Universidad Autónoma de Madrid

Moduli of Higher Connections and Holomorphic 2-Bundles

Móduli de conexiones superiores y 2-fibrados holomorfos

por

Roberto Téllez Domínguez

Dirigido por

Luis Álvarez Cónsul y Mario García Fernández

Tesis para optar al grado de
Doctor en Matemáticas

Facultad de Ciencias
Departamento de Matemáticas

Febrero 2025

Abstract

In this thesis we initiate the program of studying the geometry of moduli spaces associated to principal 2-bundles. We consider Lie 2-groups $\mathcal{G}$ that arise from a Lie group with an Ad-invariant, symmetric, bilinear form satisfying an integrality condition on its Lie algebra. In this setting, we construct derived moduli stacks of flat $\mathcal{G}$-connections over a smooth manifold, holomorphic $\mathcal{G}$-bundles over a complex manifold and holomorphic $\mathcal{G}$-bundles with holomorphic connective structure over a complex manifold. We introduce dilaton derived moduli and use them to obtain canonical shifted symplectic structures on these derived stacks, which are naturally identified with the derived critical locus of the heterotic superpotential. For this, we relate holomorphic $\mathcal{G}$-bundles with holomorphic connective structures to solutions of the Hull-Strominger system, which models supersymmetric configurations in string theory. Our results follow from a thorough study of higher connections on principal 2-bundles, unifying previous approaches in terms of trivializations of Chern-Simons 2-gerbes, adjusted connections, and splittings of Courant algebroids by introducing a new notion of Maurer-Cartan form on a Lie 2-group.

Resumen

En esta tesis iniciamos el programa de estudiar la geometría de espacios de móduli asociados a 2-fibrados principales. Consideramos 2-grupos de Lie $\mathcal{G}$ que se obtienen de un grupo de Lie junto con una forma bilineal, simétrica, Ad-invariante y satisfaciendo una condición de integralidad en su álgebra de Lie. En este contexto, construimos stacks derivados de $\mathcal{G}$-conexiones planas sobre una variedad diferenciable, $\mathcal{G}$-fibrados holomorfos sobre una variedad compleja y $\mathcal{G}$-fibrados holomorfos con estructura conectiva holomorfa sobre una variedad compleja. Introducimos modulis derivados de dilatones y los usamos para obtener formas simplécticas desplazadas canónicas en estos stacks derivados, que identificamos de manera natural con el locus crítico derivado del superpotencial heterótico. Para ello, establecemos una relación entre $\mathcal{G}$-fibrados holomorfos con estructura conectiva holomorfa y soluciones al sistema de Hull-Strominger, que modeliza configuraciones supersimétricas en teoría de cuerdas. Nuestros resultados se siguen de un cuidadoso estudio de la teoría de conexiones superiores en 2-fibrados principales, unificando enfoques anteriores en términos de trivializaciones de 2-gerbes de Chern-Simons, conexiones ajustadas y escisiones de algebroides de Courant gracias a la introducción de una nueva noción de forma de Maurer-Cartan en un 2-grupo de Lie.

Acknowledgements

Como toda ardua tarea, la redacción de esta tesis ha sido posible únicamente gracias al trabajo y apoyo de muchas personas. De entre todas ellas, han sido Luis Álvarez Cónsul y Mario García Fernández quienes han contribuido de forma más directa, prestando su ayuda por encima de las tareas convencionalmente asociadas a la figura del director de tesis. Concretamente, quiero expresar mi gratitud por haberme ofrecido más preguntas que respuestas en nuestras largas sesiones, por haberme puesto en contacto con mucha más gente de la que aprender el oficio y por haber depositado grandes dosis de confianza en mí, quizás en ocasiones incluso mayores de lo que había demostrado merecer.

Many of the ideas in this work have been influenced by discussions during my research stays and visits at other universities. In this respect, I am very grateful for the insightful comments and kind hospitality of Miquel Cueca and Chenchang Zhu, who taught me a lot about higher Lie groups and showed me around the historical city of Göttingen, Christian Sämann, with whom I had a very fruitful collaboration at Heriot-Watt University, and Markus Upmeier, who invited me to Aberdeen University and helped me understand the String group.

