φ might send a non-separating Dehn twist d_a to a proper power of a Dehn twist, $\varphi(d_a) = d_b^n$. The next lemma shows that b must be non-separating and $n \in \{1, -1\}$.

Lemma 2.21. *Let S, S' be two orientable finite-type surfaces, let $g \geq 2$ be the genus of S and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a twist-preserving map. If d_a is a non-separating Dehn twist, then $\varphi(d_a) = d_b^\epsilon$ is a non-separating Dehn twist and $\epsilon \in \{1, -1\}$.*

Proof. Let $\tilde{a}$ be a non-separating curve on S such that $i(a, \tilde{a}) = 1$. Since $d_{\tilde{a}}$ is conjugate to d_a, the image satisfies

$$\varphi(d_{\tilde{a}}) = \varphi(f) \cdot d_b^\epsilon \cdot \varphi(f)^{-1} = d_{\varphi(f)\cdot b}^\epsilon = d_{\tilde{b}}^\epsilon$$

for some element $f \in \mathrm{PMap}(S)$ and some curve $\tilde{b}$ on S'. If $b = \tilde{b}$, then φ has infinite cyclic image. However, by Theorem 2.1, the abelianization of $\mathrm{PMap}(S)$ is finite. Thus, $b \neq \tilde{b}$.

Since $d_a, d_{\tilde{a}}$ satisfy the braid relation, then $d_b^\epsilon, d_{\tilde{b}}^\epsilon$ also satisfy the braid relation. From Theorem D it follows that $i(b, \tilde{b}) = 1$ and $\epsilon \in \{1, -1\}$. □

Remark 2.22. *The lemma above ensures that $\varphi(d_a) = d_b^\epsilon$ with $\epsilon \in \{1, -1\}$. The constant ϵ does not depend on the non-separating Dehn twist and we may assume that $\epsilon = 1$ up to composing by an automorphism in the image.*

2.2.2 Reducing to the irreducible case

In this section we reduce the proof of Theorem 2.20 to the case of irreducible homomorphisms. The next proposition gives sufficient conditions on an embedding for it to induce a map between the mapping class groups (cf. [11, Proposition 9.2]).

Proposition 2.23. *Let S, S' be two orientable finite-type surfaces and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a homomorphism. If $\iota : S \to S'$ is an embedding such that $\varphi(d_a) = d_{\iota(a)}$ for every non-separating curve a, then ι induces the map φ.*

Proof. Let $\mathcal{H}$ be a Humphries generating set of $\mathrm{PMap}(S)$, let $H = \{a |\, d_a \in \mathcal{H}\}$ and let Z be a closed regular neighborhood of $\bigcup_{a\in H} a$. Note the surfaces Z and $\iota(Z)$ have no punctures. Therefore, by Remark 2.13, the three maps given by the restriction $\iota|_Z : Z \hookrightarrow \iota(Z)$, the inclusion $i : Z \hookrightarrow S$ and the inclusion $i' : \iota(Z) \hookrightarrow S'$ all induce homomorphisms between the corresponding mapping class groups. Even more, the induced homomorphisms satisfy

$$\varphi \circ \widehat{i}(d_a) = \widehat{i'} \circ \widehat{\iota|_Z}(d_a) \quad \forall d_a \in \mathcal{H}.$$

Since $\mathcal{H}$ is a generating set, the induced maps fit into the commutative diagram

$$\begin{array}{ccc} \mathrm{PMap}(Z) & \xrightarrow{\widehat{i}} & \mathrm{PMap}(S) \\ \downarrow{\scriptstyle \widehat{\iota|_Z}} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{PMap}(\iota(Z)) & \xrightarrow{\widehat{i'}} & \mathrm{PMap}(S') \end{array} .$$

The map $\widehat{\iota|_Z}$ is an isomorphism by construction. As a consequence, for every curve $c \in \partial Z$ homotopic to a puncture in S, we have that $\iota(c)$ bounds a puncture or a disk. Indeed, if $\iota(c)$ was not trivial, then we would have the contradiction

$$\mathrm{id} = \varphi \circ \widehat{i}(d_c) = \widehat{i'} \circ \widehat{\iota|_Z}(d_c) = d_{\iota(c)} \neq \mathrm{id}.$$

It follows that ι sends punctures in S to punctures or disks in S'. By Lemma 2.12, ι induces a map $\hat{\iota}$ between mapping class groups. Since $\hat{\iota}(d_a) = d_{\iota(a)} = \varphi(d_a)$ for every non-separating Dehn twist $d_a \in \mathcal{H}$, we deduce that $\hat{\iota} = \varphi$. □

Let φ be a reducible twist-preserving homomorphism. By definition, φ fixes a reducing multicurve η. Now, next lemma shows that φ actually fixes each curve in η with orientation.

Lemma 2.24. *Let S, S' be two orientable finite-type surfaces and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a homomorphism. If $\varphi(d_c)$ is a multitwist for any non-separating Dehn twist d_c and the image of φ fixes a multicurve η setwise, then the image of φ fixes each curve in η with orientation.*

Note that, in particular, Lemma 2.24 applies to twist-preserving homomorphisms.

Proof. Fix a Humphries generating set $\mathcal{H}$ of $\mathrm{PMap}(S)$. For each $d_c \in \mathcal{H}$ denote $m_C = \varphi(d_c)$ the multitwist in the image, where C is the set of components of m_C.

Looking for a contradiction, consider a curve $\eta_i \in \eta$ not fixed by $\varphi(\mathrm{PMap}(S))$. Since $\mathcal{H}$ is a generating set, there exists a $d_c \in \mathcal{H}$ such that $m_C(\eta_i) \neq \eta_i$. Thus, the multicurve C intersects η_i. It follows that the set

$$\{m_C^n(\eta_i) |\, n \in \mathbb{Z}\}$$

is infinite. However, this contradicts that $\varphi(\mathrm{PMap}(S))$ fixes the (finite) multicurve η that contains η_i. We conclude that each curve $\eta_i \in \eta$ is fixed by $\varphi(\mathrm{PMap}(S))$.

Notice that for every $d_c \in \mathcal{H}$ and $m_C = \varphi(d_c)$ we have that $i(C, \eta_i) = 0$. In particular, m_C fixes the orientation of η_i and, since $\mathcal{H}$ generates, the whole image of φ fixes the orientation of η_i. □

The next proposition reduces the proof of Theorem 2.20 to the irreducible case. This result is parallel to [11, Lemma 9.3] but we remark that in our setting we require a slightly different proof.

Proposition 2.25. *If Theorem 2.20 holds for irreducible homomorphisms, then it holds for reducible homomorphisms.*

Proof. Let S be a surface of genus $g \geq 3$ and $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a reducible twist-preserving homomorphism. Consider η a maximal reducing multicurve for φ. By Lemma 2.24, the image $\varphi(\mathrm{PMap}(S))$ fixes each component of η with orientation and so we can consider the cutting homomorphism

$$\mathrm{PMap}(S) \xrightarrow{\varphi} \mathcal{Z}_0(\eta) \xrightarrow{\mathrm{Cut}_\eta} \prod_i \mathrm{PMap}(R_i'),$$

where $R_i' \subset S'$ are surfaces such that $S' \setminus \bigcup_{\eta_i \in \eta} \eta_i = \bigsqcup_i R_i'$.

Recall that the kernel of Cut_η is free abelian and, since $\mathrm{PMap}(S)$ has trivial abelianization, the composition $\mathrm{Cut}_\eta \circ \varphi$ is not trivial. Define

$$\varphi_j = \pi_j \circ \mathrm{Cut}_\eta \circ \varphi,$$

where π_j is the projection to the jth component. Consider one j such that φ_j is non-trivial. By the maximality of η, the map φ_j is an irreducible twist-preserving homomorphism. From the hypotheses it follows that φ_j is induced by an embedding $\iota : S \to R_j'$. Composing with the inclusion $i : R_j' \hookrightarrow S'$ we obtain an embedding $i \circ \iota : S \to S'$. Moreover, $\varphi(d_a) = d_{i \circ \iota(a)}$ for every non-separating curve a. Using Proposition 2.23, we conclude that $i \circ \iota$ induces φ. □

2.2.3 Reducing to homomorphisms that do not factor

A homomorphism $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ *factors* if there is a surface $\overline{S}$, a map $\psi : \mathrm{PMap}\left(\overline{S}\right) \to \mathrm{PMap}(S')$ and a forgetful or boundary deletion map $\sigma : \mathrm{PMap}(S) \to \mathrm{PMap}\left(\overline{S}\right)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{PMap}(S) & \xrightarrow{\varphi} & \mathrm{PMap}(S') \\ {\scriptstyle\sigma}\big\downarrow & \nearrow_{\psi} & \\ \mathrm{PMap}\left(\overline{S}\right) & & \end{array}$$

Lemma 2.26. *If Theorem 2.20 holds for maps that do not factor, then it holds for maps that factor.*

Proof. Proceed by induction in the number of times it factors. Note that φ can only factor finitely many times since S is of finite-type.

Assume that $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ factors only once and let $\varphi = \psi \circ \sigma$ be such factorization. Note that the map σ is induced by an embedding s. Also, since φ factors only once, ψ does not factor. Now, by hypothesis, ψ is induced by an embedding p. This implies that φ is induced by the embedding $p \circ s$, as we wanted.

An analogous argument can be repeated for the inductive step. □

2.2.4 Reducing to surfaces with no boundary

In this section we show that if $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is an irreducible homomorphism that does not factor and satisfies the hypotheses of Theorem 2.20, then S and S' have no boundary.

The following lemma shows that if φ is an irreducible twist-preserving homomorphism, then $\varphi(\mathrm{PMap}(S))$ has trivial centralizer. This result is similar to [11, Lemma 9.5] but adapted to our context.

Lemma 2.27. *Let S, S' be two orientable finite-type surfaces, let $g \geq 3$ the genus of S and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a twist-preserving homomorphism. If φ is irreducible, then the centralizer of $\varphi(\mathrm{PMap}(S))$ is trivial.*

Proof. First, we claim that

Claim 2.28. *Let a be a non-separating curve on S and let $d_b^{\epsilon} = \varphi(d_a)$. If f centralizes $\varphi(\mathrm{PMap}(S))$, then $f(b) = b$. Even more, f preserves the orientation of b.*

Proof of Claim 2.28. Since $[f, \varphi(d_a)] = 1$, then $f(b) = b$. To show f fixes the orientation of b, consider a_1, a_2, a_3 three distinct and disjoint curves on S such that $i(a, a_i) = 1$ for $i = 1, 2, 3$. Denote the images by $d_{b_i}^{\epsilon} = \varphi(d_{a_i})$ for $i = 1, 2, 3$. Using that $\mathrm{PMap}(S)$ has trivial abelianization, it is an exercise

to see that the curves b_1, b_2, b_3 are distinct. Moreover, by Lemma 2.21, the curves b_i are disjoint and $i(b, b_i) = 1$ for $i = 1, 2, 3$. Now, we may pick a representative ϕ of f and minimally intersecting representatives β, β_i of the curves b, b_i such that $\phi(\beta) = \beta$ and $\phi(\beta_i) = \beta_i$ for $i = 1, 2, 3$. We have that ϕ fixes the graph $\beta \cup \beta_1 \cup \beta_2 \cup \beta_3$ and, since ϕ also fixes the curves, it follows that ϕ fixes the intersection points $\beta \cap \beta_i$ for $i = 1, 2, 3$. In particular, ϕ fixes the orientation of β, which implies f fixes the orientation of b. □

Continuing with the proof of Lemma 2.27, let $\mathcal{H}$ be a Humphries generating set of $\mathrm{PMap}(S)$ and let $H = \{c |\ d_c \in \mathcal{H}\}$. We denote by $\varphi_*(H)$ the set

$$\{b |\ d_b^\epsilon = \varphi(d_a) \text{ for } a \in H\}.$$

Since φ is irreducible, the set $\varphi_*(H)$ fills S'. Moreover, any two curves $b_1, b_2 \in \varphi_*(H)$ satisfy either $i(b_1, b_2) = 0$ or $i(b_1, b_2) = 1$. So, by the previous claim, f fixes every curve $b_i \in \varphi_*(H)$ with orientation. Lastly, the Alexander Method [32, Proposition 2.8] yields that f is the identity. That is, the only centralizing element of the image is the identity. □

As a consequence of Lemma 2.27, we obtain the following result.

Proposition 2.29. *Let S, S' be two orientable finite-type surfaces, let $g \geq 3$ be the genus of S and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a twist-preserving homomorphism. If φ is irreducible and does not factor, then $\partial S = \partial S' = \emptyset$.*

Proof. If $\partial S' \neq \emptyset$, then φ is reducible. Thus, $\partial S' = \emptyset$.

If φ is irreducible then the image of the centralizer is trivial and we have that $\varphi(d_a) = \mathrm{id}$ for every curve $a \in \partial S$. Therefore, φ factors through the boundary deletion homomorphism. Since φ does not factor, we conclude $\partial S = \emptyset$. □

Jointly, Proposition 2.25, Proposition 2.29 and Lemma 2.26, imply the following simplification for the proof of Theorem 2.20.

Corollary 2.30. *If Theorem 2.20 holds for surfaces $\partial S = \partial S' = \emptyset$ with irreducible maps φ that do not factor, then Theorem 2.20 holds in full generality.*

From here, the rest of the proof of Theorem 2.20 is completely analogous to the proof presented in [11, Section 10], which in reality only uses that the map is twist preserving and Corollary 2.30. We warn the reader that the terminology of Aramayona-Souto is slightly different from ours: they call *weak embeddings* to our embeddings and they call *embeddings* to what we call embeddings inducing a map between mapping class groups.

2.3 Multitwist-preserving homomorphisms

In this section we prove Theorem B. We introduce a definition and restate the theorem here for convenience.

Definition 2.31 (Multitwist-preserving homomorphism)**.** *Let S, S' be two orientable finite-type surfaces and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a homomorphism. The map φ is* multitwist preserving *if the image of a non-separating Dehn twist is a non-trivial multitwist.*

The name is justified since these maps send multitwists with non-separating components to multitwists.

Theorem B. *Let S, S' be two orientable finite-type surfaces and let $g \geq 3$ be the genus of S. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a multitwist-preserving map, then φ is induced by a multi-embedding.*

To prove Theorem B we may assume that the image of a non-separating Dehn twist is a multitwist with at least two components. Indeed, if the image has only one component, the statement is equivalent to Theorem 2.20 above.

We briefly sketch the strategy of the proof: First, we show φ is a reducible map. Then, by cutting along a maximal reducing multicurve, we obtain a set of twist-preserving maps $\{\varphi_i\}$. Using Theorem 2.20, we deduce that each φ_i is induced by an embedding ι_i. To finish, we check that φ is induced by the multi-embedding $\{\iota_i\}$.

As mentioned, the first objective is to show φ is reducible.

Lemma 2.32. *Let S, S' be two orientable finite-type surfaces and let $g \geq 1$ be the genus of S. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ sends non-separating Dehn twists to multitwists with at least two components, then φ fixes a curve. In particular, φ is reducible.*

Proof. Let $\mathcal{H}$ be a Humphries generating set for $\mathrm{PMap}(S)$ and let $H = \{c | \, d_c \in \mathcal{H}\}$. Let C be the set of components of the multitwist $\varphi(d_c)$ for $d_c \in \mathcal{H}$. Consider Z a closed regular neighborhood of the union $\bigcup_{c \in H} C$. As the image of every non-separating Dehn twist is a multitwist with at least two components, it follows from Theorem D that Z has at least two connected components. Observe, $\varphi(\mathrm{PMap}(S))$ fixes each connected component and each curve in the boundary ∂Z. □

Let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a homomorphism as in Theorem B. Further, assume φ sends non-separating Dehn twists to multitwists with at least two components. We reduce φ to produce new maps φ_i as follows:

Lemma 2.32 implies that φ is reducible and so we may choose a maximal reducing multicurve η. By Lemma 2.24, φ fixes each curve in η with

orientation. Therefore, the image of φ is contained in $\mathcal{Z}_0(\eta)$ and we may consider the composition

$$\mathrm{PMap}(S) \xrightarrow{\varphi} \mathcal{Z}_0(\eta) \xrightarrow{\mathrm{Cut}_\eta} \prod_i \mathrm{PMap}(R_i'),$$

where the surfaces R_i' are the connected components of $S' \setminus \bigcup_{\eta_i \in \eta} \eta_i$.

For each projection π_i to the ith coordinate, we define the map φ_i as

$$\varphi_i = \pi_i \circ \mathrm{Cut}_\eta \circ \varphi. \tag{2.5}$$

Since φ is multitwist preserving, every non-trivial φ_i is also multitwist preserving. Moreover, such φ_i fixes no curve by the maximality of η. So it follows from Lemma 2.32 that every non-trivial φ_i is actually twist preserving. We conclude from Theorem 2.20 that each non-trivial φ_i is induced by an embedding $\iota_i : S \hookrightarrow R_i'$. To prove Theorem B we show that φ is induced by the multi-embedding $\{\iota_i : S \hookrightarrow R_i' \subset S'\}$.

Before jumping into the proof, we require the next two lemmas.

Lemma 2.33. *Let S, S' be two orientable finite-type surfaces with S of genus $g \geq 3$, let $\mathcal{H}$ a Humphries generating set of $\mathrm{PMap}(S)$ and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a multitwist-preserving map. If $d_a, d_b \in \mathcal{H}$ are two distinct Dehn twists, then the multitwists $\varphi(d_a), \varphi(d_b)$ have no common components.*

Proof. Denote the images of d_a, d_b by $\varphi(d_a) = m_A$ and $\varphi(d_b) = m_B$. Also, denote the set of common components of m_A and m_B by $F = A \cap B$.

Claim 2.34. *If $f \in \mathrm{PMap}(S)$, then $\varphi(f)$ fixes each curve in F.*

Proof of Claim 2.34. Choose a Humphries generating set $\tilde{\mathcal{H}}$ of $\mathrm{PMap}(S)$ such that $d_a, d_b \in \tilde{\mathcal{H}}$ and for every $d_c \in \tilde{\mathcal{H}}$ either $i(c,a) = 0$ or $i(c,b) = 0$. Now, denote the image of $d_c \in \tilde{\mathcal{H}}$ by $m_C = \varphi(d_c)$. Without loss of generality assume that $i(c,a) = 0$. This implies the multitwists m_A and m_C commute, and so $i(A,C) = 0$. In particular, $i(F,C) = 0$ and therefore m_C fixes each curve in F. Since this is true for any $d_c \in \tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}$ generates $\mathrm{PMap}(S)$, the claim follows. □

We may assume that $i(a,b) = 0$. Otherwise, $i(a,b) = 1$ and we may consider $d_c \in \mathcal{H}$ such that $i(c,a) = 1$ and $i(c,b) = 0$. Since there exists $f \in \mathrm{PMap}(S)$ with $f(a) = b$ and $f(b) = c$, we have that $m_A^{\varphi(f)} = m_B$ and $m_B^{\varphi(f)} = m_C$. By the previous claim, we obtain

$$F = \varphi(f)(F) = \varphi(f)(A) \cap \varphi(f)(B) = B \cap C.$$

So, we just rename c to a and assume $i(a,b) = 0$.

Choose non-separating curves c, d, x, y, z such that $\{a, b, c, d, x, y, z\}$ forms a lantern (see [32, Chapter 5]), that is, such that

$$d_a d_b d_c d_d = d_x d_y d_z.$$

Since S has genus $g \geq 3$, we may choose the curves so that the complement $S \backslash (c_1 \cup c_2)$ is connected for any two curves $c_1, c_2 \in \{a, b, c, d, x, y, z\}$. Observe that if $i(c_1, c_2) = 0$, the images $m_{C_1} = \varphi(c_1)$ and $m_{C_2} = \varphi(c_2)$ share the same components $F = C_1 \cap C_2$. To see this, consider $f \in \mathrm{PMap}(S)$ such that $f(a) = c_1$ and $f(b) = c_2$, and repeat the argument in the previous paragraph.

For each $c_i \in \{a, b, c, d, x, y, z\}$ we may write the image of d_{c_i} as

$$\varphi(d_{c_i}) = m_{C_i} = m_F m_{C_i'},$$

where m_F is a multitwist with components F, $m_{C_i'}$ is a multitwist with components $C_i' = C \setminus F$ and $m_{C_i'}$ commutes with m_F. Since any two non-separating Dehn twists are conjugate, it follows from Claim 2.34 that m_F is independent of the curve c_i. Now, considering the lantern relation in the image we deduce that

$$m_{A'} m_{B'} m_{C'} m_{D'} m_F^4 = m_{X'} m_{Y'} m_{Z'} m_F^3.$$

Showing that $F = \emptyset$ is equivalent to showing that $m_F = \mathrm{id}$, which in turn is equivalent to showing that

$$m_{A'} m_{B'} m_{C'} m_{D'} = m_{X'} m_{Y'} m_{Z'}. \tag{2.6}$$

To prove Equation (2.6), consider a maximal reducing multicurve η containing F. Also, consider the homomorphisms $\varphi_i : \mathrm{PMap}(S) \to \mathrm{PMap}(R_i')$ defined as in Equation (2.5). Recall that each φ_i is induced by an embedding. Therefore, the image of a lantern in S is a lantern in R_i'. By identifying the Dehn twists in $\prod_i \mathrm{PMap}(R_i')$ with Dehn twists in $\mathrm{PMap}(S')$, we conclude that

$$m_{A'} m_{B'} m_{C'} m_{D'} = m_{X'} m_{Y'} m_{Z'},$$

as we wanted. □

Lemma 2.35. *Let S, S' be two orientable finite-type surfaces, let $g \geq 3$ be the genus of S and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a multitwist-preserving map. If d_a, d_b are two Dehn twists such that $i(a, b) = 1$, then every component of $\varphi(d_a)$ intersects exactly one component of $\varphi(d_b)$.*

Proof. Choose a Humphries generating set $\mathcal{H}$ such that $d_a, d_b \in \mathcal{H}$. By the previous lemma, the multitwists $\varphi(d_a)$ and $\varphi(d_b)$ share no components. Moreover, d_a, d_b satisfy the braid relation which implies that $\varphi(d_a), \varphi(d_b)$ also satisfy the braid relation. It now follows directly from Theorem D that every component of $\varphi(d_a)$ intersects one component of $\varphi(d_b)$. □

We are now ready to complete the proof of Theorem B.

Proof of Theorem B. Fix orientations on S and S'. Let η be a maximal reducing multicurve for φ. By Lemma 2.24, we know that φ fixes each curve in η with orientation. Thus, we can consider the cutting map

$$\mathrm{Cut}_\eta \circ \varphi : \mathrm{PMap}(S) \to \prod_i \mathrm{PMap}(R'_i),$$

where $R'_i \subset S'$ are the subsurfaces such that $S' \setminus \eta = \sqcup_i R'_i$. Define

$$\varphi_i = \pi_i \circ \mathrm{Cut}_\eta \circ \varphi,$$

where π_i is the projection to the ith component. Recall that φ_i is either trivial or twist preserving and, by Theorem 2.20, each non-trivial φ_i is induced by an embedding $\iota_i : S \hookrightarrow R'_i \subset S'$.

Consider the multi-embedding $\mathcal{I} = \{\iota_i |\ \varphi_i \neq \mathrm{id}\}$. The goal is to prove that $\mathcal{I}$ induces the map φ. First, we check that $\mathcal{I}$ satisfies the hypotheses of Lemma 2.18.

Let n be a curve on S homotopic to a puncture. Choose curves a, b, c and x on S such that $i(a,b) = i(b,c) = 1$, $i(a,c) = 0$ and a closed regular neighborhood of $a \cup b \cup c$ has boundary curves n and x (see Figure 2.2). Consider the multitwist

$$m_A = d^{\epsilon_1}_{\iota_1(a)} \cdots d^{\epsilon_k}_{\iota_k(a)}, \tag{2.7}$$

where $\{\iota_1, \ldots, \iota_k\} = \mathcal{I}$ and $\epsilon_i \in \{-1, 1\}$ depending on whether ι_i preserves or reverses orientation. Analogously, we define m_B, m_C, m_X and m_N. Now, to check that $\mathcal{I}$ satisfies the hypotheses of Lemma 2.18 it is enough to see that $m_N = \mathrm{id} \in \mathrm{PMap}(S')$.

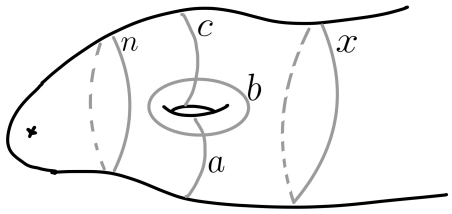


Figure 2.2: Curves a, b, c, n, x on S.

By definition $\mathrm{Cut}_\eta \circ \varphi(d_a) = m_A$. Recall that the kernel of Cut_η is

$$\ker \mathrm{Cut}_\eta = \langle \{ d_{\eta_i} |\ \eta_i \in \eta \} \rangle.$$

Thus, identifying the Dehn twists in R'_i with Dehn twists in S' via inclusion, we have $\varphi(d_a) = m_A \circ m$ for a multitwist m with components in η. By

Lemma 2.35, each component of $\varphi(d_a)$ intersects some component of $\varphi(d_b)$. However, the components of $\varphi(d_b)$ do not intersect η. We conclude $m = \mathrm{id}$ and $\varphi(d_a) = m_A$. Similarly, we deduce $\varphi(d_b) = m_B$ and $\varphi(d_c) = m_C$. Since x is a separating curve, checking that $\varphi(d_X) = m_X$ requires a little extra work. For this purpose, choose curves $\{d, e, y, z\}$ that together with $\{a, c, x\}$ form a lantern (see Figure 2.3). Since S has genus $g \geq 3$, we may choose the curves so that they are all non-separating except for x and such that a, c, d, e are disjoint and bound a four-punctured sphere. Since the curves d, e, y, z are non-separating, we can repeat the argument above and conclude that $\varphi(d_d) = m_D$, $\varphi(d_e) = m_E$, $\varphi(d_y) = m_Y$ and $\varphi(d_z) = m_Z$, where the multitwists m_* are also defined as in Equation (2.7).

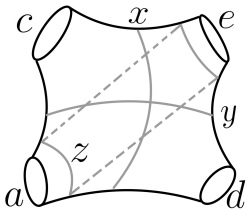


Figure 2.3: Curves a, c, d, e and x, y, z in a lantern.

Since the curves $\{a, c, d, e, x, y, z\}$ form a lantern

$$d_x = d_a d_c d_d d_e d_y^{-1} d_z^{-1}.$$

and thus

$$\varphi(d_x) = m_A m_C m_D m_E m_Y^{-1} m_Z^{-1}.$$

As every $\iota_i \in \mathcal{I}$ is an embedding, it follows from the last equality that $\varphi(d_x) = m_X$.

Now, for each $\iota_i \in \mathcal{I}$, the Dehn twists $\{d_{\iota_i(a)}, d_{\iota_i(b)}, d_{\iota_i(c)}, d_{\iota_i(x)}, d_{\iota_i(n)}\}$ satisfy the following relation known as the 3-chain relation (see [32, Proposition 4.12])

$$\left(d_{\iota_i(a)} d_{\iota_i(b)} d_{\iota_i(c)}\right)^4 = d_{\iota_i(x)} d_{\iota_i(n)}.$$

The same relation in $\mathrm{PMap}(S)$ yields that $(d_a d_b d_c)^4 = d_x$. Adding these equations for every $\iota_i \in \mathcal{I}$ we obtain that

$$\begin{aligned} m_N &= (m_A m_B m_C)^4 m_X^{-1} \\ &= (\varphi(d_a)\varphi(d_b)\varphi(d_c))^4 \varphi(d_x)^{-1} \\ &= \varphi((d_a d_b d_c)^4 d_x^{-1}) \\ &= \mathrm{id}. \end{aligned}$$

So $m_N = \mathrm{id}$, as we wanted.

So far, we used Lemma 2.18 to show the multi-embedding $\mathcal{I}$ induces a homomorphism $\widehat{\mathcal{I}} : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$. We are left to see that $\widehat{\mathcal{I}} = \varphi$. Note that for any Dehn twist $d_a \in \mathcal{H}$ in a Humphries generating set, we have that $\varphi(d_a) = \widehat{\mathcal{I}}(d_a) \circ m$ for some $m \in \ker \mathrm{Cut}_\eta$. Just as before, Lemma 2.35 implies $m = \mathrm{id}$ and so $\varphi = \widehat{\mathcal{I}}$. □

2.4 Classification of homomorphisms

In this section we prove Theorem A. Notice that by Theorem B we may restate Theorem A as follows.

Theorem A. *Let S, S' be two orientable finite-type surfaces of genus $g \geq 4$ and $g' \leq 6 \cdot 2^{g-4}$. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a non-trivial homomorphism, then φ is multitwist preserving.*

Next lemma reduces the proof of Theorem A to the case of homomorphisms fixing no curve.

Lemma 2.36. *If Theorem A holds for homomorphisms fixing no curve, then Theorem A holds for arbitrary homomorphisms.*

Proof. Let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a homomorphism fixing some curve and let $C = \{c_1, c_2, \ldots, c_k\}$ be a maximal multicurve such that φ fixes each curve $c_i \in C$. Since $\mathrm{PMap}(S)$ has trivial abelianization, the image of φ fixes the orientation of each $c_i \in C$. Therefore, we may consider the cutting homomorphism

$$\mathrm{PMap}(S) \xrightarrow{\varphi} \mathcal{Z}_0(C) \xrightarrow{\mathrm{Cut}_C} \prod_i \mathrm{PMap}(R_i'),$$

where each R_i' is a connected component of $S' \setminus \bigcup_{c_i \in C} c_i$.

We consider the maps

$$\varphi_i = \pi_i \circ \mathrm{Cut}_C \circ \varphi,$$

where π_i is the projection to the ith component. The maximality of C implies each projection φ_i fixes no curve and, by hypotheses, each non-trivial φ_i is multitwist preserving. Since the kernel of Cut_C is generated by Dehn twists, it follows that φ is also multitwist preserving. □

The rest of the section contains the proof of Theorem A for homomorphisms fixing no curve. The first step is to prove that the image of a Dehn twist has infinite order.

2.4.1 No torsion

In this section we prove the following proposition.

Proposition 2.37. *Let S, S' be two orientable surfaces of genus either $g \geq 4$ and $g' \leq 2^g - 2$, or $g = 3$ and $g' \leq 5$. Then, every non-trivial homomorphism $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ sends non-separating Dehn twists to infinite order elements.*

Proposition 2.37 is a consequence of the next two lemmas.

Lemma 2.38. *Let S be an orientable surface of genus $g \geq 3$, let G be a group and let $\varphi : \mathrm{PMap}(S) \to G$ be a homomorphism. If $\varphi(d_a)$ has order two for any non-separating Dehn twist d_a, then G contains a symmetric group of order $(2g+1)!$.*

Proof. Choose $2g+1$ non-separating chained curves $c_1, c_2, \ldots, c_{2g+1}$, that is, curves satisfying

$$i(c_i, c_j) = \begin{cases} 1 & |i-j| = 1, \\ 0 & |i-j| > 1. \end{cases}$$

Now, consider the subgroup

$$H = \langle d_{c_1}, d_{c_2} \ldots, d_{c_{2g+1}} \rangle.$$

By hypothesis, $\varphi(H) < G$ is a subgroup generated by elements $\varphi(d_{c_i})$ of order two. Note that the elements $\varphi(d_{c_i})$ and $\varphi(d_{c_j})$ commute if $|i-j| > 1$ and satisfy the braid relation if $|i-j| = 1$. Therefore, $\varphi(H)$ is a quotient of the symmetric group of order $(2g+1)!$. In other words, $\varphi(H)$ is either $\{1\}$, $\mathbb{Z}/2\mathbb{Z}$ or the entire symmetric group. Since the abelianization of $\mathrm{PMap}(S)$ is trivial, it follows that $\varphi(H) \neq 1, \mathbb{Z}/2\mathbb{Z}$. Thus, $\varphi(H) < G$ is the symmetric group of order $(2g+1)!$. □

The next result deals with the case where the image of a Dehn twist has finite order greater than two.

Lemma 2.39. *Let S be an orientable surface of genus $g \geq 3$, let G be a group and let $\varphi : \mathrm{PMap}(S) \to G$ be a homomorphism. If φ sends non-separating Dehn twists to elements of finite order $d > 2$, then G contains an abelian subgroup V of order at least $\min(2 \cdot d \cdot n^{g-1}, d^g)$, where $n \neq 1$ is some divisor of d.*

In particular, if $g > 3$ the abelian subgroup V contains at least 2^{g+2} elements. If $g = 3$, the group V has cardinality at least 27.

Proof. Choose non-separating disjoint curves $a_1, \ldots, a_g, a_{g+1}$ such that $S \setminus \bigcup_{i=1}^{g} a_i$ is connected and $S \setminus \bigcup_{i=2}^{g+1} a_i$ is also connected. For convenience, we write $v_c = \varphi(d_c)$ for any curve c.

Let $n \geq 1$ be the smallest natural number such that there are numbers $k \in \{1, 2, \dots, g\}$ and $m_i \in \mathbb{Z}$ that satisfy the equation

$$v_{a_k}^n = \prod_{i=1,\, i\neq k}^{g} v_{a_i}^{m_i}.$$

Clearly, $n \leq d$ since v_{a_k} has order d. In fact, the integer n divides d, as otherwise n would not be minimal. Without loss of generality, we assume $k = 1$.

Note that $n \geq 2$. Indeed, if $n = 1$ we can take $h \in \mathrm{PMap}(S)$ such that $h(a_1) = a_{g+1}$ and $h(a_i) = a_i$ for every $i \in \{2, \dots, g\}$. Now, conjugating

$$\begin{aligned} v_{a_{g+1}} &= \varphi(h) \cdot v_{a_1} \cdot \varphi(h)^{-1} \\ &= \varphi(h) \cdot v_{a_2}^{k_2} \dots v_{a_g}^{k_g} \cdot \varphi(h)^{-1} \\ &= v_{a_2}^{k_2} \dots v_{a_g}^{k_g} \\ &= v_{a_1}. \end{aligned}$$

By conjugation again, $n = 1$ implies $v_{a_i} = v_{a_j}$ for every i, j. In turn, this implies that the image of φ is cyclic. Since $\mathrm{PMap}(S)$ has trivial abelianization, we conclude that $n \neq 1$.

If $n \geq 2$, the same conjugation argument yields $v_a^n = v_b^n$ for any two non-separating curves a and b. This equality is of use later on in the proof.

Consider the finite abelian group $\tilde{V} = \langle v_{a_1}, \dots, v_{a_g} \rangle$. Using the minimality of n, it is an exercise to see that all the elements of the form

$$v_{a_1}^{m_1} \cdot v_{a_2}^{n_2} \dots v_{a_g}^{n_g},$$

with $m_1 \in \mathbb{Z}/d\mathbb{Z}$ and $n_i \in \mathbb{Z}/n\mathbb{Z}$, are distinct. That is, $\tilde{V}$ is a group of order at least $d \cdot n^{g-1}$. If $n = d$, then $\tilde{V}$ has d^g elements and we are done. It is left to see the case $d > n$.

We claim that if $d > n$, then there exists a non-separating curve c contained in $S \setminus \bigcup_{i=1}^{g} a_i$ such that $v_c \notin \tilde{V}$. To prove it, we assume there is no such c.

Assume that every non-separating curve c in $S \setminus \bigcup_{i=1}^{g} a_i$ satisfies $v_c \in \tilde{V}$. As $\tilde{V}$ is abelian, all the images v_c pairwise commute. Now, we can take non-separating curves $\{b, x, y, z\}$ on $S \setminus \bigcup_{i=1}^{g} a_i$ such that $\{c_1, c_2, c_3, b, x, y, z\}$ form a lantern (see [32, Chapter 5]). Considering the lantern relation in the image we have that

$$v_x v_y v_z = v_{a_1} v_{a_2} v_{a_3} v_b.$$

The elements in the equation above pairwise commute as they are contained in $\tilde{V}$. So, by taking n-th powers, we obtain

$$v_x^n = v_{a_1}^n v_{a_2}^n v_{a_3}^n v_b^n v_y^{-n} v_z^{-n}.$$

Recall that $v_x^n = v_a^n$ for any non-separating curve a. Thus, the equation above implies $v_x^n = v_x^{2n}$ and therefore $v_x^n = \mathrm{id}$. However, this contradicts that $n < d$ (recall d is the order of v_x). We conclude that our assumption was false, i.e, there exists a non-separating curve c in $S \setminus \bigcup_{i=1}^{g} a_i$ such that $v_c \notin \tilde{V}$. Fix any curve c with this property.

To finish the case $d > n$, notice that the abelian group $\langle \tilde{V}, v_c \rangle$ has elements of the form

$$v_c^{\epsilon} \cdot v_{a_1}^{m_1} \cdot v_{a_2}^{n_2} \dots v_{a_g}^{n_g},$$

where $\epsilon \in \{0, 1\}$, $m_1 \in \mathbb{Z}/d\mathbb{Z}$ and $n_i \in \mathbb{Z}/n\mathbb{Z}$. Those elements are all distinct. Therefore, $\langle \tilde{V}, v_c \rangle$ has order at least $2 \cdot d \cdot n^{g-1}$. □

We may now prove Proposition 2.37.

Proof of Proposition 2.37. Seeking a contradiction, assume there is a non-separating Dehn twist d_c whose image has order $d < \infty$.

If $d = 2$, by Lemma 2.38 the group $\mathrm{PMap}(S')$ has a finite subgroup of order $(2g+1)!$. Recall that a finite subgroup of $\mathrm{PMap}(S')$ has order at most $84(g'-1)$ (see Theorem 2.2). We deduce that $(2g+1)! \leq 84(g'-1)$ and so

$$g' \geq \frac{(2g+1)!}{84} + 1 \geq 2^g - 1.$$

However, this contradicts the hypothesis $g' \leq 2^g - 2$. Thus, $\varphi(d_c)$ has order $d > 2$.

If $d > 2$ and $g > 3$, then $\varphi(\mathrm{PMap}(S))$ has an abelian subgroup of order 2^{g+2} (see Lemma 2.39). Recall that a finite abelian subgroup of $\mathrm{PMap}(S')$ has order at most $4g'+4$ (see Theorem 2.3). As a consequence, $2^{g+2} \leq 4g'+4$ and

$$2^g - 1 \leq g'.$$

The last inequality contradicts the hypothesis $g' \leq 2^g - 2$. Thus, if $g > 3$ the element $\varphi(d_c)$ has infinite order.

Lastly, if $g = 3$ and $d > 2$ the image of φ has an abelian subgroup of finite order at least 27 (see Lemma 2.39). Again, using the bound on finite abelian subgroups, we have $27 \leq 4g' + 4$ which contradicts the hypothesis $g' \leq 5$. Thus, if $g = 3$ then the image $\varphi(d_c)$ has infinite order. □

2.4.2 Counting curves

Let S, S' be two surfaces as in Proposition 2.37 and let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a non-trivial homomorphism. Theorem 2.9 and Proposition 2.37 together guarantee the existence of an $n \in \mathbb{N}$ such that $\varphi(d_c^n)$ is a multitwist for every non-separating curve c. We write $\varphi_*(c)$ to denote the components of the multitwist $\varphi(d_c^n)$. In this section we compute an upper bound on the cardinality of $\varphi_*(c)$.

It is immediate that φ_* verifies the following properties:

- If $i(c, c') = 0$, then $i(\varphi_*(c), \varphi_*(c')) = 0$.
- The cardinality $|\varphi_*(c)|$ is independent of the non-separating curve c.
- $\varphi(f) \cdot \varphi_*(c) = \varphi_*(f(c))$ for every $f \in \mathrm{PMap}(S)$.

The next result guarantees that forgetting punctures does not change the cardinality of $\varphi_*(c)$. The result was originally stated in [11, Lemma 6.4] with different hypotheses, however their proof remains valid for the version we state here.

Lemma 2.40. *Let S, S' be two orientable surfaces of genus either $g \geq 4$ and $g' \leq 2^g - 2$, or $g = 3$ and $g' \leq 5$. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a homomorphism fixing no curve, then the composition with the forgetful map satisfies $|(\mathcal{F} \circ \varphi)_*(c)| = |\varphi_*(c)|$.*

Proof. The statement follows from the proof of [11, Lemma 6.4]. □

Before computing the upper bound, we need the following two lemmas.

Lemma 2.41. *Let S, S' be two orientable surfaces of genus either $g \geq 5$ and $g' \leq 2^g - 2$, or $g = 4$ and $g' \leq 10$. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a homomorphism fixing no curve and c is a non-separating curve on S, then $\varphi(\mathcal{Z}_0(c))$ fixes every curve in $\varphi_*(c)$ with orientation.*

Proof. Since φ fixes no curve, S' has no boundary.

We compute a rough bound for $|\varphi_*(c)|$. By Lemma 2.40, we can assume S' is closed. It follows that

$$|\varphi_*(c)| \leq 3g' - 3 \leq 3 \cdot 2^g - 9.$$

Now, let S_c be the complement of an open regular neighborhood of c. The inclusion $S_c \hookrightarrow S$ induces an epimorphism $\mathrm{PMap}(S_c) \xrightarrow{\iota} \mathcal{Z}_0(c)$. The image of $\varphi \circ \iota$ fixes the multicurve $\varphi_*(c)$ setwise and so induces an action of $\mathrm{PMap}(S_c)$ on $\varphi_*(c)$. Since S_c has genus $g-1$, Theorem 2.7 implies the action is trivial. In other words, $\varphi(\mathcal{Z}_0(c))$ fixes every curve in $\varphi_*(c)$.

Since $\mathrm{PMap}(S_c)$ has trivial abelianization, it follows that $\varphi(\mathcal{Z}_0(c))$ fixes the orientation of each curve in $\varphi_*(c)$. □

Lemma 2.42. *Let S, S' be two orientable surfaces of genus either $g \geq 5$ and $g' \leq 2^g - 2$, or $g = 4$ and $g' \leq 10$. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a homomorphism fixing no curve and c, c' are two disjoint non-separating curves such that $S \setminus (c \cup c')$ is connected, then $\varphi_*(c)$ and $\varphi_*(c')$ share no curves.*

Proof. Since φ fixes no curve, S' has no boundary. Also, by Lemma 2.40, we can assume S' has no punctures. Write $F = \varphi_*(c) \cap \varphi_*(c')$.

First, we claim that if d is a non-separating curve on S with $i(d,c)=0$ and such that $S \setminus (d \cup c)$ is connected, then $F \subset \varphi_*(d)$. Indeed, consider $f \in \mathrm{PMap}(S)$ such that $f \in \mathcal{Z}_0(c)$ and $f(c') = d$. We know that

$$\varphi(f) \cdot \varphi_*(c') = \varphi_*(f(c')) = \varphi_*(d).$$

Now, Lemma 2.41 implies $\varphi(f)$ fixes each curve in $F \subset \varphi_*(c)$. In particular, $F \subset \varphi_*(d)$.