In these years I have also benefited from conversations with many experts, who have made me discover the work of others, as well as understand my own in a deeper level. Among these, I would like to give special thanks to Severin Bunk, David Carchedi, Domenico Fiorenza, Marco Gualtieri, Pedram Hekmati, Ján Pulmann, Carlos Shahbazi, Richard Szabo, Fridrich Valach and Konrad Waldorf. I also want to acknowledge how much I learned through the activities organized by the geometry group at ICMAT, and in particular by Emilio Franco Gómez, Óscar García Prada, Tomás Gómez de Quiroga and Ángel González Prieto. Likewise, consulting the nLab has become a routine in my research and I want to express my gratitude to Urs Schreiber and to all editors that contribute to this vast project.

It has also been a pleasure both to share ideas and to enjoy some free time with my peers from other universities: Ilias, Kalin, Stefano, Dominik, Gianni, Mehran, Alejandro, Mateo, Oscar, Paula, Daniel, Francis, Lennart, Joshua, Jaime, Filip, Erfan. ¡Incluyendo la aparición sorpresa de Santi en Nueva York! De igual modo, no puedo olvidarme de las discusiones (no siempre matemáticas), los memes (casi siempre matemáticos) y las aventuras en ciudades extranjeras que he compartido con mis compañeros locales: Guille B., Guille G., Alfonso, Jesús, Bilson, Arpan, Mathieu, King, Federico, Andrés, Alfredo, Diego, Alejandro, Miguel. Reservo una línea especial para Raúl y Andoni, por los turbulentos momentos vividos juntos.

Otro tipo de ayuda, imprescindible, es la económica. Quiero agradecer a la Agencia Estatal de Investigación, dependiente del Ministerio de Ciencia de España, por financiar este trabajo, a la par que dejo constancia de que para fomentar el progreso y el desarrollo es absolutamente indispensable que mejoren los salarios y las condiciones de estabilidad y conciliación de los investigadores en los niveles profesionales iniciales. Quiero señalar también como una aportación crucial, y menos prosaica, la formación que me han dado todos los profesores de mi vida, especialmente los de matemáticas, lenguas y filosofía. Pienso sinceramente que las clases y la confianza de Antonio Molano me han permitido crecer como matemático mucho después de abandonar el instituto, y que los conceptos más abstractos con los que lidia este trabajo me habrían sido inabarcables sin la madurez intelectual adquirida en las clases de Antonio Salido.

Por otro lado, guardo un cariño especial hacia todas las personas con las que me he divertido en estos años. Fer e Irene, que estuvieron ahí en los momentos duros de la pandemia. Luis, Peio, Pablito, Koke, Adri, Mikel, Manuel y las fugaces apariciones de Isma, con los que siempre me río más de lo que creo posible. Álex, Álvaro, Edgar, Edu y Eli, máximos representantes del absurdo y del arte en general en mi vida. Adri, que, aunque le perdimos por un tiempo, es muy difćil de olvidar. Álvaro, Pablo P., Pablo T., Carlos G., Carlos I., Mario P., Mario R., Fran, Miguel y Antonio, que protagonizan conmigo una enternecedora serie de los años 80. Y Solana, claro, que solo por lo mucho que me ha aguantado se merece un honoris causa.

En cualquier caso, los máximos responsables de todo esto son mis padres. No solo por la gran educación que me han dado, sino también y principalmente por el apoyo incondicional y sin límites que siempre me han proporcionado. Nunca podré agradecerles lo suficiente su disposición y amor, con los que han construido el mejor ambiente posible para crecer, vivir y trabajar en nuestra casa, a pesar del tiempo que se diluye en esas largas sobremesas. A esto también han contribuido Lolita y Pedro, gracias a su buen humor y cariño. Y Tina, que aunque lo haga por razones egoístas nos ayuda a todos a relajarnos con sus juguetes.

Und was kann ich dir sagen, wenn du schon alles weisst? Du warst immer an meiner Seite, als ich diese Arbeit durchführte. Das meint ich wörtlich, weil wir so viel Zeit zusammen arbeiteten (Coffe Breaks inklusiv) aber auch im übertragenen Sinn, weil du mich immer zum Lachen und Entspannen bringen konntest. Du unterstütztest mich mit Wörter und mit Aktionen. Du gabst mir eine Zukunft, auf die ich mich freuen konnte. Und du bist jetzt hier, wenn die Dissertation fertig ist, und wir diese Zukunft zusammen erleben können. Danke, Tabita.

Contents

A mis abuelas y abuelos

Chapter 1

Introduction

The main result of this thesis is the construction of a moduli space parameterizing pairs consisting of a holomorphic volume form on a fixed complex compact manifold X, and holomorphic structures on a higher principal bundle $\mathcal{P} \to X$. This moduli space is presented as a simplicial derived manifold and equipped with a shifted holomorphic symplectic structure. This problem is motivated by the existence of shifted holomorphic symplectic structures on moduli spaces parameterizing holomorphic structures on ordinary principal bundles over Calabi-Yau manifolds [210] which extend classical work of Atiyah-Bott [15] and Mukai [196], their relation with the Yang-Mills equations established by the Donaldson-Uhlenbeck-Yau theorem [100, 267], and the search for an analogous complex-geometric counterpart for supersymmetric Yang-Mills equations coupled to supergravity in heterotic string theory based on [127, 245].

In Section 1.1 we motivate the language of higher derived differential geometry, which we use both to pose our problem and to present its solution. In Section 1.2 we discuss how much of ∞-category theory is needed for understanding our results. In Section 1.3 we provide a brief introduction to the use of higher derived geometry and shifted symplectic geometry for studying moduli spaces, with examples from ordinary gauge theory that motivate our work. In Section 1.4 we present our main results, relating them with constructions in generalized geometry, and motivating the study of higher gauge theory from mathematical and physical points of view.

1.1 Higher derived differential geometry

The problems and results of this thesis are expressed within the language of higher derived differential geometry. This is an extension of differential geometry which can be motivated as follows. Smooth manifolds model a notion of space which is locally equivalent to Euclidean space and which has enough global structure so that we can study it using techniques from differentiable calculus. These techniques are very powerful, but they are limited by the following facts.

1. The topological quotient X/G of a smooth manifold X by the smooth action ρ of a Lie group G does not always have structure of smooth manifold such that $X \to X/G$ is smooth. (More generally: the category of manifolds is not closed under colimits).

2. The topological preimage $f^{-1}(y)$ of a value $y \in Y$ by a smooth map $f : X \to Y$ between smooth manifolds does not always have structure of smooth manifold such that $f^{-1}(y) \to X$ is smooth . (More generally: the category of manifolds is not closed under limits).

Furthermore, even when X/G or $f^{-1}(y)$ are actually manifolds, it is not always true that they satisfy the properties that one would expect from these constructions. For example, the expected dimensions

$$\begin{aligned} \dim\, X/G &= \dim\, X - \dim\, G, \\ \dim\, f^{-1}(y) &= \dim\, X - \dim\, Y, \end{aligned} \tag{1.1}$$

fail for G a Lie group with $\dim\, G \geq 1$ acting trivially on X, or for $f : X \to Y$ a constant function and $\dim\, Y \geq 1$. The classical way to address these issues is to acknowledge them as features of the category of manifolds that our naive expectations could not foresee. An alternative approach is to replace the category of manifolds by an enhanced theory in which colimits and limits of manifolds always exist and satisfy all the combinatorial properties that one would expect. This could be useful, for example, to construct some functorial invariant for all manifolds that behaves well with respect to quotients and preimages.