We claim $F \subset \varphi_*(e)$, for every e non-separating curve on S with $i(c,e) = 1$. As $g \geq 4$, there exists a non-separating curve d on S with $i(d,c) = i(d,e) = 0$ and such that both $S \setminus (d \cup c)$ and $S \setminus (d \cup e)$ are connected. Then, considering $f \in \mathcal{Z}_0(d)$ such that $f(c) = e$, we may repeat the argument above to deduce that $F \subset \varphi_*(e)$.

From the last two paragraphs it follows there is a Humphries generating set $\mathcal{H}$ of $\mathrm{PMap}(S)$ with the property that $F \subset \varphi_*(c)$ for every element $d_c \in \mathcal{H}$. Observe that for every $d_c \in \mathcal{H}$ there exists $d_b \in \mathcal{H}$ with $d_c \in \mathcal{Z}_0(b)$. Combining these facts with Lemma 2.41, we deduce that every image $\varphi(d_c)$ fixes each curve $x \in F$. Since $\mathcal{H}$ generates, we conclude $\varphi(\mathrm{PMap}(S))$ fixes each curve $x \in F$. By hypothesis, φ fixes no curve, thus $F = \emptyset$. □

The following proposition gives the upper bound for $|\varphi_*(c)|$.

Proposition 2.43. *Let S, S' be two orientable finite-type surfaces of genus either $g \geq 5$ and $g' \leq 2^g - 2$, or $g = 4$, and $g' \leq 10$. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a homomorphism fixing no curve, then $|\varphi_*(c)| \leq \frac{g'-1}{g-1}$.*

Proof. Choose $a_1, \ldots, a_{3g-3}$ disjoint non-separating curves such that each $S \setminus (a_i \cup a_j)$ is connected for any i, j. From Lemma 2.42, we deduce that $\bigcup_{i=1}^{3g-3} \varphi_*(a_i)$ has $K(3g-3)$ distinct curves, where $K = |\varphi_*(c)|$. Since we may assume S' is closed (see Lemma 2.40), we have that

$$K(3g-3) \leq 3g' - 3$$

and the statement follows. □

2.4.3 Dehn twists to multitwists

In this section we complete the proof of Theorem A. So far, we showed that the image of a non-separating Dehn twist is an infinite order element, root of a multitwist and that the associated multicurve has at most $\frac{g'-1}{g-1}$ components. The last step is to show that the image cannot be a proper root of a multitwist, and hence it is actually a multitwist. The argument is by induction.

Proposition 2.44. *(Base case) Let S, S' be two orientable finite-type surfaces of genus $g \geq 4$ and $g' \leq 6$, respectively. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a homomorphism fixing no curve, then φ is multitwist preserving.*

Proof. Let c be a non-separating curve on S. Looking for a contradiction, assume $\varphi(d_c)$ is a proper root of a multitwist of degree $k > 1$.

By Proposition 2.43, we have that $|\varphi_*(c)| \leq \frac{g'-1}{g-1} < 2$ and so $\varphi(d_c^k)$ is a power of a Dehn twist. The curve $\varphi_*(c)$ is either separating or non-separating; we treat these two cases separately.

If $\varphi_*(c)$ is non-separating, then $R' = S' \setminus \varphi_*(c)$ is a connected surface. Let $S_c \subset S$ be the complement of an open regular neighborhood of c. The inclusion $S_c \hookrightarrow S$ induces an epimorphism $\mathrm{PMap}(S_c) \xrightarrow{\iota} \mathcal{Z}_0(c)$. By Lemma 2.41, we may consider the cutting homomorphism

$$\gamma = \mathrm{Cut}_{\varphi_*(c)} \circ \varphi \circ \iota : \mathrm{PMap}(S_c) \to \mathrm{PMap}(R').$$

Notice that R' is a surface with at least two punctures and of genus at most five. Also, the image of γ commutes with the element $h = \gamma(d_c)$ of order k. Up to taking a power of h, we can assume h has primer order. Thus, we can consider the Birman-Hilden map defined in Equation (2.4): Let Q' be the surface obtained by removing the singular points from the quotient $R'/\langle h \rangle$ and let ψ be the Birman-Hilden map induced by h, we have that

$$\mathrm{PMap}(S_c) \xrightarrow{\gamma} \mathcal{Z}(h) \xrightarrow{\psi} \mathrm{Map}^{\pm}(Q').$$

Recall that the kernel of ψ is $\langle h \rangle$. Since $\mathrm{PMap}(S_c)$ has trivial abelianization as $g \geq 4$, the composition $\psi \circ \gamma$ is a non-trivial homomorphism. Even more, the image of $\psi \circ \gamma$ is contained in $\mathrm{Map}(Q')$.

As a consequence of [11, Lemma 7.3] the surface Q' has genus $\tilde{g} < 3$ and $\tilde{p} \leq 12$ punctures. By Theorem 2.7, the image of $\psi \circ \gamma$ fixes each puncture in Q'. This implies that

$$\psi \circ \gamma : \mathrm{PMap}(S_c) \to \mathrm{PMap}(Q')$$

is a non-trivial map that reduces the genus of the involved surfaces, however this contradicts [11, Proposition 7.1].

We are left with the case where $\varphi_*(c)$ is a separating curve. We denote R'_1 and R'_2 the connected components of $S' \setminus \varphi_*(c)$. Lemma 2.40 implies that $\varphi_*(c)$ does not bound a punctured disk, and so the surfaces R'_1, R'_2 both have positive genus at most five. Moreover, both surfaces R'_1, R'_2 have at least one puncture. As before, consider the composition

$$\mathrm{PMap}(S_c) \xrightarrow{\iota} \mathcal{Z}_0(c) \xrightarrow{\mathrm{Cut}_{\varphi_*(c)} \circ \varphi} \mathrm{PMap}(R'_1) \times \mathrm{PMap}(R'_2).$$

We denote the projections $\gamma_i = \pi_i \circ \mathrm{Cut}_{\varphi_*(c)} \circ \varphi \circ \iota$ for $i = 1, 2$, where π_1, π_2 are the projections to the first and second component. Note that at least one of the two projections is non trivial. Without loss of generality, assume $h = \gamma_1(d_c)$ is non trivial. We have that $h = \gamma_1(d_c)$ is an element of

finite order $k' > 1$ and commutes with the image of γ_1. As in the previous paragraph, we have a map

$$\mathrm{PMap}(S_c) \xrightarrow{\gamma_1} \mathcal{Z}_0(h) \xrightarrow{\psi} \mathrm{Map}(Q'),$$

where Q' is the surface obtained by removing the singular points from the quotient $R_1'/\langle h \rangle$.

Again, we know $\psi \circ \gamma_1$ is a non-trivial map. Also, by Theorem 2.7 the image of $\psi \circ \gamma_1$ is contained in $\mathrm{PMap}(Q')$, since the number of punctures of Q' is $\tilde{p} \leq 12$ (see [11, Lemma 7.3]). By [11, Lemma 7.3], the surface Q' has genus at most two. We conclude that $\psi \circ \gamma_1$ is a non-trivial homomorphism between pure mapping class groups reducing the genus of the involved surfaces. Again, this contradicts [11, Proposition 7.1]. □

Before finishing the proof of Theorem A with the inductive step, we state the last lemma we require.

Lemma 2.45. *Let X be an orientable finite-type surface, let $C = \{c_1, c_2, \dots\}$ be a multicurve on X and let Y_i for $i = 1, \dots, k$ be the connected components of $S \setminus C$. If $h \in \mathrm{PMap}(X)$ is an element of finite order $k \in \mathbb{N}$ and h fixes each curve c_i with orientation, then $h|_{Y_i}$ induces an element of order k in $\mathrm{PMap}(Y_i)$ for every $i = 1, \dots, k$.*

Proof. We may realize h by an isometry ϕ of a hyperbolic structure in X (see [55, Theorem 5]). Realizing the curves in C by geodesics, ϕ restricts to each component Y_i as a finite order element. Note the isometry ϕ is locally non-trivial, since the only locally trivial isometry is the identity. Therefore, $\phi|_{Y_i}$ is a diffeomorphism of finite order in each Y_i and so induces a non-trivial element of finite order in each $\mathrm{PMap}(Y_i)$. Since the argument can be repeated for any non-trivial power of ϕ, we conclude that the diffeomorphism $\phi|_{Y_i}$ has order k and hence also does the induced mapping class. □

Finally, we complete the proof of Theorem A.

Proof of Theorem A (inductive step). Let $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be a homomorphism fixing no curve, where S, S' are finite-type orientable surfaces of genus $g \geq 5$ and $g' \leq 6 \cdot 2^{g-4}$, respectively.

Let c_1 be a non-separating curve in S and let S_{c_1} be the complement of an open regular neighborhood of c_1. The inclusion $S_{c_1} \hookrightarrow S$ induces the epimorphism $\iota : \mathrm{PMap}(S_{c_1}) \to \mathcal{Z}_0(c_1)$. By Lemma 2.41, $\mathcal{Z}_0(c_1)$ is a subgroup of $\mathcal{Z}_0(\varphi_*(c_1))$. Therefore, we may consider the following composition of maps

$$\mathrm{PMap}(S_{c_1}) \xrightarrow{\varphi \circ \iota} \mathcal{Z}_0(\varphi_*(c_1)) \xrightarrow{\mathrm{Cut}_{\varphi_*(c_1)}} \prod_i \mathrm{PMap}(R_i'),$$

where R_i' are the connected components of $S \setminus \varphi_*(c_1)$.

We define the maps

$$\varphi_j = \pi_j \circ \mathrm{Cut}_{\varphi_*(c_1)} \circ \varphi \circ \iota,$$

where π_j is the projection to the jth component. If all the projections φ_j are multitwist preserving, then φ is also multitwist preserving. Seeking a contradiction, we assume that at least one projection φ_j is not multitwist preserving.

First, suppose there are two projections $\varphi_j, \varphi_{j'}$ (with $j \neq j'$) that are not multitwist preserving. In this case, we may use the inductive hypothesis on the surfaces R'_j and $R'_{j'}$ to conclude that both surfaces have genus greater than $6 \cdot 2^{g-5}$. It follows that $g' > 6 \cdot 2^{g-4}$, which contradicts the hypothesis of the statement.

To complete the proof, we rule out the case where there is a unique projection φ_j that is not multitwist preserving. Without loss of generality, we assume that φ_1 is the unique non-multitwist preserving projection.

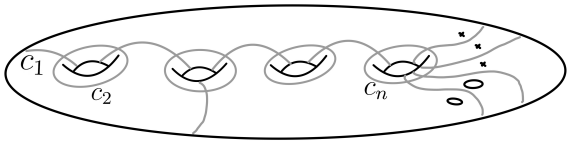


Figure 2.4: The set of curves H.

The contradiction follows easily from the next two claims. Choose $\mathcal{H}$ a Humphries generating set with $d_{c_1} \in \mathcal{H}$ and let $H = \{c |\, d_c \in \mathcal{H}\}$.

Claim 2.46. *Consider $c_i \in H$. We can decompose $\varphi(d_{c_i})$ as*

$$\varphi(d_{c_i}) = M_i h_i,$$

where M_i is a multitwist with components contained in $\varphi_*(c_i)$ and h_i is a finite order element that fixes every curve $x \in \varphi_*(c_j)$ for every $c_j \in H$.

Claim 2.47. *Consider $c_i, c_j \in H$. If $i(d_{c_i}, d_{c_j}) = 1$, then $h_i h_j h_i = h_j h_i h_j$ and $M_i M_j M_i = M_j M_i M_j$.*

We assume both claims and defer their proof for later.

Note the finite order element h_i fixes the curves in $\varphi_*(c_i)$. In particular, h_i commutes with M_i, which implies that $\varphi(d_{c_i}^k) = M_i^k$. Thus, the components of the multitwist M_i are precisely the curves $\varphi_*(c_i)$.

Using Claim 2.47 and Theorem D, the curves $\varphi_*(c_i)$ admit the following description. Let Z be a closed regular neighborhood of $\bigcup_{c_i \in H} c_i$ (see Figure 2.4). A closed regular neighborhood of the curves $\bigcup_{c_i \in H} \varphi_*(c_i)$ is a disjoint union $\bigsqcup Z_i$ of subsurfaces $Z_i \subset S'$ each homeomorphic to Z.

If the curves $\bigcup_{c_i \in H} \varphi_*(c_i)$ fill S', then there is a single component Z_1. Moreover, the Alexander method [32, Proposition 2.8] yields that h_1 is trivial and so φ is multitwist preserving as $\varphi(d_{c_i}) = M_i$.

If the curves $\bigcup_{c_i \in H} \varphi_*(c_i)$ do not fill S', then there is a boundary curve x of $\bigsqcup Z_i$ which is non-trivial on S'. Since h_1 fixes each curve in $\varphi_*(c_1)$, then h_1 fixes each component of $\bigsqcup Z_i$. Even more, the Alexander method implies that h_1 fixes the boundary curve x. Repeating the argument for every h_j, we deduce that any h_j also fixes the curve x. It follows that the image of φ fixes the curve x, but this contradicts that φ fixes no curve.

We are only left to prove the claims.

Proof of Claim 2.46. Let c_2 be a curve on S with $i(c_1, c_2) = 1$. Since $g \geq 5$, we may choose a curve c_n on S with $i(c_1, c_n) = i(c_2, c_n) = 0$ and such that both $S \setminus (c_1 \cup c_n)$ and $S \setminus (c_2 \cup c_n)$ are connected (see Figure 2.4).

Note the element $\varphi_1(d_{c_1})$ has order k, where k is the degree of the proper root $\varphi(d_{c_1})$. Indeed, Theorem B implies that all non-trivial projections φ_i with $i \neq 1$ are induced by multi-embeddings and so the elements $\varphi_i(d_{c_1})$ are either trivial or have infinite order. Since $\mathrm{Cut}_{\varphi_*(c_1)} \circ \varphi(d_{c_1})$ has finite order, the elements $\varphi_i(d_{c_1})$ are trivial for $i \neq 1$. It follows $\varphi_1(d_{c_1})$ has order k.

Now, we cut the surface R_1' along the curves in $\varphi_{1*}(c_n)$. Denote by $S_{c_1 \cup c_n}$ the complement to an open regular neighborhood of $c_1 \cup c_n$. The inclusion $S_{c_1 \cup c_n} \to S_{c_1}$ induces a homomorphism $\sigma : \mathrm{PMap}(S_{c_1 \cup c_n}) \to \mathrm{PMap}(S_{c_1})$. Since the restriction $(\varphi_1 \circ \sigma)|_{\mathrm{PMap}(S_{c_1 \cup c_n})}$ is contained in $\mathcal{Z}_0(\varphi_{1*}(c_n)) \subset \mathrm{PMap}(R_1')$, we can consider the composition

$$\mathrm{PMap}(S_{c_1 \cup c_n}) \xrightarrow{\varphi_1 \circ \sigma} \mathcal{Z}_0(\varphi_{1*}(c_n)) \xrightarrow{\mathrm{Cut}_{\varphi_{1*}(c_n)}} \prod_i \mathrm{PMap}(O_i'),$$

where the O_i' are the connected components of $R_1' \setminus \varphi_{1*}(c_n)$. Define the projection map to each component $\mathrm{PMap}(O_i')$ as

$$\psi_i = \pi_i \circ \mathrm{Cut}_{\varphi_{1*}(c_n)} \circ \varphi_1 \circ \sigma.$$

It follows from Lemma 2.45 that $\psi_i(d_{c_1})$ has finite order k for every i.

The rest of the proof is a bit technical and so we divide it into four steps. In the first step, we show that there exists $m_i \in \mathbb{N}$ such that $\psi_i(d_{c_1}) = \psi_i(d_{c_n})^{m_i}$. In the second step, we show that the element $\varphi(d_{c_1})$ fixes the curves $\varphi_*(c_2)$ outside an open neighborhood of $\varphi_*(c_1)$. In the third step, we show that every curve of $\varphi_*(c_1)$ intersects at least one curve of $\varphi_*(c_2)$ (and vice versa). Finally, in the four step we show that $\varphi(d_{c_i}) = M_i h_i$.

Step 1. First, we show $\psi_i(d_{c_n})$ has finite order k for every i.

Let $A = \varphi_*(c_1) \cup \varphi_*(c_n)$. Note $\varphi(\mathcal{Z}_0(c_1) \cap \mathcal{Z}_0(c_n))$ is contained in $\mathcal{Z}_0(A)$. Therefore, we may consider the composition

$$\mathcal{Z}_0(c_1) \cap \mathcal{Z}_0(c_n) \xrightarrow{\varphi} \mathcal{Z}_0(A) \xrightarrow{\mathrm{Cut}_A} \prod_i \mathrm{PMap}(W_i'),$$

where the W_i' are the connected components of $S' \setminus A$. To simplify notation, for each projection π_i to the ith coordinate we write

$$\varphi|_{W_i'}(f) = \pi_i \circ \mathrm{Cut}_A \circ \varphi(f) \text{ for any } f \in \mathcal{Z}_0(c_1) \cap \mathcal{Z}_0(c_n).$$

Recall that

$$S' \setminus \varphi_*(c_1) = \sqcup_i R_i' \text{ and } R_1' \setminus \varphi_{1*}(c_n) = \sqcup_i O_i'.$$

And observe that the surfaces $\{O_i'\}$ are a subset of the surfaces $\{W_i'\}$. Since φ_i is non-multitwist preserving only for $i = 1$, if $W_i' \notin \{O_k'\}$ then the element $\varphi|_{W_i'}(d_{c_n})$ is trivial. Additionally, by Lemma 2.45, $\psi_i(d_{c_1})$ has order k for every O_i'. These two facts imply

$$\{W_i' |\, \varphi(d_{c_n})|_{W_i'} \neq \mathrm{id}\} = \{O_i' |\, \psi_i(d_{c_n}) \neq \mathrm{id}\} \subset \{O_i' |\, \psi_i(d_{c_1}) \neq \mathrm{id}\}.$$

In words, after cutting by A, the set of subsurfaces for which $\varphi|_{W_j'}(d_{c_n})$ is non-trivial is a subset of the subsurfaces for which $\varphi|_{W_j'}(d_{c_1})$ is non-trivial. However, the argument is symmetric, we may cut first along the multicurve $\varphi_*(c_n)$ and then along the multicurve $\varphi_*(c_1)$. This implies that, after cutting by A, the set of subsurfaces for which $\varphi|_{W_j'}(d_{c_n})$ is non-trivial is precisely the set of subsurfaces for $\varphi|_{W_j'}(d_{c_1})$ is non-trivial, that is, both sets coincide with

$$\{O_i' |\, O_i' \text{ is a connected component of } R_1' \setminus \varphi_{1*}(c_n)\}.$$

Tracing back the definitions of the maps, we have that $\varphi|_{O_i'}(d_{c_1}) = \psi_i(d_{c_1})$ and $\varphi|_{O_i'}(d_{c_n}) = \psi_i(d_{c_n})$. Observe $\psi_i(d_{c_1}), \psi_i(d_{c_n})$ are commuting finite order elements and together generate a finite abelian subgroup of $\mathrm{PMap}(O_i')$. Also, since O_i' is a punctured surface, all finite abelian subgroups of $\mathrm{PMap}(O_i')$ are cyclic. It follows that there exists $m_i \in \mathbb{N}$ such that

$$\psi_i(d_{c_1}) = \psi_i(d_{c_n})^{m_i}. \tag{2.8}$$

Step 2. After cutting S' by the multicurve $\varphi_*(c_1)$, each curve in $\varphi_*(c_2)$ projects onto R_i' either to a curve or to an arc joining punctures. The set containing such curves and arcs is denoted $\varphi_{i*}(c_2)$. We start by showing that $\varphi_i(d_{c_1}) \cdot a = a$ for every curve or arc $a \in \varphi_{i*}(c_2)$ and that $\varphi_i(d_{c_1})$ also fixes the orientation of a.

For $i \neq 1$, the element $\varphi_i(d_{c_1})$ is the identity. So, we trivially obtain $\varphi_i(d_{c_1}) \cdot a = a$ for every $a \in \varphi_{i*}(c_2)$.

Assume $i = 1$. The curves $\varphi_*(c_n)$ are disjoint from the curves in $\varphi_*(c_2)$. Therefore, the curves $\varphi_{1*}(c_n)$ are disjoint from the curves/arcs in $\varphi_{1*}(c_2)$. Denote by $\psi_{i*}(c_2)$ the set of curves/arcs of $\varphi_{1*}(c_2)$ that are entirely contained in O_i'. To show that $\varphi_1(d_{c_1})$ fixes $a \in \varphi_{1*}(c_2)$ with orientation, it is enough to see that $\psi_i(d_{c_1})$ fixes each curve/arc in $\psi_{i*}(c_2)$ with orientation.

By Step 1, we have that $\psi_i(d_{c_1}) = \psi_i(d_{c_n})^{m_i}$ for certain $m_i \in \mathbb{N}$. Also, by Lemma 2.41, $\varphi(d_{c_n})$ fixes every curve in $\varphi_*(c_2)$ with orientation. We conclude that $\psi_i(d_{c_n})$ fixes each curve/arc in $\psi_{i*}(c_2)$ with orientation. Now, from Step 1 we deduce that $\psi_i(d_{c_1})$ also fixes each curve/arc in $\psi_{i*}(c_2)$, as we wanted.

Since $\varphi_i(d_{c_1})$ fixes every curve and arc in $\varphi_{i*}(c_2)$ with orientation, it follows that $\varphi(d_{c_1})$ fixes the curves $\varphi_*(c_2)$ outside an open regular neighborhood of $\varphi_*(c_1)$.

Step 3. Every curve $x \in \varphi_*(c_2)$ intersects a curve in $\varphi_*(c_1)$ (and vice versa). Indeed, from Step 2 it follows that if $x \in \varphi_*(c_2)$ is disjoint from every curve in $\varphi_*(c_1)$, then $\varphi(d_{c_1}) \cdot x = x$. By completing $\{d_{c_1}, d_{c_2}\}$ to a Humphries generating set of $\mathrm{PMap}(S)$ and using Lemma 2.41, we deduce that φ fixes the curve x, which contradicts that φ fixes no curve.

Step 4. It follows from Step 2 that there exists a multitwist M_1 with components in the multicurve $\varphi_*(c_1)$ such that $\varphi(d_{c_1})M_1^{-1} \in \mathrm{PMap}(S')$ fixes every curve in $\varphi_*(c_2)$. Denote $h_1 = \varphi(d_{c_1})M_1^{-1}$.

Choose a Humphries generating set $\mathcal{H}$ such that $d_{c_1}, d_{c_2}, d_{c_n} \in \mathcal{H}$ and such that $i(c_1, c) = 0$ for every $d_c \in \mathcal{H} \setminus \{d_{c_2}\}$ (see Figure 2.4). Let $H = \{c |\, d_c \in \mathcal{H}\}$. By definition, h_1 fixes every curve $x \in \varphi_*(c_2)$. And, since d_{c_1} commutes with every Dehn twist $d_c \in \mathcal{H} \setminus \{d_{c_2}\}$, we have that h_1 fixes every curve $x \in \varphi_*(c)$ for every $c \in H$.

We show h_1 has finite order. Let Y be the surface filled by the curves in $\bigcup_{c_i \in H} \varphi_*(c_i)$. If $Y = S'$, then the Alexander method yields that h_1 has finite order.

Looking for a contradiction, assume $Y \neq S'$ and h_1 has infinite order. In this case, the Alexander method implies there is an $m \in \mathbb{N}$ such that h_1^m is supported on the complement of Y. Note that $h_1^k = \varphi(d_{c_1}^k)M_1^{-k}$ is a multitwist with components in $\varphi_*(c_1)$. Thus, h_1^{km} is a multitwist supported on both $\varphi_*(c_1)$ and the complement of Y. That is, the components of h_1^{km} are contained in both the multicurve $\varphi_*(c_1)$ and in the boundary ∂Y. However, by Step 3, no curve is both in $\varphi_*(c_1)$ and ∂Y. Thus, h_1 has finite order.

We write $\varphi(d_{c_1}) = M_1 h_1$ and observe that h_1 is an element of order k, since $\varphi(d_{c_1})$ is a root of degree k of a multitwist.

Finally, the elements h_1 and M_1 do not depend on the choice of curve c_2 and the choice of generating set $\mathcal{H}$. To see this, consider another curve c_2' and a different generating set $\mathcal{H}'$, we obtain h_1' and M_1' with $M_1 h_1 = M_1' h_1'$. By taking k-th powers on both sides of the equation, we conclude that $M_1^k = (M_1')^k$. Since M_1, M_1' are multitwists, it follows that $M_1 = M_1'$. Then, $h_1 = h_1'$.

For the curves $c_2, c_3, \ldots$ in H we may similarly define an element of finite order h_i and a multitwist M_i such that

$$\varphi(d_{c_i}) = M_i h_i.$$

□

Proof of Claim 2.47. Since $d_{c_i}^{d_{c_i}d_{c_j}} = d_{c_j}$, by taking k-th powers and taking the image by φ we obtain

$$\left(M_i^{\varphi(d_{c_i}d_{c_j})}\right)^k = M_j^k.$$

Both M_j and $M_i^{\varphi(d_{c_i}d_{c_j})}$ are multitwists, and so $M_j = M_i^{\varphi(d_{c_i}d_{c_j})}$. Again, using $d_{c_i}^{d_{c_i}d_{c_j}} = d_{c_j}$ and taking the image by φ, we have

$$M_i^{\varphi(d_{c_i}d_{c_j})} h_i^{\varphi(d_{c_i}d_{c_j})} = M_j h_j.$$

It follows that $h_i^{\varphi(d_{c_i}d_{c_j})} = h_j$.

Lastly, observe that h_m fixes the curves in $\varphi_*(c_n)$ for every m, n. Thus, h_m commutes with M_n for every m, n. Now, from the equality $h_i^{\varphi(d_{c_i}d_{c_j})} = h_j$ we obtain

$$(h_i h_j) h_i (h_i h_j)^{-1} = h_j,$$

and we conclude that $h_i h_j h_i = h_j h_i h_j$. Similarly, from $M_j = M_i^{\varphi(d_{c_i}d_{c_j})}$ one deduces $M_i M_j M_i = M_j M_i M_j$. □

The proofs of Claim 2.46 and Claim 2.47 finishes the argument for Theorem A. □

2.5 Lifting to the diffeomorphism group

In this section we consider surfaces to be endowed with a differentiable structure. Recall that $\mathrm{Diff}_c^+(S)$ is the group of orientation preserving diffeomorphisms of S with compact support on $S \setminus \partial S$. In particular, the elements of $\mathrm{Diff}_c^+(S)$ fix the boundary of S pointwise and a neighborhood of each puncture.

Every element in the pure mapping class group $[\phi] \in \mathrm{PMap}(S)$ admits a representative in $\mathrm{Diff}_c^+(S)$. Indeed, this follows from the Dehn-Lickorish theorem and the fact that every Dehn twist admits such representative. As a consequence, the map $\pi : \mathrm{Diff}_c^+(S) \to \mathrm{PMap}(S)$ sending each diffeomorphism to its homotopy class is a surjective homomorphism.

In this section we show that every homomorphism $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ induced by a multi-embedding can be lifted to a map between the diffeomorphism groups. In other words, there exists $\overline{\varphi}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Diff}_c^+(S) & \xrightarrow{\overline{\varphi}} & \mathrm{Diff}_c^+(S') \\ \downarrow \pi & & \downarrow \pi \\ \mathrm{PMap}(S) & \xrightarrow{\varphi} & \mathrm{PMap}(S'). \end{array}$$

Proposition 2.48. *Let S, S' be finite-type orientable smooth surfaces and let $g \geq 3$ be the genus of S. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a homomorphism induced by a multi-embedding, then φ admits a lift $\overline{\varphi} : \mathrm{Diff}_c^+(S) \to \mathrm{Diff}_c^+(S')$.*

Proof. Let $\mathcal{I} = \{\iota_i : S \hookrightarrow S' | \ \ i = 1, 2, \ldots, n\}$ be a multi-embedding inducing the map φ. We require that the images of the embeddings have disjoint closures, i.e, $\overline{\iota_i(S)} \cap \overline{\iota_j(S)} = \emptyset$ for every $i \neq j$.

Claim 2.49. *Each embedding $\iota_i : S \to S'$ is isotopic to a map $\epsilon_i : S \to \iota_i(S) \subset S'$, where ϵ_i is smooth in the interior of S and is a diffeomorphism onto its image.*

Proof of Claim 2.49. The embedding $\iota_i : S \to S'$ is a homeomorphism onto its image. By [39, Theorem B] there exists an isotopy $H : S \times [0, 1] \to \iota_i(S)$ such that $H(\cdot, 0) = \iota_i(\cdot)$, $H(\cdot, t) = h_t(\cdot)$ is a homeomorphism for every t and $H(\cdot, 1) = \epsilon_i(\cdot)$ is a diffeomorphism. □

Now, we replace $\mathcal{I} = \{\iota_i : S \hookrightarrow S' | \ \ i = 1, 2, \ldots, n\}$ with the multi-embedding $\mathcal{E} = \{\epsilon_i : S \hookrightarrow S' | \ \ i = 1, 2, \ldots, n\}$. Since we only changed the embeddings by isotopies, $\mathcal{E}$ induces the same homomorphism φ. Moreover, for any $\phi \in \mathrm{Diff}_c^+(S)$ we can define

$$\overline{\phi}(x) = \begin{cases} x & x \notin \epsilon_i(S) \ \forall i, \\ \epsilon_i \circ h \circ \epsilon_i^{-1}(x) & x \in \epsilon_i(S). \end{cases}$$

The map $\overline{\phi}$ is a compactly supported diffeomorphism. It follows that the assignation $\phi \mapsto \overline{\phi}$ defines a homomorphism

$$\overline{\varphi} : \mathrm{Diff}_c^+(S) \to \mathrm{Diff}_c^+(S')$$

and the map $\overline{\varphi}$ is a lift of φ. □

Proof of Theorem C. The statement follows immediately from Theorem A and Proposition 2.48. □

2.6 Maps not induced by multi-embeddings

Denote S_g the closed orientable surface of genus $g \in \mathbb{N}$ and denote $S_{g,1}$ the surface S_g minus a point.

Theorem 2.50. *For every $g \geq 2$ and $g' = (g-1) \cdot 2^{2g} + 1$, there exists an injective homomorphism $\mathrm{PMap}(S_{g,1}) \to \mathrm{PMap}(S_{g',1})$ not induced by multi-embeddings.*

Proof. Every characteristic cover $p : S_{g'} \to S_g$ induces an injective homomorphism $\varphi : \mathrm{PMap}(S_{g,1}) \to \mathrm{PMap}(S'_{g',1})$ (see [53, Section 2]). To construct φ we choose an appropriate cover. Then, we show that φ is not induced by a multi-embedding.

For the cover, consider the abelianzation of $\pi_1(S_g)$ composed with the map that quotients each factor modulo two

$$\pi_1(S_g) \xrightarrow{ab} \mathbb{Z}^{2g} \xrightarrow{\mathrm{mod}2} (\mathbb{Z}/2\mathbb{Z})^{2g}.$$

The kernel of the composition produces a characteristic cover $p : \widetilde{S} \to S_g$. The Riemann-Hurwitz formula implies $\widetilde{S}$ has genus $g' = (g-1) \cdot 2^{2g} + 1$. In other words, we may write the cover as $p : S_{g'} \to S_g$. Let $\varphi : \mathrm{PMap}(S_{g,1}) \to \mathrm{PMap}(S_{g',1})$ be the injective map induced by p. We are left to show that such φ is not induced by a multi-embedding.

It is a consequence of Lemma 2.18 that if $\mathrm{PMap}(S_{g,1}) \to \mathrm{PMap}(S')$ is a homomorphism induced by a multi-embedding, then S' is homeomorphic to either S_g, $S_{g,1}$ or S_{2g}. Since $S_{g'}$ has genus $g' > 2g$, then φ is not induced by a multi-embedding. □

Chapter 3

Maps between moduli spaces

The goal of this chapter is to study holomorphic maps between moduli spaces. The results on this chapter were obtained in joint work with Juan Souto.

Denote by $S_{g,r}$ the oriented surface of genus g with r punctures and let $\mathcal{T}(S_{g,r})$ be the Teichmüller space of $S_{g,r}$. The action $\mathrm{PMap}(S_{g,r}) \curvearrowright \mathcal{T}(S_{g,r})$ preserves the standard complex structure of Teichmüller space [72]. We always think of moduli space as the fine moduli space, that is, the complex orbifold

$$\mathcal{M}(S_{g,r}) = \mathcal{T}(S_{g,r})/\,\mathrm{PMap}(S_{g,r}),$$

meaning that maps $F : M \to \mathcal{M}(S_{g',r'})$ from a manifold, or more generally an orbifold, are induced by maps $\tilde{F} : \tilde{M} \to \mathcal{T}(S_{g',r'})$ equivariant under a homomorphism $F_* : \pi_1(M) \to \mathrm{PMap}(S_{g',r'})$. Here $\tilde{M}$ and $\pi_1(M)$ are the (orbifold) universal cover and fundamental group of M. We refer to [81] for more on orbifolds.

The basic examples of holomorphic maps between moduli spaces are the *forgetful maps*, that is, the maps $\mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g,r-k})$ obtained by unmarking $k \leq r$ marked points. In fact, it was proved in [4] that forgetful maps are, as long as $g \geq 6$ and $g' \leq 2g-2$, the only non-constant holomorphic maps $F : \mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g',r'})$ between fine moduli spaces. The goal of the chapter is to show that the same holds true when we replace the linear bound on g' by an exponential bound:

Theorem E. *Suppose that $g \geq 4$ and that $g' \leq 3 \cdot 2^{g-3}$, and let $r, r' \geq 0$. Then every non-constant holomorphic map $F : \mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g',r'})$ is a forgetful map. In particular, if such a map exists, then $g' = g$ and $r' \leq r$.*

Note that some exponential bound for g' is needed. Indeed, there is $\mathcal{M}_{g,1} \to \mathcal{M}_{(g-1)\cdot 2^{2g}-1,1}$ non-constant, holomorphic, and not a forgetful map. Such examples are obtained via covers (see Example 3.14 in the last section of this chapter).

Let us briefly recall the strategy followed in [4]. First, one derives from [11] that if $g \geq 6$ and $g' \leq 2g-2$ then every non-trivial homomorphism $\mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g',r'})$ is induced by forgetting marked points. It follows that, still in the same range, every non-constant holomorphic map $\mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g',r'})$ is homotopic to a forgetful map. The claim then follows from the fact that moduli space is, as an analytic space, a quasi-projective variety [28], and that any two non-constant homotopic holomorphic maps from a quasi-projective variety to moduli space agree [4, Prop 3.2].

The strategy of the proof of Theorem E basically follows the same lines. The first and main ingredient is the classification of homomorphisms $\rho : \mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g',r'})$ for g, g' given in Theorem A. The new difficulty we encounter is that it is no longer true that all such homomorphisms are induced by forgetful maps—see Example 3.15. We get however that all *irreducible* homomorphisms are induced by forgetful maps, where irreducible means that there is no simple (non-trivial) multicurve $\gamma \subset S_{g',r'}$ which is fixed by the image $\rho(\mathrm{PMap}(S_{g,r}))$. What we need to prove is that homomorphisms F_* induced by holomorphic maps $F : \mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g',r'})$ are indeed irreducible.

Theorem 3.1. *If M is an irreducible quasi-projective variety and $F : M \to \mathcal{M}(S_{g,r})$ is a non-constant holomorphic map, then the homomorphism $F_* : \pi_1(M) \to \mathrm{PMap}(S_{g,r})$ is irreducible.*

Let us be clear about the terminology in Theorem 3.1. Per se, a quasi-projective variety is nothing other than a Zariski open subset of a projective variety over some field, which we will always assume to be $\mathbb{C}$. We can thus consider quasi-projective varieties as analytic spaces, referring to analytic maps between such spaces as holomorphic maps. For example, it is due to Deligne and Mumford [28] that moduli space $\mathcal{M}(S_{g,r})$ is a quasi-projective variety. Moreover, since we are considering the moduli space as an orbifold (that is, we are considering it as a fine moduli space, not as a coarse one), we also want to allow the case that our variety M is an orbifold and that the map $F : M \to \mathcal{M}(S_{g,r})$ is a map between orbifolds. Let us stress this point: the quasi-projective variety M in Theorem 3.1 is allowed to be an orbifold. If M is an orbifold, then we assume that the holomorphic map $F : M \to \mathcal{M}(S_{g,r})$ is induced by a holomorphic map $\tilde{F} : \tilde{M} \to \mathcal{T}(S_{g,r})$ which is equivariant under a homomorphism $F_* : \pi_1(M) \to \mathrm{PMap}(S_{g,r})$, where $\tilde{M}$ is the orbifold universal cover $\pi_1(M)$ the orbifold fundamental group.

Once the statement of Theorem 3.1 is clarified, let us add a few comments on its proof. First note that since every quasi-projective variety of complex dimension at least 2 contains a wealth of algebraic curves, it suffices to prove Theorem 3.1 in the case that the domain M is 1-dimensional,

and more specifically a Riemann surface of finite analytic type. If the domain of $F : M \to \mathcal{M}(S_{g,r})$ is closed, something we cannot assume here because the moduli space is not closed, then the irreducibility of F_* is due to McMullen [69]. His basic observation is that if $F_*(\pi_1(M))$ were to fix a multicurve γ then the map $F : M \to \mathcal{M}(S_{g,r})$ would lift to a map $F' : M \to \mathcal{T}(S_{g,r})/\operatorname{Stab}(\gamma)$. The function ℓ_γ associating to a point in Teichmüller space X the length of γ with respect to the hyperbolic structure of X descends to a well-defined function $\hat{\ell}_\gamma : \mathcal{T}(S_{g,r})/\operatorname{Stab}(\gamma) \to \mathbb{R}_{>0}$. Wolpert [88] has proved that ℓ_γ is plurisubharmonic, and from this fact McMullen obtains that $\hat{\ell}_\gamma \circ F'$ would be constant. He then gets a contradiction from what he describes, without overwhelming the reader with unnecessary details, as 'an argument with quasifuchsian groups' [69, p. 136].

In the setting we care mostly about, that is the case that the domain M is not compact, we cannot argue like that: plurisubharmonicity is not enough to show that $\hat{\ell}_\gamma \circ F'$ would be constant. Indeed, neither plurisubharmonicity nor quasifuchsian groups will play any role here. For our argument we rely directly on what one can think are the essential qualities of the Weil-Petersson geometry: $(\mathcal{T}(S_{g,r}), g_{\mathrm{WP}})$ is Kähler, incomplete, negatively curved, geodesically convex, and the metric is dominated by the Kobayashi metric, that is the Teichmüller metric. Consider namely the function $h_\gamma : \mathcal{T}(S_{g,r}) \to \mathbb{R}_{>0}$ sending $X \in \mathcal{T}(S_{g,r})$ to the square of its distance to the stratum $\mathcal{S}_\gamma \subset \overline{\mathcal{T}}(S_{g,r})$ consisting of those points with nodal locus γ. Negative curvature implies that h_γ is strictly convex. Moreover, when it exists, the Weil-Petersson norm of its differential $dh_\gamma|_X$ is at most linear in the distance $d_{\mathrm{WP}}(X_0, X)$ to some arbitrary base point X_0. This implies the function $\|dh_\gamma|_X\|$ is subexponential and so we say that h_γ has *subexponential growth*. We get thus a contradiction from the following result:

Theorem 3.2. *Suppose that N is a Kähler manifold whose metric is dominated by a multiple of the Kobayashi metric and which admits a strictly convex function with subexponential growth. Let also M be either a closed connected Kähler manifold or an irreducible quasi-projective variety. Then there are no non-constant holomorphic maps $F : M \to N$.*

We stress that we are neither assuming that N is complete, nor that the convex function f is smooth. Let us however suppose for a moment that f is smooth. The basic idea of the proof of Theorem 3.2 is that convexity implies that flowing any non-constant map $F : M \to N$ in the direction of the negative gradient of f reduces the energy of F. On the other hand, the standard Wirtinger inequality implies that holomorphic maps are absolute energy minimizers in their homotopy class. A contradiction.

This chapter is organized as follows. After recalling some basic properties of Teichmüller space and the Weil-Petersson metric in Section 3.1, we prove Theorem 3.2 in section 3.2. Theorem 3.1 and Theorem E are dealt with in Section 3.3.

3.1 Teichmüller space and the Weil-Petersson metric

In this section, let us fix $g, r \geq 0$ satisfying $2g+r \geq 3$. Let us also denote by $S_{g,r}$ the smooth oriented surface of genus g with r ends and empty boundary. We remark that throughout the chapter we often regard punctures as marked points on the surface.

3.1.1 Preliminaries

Recall that a Riemann surface Σ is of finite analytic type if it is biholomorphic to the complement of finitely many points in a closed Riemann surface. As long as it has negative Euler-characteristic, Σ admits a unique conformal complete hyperbolic metric (that is, of constant curvature -1) which automatically has finite volume. Here, conformal means that, for all $x \in \Sigma$, the scalar product induced on $T_x\Sigma$ is invariant under multiplication by the imaginary unit $i \in \mathbb{C}$.

Under a *marked Riemann surface* we understand a pair $(\Sigma, \phi : S_{g,r} \to \Sigma)$ where ϕ is a orientation preserving homeomorphism. The *Teichmüller space* $\mathcal{T}(S_{g,r})$ is the space of all equivalence classes of marked Riemann surfaces, where $(\Sigma, \phi : S_{g,r} \to \Sigma)$ and $(\Sigma', \phi' : S_{g,r} \to \Sigma')$ are equivalent if and only if $\phi' \circ \phi^{-1} : \Sigma \to \Sigma'$ is isotopic to a biholomorphism.

The mapping class group also acts on Teichmüller space, and the quotient

$$\mathcal{M}(S_{g,r}) = \mathcal{T}(S_{g,r}) / \operatorname{PMap}(S_{g,r})$$

under the action of the pure mapping class group is the *moduli space*. Dividing by the whole mapping class group amounts to forgetting the marking. Since we are dividing by the pure mapping class group, the quotient $\mathcal{M}(S_{g,r})$ is the space of biholomorphism classes of Riemann surfaces of genus g with r labelled points. Teichmüller space admits a structure as simply connected complex manifold, and the mapping class group acts on it by biholomorphisms. Hence, $\mathcal{M}(S_{g,r})$ can be seen as a complex orbifold with universal cover $\mathcal{T}(S_{g,r})$. It is well known that $\mathcal{M}(S_{g,r})$ admits an algebraic structure. Indeed, $\mathcal{M}(S_{g,r})$ is an irreducible quasi-projective variety, that is a Zariski open subset of a projective variety [28].