Higher differential geometry is an enhancement of differential geometry in which the notion of smooth manifold is generalized in such a way that finite colimits always exist and behave well. Similarly, *derived differential geometry* generalizes the notion of smooth manifold in such a way that finite limits always exist and behave well; when both theories are combined we are working with *higher derived differential geometry.*

The basic idea behind these theories is the following. While quotients and preimages of manifolds do not behave well in general, two classical results provide conditions under which they do.

1. If G is a Lie group acting freely and properly on a manifold X, then there is a canonical structure of manifold on X/G such that $X \to X/G$ is smooth and dim $X/G = dim\, X - dim\, G$.

2. If $f : X \to Y$ is a smooth map between manifolds such that $df_{|x} : T_x X \to T_{f(x)} Y$ is surjective for every $x \in X$ with $f(x) = y$, then $f^{-1}(y) \subset X$ is a submanifold with dim $f^{-1}(Y) =$ dim $X -$ dim Y.

Thus, one way to solve our problem would be to find a category C, including the category Man of manifolds as a fully faithful subcategory, and whose objects can be studied with differential geometric tools, such that the following conditions are satisfied.

1. The notions of free, proper action and submersion can be generalized to C in such a way that the analogs of 1 and 2 remain true.

2. There is a distinguished class of arrows in C, called *quasi-isomorphisms*, and such that objects related by a quasi-isomorphism are considered to represent the same space.

3. Every smooth action $\rho : X \times G \to X$ can be lifted to a free and proper action $\tilde{\rho} : \tilde{X} \times G \to \tilde{X}$ along a quasi-isomorphism $q : \tilde{X} \to X$.

4. Every smooth map $f : X \to Y$ factorices as $f : X \xrightarrow{q} \tilde{X} \xrightarrow{\tilde{f}} Y$, where q is a quasi-isomorphism and $\tilde{f}$ is a submersion.

If these conditions are satisfied, then we can construct the preimage of a value by an arbitrary smooth function $f : X \to Y$ by factorizing it as in 4 and taking $\tilde{f}^{-1}(y)$ instead of $f^{-1}(y)$. Since $\tilde{f}$ is a submersion, $\tilde{f}^{-1}(y)$ is an object of C, and since q is a quasi-isomorphism, f and $\tilde{f}$ should represent the same map of spaces, hence $\tilde{f}^{-1}(y)$ should be a good geometric model for the topological space $f^{-1}(y)$. For quotients we could proceed in a similar way.

As we will see in Section 2.1.1, following [282], the category of simplicial manifolds allows us to perform arbitrary quotients by smooth actions using this idea. Roughly speaking, while any two points in a manifold are either distinct or equal, a simplicial manifold is a manifold such that any two points determine a smooth manifold of *arrows* between them, while any three arrows determine a smooth manifold of *2-cells* whose boundary is given by the three arrows, etc. For $m \in \mathbb{N}$, a simplicial manifold is called a *Lie m-groupoid* if the spaces of k-cells for $k > m$ are determined by the spaces of k-cells for $k \leq m$. While the tangent space of a manifold at a point is a vector space, the tangent space of a simplicial manifold at a point is a $\mathbb{Z}^{\leq 0}$-graded cochain complex of vector spaces.

One possible model for the quotient of X by G as a simplicial manifold $X//G$, which can be obtained resolving X as above, is given simply by considering X as the manifold of points, and adding an arrow from x to $x \cdot g$, for every $x \in X$ and $g \in G$. Its tangent space at $x \in X$ is the cochain complex $\mathfrak{g}[1] \overset{\rho_*}{\to} T_xX$, where ρ_* denotes the infinitesimal action; note that its Euler characteristic is the expected dimension dim $X -$ dim G. A more general simplicial manifold, in which arrows could have non-trivial 2-cells between them, would be suitable, for example, for modelling iterated quotients.

As we will see in Section 2.2.2, following [32], the category of derived manifolds allows us to take arbitrary preimages of smooth maps using the preceding idea. Roughly speaking, a derived manifold is a manifold M equipped with a $\mathbb{N}^{\geq 1}$-graded vector bundle $E \to M$ and an algebraic structure encoding a sequence of smooth maps $M \overset{s_0}{\to} E_1 \overset{s_1}{\to} E_2 \overset{s_2}{\to} \ldots$, smooth over M and polynomial on the fibers, such that $s_j \circ s_{j-1} = 0$. While the tangent space of a manifold at a point is a vector space, the tangent space of a derived manifold at a point is a $\mathbb{Z}^{\geq 0}$-graded cochain complex of vector spaces.

One possible model for the preimage of $y \in Y$ by $f : X \to Y$ as a derived manifold, which can be obtained resolving X as above, is given by taking M an open neighborhood of $f^{-1}(y) \subset X$ where $f_{|M} : M \to Y$ can be identified with a function $f'_{|M} : M \to T_yY$ after taking a chart of Y around y, and letting $E \to M$ be the trivial vector bundle with fiber T_yY on degree 1 and section $f'_{|M} : M \to E$. Its tangent space at a point $x \in M$ is the cochain complex $T_xM \overset{df'_{|x}}{\to} T_{f(x)}Y[-1]$; note that its Euler characteristic is the expected dimension dim $X -$ dim Y. A more general derived manifold, in which E could have coordinates of higher degrees, would be suitable, for example, for modelling iterated preimages.

In Section 2.2.3 we will combine both approaches in the notion of *simplicial derived manifolds*, whose 'tangent space' at each point is a $\mathbb{Z}$-graded cochain complex of vector spaces. In any case, the punchline is that simplicial derived manifolds are differential geometric objects, in the sense that one can develop differentiable calculus and define notions such as symplectic structures, Riemannian metrics, complex structures, etc. over them, which have good combinatorial properties, and which can model possibly very singular spaces. See the introductions of Chapters 2 and 7 for historical accounts of simplicial derived manifolds and higher derived differential geometry.

1.2 The need for ∞-categories

A proper treatment of higher derived differential geometry requires familiarity with the quite technical notion of ∞-categories. However, this can be avoided for treating some problems and results. We proceed to explain why and when ∞-category theory is necessary with some motivating examples. First, the idea of enhancing the category of manifolds to a category C satisfying properties 1-4 is inspired by homological algebra and homotopy theory, where we find the following situations.

1. Let $f : x_\bullet \to y_\bullet$ be a morphism of $\mathbb{N}$-graded cochain complexes in an abelian category. Then the *mapping cone* of f is computed by choosing a factorization of f as $x_\bullet \xrightarrow{q} \tilde{x}_\bullet \xrightarrow{\tilde{f}} y_\bullet$, where q is a quasi-isomorphism and $\tilde{f}$ is degreewise injective, and taking the degreewise cokernel *coker* $\tilde{f}_\bullet$. For any two factorizations $x \xrightarrow{q^i} \tilde{x}^i \xrightarrow{\tilde{f}^i} y_\bullet$, $i = 1, 2$, there exists a quasi-isomorphism *coker* $\tilde{f}^1_\bullet \to$ *coker* $\tilde{f}^2_\bullet$, unique up to homotopy.
2. Let $f_i : X_i \to Y$, $i = 1, 2$ be continuous maps of topological spaces. Then the *homotopy fibered product* $X_1 \times^h_Y X_2$ is computed by choosing a factorization of f_1 as $X_1 \xrightarrow{q} \tilde{X}_1 \xrightarrow{\tilde{f}_1} Y$, where q is a weak homotopy equivalence and $\tilde{f}_1$ is a Serre fibration, and taking $\tilde{X}_1 \times_Y X_2$. For any two factorizations $X_1 \xrightarrow{q^i} \tilde{X}^i_1 \xrightarrow{\tilde{f}^i_1} Y$, $i = 1, 2$, there exists a weak homotopy equivalence $\tilde{X}^1_1 \times_Y X_2 \to \tilde{X}^2_1 \times_Y X_2$, unique up to homotopy.