Recall now that the *Kobayashi metric* of a complex manifold M is the largest pseudo-metric with the property that every holomorphic map $F : \mathbb{H}^2 \to M$ is 1-Lipschitz, where $\mathbb{H}^2$ is the hyperbolic plane endowed with the hyperbolic metric. In the case of Teichmüller space, the Kobayashi metric is an actual metric, which moreover agrees with the well-known Teichmüller metric [77]. We refer to [32] and [46] for more on mapping class groups and Teichmüller spaces, and to [56] for more on the Kobayashi metric on general complex manifolds.

3.1.2 The Weil-Petersson metric

Teichmüller space admits many natural mapping class group invariant metrics—we already mentioned the Teichmüller metric. Here we will be working with the so-called Weil-Petersson metric g_{WP}. This Riemannian metric was introduced by Weil in [84], and Ahlfors [2] proved that it is Kähler. Since this property will be key in this paper, let us recall that a Kähler metric on a complex manifold N is nothing other than a Riemannian metric $\langle \cdot, \cdot \rangle$ with $\langle iX_p, iY_p \rangle_p = \langle X_p, Y_p \rangle_p$ for all $p \in N$ and $X_p, Y_p \in T_pN$, and so that the Kähler form

$$\omega_p(X_p, Y_p) = \langle X_p, iY_p \rangle_p$$

is closed: $d\omega = 0$. Note in particular that every conformal metric on a Riemann surface is automatically Kähler. We refer to for example [14] for facts about Kähler manifolds.

A first key fact about the Weil-Petersson metric is that it is dominated, up to a factor $|2\pi\chi(S_{g,r})|^{\frac{1}{2}}$, by the Teichmüller metric (see for example [70, Prop. 2.4]). Since the Teichmüller metric agrees with the Kobayashi metric, we have:

Lemma 3.3. *The Weil-Petersson metric on $\mathcal{T}(S_{g,r})$ is dominated by a multiple of the Kobayashi metric.* □

From a metric point of view, what was maybe first noted about the Weil-Petersson metric is that it is incomplete [86]. To see that this is the case, Wolpert proved that multicurves in $S_{g,r}$ can be pinched in finite time. Indeed, Yamada [91] identified the completion $\overline{\mathcal{T}}(S_{g,r})$ of $(\mathcal{T}(S_{g,r}), g_{\mathrm{WP}})$ with what is known as *augmented Teichmüller space*. In other words, the completion of $(\mathcal{T}(S_{g,r}), g_{\mathrm{WP}})$ consists of *nodal curves*, that is Riemann surfaces where some multicurve has been pinched to have length 0. Given a simple (always non-trivial) multicurve γ in $S_{g,r}$, we denote by

$$\mathcal{S}_\gamma = \{X \in \overline{\mathcal{T}}(S_{g,r}) \text{ with } \ell_X(\gamma) = 0\} \tag{3.1}$$

the (closed) subset of the completion of $(\mathcal{T}(S_{g,r}), g_{\mathrm{WP}})$ where the components of γ have been pinched. We stress the fact that $\mathcal{S}_\gamma \neq \emptyset$.

From a Riemannian point of view, what is maybe best known about the Weil-Petersson metric is that its sectional curvature is negative, a fact due to Royden, Wolpert [87] and Tromba [82]. However, incomplete Riemannian manifolds, even negatively curved ones, can in general be pretty nasty to work with. In this case one is saved by a landmark result of Wolpert [88] asserting that $(\mathcal{T}(S_{g,r}), g_{WP})$ is a geodesic metric space, or in Riemannian terms that g_{WP} is geodesically convex in the sense that any two points are joined by a Weil-Petersson geodesic. In the same paper, he proved that the length function $\ell_\gamma : \mathcal{T}(S_{g,r}) \to (0, \infty)$ is convex, meaning that $\ell_\gamma \circ \alpha : (a, b) \to \mathbb{R}$ is convex for any Weil-Petersson geodesic $\alpha : (a, b) \to \mathcal{T}(S_{g,r})$.

As a consequence of the convexity of ℓ_γ and of the geodesic convexity of $\mathcal{T}(S_{g,r})$ one gets that the latter is a nested union of compact convex sets: pick two filling curves γ_1 and γ_2, that is two curves which have positive geometric intersection number with every curve, and consider compact sets of the form

$$K(L) = \{X \in \mathcal{T}(S_{g,r}) \text{ with } \ell_{\gamma_1}(X) + \ell_{\gamma_2}(X) \leq L\}. \tag{3.2}$$

Since the sectional curvature is overall negative, one has that on $K(L)$ it is bounded from above by some constant $\kappa_L < 0$. This implies that $K(L)$ is CAT(κ_L), and hence that $\mathcal{T}(S_{g,r}) = \cup_L K(L)$ is CAT(0). Since being CAT(0) is a property inherited by metric completions of geodesic metric spaces, we get that $\overline{\mathcal{T}}(S_{g,r})$ is also CAT(0). As a consequence, any two points in the completion $\overline{\mathcal{T}}(S_{g,r})$ of $(\mathcal{T}(S_{g,r}), g_{\mathrm{WP}})$ are joined by a unique geodesic segment. We refer to [22] for facts on CAT(0)-spaces.

3.1.3 A strictly convex function

Convex functions, in fact strictly convex functions, play a key role in this note.

Definition 3.4. *A continuous function $f \in C^0(M)$ on a Riemannian manifold is* strictly convex*, if for every compact set $K \subset M$ there are $\delta, C > 0$ such that for any $\epsilon < \delta$ and geodesic segment $\alpha : [-\epsilon, \epsilon] \to M$ parameterized by arc length and with $\alpha(0) \in K$ we have $f(\alpha(\epsilon)) + f(\alpha(-\epsilon)) - 2f(\alpha(0)) \geq C \cdot \epsilon^2$.*

Remark. *Note that a smooth function $f \in C^\infty(M)$ is strictly convex if and only if we have $\frac{d^2}{dt^2}(f \circ \alpha)|_{t=0} > 0$ for every geodesic $\alpha : (\epsilon, \epsilon) \to M$ parameterized by arc length.*

Fixing a simple multicurve $\gamma \subset S_{g,r}$ let $\mathcal{S}_\gamma \subset \overline{\mathcal{T}}(S_{g,r})$ be as in (3.1) the set of points in the completion where γ has been pinched. Convexity of ℓ_γ on $\mathcal{T}(S_{g,r})$ implies that the set $\mathcal{S}_\gamma$ is convex in $\overline{\mathcal{T}}(S_{g,r})$. Now, since $\overline{\mathcal{T}}(S_{g,r})$ is CAT(0), and since in such a space the distance to a convex set is a convex function, we get that

$$\mathrm{dist}_\gamma : \mathcal{T}(S_{g,r}) \to \mathbb{R}_{>0},\ \mathrm{dist}_\gamma(X) = \min_{Y \in \mathcal{S}_\gamma} d_{\overline{\mathcal{T}}(S_{g,r})}(X, Y)$$

is convex. We claim that its square is strictly convex:

Lemma 3.5. *Let $\gamma \subset S_{g,r}$ be a simple multicurve. The function*

$$h_\gamma : \mathcal{T}(S_{g,r}) \to \mathbb{R}_{>0},\ h_\gamma(X) = \min_{Y \in \mathcal{S}_\gamma} d_{\overline{\mathcal{T}}(S_{g,r})}(X, Y)^2 \tag{3.3}$$

is strictly convex.

It is well-known that in a CAT(0)-space, the square of the distance function to a given point is strictly convex. This is however not true for the square of the distance function to a general convex set—think of the square of the distance function to a line in $\mathbb{R}^2$. This is why we have to work a bit to prove Lemma 3.5. The proof will rely on a standard argument in comparison geometry together with the elementary fact that if $a, b, c, d \in \mathbb{H}^2$ are 4 points such that the geodesic segments $[a, b]$ and $[c, d]$ are non-degenerate and disjoint, then the function $d_{\mathbb{H}^2}(\cdot, [c, d])^2$ is strictly convex along $[a, b]$. Although this seems pretty much evident and is surely well-known, we didn't find a handy reference. Thus, we give an argument. Well, note that every point in $[a, b]$ is contained in a closed interval $[a', b'] \subset [a, b]$ on which $d_{\mathbb{H}^2}(\cdot, [c, d])^2$ agrees with one of the three functions

$$p \mapsto d_{\mathbb{H}^2}(p, c)^2, \ p \mapsto d_{\mathbb{H}^2}(p, d)^2, \text{ or } p \mapsto d_{\mathbb{H}^2}(p, \gamma)^2 \tag{3.4}$$

where γ is the unique bi-infinite geodesic containing $[c, d]$—to get a clean picture note that in the last case we have that $[a', b'] \cap \gamma = \emptyset$. Anyways, since the first two functions in (3.4) are strict convex and since $d_{\mathbb{H}^2}(\cdot, [c, d])^2$ is C^1, it suffices to argue that the function $t \mapsto d_{\mathbb{H}^2}(\alpha(t), \gamma)^2$ is strictly convex whenever $t \to \alpha(t)$ parameterizes by arc length an (open) geodesic segment disjoint of γ. Since this function is smooth, it suffices to prove that its second derivative

$$2\left(\frac{d}{dt} d_{\mathbb{H}^2}(\alpha(t), \gamma)\right)^2 + 2 \cdot d_{\mathbb{H}^2}(\alpha(t), \gamma) \cdot \frac{d^2}{dt^2} d_{\mathbb{H}^2}(\alpha(t), \gamma)$$

is positive. Well, the second summand is non-negative because the distance function is convex and the first is positive unless $d_{\mathbb{H}^2}(\alpha(t), \gamma)$ attains its minimum at t. Now, when we are in the minimum, it follows from a calculation (using for example [24, p.454, 2.3.1 (v)]) that the second factor is positive at that t. Having proved the strict convexity of h, we get that also the third function in (3.4) is strict convex, and hence that the function $d_{\mathbb{H}^2}(\cdot, [c, d])^2$ itself is strict convex on $[a, b]$. In fact, since everything, including lower bounds for the second derivatives, depend smoothly on a, b, c, d we get that there is a constant C which depends only on upper and lower bounds[1] for the distance to $[c, d]$ from points in $[a, b]$ so that we have

$$d_{\mathbb{H}^2}(p^-, [c, d])^2 + d_{\mathbb{H}^2}(p^+, [c, d])^2 - 2 \cdot d_{\mathbb{H}^2}(p^0, [c, d])^2 \geq C \cdot d_{\mathbb{H}^2}(p^-, p^0) \tag{3.5}$$

for all tuples (p^-, p^0, p^+) of distinct points in $[a, b]$ with $d_{\mathbb{H}^2}(p^-, p^0) = d_{\mathbb{H}^2}(p^0, p^+)$.

Well, after all this preparation, we are ready to prove Lemma 3.5.

Proof of Lemma 3.5. Suppose that $K \subset \mathcal{T}(S_{g,r})$ is compact, let $\delta_0 > 0$ be so that the distance between K and $\overline{\mathcal{T}}(S_{g,r}) \setminus \mathcal{T}(S_{g,r})$ is at least $10\delta_0$, and let

[1] Actually, a lower bound suffices.

$\mathcal{N}_{\delta_0}(K)$ be the set of points at distance at most δ_0 of K. Finally, fix $L > 0$ so that the compact convex set $K(L)$ from (3.2) contains both $\mathcal{N}_{\delta_0}(K)$ and every point $y \in \mathcal{T}(S_{g,r})$ with $d_{\mathcal{T}}(y, \mathcal{S}_\gamma) = \delta_0$ and which lies in a minimizing geodesic segment joining $\mathcal{S}_\gamma$ to some $x \in \mathcal{N}_{\delta_0}(K)$. All points considered below will belong to $K(L)$.

Anyways, suppose now that we have a geodesic

$$\alpha : [-\delta_0, \delta_0] \to \mathcal{T}(S_{g,r})$$

parameterized by arc length and with $\alpha(0) \in K$ and take $\epsilon \leq \delta_0$.

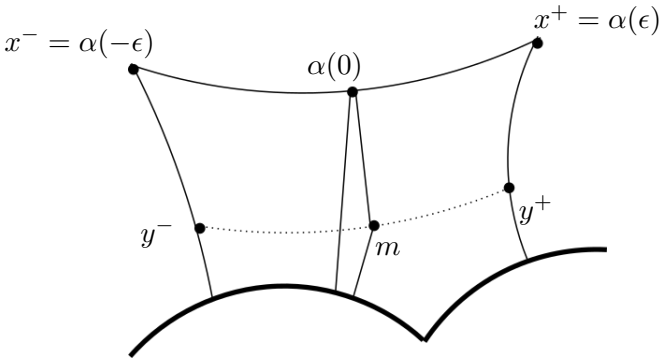


Figure 3.1: The bold printed lines represent $\mathcal{S}_\gamma$. The lines joining $x^\pm$ to $\mathcal{S}_\gamma$ represent the segments $I^\pm$. The points $y^\pm$ are chosen so that $d_{\mathcal{T}}(y^\pm, \mathcal{S}_\gamma) = \delta_0$ and the point m is the projection of $\alpha(0)$ to $[y^{-1}, y^+]$. Since $\overline{\mathcal{T}}(S_{g,r})$ is CAT(0) we have that $d_{\mathcal{T}}(m, \mathcal{S}_\gamma) \leq \delta_0$.

As in Figure 3.1 consider the shortest geodesic segments I^-, I^+ joining $x^- = \alpha(-\epsilon)$ and $x^+ = \alpha(\epsilon)$ to $\mathcal{S}_\gamma$. Let also $y^\pm \in I^\pm$ be the point at distance δ_0 from $\mathcal{S}_\gamma$, and finally let $m \in [y^-, y^+]$ be the point in there closest to $\alpha(0)$ (compare with Figure 3.1). Convexity in the CAT(0)-space $\overline{\mathcal{T}}(S_{g,r})$ of the distance to the convex set $\mathcal{S}_\gamma$ implies that

$$d_{\mathcal{T}}(m, \mathcal{S}_\gamma) \leq \frac{1}{2}\left(d_{\mathcal{T}}(y^-, \mathcal{S}_\gamma) + d_{\mathcal{T}}(y^\pm, \mathcal{S}_\gamma)\right) = \delta_0$$

and hence that

$$d_{\mathcal{T}}(\alpha(0), \mathcal{S}_\gamma) \leq d_{\mathcal{T}}(\alpha(0), m) + d_{\mathcal{T}}(m, \mathcal{S}_\gamma) \leq d_{\mathcal{T}}(\alpha(0), m) + \delta_0.$$

Convexity of the distance function on $\mathcal{T}(S_{g,r})$ implies also that

$$d_{\mathcal{T}}(x^-, y^-) + d_{\mathcal{T}}(x^+, y^+) - 2 \cdot d_{\mathcal{T}}(\alpha(0), m) \geq 0.$$

Multiplying out and invoking the last two displayed equations we get that

$$\begin{aligned} &h(\alpha(-\epsilon)+h(\alpha(\epsilon))-2h(\alpha(0))\geq \\ &\quad = d_{\mathcal{T}}(x^-,\mathcal{S}_\gamma)^2+d_{\mathcal{T}}(x^+,\mathcal{S}_\gamma)^2-2\cdot d_{\mathcal{T}}(\alpha(0),\mathcal{S}_\gamma)^2 \\ &\quad \geq (d_{\mathcal{T}}(x^-,y^-)+\delta_0)^2+(d_{\mathcal{T}}(x^+,y^+)+\delta_0)^2-2\cdot(d_{\mathcal{T}}(\alpha(0),m)+\delta_0)^2 \\ &\quad \geq d_{\mathcal{T}}(x^-,y^-)^2+d_{\mathcal{T}}(x^+,y^+)^2-2\cdot d_{\mathcal{T}}(\alpha(0),m)^2 \end{aligned}$$

Recall at some point that all of this is happening is a fixed compact convex set $K(L)$ as in (3.2)—more specifically that $x^\pm, y^\pm \in K(L)$. Let then $\kappa<0$ be an upper bound for the sectional curvature on $K(L)$. As we mentioned earlier, convexity of $K(L)$ and the upper bound on the curvature imply that $K(L)$ is a CAT(κ)-space. It follows in particular that distances between points in the boundary of the triangles (x^-,y^-,y^+) and (x^-,y^+,x^+) are smaller than in the comparison triangle $(\bar{x}^-,\bar{y}^-,\bar{y}^+)$ and $(\bar{x}^-,\bar{y}^+,\bar{x}^+)$ in the space $\mathbb{H}^2_\kappa$ of constant curvature κ. When we glue, as suggested by the notation, those two comparison triangles along the side $[\bar{x}^-,\bar{y}^+]$ we get a quadrilateral satisfying that

$$\begin{aligned} &d_{\mathcal{T}}(x^\pm,y^\pm)=d_{\mathbb{H}^2_\kappa}(\bar{x}^\pm,\bar{y}^\pm) \text{ and} \\ &d_{\mathcal{T}}(\alpha(0),m)\leq d_{\mathbb{H}^2_\kappa}(\text{mid}(\bar{x}^-,\bar{x}^+)),[\bar{y}^-,\bar{y}^+]). \end{aligned}$$

We thus have that

$$\begin{aligned} &h(\alpha(-\epsilon)+h(\alpha(\epsilon))-2h(\alpha(0))\geq \\ &\qquad \geq d_{\mathbb{H}^2_\kappa}(\bar{x}^-,\bar{y}^-)^2+d_{\mathbb{H}^2_\kappa}(\bar{x}^+,\bar{y}^+)^2-2d_{\mathbb{H}^2_\kappa}(\text{mid}(\bar{x}^-,\bar{x}^+),[\bar{y}^-,\bar{y}^+])^2 \end{aligned}$$

Noting now that, by construction and the CAT(κ) property, the segments $[\bar{x}^-,\bar{x}^+]$ and $[\bar{y}^-,\bar{y}^+]$ are at least at distance $8\delta_0$ of each other, and that their distance is bounded from above by $\text{diam}_{\mathcal{T}}(K(L))$, we get that the right side of the last inequality is bounded from below by $C_K\cdot\epsilon^2$ for some $C_K>0$ which depends only on $K(L)$, and hence indirectly only on K. Putting things together, we get that

$$h(\alpha(-\epsilon)+h(\alpha(\epsilon))-2h(\alpha(0))\geq C_K\cdot\epsilon^2 \text{ for all } \epsilon\in(0,\delta_0).$$

Since $\alpha:(-\delta_0,\delta_0)\to\mathcal{T}(S_{g,r})$ was an arbitrary geodesic parameterized by arc length with $\alpha(0)\in K$, we have proved what we had to prove. □

Continuing with the properties of the function h_γ given by (3.3) note that that distance function $\text{dist}_\gamma(\cdot)$ is Lipschitz with Lipschitz constant 1. It follows from Rademacher's theorem that $\text{dist}_\gamma(\cdot)$ is almost everywhere differentiable with gradient of norm 1. From here we get that the function h_γ is almost everywhere differentiable with gradient of norm $2\cdot\text{dist}_\gamma(\cdot)$, meaning that it has what we call *subexponential growth*:

Definition 3.6. *We will say that an almost everywhere differentiable function $f : M \to \mathbb{R}$ on a connected Riemannian manifold M* has subexponential growth *if for some, and hence any point $p_0 \in M$ we have*

$$\lim_{\substack{M \ni p \to \infty \\ df_p \text{ exists}}} \|df_p\| \cdot e^{-\epsilon \cdot d_M(p,p_0)} = 0$$

for all $\epsilon > 0$.

Finally, note that $\mathcal{S}_\gamma$ is $\mathrm{Stab}_{\mathrm{PMap}(S_{g,r})}(\gamma)$-invariant, and that this implies that our function h_γ is also $\mathrm{Stab}_{\mathrm{PMap}(S_{g,r})}(\gamma)$-invariant. We collect all the properties we have established for h_γ in the following statement:

Proposition 3.7. *Let $\gamma \subset S_{g,r}$ be a simple multicurve. The function $h_\gamma : \mathcal{T}(S_{g,r}) \to \mathbb{R}_{>0}$ given by* (3.3) *is $\mathrm{Stab}_{\mathrm{PMap}(S_{g,r})}(\gamma)$-invariant, strictly convex and has subexponential growth.* □

Lemma 3.3 and Proposition 3.7, together with the fact that g_{WP} is Kähler, encapsulate everything we will need about the Weil-Petersson metric in this paper. We refer however to [85, 89, 90] for more about the Weil-Petersson geometry of Teichmüller space.

3.2 Maps to Kähler manifolds

We devote the first 3 parts of this section to recall a few facts on Riemannian geometry, more specifically on gradient flows of smooth convex functions and on the fact that holomorphic maps are critical points of the energy functional. These results are well-known, and readers familiar with tools from complex geometry may choose to skim over the statements before seeing how these ingredients are combined to prove Theorem 3.2.

Throughout this section, N will be a Riemannian manifold, which we will eventually assume to be Kähler, and a fortiori complex. We denote the Riemannian metric by $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_M$ and the associated Levi-Civita connection by ∇. We stress that we are not assuming that our manifold is complete.

A comment on notation. By $X_p, \langle \cdot, \cdot \rangle_p, \omega_p$ and such we denote a tangent vector at $p \in M$, the scalar product or an alternating form on T_pM. When these pointwise objects are the restriction of a global object such as a vector field $X \in \Gamma(TM)$, and when we want to stress its evaluation at p, then we might write $X|_p$ instead of simply X_p.

3.2.1 Convexity

Recall that the *Hessian* at a point $p \in N$ of a smooth function $f \in C^\infty(N)$ is the symmetric bilinear form given by

$$\begin{aligned} &Hf|_p : T_pM \times T_pM \to \mathbb{R} \\ &Hf|_p(X_p, Y_p) = \langle \nabla_X \operatorname{grad}(f), Y \rangle|_p, \end{aligned}$$

where $\operatorname{grad}(f)$ is the gradient of f and where $X, Y \in \Gamma(TM)$ are arbitrary smooth vector fields on M with $X|_p = X_p$ and $Y|_p = Y_p$. The reason we are interested in the Hessian is that it detects convexity of smooth functions. Indeed, noting that for every geodesic $\alpha : (-\epsilon, \epsilon) \to M$ we have

$$\begin{aligned} \frac{d^2}{dt^2}(f \circ \alpha) &= \frac{d}{dt}\langle \operatorname{grad}(f), \alpha'(t) \rangle = \langle \nabla_{\alpha'(t)} \operatorname{grad}(f), \alpha' \rangle + \langle \nabla f, \nabla_{\alpha'(t)} \alpha'(t) \rangle \\ &= \langle \nabla_{\alpha'(t)} \operatorname{grad}(f), \alpha' \rangle = Hf|_{\gamma(t)}(\alpha'(t), \alpha'(t)), \end{aligned} \tag{3.6}$$

we get:

Lemma 3.8. *Let M a Riemannian manifold. A smooth function $C^\infty(M)$ is strictly convex if and only if its Hessian $Hf|_p$ is positive definite at every $p \in M$.* □

Another well-known fact is that the Hessian of a function controls the metric behavior of the flow associated to its negative gradient. We make this formal in the setting we will need it:

Lemma 3.9. *Let $F : M \to N$ be a smooth map between Riemannian manifolds, let $f \in C^\infty(N)$ and $\rho \in C_c^\infty(M)$ be two smooth functions, the latter non-negative and with compact support, and for $t \in \mathbb{R}$ set*

$$F_t : M \to N, \ F_t(x) = \phi_{\rho(x)\cdot t}(F(x))$$

where (ϕ_t) is the local flow of $-\operatorname{grad} f$. We have

$$\frac{d}{dt}\|dF_t|_x X_x\|^2|_{t=0} = -2\rho(Hf|_{F(x)})(dF_x X_x, dF_x X_x) - 2d\rho(X_x) \cdot df_{F(x)}(dF_x X_x)$$

for every $x \in M$ and $X_x \in T_xM$.

Proof. Before launching the proof, note that, while $\phi_t(p)$ might be only defined for $t \in (\epsilon_p, \epsilon_p)$ for some $\epsilon_p > 0$ depending on $p \in N$, the fact that ρ has compact support implies that there is some $\epsilon > 0$ such that $F_t : M \to N$ is defined for all $t \in (-\epsilon, \epsilon)$. This means in particular that the claim in the lemma does actually make sense as stated. Now, once we have said this we can forget about the condition that ρ is compactly supported and since the statement is purely local we can replace M by a small neighborhood of the

point x we are interested in. In particular, we may assume $\rho > 0$. Note also that if needed, we might replace the target by $N \times \mathbb{R}^d$ for whatever d we want, replacing then the function f by the function $(p, z) \to f(p)$. It follows that we might assume without loss of generality that the dimension of N is much larger than that of M.

Now, since everything depends continuously on ρ and F when we perturb in the C^∞-topology, we might suppose without loss of generality that dF_x is injective and that $\operatorname{grad} f|_{F(x)} \notin dF_x(T_xM)$. This together with the fact that $\rho > 0$ locally implies that, up to reducing M to a neighborhood of x, we might assume that the map

$$(-\epsilon, \epsilon) \times M \to N,\ (t, x) \mapsto \phi_t(F(x))$$

is an embedding. This has two important consequences:

1. First, there is a function $\hat{\rho} \in C^\infty(N)$ with

   $$\hat{\rho}(\phi_t(F(x))) = \rho(x) \text{ for all } (t, x) \in (-\epsilon, \epsilon) \times M$$

   and hence with

   $$\frac{\partial}{\partial t} F_t(x) = -\hat{\rho}(F_t(x)) \cdot \operatorname{grad}(f)|_{F_t(x)}.$$

2. Second, whenever $X \in \Gamma(TM)$ is a vector field with $X|_x = X_x$ then there is a global vector field $Y \in \Gamma(TN)$ with

   $$Y|_{F_t(x)} = dF_t|_x(X|_x) \text{ for all } (t, x) \in (-\epsilon, \epsilon) \times M$$

   Note that the Lie bracket $[Y, -\hat{\rho} \cdot \operatorname{grad}(f)]_{F_t(x)} = 0$ vanishes for all $(t, x) \in (-\epsilon, \epsilon) \times M$, and hence that

   $$\nabla_{\frac{\partial}{\partial t} F_t(x)} Y = \nabla_Y \left(-\hat{\rho} \cdot \operatorname{grad}(f)\right)|_{F_t(x)},$$

   again for all such (t, x).

All what is left is a calculation. Taking all derivatives at $t = 0$ we have:

$$\begin{aligned}
\frac{d}{dt}\|dF_t|_x X_x\|^2 = \frac{d}{dt}\|Y|_{F_t(x)}\|^2 &= \left(\frac{\partial}{\partial t} F_t(x)\right) \langle Y, Y\rangle \\
&= 2 \cdot \langle \nabla_{\frac{\partial}{\partial t} F_t(x)} Y, Y\rangle = 2 \cdot \langle \nabla_Y \left(-\hat{\rho} \cdot \operatorname{grad}(f)\right), Y\rangle \\
&= -2 \cdot \langle \hat{\rho} \cdot \nabla_Y \operatorname{grad} f + (Y\hat{\rho}) \cdot \operatorname{grad}(f), Y\rangle \\
&= -2 \cdot \hat{\rho}(Hf|_{F(x)})(Y_{F(x)}, Y_{F(x)}) - 2 \cdot d\hat{\rho}(Y_{F(x)}) \cdot df_{F(x)}(Y_{F(x)}).
\end{aligned}$$

Recalling that $Y_{F(x)} = dF_x X_x$ and that $\rho = \hat{\rho} \circ F$ we get that $d\hat{\rho}(Y_{F(x)}) = d\rho(X_x)$, and the claim follows. □

3.2.2 Energy and its first variation

Recall that if V and W are Euclidean vector spaces, that is vector spaces endowed with a scalar product, then there is a unique scalar product on $\mathrm{Hom}(V,W)$ with the property that if $(v_1,\dots,v_n)$ and $(w_1,\dots,w_m)$ are orthonormal bases of V and W, then $(v_i^*\otimes w_j)_{i,j}$ is an orthonormal basis of $\mathrm{Hom}(V,W)\simeq V^*\otimes W$ where $(v_1^*,\dots,v_n^*)$ is the dual basis to $(v_1,\dots,v_n)$. The *energy* of a linear map $L:V\to W$ is nothing other than the square of its norm

$$e(L)=\|L\|^2_{\mathrm{Hom}(V,W)}=\sum_i\|L(v_i)\|^2$$

with respect to this scalar product. Before moving on, note also that the isomorphism $V^*\simeq V$ given by the scalar product induces an isomorphism $\mathrm{Bil}(V\times V,\mathbb{R})\simeq\mathrm{Hom}(V,V)$, meaning that we can interpret bi-linear forms as endomorphisms. This explains why we refer to the quantity $\mathrm{Tr}(H)=\sum_i H(v_i,v_i)$ as the *trace* of $H\in\mathrm{Bil}(V\times V,\mathbb{R})$.

Moving on now to the world of manifolds, recall that the *total energy*, or just simply the *energy*, of a smooth map $F:M\to N$ between two Riemannian manifolds is the integral

$$E(F)=\int_M e(dF_p)d\,\mathrm{vol}_M(p)$$

of the pointwise energy $e(dF_p)$ of the differential $dF_p:T_pM\to T_{F(p)}N$. Here, the integral is taken with respect to the Riemannian measure on the domain M of F. We will be interested in how the energy changes when we perturb a map $F:M\to N$ as in Lemma 3.9. The following follows directly from the said lemma:

Lemma 3.10 (First variation of energy)**.** *Let $F:M\to N$ be a smooth map between Riemannian manifolds, let $f\in C^\infty(N)$ and $\rho\in C^\infty_c(M)$ be two smooth functions, the latter non-negative and with compact support, and for $t\in\mathbb{R}$ set*

$$F_t:M\to N,\ F_t(x)=\phi_{\rho(x)\cdot t}(F(x))$$

where (ϕ_t) is the local flow of $-\operatorname{grad} f$. We have

$$\frac{d}{dt}E(F_t)|_{t=0}=-2\int_M\mathrm{Tr}\left(\rho\cdot F^*(Hf)+d\rho\otimes F^*(df)\right)_p d\,\mathrm{vol}_M(p).\qquad\square$$

The statement of Lemma 3.10 is rendered more complicated than it would be desirable by the presence of the compactly supported function ρ: if M is compact, and we take $\rho\equiv 1$, then it simplifies to

$$\frac{d}{dt}E(F_t)|_{t=0}=-2\int_N\mathrm{Tr}(F^*(Hf))_p d\,\mathrm{vol}_M(p).$$

It thus follows that if F is non-constant and Hf is positive definite, that is if f is strictly convex, then $\frac{d}{dt}E(F_t)|_{t=0} < 0$, meaning that F is not a critical point for the energy functional. Said differently, *if M is compact and $F : M \to N$ is non-constant, then the energy decreases if we flow F in the direction of the negative gradient of a smooth strictly convex function.*

3.2.3 Energy of holomorphic maps and the Wirtinger inequality

A well known fact, key for us, is that holomorphic maps between Kähler manifolds are critical points of the energy functional. This follows basically from the *Wirtinger inequality* (see [31]):

Wirtinger inequality. *Let M and N be Kähler manifolds and denote by ω_M and ω_N their Kähler forms. For any smooth finite energy map $F : M \to N$ we have*

$$E(F) \geq \int_M (F^*\omega_N) \wedge \overbrace{\omega_M \wedge \cdots \wedge \omega_M}^{\dim_{\mathbb{C}} M - 1}$$

with equality if and only if F is holomorphic.

Note that in the case that the domain M is a Riemann surface, the Wirtinger inequality simplifies to $E(F) \geq \int_M F^*\omega_N$, once again with equality if and only if F is holomorphic.

Anyways, the key observation now is that, since the Kähler forms ω_M and ω_N are closed, we get from Stokes' theorem that

$$\int_M (F_t^*\omega_N) \wedge (\omega_M^{\dim_{\mathbb{C}} M - 1}) = \int_M (F^*\omega_N) \wedge (\omega_M^{\dim_{\mathbb{C}} M - 1})$$

for any compactly supported deformation F_t of F. This observation, together with the Wirtinger inequality, implies that holomorphic maps are critical points of the energy functional. Indeed, if M and N are Kähler and if $F_t : M \to N$ is a compactly supported perturbation of a holomorphic map $F : M \to N$ then, applying twice the Wirtinger inequality we have

$$E(F) = \int_M (F^*\omega_N) \wedge (\omega_M^{\dim_{\mathbb{C}} M - 1}) = \int_M (F_t^*\omega_N) \wedge (\omega_M^{\dim_{\mathbb{C}} M - 1}) \leq E(F_t)$$

with equality if and only if F_t is also holomorphic. We record this fact:

Proposition 3.11. *Let M and N be Kähler manifolds with Kähler forms ω_M and ω_N and let $F : M \to N$ be holomorphic. Then we have*

$$E(F_t) \geq E(F)$$

for any compactly supported perturbation F_t. Moreover, we have equality if and only if F_t is also holomorphic. □

We refer to [59] for more results along the lines discussed here and many beautiful applications.

3.2.4 Non-existence of certain holomorphic maps

In this section we prove Theorem 3.2, which we restate here for convenience of the reader:

Theorem 3.2. *Suppose that N is a Kähler manifold whose metric is dominated by a multiple of the Kobayashi metric and which admits a strictly convex function with subexponential growth. Let also M be either a closed connected Kähler manifold or an irreducible quasi-projective variety. Then there are no non-constant holomorphic maps $F : M \to N$.*

Let us get a pesky issue directly out of the way: while every convex function on a manifold without boundary is continuous and even locally Lipschitz, such functions do not need to be smooth. However, in the course of the proof of the theorem, we might assume without loss of generality that the convex function $f : N \to \mathbb{R}$ is actually smooth. Indeed, Greene and Wu [33] proved that every strictly convex function on a non-necessarily complete manifold can be approximated by smooth strictly convex functions. They moreover prove that if the original function is L-Lipschitz then the approximating functions can, for any $\epsilon > 0$, be taken to be $(L+\epsilon)$-Lipschitz. What their argument indeed shows is that if the original function is L-Lipschitz on some set, then the approximating function can be taken to be $(L + \epsilon)$-Lipschitz on that set. It follows that if the original strictly convex function has subexponential growth, then the approximating smooth strictly convex function can be chosen to also have subexponential growth. In other words, when proving Theorem 3.2 we might assume without loss of generality that the function f is smooth. After this comment, we are now ready to prove the theorem.

Proof. Since this is the case we really care about, we will prove the claim in the case that M is a quasi-projective variety. The argument for a closed Kähler manifold follows the same lines and is actually a bit simpler—we leave it to the reader.

Well, seeking a contradiction suppose that $F : M \to N$ is a non-constant holomorphic map where $M \subset \mathbb{C}P^k$ is a quasi-projective variety of dimension d and note that there is some copy $L \subset \mathbb{C}P^k$ of $\mathbb{C}P^{k-d+1}$ such that the restriction of F to $L \cap M$ is not locally constant: every 1-dimensional complex subspace of T_pM is the tangent space of such an intersection. The intersection $L \cap M$ might well be singular, but it is dominated by some Riemann surface of finite analytic type. It follows that it suffices to prove the claim in the case that the domain M is a Riemann surface of finite analytic type. From now on, we work in this specific setting.

Although it is not strictly necessary for the argument, note also that we can remove a few points of M to ensure that $F : M \to N$ has maximal rank everywhere and that M has negative Euler characteristic. This latter

property implies that M admits a complete conformal hyperbolic metric, which is automatically Kähler because M has complex dimension one. Note moreover that, since we are assuming that the metric of N is dominated by the Kobayashi metric, we get that the holomorphic map $F : M \to N$ is Lipschitz.

After all these preparations, we come to the meat of the argument. Let $f : N \to \mathbb{R}$ be our strictly convex function of subexponential growth. As we discussed just prior to the proof, we might assume without loss of generality that f is smooth. Let then (ϕ_t) be the (local) flow associated to $-\operatorname{grad} f$, let $\rho \in C_c^\infty(M)$ be a compactly supported smooth function, and consider for small $t > 0$ the compact deformation of F given by

$$F_t : M \to N,\ F_t(p) = \phi_{\rho(p)\cdot t}(F(p))$$

We get from the Wirtinger inequality that $E(F_t) \geq E(F)$. Invoking Lemma 3.10 we thus get a contradiction when we show that we can choose $\rho \in C_c^\infty(M)$ with

$$\int_M \operatorname{Tr}\left(\rho \cdot F^*(Hf) + d\rho \otimes F^*(df)\right)_p d\operatorname{vol}_M(p) > 0. \tag{3.7}$$

The remaining of the proof is devoted to construct such a function ρ.

To begin with, note that from the assumption that f is smooth and strictly convex, we get that the Hessian $Hf|_p$ is positive definite at every $p \in M$. We in particular get that $\operatorname{Tr}(F^*(H_f))_x > 0$ for every $x \in M$, and hence that the integral

$$\int_M \rho \cdot \operatorname{Tr}(F^*(H_f))_p d\operatorname{vol}_M(p) > 0 \tag{3.8}$$

is positive. We will not care whether the integral is finite or not.

Now recall that each cusp of M has a standard neighborhood, which abusing terminology we call simply *a cusp*, isometric to the quotient

$$U = \{z \in \mathbb{H}^2 \text{ with } \Im(z) \geq 1\}/(z \sim z+1)$$

where $\mathbb{H}^2$ is the hyperbolic plane. Fix $\beta : [0,1] \to [0,1]$ a smooth bump function which sends a neighborhood of 0 (resp. 1) to 1 (resp. 0) and consider for $L \geq 0$ the function

$$B_L : U \to [0,1],\ B_L(z) = \begin{cases} 1 & \text{if } \Im(z) \leq e^L \\ \beta(\frac{1}{L}\log(\Im(z)) - 1) & \text{if } \Im(z) \in [e^L, e^{2L}] \\ 0 & \text{otherwise} \end{cases}$$

If $V \subset M$ is a cusp, that is the standard neighborhood of a cusp, then we will also denote by B_L the function on V obtained by composing the isometry

$V \simeq U$ with the actual function B_L. With this notation, consider a smooth and compactly supported function

$$\rho_L : M \to [0,1],\ \rho_L(z) = \begin{cases} B_L(z) & \text{if } z \text{ belong to a cusp} \\ 1 & \text{otherwise} \end{cases}$$

The norm of $d\rho_L$ is bounded independently of the point and independently of L and this implies that there is a constant C with

$$|\operatorname{Tr}(d\rho_L \otimes F^* df)|_p| \leq C \cdot \|F^* df|_p\|$$

for every L and every point $p \in M$. Picking a base point $p_0 \in M$ outside the cusps, note that $d\rho_L|_p = 0$ for all $p \in M$ with $d_M(p, p_0) \leq L$. Combining these two facts, and denoting by $B^M(p_0, L)$ the ball in M of radius L, we get

$$\left| \int_M \operatorname{Tr}(d\rho_L \otimes F^* df)|_p d\operatorname{vol}_M(p) \right| \leq C \cdot \int_{M \setminus B^M(p_0,L)} \|F^* df|_p\| \, d\operatorname{vol}_M(p)$$

for all L. Now, the assumption that f has subexponential growth and the fact that F is 1-Lipschitz imply that $\|F^* df|_p\|$ grows subexponentially, meaning in particular that for every $\epsilon \in (0,1)$ there is some other constant c with

$$\|F^* df|_p\| \leq c \cdot e^{(1-\epsilon) \cdot d_M(p_0,p)}.$$

Up to replacing C by $C \cdot c$ we thus get for all L that

$$\begin{aligned} \left| \int_M \operatorname{Tr}(d\rho_L \otimes F^* df)|_p d\operatorname{vol}_M(p) \right| &\leq C \cdot \int_{M \setminus B^M(p_0,L)} e^{(1-\epsilon) \cdot d_M(p_0,p)} \, d\operatorname{vol}_M(p) \\ &= C \cdot \int_L^\infty e^{(1-\epsilon)t} \ell_M(\partial B^M(p_0,t)) \, dt. \end{aligned}$$

Here $\ell_M(\partial B^M(p_0, L))$ is the length (or if you so wish, the 1-dimensional Hausdorff measure) of the set of points in M which are exactly at distance L from p_0, and the integral is with respect to the standard Lebesgue measure on $\mathbb{R}$. Noting that there is some $c > 0$ with $\ell_M(\partial B^M(p_0,t)) \leq c \cdot e^{-t}$ for all t. Replacing again C by $C \cdot c$ we get that

$$\left| \int_M \operatorname{Tr}(d\rho_L \otimes F^* df)|_p d\operatorname{vol}_M(p) \right| \leq C \cdot \int_L^\infty e^{-\epsilon \cdot t} dt \to 0 \text{ as } L \to \infty.$$

Since we also evidently have

$$\lim_L \int_M \rho_L \cdot \operatorname{Tr}(F^* Hf)|_p d\operatorname{vol}_M(p) \to \int_M \operatorname{Tr}(F^*(H_f))_p d\operatorname{vol}_M(p) \text{ as } L \to \infty,$$

we get from (3.8) that

$$\int_M \mathrm{Tr}\left(\rho_L \cdot F^*(Hf) + d\rho_L \otimes F^*(df)\right)_p d\,\mathrm{vol}_M(p) > 0$$

for all large enough L. It follows that for any such L, the function $\rho = \rho_L$ satisfies (3.7). This concludes the proof of Theorem 3.2. □

Note at this point that, as long as the target N is a 'good' orbifold then the statement of Theorem 3.2 also holds true in the category of orbifolds, where an orbifold is *good* if it has a manifold as a finite cover. Let us explain why. First note, as in the proof, that it suffices to consider the case that M is a Riemann surface of finite analytic type. Now, the assumption that there is a manifold N' and a finite orbifold cover $\pi : N' \to N$ implies that any map $F : M \to N$ lifts to a map $F' : M' \to N'$ from a finite cover M' of M. Evidently, if N satisfies that conditions in Theorem 3.2, and if we lift the structure of N to N', then so does N'. Moreover, if F is holomorphic then so is F' and if $f : N \to \mathbb{R}$ is strictly convex and has subexponential growth, then the function $f \circ \pi : N' \to \mathbb{R}$ has those same properties. The theorem, as stated, implies thus that the lifted holomorphic map F' is constant, from where we get that also F is constant, as we wanted to prove.

To conclude, let us stress that in the proof of Theorem 3.2 we did not use completeness of N.