The classical way to deal with the dependence of *coker* $\tilde{f}_\bullet$ (resp. $\tilde{X}_1 \times_Y X_2$) on the factorization is to *localize*; i.e., to regard *coker* $\tilde{f}_\bullet$ as an object in the *derived* category of complexes instead of as an object in the category of complexes (resp. to regard $\tilde{X}_1 \times_Y X_2$ as a homotopy type instead of as a topological space). This approach is technically flawless and sufficient for understanding what the mapping cone of a specific morphism of cochain complexes is (resp. what the homotopy fibered product of a specific pair of continuous maps is).

However, when regarded as a construction in the localized category, the mapping cone (resp. the homotopy fibered product) is not functorial. This is a consequence of the fact that different factorizations yield objects that are related by a non-unique quasi-isomorphism (resp. weak homotopy equivalence). But, since this quasi-isomorphism is unique up to homotopy, we might ask for a generalization of the notion of category that also contains the information of some notion of homotopy between arrows, and in which constructions such as the mapping cone or the homotopy fibered product are naturally understood as 'functorial up to homotopy', for a precise meaning of this concept.

Properties 1-4 are formalized in Quillen's notion of a *model category* [214], of which the category of $\mathbb{N}$-graded cochain complexes and the category of topological spaces are the main examples. The problem of formalizing mapping cones and homotopy fibered products (more generally: homotopy limits and colimits in model categories) as functorial constructions was one of the inspirations for the development of ∞-categories envisioned by Grothendieck [138] and carried out later by many authors (e.g. [17, 26, 29, 104, 157, 181, 218, 255]). For a more detailed history of this theory see the introduction of Chapter 7.

The same technical problems apply to the setting of Section 1.1 that is of interest for this thesis, and so we can summarize our conclusions as follows. The categories of simplicial manifolds, derived manifolds and simplicial derived manifolds that we present in Sections 2.1.1, 2.2.2 and 2.2.3, respectively, are sufficient for the following purposes.

1. Presenting specific examples of geometric spaces within the context of higher, derived and higher derived differential geometry.
2. Presenting specific examples of maps between geometric spaces within the context of higher, derived and higher derived differential geometry.
3. Computing specific examples of quotients (resp. fibered products) of manifolds by non-free smooth actions (resp. along non-transversal smooth maps) as geometric spaces within the context of higher (resp. derived) differential geometry.

On the other hand, the ∞-categories of differentiable ∞-stacks, derived manifolds and derived differentiable ∞-stacks from Sections 7.2.1 and 7.2.3 are necessary for the following applications.

1. Understand all maps between geometric spaces within the context of higher, derived and higher derived differential geometry.
2. Understand the symmetries of a geometric space within the context of higher, derived and higher derived differential geometry, such as dualities in certain field theories.
3. Perform functorial constructions over manifolds that behave well under quotients and fibered products, such as certain enumerative invariants.

For each $m \in \mathbb{N}$, the full sub-∞-category of the ∞-category of differentiable ∞-stacks spanned by Lie m-groupoids is actually an $(m+1)$-category, which is a simplified version of an ∞-category. In particular, for $m = 1$ we obtain the 2-category of Lie 1-groupoids that is described completely in Section 3.1.1, using the classical algebraic approach to bicategories [35]. Insight into this bicategory is necessary for a good understanding of the symmetries of *differentiable* 1-*stacks*.

The main results of this thesis consist on presenting specific examples of geometric objects in higher derived differential geometry that represent moduli spaces parameterizing structures associated to certain differentiable 1-stacks. Thus, these results require familiarity with the category of simplicial derived manifolds, and with the bicategory of Lie groupoids, but not with the ∞-category of derived differentiable ∞-stacks. Chapter 7 is nevertheless included in order to put our results into the right context, and to relate them with possible future applications that might need this machinery.