3.3 Irreducibility of maps to moduli spaces

The main goal of this section is to prove Theorem E and Theorem 3.1 from the introduction of this chapter.

3.3.1 Irreducibility

Recall that we are thinking of the moduli space as the complex orbifold

$$\mathcal{M}(S_{g,r}) = \mathcal{T}(S_{g,r})/\,\mathrm{PMap}(S_{g,r})$$

and that maps $F : M \to \mathcal{M}(S_{g,r})$ are always in the category of orbifolds: they are induced by maps $\tilde{F} : \tilde{M} \to \mathcal{T}(S_{g,r})$ on the universal cover of M which are equivariant under some homomorphism $F_* : \pi_1(M) \to \mathrm{PMap}(S_{g,r})$.

Remark. *Although we are not making it explicit with our notation, if M is a good orbifold instead of a manifold, we denote by $\pi_1(M)$ the fundamental group in the orbifold category.*

Theorem 3.1 asserts that, under suitable conditions on M, the homomorphism $F_* : \pi_1(M) \to \mathrm{PMap}(S_{g,r})$ induced by a non-constant holomorphic

map $F : M \to \mathcal{M}(S_{g,r})$ is *irreducible* in the sense that its image $F_*(\pi_1(M))$ is not contained in the stabilizer of any simple multicurve γ in the surface $S_{g,r}$ of genus g and with r punctures. We restate the theorem for convenience of the reader:

Theorem 3.1. *If M is an irreducible quasi-projective variety and $F : M \to \mathcal{M}(S_{g,r})$ is a non-constant holomorphic map, then the homomorphism $F_* : \pi_1(M) \to \mathrm{PMap}(S_{g,r})$ is irreducible.*

Proof. Suppose that the orbifold M is a quasi-projective variety with universal cover $\tilde{M}$ and fundamental group $\pi_1(M)$ and that $F : M \to \mathcal{M}(S_{g,r})$ is a non-constant holomorphic map. Seeking a contradiction, suppose that the image of $F_*(\pi_1(M)) \subset \mathrm{PMap}(S_{g,r})$ fixes a multicurve γ. This implies that $F_*(\pi_1(M)) \subset \mathrm{Stab}_{\mathrm{PMap}(S_{g,r})}(\gamma)$ and hence that F lifts to a map

$$F' : M \to \mathcal{T}(S_{g,r})/\operatorname{Stab}_{\mathrm{PMap}(S_{g,r})}(\gamma).$$

We endow the target with the Weil-Petersson metric, which we recall is Kähler and dominated by a multiple of the Kobayashi metric. As we pointed out after the proof of Theorem 3.2, the theorem holds true in the category of orbifolds, as long as the target is a good orbifold, that is an orbifold which is finitely covered by a manifold. The orbifold $\mathcal{T}(S_{g,r})/\operatorname{Stab}_{\mathrm{PMap}(S_{g,r})}(\gamma)$ is good because $\mathrm{PMap}(S_{g,r})$ has a finite index torsion free subgroup.

It follows from all of this that to be able to get a contradiction from Theorem 3.2 we just need to exhibit a strictly convex function with subexponential growth on $\mathcal{T}(S_{g,r})/\operatorname{Stab}_{\mathrm{PMap}(S_{g,r})}(\gamma)$. Luckily for us, we get from Proposition 3.7 that the function

$$h_\gamma : \mathcal{T}(S_{g,r}) \to \mathbb{R}_{\geq 0}$$

defined in (3.3) is strictly convex, has subexponential growth and is invariant under $\mathrm{Stab}_{\mathrm{PMap}(S_{g,r})}(\gamma)$. It thus descends to a strictly convex function

$$\hat{h}_\gamma : \mathcal{T}(S_{g,r})/\operatorname{PMap}(S_{g,r})(\gamma) \to \mathbb{R}_{>0}$$

with sub-exponential growth. Having found our function, we just got a contradiction to the assumption that $F_*(\pi_1(M))$ fixes γ. We are done. □

Remark. *Note that the same statement and proof apply if M is a general, but closed, Kähler manifold. Remark also that the reader that feels uneasy with using the fact that moduli space is a quasi-projective variety could avoid it as follows. In the proof of Theorem 3.2 we only used that the domain was quasi-projective to reduce to the case that it was a Riemann surface. We did that using that in quasi-projective varieties there are plenty of algebraic curves. In the specific setting of moduli space, one can instead use the fact that there are plenty of Teichmüller curves.*

3.3.2 Homomorphisms between mapping class groups

Our next and last goal is to prove Theorem E, but first we must recall a few things about homomorphisms between mapping class groups. In general, group homomorphisms $\mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g',r'})$ can be quite diverse. Some well-known maps are induced by covers, inclusions, and forgetting punctures. Let us give some examples:

Example 3.12 (Automorphisms)**.** *Ivanov proved that all automorphisms of the mapping class group* $\mathrm{PMap}(S_{g,r})$ *are of the form* $[\psi] \mapsto [\Psi \circ \psi \circ \Psi^{-1}]$ *where* Ψ *is a homeomorphism of* $S_{g,r}$*. Here and in the sequel* $[\cdot]$ *stands for the class of. Anyways, if* Ψ *is orientation preserving, then the map*

$$\mathcal{T}(S_{g,r}) \to \mathcal{T}(S_{g,r}),\ [\phi : S_{g,r} \to X] \mapsto [\phi \circ \Psi^{-1} : S_{g,r} \to X]$$

is equivariant under this homomorphism and hence descends to the map $\mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g,r})$*. This map is however not very interesting: it is just the identity. When* Ψ *is orientation reversing, things are more intriguing. In this case one also has an equivariant map, namely*

$$\mathcal{T}(S_{g,r}) \to \mathcal{T}(S_{g,r}),\ [\phi : S_{g,r} \to X] \mapsto [\phi \circ \Psi^{-1} : S_{g,r} \to \bar{X}]$$

where $\bar{X}$ *is the Riemann surface with the complex conjugated structure. The induced map* $\mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g,r})$ *depends on the individual element* Ψ*, but in all cases we have that* Ψ *is not holomorphic, but rather anti-holomorphic.*

Example 3.13 (Forgetful maps)**.** *Think of* $S_{g,r}$ *as being* $S_{g,0}$ *with* r *marked points and, accordingly, think of* $\mathrm{Homeo}_+(S_{g,r})$ *as a subgroup of* $\mathrm{Homeo}(S_{g,0})$*. For some* $r' \leq r$*, let* $S_{g,r'}$ *be obtained from* $S_{g,r}$ *by forgetting* $r - r'$ *marked points. The inclusion* $\mathrm{Homeo}_+(S_{g,r}) \subset \mathrm{Homeo}_+(S_{g,r'})$ *induces a homomorphism* $\mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g,r'})$*. It is namely the homomorphism associated to the forgetful map* $\mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g,r'})$*, which we recall is holomorphic.*

Example 3.14 (Constructions via covers)**.** *For some* $g \geq 2$*, let* $* \in S_{g,0}$ *be a base point and let* $\Gamma \subset \pi_1(S_{g,0}, *)$ *the subgroup consisting of elements which are trivial in the* $\mathbb{Z}/2\mathbb{Z}$*-homology. The subgroup* Γ *is the fundamental group of a closed surface, and indeed it follows from the Riemann-Hurwitz formula that we have* $\Gamma \simeq \pi_1(S_{(g-1)\cdot 2^{2g}-1,0}, *)$*. Noting that* $\mathrm{Aut}(\pi_1(S_{g,0}, *))$ *preserves* Γ *we get thus a homomorphism*

$$\mathrm{Aut}(\pi_1(S_{g,0}, *)) \to \mathrm{Aut}(\Gamma) \simeq \mathrm{Aut}(\pi_1(S_{(g-1)\cdot 2^{2g}-1,0}), *)).$$

Since $\mathrm{PMap}(S_{g,1}) \simeq \mathrm{Aut}(\pi_1(S_{g,0}, *))$ *we get a homomorphism*

$$\mathrm{PMap}(S_{g,1}) \to \mathrm{PMap}(S_{(g-1)\cdot 2^{2g}-1,1})$$

By construction, this homomorphism is associated to a holomorphic map $\mathcal{M}(S_{g,1}) \to \mathcal{M}(S_{(g-1)\cdot 2^{2g}-1,1})$. *Indeed, there is an equivariant map* $\mathcal{T}(S_{g,1}) \to \mathcal{T}(S_{(g-1)\cdot 2^{2g}-1,1})$ *given by lifting each complex structure on* $S_{g,1}$ *to a complex structure on* $S_{(g-1)\cdot 2^{2g}-1,2^{2g}}$ *and then forgetting all but one of the marked points.*

In the absence of marked points it is a bit harder to use covers to get homomorphisms between mapping class groups, but it was shown in [6] that for some suitable g' *there is a cover* $\pi : S_{g',0} \to S_{g,0}$ *which induces a homomorphism* $\mathrm{PMap}(S_{g,0}) \to \mathrm{PMap}(S_{g',0})$. *This homomorphism is once again induced by the holomorphic map* $\mathcal{M}(S_{g,0}) \to \mathcal{M}(S_{g',0})$ *given lifting to* $S_{g',0}$ *via* π *complex structures on* $S_{g,0}$.

Example 3.15 (Multi-embeddings)**.** *Thinking now of* $S_{g,r}$ *as an open surface, that is as a surface with cusps instead of marked points, suppose that for some* (g', r') *we have an (automatically finite) collection* $\mathcal{I} = \{\iota_i : S_{g,r} \to S_{g',r'}\}$ *of embeddings with disjoint image. Every multi-embedding induces a (diagonal) homomorphism*

$$\mathcal{I}_* : \mathrm{Homeo}_c(S_{g,r}) \to \mathrm{Homeo}_c(S_{g',r'})$$

between the groups of compactly supported homeomorphisms of the domain and the target. If the homomorphism $\mathcal{I}_*$ *induces a homomorphism* $\mathcal{I}_* : \mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g',r'})$, *then we say that the latter is* induced by the multi-embedding $\mathcal{I}$.

It is important to keep in mind that not every multi-embedding induces a homomorphism between mapping class groups. Indeed, Lemma 2.18 asserts that a multi-embedding $\mathcal{I}$ *as above induces a homomorphism between mapping class groups if and only if for every* i *and every oriented curve* $\gamma \subset S_{g,r}$ *which can be homotoped into a cusp, one of the following holds:*

- *c1) Either* $\iota_i(\gamma)$ *is homotopically trivial,*
- *c2) or* $\iota_i(\gamma)$ *can be homotoped into a cusp of* $S_{g',r'}$,
- *c3) or there is another* $i' \neq i$ *with* $\iota_{i'}(\gamma)$ *homotopic to* $\iota_i(\gamma)$ *such that the two embeddings* ι_i *and* $\iota_{i'}$ *pullback opposite orientations.*

For example, let Σ *be a compact surface with interior homeomorphic to* $S_{g,1}$ *and identify* $S_{2g,0}$ *with the double of* Σ. *The two tautological embeddings of* $S_{g,1} = \Sigma \setminus \partial\Sigma$ *into the double yield a multi-embedding* $\mathcal{I}$ *of* $S_{g,1}$ *satisfying the conditions above and hence inducing a homomorphism* $\mathcal{I}_* : \mathrm{PMap}(S_{g,1}) \to \mathrm{PMap}(S_{2g,0})$. *Noting that this homomorphism is not irreducible, we get from Theorem 3.1 that* $\mathcal{I}_*$ *is not induced by any holomorphic (or anti-holomorphic) map* $\mathcal{M}(S_{g,1}) \to \mathcal{M}(S_{2g,0})$.

As the reader might have noticed, there is some redundancy in the above list of examples: both automorphisms and homomorphisms induced by forgetful maps are also induced by multi-embeddings. Indeed, a homomorphism

$\mathcal{I}_*$ induced by a multi-embedding $\mathcal{I}$ is the composition of an automorphism and a forgetful map if and only if the collection $\mathcal{I}$ is just an embedding, in the sense that it has a single element.

As we mentioned, some homomorphisms induced by multi-embeddings are reducible and hence, by Theorem 3.1, don't come from any holomorphic map between moduli spaces. The following proposition characterizes the irreducible homomorphisms $\varphi : \mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g',r'})$ in the range we are interested in:

Proposition 3.16. *A homomorphism $\varphi : \mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g',r'})$ induced by a multi-embedding is irreducible if and only if it is the composition of an automorphism and of a forgetful map.*

Proof. If φ is the composition of automorphisms and forgetful maps, then it is easy to see that φ is irreducible. We show the other direction.

Let $\mathcal{I}$ be a multi-embedding inducing φ. Observe that φ is the composition of automorphisms and forgetful maps if and only if $\mathcal{I}$ contains a single embedding. Thus, to prove the statement we assume $\mathcal{I}$ contains at least two embeddings and show that φ is reducible.

If $\mathcal{I}$ contains two embeddings, then $r > 0$ and $g' > g$. Even more, by Lemma 2.18 there is a curve $\gamma \subset S_{g,r}$ homotopic to a cusp and an embedding $\iota \in \mathcal{I}$ such that $\iota(\gamma) \subset S_{g',r'}$ is a non-trivial curve non-homotopic to a cusp. Since every element of $\mathrm{PMap}(S_{g,r})$ fixes the homotopy class of γ and φ is induced by a multi-embedding, then the image of φ fixes the homotopy class of $\iota(\gamma)$. In other words, φ is reducible. □

3.3.3 Holomorphisms between moduli spaces

We come now to the final act of this chapter, the proof of Theorem E:

Theorem E. *Suppose that $g \geq 4$ and that $g' \leq 3 \cdot 2^{g-3}$, and let $r, r' \geq 0$. Then every non-constant holomorphic map $F : \mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g',r'})$ is a forgetful map. In particular, if such a map exists, then $g' = g$ and $r' \leq r$.*

Proof. Let $F_* : \mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g',r'})$ be the homomorphism corresponding to a non-constant holomorphic map $F : \mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g',r'})$. From Theorem 3.1 we get that $F_*(\mathrm{PMap}(S_{g,r}))$ does not fix any non-trivial simple multi-curve. It thus follows from Theorem A and Proposition 3.16 that F_* is induced by the composition of an automorphism

$$\mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g,r}), [\psi] \mapsto [\Psi \circ \psi \circ \Psi^{-1}]$$

as in Example 3.12 and of a forgetful homomorphism

$$\mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g',r'})$$

as in Example 3.13. In particular, $g = g'$ and $r' \leq r$. Moreover, since the composition of an anti-holomorphic map and of a holomorphic map is anti-holomorphic we get that the homeomorphism $\Psi \in \mathrm{Homeo}(S_{g,r})$ inducing our automorphism has to be orientation preserving. From here, it follows that $F_* : \mathrm{PMap}(S_{g,r}) \to \mathrm{PMap}(S_{g,r'})$ after conjugacy agrees with the homomorphism $\hat{F}_*$ induced by the forgetful map $\hat{F} : \mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g,r'})$. Now, since Teichmüller space is a classifying space for proper actions of the mapping class group [54] we get that F and $\hat{F}$ are homotopic. To conclude the proof, it suffices to invoke a result from [4, Proposition 3.2]: any two non-constant holomorphic and homotopic maps from a quasi-projective variety to the moduli space agree. We are done. □

Chapter 4

Hopfian big mapping class groups

In this chapter we study the Hopfian property of certain subgroups of big mapping class groups. The results presented in this chapter were obtained in joint work with Javier Aramayona, Rachel Skipper, Jing Tao, Nicholas G. Vlamis and Xiaolei Wu.

Let M be an orientable surface with no boundary and let $\mathcal{E}(M)$ be the space of ends of M. We equip $\mathrm{Map}(M)$ with the compact-open topology, that is, the quotient topology coming from the compact-open topology on $\mathrm{Homeo}^+(M)$. With this topology, $\mathrm{Map}(M)$ is a topological group. Throughout the chapter, subgroups of $\mathrm{Map}(M)$ are assumed to be equipped with the subspace topology. We let $\Gamma_c(M)$ denote the subgroup of $\mathrm{Map}(M)$ consisting of compactly supported mapping classes (i.e., the mapping classes with a representative that restricts to the identity outside of a compact set), and we let $\Gamma(M)$ denote the closure of $\Gamma_c(M)$.

Definition 4.1. *Let M be an infinite-type 2-manifold, and let H be a subgroup of* $\mathrm{Map}(M)$.

- *H is* large *if it contains* $\Gamma(M)$.
- *H is* mostly pure *if $H \cap \mathrm{PMap}(M)$ has countable index in H.*

For example, $\mathrm{PMap}(M)$ and $\Gamma(M)$ are both large and mostly pure. On the other hand, $\mathrm{Map}(M)$ is large, but need not be mostly pure; in particular, $\mathrm{Map}(M)$ is mostly pure if and only if M has finitely many ends.

Now, in this chapter we present a proof of Theorem F, that we restate here in a more general version.

Theorem F. *Let M and M' be two orientable surfaces with no boundary, and suppose M has infinite genus and has no planar ends. If G is a mostly pure large subgroup of* $\mathrm{Map}(M)$ *and G' is a large subgroup of* $\mathrm{Map}(M')$,

then every continuous epimorphism $G \to G'$ is induced by a homeomorphism $M \to M'$.

Here, it is worth stressing that, to the best of our knowledge, the analog of Theorem F is not known in the case of finite-type surfaces. In light of the forgetful and capping homomorphisms, a possible version of this problem is:

Question 4.2. *Let M and M' be finite-type surfaces of genus g and g', respectively, where $g \geq 3$ and $g' \neq g$. Are there any epimorphisms $\mathrm{PMap}(M) \to \mathrm{PMap}(M')$?*

Next, recall that an object A in a category is *Hopfian* if every epimorphism $A \to A$ is an automorphism. It is a standard fact that every finitely generated residually finite group is Hopfian (in the category of abstract groups). As the mapping class group of a finite-type surface is residually finite (see [32, §6.4]) and finitely generated (see [32, §4.3]), every finitely generated subgroup of the mapping class group of a finite-type orientable surface is Hopfian. One of the motivating problems for this article is to understand when $\mathrm{Map}(M)$ and $\mathrm{PMap}(M)$ are Hopfian groups. As a corollary of Theorem F, we establish a class of big mapping class groups that are Hopfian in the category of topological groups.

Corollary 4.3. *If M is an orientable infinite-genus surface with no boundary no planar ends, then every mostly pure large subgroup of $\mathrm{Map}(M)$ is a Hopfian topological group; in particular, $\mathrm{PMap}(M)$ is a Hopfian topological group, and if M has finitely many ends, then so is $\mathrm{Map}(M)$.* □

Note that Corollary 4.3 in particular yields Corollary 0.4 from the introduction.

Our proof of the Theorem F relies crucially on continuity in several locations. In the final section of this chapter, we give an accounting of how continuity is used and an a priori weaker condition that suffices for our arguments. This weaker condition is motivated by the definition of *infinitely multiplicative homomorphisms* given by Cannon-Conner [26]. We leave as a question whether continuity can be removed all together from the hypotheses in Theorem F.

Question 4.4. *If M is a non-compact orientable surface with no boundary and no planar ends, is every mostly pure large subgroup of $\mathrm{Map}(M)$ a Hopfian group?*

When discussing forgetting ends, it is natural to work in the setting of pure mapping class groups, but we expect Theorem F to hold more generally for large subgroups. In the specific case of generalizing Theorem F to $\mathrm{Map}(M)$, we can reduce the problem to asking if there exists an epimorphism $\mathrm{Homeo}(\mathcal{E}(M)) \to G'$, where $\mathcal{E}(M)$ denotes the end space of M. We have an approach, based on the work of Afton-Calegari-Chen-Lyman [1],

for answering this question in the negative, but it is not clear if it can be applied to all cases. In the next theorem, we provide a class of surfaces for which this approach is successful.

A surface M is *perfectly self-similar* if $M\#M$ is homeomorphic to M and for every proper compact subset K of M there exists $f \in \mathrm{Homeo}(M)$ such that $f(K) \cap K = \varnothing$.

Theorem 4.5. *Let M and M' be orientable surfaces with no boundary, and suppose M is infinite genus with no planar ends. If M is perfectly self-similar and G' is a large subgroup of $\mathrm{Map}(M')$, then every continuous epimorphism from $\mathrm{Map}(M)$ to G' is induced by a homeomorphism $M \to M'$.*

And as a corollary, we have the Hopfian property.

Corollary 4.6. *If M is a perfectly self-similar infinite-genus surface with no planar ends, then $\mathrm{Map}(M)$ is a Hopfian topological group.* □

In the special case that M is as in the corollary and the end space of M is a Cantor set (i.e., M is the blooming Cantor tree surface), Vlamis [83] showed that $\mathrm{Map}(M)$ has the automatic continuity property, which implies that every endomorphism of M is continuous. Therefore, we can upgrade the corollary from the topological category to the abstract category.

Corollary 4.7. *The mapping class group of the blooming Cantor tree surface is Hopfian.* □

Motivation and context

Motivated by Margulis's Superrigidity for lattices in Lie groups, Ivanov [50] proved that automorphisms of finite-type mapping class groups are induced by surface homeomorphisms. Since then, there has been a significant amount of work dedicated to classifying homomorphisms between mapping class groups; we refer the reader to the survey article [12] and the references therein for an overview.

In light of this, a natural question to ask is whether the same types of phenomena occur in the context of big mapping class groups. In this direction, Bavard-Dowdall-Rafi [15] proved the analog of Ivanov's theorem for big mapping class groups, showing that any isomorphism between finite-index subgroups of big mapping class groups is induced by a homeomorphism. On the other hand, Leininger, McLeay and Aramayona [9] constructed infinite families of (pure) big mapping class groups that are not *co-Hopfian* in the category of topological groups, that is, they admit continuous injective homomorphisms that are not surjective. On the other hand, subject to certain topological restrictions on the surfaces, the same authors proved that if a continuous injective homomorphism sends Dehn twists to Dehn twists, then it is induced by a proper surface embedding [9, Theorem 3]

For more general types of homomorphisms, the situation is even wilder. For instance, if M is an orientable infinite-genus surface with no boundary and no planar ends, then $\mathrm{PMap}(M)$ admits non-trivial endomorphisms: If M has at least two ends, then by a result of the Aramayona, Patel, and Vlamis [10], $\mathrm{PMap}(M)$ surjects onto $\mathbb{Z}$, and hence, composing this surjection with a homomorphism $\mathbb{Z} \to \mathrm{PMap}(M)$ gives a nontrivial endomorphism. In the case M has one end, i.e., M is the Loch Ness monster surface, Domat-Dickmann [29] showed that $\mathrm{Map}(M)$ surjects onto $\mathbb{Q}$. Aougab, Patel, and Vlamis [5] showed that $\mathrm{Map}(M)$ contains a copy of every countable group; in particular, it contains a copy of $\mathbb{Q}$, and hence taking the composition $\mathrm{Map}(M) \to \mathbb{Q} \to \mathrm{Map}(M)$ yields a nontrivial endomorphism.

Finally, it is also not difficult to see that the non-planar hypothesis is necessary. If M is a surface with infinitely many isolated planar ends, then forgetting a single isolated planar end yields a continuous epimorphism $\mathrm{PMap}(M) \to \mathrm{PMap}(M)$ that is not an isomorphism. For the full mapping class group, consider the surface M obtained by removing a copy of the ordinal space $\omega^\omega + 1$ from the 2-sphere. The derived set of $\omega^\omega + 1$ is homeomorphic to $\omega^\omega + 1$; therefore, forgetting all the isolated ends of M yields a continuous epimorphism $\mathrm{Map}(M) \to \mathrm{Map}(M)$ that is not an isomorphism. These examples show that $\mathrm{Map}(M)$ and $\mathrm{PMap}(M)$ need not be Hopfian when M has planar ends. However, despite these examples failing to be induced by a homeomorphism, they do arise from geometric constructions, i.e., by embedding the manifold into itself, which leads to the natural question:

Question 4.7. *Does every (continuous) epimorphism between (pure) mapping class groups come from a geometric construction?*

4.1 Preliminaries

4.1.1 Infinite-type surfaces and curves

A surface is *planar* if it can be embedded in $\mathbb{R}^2$. A subsurface of a surface S is a closed subset of S that is a surface with respect to the subspace topology. We will usually consider subsurfaces up to isotopy. The *space of ends* of S is

$$\mathcal{E}(S) = \lim_{\leftarrow} \pi_0(S \smallsetminus K),$$

where the inverse limit is taken over the collection of all compact subsets $K \subset S$. Equipping $\pi_0(S \smallsetminus K)$ with the discrete topology, the limit topology on $\mathcal{E}(S)$ is totally disconnected, separable, and metrizable. An element $e \in \mathcal{E}(S)$ is called an *end* of S. An end is *planar* if it is the end of a planar subsurface of S.

The classification of surfaces (see [76]) says that the homeomorphism type of an orientable surface with no boundary is determined by the following

data: its genus (possibly infinite) and, up to homeomorphism, its space of ends together with the closed subset of its set of non-planar ends.

Recall that a *multicurve* is a set of pairwise-disjoint curves. Two multicurves A and B are *disjoint* if $A \cap B = \varnothing$ and $i(a, b) = 0$ for all $a \in A$ and for all $b \in B$. A set of curves A is *locally finite* if, given a curve b, all but finitely many curves in A are disjoint from b.

A *pants decomposition* of a surface S is a locally finite multicurve P such that each component of $S \setminus P$ is homeomorphic to the thrice-punctured sphere (i.e., a *pair of pants*). A surface admits a pants decomposition if and only if its fundamental group is non-abelian and every boundary component (if there are any) is compact.

On several occasions, we will have to consider arcs in addition to curves. In this chapter we only discuss arcs in the context of surfaces with no boundary. A *simple proper arc* in a surface with no boundary M refers to the image of a proper embedding $\mathbb{R} \to M$; it is *essential* if it does not bound an open disk in M. We will use the term *arc* to refer to the isotopy class of an essential simple proper arc. The definition of the geometric intersection number extends to the setting of arcs and curves.

4.1.2 Mapping class groups and their topology

The group $\mathrm{Map}(S)$ naturally inherits the quotient topology coming from the compact-open topology on the homeomorphism group of S. It will be useful to have a more workable description of this topology; this involves using a corollary to the Alexander method (see [32, §2.3] for finite-type surfaces and see [41] for infinite-type surfaces).

Proposition 4.8. *Let M be a orientable surface of genus at least three with no boundary, and let $f \in \mathrm{Map}(M)$. If $f(c) = c$ for every curve c in M, then f is the identity.* □

The genus restriction above is to rule out the finite number of surfaces with no boundary for which the statement fails. This is exactly when the mapping class group has nontrivial center.

The Alexander method implies that $\mathrm{Map}(M)$ acts faithfully on the set of curves in M, and therefore, a mapping class is completely determined by the permutation it induces on the set of curves. Given a set of curves A, set

$$U_A = \{f \in \mathrm{Map}(M) : f(a) = a \text{ for all } a \in A\}.$$

Then the U_A form a neighborhood basis for identity in the compact-open topology. It follows that the sets of the form $f\,U_A = \{g \in \mathrm{Map}(M) : g(a) = f(a) \text{ for all } a \in A\}$ form a basis for the compact-open topology (see [13, §4]).

4.1.3 Multitwists along infinite multicurves

It is an exercise in the compact-open topology to show that if A is a locally finite multicurve containing infinitely many curves, $\{k_a\}_{a\in A}$ is a sequence of integers, and $\{a_n\}_{n\in\mathbb{N}}$ is an enumeration of the elements of A, then

$$\lim_{n\to\infty}\prod_{j=1}^{n} d_{a_n}^{k_{a_n}}$$

exists and does not depend on the choice of enumeration; we write this limit as $\prod_{a\in A} d_a^{k_a}$. For a locally finite multicurve A, a *multitwist* along A is a mapping class of the form $\prod_{a\in A} d_a^{k_a}$, where $k_a \in \mathbb{Z}$. For simplicity, we will usually write m_A for a multitwist along A.

Lemma 4.9. *Let $A = \{a_n\}_{n\in\mathbb{N}}$ be a multicurve on a surface S, and let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence of non-zero integers. Then $\lim_{n\to\infty}\prod_{j=1}^{n} d_{a_n}^{k_n}$ exists if and only if A is locally finite.*

Proof. We already established the backwards direction above, so we only need to show the forwards direction. We will argue the contrapositive: let A fail to be locally finite. Then there exists a curve c and a subsequence of $\{a_{n_j}\}$ such that $i(c, a_{n_j}) \neq 0$ for all $j \in \mathbb{N}$. Let $T_m = \prod_{j=1}^{n_m} d_{a_j}^{k_j}$, and let $F_m = T_m^{-1}T_{m+1}$. Note that if $\lim_{n\to\infty}\prod_{j=1}^{n} d_{a_n}^{k_n}$ exists, then the sequence $\{T_m\}_{m\in\mathbb{N}}$ converges, and hence $\{F_m\}_{m\in\mathbb{N}}$ converges to the identity. We will show that the sequence $\{F_m\}$ fails to converge to the identity, which will finish the proof.

Observe that $F_m(c) = d_{a_{n_{m+1}}}^{k_{n_{m+1}}}(c)$. For each $m \in \mathbb{N}$, choose a curve b_m satisfying $i(c, b_m) = 0$ and $i(a_{n_{m+1}}, b_m) > 0$. Then, by [32, Proposition 3.4],

$$i(F_m(c), b_m) = i(d_{a_{n_{m+1}}}^{k_{n_{m+1}}}(c), b_m) \geq |k_{n_{m+1}}|i(a_{n_{m+1}}, c)i(a_{n_{m+1}}, b_m) - i(c, b_m),$$

and hence $i(F_m(c), b_m) > 0$, as $i(c, b_m) = 0$ and all the other terms on the right are positive. This implies that $F_m(c) \neq c$ for all $m \in \mathbb{N}$, and in particular, there exists a neighborhood of the identity in $\mathrm{Map}(S)$ that the sequence $\{F_m\}_{m\in\mathbb{N}}$ never enters, namely $\{f \in \mathrm{Map}(S) : f(c) = c\}$. Therefore, the sequence $\{F_m\}$ does not converge. □

From the above lemmas, we can readily deduce the following, using a limiting argument in the case of infinite multitwists:

Lemma 4.10. *Let S be an orientable surface, and let $m_A, m_B \in \mathrm{Map}(S)$ be non-trivial multitwists along the locally finite multicurves A and B, respectively. Then m_A and m_B commute if and only if $i(a, b) = 0$ for every $a \in A$ and $b \in B$.* □

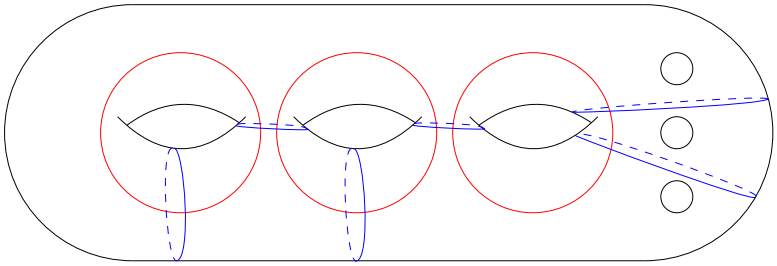


Figure 4.1: The Dehn twists about the curves shown generate $\mathrm{PMap}(S)$, where S is the surface shown. Here, S is a genus three surface with three boundary components.

4.2 Generating compactly supported mapping classes

A mapping class is *compactly supported* if it has a representative that restricts to the identity outside a compact set. Given a surface M, let $\Gamma_c(M)$ be the subgroup of $\mathrm{Map}(M)$ consisting of compactly supported mapping classes. In the proof of Theorem F, we will need to have a generating set for $\Gamma_c(M)$ that is analogous to the Humphries generating set in the finite-type setting. We separate this goal into its own section as it may be of independent interest.

Let us begin by recalling the Humphries generating set (see [32, §4.4.3 and §4.4.4]). Humphries's theorem tells us that the Dehn twists about the curves shown in Figure 4.1 generate $\mathrm{Map}(S)$; in particular, for a compact surface with nonempty boundary, the Humphries generating set consists of $2g + b$ non-separating Dehn twists (the figure shows a genus three surface with three boundary components, but it should be clear how to generalize to higher genus and more boundary components). We note that Humphries's theorem requires the surface to be orientable and nonplanar. We will build on Humphries's theorem to construct a generating set for $\Gamma_c(M)$ when M is an orientable infinite-genus surface with no boundary and no planar ends.

Lemma 4.11. *Let M be an orientable infinite-genus surface with no boundary. If M has no planar ends, then there exists a sequence of compact subsurfaces $\{\Sigma_n\}_{n\in\mathbb{N}\cup\{0\}}$ in M such that*

(i) Σ_0 is a genus one surface with one boundary component,

(ii) each component of $\partial\Sigma_n$ is separating,

(iii) $\Sigma_n \subset \Sigma_{n+1}$,

(iv) $M = \bigcup_{n\in\mathbb{N}} \Sigma_n$, and

(v) the closure of $\Sigma_{n+1} \smallsetminus \Sigma_n$ is a genus one surface with either two or three boundary components.

Sketch of proof. Let T be a rooted tree whose end space is homeomorphic to the end space of M and such that the valence at the root is one but at all other vertices is either two or three (such a tree can be obtained by appropriately pruning an infinite binary tree). For $i \in \{2, 3\}$, let F_i be a genus one surface with i boundary components. We can then build a surface using T as a blueprint as follows: if v is the root of T, let F_v be a genus one surface with one boundary component, and for every other vertex v in T, let F_v be a copy of F_i, where i is the valence of v. Take the disjoint union of all the F_v and glue them along their boundaries according to the adjacency relation in T. By the classification of surfaces, the resulting surface is homeomorphic to M. The vertices of a binary tree have a canonical enumeration (in base two), inducing an order on its vertices. As T is obtained from pruning a binary tree, its vertices inherit an enumeration $\{v_i\}_{i\in\mathbb{N}}$, labeled by the order in which they appear. The surfaces of the form $\Sigma_n = \bigcup_{i=0}^{n} F_{v_i}$ yield the desired sequence. □

A *chain* of curves in a surface is a locally finite set of curves such that any two curves in the set have geometric intersection number at most one; a chain is *filling* if every curve and arc on the surface has nontrivial geometric intersection with at least one curve in the chain. Associated to any chain $\mathcal{C}$, we can define a graph $T(\mathcal{C})$ as follows: the vertices are the curves in the chain, and two vertices are adjacent if their geometric intersection is one. If $T(\mathcal{C})$ is a tree, we say the chain is *tree-like*. An *Alexander chain* is a filling tree-like chain consisting of non-separating curves. Observe that the curves in Figure 4.1 form an Alexander chain.

In what follows, we will require a generating set for the mapping class group of a four-holed sphere. Let R be a compact four-holed sphere. If a and b are curves with $i(a, b) = 2$, then d_a and d_b together with the Dehn twists along any three of the four boundary components of R generate $\mathrm{Map}(R)$ (this is an exercise combining the computation of generators for the mapping class group of a pair of pants [32, §3.6.4] and a four-punctured sphere [32, §4.2.4], the inclusion homomorphism [32, Proposition 3.18], and the lantern relation [32, Proposition 5.1]).

Given a compact subsurface Σ of a surface M such that each component of $\partial\Sigma$ is separating and essential, the inclusion of Σ into M induces a monomorphism $\mathrm{Map}(\Sigma) \to \mathrm{Map}(M)$, see [32, Theorem 3.18]. Abusing notation, we will identify $\mathrm{Map}(\Sigma)$ with its image.

Theorem 4.12. *Let M be an orientable infinite-genus surface with no boundary. If M has no planar ends, then there exists an Alexander chain $\mathcal{A}$ in M such that the set of Dehn twists $\{d_a\}_{a\in\mathcal{A}}$ generates $\Gamma_c(M)$. Moreover, there exists an exhaustion of M by compact surfaces $\{\Sigma_n\}_{n\in\mathbb{N}}$ such that each component of $\partial\Sigma_n$ is separating and essential, such that $\mathcal{A}_n = \{a \in \mathcal{A} : a \subset \Sigma_n\}$ is a an Alexander chain in Σ_n, and such that the set $\{d_a : a \in \mathcal{A}_n\}$ generates* $\mathrm{Map}(\Sigma_n)$.

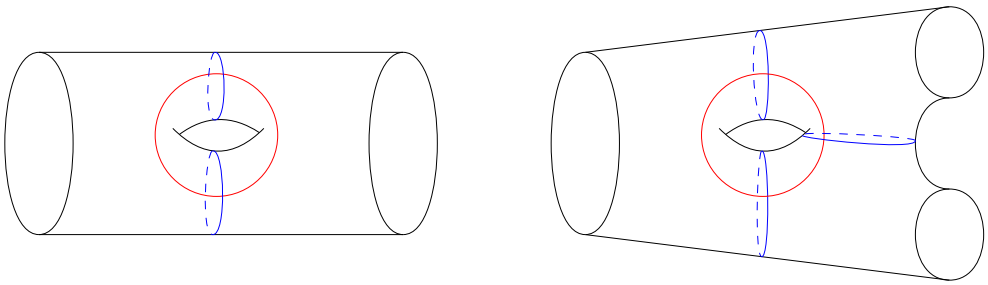


Figure 4.2: The surfaces F_2 (left) and F_3 (right) and their corresponding Alexander chains, C_2 and C_3.

Proof. Let $\{\Sigma_n\}_{n\in\mathbb{N}}$ be the compact exhaustion of M given by Lemma 4.11. Then $\Gamma_c(M) = \bigcup_{n\in\mathbb{N}} \mathrm{Map}(\Sigma_n)$. We therefore need to give an Alexander chain $\mathcal{A}$ such that the Dehn twists about the curves in $\mathcal{A}$ contained in Σ_n generate $\mathrm{Map}(\Sigma_n)$. Let us construct $\mathcal{A}$ and then argue that it satisfies this property.

Let $S_0 = \Sigma_0$, and for $n \in \mathbb{N}$, let S_n be the closure of $\Sigma_n \smallsetminus \Sigma_{n-1}$. Let $\mathcal{A}_0$ consist of any two simple closed curves with geometric intersection number one; hence, the curves in $\mathcal{A}_0$ generate $\mathrm{Map}(\Sigma_0, \partial\Sigma_0)$ (see [32, §3.6.4]). For $i \in \{2, 3\}$, let F_i be the genus one compact surface with i boundary components. Let C_i be the Alexander chain in F_i shown in Figure 4.2. For $n \in \mathbb{N}$, let $i(n) \in \{2, 3\}$ such that S_n is homeomorphic to $F_{i(n)}$, and fix a homeomorphism $f_n : F_{i(n)} \to S_n$. Let $\mathcal{A}'$ be the chain defined by $\mathcal{A}' = \mathcal{A}_0 \cup \bigcup_{n=1}^{\infty} f_n(C_{i(n)})$.

We now add curves to the collection $\mathcal{A}'$ so that it forms an Alexander chain. Each of the curves in Figure 4.2 are colored either blue or red; each surface in the figure contains a single red curve that intersects each of the remaining curves—the blue curves—exactly once. Note that, in each surface, the blue curves give a pants decomposition. Designate one curve in $\mathcal{A}_0$ to be blue. Now, let $\mathcal{A}'_b$ denote the subset of $\mathcal{A}'$ containing all the blue curves. Each component of $M \smallsetminus \mathcal{A}'_b$ is a four-holed sphere. Enumerate the components as follows: define Ω_n to be the unique component that is contained in Σ_n but not in Σ_{n-1}. Choose a non-separating curve a_n in M that is contained in Ω_n and such that $\mathcal{A}' \cup \{a_n\}$ is a chain. Let $\mathcal{A} = \mathcal{A}' \cup \{a_n : n \in \mathbb{N}\}$. By construction, $\mathcal{A}$ is an Alexander chain.

For $n \in \mathbb{N} \cup \{0\}$, let $\mathcal{A}_n$ be the subset of $\mathcal{A}$ consisting of curves contained in Σ_n. We claim that $\{d_a\}_{a\in\mathcal{A}_n}$ generates $\mathrm{Map}(\Sigma_n)$. We proceed by induction. By construction, $\{d_a\}_{a\in\mathcal{A}_0}$ is a generating set for $\mathrm{Map}(\Sigma_0)$. Now, assume that $\{d_a\}_{a\in\mathcal{A}_n}$ generates $\mathrm{Map}(\Sigma_n)$. Recall that $\Sigma_{n+1} = \Sigma_n \cup S_{n+1}$, where S_{n+1} is a compact genus one surface with either two or three boundary components.

Let H be the subgroup generated by $\{d_a\}_{a\in\mathcal{A}_{n+1}}$. In order to show that $H = \mathrm{Map}(\Sigma_{n+1}$, it is enough to show that $d_{b_i} \in H$ for $i \in \{1, \ldots, k\}$, where

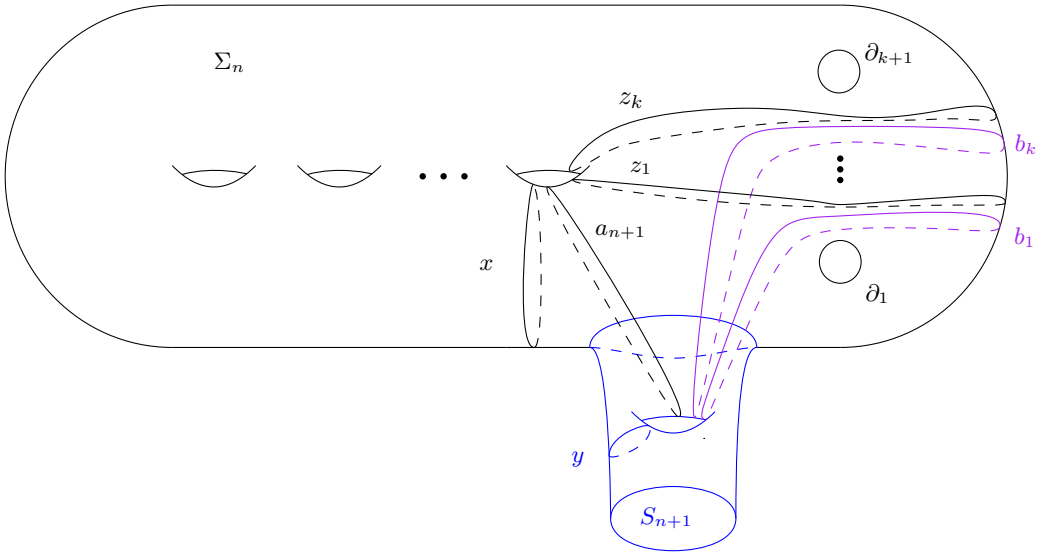


Figure 4.3: The curves involved in the inductive step in the proof of Theorem 4.12.

b_i is as shown in Figure 4.3. As the set $\{d_a\}_{a\in\mathcal{A}_{n+1}} \cup \{d_{b_i}\}_{i=1}^{k}$ contains a Humphries generating set for $\mathrm{Map}(\Sigma_{n+1})$. For the rest of the argument, we will use the curves as labelled in Figure 4.3. The figure is drawn with S_{n+1} having two boundary components, but the same labeling and the following argument directly apply to the case where S_{n+1} has three boundary components.