1.3 Higher derived geometry and moduli spaces

One of the reasons why we consider higher derived differential geometry in this thesis is because it provides a framework to construct well-behaved moduli spaces. When studying the problem of parameterizing a certain class of geometric objects with complicated symmetries, it is often necessary to restrict these objects by imposing conditions such as irreducibility or stability in order to obtain a non-singular moduli space [101, 112]. While these notions yield important theories with applications in mathematical physics and the construction of invariants, the fundamental nature of the moduli space parameterizing all (i.e., possibly non-stable) geometric objects is singular, and so it is also desirable to have tools for handling this sort of spaces.

As discussed in Section 1.1, higher derived geometry is precisely the context in which the notion of a geometric object is generalized to deal with possibly singular spaces, making it an appropriate framework for studying fundamental aspects of moduli spaces. For a historical review of the development of higher derived geometry applied to the study of moduli spaces, see the introduction to Chapter 7. Here we will comment on a specific example that motivates our work.

Consider the space $\mathcal{B}^\flat(P)$ of flat connections modulo gauge on a principal G-bundle $P \to M$, for G a compact Lie group and M a smooth compact manifold of dimension n. Classically,

$$\mathcal{B}^\flat(P) = \{A \in \mathcal{A}(P) \mid F_A = 0\}/Gauge(P), \tag{1.2}$$

for $\mathcal{A}(P)$ the space of connections on P and $Gauge(P)$ the group of automorphisms of P covering the identity on M. Note $\mathcal{A}(P)$ is an affine space modelled on $\Omega^1(ad\, P)$, the curvature F_A of a connection A lives in the vector space $\Omega^2(ad\, P)$ and $Gauge(P)$ is a Lie group with Lie algebra $\Omega^0(ad\, P)$. Then, by differentiating the map $A \mapsto F_A$ and computing the infinitesimal action map we see that a manifold structure on $\mathcal{B}^\flat(P)$ should be modelled around $[A] \in \mathcal{B}^\flat(P)$ on the middle cohomology of the complex

$$\Omega^0(ad\, P) \xrightarrow{d^A} \Omega^1(ad\, P) \xrightarrow{d^A} \Omega^2(ad\, P). \tag{1.3}$$

However, since the gauge action is in general not free, and the map $A \mapsto F_A$ is in general not transversal to the zero section, it is not true in general that $\mathcal{B}^\flat(P)$ is a manifold. One way of interpreting the philosophy of higher derived geometry is that it replaces the expected tangent space

$$Ker(d^A : \Omega^1(ad\,P) \to \Omega^2(ad\,P))/d^A\Omega^0(ad\,P) \tag{1.4}$$

by the elliptic complex

$$\Omega^0(ad\,P) \xrightarrow{d^A} \Omega^1(ad\,P) \xrightarrow{d^A} \Omega^2(ad\,P) \xrightarrow{d^A} \Omega^3(ad\,P) \xrightarrow{d^A} \ldots \xrightarrow{d^A} \Omega^n(ad\,P). \tag{1.5}$$

More precisely, there exists a simplicial derived manifold $\mathcal{B}^{\flat,d}(P)$ (see Section 2.3.3 for the construction) having (1.5) as its 'tangent complex' at each point. The fact that the cohomology groups of this complex are finite-dimensional means that $\mathcal{B}^{\flat,d}(P)$ can be represented locally by finite-dimensional data. While this sort of construction has been performed using differential graded supermanifolds [240, 241] (inspired by the BV-BRST approach to quantizing gauge theories [28, 30]), we emphasize that these can be interpreted as the tangent bundle of $\mathcal{B}^{\flat,d}(P)$, but not as $\mathcal{B}^{\flat,d}(P)$ itself.

An important observation from [240, 241] is that the differential graded supermanifolds modelling the BV-BRST formalism are naturally equipped with a certain symplectic-like structure which is crucial in the quantization procedure from [28]. We can see a shadow of this symplectic structure in the elliptic complex (1.5): the fact that $\Omega^j(ad\,P)$ is naturally dual to $\Omega^{n-j}(ad\,P)$ (after choosing an Ad-invariant, non-degenerate, symmetric bilinear form on the Lie algebra of G and a volume form on M) can be interpreted as an isomorphism between the tangent complex of $\mathcal{B}^{\flat,d}(P)$ and its (shifted) cotangent complex, behaving similarly as the isomorphism between the tangent and the cotangent bundles of a manifold induced by a symplectic structure on it.