For $i \in \{1, \dots, k\}$, consider the four-holed sphere R_i embedded in Σ_{n+1} with boundary components x, y, b_i, and z_i. Observe that there exists an essential simple closed curve w_i in R_i that is contained in Σ_n and satisfies $i(a_{n+1}, w_i) = 2$. Then $\{d_{a_{n+1}}, d_{w_i}, d_x, d_y, d_{z_i}\}$ generates $\mathrm{Map}(R_i)$. Therefore, as $d_{b_i} \in \mathrm{Map}(R_i)$ and $\{d_{a_{n+1}}, d_{w_i}, d_x, d_y, d_{z_i}\} \subset H$, we have that $d_{b_i} \in H$, as desired. □

Remark 4.13. *Above we describe the Humphries generating set for an orientable non-planar compact surface; however, the Humphries theorem holds more generally for orientable non-planar surfaces of finite type, with* $\mathrm{Map}(S)$ *replaced by* $\mathrm{PMap}(S)$*. As a consequence, Theorem 4.12 can be readily extended to orientable infinite-genus 2-manifolds in which every planar end is isolated.*

4.3 Irreducible epimorphisms

As $\Gamma(M)$ is the smallest mostly pure large subgroup of $\mathrm{Map}(M)$, we will begin by working with this group and later reduce the main theorem to this

case.

A homomorphism $\varphi : \Gamma(M) \to \Gamma(M')$ is *compact* if $\varphi(\Gamma_c(M)) \subset \Gamma_c(M')$; it is *reducible* if there exists an arc or curve a in M' such that $\varphi(g)(a) = a$ for all $g \in \Gamma(M)$, and otherwise it is *irreducible*. The goal of the section is to prove the following:

Theorem 4.14. *Let M and M' be non-compact orientable surfaces with no boundary and no planar ends. Every continuous compact irreducible homomorphism $\Gamma(M) \to \Gamma(M')$ is induced by a homeomorphism.*

The key ingredient in the proof is to establish that such a homomorphism sends Dehn twists to Dehn twists and preserves geometric intersection numbers 0 and 1 between the associated curves. From this, we can build the desired homeomorphism. To accomplish this, we proceed with a sequence of lemmas. As we proceed through the lemmas, we will put as few hypotheses on each lemma as possible in hopes that they will be useful to others.

Lemma 4.15. *If M is an infinite-type orientable surface with no boundary, then every nontrivial element of $\Gamma_c(M)$ has infinite order.*

Proof. If $g \in \Gamma_c(M)$ is nontrivial, then there exists a finite-type subsurface $\Sigma \subset M$ containing the support of g such that each boundary component of Σ is an essential simple closed curve. In this case, the inclusion $\Sigma \hookrightarrow M$ induces a monomorphism $\mathrm{Map}(\Sigma) \to \Gamma_c(M)$ whose image contains g. Therefore, we can view g as an element of $\mathrm{Map}(\Sigma)$, which is torsion free since Σ has non-empty boundary (see [32, Corollary 7.3]). □

Next, under the additional assumptions of non-triviality and continuity, we rule out the case that the image of a Dehn twist is trivial. As discussed above, pure mapping class groups of finite-type nonplanar surfaces are generated by Dehn twists about non-separating curves, and as all such Dehn twists are conjugate, these pure mapping class groups are normally generated by any such Dehn twist. It readily follows that $\Gamma(M)$ is topologically normally generated by any non-separating Dehn twist.

Lemma 4.16. *Let M be a non-planar orientable surface with no boundary, let M' be an infinite-type orientable surface with no boundary, and let $d \in \Gamma(M)$ be a non-separating Dehn twist. If $\varphi : \Gamma(M) \to \Gamma(M')$ is continuous, compact, and nontrivial, then $\varphi(d)$ has infinite order.*

Proof. By Lemma 4.15, $\varphi(d)$ is either trivial or infinite order. If $\varphi(d)$ is trivial, then φ is trivial, as d is a topological normal generator of $\Gamma(M)$ and φ is continuous; hence, $\varphi(d)$ has infinite order. □

Next, we extend a result of Bridson [21] to our setting.

Lemma 4.17. *Let M and M' be orientable surfaces with no boundary such that M has genus at least three, and let $d \in \mathrm{Map}(M)$ be a Dehn twist. If $\varphi : \Gamma(M) \to \Gamma(M')$ is a compact homomorphism, then $\varphi(d)$ is the root of a multitwist.*

Proof. Let d_a be a Dehn twist about the curve a in M. We can then find a compact surface Σ embedded in M such that $a \subset \Sigma$, the inclusion of Σ into M induces a monomorphism $\mathrm{Map}(\Sigma) \to \Gamma(M)$, and the genus of Σ is at least three. In particular, $d_a \in \mathrm{Map}(\Sigma)$. As φ is compact and as $\mathrm{Map}(\Sigma)$ is finitely generated, there exists a compact surface Σ' of M' whose inclusion induces a monomorphism $\mathrm{Map}(\Sigma') \to \Gamma(M')$ such that $\mathrm{Map}(\Sigma')$ contains $\varphi(\mathrm{Map}(\Sigma))$. We can then apply a result of Bridson [21, Remark 3.1(3)] to $\varphi|_{\mathrm{Map}(\Sigma)} : \mathrm{Map}(\Sigma) \to \mathrm{Map}(\Sigma')$ to see that $\varphi(d_a)$ is a root of a multitwist. □

Note that we allow for the identity to be considered a multitwist; however, Lemma 4.16 and Lemma 4.17 imply that the image of a non-separating Dehn twist under a compact, continuous, nontrivial homomorphism is a root of a multitwist of infinite order. We will now proceed to show that such an image is a nontrivial multitwist, i.e., it is not a root. But first, we introduce the following notation.

Notation. Let M and M' be orientable surfaces with the genus of M being at least three, and let $\varphi : \Gamma(M) \to \Gamma(M')$ be a continuous compact nontrivial homomorphism. If a is a non-separating simple closed curve in M, Lemma 4.16 and Lemma 4.17 imply there exists a multicurve, which we denote $\varphi_*(a)$, such that $\varphi(d_a)$ is a root of a multitwist along $\varphi_*(a)$.

A multicurve is *non-separating* if the union of all its elements is non-separating; it is *infinite* if it is an infinite set. Note that every subset of a non-separating multicurve is itself a non-separating multicurve; in particular, each curve in a non-separating multicurve is non-separating. Let us record a readily verifiable topological fact.

Lemma 4.18. *Let M be an infinite-genus orientable surface with compact boundary. If $\{a, b\}$ is a non-separating multicurve, then there exists a locally finite non-separating infinite multicurve containing a and b.* □

We use this topological observation to show that the images of two Dehn twists about curves that together form a non-separating multicurve are multitwists along disjoint multicurves. Recall that two multicurves A and B are disjoint if they have no curves in common and $i(a, b) = 0$ for all $a \in A$ and $b \in B$.

Lemma 4.19. *Let M and M' be orientable surfaces with no boundary such that M has infinite genus, and let $\varphi : \Gamma(M) \to \Gamma(M')$ be a continuous compact nontrivial homomorphism. If $\{a, b\}$ is a non-separating multicurve in M, then $\varphi_*(a)$ and $\varphi_*(b)$ are disjoint.*

Proof. First observe that d_a and d_b commute, so $\varphi(d_a)$ and $\varphi(d_b)$ also commute, as do their powers. Since $\varphi(d_a)$ and $\varphi(d_b)$ are roots of multitwists along $\varphi_*(a)$ and $\varphi_*(b)$, it follows then, by Lemma 4.10, that $i(a', b') = 0$ for all $a' \in \varphi_*(a)$ and $b' \in \varphi_*(b)$. So it is left to show that $\varphi_*(a) \cap \varphi_*(b) = \varnothing$.

By Lemma 4.18, there exists a locally finite non-separating infinite multicurve $\{a, b, c_1, c_2, \ldots\}$. Let $A = \varphi_*(a)$, $B = \varphi_*(b)$, and $C_n = \varphi_*(c_n)$ for each $n \in \mathbb{N}$. By the change of coordinates principle, for each $n \in \mathbb{N}$, there exists $f_n \in \Gamma_c(M)$ such that $f_n(a) = a$ and $f_n(b) = c_n$. Let $m \in \mathbb{N}$ such that $\varphi(d_a^m)$ is a multitwist. It follows that $\varphi(d^m)$ is a multitwist for any non-separating Dehn twist d. As f_n commutes with d_a^m and conjugates d_b^m to $d_{c_n}^m$, it follows that $\varphi(f_n)(A) = A$ and $\varphi(f_n)(B) = C_n$. Moreover,

$$\varphi(f_n)(A \cap B) = \varphi(f_n)(A) \cap \varphi(f_n)(B) = A \cap C_n,$$

where the first equality uses that fact that $\varphi(f_n)$ gives a bijection on the set all curves of M'.

Let us assume that $A \cap B$ is non-empty, so that $A \cap C_n$ is non-empty for all $n \in \mathbb{N}$. As the power set of A is finite, by passing to a subsequence, we may assume that $A \cap C_k = A \cap C_j$ for all $j, k \in \mathbb{N}$. This allows us to define $X = \varphi(f_k)(A \cap B)$ to be independent of k. Now observe that given $j \neq k \in \mathbb{N}$, there exists $g_{j,k} \in \Gamma(M)$ such that $g_{j,k}(a) = c_j$ and $g_{j,k}(b) = c_k$. It follows that $\varphi(g_{j,k})(A \cap B) = C_j \cap C_k$, and hence $|A \cap B| = |C_j \cap C_k|$. As $|X| = |A \cap B|$ and as $X \subset C_j \cap C_k$, we have that $C_j \cap C_k = X$.

Let d be a Dehn twist about a non-separating simple closed curve. Note that the set $\{c_k\}_{k \in \mathbb{N}}$ is locally finite, so d commutes with infinitely many of the c_k, and hence $\varphi(d)$ must commute with any multitwist about X, implying $\varphi(t)(X) = X$. As d was arbitrary, the previous statement holds for all non-separating Dehn twists, and as the non-separating Dehn twists generate $\Gamma_c(M)$, we have that $\varphi(f)(X) = X$ for all $f \in \Gamma_c(M)$.

For $k \in \mathbb{N}$, let d_k be the Dehn twist about c_k. Let $m \in \mathbb{N}$ be as above, so that $\varphi(d_k^m)$ is a multitwist, denoted m_k, about the multicurve C_k. As $X \subset C_k$, there exist multitwists $m_{X,k}$ and m'_k about X and $C_k \smallsetminus X$, respectively, such that $m_k = m_{X,k} m'_k$. Choose $h_k \in \Gamma_c(M)$ so that $h_k(c_1) = c_k$. It follows that $\varphi(h_k)$ conjugates m_1 to m_k. From the previous paragraph, we know that $\varphi(h_k)$ must fix X (setwise), and so $\varphi(h_k)$ conjugates $m_{X,1}$ to $m_{X,k}$. Since X is finite, its permutation group is finite; hence, by passing to a subsequence, we may assume that $\varphi(h_k)$ induces the same permutation on X for all k. Therefore, we have $m_{X,i} = m_{X,j}$ for all $i, j \in \mathbb{N}$. This allows us to define $m_X = m_{X,1}$ and to write

$$m_k = m_X m'_k$$

for every $k \in \mathbb{N}$.

To finish, for $n \in \mathbb{N}$, consider the mapping class $F_n = \prod_{k=1}^{n} d_k$. Then

$$\varphi(F_n) = \prod_{k=1}^{n} m_k = m_X^n \prod_{k=1}^{n} m'_k.$$

From the fact that the multicurve $\{c_k\}_{k\in\mathbb{N}}$ is locally finite, the limit of the F_n exists; however, the sequence $\{\varphi(F_n)\}_{n\in\mathbb{N}}$ diverges. This is not possible as φ is continuous, and therefore, $A \cap B$ must be empty. □

From Lemma 4.19, we readily deduce the following.

Lemma 4.20. *Let M and M' be orientable surfaces with no boundary such that M has infinite genus, and let $\varphi : \Gamma(M) \to \Gamma(M')$ be a continuous compact nontrivial homomorphism. If $A = \{a_i\}_{i\in I}$ is a locally finite multicurve in M and each a_i is a non-separating curve, then $\varphi_*(A) := \bigcup_{i\in I} \varphi_*(a_i)$ is a locally finite multicurve in M'.*

Proof. It readily follows from Lemma 4.19 that $\varphi_*(A)$ is a multicurve. Choose $m \in \mathbb{Z}$ such that $\varphi(d_a^m)$ is a multitwist for each $a \in A$, choose an enumeration $\{a_n\}_{n\in\mathbb{N}}$ of A, and let $m_n = \prod_{i=1}^{n} \varphi(d_{a_n}^m)$. Then, there exists an enumeration $\{b_i\}_{i\in\mathbb{N}}$ of $\varphi_*(A)$ such that, for each $n \in \mathbb{N}$, there exist $j_n \in \mathbb{N}$ and $\{k_1, \dots, k_{j_n}\} \subset \mathbb{Z} \setminus \{0\}$ satisfying $\varphi(m_n) = \prod_{i=1}^{j_n} d_{b_i}^{k_i}$. By continuity, $\lim \varphi(m_n)$ exists, as $\lim d_{a_n}^m$ exists. By Lemma 4.9, the existence $\lim \varphi(m_n)$ implies that $\varphi_*(A)$ is locally finite. □

It is worth noting that we have not yet put any restriction on the surface in the codomain in any of the above lemmas, but the previous lemma implies that it must be of infinite type.

Thus far, we have established that the image of two commuting non-separating Dehn twists have powers that are multitwists about disjoint multicurves; in other words, their images have powers with disjoint support. The next lemma upgrades this to saying that their images have disjoint supports. Recall that the *support* of a homeomorphism $f : M \to M$ is the closure of the set $\{x \in M : f(x) \neq x\}$, and we say that two mapping classes have *disjoint supports* if they have representative homeomorphisms with disjoint supports.

Given a locally finite multicurve A on a surface S, we abuse notation and write $S \setminus A$ to refer to the surface obtained by removing the union of representatives of the curves in A that are chosen to be pairwise disjoint (this can always be done by choosing a hyperbolic metric and taking geodesic representatives).

Lemma 4.21. *Let M and M' be orientable surfaces with no boundary such that M has infinite genus, and let $\varphi : \Gamma(M) \to \Gamma(M')$ be a continuous compact nontrivial homomorphism. If $\{a, b\}$ is a non-separating multicurve, then $\varphi(d_a)$ and $\varphi(d_b)$ have disjoint supports.*

Proof. Let $A' = \varphi_*(a)$, let $B' = \varphi_*(b)$, and let $M'_a = M' \setminus A'$. By Lemma 4.19, we know that A' and B' are disjoint multicurves. As A' is a finite multicurve, M'_a has finitely many components. Let $f \in \Gamma_c(M)$ commute with d_a. It follows that $\varphi(f)(A') = A'$, and hence permutes the components of M'_a. As $\varphi(f) \in \Gamma_c(M')$, it must be the case that $\varphi(f)$ fixes each infinite-type component of M'_a.

Let Σ be an infinite-type component of M'_a. Letting $\mathrm{Stab}(\mathrm{A}')$ denote the stabilizer of A' in $\Gamma(M')$, we see that there exists a restriction homomorphism $r : \mathrm{Stab}(\mathrm{A}') \to \Gamma(\Sigma)$ given by taking the restriction of an element of $\mathrm{Stab}(\mathrm{A}')$ to Σ. Under this homomorphism, any multitwist about A' is trivial, and hence, $(r \circ \varphi)(d_a)$ has finite order in $\Gamma(\Sigma)$, as $\varphi(d_a)$ is a root of a multitwist about A'. But, $(r \circ \varphi)(d_a) \in \Gamma_c(\Sigma)$, and hence is trivial by Lemma 4.15, allowing us to conclude that $\varphi(d_a)$ has a representative that restricts to the identity on the interior of any infinite-type component of M'_a.

From the above, we observe that if each component of M'_a has infinite type, then $\varphi(d_a)$ is a multitwist. It then follows that $\varphi(d_b)$ is also a multitwist, and as A' and B' are disjoint multicurves, $\varphi(d_a)$ and $\varphi(d_b)$ have disjoint supports, as desired.

We may therefore assume that M'_a contains at least one finite-type component. Let F'_a be the closures in M' of the finite-type components of M'_a. Then there exists a representative of $\varphi(d_a)$ supported in the union of F'_a with annuli about the curves in A'.

Let $b' \in B'$. As A' and B' are disjoint, b' is either an essential curve in F'_a or in $M \setminus F'_a$. Let $C = \{a, b, c_1, c_2, \ldots\}$ be a locally finite non-separating multicurve, which is guaranteed to exist by Lemma 4.18. By Lemma 4.20, $\varphi_*(C)$ is a locally finite multicurve in M'. So, there exists $c_i \in C$ such that $\varphi_*(c_i)$ is disjoint from F'_a. We can then choose $f \in \Gamma_c(M)$ such that $f(a) = a$ and $f(b) = c_i$, so that $\varphi(f)(B') = \varphi_*(c_i)$; in particular, $\varphi(f)(b') \subset M \setminus F'_a$. As $f(a) = a$, we have that f commutes with d_a. From above, it follows that $\varphi(f)(F'_a) = F'_a$. Therefore, b' cannot be an essential curve in F'_a, as $\varphi(f)(F'_a) = F'_a$ and $\varphi(f)(b') \subset M \setminus F'_a$, implying no curve in B' is contained in F'_a.

Defining F'_b analogously, a similar argument shows that no curve in A' is contained in F'_b. As A' and B' are disjoint multicurves, this forces F'_a and F'_b to be disjoint, and hence, the supports of $\varphi(d_a)$ and $\varphi(d_b)$ are disjoint. □

We can now eliminate the possibility of roots.

Lemma 4.22. *Let M and M' be orientable surfaces with no boundary such that M has infinite genus, and let $\varphi : \Gamma(M) \to \Gamma(M')$ be a continuous compact nontrivial homomorphism. If $d \in \Gamma(M)$ is a non-separating Dehn twist, then $\varphi(d)$ is a multitwist.*

Proof. Every surface of genus at least three contains an embedded 4-holed

sphere Σ in which each of its boundary components and each curve contained in Σ is non-separating in the ambient surface. Let a_1, a_2, a_3, and a_4 be the boundary components of Σ. By the lantern relation (see [32, Proposition 5.1]), there exist curves a_5, a_6, and a_7 in Σ such that

$$d_1 = d_5 d_6 d_7 (d_2 d_3 d_4)^{-1},$$

where d_i is the Dehn twist about a_i and such that $\{a_1, a_i\}$ is a non-separating multicurve for each $i \neq 1$.

By Lemma 4.21, for each i, we can choose a representative of $\varphi(d_i)$ in M' supported in a subsurface Σ'_i such that Σ'_1 and Σ'_i are disjoint whenever $i \neq 1$. In particular, by the lantern relation, $\varphi(d_1)$ has a representative supported in $\bigcup_{i=2}^{7} \Sigma'_i$, which is disjoint from Σ'_1. The only way this is possible is if $\varphi(d_1)$ is supported in a union of annuli, implying $\varphi(d_1)$ is a multitwist. But, d_1 is conjugate to d, and hence $\varphi(d)$ is a multitwist. □

So far, we have established that the image of a Dehn twist is a multitwist under the given hypotheses. We now want to show that the image is in fact a Dehn twist. To accomplish this, we will need to use all the hypotheses in the statement of Theorem 4.14, which has not been the case up to this point. Also, we require the characterization of braided multitwists given in Theorem D and the following lemma.

Lemma 4.23. *Let M be a nonplanar orientable surface with no boundary, and let $\varphi : \Gamma_c(M) \to H$ be a homomorphism to a group H. If there exist curves a and b in M such that $i(a, b) = 1$ and $\varphi(d_a) = \varphi(d_b)$, then $\varphi(\Gamma_c(M))$ is cyclic. Moreover, if M has genus at least three, then φ is trivial.*

Proof. If c and e are curves such that $i(c, e) = 1$, then there exists $g \in \Gamma_c(M)$ such that $g(a) = c$ and $g(b) = e$. Therefore, $\varphi(d_c) = \varphi(g\, d_a\, g^{-1}) = \varphi(g\, d_b\, g^{-1}) = \varphi(d_e)$. Now, given any two non-separating curves a' and b', there exist curves $c_0, c_1, \ldots, c_k$ such that $i(c_i, c_{i+1}) = 1$, $c_0 = a'$, and $c_k = b'$ (see [32, Lemma 4.5]). It follows that $\varphi(d_{c_i}) = \varphi(d_{c_{i+1}})$, and hence $\varphi(d_{a'}) = \varphi(d_{b'})$. As $\Gamma_c(M)$ is generated by non-separating Dehn twists, we have that $\varphi(\Gamma_c(M))$ is cyclic. Now, if the genus of M is at least three, then $\Gamma_c(M)$ has trivial abelianization—this is an application of the lantern relation, [32, Section 5.1] for a proof in the finite-type setting that extends to the infinite-type setting verbatim. Now, the image of φ is abelian, as it is cyclic, and hence φ is trivial. □

We say two chains $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic if there exists a bijection $\psi : \mathcal{C} \to \mathcal{C}'$ such that $i(\psi(a), \psi(b)) = i(a, b)$ for all $a, b \in \mathcal{C}$; we call ψ a *chain isomorphism.*

Proposition 4.24. *Let M and M' be orientable surfaces with no boundary such that M has infinite genus and no planar ends, and let $\varphi : \Gamma(M) \to \Gamma(M')$ be a continuous compact irreducible homomorphism.*

(1) *If $\mathcal{A}$ is an Alexander chain in M such that $\{d_a\}_{a\in\mathcal{A}}$ generates $\Gamma_c(M)$, then $\mathcal{A}' = \bigcup_{a\in A}\varphi_*(a)$ is an Alexander chain in M'. Moreover, the map $a \mapsto \varphi_*(a)$ is a chain isomorphism from $\mathcal{A}$ to $\mathcal{A}'$.*

(2) *If $d \in \Gamma(M)$ is a non-separating Dehn twist, then $\varphi(d)$ is a non-separating Dehn twist.*

Proof. Let $\mathcal{A}$ be an Alexander chain in M such that $\{d_a\}_{a\in\mathcal{A}}$ generates $\Gamma_c(M)$, and let $\mathcal{A}' = \bigcup_{a\in A}\varphi_*(a)$. Note that such a chain exists by Theorem 4.12 and, by Lemma 4.22, the images $\varphi(d_a)$ are multitwists.

Let $a, b \in \mathcal{A}$, let $a' \in \varphi_*(a)$, and let $b' \in \varphi_*(b)$. If $i(a,b) = 0$, then d_a and d_b commute and so do their images $\varphi(d_a)$ and $\varphi(d_b)$. Therefore, by Lemma 4.21, $i(a',b') = 0$. We will use the contrapositive of this fact multiple times, so we record it here for reference:

$$\text{if } i(a',b') = 1 \text{ then } i(a,b) = 1. \tag{$\star$}$$

Now, if $i(a,b) \neq 0$, then $i(a,b) = 1$, implying d_a and d_b are braided. It follows that the multitwists $\varphi(d_a)$ and $\varphi(d_b)$ are braided; hence, by Theorem D, $\varphi_*(a) \cup \varphi_*(b)$ is a chain, so $i(a',b') \in \{0,1\}$. In either case, we have that $i(a',b') \in \{0,1\}$.

To show that $\mathcal{A}'$ is a chain, it is left to show that it is locally finite. First observe that, for $a' \in \mathcal{A}'$, there are only finitely many curves in $\mathcal{A}'$ that have nontrivial geometric intersection with a'; this follows from the local finiteness of $\mathcal{A}$ together with ($\star$). Suppose $\mathcal{A}'$ fails to be locally finite, so that there exits a curve c and a sequence $\{a'_n\}_{n\in\mathbb{N}} \subset \mathcal{A}'$ such that $i(a'_n, c) > 0$ for all $n \in \mathbb{N}$. All but finitely many of the a'_n have trivial geometric intersection with a'_1, so passing to a subsequence, we may assume that $i(a'_1, a'_n) = 0$ for all $n \in \mathbb{N}$. Recursively passing to subsequences, we can assume that $i(a'_n, a'_m) = 0$ for all $n, m \in \mathbb{N}$. Let $a_n \in \mathcal{A}$ such that $a'_n \in \varphi_*(a)$. Again recursively passing to subsequences as above, we can assume that $i(a_n, a_m) = 0$ for all $n, m \in \mathbb{N}$. Therefore, as $\{a_n\}_{n\in\mathbb{N}}$ is a locally finite multicurve, Lemma 4.20 implies that $\bigcup_{n\in\mathbb{N}}\varphi_*(a_n)$ is a locally finite multicurve. It follows that $\{a'_n\}_{n\in\mathbb{N}}$ is locally finite as it is contained in $\{\varphi_*(a_n)\}$, implying that only finitely many of the a'_n have nonzero geometric intersection with c, a contradiction. We can conclude that $\mathcal{A}'$ is a chain.

We now need to show that $\mathcal{A}'$ is filling and tree-like; first, we show it is filling. If there exists a curve or arc α that is disjoint from from every element of $\mathcal{A}'$, then α must be fixed by every element of $\varphi(\Gamma_c(M))$, as the image of $\Gamma_c(M)$ is generated by $\{\varphi(d_a)\}_{a\in\mathcal{A}}$. By continuity, α must then be fixed by every element in the image of φ, but φ is irreducible, so no such curve or arc exists. Therefore, $\mathcal{A}'$ is filling.

Recall that $T(\mathcal{A}')$ is the graph whose vertices are the curves of $\mathcal{A}'$ and that two vertices are adjacent if they have geometric intersection one; to show that $\mathcal{A}'$ is tree-like, we must show that $T(\mathcal{A}')$ is a tree, i.e., that it

is connected and has no cycles. Suppose $a'_1, a'_2, \ldots, a'_k$ are distinct curves $\mathcal{A}'$ forming a cycle in $T(\mathcal{A}')$. Let $a_1, \ldots, a_k \in \mathcal{A}$ such that $a'_i \in \varphi_*(a_i)$. Reading indices modulo k, we have by $(\star)$ that $i(a_i, a_{i+1}) = 1$, and using Theorem D, we can conclude that a_i, a_{i+1}, and a_{i+2} are distinct. Therefore, if $T(\mathcal{A}')$ contains a cycle, so does $T(\mathcal{A})$. It follows that $T(\mathcal{A}')$ cannot have any cycles, as $T(\mathcal{A})$ is a tree. Finally, $T(\mathcal{A}')$ is connected as $\mathcal{A}'$ is filling. This establishes that $T(\mathcal{A}')$ is a tree and that $\mathcal{A}'$ is an Alexander chain.

Next, we claim that if $a, b \in \mathcal{A}$ such that $i(a, b) = 1$, then $\varphi_*(a)$ and $\varphi_*(b)$ are non-separating curves that have geometric intersection one. First, we show that $\varphi_*(a)$ is a curve for each $a \in \mathcal{A}$, or equivalently, that $|\varphi_*(a)| = 1$. Suppose $|\varphi_*(a)| > 1$, and let $a', a'' \in \varphi_*(a)$. As $T(\mathcal{A}')$ is connected, we can choose a path $a' = c'_0, \ldots, c'_m = a''$ in $T(\mathcal{A}')$ connecting a' and a''. Let $c_i \in \mathcal{A}$ such that $c'_i \in \varphi_*(c_i)$. Arguing as we did above, $c_0, \ldots, c_m$ is a path in $T(\mathcal{A})$ that must contain a cycle, contradicting the fact that $T(\mathcal{A})$ is a tree. Therefore, $|\varphi_*(a)| = 1$, and $\varphi_*(a)$ is a curve.

Now, let $a, b \in \mathcal{A}$ such that $i(a, b) = 1$. Suppose that $i(\varphi_*(a), \varphi_*(b)) = 0$. Then $\varphi(d_a)$ and $\varphi(d_b)$ commute and are braided. It is a quick exercise to show that this implies that $\varphi_*(d_a) = \varphi_*(d_b)$. By Lemma 4.23, $\varphi(\Gamma_c(M)) < \ker \varphi$, and hence by continuity, φ is trivial. But this is not the case, as φ is irreducible. Therefore, $i(\varphi_*(a), \varphi_*(b)) = 1$. At this point, we have shown that $\varphi_* : \mathcal{A} \to \mathcal{A}'$ is a bijection and preserves intersection number; hence, it is a chain isomorphism.

Now, for any $a \in \mathcal{A}$, there exists $b \in \mathcal{A}$ such that $i(a, b) = 1$. Hence, from above, we may conclude that $\varphi_*(a)$ and $\varphi_*(b)$ are curves and $i(\varphi_*(a), \varphi_*(b)) = 1$, implying that $\varphi_*(a)$ and $\varphi_*(b)$ are non-separating curves. It follows then that both $\varphi(d_a)$ and $\varphi(d_b)$ are powers of non-separating Dehn twists. Moreover, as $\varphi(d_a)$ and $\varphi(d_b)$ are braided, Theorem D implies that they are in fact each non-separating Dehn twists. As any two non-separating Dehn twists are conjugate, $\varphi(d)$ is a non-separating Dehn twist for every non-separating Dehn twist $d \in \Gamma(M)$. This completes the proof. □

The idea for the final step is to realize that if M and M' are infinite-genus surfaces admitting isomorphic Alexander chains, then M and M' are homeomorphic and the isomorphism between the chains is induced by a homeomorphism between the surfaces. To do this, we will realize the homeomorphism as a direct limit of homeomorphisms between compacts sets in an exhaustion, which we will obtain by appealing to the work of the Aramayona-Souto [11].

Given a chain $\mathcal{C}$ in a surface M, fix a hyperbolic metric on M, and for each $c \in \mathcal{C}$, let γ_c be its geodesic representative. We call $\bigcup_{c \in \mathcal{C}} \gamma_c$ a *realization of the chain*.

Proof of Theorem 4.14. By Theorem 4.12, we can choose a Alexander chain $\mathcal{A}$ such that $\{d_a\}_{a \in \mathcal{A}}$ generates $\Gamma_c(M)$. Theorem 4.12 also provides a se-

quence of compact subsurfaces $\{\Sigma_n\}_{n\in\mathbb{N}}$ of M such that

(i) each component of $\partial\Sigma_n$ is separating and essential,

(ii) $\Sigma_n \subset \Sigma_{n+1}$,

(iii) $M = \bigcup_{n\in\mathbb{N}} \Sigma_n$, and

(iv) $\mathcal{A}_n = \{a \in \mathcal{A} : a \subset \Sigma_n\}$ is an Alexander chain in Σ_n, and

(v) $\{d_a : a \in \mathcal{A}_n\}$ generates $\mathrm{Map}(\Sigma_n, \partial\Sigma_n)$.

By possibly forgetting the first few surfaces in the sequence, we may assume that the genus of Σ_n is at least six for all $n \in \mathbb{N}$.

Let $\mathcal{A}' = \varphi_*(\mathcal{A})$, and for $n \in \mathbb{N}$, let $\mathcal{A}'_n = \varphi_*(\mathcal{A}_n)$. By Proposition 4.24, $\mathcal{A}'$ is an Alexander chain, and φ_* induces an isomorphism between $\mathcal{A}$ and $\mathcal{A}'$, as well as $\mathcal{A}_n$ and $\mathcal{A}'_n$. Let Σ'_n be the subsurface of M' filled by the chain $\mathcal{A}'_n$, that is, Σ'_n is obtained by taking a regular neighborhood of a realization of $\mathcal{A}'_n$ and then taking the union of this neighborhood with each component of its complement that is a disk. Note that Σ_n is the surface filled by $\mathcal{A}_n$.

Given a pair of simple closed curves that have geometric intersection one, a regular neighborhood of their union is a once-punctured torus. This motivates the following definition: the *lower genus* of a finite chain is the cardinality of a maximal collection of pairs of curves in the chain in which the curves in each pair intersect once and the curves in distinct pairs are pairwise disjoint. It follows that the genus of a regular neighborhood of a realization of a finite chain is bounded below by the lower genus of the chain.

By the construction of Σ_n, the lower genus of $\mathcal{A}_n$ agrees with the genus of Σ_n. Moreover, there are pairs $(a_1, b_1), \ldots, (a_g, b_g) \in \mathcal{A}_n^2$ realizing the lower genus of $\mathcal{A}_n$ such that any two distinct curves in $\mathcal{A}_n \smallsetminus \{a_1, b_1, \ldots, a_g, b_g\}$ are disjoint (e.g., take the a_i to be the red curves from Figure 4.2). As the lower genus of $\mathcal{A}'_n$ is equal to that of $\mathcal{A}_n$, the genus of Σ'_n is at least the genus of Σ_n. Now, suppose that the genus of Σ'_n is greater than the genus of Σ_n. Let $a'_i = \varphi_*(a_i)$ and $b'_i = \varphi_*(b_i)$, so $(a'_1, b'_1), \ldots, (a'_g, b'_g)$ is a maximal collection of pairs of curves in $\mathcal{A}'_n$ realizing its lower genus. Let F'_i be a closed regular neighborhood of a realization of $\{a'_i, b'_i\}$, so that F'_i is a one-hole torus. We can assume that $F'_i \cap F'_j = \varnothing$ if $i \neq j$. Then $\Sigma'_n \smallsetminus \bigcup F'_i$ has positive genus, implying that there exists a pair of curves in $\mathcal{A}'_n \smallsetminus \{a'_1, b'_1, \ldots, a'_g, b'_g\}$ having nontrivial geometric intersection, a contradiction. Therefore, Σ_n and Σ'_n have the same genus.

Given the setup at hand, for each $n \in \mathbb{N}$, the restriction of φ to $\mathrm{Map}(\Sigma_n)$ induces a homomorphism $\varphi_n : \mathrm{Map}(\Sigma_n) \to \mathrm{Map}(\Sigma'_n)$. We claim that φ_n is an isomorphism. As Σ_n and Σ'_n have the same genus (and it is at least six), we can apply a result of Aramayona-Souto [11, Theorem 1.1] to see that φ_n is induced by an embedding $h_n : \Sigma_n \to \Sigma'_n$.

As φ_m restricts to φ_n on $\mathrm{Map}(\Sigma_n)$, we can choose the embeddings such that h_m restricts to h_n on Σ_n. We now claim that h_n can be chosen to be a homeomorphism. As Σ'_n is filled by curves contained in $h(\Sigma_n)$, each component of $\Sigma'_n \smallsetminus h(\Sigma_n)$ is either a disk or annulus. We claim that each such component is a boundary annulus, i.e., is homotopic to a component of $\partial\Sigma'_n$. This implies that h_n be chosen to be a homeomorphism. Let $m \in \mathbb{N}$ such that $m > n$ and each component of $\Sigma_m \smallsetminus \Sigma_n$ has positive genus. It follows that each component of the complement of $\Sigma'_m \smallsetminus h_m(\partial\Sigma_n)$ has positive genus; in particular, if δ_1 and δ_2 are any two components of $\partial\Sigma_n$, then $h_m(\delta_1)$ does not bound a disk and $h_m(\delta_1)$ and $h_m(\delta_2)$ do not co-bound an annulus. The claim now follows, that is, each component of $\Sigma'_n \smallsetminus h(\Sigma_n)$ is a boundary annulus, and h_n can be chosen to be a homeomorphism. Therefore, $\varphi_n : \mathrm{Map}(\Sigma_n) \to \mathrm{Map}(\Sigma'_n)$ is an isomorphism.

Taking the direct limit of the h_n, we obtain an embedding $h : M \to M'$. As $\mathcal{A}'$ is an Alexander chain, we have that $M' = \bigcup_{n\in\mathbb{N}} \Sigma'_n$, and hence h is a homeomorphism. Let $h_* : \Gamma(M) \to \Gamma(M')$ be the isomorphism induced by h. By continuity, as h_* and φ agree on $\Gamma_c(M)$, they must agree everywhere, so $\varphi = h_*$. □

4.4 From epimorphisms to isomorphisms

The goal of this section is to prove Theorem F by showing that it follows from Theorem 4.14. Let us recall the statement.

Theorem F. *Let M and M' be two orientable surfaces with no boundary, and suppose M has infinite genus and has no planar ends. If G is a mostly pure large subgroup of $\mathrm{Map}(M)$ and G' is a large subgroup of $\mathrm{Map}(M')$, then every continuous epimorphism $G \to G'$ is induced by a homeomorphism $M \to M'$.*

Given a continuous epimorphism $\varphi : G \to G'$ with G and G' as in the theorem, we will see that the surjectivity will allow us to conclude that φ restricted to $\Gamma(M)$ induces a continuous compact irreducible homomorphism $\Gamma(M) \to \Gamma'(M)$, which will allow us to apply Theorem 4.14. We will take the same tact to prove Theorem 4.5 below. As the proof of these two theorems share the same core idea, we proceed with a sequence of lemmas/propositions that will be required for both theorems. The first required proposition is a result from Bavard–Dowdall–Rafi [15].

Proposition 4.25 ([15, Proposition 4.2])**.** *Let M be an orientable infinite-type surface with no boundary, and let G be a large subgroup of $\mathrm{Map}(M)$. A mapping class in G has finite-type support if and only if its conjugacy class in G is countable.* □

The statement in [15] is about finite-index subgroups of $\mathrm{Map}(M)$ and $\mathrm{PMap}(M)$, but the proof works just as well for large subgroups.

Proposition 4.26. *Let M be an infinite-genus orientable surface with no boundary and no planar ends, let M' be an orientable surface with no boundary, and let G and G' be large subgroups of* $\mathrm{Map}(M)$ *and* $\mathrm{Map}(M')$*, respectively. If $\varphi : G \to G'$ is a continuous epimorphism such that $\Gamma_c(M)$ is not contained in the kernel of φ, then φ is induced by a homeomorphism $M \to M'$.*

Proof. Let $\varphi : G \to G'$ be a continuous epimorphism. An epimorphism maps conjugacy classes in the domain onto conjugacy classes in the codomain. In particular, if $g \in G$ has a countable conjugacy class, then so does $\varphi(g)$. Therefore, by Proposition 4.25, $\varphi(\Gamma_c(M)) < \Gamma_c(M')$, and by continuity, $\varphi(\Gamma(M)) < \Gamma(M')$. We can then restrict φ to $\Gamma(M)$ to get a continuous compact homomorphism $\psi : \Gamma(M) \to \Gamma(M')$. We claim that ψ is irreducible.

Let a and b be simple closed curves in M satisfying $i(a, b) = 1$. By Lemma 4.22, $\psi(a)$ and $\psi(b)$ are multitwists; let A (resp., B) be the multicurve such that $\psi(a)$ (resp., $\psi(b)$) twists about. By Theorem D, either there exist $a' \in A$ and $b' \in B$ such that $i(a', b') = 1$ or $\psi(a) = \psi(b)$. If the latter case holds, then Lemma 4.23 implies that $\Gamma_c(M) < \ker \psi$, which is not the case; hence, the former case holds, implying that A contains a non-separating simple closed curve.

As $\Gamma(M)$ is normal in G and as φ is surjective, $\varphi(\Gamma(M))$ is normal in G'; in particular, $\psi(\Gamma(M))$ is normal in $\Gamma(M')$. Let α be a curve or arc in M'. Let A be as above. As A contains a non-separating curve, there exists $h \in G'$ such that $h(A)$ has nontrivial geometric intersection with α. It follows that $(h\psi(d_a)h^{-1})(\alpha) \neq \alpha$, and as $h\psi(d_a)h^{-1} \in \psi(\Gamma(M))$, we can conclude that ψ is irreducible.

We have established that $\psi : \Gamma(M) \to \Gamma(M')$ is a continuous compact irreducible homomorphism; hence, by Theorem 4.14, there exists a homeomorphism $h : M \to M'$ such that $h_* = \psi$, where h_* is the map from $\Gamma(M)$ induced by h. It is left to check that that φ is also induced by h, or that, abusing notation, $h_* = \varphi$. Let φ_0 be given by $\varphi_0 = h_*^{-1} \circ \varphi$. Observe that for every simple closed curve c in M, $\varphi(d_c) = \psi(d_c)$ by definition, and so $\varphi_0(d_c) = d_c$. Let $g \in G$, and let c be a simple closed curve in M. Then

$$\begin{aligned} d_{\varphi_0(g)(c)} &= \varphi_0(g) d_c \varphi_0(g)^{-1} \\ &= \varphi_0(g)\varphi_0(d_c)\varphi_0(g)^{-1} \\ &= \varphi_0(g d_c g^{-1}) \\ &= \varphi_0(d_{g(c)}) \\ &= d_{g(c)}. \end{aligned}$$

Therefore, $\varphi_0(g)(c) = g(c)$ for every simple closed curve c in M. By Proposition 4.8, $\varphi_0(g) = g$. It follows that $\varphi_0 = h_*^{-1} \circ \varphi$ is the identity isomorphism, implying $\varphi = h_*$. Therefore, φ is induced by a homeomorphism. □

As we will see below, Proposition 4.26 will imply that either φ fails to be surjective or G' contains a nontrivial normal abelian subgroup; the next lemma says the latter case is impossible.

Lemma 4.27. *Let M be an infinite-type orientable surface with no boundary, and let G be a large subgroup of* $\mathrm{Map}(M)$*. The trivial subgroup is the only normal abelian subgroup of G.*

Proof. Let H be a nontrivial normal subgroup of G. If $h \in H$ is nontrivial, then there exists a curve a such that $h(a) \neq a$. Let $g_1 = [d_a, h]$, so

$$g_1 = d_a\, h\, d_a^{-1}\, h^{-1} = (d_a\, h\, d_a^{-1})\, h^{-1} = d_a\, (h\, d_a^{-1}\, h^{-1}) = d_a\, d_{h(a)}^{-1}.$$

The second equality implies that $g_1 \in H$, and the last equality guarantees that g_1 is nontrivial, as $h(a) \neq a$. Let S_1 be the surface filled by a realization of $\{a, h(a)\}$; note that S_1 has finite type and contains the support of a representative of g_1. Thurston's construction (see [32, Theorem 14.1]) implies that g_1 is the image of a pseudo-Anosov element under the homomorphism $\mathrm{Map}(S_1) \to G$ induced by the embedding $S_1 \hookrightarrow M$ (if a and $h(a)$ are disjoint, then S_1 is a union of disjoint annuli implying that g_1 is a multitwist, which we will also consider pseudo-Anosov in this disjoint union of annuli). It follows that if a curve c has nontrivial geometric intersection with ∂S_1, then $g_1(c) \neq c$; in particular, $g_1(c) = c$ if and only if $i(c, \partial S) = 0$.

Let e be the isotopy class of a component of ∂S_1, and choose a curve c such that $i(c, e) > 0$. Set $g_2 := d_c\, g_1\, d_c^{-1} \in H$, and let S_2 be the image of S_1 under a representative of d_c. Then g_2 is the image of a pseudo-Anosov element of $\mathrm{Map}(S_2)$ under the homomorphism into $\mathrm{Map}(M)$ and $i(e, \partial S_2) \neq 0$. It follows that $g_2(e) \neq e$ and $g_1(e) = e$. Therefore, g_1 and g_2 do not commute; indeed, if they commuted, then g_2 would preserve the fix set of g_1, which is not the case. We can now conclude that H is not abelian, and the result follows, as H was an arbitrary nontrivial normal subgroup of G. □

We can now proof Theorem F.

Proof of Theorem F. Let $\varphi : G \to G'$ be a continuous epimorphism. If $\Gamma_c(M)$ is contained in the kernel of φ, then by continuity, $\Gamma(M)$ is in the kernel of φ. In this case, φ factors through $G/\Gamma(M)$. By [10], $\mathrm{PMap}(M)/\Gamma(M)$ is abelian, so the image of $(G \cap \mathrm{PMap}(M))/\Gamma(M)$ is an abelian normal subgroup of G' and hence is trivial by Lemma 4.27. But, as G is mostly pure, it follows that the image of φ is countable, and hence cannot surject onto G', a contradiction. Therefore, $\Gamma_c(M)$ is not contained in the kernel of φ. By Proposition 4.26, φ is induced by a homeomorphism. □

4.4.1 Moving to the full mapping class group

If M has infinitely many ends, then $\mathrm{Map}(M)$ is not mostly pure, and hence, Theorem F does not apply. The place where the proof is incomplete in this case is in establishing that $\Gamma_c(M)$ is not in the kernel of the given epimorphism. Here we provide a strategy for dealing with this that works in some cases, but it is not clear if it can be adapted to work in all cases. To exhibit the technique, we extend Theorem F to $\mathrm{Map}(M)$ when M is perfectly self-similar.

Theorem 4.5. *Let M and M' be orientable surfaces with no boundary, and suppose M is infinite genus with no planar ends. If M is perfectly self-similar and G' is a large subgroup of $\mathrm{Map}(M')$, then every continuous epimorphism from $\mathrm{Map}(M)$ to G' is induced by a homeomorphism $M \to M'$.*

The main tool in proving Theorem 4.5 is a result of Afton-Calegari-Chen-Lyman.

Proposition 4.28 ([1, Proposition 9])**.** *If M is an orientable surface, then there exists an open neighborhood of the identity in $\mathrm{Map}(M)$ such that the identity is the only torsion element in the neighborhood.* □

The idea behind the proof of Theorem 4.5 is that if $\Gamma_c(M)$ is in the kernel of an epimorphism, then there must be torsion elements arbitrarily close to the identity that normally generate G', contradicting Proposition 4.28. The existence of these generators come from the following lemma.

Lemma 4.29. *If E is the end space of a perfectly self-similar 2-manifold, then every neighborhood of the identity in $\mathrm{Homeo}(E)$ contains an involution that normally generates $\mathrm{Homeo}(E)$.*

Proof. By [83, Proposition 5.4.6], there exists $x \in E$ and a sequence $\{E_n\}_{n\in\mathbb{N}}$ of pairwise-homeomorphic and disjoint clopen subsets of E such that $E \smallsetminus \{x\} = \bigcup_{n\in\mathbb{N}} E_n$. From this decomposition, for each $n \in \mathbb{N}$, we can find a involution $\tau_n : E \to E$ such that $\tau_n(E_n) = E_{n+1}$ and τ_n restricts to the identity on E_k for $k \neq n, n+1$. Observe that $\lim_{n\to\infty} \tau_n$ is the identity. Moreover, [83, Corollary 5.2.3] implies that τ_n is a normal generator for $\mathrm{Homeo}(E)$. □

Proof of Theorem 4.5. Let $\varphi : \mathrm{Map}(M) \to G'$ be a continuous epimorphism, and suppose that $\Gamma_c(M)$ is contained in $\ker\varphi$. Arguing as in the proof of Theorem F, it follows that φ induces an epimorphism $\mathrm{Map}(M)/\,\mathrm{PMap}(M) \to G'$. Richards's proof of the classification of surfaces [76], together with the fact that all the ends are nonplanar, implies that $\mathrm{Map}(M)/\,\mathrm{PMap}(M)$ is isomorphic to $\mathrm{Homeo}(\mathcal{E}(M))$, where $\mathcal{E}(M)$ is the end space of M. Let $\bar{\varphi}$ denote the induced epimorphism $\mathrm{Homeo}(\mathcal{E}(M)) \to G'$. By Lemma 4.29, there exists a sequence of involutions $\{\tau_n\}_{n\in\mathbb{N}}$ in $\mathrm{Homeo}(\mathcal{E}(M))$ that limit to the

identity and such that each involution normally generates Homeo($\mathcal{E}(M)$). But Proposition 4.28 implies that $\bar{\varphi}(\tau_n)$ must be the identity for all large n. As the τ_n are normal generators, $\bar{\varphi}$ is trivial, a contradiction. Therefore, $\Gamma_c(M)$ is not in the kernel of φ, and hence φ is induced by a homeomorphism by Proposition 4.26. □

The argument in the proof of Theorem 4.5 works for a larger class of surfaces; however, it is not clear how to naturally define the class. But, for example, the proof also implies that if M is an orientable infinite-genus surface with no planar ends and with $\mathcal{E}(M)$ homeomorphic to $\omega + 1$, then Theorem 4.5 holds for M. In this case, Homeo($\mathcal{E}(M)$) is topologically isomorphic to Sym($\mathbb{N}$), and so contains a dense subgroup generated by involutions arbitrary close to the identity (namely, the subgroup consisting of finitely supported permutations).

4.5 Relaxing continuity

In this final section, we weaken the continuity assumption in Theorem F. The idea is to replace continuity with a version of "algebraic continuity" that preserves certain limits that can be described in terms of infinite multiplication. We will work with a version of *infinitely multiplicative* homomorphisms—as introduced by Cannon-Conner [26]—suited to our purposes.

Let us begin by expanding our discussion of infinite products from Section 4.1. A sequence of mapping classes $\{f_n\}_{n\in\mathbb{N}}$ in Map(M) is *locally finitely supported* if $\{n \in \mathbb{N} : f_n(a) \neq a\}$ is finite for every curve a in M. If $\{f_n\}_{n\in\mathbb{N}}$ is locally finitely supported, then we can define $f = \prod_{n\in\mathbb{N}} f_n \in \mathrm{Map}(M)$ as follows: given a curve a in M there exists $N_a \in \mathbb{N}$ such that $f_n(a) = a$ for all $n > N_a$, allowing us to define $f(a) = (f_1\, f_2 \cdots f_{N_a})(a)$. Note that, in the compact-open topology, $f = \lim \prod_{i=1}^{n} f_i$, where we write the product left to right, i.e., $\prod_{i=1}^{n} f_i = f_1\, f_2 \cdots f_n$.

Definition 4.30. *Let M and M' be surfaces with M nonplanar, and let G and G' be subgroups of* Map(M) *and* Map(M')*, respectively. A homomorphism $\varphi : G \to G'$ is* infinitely multiplicative on twists *if, for any locally finitely supported sequence of non-separating Dehn twists $\{d_n\}_{n\in\mathbb{N}}$, the infinite product $\prod_{n\in\mathbb{N}} \varphi(d_n)$ exists.*

We only need one lemma to replace the continuity assumption in Theorem F with requiring the epimorphisms to be infinitely multiplicative on twists.

Lemma 4.31. *Let M be an orientable infinite-genus surface in which every planar end is isolated. If $f \in \Gamma(M) \smallsetminus \Gamma_c(M)$, then there exists a sequence $\{d_n\}_{n\in\mathbb{N}}$ of non-separating Dehn twists such that $\prod_{n\in\mathbb{N}} d_n$ exists and is equal to f.*

Proof. Let $\{\Sigma_n\}_{n\in\mathbb{N}}$ be an exhaustion of M by finite-type subsurfaces. Let $U_n \subset \mathrm{Map}(M)$ be defined as follows: $f \in U_n$ if and only if $f(a) = a$ for every curve a with representative contained in Σ_n. Then $\{U_n\}_{n\in\mathbb{N}}$ is a neighborhood basis for the identity in $\mathrm{Map}(M)$.

As $\Gamma_c(M)$ is generated by non-separating Dehn twists and is dense in $\Gamma(M)$, there exists $f_1 \in \Gamma_c(M)$ such that $f_1 \in f\,U_1$ and such that f_1 is a finite product of non-separating Dehn twists. Therefore, $f_1^{-1} f \in U_1$. Arguing as above, there exists $f_2 \in f_1^{-1} f\,U_2$ such that f_2 is a finite product of non-separating Dehn twists. Moreover, $f_2 \in U_1$. Proceeding recursively, we obtain a sequence of mapping class $\{f_n\}_{n\in\mathbb{N}}$ such that

- $f_{n+1} \in U_n$,
- $f_n^{-1} \cdots f_1^{-1} f \in U_n$, and
- f_n is a finite product of non-separating Dehn twists.

It follows that $\{f_n\}$ is locally finitely supported, $\lim f_n = f$, and $f = \prod_{n\in\mathbb{N}} d_n$ for some locally finitely supported sequence of non-separating Dehn twists $\{d_n\}_{n\in\mathbb{N}}$. □

Tracing through the various results in the prior sections, one readily checks that every instance of continuity can be replaced with infinite multiplicativity on twists using Lemma 4.31, which yields a more algebraic version of Theorem F.

Chapter 5

Finite rigidity of the non-separating curve complex

Let S be a orientable, finite-type surface with no boundary. The *curve complex* is the simplicial complex $\mathcal{C}(S)$ whose k-simplices are sets of $k+1$ distinct isotopy classes of essential simple curves on S that are pairwise disjoint.

The extended mapping class group, denoted $\mathrm{Map}^{\pm}(S)$, acts naturally on the set of curves up to isotopy on S. This action preserves disjointness of curves, and therefore extends to an action on the complex $\mathcal{C}(S)$. Via this action, the curve complex is regarded as a combinatorial model for $\mathrm{Map}^{\pm}(S)$. In fact, Ivanov showed in [51] that $\mathcal{C}(S)$ is *simplicially rigid*, meaning that the action induces an isomorphism $\mathrm{Map}^{\pm}(S) \cong \mathrm{Aut}(\mathcal{C}(S))$. In turn, this result was a key ingredient in establishing that $\mathrm{Aut}(\mathrm{Map}^{\pm}(S)) \cong \mathrm{Map}^{\pm}(S)$.

In this chapter we study a stronger notion known as *finite rigidity*: the simplicial complex $\mathcal{C}(S)$ is said to be finitely rigid if there exists a finite subcomplex X such that any locally injective simplicial map

$$\phi : X \to \mathcal{C}(S)$$

is induced by a *unique* mapping class, that is, there exists a unique $h \in \mathrm{Map}^{\pm}(S)$ such that the simplicial action $h : \mathcal{C}(S) \to \mathcal{C}(S)$ satisfies $h|_X = \phi$. Such X is called a *finite rigid set* of $\mathcal{C}(S)$ with *trivial pointwise stabilizer*.

The finite rigidity of the curve complex was proven by Aramayona and Leininger in [8]. Even more, they constructed in [7] an exhaustion of $\mathcal{C}(S)$ by finite rigid sets, which in particular implies simplicial rigidity. After their result, finite rigidity has been studied for other complexes: Shinkle proved it for the arc complex [78] and the flip graph [79]; Hernández, Leininger and Maungchang proved a slightly different notion for the pants graph [42], [67];

and Huang and Tshishiku proved a weaker notion for the separating curve complex [43].

The main goal of this chapter is to prove the finite rigidity of the *non-separating curve complex* $\mathcal{N}(S)$, which is the subcomplex of $\mathcal{C}(S)$ spanned by the non-separating curves. To prove the finite rigidity of $\mathcal{N}(S)$, one would like to restrict the finite rigid set of $\mathcal{C}(S)$ in [8] to $\mathcal{N}(S)$, however it is not clear why this restriction would yield a finite rigid set in $\mathcal{N}(S)$ as their proof uses separating curves in a fundamental way. Below, we construct a different subcomplex of $\mathcal{N}(S)$ and prove its rigidity.

The main result of this chapter is compiled in the next theorem.

Theorem 5.1. *Let S be a orientable, finite-type surface of genus $g \geq 3$ with no boundary. There exists a finite simplicial complex $X \subset \mathcal{N}(S)$ such that any locally injective simplicial map*

$$\phi : X \to \mathcal{N}(S)$$

is induced by a unique $h \in \mathrm{Map}^{\pm}(S)$.

Then, we produce an exhaustion by finite-rigid sets with trivial pointwise stabilizers.

Theorem 5.2. *Let S be an orientable finite type surface of genus $g \geq 3$. There exist subcomplexes $X_1 \subset X_2 \subset \cdots \subset \mathcal{N}(S)$ such that*

$$\bigcup_{i=1}^{\infty} X_i = \mathcal{N}(S)$$

and each X_i is a finite rigid set with trivial pointwise stabilizer.

Note that Theorem G in the introduction is precisely the statements of Theorem 5.1 and Theorem 5.2 combined. Also, we remark that from Theorem 5.2 we can recover the simplicial rigidity of $\mathcal{N}(S)$ (see [48, Theorem 1.1]):

Corollary 5.3. *Let S be a connected, orientable, finite-type surface of genus $g \geq 3$. Any locally injective simplicial map $\phi : \mathcal{N}(S) \to \mathcal{N}(S)$ is induced by a unique $h \in \mathrm{Map}^{\pm}(S)$. In particular, this yields an isomorphism between $\mathrm{Map}^{\pm}(S)$ and the group of simplicial automorphisms of $\mathcal{N}(S)$.*

5.1 Preliminaries

Throughout the chapter S is a connected, orientable finite-type surface without boundary. Recall that we write $S_{g,n}$ for a genus g surface with n punctures.

Before fervently jumping into the proofs, we warn the reader that the classification of surfaces, the change of coordinates principle and the Alexander method will be frequently used in proofs, sometimes without mention. For these and other fundamental results on mapping class groups, we refer the reader to [32].

5.1.1 Curves

Recall that a curve c is non separating if any representative γ of c has connected complement in S. Given two representatives $\alpha \in a$ and $\beta \in b$, we say they are in *minimal position* if $i(a,b) = |\alpha \cap \beta|$. Often we will use the fact that for any set of curves we may pick a single representative for each curve, such that the representatives are pairwise in minimal position (see [32, Chapter 1.2]).

Given a set of curves $\{c_1, \dots, c_k\}$, consider representatives $\gamma_i \in c_i$ pairwise in minimal position. We will denote a regular neighborhood of $\bigcup \gamma_i$ by $N(\bigcup \gamma_i)$. The set of curves in the boundary of $N(\bigcup \gamma_i)$ will be denoted

$$\partial(c_1, \dots, c_k).$$

We emphasize that it is implicit in the definition that $b \in \partial(c_1, \dots, c_k)$ an isotopy class of a simple closed curve which does not bound a disk or a punctured disk.

5.1.2 Non-separating curve complex

The *non-separating curve complex* $\mathcal{N}(S)$ is the simplicial complex whose k-simplices are sets of $k+1$ isotopy classes of pairwise disjoint non-separating curves.

Note that we can endow $\mathcal{N}(S)$ with a metric by declaring each k-simplex to have the standard euclidean metric and considering the resulting path metric on $\mathcal{N}(S)$.

Pants decompositions

The dimension of $\mathcal{N}(S_{g,n})$ is $3g-3+n$ and the vertex set of a top dimensional simplex in $\mathcal{N}(S_{g,n})$ is called a *non-separating pants decomposition* of $S_{g,n}$. If $P = \{c_1, \dots, c_k\}$ is a pants decomposition of S, then $S \setminus \bigcup c_i$ is a union of *pairs of pants* (i.e, a union of subsurfaces homeomorphic to $S_{0,3}$).

Let $P = \{c_1, \dots, c_k\}$ be a pants decomposition of the surface S. Two curves $c_i, c_j \in P$ are said to be *adjacent rel to* P if there exist a curve $c_k \in P$ such that c_i, c_j, c_k bound a pair of pants in S; see Figure 5.1a for an example.

We record the following observation for future use.

Remark 5.4. *Let P be a non-separating pants decomposition of $S_{g,n}$, where $g \geq 3$:*

- *If $n \leq 1$, then every curve in P is adjacent rel to P to at least three other curves.*
- *If $n > 1$, then every curve is adjacent to at least two other curves.*

Consider $A \subset P$, where P is a pants decomposition of S. We say that a set of curves $\tilde{A}$ *substitutes* A *in* P if

$$(\tilde{A} \cup P) \setminus A$$

is a pants decomposition. In words, we say $\tilde{A}$ substitutes A in P if both sets have no curves in common and we can replace the curves in A by the curves in $\tilde{A}$ and still get a pants decomposition.

5.2 Finite rigid sets

For a simplicial subcomplex $X \subset \mathcal{N}(S)$, a map $\phi : X \to \mathcal{N}(S)$ is said to be a *locally injective simplicial map* if ϕ is simplicial and injective on the star of each vertex. A first elementary observation is the following:

Lemma 5.5. *Let $\phi : X \to \mathcal{N}(S)$ be a locally injective simplicial map. If $P \subset X$ is a pants decomposition, then $\phi(P)$ is a pants decomposition.*

Proof. Take a vertex $p \in P$. Since ϕ is injective in the star of p and P is a simplex, then ϕ is injective on P. Thus, $\phi(P)$ is a maximal dimensional simplex, i.e, $\phi(P)$ is a pants decomposition. □

As mentioned in the introduction, the main goal of this article is to construct a finite subcomplex $X \subset \mathcal{N}(S)$ with the following properties:

Definition 5.6 (Finite rigid set)**.** *A* finite rigid set *X of $\mathcal{N}(S)$ is a finite subcomplex such that any locally injective simplicial map $\phi : X \to \mathcal{N}(S)$ is induced by a mapping class, i.e, there exists $h \in \mathrm{Map}^{\pm}(S)$ with $h|_X = \phi$.*

In addition, if h is unique, we say X has trivial pointwise stabilizer*.*

Observe that a subcomplex $X \subset \mathcal{N}(S)$ has trivial pointwise stabilizer if and only if the inclusion $X \hookrightarrow \mathcal{N}(S)$ is induced uniquely by the identity $1 \in \mathrm{Map}^{\pm}(S)$, hence the name.

Remark 5.7. *By the change of coordinates principle (see [32, Chapter 1.3]), every vertex $\{v\} \subset \mathcal{N}(S)$ is a finite rigid set. However, the stabilizer of $\{v\}$ is not trivial.*

Remark 5.8. *If X is a finite rigid set and $X \subset Y$, then Y may not be a finite rigid set. For example:*

Consider two disjoint curves v_1, v_2 such that $S \setminus \bigcup v_i$ is connected and let be $X = \{v_1\}$ and $Y = \{v_1, v_2\}$. Now, X is a finite-rigid set but Y is not. To

see this consider two disjoint curves v_1', v_2' such that $S \setminus \bigcup v_i'$ is disconnected and the map $\phi(v_i) = v_i'$. Clearly, ϕ is locally injective simplicial but is not induced by a mapping class, thus Y is not a finite rigid set of $\mathcal{N}(S)$.

Following Aramayona and Leininger in [8], we will say that a subcomplex $X \subset \mathcal{N}(S)$ *detects the intersection* of two curves $a, b \in X$, if every locally injective simplicial map $\phi : X \to \mathcal{N}(S)$ satisfies

$$i(a,b) \neq 0 \Leftrightarrow i(\phi(a), \phi(b)) \neq 0.$$

5.3 Finite rigid sets for closed surfaces

In this section we construct finite rigid sets for closed surfaces and prove their rigidity. This will establish Theorem 5.1 for closed surfaces.

5.3.1 Constructing the finite rigid set

Let S be a closed surface of genus $g \geq 3$. We will start by defining the curves in the finite rigid set. The reader should keep figures 5.1a, 5.1b, 5.1c, 5.1d, 5.1f and 5.1e in mind throughout the section.

Fix a set $\{p_1, c_1, \ldots, p_g, c_g, p_{g+1}\}$ of non-separating curves such that $i(c_i, p_i) = i(c_i, p_{i+1}) = 1$ and the rest of the curves are pairwise disjoint (see figures 5.1a and 5.1b). Such a set of curves is unique up to homeomorphism. Let c_{g+1} be a curve such that $i(p_1, c_{g+1}) = i(p_{g+1}, c_{g+1}) = 1$ and is disjoint from every other curve in the set above. We define

$$C = \{c_1, \ldots, c_{g+1}\}.$$

Notice that $S \setminus \bigcup p_i$ has two connected components $(S \setminus \bigcup p_i)^+$ and $(S \setminus \bigcup p_i)^-$, we will call $(S \setminus \bigcup p_i)^+$ the *top component* and $(S \setminus \bigcup p_i)^-$ the *bottom component*. In the same fashion, $S \setminus \bigcup c_i$ has two connected components $(S \setminus \bigcup c_i)^+$ and $(S \setminus \bigcup c_i)^-$, we will call $(S \setminus \bigcup c_i)^+$ the *front component* and $(S \setminus \bigcup c_i)^-$ the *back component*.

For each $k = 2, \ldots, g-1$, the set $\partial(p_1, c_1, \ldots, p_k, c_k)$ consists of two curves: one of them in $(S \setminus \bigcup p_i)^+$ and the other one in $(S \setminus \bigcup p_i)^-$. We will call p_k^+ the curve of $\partial(p_1, c_1, \ldots, p_k, c_k)$ contained in $(S \setminus \bigcup p_i)^+$ and by p_k^- the curve of $\partial(p_1, c_1, \ldots, p_k, c_k)$ in $(S \setminus \bigcup p_i)^-$. We set

$$P = \{p_1 \ldots, p_{g+1}\} \cup \{p_2^+, p_2^-, \ldots, p_{g-1}^+, p_{g-1}^-\}.$$

Notice that P is a pants decomposition (see Figure 5.1a).

For each $k = 2, \ldots, g-1$, the set $\partial(p_{k-1}, c_k, p_k)$ has two curves, one in $(S \setminus \bigcup p_i)^+$ and the other one in $(S \setminus \bigcup p_i)^-$. We will denote by u_k the curve in $(S \setminus \bigcup p_i)^+$ and by d_k the curve in $(S \setminus \bigcup p_i)^-$ (see Figure 5.1c). We set

$$U = \{u_2, \ldots, u_{g-1}\}$$

and

$$D = \{d_2, \dots, d_{g-1}\}.$$

Given $k \in \{2, \dots, g-1\}$ the set $\partial(p_k, c_k, p_k^+)$ contains two curves and only one of them is also a curve in P. We will denote by l_k the curve in $\partial(p_k, c_k, p_k^+)$ not already in P (see Figure 5.1d). We set

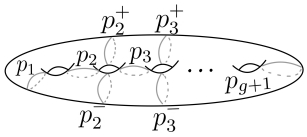
$$L = \{l_2, \dots, l_{g-2}\}.$$

Analogously, let $R = \{r_2, \dots, r_{g-2}\}$ be the set of curves where r_k is the unique curve in $\partial({p_{k+1}}^+, c_{k+1}, p_{k+2})$ that is not in P (see Figure 5.1d).

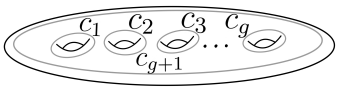
The set $\partial(p_2, c_2, \dots p_{g-1}, c_{g-1}, p_g)$ has two curves, one curve in each component of $S \setminus \bigcup p_i$. Let b be the curve contained in the bottom component $(S \setminus \bigcup p_i)^-$. Then, the set $\partial(c_1, b, c_g)$ has exactly two curves, one curve contained in $(S \setminus \bigcup c_i)^+$ and the other in $(S \setminus \bigcup c_i)^-$. Denote by nd the curve in $(S \setminus \bigcup c_i)^+$ (see Figure 5.1e).

Lastly, consider the torus T_1 that contains p_1 and is bounded by the curves p_2^+, p_2^-. Let nl be the unique curve contained in $T_1 \setminus (nd \cup c_1)$ distinct from c_1 and p_2^+. In the same way, p_{g-1}^+, p_{g-1}^- bound a torus T_g such that $p_g \subset T_g$, let nr be the unique curve in $T_g \setminus (nd \cup c_g)$ distinct from c_g and p_{g-1}^+ (see Figure 5.1f). We set

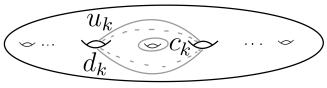
$$N = \{nl, nr, nd\}.$$



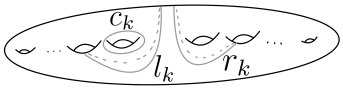
(a) Pants decomposition P of surface S_g. As an example, note that p_1 and p_2 are adjacent rel to P; while p_1 and p_3 are not adjacent rel to P.



(b) Circular curves C.



(c) Up and down curves u_k, d_k.



(d) Left and right curve l_k, r_k.



(e) Non symmetrical down curve nd.



(f) Non symmetrical left and right curves nl, nr.

Figure 5.1: Curves in F_R for a closed surface.

We set F_R to be the subcomplex of $\mathcal{N}(S)$ spanned by the vertices in

$$P \cup C \cup U \cup D \cup L \cup R \cup N.$$

Remark 5.9. *Note that the subcomplex F_R has diameter two. Thus, any locally injective simplicial map $\phi : F_R \to \mathcal{N}(S)$ is injective.*

5.3.2 Proving the rigidity of F_R

The first step is to check that the locally injective simplicial map $\phi : F_R \to \mathcal{N}(S)$ preserves the *non* adjacency rel to P. To do so, we require the following technical lemma:

Lemma 5.10. *Let S be a finite-type orientable surface, $F_R \subset \mathcal{N}(S)$ a subcomplex, $P \subset F_R$ a pants decomposition and $a, b \in P$ two curves. If there exist subsets $A, B \subset P$ and $\tilde{A}, \tilde{B} \subset F_R$ satisfying the following:*

- *$a \in A$ and $b \in B$,*
- *$\tilde{A}$ substitutes A in P,*
- *$\tilde{B}$ substitutes B in P,*
- *$\tilde{A} \cup \tilde{B}$ substitutes $A \cup B$ in P, and*
- *$A \cap B = \emptyset$.*

Then a, b are not adjacent rel to P.

Proof. We will proceed by contradiction. Suppose the curves a, b are adjacent in a pair of pants Q bound by a, b, c. Since $A \cap B = \emptyset$ either $c \notin A$ or $c \notin B$. Without loss of generality, suppose $c \notin A$.

If $\tilde{A}$ is a substitution of A and $A \cap B = \emptyset$, then there is a curve $\tilde{a} \in \tilde{A}$ such that $i(\tilde{a}, a) \neq 0$, $i(\tilde{a}, b) = 0$ and $i(\tilde{a}, c) = 0$. Now, note that since $\tilde{B}$ and $\tilde{A} \cup \tilde{B}$ are substitutions, there exists $\tilde{b} \in \tilde{B}$ with $i(\tilde{b}, b) \neq 0$, $i(\tilde{b}, a) = 0$ and $i(\tilde{b}, \tilde{a}) = 0$. However, is impossible to have arcs $\tilde{a} \cap Q$ and $\tilde{b} \cap Q$ satisfying the intersections above (see Figure 5.2). □

Now, we prove that ϕ preserves non adjacency rel to P:

Lemma 5.11. *Let S be a closed orientable surface of genus $g \geq 3$ and $\phi : F_R \to \mathcal{N}(S)$ a locally injective simplicial map. If $a, b \in P$ are not adjacent rel to P, then $\phi(a), \phi(b)$ are not adjacent rel to $\phi(P)$.*

Proof. Assume that for two curves $a, b \in P$ we have subsets $A, B \subset P$ and $\tilde{A}, \tilde{B} \subset F_R$ as in Lemma 5.10. Under these conditions, the lemma ensures that a and b are not adjacent rel to P. Moreover, these properties are carried to the image, that is, the curves $\phi(a), \phi(b) \in \phi(P)$ satisfy the conditions of Lemma 5.10 for the sets $\phi(A), \phi(B) \subset \phi(P)$ and $\phi(\tilde{A}), \phi(\tilde{B})$.

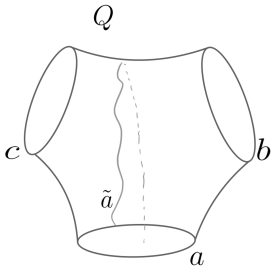


Figure 5.2: There can be no arc $\tilde{b} \cap Q$ that intersects b and does not intersect $\tilde{a} \cup a$.

As a consequence, we deduce that $\phi(a)$ and $\phi(b)$ are not adjacent rel to $\phi(P)$.

By means of the method above, we are only left to find appropriate subsets $A, \tilde{A}, B, \tilde{B}$ for any non-adjacent curves $a, b \in P$. We will find such subsets for certain $a, b \in P$, as the rest of the cases are similar.

If $a = p_k$ and $b = p_{k+1}$ for $k \in \{3, \dots, g-3\}$. We can consider $A = \{p_{k-1}^-, p_k\}$ and $\tilde{A} = \{l_{k-1}, d_{k-1}\}$, $B = \{p_{k+1}, p_{k+1}^-\}$ and $\tilde{B} = \{r_k, d_{k+1}\}$. It is straightforward to check that these subsets satisfy conditions of Lemma 5.10 and so $\phi(a)$, $\phi(b)$ are not adjacent rel to $\phi(P)$.

If $a = p_1$. Consider $A = \{p_1, p_2\}$ and $\tilde{A} = \{nl, c_1\}$, then

- If $b \in \{p_g, p_{g+1}\}$, take $B = \{p_g, p_{g+1}\}$ and $\tilde{B} = \{nr, c_g\}$.
- If $b \in \{p_k, p_k^-\}$ for $k \in \{3, \dots, g-1\}$, consider $B = \{p_k, p_k^-\}$ and $\tilde{B} = \{c_k, r_{k-1}\}$.
- If $b = p_k^+$ for $k \in \{3, \dots, g-1\}$, consider $B = \{p_k^+\}$ and $\tilde{B} = \{u_k\}$.

□

Using the previous result, we prove that ϕ preserves adjacency rel to P:

Lemma 5.12. *Let S be a closed orientable surface of genus $g \geq 3$ and $\phi : F_R \to \mathcal{N}(S)$ a locally injective simplicial map. If $a, b \in P$ are adjacent rel to P, then $\phi(a), \phi(b)$ are adjacent rel to $\phi(P)$.*

Proof. Take $\phi(p_1) \in \phi(P)$. From the non adjacency rel to $\phi(P)$, it follows that $\phi(p_1)$ has at most three adjacent curves. On the other hand, Remark 5.4 implies that $\phi(p_1)$ has at least three adjacent curves. Thus, we conclude that $\phi(p_1)$ has exactly three adjacent curves, namely $\phi(p_2), \phi(p_2^+), \phi(p_2^-)$. The same argument applies to $\phi(p_2)$, so it is adjacent rel to $\phi(P)$ to exactly three curves, namely $\phi(p_1), \phi(p_2^+), \phi(p_2^-)$.

We now determine the curves adjacent to $\phi(p_2^+)$ and $\phi(p_2^-)$. First, note that the adjacency rel to $\phi(P)$ of $\phi(p_1)$ and $\phi(p_2)$ implies that $\phi(p_2^+)$ and

$\phi(p_2^-)$ bound a subsurface homeomorphic to $S_{1,2}$. Since both curves $\phi(p_2^+)$ and $\phi(p_2^-)$ are non separating, it follows that both have four adjacent curves rel to $\phi(P)$. Finally, considering the non adjacency rel to $\phi(P)$, it follows that the curves adjacent to $\phi(p_2^+)$ are

$$\{\phi(p_1), \phi(p_2), \phi(p_3), \phi(p_3^+)\}$$

and the curves adjacent to $\phi(p_2^-)$ are

$$\{\phi(p_1), \phi(p_2), \phi(p_3), \phi(p_3^-)\}.$$

In the same style, we can argue inductively to determine the adjacency of each curve in $\phi(P)$. □

As a corollary we obtain that ϕ preserves the topological type of P:

Corollary 5.13. *Let S be a finite-type orientable surface and $\phi : F_R \to \mathcal{N}(S)$ a map that preserves adjacency rel to P. There exists $h \in \mathrm{Map}^{\pm}(S)$ such that $h|_P = \phi|_P$.*

Proof. We construct a homeomorphism h inductively by gluing abstract pairs of pants.

Consider the pairs of pants $Q_1, \dots, Q_k \subset S$ bound by curves in P and denote $Q_1', \dots, Q_k' \subset S$ the pairs of pants satisfying $\partial Q_i' = \phi(\partial Q_i)$. Any two pairs of pants are homeomorphic, thus we can consider homeomorphisms $h_i(Q_i) = Q_i'$. Since ϕ preserves the adjacency rel to P, we can ensure these homeomorphisms agree on the boundary curves. Then, by gluing the maps h_i we obtain a homeomorphism h of the surface such that $h|_P = \phi|_P$. □

The next three lemmas prove that F_R detects intersection among certain curves. Recall that F_R detects the intersection between a and b if for any locally injective simplicial map $\phi : F_R \to \mathcal{N}(S)$ we have that

$$i(a, b) \neq 0 \Leftrightarrow i(\phi(a), \phi(b)) \neq 0.$$

Lemma 5.14. *The subcomplex $F_R \subset \mathcal{N}(S)$ detects the following intersections:*

1. *u_k with p_k^+ and*
2. *d_k with p_k^-, for every $k = 2, \dots, g-1$.*

Proof. Let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. We need to check that $i(\phi(u_k), \phi(p_k^+)) \neq 0$ and $i(\phi(d_k), \phi(p_k^-)) \neq 0$.

Seeking a contradiction to case (1), we assume $i(\phi(u_k), \phi(p_k^+)) = 0$. Since ϕ is locally injective, it sends disjoint curves to disjoint curves. Thus, $\phi(u_k)$ is disjoint from every curve in the pants decomposition $\phi(P)$, which implies $\phi(u_k) \in \phi(P)$. However, this contradicts the injectivity of ϕ (see Remark 5.9).

To prove the case (2), the same argument works. □

Remark 5.15. *Notice that from the previous lemma we actually know that F_R detects the intersection of u_k with every curve in P. Indeed, u_k is disjoint from any curve in $P \setminus \{p_k^+\}$ and ϕ preserves disjointness. In the same way, F_R detects the intersection of d_k with every curve in P.*

Lemma 5.16. *The subcomplex $F_R \subset \mathcal{N}(S)$ detects the intersection of c_k with p_k^- and p_k^+, for every $k \in \{2, \ldots, g-1\}$.*

Proof. Let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. By Corollary 5.13, there exists $h \in \mathrm{Map}^{\pm}(S)$ such that $h \circ \phi$ fixes the pants decomposition P. Observe that detecting intersection is equivalent for ϕ and for $h \circ \phi$, since we have

$$i(\phi(a), \phi(b)) \neq 0 \Leftrightarrow i(h \circ \phi(a), h \circ \phi(b)) \neq 0.$$

And so we can rename $h \circ \phi$ to ϕ, and prove the statement assuming ϕ fixes every $p \in P$.

With the previous simplification, the proof boils down to check that $\phi(c_k)$ intersects p_k^+ and p_k^-. In this direction, consider the torus T_k bounded by the curves p_{k-1}^-, p_{k-1}^+, p_{k+1}^- and p_{k+1}^+ (see Figure 5.3). Further, define the top of the torus as $T_k^+ = T_k \cap (S \setminus \bigcup p_i)^+$ and the bottom of the torus as $T_k^- = T_k \cap (S \setminus \bigcup p_i)^-$.

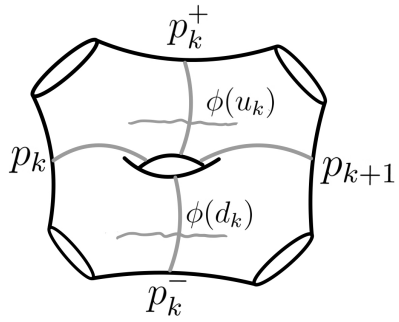


Figure 5.3: Torus T_k.

By Lemma 5.14, $\phi(u_k)$ is a curve in T_k^+ intersecting p_k^+ and $\phi(d_k)$ is a curve in T_k^- intersecting p_k^- (see Figure 5.3). Notice $\phi(c_k)$ is a curve in T_k distinct and disjoint from $\phi(d_k)$ and $\phi(u_k)$. It follows that $\phi(c_k)$ intersects both T_k^+ and T_k^-.

To finish, note $\phi(c_k) \cap T_k^+$ is disjoint from $\phi(u_k)$, so $\phi(c_k)$ must intersect p_k^+. Indeed, an arc disjoint from a curve in a sphere with four boundary components, must intersect every other curve in the sphere. Similarly, $\phi(c_k) \cap T_k^-$ is disjoint from $\phi(d_k)$, so $\phi(c_k)$ must intersect p_k^-. □

Lemma 5.17. *The subcomplex $F_R \subset \mathcal{N}(S)$ detects the intersection of $c_k \in C$ with every curve in P.*

Proof. Let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. As in the previous proof, we may assume that ϕ fixes every curve in P.

Now, we start by proving the cases $2 \leq k \leq g-1$. Note that with the simplification above and Lemma 5.16, we only need to check that $\phi(c_k)$ intersects p_k and p_{k+1}:

To prove $\phi(c_k)$ intersects p_k, consider the pair of pants Q bounded by the curves p_{k-1}^+, p_k and p_k^+. By Lemma 5.16, there are disjoint arcs $Q \cap \phi(c_{k-1})$ and $Q \cap \phi(c_k)$ with at least one endpoint in p_{k-1}^+ and p_k^+, respectively (see Figure 5.4). Using that $\phi(c_{k-1})$, p_k^+ are disjoint and $\phi(c_k)$, p_{k-1}^+ are also disjoint, we conclude that any such arc configuration requires $\phi(c_k)$ to intersect p_k. The same argument yields that $\phi(c_k)$ intersects p_{k+1}.

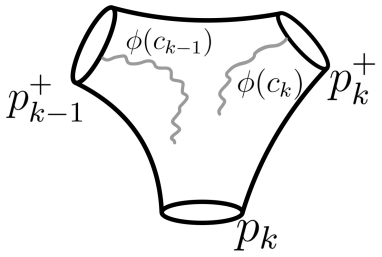


Figure 5.4: Pair of pants bounded by p_{k-1}^+, p_k and p_k^+.

We are left to prove the cases $k \in \{1, g, g+1\}$. First, we prove that $\phi(c_1)$ intersects p_2. Consider the torus T_1 bounded by the curves p_2^+ and p_2^-, and denote by T_1^+ the pair of pants bounded by p_1, p_2 and p_2^+ (see Figure 5.5). Note that $\phi(c_1)$ is a curve in T_1 distinct from p_1, p_2, p_2^- and p_2^+, it follows that $\phi(c_1)$ intersects T_1^+. Thus, we have disjoint arcs $\phi(c_1) \cap T_1^+$ and $\phi(c_2) \cap T_1^+$, the last one having an endpoint in p_2^+ (see Figure 5.5). Since $\phi(c_1)$, p_2^+ are disjoint and $\phi(c_2)$, p_1 are disjoint, we conclude that $\phi(c_1)$ must intersect p_2. Again, the same argument with slight changes yields that $\phi(c_g)$ intersects p_g.

Before proving that $\phi(c_1)$ intersects p_1 and $\phi(c_g)$ intersects p_{g+1}, we need to check that $\phi(c_{g+1})$ intersects both p_1, p_g and every p_k^+, p_k^-. Surely, $\phi(c_{g+1})$ intersects at least one of these curves, since otherwise $\phi(c_{g+1})$ would be disjoint and distinct from every curve in the pants decomposition $\phi(P)$. Suppose $\phi(c_{g+1})$ intersects p_k^+, and consider the pair of pants Q bound by p_{k-1}^+, p_k and p_k^+. Using the intersections above, we deduce that there are disjoint arcs $\phi(c_{k-1}) \cap Q$ and $\phi(c_{g+1}) \cap Q$, so that $\phi(c_{g+1})$ must also intersect p_{k-1}^+. Repeating the argument iteratively, we can detect the intersection of $\phi(c_{g+1})$ with every curve in P.

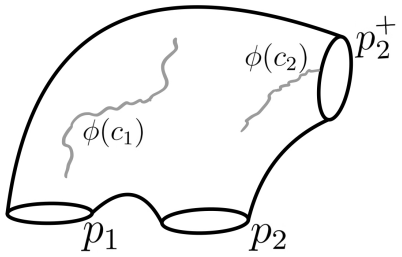


Figure 5.5: Pair of pants T_1^+.

To finish the proof, we check that $\phi(c_1)$ intersects p_1. Consider the disjoint arcs $a_2 = \phi(c_2) \cap T_1^+$ and $a_{g+1} = \phi(c_{g+1}) \cap T_1^+$, where a_2 has an endpoint in p_2^+ and a_{g+1} has an endpoint in p_1. These two arcs exist by the above intersections. Moreover, a_2 is disjoint from p_1, and a_{g+1} is disjoint from p_2. Recall that $\phi(c_1)$ intersects T_1^+ and, since it is disjoint from both a_1, a_2, it follows that $\phi(c_1)$ intersects p_1. A similar argument yields that $\phi(c_g)$ intersects p_{g+1}. □

So far we have seen that the map $\phi : F_R \to \mathcal{N}(S)$ can be taken to agree with a homeomorphism on $P \subset F_R$, and that it detects some intersections. In the next lemma, we extend it so ϕ agrees with a homeomorphism on $F_R \setminus N$.

Lemma 5.18. *Let S be a closed orientable surface of genus $g \geq 3$ and $\phi : F_R \to \mathcal{N}(S)$ a locally injective simplicial map. There exists $h \in \mathrm{Map}^{\pm}(S)$ such that $h|_{F_R \setminus N} = \phi|_{F_R \setminus N}$.*

Proof. By Corollary 5.13 there exists $h \in \mathrm{Map}^{\pm}(S)$ such that $h \circ \phi$ fixes every $p \in P$; we rename $h \circ \phi$ to ϕ. Recall that by Remark 5.9 we know ϕ is injective.

First, we find a homeomorphism that agrees with ϕ on $c_1 \in C$. Observe that $\phi(c_1)$ is contained in the torus T_1 bounded by p_2^+ and p_2^- (see Figure 5.6). Moreover, Lemma 5.17 imply there exist disjoint arcs $a \in \phi(c_2) \cap T_1$ and $\tilde{a} \in \phi(c_{g+1}) \cap T_1$. Notice both arcs have endpoints in p_2^+ and p_2^-, as otherwise $\phi(c_1)$ would not intersect p_1, contradicting Lemma 5.17. It follows that $\phi(c_1)$ is the curve contained in the annulus $T_1 \setminus (a \cup \tilde{a})$. Even more, there exists a twist $h' \in \mathrm{Map}^{\pm}(S)$ along p_1 and p_2, such that $h' \circ \phi(c_1) = c_1$. We rename $h' \circ \phi$ to ϕ.

The same argument with minor changes yield homeomorphisms that agree with ϕ on every $c_k \in C \setminus \{c_{g+1}\}$. Thus, we may assume that ϕ fixes every curve in $P \cup C \setminus \{c_{g+1}\} \subset F_R$.

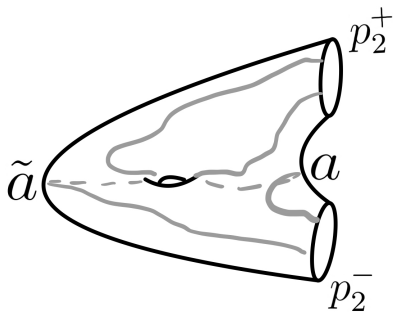


Figure 5.6: Torus T_1.

To finish the proof, we are going to check that ϕ is fixing c_{g+1} and every curve in $U \cup D \cup L \cup R$:

- Notice that $\phi(c_{g+1})$ is contained in the torus with boundary $T' = S \setminus \bigcup_{i=2}^{g} p_i$. In fact, $\phi(c_{g+1})$ is the unique curve contained in the annulus $S' \setminus \bigcup_{k=1}^{g} c_k$, i.e, $\phi(c_{g+1}) = c_{g+1}$.

- To prove ϕ fixes U, consider $u_k \in U$. Observe that $\phi(u_k)$ is contained in the sphere S_k^+ bounded by the curves p_{k-1}^+, p_k, p_{k+1} and p_{k+1}^+. Moreover, $\phi(u_k)$ must be the only curve in the annulus $S_k^+ \setminus (c_k \cup c_{g+1})$, i.e, $\phi(u_k) = u_k$.

- To prove ϕ fixes D, consider $d_k \in U$. Notice that $\phi(d_k)$ is contained in the sphere S_k^- bounded by the curves p_{k-1}^-, p_k, p_{k+1} and p_{k+1}^-. Now, $\phi(d_k)$ must be the only curve in the annulus $S_k^- \setminus \{c_k, c_{g+1}\}$, i.e, $\phi(d_k) = d_k$.

- To prove ϕ fixes L, consider the curve $l_k \in L$. The image $\phi(l_k)$ is a curve in the subsurface S_l bounded by p_{k-1}^-, p_k, p_k^+, p_{k+1}^- and p_{k+1}^+. Since $\phi(l_k)$ is disjoint from curves in C, we know that $\phi(l_k)$ is contained in the pair of pants $Q = S_l \setminus (c_k \cup c_{k+1})$. Note that Q has only one boundary component not already in P, and so we conclude $\phi(l_k) = l_k$. Naturally, a similar argument yields that ϕ fixes $r_k \in R$.

Summarizing, we have proven that there exists a mapping class $f \in \mathrm{Map}^{\pm}(S)$ such that $f \circ \phi$ fixes $F_R \setminus N$. In other words, $f^{-1}|_{F_R \setminus N} = \phi|_{F_R \setminus N}$. □

To finish the proof of Theorem 5.1, we need to check that we can take ϕ to agree with a homeomorphism on $N \subset F_R$, and that such homeomorphism is unique. Before doing so, we require one more definition:

The *back-front (orientation reversing) involution* is the unique non-trivial mapping class $\iota \in \mathrm{Map}^{\pm}(S)$ fixing every curve $c \in P \cup C$.

Proof of Theorem 5.1 for closed surfaces. Let $\phi : F_R \to \mathcal{N}(S)$ be a locally injective simplicial map. By Lemma 5.18 there exists $h \in \mathrm{Map}^{\pm}(S)$ such that $h \circ \phi$ fixes $F_R \setminus N$, we rename $h \circ \phi$ as ϕ.

Consider $nd \in N$ and notice $\phi(nd)$ is a curve in the genus two surface T bounded by the curves p_2^+, p_3, p_4, $\ldots, p_{g-1}$, p_{g-1}^+. Since $\phi(nd)$ is disjoint from C, then $\phi(nd)$ is a curve in $Q = T \setminus \bigcup_{c \in C} c$. Note that Q is the disjoint union of two pairs of pants and it has only two boundary components not contained in C, namely, nd and $\iota(nd)$, where ι is the back-front involution. It follows that $\phi(nd) \in \{nd, \iota(nd)\}$. Thus, by precomposing by ι if necessary, we may assume ϕ fixes $F_R \setminus \{nl, nr\}$.

We continue by proving that ϕ fixes nl. Note that $\phi(nl)$ is a curve in the torus T_1 bounded by p_2^- and p_2^+. Even more, $\phi(nl)$ is contained in $Q = T_1 \setminus (c_1 \cup nd)$, which is the union of an annulus and a pair of pants. Note that only one curve in Q is not a curve in $P \cup C$, and so we deduce $\phi(nl) = nl$. An analogous argument leads to the conclusion $\phi(nr) = nr$.

So far, we have found a composition of mapping classes $f \in \mathrm{Map}^{\pm}(S)$ such that $f \circ \phi$ is the identity in F_R. Therefore, ϕ is induced by the mapping class f^{-1}.

To prove the uniqueness of the inducing mapping class, suppose $f, \tilde{f} \in \mathrm{Map}^{\pm}(S)$ both induce the same $\phi : F_R \to \mathcal{N}(S)$. Since $g = \tilde{f} \circ f^{-1}$ fixes the curves $(P \cup C) \subset F_R$, the Alexander method implies that g is either the identity or the back-front orientation reversing involution. However, since g also fixes the curve nd, then g is the identity and $f = \tilde{f}$. □

5.4 Finite rigid exhaustion for closed surfaces

In this section we prove that for any closed surface S of genus $g \geq 3$, there exist subcomplexes $F_1 \subset F_2 \subset \cdots \subset \mathcal{N}(S)$ such that each F_i is a finite rigid set with trivial pointwise stabilizer and

$$\bigcup_{i=1}^{\infty} F_i = \mathcal{N}(S).$$

The strategy to produce an exhaustion is to first extend F_R to a larger finite rigid set F_1 with desirable properties. Then, the set F_1 will work as base case for an induction that enlarges F_i into F_{i+1}. This method heavily resembles the proof given by Aramayona and Leininger in [7] to produce an exhaustion of the curve complex.

The plan of the proof is summarized in the following lemma (cf. [7, Lemma 3.13]):

Lemma 5.19. *Let S be a finite-type orientable surface and let $F_1 \subset \mathcal{N}(S)$ be a a finite rigid set with trivial pointwise stabilizer and $\{h_1, \ldots, h_k\}$ a set*

of generators of $\mathrm{Map}^{\pm}(S)$. *If* $F_1 \cup h_j(F_1)$ *is a finite rigid set with trivial pointwise stabilizer for every* j, *then the sets*

$$F_{i+1} = F_i \cup \bigcup_{j=1}^{k} \left(h_j(F_i) \cup h_j^{-1}(F_i) \right),$$

satisfy that $F_1 \subset F_2 \subset \cdots \subset \mathcal{N}(S)$, $\bigcup F_i = \mathcal{N}(S)$ *and every* F_i *is finite rigid with trivial pointwise stabilizer.*

Proof. First, notice that if $F_1 \cup h_j(F_1)$ is a finite rigid set with trivial stabilizer, then the same holds for $F_1 \cup h_j^{-1}(F_1)$.

We will now check that F_2 is finite rigid with trivial stabilizer: take $\phi : F_2 \to \mathcal{N}(S)$ and observe that by hypothesis there exists $f_j \in \mathrm{Map}^{\pm}(S)$ such that

$$\phi|_{F_1 \cup h_j(F_1)} = f_j|_{F_1 \cup h_j(F_1)},$$

for every $j \in \{1, \ldots, k\}$. Since F_1 has trivial pointwise stabilizer and

$$f_j|_{F_1} = \phi|_{F_1} = f_k|_{F_1},$$

we deduce $f_j = f_k$ for all j, k. Clearly, the same argument holds when considering $F_1 \cup h_j^{-1}(F_1)$. It follows that $\phi = f$ for $f \in \mathrm{Map}^{\pm}(S)$ and F_2 is a finite rigid set with trivial pointwise stabilizer.

For F_{i+1} we proceed by induction. Consider $\phi : F_{i+1} \to \mathcal{N}(S)$, by induction we have $\phi|_{F_i} = f|_{F_i} \in \mathrm{Map}^{\pm}(S)$ and

$$\phi|_{h_j(F_i)} = f_j|_{h_j(F_i)}$$

for some $f_j \in \mathrm{Map}^{\pm}(S)$. Observe that $F_1 \subset F_i \cap h_j(F_i)$ and F_1 is trivially stabilized, thus $f = f_j$ for every j and so F_{i+1} is finite rigid with trivial stabilizer. □

To produce the finite rigid exhaustion of $\mathcal{N}(S)$ we are going to use the previous lemma. In this direction, we enlarge F_R into a set F_1 and provide a set of generators for $\mathrm{Map}^{\pm}(S)$.

5.4.1 Enlarging the finite rigid set

Let $\iota \in \mathrm{Map}^{\pm}(S)$ be the back-front orientation reversing involution. We define the set of curves $A := A_1 \cup A_2$, where:

$$A_1 := \bigcup_{k=1}^{g-1} \partial(c_k, p_{k+1}, c_{k+1}).$$

Note that the set $\partial(c_k, p_{k+1}, c_{k+1})$ has two curves, one curve in the front component and one curve in the back component of the surface. We will call $c_{k,k+1}$ the curve in the front component, so

$$\partial(c_k, p_{k+1}, c_{k+1}) = \{c_{k,k+1},\, \iota(c_{k,k+1})\}.$$

We proceed to define A_2. Consider the torus T bounded by p_{k-1}^-, p_{k+1}^- and u_k. Now, there is only one curve in the pair of pants $T \setminus (c_{k-1} \cup c_k \cup \iota(c_{k-1,k}))$ that is not already in $P \cup C$, denote this curve by nl_k (see Figure 5.7b). Analogously, let nr_k be the unique curve in $T \setminus (c_k \cup c_{k+1} \cup \iota(c_{k,k+1}))$ that is not already a curve in $P \cup C$ (see Figure 5.7b). We set

$$A_2 = \{nl_k, nr_k |\ k \in \{2, \dots, g-1\}\}.$$



(a) Curves in A_1. (b) Curves in A_2.

Figure 5.7: Curves in A for a closed surface.

Lemma 5.20. *Let S be a closed orientable surface of genus $g \geq 3$. The subcomplex spanned by $F_R \cup A \subset \mathcal{N}(S)$ is a finite rigid set with trivial pointwise stabilizer.*

Remark 5.21. *In the argument below we sometimes abuse notation using $p_1^+ = p_1^- = p_1$ and $p_g^+ = p_g^- = p_{g+1}$.*

Proof. Take $\phi : F_R \cup A \to \mathcal{N}(S)$ a locally injective simplicial map. By Theorem 5.1 in the closed case, there is an $h \in \mathrm{Map}^{\pm}(S)$ such that $h \circ \phi$ fixes F_R; we rename $h \circ \phi$ to ϕ. Furthermore, the subcomplex $F_R \cup A$ has diameter two, so ϕ is injective.

First, we check ϕ fixes A_1. Note that $\phi(\iota(c_{1,2}))$ is contained in the torus T bounded by the curves p_2, p_3^+ and p_3^-. Since $\phi(\iota(c_{1,2}))$ is disjoint from the curves in C, we have that $\phi(\iota(c_{1,2}))$ is contained in the pair of pants $Q = T \setminus (c_1 \cup c_2 \cup c_3)$. Notice that only one curve in Q is non-separating and disjoint from nd, namely $\iota(c_{1,2})$ and so $\phi(\iota(c_{1,2})) = \iota(c_{1,2})$. Now, $\phi(c_{1,2})$ is the unique non-separating curve in Q distinct from $\iota(c_{1,2})$, that is, $\phi(c_{1,2}) = c_{1,2}$.

By an obvious modification of the above argument one checks that

$$\phi(c_{k,k+1}) = c_{k,k+1} \text{ and } \phi(\iota(c_{k,k+1})) = \iota(c_{k,k+1}),$$

for $k \leq g-1$.

We are left to check that A_2 is also fixed by ϕ. Take $nr_k \in A_2$; we have that $\phi(nr_k)$ is contained in the torus T bounded by the curves p_{k-1}^-, p_{k+1}^- and u_k. Also, $\phi(nr_k)$ is disjoint from c_k, c_{k+1} and $\iota(c_{k,k+1})$, thus $\phi(nr_k)$ is a curve in the pair of pants $Q = T \setminus (c_k \cup c_{k+1} \cup \iota(c_{k,k+1}))$. Notice that the only curve in Q distinct from curves in $P \cup C$ is the curve nr_k, thus we have $\phi(nr_k) = nr_k$. For $nl_k \in A_2$ the argument is analogous.

We have proven that if ϕ fixes F_R, then it fixes $F_R \cup A$. The statement follows immediately. □

We define $F_1 := F_R \cup A$ and proceed to prove that it satisfies hypothesis of Lemma 5.19.

5.4.2 Constructing the exhaustion for closed surfaces

Let ι be the back-front orientation reversing involution and let δ_α be the right Dehn twist along the curve α. The set of Dehn twists

$$H := \{\delta_{p_1}, \delta_{p_2}, \ldots, \delta_{p_{g+1}}, \delta_{p_2^-}\} \cup \{\delta_{c_1}, \ldots, \delta_{c_k}\}$$

are known as the *Humphries generators*, which are known to generate the group of orientation preserving mapping classes $\mathrm{Map}(S)$ [44]. Hence, $H \cup \{\iota\}$ generates $\mathrm{Map}^{\pm}(S)$. We stress that for convenience we are using δ_a to denote a Dehn twist, instead of our earlier notation of d_a, since we need to name large sets of curves.

Lemma 5.22. *Let $h \in H \cup \{\iota\}$, the set $F_1 \cup h(F_1)$ is a finite rigid set with trivial pointwise stabilizer.*

Proof. First, we prove it for $h = \delta_{p_1}$. Let $\phi : F_1 \cup h(F_1) \to \mathcal{N}(S)$ be any locally injective simplicial map.

Since F_1 is a finite rigid set, we may assume that ϕ fixes F_1 by precomposing with a mapping class. Moreover, since $h(F_1)$ is also finite rigid, there exists a mapping class $f \in \mathrm{Map}^{\pm}(S)$ such that

$$\phi|_{h(F_1)} = f|_{h(F_1)}.$$

Proving $F_1 \cup h(F_1)$ is a finite rigid set with trivial pointwise stabilizer boils down to prove that $f = 1 \in \mathrm{Map}^{\pm}(S)$.

Recall that p_1 is the associated curve to the Dehn twist $h = \delta_{p_1}$. Now, let $S' = S \setminus p_1$ and consider the well-known cutting homomorphism (see [32, Proposition 3.20])

$$1 \to \langle \delta_{p_1} \rangle \to \mathrm{Map}(S, p_1) \to \mathrm{Map}(S'),$$

where $\mathrm{Map}(S') \subset \mathrm{Map}^{\pm}(S')$ is the subgroup of orientation preserving classes and $\mathrm{Map}(S, p_1) \subset \mathrm{Map}^{\pm}(S)$ is the subgroup of orientation preserving classes that fix p_1. Notice that f fixes all the curves in $F_1 \cap h(F_1)$, which fill S'. In particular, f is orientation preserving and fixes p_1. Furthermore, by means of the Alexander method, the image of f by the cutting homomorphism is trivial and the above sequence implies that $f = \delta_{p_1}^k$, for some $k \in \mathbb{Z}$. We are left to see that $k = 0$.

Note that $i\left(nl, \delta_{p_1}(c_{g+1})\right) = 0$. As ϕ is locally injective, we know that $i\left(\phi(nl), \phi\left(\delta_{p_1}(c_{g+1})\right)\right) = 0$, which is equivalent to

$$i\left(nl, \delta_{p_1}^{k+1}(c_{g+1})\right) = 0, \tag{5.1}$$

since $\phi|_{F_1} = \mathrm{id}$ and $\phi|_{h(F_1)} = \delta_{p_1}^k$.

It can be directly checked that Equation (5.1) is satisfied if and only if $k = 0$, thus proving the statement of the lemma for $h = \delta_{p_1}$.

The rest of the cases $h \in H$ are proved in exactly the same way, but changing the curves in Equation (5.1). We list all the cases for completeness:

- If $h = \delta_{p_{g+1}}$, change (nl, c_{g+1}) in (5.1) by (c_{g+1}, nr);
- If $h = \delta_{p_i}$ for $i \in \{2, \dots, g-2\}$, change (nl, c_{g+1}) by $(nl_i, c_{i,i+1})$;
- If $h = \delta_{p_{g-1}}$, change (nl, c_{g+1}) by $(c_{g-2,g-1}, nr_{g-1})$;
- If $h = \delta_{p_2^-}$, change (nl, c_{g+1}) by (c_{g+1}, nr_2);
- If $h = \delta_{c_k}$ for $k \in \{1, \dots, g-1\}$, change (nl, c_{g+1}) by (p_{k+1}, nr_{k+1});
- If $h = \delta_{c_g}$, change (nl, c_{g+1}) by (nl_{g-1}, p_g).

To finish the proof, we need to consider the case $h = \iota$. In this case, $F_1 \cap h(F_1)$ contains the trivially pointwise stabilized set $P \cup C \cup A_1$, so it follows immediately that $f = 1 \in \mathrm{Map}^{\pm}(S)$. □

Proof of Theorem 5.2 for closed surfaces. Let S be a closed surface of genus $g \geq 3$. Consider $F_1 := F_R \cup A$ and the set of generators $H \cup \{\iota\}$ of the extended mapping class group $\mathrm{Map}^{\pm}(S)$. By Lemma 5.19 we have the desired exhaustion, where the hypotheses have been checked in lemmas 5.20 and 5.22. □

5.5 Finite rigid sets for punctured surfaces

Let $S = S_{g,n}$ with $n \geq 1$ and $g \geq 3$. In this section we construct a finite rigid set $F_R' \subset \mathcal{N}(S)$ with trivial pointwise stabilizer. As we will see, the argument to prove the rigidity of F_R' is morally the same as the one in Section 5.3, although more involved as it requires to manipulate more curves.

5.5.1 Constructing the finite rigid set

We start by naming some curves in the punctured surface S.

Consider the closed surface S_g and the pair of pants Q bounded by p_1, p_2, p_2^+. Note that $Q \setminus d_2$ has two connected components, name A the connected component intersected by c_{g+1}. Now, $A \setminus (c_1 \cup c_{g+1})$ is the union of two disks one in the front component and one in the back component of S_g, name B the disk in the back component $(S_g \setminus \bigcup c_i)^-$. By removing n points from the interior of B we obtain the punctured surface S. Via this procedure we get natural analogues in S of the sets of curves P, C, U, D, L, R and N.

After this, we define top of the surface $(S\setminus\bigcup p_i)^+$, the bottom $(S\setminus\bigcup p_i)^-$, the front $(S\setminus\bigcup c_i)^+$ and the back $(S\setminus\bigcup c_i)^-$, in the same way we did in Section 5.3.

The set P is no longer a pants decomposition in S; to fix this we add some curves. First, rename p_2^+ to $p_{2,n}^+$ and consider the pair of pants Q bounded by p_1, p_2 and $p_{2,n}^+$. Observe $Q\setminus c_1$ is an annulus with n punctures. Now, fix a set of non-separating curves $\{p_{2,0}^+, p_{2,1}^+, \ldots, p_{2,n-1}^+, p_{2,n}^+\}$ in $Q\setminus c_1$ satisfying that $p_{2,i}^+$ and $p_{2,i+1}^+$ bound a once punctured annulus. Adding the curves $\{p_{2,0}^+, p_{2,1}^+, \ldots, p_{2,n-1}^+, p_{2,n}^+\}$ to the set P, makes it a pants decomposition (see Figure 5.8a). We will say that the puncture contained in the annulus bounded by $p_{2,k-1}^+$ and $p_{2,k}^+$ is the kth puncture.

In this setting, we also define the back-front (orientation reversing) involution as the unique non-trivial element $\iota \in \mathrm{Map}^{\pm}(S)$ that fixes every curve $P\cup C$.

Let pl_1 be the unique curve in the punctured pair of pants bounded by p_1, p_2, $p_{2,1}^+$ that is disjoint from c_2 and is not a curve already in P. Similarly, let pl_k be the unique curve in the punctured pair of pants bounded by pl_{k-1}, p_2, $p_{2,k}^+$ that is disjoint from c_2 and is not already a curve in P (see Figure 5.8d). We set

$$Pl := \{pl_1, \ldots, pl_n\}.$$

Let pr_n be the unique curve in the punctured pair of pants bounded by $p_{2,n-1}^+$, p_3, p_3^+ that is disjoint from c_2 and is not already a curve in P. Inductively, define pr_k to be the unique curve in the punctured pair of pants bounded by $p_{2,k-1}^+$, p_3, p_3^+ that is disjoint from c_2 and not already in P (see Figure 5.8e). We set

$$Pr := \{pr_1, \ldots, pr_n\}.$$

Lastly, we define the set

$$C_* = \partial(c_1,\, p_2,\, c_2) \cup \partial(c_2,\, p_2, \ldots,\, p_g,\, c_g).$$

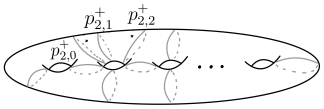
Note the set $\partial(c_1,\, p_2,\, c_2)$ has two curves, one curve in the front component which we will denote by $c_{1,\,g}$ and one curve in the back component that is $\iota(c_{1,g})$. Also, the set $\partial(c_2,\, p_2, \ldots,\, p_g,\, c_g)$ has two curves, denote by $c_{2,\,g}$ the curve in the front component and by $\iota(c_{2,\,g})$ the curve in the back component (See Figure 5.8f).

The finite rigid set F_R' is the maximal subcomplex of $\mathcal{N}(S)$ spanned by the vertices

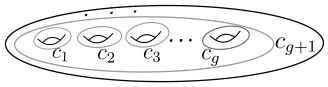
$$P\cup C\cup U\cup D\cup L\cup R\cup N\cup Pl\cup Pr\cup C_*.$$

Remark 5.23. *Note that the subcomplex F_R' has diameter two. Thus, any locally injective simplicial map $\phi : F_R' \to \mathcal{N}(S)$ is injective.*

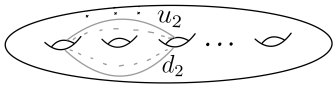
The following lemmas prove the finite rigidity of F_R'.



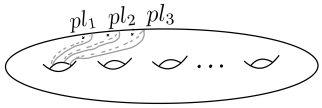
(a) Pants decomposition P for $S_{g,2}$.



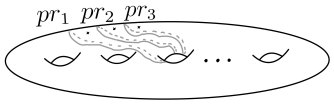
(b) Curves in C for $S_{g,3}$.



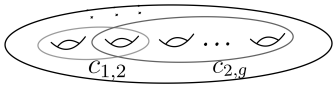
(c) Up and down curves u_2, d_2 for $S_{g,3}$.



(d) Curves pl_1, pl_2, pl_3 for $S_{g,3}$.



(e) Curves pr_1, pr_2, pr_3 for $S_{g,3}$.



(f) Curves $c_{1,2}$ and $c_{2,g}$ for $S_{g,3}$.

Figure 5.8: Curves in F'_R. Punctures are marked as small crosses on the top of the surface.

Lemma 5.24. *Let S be a punctured orientable surface of genus $g \geq 3$. For any locally injective simplicial map $\phi : F'_R \to \mathcal{N}(S)$, there exists $h \in \mathrm{Map}^{\pm}(S)$ such that $h|_P = \phi|_P$.*

Proof. First, notice that we can extend Lemma 5.11 to the case of punctured surfaces, and the proof works with minor changes.

Lemma 5.12 can also be extended to punctured surfaces. The proof for once punctured surfaces is similar to the closed case. Here we prove it for surfaces with $n > 1$ punctures:

Using Remark 5.4, it is straightforward to deduce that the curve $\phi(p^+_{2,i})$ is adjacent rel to P to $\phi(p^-_{2,i-1})$ and $\phi(p_{2,i+1})$, for $i \in \{1, \ldots, n-1\}$. As a consequence of these adjacencies, we deduce that $\phi(p^+_{2,0})$ and $\phi(p^+_{2,n})$ bound a subsurface with two boundary components and n punctures. It follows that $\phi(p^+_{2,0})$ is adjacent rel to P to at least three curves. On the other hand, the non adjacency rel to P of $\phi(p^+_{2,0})$ implies that it is adjacent to at most three curves. Thus, $\phi(p^+_{2,0})$ is adjacent to exactly three curves, namely $\phi(p^+_{2,1})$, $\phi(p_1)$ and $\phi(p_2)$. The rest of the adjacencies follow just as in the proof of Lemma 5.12.

Once we know ϕ preserves adjacency rel to P, we can use Corollary 5.13 to produce a mapping class $h \in \mathrm{Map}^{\pm}(S)$ with $h|_P = \phi|_P$. □

The next lemma proves that we may choose $h \in \mathrm{Map}^{\pm}(S)$ to coincide with ϕ on all the curves of F'_R.

Lemma 5.25. *Let S be a punctured orientable surface of genus $g \geq 3$. For any locally injective simplicial map $\phi : F'_R \to \mathcal{N}(S)$ there exists $h \in \mathrm{Map}^{\pm}(S)$ such that $\phi = h|_{F'_R}$.*

Proof. The idea is to progressively detect intersections between curves and, by composing with Dehn twists along P, construct a mapping class that coincides with ϕ on F'_R.

First, by Lemma 5.24, there exists $h \in \mathrm{Map}^{\pm}(S)$ such that $h \circ \phi$ fixes P pointwise; rename $h \circ \phi$ as ϕ.

Next, using the same arguments as in lemmas 5.14, 5.16 and 5.17, we detect the following intersections:

- intersection of curves in $U \cup D$ with curves in P,
- intersections of curves in Pl with curves in P,
- intersections of curves in Pr with curves in P, and
- c_k with curves in P for $k \in \{2, \ldots, g-1\}$.

Notice that we can now use the proof of Lemma 5.18 to produce a mapping class that coincides with ϕ on the curves of U, D and $\{c_2, \ldots, c_{g-1}\}$. Thus, we may assume these curves are fixed by ϕ.

It follows that ϕ fixes the curves in Pl. To see this, note that $\phi(pl_1)$ is a curve contained in the punctured sphere S' bounded by p_1, u_2, p_3^+, and is disjoint from $p_{2,i}^+$ for $i > 0$. Thus, $\phi(pl_1)$ is contained in the once punctured annulus which is a component of $S' \setminus p_{2,1}^+$. As there is only one curve in that annulus that is not already in P, it follows that $\phi(pl_1) = pl_1$. In the same way, $\phi(pl_k)$ is contained in the punctured sphere S'_k bounded by pl_{k-1}, u_2, p_3^+, and is disjoint from $p_{2,i}^+$ for $i > k-1$. Thus, $\phi(pl_k)$ is contained in the once punctured annulus which is a component of $S'_k \setminus p_{2,k}^+$, and since there is only one curve in that annulus which is not in P we conclude that $\phi(pl_k) = pl_k$. Naturally, an analogous argument works to prove that ϕ fixes every curve in Pr.

Now, note that ϕ detects the following intersections:

- curves in C_* with curves in P,
- curves in $\{c_1, c_g\}$ with curves in P.

For instance, consider $c_{1,2} \in C_*$ and $p \in P$ disjoint from $c_{1,2}$. If $\phi(c_{1,2})$ was disjoint from p, then $\phi(c_{1,2})$ would have to intersect either the curve $\phi(c_1), \phi(c_2)$ or $\phi(\iota(c_{2,g}))$, leading to a contradiction. Thus, $\phi(c_{1,2})$ and p intersect.

Using the above intersections and fixed curves, we now focus on finding a homeomorphism that agrees with ϕ on the curves in C_*.

Observe $\phi(c_1)$ (resp. $\phi(c_g)$) is contained in the subsurface $S' = S_{1,2}$ bound by the curves p_2^-, p_2^+ (resp. by p_{g-1}^-, p_{g-1}^+). Additionally, since $\phi(c_1)$ is disjoint from the arcs $a_1 = S' \cap \phi(c_{1,2})$ and $a_2 = S' \cap \phi(c_2)$ (resp. $S' \cap \phi(c_{2,g})$ and $S' \cap \phi(c_{g-1})$), we have that $\phi(c_1)$ (resp. $\phi(c_g)$) is contained in the annulus $S' \setminus (a_1 \cup a_2)$. It follows that $\phi(c_1)$ (resp. $\phi(c_g)$) is the unique curve in the annulus. We may again consider a mapping class $h \in \mathrm{Map}^{\pm}(S)$ that is a composition of twists along curves in P such that $h \circ \phi(c) = c$ for $c \in \{c_1, c_g\}$; we rename $h \circ \phi$ to ϕ (the same argument with more details is given in Lemma 5.18).

For $c_{1,2} \in C_*$, note that $\phi(c_{1,2})$ is contained in a subsurface $S' = S_{2,2}$ bound by pr_1 and p_3^- (or p_4 if $g = 3$). Even more, $\phi(c_{1,2})$ is contained in the pair of pants $S' \setminus (c_1 \cup c_2 \cup c_3 \cup p_2)$, therefore it is one of the boundary components. One of the boundaries is a separating curve, so either $\phi(c_{1,2}) = c_{1,2}$ or $\phi(c_{1,2}) = \iota(c_{1,2})$. These two alternatives are related by the involution ι. Thus, by precomposing with ι we can assume ϕ fixes $c_{1,2}$.

It is now easy (but lengthy) to check that the curves in

$$\{\iota(c_{1,2}), \iota(c_{2,g}), nl, nr, nd\} \cup L \cup R$$

are fixed by ϕ. Thus, we have found a mapping class $h \in \mathrm{Map}^{\pm}(S)$ such that $h \circ \phi$ fixes F'_R and the statement follows. □

We have essentially completed the proof of Theorem 5.1.

Proof of Theorem 5.1 for surfaces with punctures. Let $\phi : F'_R \to \mathcal{N}(S)$ be a locally injective simplicial map, Lemma 5.25 provides the mapping class inducing ϕ.

The uniqueness of the mapping class follows as in the closed case. □

5.6 Finite rigid exhaustion for punctured surface

Let S be a punctured surface of genus $g \geq 3$. In this section we construct a sequence $F'_1 \subset F'_2 \subset \cdots \subset \mathcal{N}(S)$ such that each F'_i is a finite rigid set with trivial pointwise stabilizer and

$$\bigcup_{i=1}^{\infty} F'_i = \mathcal{N}(S).$$

The strategy to construct the exhaustion is the same as in the closed case (see Section 5.4). First, we are going to enlarge the finite rigid set F'_R to F'_1 and then use Lemma 5.19 to construct the exhaustion.

5.6.1 Enlarging the finite rigid set

First, we enlarge F'_R:

Consider the set of curves A_1 and A_2 in the closed surface. Via the same procedure described in Section 5.5.1, we remove n points from the interior of the closed surface S_g and so we obtain the punctured surface S. This produces natural analogues of the set of curves A_1 and A_2 in the surface S.

We define the set of curves $A_3 := \{a_1, \dots, a_n\}$, where a_k is the unique curve in the torus bounded by pl_{k-1}, pr_{k+1} and d_2, which is disjoint from c_1, c_2, c_3 and $\iota(c_{1,2})$ (see Figure 5.9).

We set $A' := A_1 \cup A_2 \cup A_3$.

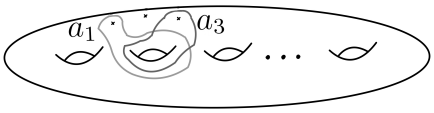


Figure 5.9: Curves a_1, a_3 in A_3 for $S_{g,3}$.

Lemma 5.26. *The set $F'_R \cup A'$ is finite rigid with trivial pointwise stabilizer.*

Proof. Let $\phi : F'_R \cup A' \to \mathcal{N}(S)$ be a locally injective simplicial map. By precomposing with a mapping class we may assume that ϕ fixes F'_R pointwise.

First, we prove that A_1 is also fixed by ϕ. The curves $c_{1,2}, \iota(c_{1,2}) \in A_1$ are already in $C_* \subset F'_R$, so they are fixed by ϕ. For $c_{2,3} \in A_1$, notice that $\phi(c_{2,3})$ is contained in the sphere S' bounded by p_1, pl_n, c_2, c_3, p_4^+ and p_4^-. Moreover, $\phi(c_{2,3})$ is contained in the pair of pants $S' \setminus c_1 \cup p_3 \cup \iota(c_{2,g})$. But there is only one curve in that pair of pants that is non-separating, i.e, $\phi(c_{2,3}) = c_{2,3}$. Slight modifications yield that $\phi(\iota(c_{2,3})) = \iota(c_{2,3})$. For the rest of A_1, one can proceed as in the closed case (see the proof of Lemma 5.20).

To prove A_2 is fixed, we can just repeat the argument as in the closed case (see Lemma 5.20).

To finish the proof we must show that $a_k \in A_3$ is fixed. To check this, note that $\phi(a_k)$ is contained in the torus T bounded by pl_{k-1}, pr_{k+1} and d_2. Note that $\phi(a_k)$ is the unique curve in $T \setminus (c_1 \cup c_2 \cup c_3 \cup \iota(c_{1,2}))$ that is not c_2, that is, $\phi(a_k) = a_k$. Thus, A' is fixed and $F'_R \cup A'$ is a finite rigid set with trivial pointwise stabilizer. □

We define $F'_1 := F'_R \cup A'$.

5.6.2 Constructing the exhaustion for punctured surfaces

The goal of this section is to construct an exhaustion of $\mathcal{N}(S)$ by finite rigid sets with trivial pointwise stabilizers. In this direction, we will consider a set

of generators of $\mathrm{Map}^{\pm}(S)$ such that the subcomplex F_1' satisfies hypotheses of Lemma 5.19. We are going to assume $S = S_{g,n}$ with $n > 0$ punctures.

Let ι be the back-front orientation reversing involution. Consider the usual Humphries generators

$$H' = \{\delta_{p_1}, \dots, \delta_{p_{g-1}}, \delta_{p_2^-}\} \cup \{\delta_{p_{2,i}^+} |\ 0 \leq i \leq n\}\} \cup \{\delta_{c_1}, \dots, \delta_{c_k}\}$$

and the half twists

$$\{h_{(k,k+1)} |\ 1 \leq k \leq n-1\},$$

where $h_{(k,k+1)}$ is the half twist that permutes the puncture k with the puncture $k+1$. It is well known that $H' \cup \{\iota\} \cup \{h_{(k,k+1)} |\ 1 \leq k \leq n-1\}$ generates $\mathrm{Map}^{\pm}(S)$ (see [32, Chapter 4.4.4]).

Lemma 5.27. *For every $h \in H' \cup \{\iota\} \cup \{h_{(k,k+1)} |\ 1 \leq k \leq n-1\}$, the set $F_1' \cup h(F_1')$ is finite rigid with trivial pointwise stabilizer.*

Proof. The proof is analogous to the closed case (see Lemma 5.22) and works directly for $h = \iota$, $h = \delta_{p_j}$, $h = \delta_{c_j}$ and $h = \delta p_2{}^-$. We consider here the rest of the cases:

Recall that given $h = \delta_x$ and $\phi : F_1' \cup h(F_1') \to \mathcal{N}(S)$, we can assume that ϕ fixes F_1' and $\phi|_{h(F_1)} = \delta_x^k|_{h(F_1)}$. Thus, the proof is a matter of checking that $k = 0$.

For $h = \delta_{p_{2,j}^+}$ with $j \leq n-1$, we can consider the curves l_j, a_j and plug them into Equation (5.1). It follows that this is satisfied if and only if $k = 0$. For $h = \delta_{p_{2,n}^+}$, one uses the curves r_n, a_n in the same equation.

Th last case to prove is $h = h_{(k-1,k)}$, which can be proved using the curves a_k, r_k in Equation (5.1). □

We now complete the proof of Theorem 5.2.

Proof of Theorem 5.2 for punctured surfaces. The Lemmas 5.26 and 5.27 ensure that F_1' satisfies the hypothesis of Lemma 5.19, which in turn produces the desired exhaustion. □

Chapter 6

Matching arc complexes

In this chapter, we concentrate on a variation of the arc complex called the *matching arc complex*. Matching arc complexes, especially those associated to planar surfaces, have recently played a central role in the study of finiteness properties of *braided Thompson groups* [25].

Let S be a connected orientable surface with no boundary. Zaremsky [92] asked the question of determining for which graphs Γ is the associated matching arc complex $\mathcal{MA}(S, \Gamma)$ connected and hyperbolic; the aim of this short chapter is to deal with this problem.

The first result we present characterizes those graphs Γ for which $\mathcal{MA}(S, \Gamma)$ is connected. Let Γ_0 be the simplicial graph obtained from Γ by deleting every isolated vertex of Γ; abusing notation, we will write $|\Gamma_0|$ for the number of vertices of Γ_0. We will prove:

Theorem 6.1. *Let Γ be a finite simplicial graph with $|\Gamma_0| \geq 5$. Then $\mathcal{MA}(S, \Gamma)$ is connected if and only if the complement of every edge of Γ contains at least one edge.*

We remark that, in the special case when Γ is a *complete* or *linear* graph on $n \geq 5$ vertices, Theorem 6.1 follows from Theorem 3.8 and Corollary 3.11 of [25].

Next, we give a condition on Γ that ensures the hyperbolicity of the associated matching arc complex:

Theorem 6.2. *Let Γ be a finite simplicial graph for which $\mathcal{MA}(S, \Gamma)$ is connected. If Γ_0 is connected and contains a triangle, then $\mathcal{MA}(S, \Gamma)$ is hyperbolic.*

We stress that if Γ is the complete graph on at least five vertices, then Theorem 6.2 follows from the combination of [30, Theorem 5.2] and Lemma 6.13 below.

Remark 6.3. *The inclusion of the matching arc complex into the arc complex is never a quasi-isometry, and hence Theorem 6.2 does not follow (or not immediately, at least) from the fact that arc complexes are hyperbolic.*

Remark 6.4. *Not all finite simplicial graphs yield a hyperbolic matching arc complex. Indeed, the* Disjoint Holes Principle *of Masur–Schleimer [66, Lemma 5.12] implies that if* Γ *is a bipartite graph whose parts have at least four vertices each, then* $\mathcal{MA}(S, \Gamma)$ *contains a quasi-isometrically embedded copy of* $\mathbb{Z}^2$*, and therefore cannot be hyperbolic. In particular, matching arc complexes associated to trees or even-sided polygons are never hyperbolic. As a special case, matching arc complexes associated to linear graphs (one of the family of matching arc complexes studied in [25]) are never hyperbolic. Of course, there are connected graphs that are triangle-free but not bipartite, and it remains an interesting problem to determine if the associated matching arc complexes are hyperbolic or not.*

Our proof of Theorem 6.2 adapts the *unicorn path* machinery of Hensel-Przytycki-Webb [40], which they used to prove the *uniform hyperbolicity* of arc graphs (the 1-skeleta of arc complexes). The difficulties we encounter boil down to the fact that a unicorn path between two vertices of $\mathcal{MA}(S, \Gamma)$ may contain edges that do not belong to $\mathcal{MA}(S, \Gamma)$; in order to overcome this problem, we will use the structure of Γ, plus techniques similar in spirit to those of [30, Section 5].

Lastly, we note that the statement of Theorem H in the introduction is precisely the combination of Theorem 6.2 and Theorem 6.1.

6.1 Preliminaries

Throughout this chapter, S will be a orientable surface with empty boundary and negative Euler characteristic. Let g and n be, respectively, the genus and number of punctures of S; as usual, we will treat punctures either as topological ends or as marked points on S.

6.1.1 Arcs

An *arc* is the image of a continuous map $a : [0, 1] \to S$ such that its restriction to $(0, 1)$ is injective, $a(0)$ and $a(1)$ are marked points of S, and $a((0, 1))$ is disjoint from the set of marked points of S. We say that an arc is *essential* if it does not bound a disk on S. In order to keep notation light, we will refer to the isotopy class (relative to endpoints) of an essential arc simply as an *arc*, and will blur the difference between arcs and their representatives.

As for curves, given arcs a and b, their *geometric intersection number* $i(a, b)$ is defined as the minimum number of intersection points (not counting endpoints) between representatives of a and b. We say that representatives of a and b are in *minimal position* if they realize the intersection number between a and b. Note that, given any set of arcs, it is possible to find representatives in minimal position by endowing S with a hyperbolic metric and choosing geodesic representatives.

We will say that two arcs a and b are *disjoint* if $i(a,b) = 0$. If a and b are not disjoint, we say they *intersect*. Finally, if $i(a,b) = 0$ and a and b have no endpoints in common, we say they are *completely disjoint*.

Definition 6.5 (Arc complex)**.** *The* arc complex $\mathcal{A}(S)$ *is the simplicial complex whose k-simplices are sets of $k+1$ (pairwise) distinct and disjoint arcs.*

Unicorn paths

We now briefly recall the main features of the *unicorn arc* machinery of [40]. Let a and b be arcs in minimal position, and choose endpoints u and v of a and b, respectively. Consider subarcs $a' \subset a$ and $b' \subset b$ which have one endpoint at v and u, respectively, and the other a common endpoint $p \in a \cap b$. If $a' \cup b'$ has no self-intersections, then it is called a *unicorn arc* obtained from a^v and b^u. Observe that $a' \cup b'$ is automatically essential, since a and b are in minimal position. Also, note that the isotopy class of an unicorn arc is completely determined by the point p; however not every such point defines an unicorn arc, as for some choices the subarcs a' and b' may intersect each other. We define a total order for unicorns arcs obtained from a^v and b^u, namely:

$$a' \cup b' \leq a'' \cup b'' \iff a'' \subset a'.$$

Let

$$\mathcal{U}(a^v, b^u) := (c_0 = a, c_1, \ldots, c_{n-1}, c_n = b)\,,$$

be the ordered tuple of unicorn arcs defined by a^v and b^u; we call this tuple a *unicorn path* between a and b. The following is Remark 3.4 of [40]:

Lemma 6.6. *Any two consecutive elements of $\mathcal{U}(a^v, b^u)$ are disjoint.*

As a consequence, $\mathcal{U}(\alpha^v, \beta^u)$ is really a path in $\mathcal{A}(S)$. The next lemma is [40, Lemma 3.3.]:

Lemma 6.7. *Let a^v, b^u and c^w based at v, u and w, respectively. For every $x \in \mathcal{U}(a^v, b^u)$ there is $x^* \in \mathcal{U}(a^v, c^w) \cup \mathcal{U}(c^w, b^u)$ with $i(x, x^*) = 0$.*

6.1.2 Matching arc complex

Let Γ be a finite simplicial graph. We fix an embedding $\Gamma \hookrightarrow S$ such that the image of every vertex of Γ is a marked point of S; in particular, we are implicitly assuming that S has at least as many punctures as Γ has vertices. In what follows, we will blur the difference between Γ and its image under this embedding. An arc α is *compatible* with Γ if there is an edge e of Γ with the same endpoints; we say that a is compatible with e.

Definition 6.8 (Matching arc complex)**.** *The* matching arc complex $\mathcal{MA}(S, \Gamma)$ *is the simplicial complex whose k-simplices are sets of $k+1$ arcs compatible with Γ and which are pairwise completely disjoint.*

Throughout, we will write $d(\cdot, \cdot)$ for the natural path-metric on the vertex set of $\mathcal{MA}(S, \Gamma)$.

6.2 Connectivity of $\mathcal{MA}(\Gamma)$

In this section we prove Theorem 6.1. Before doing so, we need a couple of technical lemmas that will be used to (uniformly) bound the distance between consecutive vertices in a unicorn path.

Lemma 6.9. *Let Γ as in Theorem 6.1. If a and b are disjoint arcs on S with exactly one endpoint in common, then $d(a, b) \leq 4$.*

Proof. Let $e, f \in \Gamma$ be edges compatible with a and b, respectively. If there exists an edge $e'' \in \Gamma$ such that $e'' \cap (e \cup f) = \emptyset$, then we can choose an arc in $\mathcal{MA}(S, \Gamma)$ compatible with e'' and which is completely disjoint from $a \cup b$. In particular, $d(a, b) = 2$.

Suppose now that no such edge exists. By hypothesis, there exists an edge e' with $e \cap e' = \emptyset$; we may assume that e' and f have exactly one endpoint in common, for otherwise we are back in the situation above. Similarly, there exists another edge f', disjoint from f and with exactly one endpoint in common with e. There are two cases to consider, depending on whether e' and f' are disjoint or not.

Assume first that $e' \cap f' = \emptyset$. Then, we can choose completely disjoint arcs $c_1, c_2 \in \mathcal{MA}(S, \Gamma)$, compatible with e' and f' respectively, such that $i(c_i, a) = i(c_i, b) = 0$. Then the sequence a, c_1, c_2, b gives a path in $\mathcal{MA}(S, \Gamma)$ of length three between a and b.

It remains to consider the case where $e' \cap f' \neq \emptyset$; as Γ is simplicial, this means that e' and f' have exactly one endpoint in common. Since $|\Gamma_0| \geq 5$, there exists an edge e'' that shares at most one endpoint with $e \cup f \cup e' \cup f'$. Moreover, since we are not in any of the cases considered previously, we may assume that e'' shares exactly one endpoint with both e and f, but is completely disjoint from e' and f'. Then we may choose pairwise disjoint arcs c_1, c_2, c_3 such that c_1 (resp. c_2, c_3) is compatible with e' (resp. e'', f'), and $i(c_i, a) = i(c_i, b) = 0$. Thus, a, c_1, c_2, c_3, b yield a path of length four in $\mathcal{MA}(S, \Gamma)$ between a and b. □

Next, we deal with arcs sharing two endpoints:

Lemma 6.10. *Let Γ as in Theorem 6.1. If $a, b \in \mathcal{MA}(\Gamma)$ are disjoint arcs with the same endpoints, then $d(a, b) \leq 6$.*

Proof. If the complement of $a \cup b$ is connected, then there is an arc completely disjoint from both, as we are assuming $|\Gamma_0| \geq 5$. Thus, assume $a \cup b$ separate S. If any of the connected components of the complement of $a \cup b$ contains a pair of punctures spanning an edge of Γ, then again there exists an arc completely disjoint from a and b, and we are done.

Thus suppose we are not in any of the above situations. Let e be the edge of Γ which a and b are compatible with. By the hypothesis on Γ, there exists an edge f_1 of Γ disjoint from e; similarly, as $|\Gamma_0| \geq 5$, there exists an edge f_2 which shares at most one endpoint with $e \cup f_1$. We may choose disjoint arcs $c_1, c_2 \in \mathcal{MA}(S, \Gamma)$ such that c_i is compatible with f_i, and $i(a, c_1) = i(b, c_2) = 0$. At this point, there are two cases to consider.

Suppose first that $f_2 \cap e = \emptyset$. If the arcs c_1 and c_2 are completely disjoint, then the sequence a, c_1, c_2, b yields a path of length three in $\mathcal{MA}(S, \Gamma)$. Otherwise, c_1 and c_2 satisfy the hypotheses of Lemma 6.9 and therefore are at distance at most four, which implies the desired result.

If, on the other hand, f_2 shares one endpoint with e, then $d(b, c_2) \leq 4$, by Lemma 6.9. Since $d(c_1, a) = 1$, we are done. □

We are finally ready to prove our first main result:

Proof of Theorem 6.1. If there existed one edge whose complement contains no edges, then we would be able to find a vertex of $\mathcal{MA}(S, \Gamma)$ with no neighbors, as any other arc would share at least one endpoint with it.

For the other implication, suppose that the complement of every edge contains at least one edge. Let a and b be any arcs in $\mathcal{MA}(S, \Gamma)$. Choose an arc b', disjoint from b and with the same endpoints as a. Appealing to Lemmas 6.9 or 6.10 if necessary, we know that $d(b, b') \leq 6$. Now, the tuple $\mathcal{U}(a^{v_1}, b'^{v_2})$ defines a sequence of vertices in $\mathcal{MA}(S, \Gamma)$; by Lemma 6.10, any two consecutive arcs of this sequence are at distance at most 6 in $\mathcal{MA}(S, \Gamma)$, and thus we are done. □

6.2.1 Graphs with few vertices

Before we end this section, we make some comments on the need for a lower bound on $|\Gamma_0|$. First, observe that if $\mathcal{MA}(S, \Gamma)$ is connected then $|\Gamma_0| \geq 4$. Indeed, as was the case in the proof of Theorem 6.1, if this were not the case we would be able to find a vertex of $\mathcal{MA}(S, \Gamma)$ with no neighbors, since any other arc on $\mathcal{MA}(S, \Gamma)$ would share at least one endpoint with it. In other words, we have proved:

Corollary 6.11. *Let Γ be a finite simplicial graph with $|\Gamma_0| \leq 3$. Then $\mathcal{MA}(S, \Gamma)$ is not connected.*

Combining the above lemma with Theorem 6.1, the only outstanding case to consider is $|\Gamma_0| = 4$. We will prove:

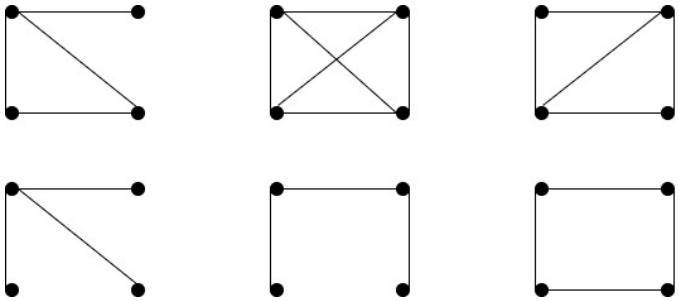


Figure 6.1: Connected graphs with four non-isolated vertices and at least three edges.

Proposition 6.12. *Let Γ be a finite simplicial graph with $|\Gamma_0| = 4$. Then $\mathcal{MA}(S, \Gamma)$ is connected if and only if S has positive genus and Γ_0 has exactly two edges, which are disjoint.*

Proof. Assume first that $\mathcal{MA}(S, \Gamma)$ is connected, and assume that Γ_0 does not satisfy the conclusion of the theorem. Since $|\Gamma_0| = 4$, the graph Γ_0 is one of the graphs depicted in Figure 6.1. For any of them, there are two edges e_1, e_2 of Γ sharing an endpoint and such that any other edge in Γ shares one endpoint with (at least) one of them. If a_1 and a_2 are arcs compatible with e_1 and e_2, respectively, there is no path between a_1 and a_2 in $\mathcal{MA}(S, \Gamma)$.

Therefore, Γ has exactly two edges, which are disjoint. Let p, p' be two marked points of S spanning an edge of Γ. If the genus of S is zero, there are two disjoint arcs a, b on S, with the same endpoints, such that p and p' lie on different components of the complement of $a \cup b$. We claim that there is no path in $\mathcal{MA}(S, \Gamma)$ between a and b. To see this, first observe that the Jordan Curve Theorem implies that a and b are not at distance two. Suppose we had a path $a = c_0, c_1, \ldots, c_{2k} = b$ in $\mathcal{MA}(S, \Gamma)$. Since c_{2i} and c_{2i+2} are at distance 2, then p and p' cannot lie in different components of the complement of $c_{2i} \cup c_{2i+2}$. The same is true for c_{2i} and c_{2i+2k} for all i and k; in particular, it holds for a and b, which is a contradiction.

For the reverse implication, let a and b be two arbitrary vertices of $\mathcal{MA}(S, \Gamma)$. Using a unicorn path argument implies that it suffices to check the case when a and b are disjoint and have the same endpoints. If $a \cup b$ does not separate S, then a and b are at distance two. Otherwise, they are at distance four in $\mathcal{MA}(S, \Gamma)$, see Figure 6.2.

□

6.3 Gromov hyperbolicity of $\mathcal{MA}(\Gamma)$

In this section we prove Theorem 6.2. We commence by passing to a related complex that is easier to analyze. Let $\mathcal{A}(S, \Gamma)$ be the complex with the same vertex set as $\mathcal{MA}(S, \Gamma)$, but where simplices are spanned by disjoint (but

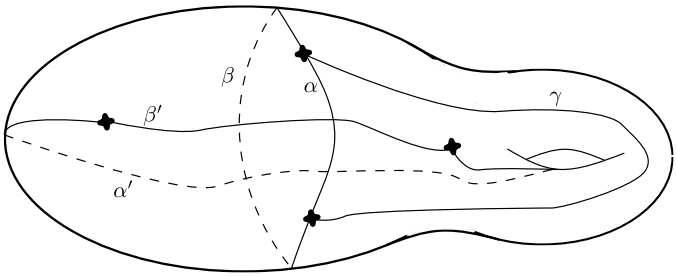


Figure 6.2: The arcs a, b are connected via the path a, a', c, b', b, on a surface of genus one with four marked points. The general situation is analogous.

not necessarily completely disjoint) arcs; in other words, we are allowing arcs intersect in their endpoints. A direct consequence of Lemmas 6.9 and 6.10 is the following:

Lemma 6.13. *If $\mathcal{MA}(S,\Gamma)$ is connected, then the inclusion map $\mathcal{MA}(S,\Gamma) \hookrightarrow \mathcal{A}(S,\Gamma)$ is a quasi-isometry.*

In light of the above lemma, it suffices to establish the hyperbolicity of $\mathcal{A}(S,\Gamma)$. The main tool will be following result of Masur–Schleimer[66]; here, we use a similar form, as stated by Bowditch in [19]. Given a subset C of a metric space X, denote by $N_k(C)$ the k-neighborhood of C in X, i.e., the set of points of X at distance at most k from C.

Lemma 6.14 (Guessing geodesics lemma)**.** *Let G be a connected graph, and write d for its usual distance function. Let $M > 0$ be a constant, and suppose that for any vertices $x, y \in G$ there is a connected subgraph $A(x,y)$, with $x, y \in A(x,y)$, such that:*

1. *If $d(x,y) \leq 1$, then $\operatorname{diam}(A(x,y)) \leq M$.*
2. *For any vertices $x, y, z \in G$, $A(x,y) \subset N_M(A(x,z) \cup A(z,y))$.*

Then G is hyperbolic, with hyperbolicity constant that depends only on M.

In order to apply Lemma 6.14, we use the unicorn path machinery of Hensel-Przytycki-Webb [40]. As mentioned in the introduction, the main hurdle in this direction is that the unicorn path between two vertices of $\mathcal{A}(S,\Gamma)$ need not be contained in $\mathcal{A}(S,\Gamma)$.

6.3.1 A family of subpaths in $\mathcal{A}(S,\Gamma)$

From now on, we will assume that Γ_0 is a finite, connected simplicial graph that contains a triangle Δ, and such that $\mathcal{MA}(S,\Gamma)$ is connected. Denote by e_i the three edges of Δ. To any arc $a \in \mathcal{A}(S,\Gamma)$ we associate, once and for all, three pairwise disjoint arcs a_i, called the *triangle arcs* associated to a, such that:

- a_i is compatible with e_i;
- $a_1 \cup a_2 \cup a_3$ bound a disk on S;
- $i(a, a_i) = 0$ for $i = 1, 2, 3$.

In general, we are going to use subindices to denote the triangle arcs associated to a given arc. In the particular case when a is already compatible with some e_i, then we simply set $a_i := a$. Note that the existence of the arcs a_i is easily verified using the Change of Coordinates Principle [32, Section 1.3].

Now, given two arcs $a, b \in \mathcal{A}(S, \Gamma)$, we define $A'(a, b)$ to be the set of all those unicorn arcs defined by a_i and b_j that are elements of $\mathcal{A}(S, \Gamma)$. Then, we define

$$A(a, b) := N_1(A'(a, b)).$$

Clearly, $A(a, b)$ is connected and $a, b \in A(a, b)$. Now, we are left to prove that both conditions in Lemma 6.14 are satisfied. In what follows, we will write v_1, v_2, v_3 for the vertices of the triangle Δ, where the edge e_i has vertices v_i and v_{i+1}.

Lemma 6.15. *For every $a, b, c \in \mathcal{A}(S, \Gamma)$,*

$$A(a, b) \subset N_2(A(a, c) \cup A(c, b)).$$

Proof. It suffices to prove that $A'(a, b) \subset N_1(A(a, c) \cup A(c, b))$. To this end, let $d \in A'(a, b)$; without loss of generality we may assume that $d \in \mathcal{U}(a_i^{v_1}, b_j^{v_2})$. By Lemma 6.7, there is a $d^* \in \mathcal{U}(a_i^{v_1}, c_3^{v_3}) \cup \mathcal{U}(c_3^{v_3}, b_j^{v_2})$ such that $d(d, d^*) = 1$. Observe that $d^* \in A(a, c) \cup A(c, b)$, which gives the result. □

Lemma 6.16. *Let $a, b \in \mathcal{A}(S, \Gamma)$ with $d(a, b) = 1$. Then*

$$\operatorname{diam}(A(a, b)) \leq 2 \operatorname{diam}(\Gamma_0) + 9.$$

Proof. We will write $k = \operatorname{diam}(\Gamma_0)$, for compactness. By definition, it suffices to see that

$$\operatorname{diam}(A'(a, b)) \leq 2k + 7;$$

for this, we will show that $d(c, \{a, b\}) \leq k + 3$ for any $c \in A'(a, b)$. Let $c \in A'(a, b)$, and recall that a_i and b_j are the triangle arcs of a and b, respectively. Up to renaming, we may assume that $c \in \mathcal{U}(a_m^{v_1}, b_n^{v_2})$. Let $\partial a = \{x_1, x_2\}$ and $\partial b = \{y_1, y_2\}$. Choose a simple path

$$X : w_0 = v_1, w_1 = v_2, w_2, \ldots, w_{k-1}, w_k = x_1$$

in Γ; however, if $w_l \in \{y_1, y_2\}$ for some $l \leq k$, then we truncate X at w_l.

Next, the triangle arcs a_j bound a topological disk on S, and hence we may choose pairwise disjoint arcs α_i, for $i = 1, \ldots, k-1$, such that α_i is compatible with the edge spanned by w_i and w_{i+1}, and

$$i(\alpha_i, a) = i(\alpha_i, a_j) = 0,$$

for $j = 1, 2, 3$. Analogously, we select pairwise disjoint arcs $\beta_1, \ldots, \beta_{k-1}$ such that β_i is compatible with the edge spanned by w_i and w_{i+1}, and

$$i(\beta_i, b) = i(\beta_i, b_j) = 0,$$

for $j = 1, 2, 3$. Finally, we set $\alpha_0 = a_m$ and $\beta_0 = b_n$.

Claim. Either $d(c, \{a, b\}) \leq k + 1$, or there exists $c' \in \mathcal{U}(\alpha_{k-1}^{w_{k-1}}, \beta_{k-1}^{w_k}) \cup \mathcal{U}(\alpha_{k-1}^{w_k}, \beta_{k-1}^{w_{k-1}})$ with $d(c, c') \leq k$.

We accept the claim as true for the time being and continue with our argument; we will establish the claim at the end of the proof. If we are in the first case of the claim, then we are done; hence, assume that we are in the second case. We distinguish two subcases:

Case 1: $c' \in \mathcal{U}(\alpha_{k-1}^{w_{k-1}}, \beta_{k-1}^{w_k})$. We apply Lemma 6.7, obtaining $c'' \in \mathcal{A}(S, \Gamma)$, with $d(c', c'') \leq 1$, such that

$$c'' \in \mathcal{U}(\alpha_{k-1}^{w_{k-1}}, a^{x_2}) \cup \mathcal{U}(a^{x_2}, \beta_{k-1}^{w_k}).$$

Again, there are two cases to consider:

Subcase (1a): Suppose first that $c'' \in \mathcal{U}(\alpha_{k-1}^{w_{k-1}}, a^{x_2})$, and note that $\mathcal{U}(\alpha_{k-1}^{w_{k-1}}, a^{x_2}) = \{a, \alpha_{k-1}\}$. As $d(c, c') \leq k$, $d(c', c'') \leq 1$ and $d(c'', a) \leq 1$, then $d(c, \{a, b\}) \leq k + 2$, and we are done.

Subcase (1b): Assume now that $c'' \in \mathcal{U}(a^{x_2}, \beta_{k-1}^{w_k})$. Lemma 6.7 yields the existence of an arc $c''' \in \mathcal{A}(S, \Gamma)$, with $d(c'', c''') \leq 1$, such that $c''' \in \mathcal{U}(a^{x_2}, b^v) \cup \mathcal{U}(b^v, \beta_{k-1}^{w_k})$. Note that $\mathcal{U}(a^{x_2}, b^v) = \{a, b\}$ and $\mathcal{U}(b^v, \beta_{k-1}^{w_k}) = \{b, \beta_{k-1}\}$. In either situation, $d(c''', \{a, b\}) \leq 1$ and hence $d(c, \{a, b\}) \leq k+3$, as desired.

Case 2: $c' \in \mathcal{U}(\alpha_{k-1}^{w_k}, \beta_{k-1}^{w_{k-1}})$. Again, by Lemma 6.7 there is $c'' \in \mathcal{A}(S, \Gamma)$, with $d(c', c'') \leq 1$, such that

$$c'' \in \mathcal{U}(\alpha_{k-1}^{w_k}, a^{w_k}) \cup \mathcal{U}(a^{w_k}, \beta_{k-1}^{w_{k-1}}),$$

and we proceed as in the previous case.

At this point, all that remains is to prove the above Claim. To this end, we will apply an inductive argument based on the structure of Γ. First, since $c \in \mathcal{U}(a_m^{v_1}, b_n^{v_2})$, Lemma 6.7 implies that there is

$$c_1 \in \mathcal{U}(\alpha_0^{w_0}, \alpha_1^{w_2}) \cup \mathcal{U}(\alpha_1^{w_2}, \beta_0^{w_1})$$

with $d(c, c_1) \leq 1$. If $c_1 \in \mathcal{U}(\alpha_0^{w_0}, \alpha_1^{w_2})$ then $d(c_1, a) \leq 1$, as $\mathcal{U}(\alpha_0^{w_0}, \alpha_1^{w_2}) = \{\alpha_0, \alpha_1\}$; hence $d(c, \{a, b\}) \leq 2$ and the result follows. If, on the contrary, $c_1 \in \mathcal{U}(\alpha_1^{w_2}, \beta_0^{w_1})$, we continue this process inductively, and obtain a sequence of arcs γ_i such that $d(c, \gamma_i) \leq i$ and

$$\gamma_i \in \mathcal{U}(\beta_i^{w_{i+1}}, \alpha_{i-1}^{w_i}) \cup \mathcal{U}(\alpha_i^{w_{i+1}}, \beta_{i-1}^{w_i}).$$

We want to find a suitable arc γ_{i+1}; there are two possible cases. Suppose first that $\gamma_i \in \mathcal{U}(\beta_i^{w_{i+1}}, \alpha_{i-1}^{w_i})$. Then, there exists

$$\gamma_{i+1} \in \mathcal{U}(\beta_i^{w_{i+1}}, \alpha_{i+1}^{w_{i+2}}) \cup \mathcal{U}(\alpha_{i+1}^{w_{i+2}}, \alpha_{i-1}^{w_i})$$

with $d(\gamma_i, \gamma_{i+1}) \leq 1$. Once again, if $\gamma_{i+1} \in \mathcal{U}(\alpha_{i+1}^{w_{i+2}}, \alpha_{i-1}^{w_i})$, then

$$d(c, \{a, b\}) \leq d(c, \gamma_i) + d(\gamma_i, \gamma_{i+1}) + d(\gamma_{i+1}, a) \leq i + 2,$$

as $d(\alpha_{i+1}, a) = d(\alpha_{i-1}, a) = 1$. Otherwise, $\gamma_{i+1} \in \mathcal{U}(\beta_i^{w_{i+1}}, \alpha_{i+1}^{w_{i+2}})$ and the induction process continues.

The case where $\gamma_i \in \mathcal{U}(\alpha_i^{w_{i+1}}, \beta_{i-1}^{w_i})$ is analogous, and follows using the same argument as in the previous case, using b instead of a.

After $k-1$ steps, we have that either $d(c, \{a, b\}) \leq k$, or else there is γ_{k-1} with $d(c, \gamma_{k-1}) \leq k-1$ and such that

$$\gamma_{k-1} \in \mathcal{U}(\beta_{k-1}^{w_k}, \alpha_{k-2}^{w_{k-1}}) \cup \mathcal{U}(\alpha_{k-1}^{w_k}, \beta_{k-2}^{w_{k-1}}).$$

If γ_{k-1} is in the first term of the union, there is

$$c' \in \mathcal{U}(\beta_{k-1}^{w_k}, \alpha_{k-1}^{w_{k-1}}) \cup \mathcal{U}(\alpha_{k-1}^{w_{k-1}}, \alpha_{k-2}^{w_{k-1}})$$

with $i(c', \gamma_{k-1}) = 0$. If $c' \in \mathcal{U}(\alpha_{k-1}^{w_{k-1}}, \alpha_{k-2}^{w_{k-1}})$, then $d(c, a) \leq k+1$. Otherwise, $c' \in \mathcal{U}(\beta_{k-1}^{w_k}, \alpha_{k-1}^{w_{k-1}})$ with $d(c, c') \leq k$. The second case is analogous, and produces either $d(c, b) \leq k+1$ or $c' \in \mathcal{U}(\alpha_{k-1}^{w_k}, \beta_{k-1}^{w_{k-1}})$, proving the Claim and thus the Lemma also. □

We are finally in a position for proving our result about the hyperbolicity of matching arc complexes:

Proof of Theorem 6.2. The result follows from the combination of Lemmas 6.14, 6.15 and 6.16 □

Conclusions

The main results proven in this thesis are:

Theorem A. *Let S, S' be two orientable finite-type surfaces of genus $g \geq 4$ and $g' \leq 6 \cdot 2^{g-4}$. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a non-trivial homomorphism, then φ is induced by a multi-embedding.*

Theorem B. *Let S, S' be two orientable finite-type surfaces and let $g \geq 3$ be the genus of S. If $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is a multitwist-preserving map, then φ is induced by a multi-embedding.*

Theorem C. *Let S, S' be two orientable finite-type surfaces of genus $g \geq 4$ and $g' \leq 6 \cdot 2^{g-4}$. Any homomorphism $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ admits a lift $\overline{\varphi} : \mathrm{Diff}_c^+(S) \to \mathrm{Diff}_c^+(S')$ to the diffeomorphism groups.*

Theorem D. *Let S be an orientable surface. If $m_A, m_B \in \mathrm{Map}(S)$ are two multitwists that satisfy $m_A m_B m_A = m_B m_A m_B$, then*

$$m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k} m_C$$
$$m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k} m_C,$$

where $n_i \in \{1, -1\}$, the curves $a_1, \dots, a_k, b_1, \dots, b_k$ are pairwise disjoint except for $i(a_i, b_i) = 1$ and m_C is a common multitwist fixing every curve a_i and b_j.

Theorem E. *Suppose that $g \geq 4$ and that $g' \leq 3 \cdot 2^{g-3}$, and let $r, r' \geq 0$. Then every non-constant holomorphic map $F : \mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g',r'})$ is a forgetful map. In particular, if such a map exists, then $g' = g$ and $r' \leq r$.*

Theorem F. *Let M and M' be two orientable surfaces with no boundary, and suppose M has infinite genus and has no planar ends. If G is a mostly pure large subgroup of $\mathrm{Map}(M)$ and G' is a large subgroup of $\mathrm{Map}(M')$, then every continuous epimorphism $G \to G'$ is induced by a homeomorphism $M \to M'$.*

Conclusiones

En este trabajo se han demostrado los siguientes resultados:

Teorema. *Sean S, S' dos superficies orientables de tipo finito con géneros $g \geq 4$ y $g' \leq 6 \cdot 2^{g-4}$. Todo homomorfismo no trivial $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ es inducido por un multi-embebimiento.*

Teorema. *Sean S, S' dos superficies orientables de tipo finito y sea $g \geq 3$ el género de S. Si $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ es un homomorfismo que preserva multigiros, entonces φ es inducido por un multi-embebimiento.*

Teorema. *Sean S, S' dos superficies orientables de tipo finito con géneros $g \geq 4$ y $g' \leq 6 \cdot 2^{g-4}$. Todo homomorfismo $\varphi : \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ admite un levantamiento $\overline{\varphi} : \mathrm{Diff}_c^+(S) \to \mathrm{Diff}_c^+(S')$ al grupo de difeomorfismos.*

Teorema. *Sea S una superficie orientable. Si $m_A, m_B \in \mathrm{Map}(S)$ son dos multigiros que satisfacen $m_A m_B m_A = m_B m_A m_B$, entonces*

$$m_A = d_{a_1}^{n_1} \dots d_{a_k}^{n_k} m_C$$
$$m_B = d_{b_1}^{n_1} \dots d_{b_k}^{n_k} m_C,$$

donde $n_i \in \{1, -1\}$, las curvas $a_1, \dots, a_k, b_1, \dots, b_k$ son disjuntas dos a dos excepto por $i(a_i, b_i) = 1$ y m_C es un multigiro común que fija todas las curvas a_i y b_j.

Teorema. *Sean g, g', r y r' enteros tal que $g \geq 4$, $g' \leq 3 \cdot 2^{g-3}$ y $r, r' \geq 0$. Entonces toda función holomorfa no constante $F : \mathcal{M}(S_{g,r}) \to \mathcal{M}(S_{g',r'})$ es un mapa de olvido. En particular, si existe una tal función, entonces $g = g'$ y $r' \leq r$.*

Teorema. *Sean M y M' dos superficies orientables sin frontera, y supón que M es de género infinito sin finales planos. Entonces, si G es un subgrupo principalmente puro de $\mathrm{Map}(M)$ y G' es un subgrupo grande de $\mathrm{Map}(M')$, entonces todo epimorfismo continuo $G \to G'$ está inducido por un homeomorfismo $M \to M'$.*

Bibliography

[1] S. Afton et al. Nielsen realization for infinite-type surfaces. *Proceedings of the American Mathematical Society* vol. 149, no. 4 (2021).

[2] L. V. Ahlfors. Some Remarks on Teichmüller's Space of Riemann Surfaces. *Annals of Mathematics* vol. 74, no. 1 (1961).

[3] AimPL: Surfaces of infinite type, available at http://aimpl.org/genusinfinity.

[4] S. Antonakoudis, J. Aramayona, and J. Souto. Holomorphic maps between moduli spaces. *Annales de l'Institut Fourier* vol. 68, no. 1 (2018).

[5] T. Aougab, P. Patel, and N. G. Vlamis. Isometry groups of infinite-genus hyperbolic surfaces. *Mathematische Annalen* vol. 381, no. 1 (2021).

[6] J. Aramayona, C. J. Leininger, and J. Souto. Injections of mapping class groups. *Geometry & Topology* vol. 13, no. 5 (2009).

[7] J. Aramayona and C. J. Leininger. Exhausting curve complexes by finite rigid sets. *Pacific Journal of Mathematics* vol. 282, no. 2 (2016).

[8] J. Aramayona and C. J. Leininger. Finite rigid sets in curve complexes. *Journal of Topology and Analysis* vol. 5, no. 2 (2013).

[9] J. Aramayona, C. J. Leininger, and A. McLeay. Big Mapping Class Groups and the Co-Hopfian Property. *Michigan Mathematical Journal* vol. 74, no. 2 (2024).

[10] J. Aramayona, P. Patel, and N. G. Vlamis. The First Integral Cohomology of Pure Mapping Class Groups. *International Mathematics Research Notices* vol. 2020, no. 22 (2020).

[11] J. Aramayona and J. Souto. Homomorphisms between mapping class groups. *Geometry & Topology* vol. 16, no. 4 (2013).

[12] J. Aramayona and J. Souto. Rigidity phenomena in the mapping class group. *Handbook of Teichmüller theory. Vol. VI.* vol. 27. IRMA Lect. Math. Theor. Phys. Eur. Math. Soc., Zürich, 2016.

[13] J. Aramayona and N. G. Vlamis. Big Mapping Class Groups: An Overview. *In the Tradition of Thurston: Geometry and Topology*. Ed. by K. Ohshika and A. Papadopoulos. Springer International Publishing, 2020.

[14] W. Ballmann. Lectures on Kähler Manifolds. 1st ed. vol. 2. ESI Lectures in Mathematics and Physics. EMS Press, 2006.

[15] J. Bavard, S. Dowdall, and K. Rafi. Isomorphisms Between Big Mapping Class Groups. *International Mathematics Research Notices* vol. 2020, no. 10 (2020).

[16] R. W. Bell and D. Margalit. Braid groups and the co-Hopfian property. *Journal of Algebra* vol. 303, no. 1 (2006).

[17] A. J. Berrick, V. Gebhardt, and L. Paris. Finite index subgroups of mapping class groups. *Proceedings of the London Mathematical Society* vol. 108, no. 3 (2014).

[18] J. S. Birman and H. M. Hilden. On Isotopies of Homeomorphisms of Riemann Surfaces. *Annals of Mathematics* vol. 97, no. 3 (1973).

[19] B. Bowditch. Uniform hyperbolicity of the curve graphs. *Pacific Journal of Mathematics* vol. 269, no. 2 (2014).

[20] T. E. Brendle and D. Margalit. Commensurations of the Johnson kernel. *Geometry & Topology* vol. 8, no. 3 (2004).

[21] M. R. Bridson. Semisimple actions of mapping class groups on CAT(0) spaces. *Geometry of Riemann surfaces*. vol. 368. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2010.

[22] M. R. Bridson and A. Haefliger. Metric spaces of non-positive curvature. Grundlehren der mathematischen Wissenschaften no. 319. Springer, 1999.

[23] J. Brock, R. Canary, and Y. Minsky. The classification of Kleinian surface groups, II: The Ending Lamination Conjecture. *Annals of Mathematics* vol. 176, no. 3 (2012).

[24] P. Buser. Geometry and Spectra of Compact Riemann Surfaces. Birkhäuser Boston, 2010.

[25] K.-U. Bux et al. The braided Thompson's groups are of type F_∞. *Journal für die reine und angewandte Mathematik* vol. 2016, no. 718 (2016).

[26] J. W. Cannon and G. R. Conner. The combinatorial structure of the Hawaiian earring group. *Topology and its Applications* vol. 106, no. 3 (2000).

[27] L. Chen and J. Lanier. Constraining mapping class group homomorphisms using finite subgroups. *Geometriae Dedicata* vol. 218, no. 5 (2024).

[28] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Publications mathématiques de l'IHÉS* vol. 36, no. 1 (1969).

[29] G. Domat and R. Dickmann. Big pure mapping class groups are never perfect. *Mathematical Research Letters* vol. 29, no. 3 (2022).

[30] M. G. Durham, F. Fanoni, and N. G. Vlamis. Graphs of curves on infinite-type surfaces with mapping class group actions. *Annales de l'Institut Fourier* vol. 68, no. 6 (2018).

[31] J. Eells and J. H. Sampson. Harmonic Mappings of Riemannian Manifolds. *American Journal of Mathematics* vol. 86, no. 1 (1964).

[32] B. Farb and D. Margalit. A primer on mapping class groups. vol. 49. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 2012.

[33] R. E. Greene and H. Wu. C_∞ convex functions and manifolds of positive curvature. *Acta Mathematica* vol. 137 (1976).

[34] E. K. Grossman. On the Residual Finiteness of Certain Mapping Class Groups. *Journal of the London Mathematical Society* vol. s2-9, no. 1 (1974).

[35] H. Hamidi-Tehrani. Groups generated by positive multi-twists and the fake lantern problem. *Geometric Topology* vol. 2 (2002).

[36] J. Harer. The second homology group of the mapping class group of an orientable surface. *Inventiones mathematicae* vol. 72, no. 2 (1983).

[37] J. L. Harer. The virtual cohomological dimension of the mapping class group of an orientable surface. *Inventiones Mathematicae* vol. 84, no. 1 (1986).

[38] W. J. Harvey and M. Korkmaz. Homomorphisms From Mapping Class Groups. *Bulletin of the London Mathematical Society* vol. 37, no. 2 (2005).

[39] A. Hatcher. The Kirby torus trick for surfaces. (2022). arXiv: `1312.3518[math]`.

[40] S. Hensel, P. Przytycki, and R. C. H. Webb. 1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs. *Journal of the European Mathematical Society* vol. 17, no. 4 (2015).

[41] J. H. Hernández, I. Morales, and F. Valdez. The Alexander Method for Infinite-Type Surfaces. *Michigan Mathematical Journal* vol. 68, no. 4 (2019).

[42] J. Hernández Hernández, C. J. Leininger, and R. Maungchang. Finite rigid subgraphs of pants graphs. *Geometriae Dedicata* vol. 212, no. 1 (2021).

[43] J. Huang and B. Tshishiku. Finite rigid sets and the separating curve complex. *Topology and its Applications* vol. 312 (2022).

[44] S. P. Humphries. Generators for the mapping class group. *Topology of Low-Dimensional Manifolds*. Lecture Notes in Mathematics. Springer, 1979.

[45] S. Hurtado. Continuity of discrete homomorphisms of diffeomorphism groups. *Geometry & Topology* vol. 19, no. 4 (2015).

[46] Y. Imayoshi and M. Taniguchi. An Introduction to Teichmüller Spaces. Springer Science & Business Media, 2012.

[47] E. Irmak. Complexes of nonseparating curves and mapping class groups. *Michigan Mathematical Journal* vol. 54, no. 1 (2006).

[48] E. Irmak. Edge-preserving maps of the nonseparating curve graphs, curve graphs and rectangle preserving maps of the Hatcher–Thurston graphs. *Journal of Knot Theory and Its Ramifications* vol. 29, no. 11 (2020).

[49] E. Irmak and J. D. Mccarthy. Injective simplicial maps of the arc complex. *Turkish Journal of Mathematics* (2010).

[50] N. V. Ivanov. Automorphisms of Teichmüller modular groups. *Topology and Geometry — Rohlin Seminar*. Springer, 1988.

[51] N. V. Ivanov. Automorphism of complexes of curves and of Teichmüller spaces. *International Mathematics Research Notices* vol. 1997, no. 14 (1997).

[52] N. V. Ivanov. Subgroups of Teichmüller modular groups. vol. 115. Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 1992.

[53] N. V. Ivanov and J. D. McCarthy. On injective homomorphisms between Teichmüller modular groups I. *Inventiones mathematicae* vol. 135, no. 2 (1999).

[54] L. Ji and S. A. Wolpert. A cofinite universal space for proper actions for mapping class groups. In the tradition of Ahlfors–Bers, V. Contemp. Math. vol. 510, Amer. Math. Soc. (2010). Ed. by M. Bonk et al.

[55] S. P. Kerckhoff. The Nielsen Realization Problem. *Annals of Mathematics* vol. 117, no. 2 (1983).

[56] S. Kobayashi. Hyperbolic Complex Spaces. Springer Science & Business Media, 1998.

[57] M. Korkmaz. On endomorphisms of surface mapping class groups. *Topology* vol. 40, no. 3 (2001).

[58] R. Lamy. Roots of hyperelliptic involutions and braid groups modulo their center inside mapping class groups. (2025). arXiv: 2503.09552[math].

[59] H. B. Lawson. Lectures on Minimal Submanifolds. Vol. I, Math. Lect. Ser. 9, Publish or Perish, 1980.

[60] C. J. Leininger. On groups generated by two positive multi-twists: Teichmueller curves and Lehmer's number. *Geometry & Topology* vol. 8, no. 3 (2004).

[61] W. B. R. Lickorish. A finite set of generators for the homeotopy group of a 2-manifold. *Mathematical Proceedings of the Cambridge Philosophical Society* vol. 60, no. 4 (1964).

[62] C. Maclachlan. Abelian Groups of Automorphisms of Compact Riemann Surfaces. *Proceedings of the London Mathematical Society* vol. s3-15, no. 1 (1965).

[63] K. Mann. Homomorphisms between diffeomorphism groups. *Ergodic Theory and Dynamical Systems* vol. 35, no. 1 (2015).

[64] D. Margalit. Automorphisms of the pants complex. *Duke Mathematical Journal* vol. 121, no. 3 (2004).

[65] D. Margalit and S. Schleimer. Dehn twists have roots. *Geometry & Topology* vol. 13, no. 3 (2009).

[66] H. Masur and S. Schleimer. The geometry of the disk complex. *Journal of the American Mathematical Society* vol. 26, no. 1 (2012).

[67] R. Maungchang. Finite rigid subgraphs of the pants graphs of punctured spheres. *Topology and its Applications* vol. 237 (2018).

[68] J. McCarthy and A. Papadopoulos. Simplicial actions of mapping class groups. *Handbook of Teichmüller Theory, Volume III*. Ed. by A. Papadopoulos. European Mathematical Society Publishing House, 2012.

[69] C. McMullen. From dynamics on surfaces to rational points on curves. *Bulletin of the American Mathematical Society* vol. 37, no. 2 (1999).

[70] C. T. McMullen. The Moduli Space of Riemann Surfaces is Kahler Hyperbolic. *The Annals of Mathematics* vol. 151, no. 1 (2000).

[71] D. Mumford. Abelian quotients of the Teichmüller modular group. *Journal d'Analyse Mathématique* vol. 18, no. 1 (1967).

[72] S. Nag. The Complex Analytic Theory of Teichmuller Spaces. Wiley, 1988.

[73] S. Nakajima. On Abelian Automorphism Groups of Algebraic Curves. *Journal of the London Mathematical Society* vol. s2-36, no. 1 (1987).

[74] J. Powell. Two Theorems on the Mapping Class Group of a Surface. *Proceedings of the American Mathematical Society* vol. 68, no. 3 (1978).

[75] K. Rajeevsarathy and P. Vaidyanathan. Roots of Dehn twists about multicurves. *Glasgow Mathematical Journal* vol. 60, no. 3 (2018).

[76] I. Richards. On the classification of noncompact surfaces. *Transactions of the American Mathematical Society* vol. 106 (1963).

[77] H. L. Royden. Report on the Teichmuller Metric. *Proceedings of the National Academy of Sciences of the United States of America* vol. 65, no. 3 (1970).

[78] E. Shinkle. Finite rigid sets in arc complexes. *Algebraic & Geometric Topology* vol. 20, no. 6 (2020).

[79] E. Shinkle. Finite Rigid Sets in Flip Graphs (2021). arXiv: 2010.05363.

[80] B. Szepietowski. Embedding the Braid Group in Mapping Class Groups. *Publicacions Matemàtiques* vol. 54, no. 2 (2010).

[81] W. P. Thurston. The Geometry and Topology of Three-manifolds. Princeton University, 1979.

[82] A. J. Tromba. On a natural algebraic affine connection on the space of almost complex structures and the curvature of teichmüller space with respect to its Weil-Petersson metric. *manuscripta mathematica* vol. 56, no. 4 (1986).

[83] N. G. Vlamis. Homeomorphism Groups of Self-Similar 2-Manifolds. *In the Tradition of Thurston III: Geometry and Dynamics*. Ed. by K. Ohshika and A. Papadopoulos. Springer International Publishing, 2024.

[84] A. Weil. Modules des surfaces de Riemann. *Séminaire Bourbaki* vol. 4 (1956).

[85] S. A. Wolpert. Geometry of the Weil-Peterson completion of Teichmüller space. *Surveys in Differential Geometry* vol. 8, no. 1 (2003).

[86] S. Wolpert. Noncompleteness of the Weil-Petersson metric for Teichmüller space. *Pacific Journal of Mathematics* vol. 61, no. 2 (1975).

[87] S. A. Wolpert. Chern forms and the Riemann tensor for the moduli space of curves. *Inventiones mathematicae* vol. 85, no. 1 (1986).

[88] S. A. Wolpert. Geodesic length functions and the Nielsen problem. *Journal of Differential Geometry* vol. 25, no. 2 (1987).

[89] S. A. Wolpert. The Weil–Petersson metric geometry. *Handbook of Teichmüller Theory* vol. 2 (2009).

[90] S. A. Wolpert. Weil-Petersson perspectives. *Proceedings of Symposia in Pure Mathematics* (2006).

[91] S. Yamada. On the Geometry of Weil-Petersson Completion of Teichmüller Spaces. *Mathematical Research Letters* vol. 11, no. 3 (2004).

[92] M. C. B. Zaremsky. Geometric structures related to the braided Thompson groups. (2018). arXiv: `1803.02717[math]`.