This structure is formalized in [210] as a $(2-n)$-*shifted symplectic structure* on (the algebraic analog of) the simplicial derived manifold $\mathcal{B}^{\flat,d}(P)$. For $n = 2$, it extends the Atiyah-Bott symplectic structure on the smooth locus of the moduli space of G-local systems over a Riemann surface [15]. For general n, this structure is interpreted in terms of the AKSZ formalism, a construction of field theories based on sigma-models whose target is a symplectic differential graded supermanifold and which had been introduced in [1, 242]. The improved language of shifted symplectic structures on higher derived stacks has recently lead to a proof [69] that the field theories from [1] can be modelled as extended topological field theories in the sense of [184], a notion that is itself inspired by the literature on mathematical physics [113].

Similarly, the moduli space of holomorphic G-bundles over a compact Calabi-Yau n-fold X, for G a reductive complex Lie group, has a $(2-n)$-shifted holomorphic symplectic structure [210]. Again, we can see a shadow of this structure by looking at the elliptic complex associated to a holomorphic structure on a principal G-bundle $P \to X$

$$\Omega^0(ad\, P) \xrightarrow{\overline{\partial}^A} \Omega^{0,1}(ad\, P) \xrightarrow{\overline{\partial}^A} \Omega^{0,2}(ad\, P) \xrightarrow{\overline{\partial}^A} \Omega^{0,3}(ad\, P) \xrightarrow{\overline{\partial}^A} \ldots \xrightarrow{\overline{\partial}^A} \Omega^{0,n}(ad\, P) \tag{1.6}$$

and noting that a holomorphic volume form on the base manifold (with a non-degenerate, Ad-invariant, symmetric $\mathbb{C}$-bilinear form on the Lie algebra of G) allows us to see $\Omega^{0,j}(ad\, P)$ as the complex dual of $\Omega^{0,n-j}(ad\, P)$, in a way which is compatible with the differential $\overline{\partial}^A$. For $n = 2$, this extends Mukai's holomorphic symplectic structure on the smooth locus of the moduli space of holomorphic G-bundles over a $K3$ surface [196]. For $n = 3$, [34] argues that this shifted holomorphic symplectic structure is responsible for the existence of Donaldson-Thomas invariants [102, 259], presenting a point of view that has lead to a categorification of these invariants [160, 164] and a generalization for Calabi-Yau fourfolds [47, 208, 209]. To sum up, we can conclude that shifted-symplectic structures on higher derived stacks parameterizing geometric objects seem to be useful for constructing invariants of manifolds.

1.4 Main results

In Section 1.3 we motivated higher derived differential geometry as a tool for studying moduli spaces parameterizing structures in ordinary differential geometry. The main goal of this thesis is to use higher derived differential geometry for studying moduli spaces parameterizing structures that are themselves best described within higher differential geometry. To be more precise, we construct moduli spaces parameterizing geometric structures on fibrations of the form $\mathfrak{P} \to M$, where M is a manifold and $\mathfrak{P}$ is a *principal 2-bundle* for a *Lie 2-group* $\mathfrak{G}$.

Lie 2-groups are analogs of Lie groups in higher differential geometry: geometric objects that describe symmetries of differentiable stacks (i.e. the differentiable 1-stacks from Definition 7.22) as Lie groups describe symmetries of manifolds (i.e. differentiable 0-stacks). One way to define a Lie 2-group is as a differentiable stack $\mathfrak{G}$ equipped with a morphism $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ which may not be strictly associative, but which is equipped with a natural transformation $\alpha : m \circ (m \times id) \Rightarrow m \circ (id \times m) : \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$, expressing a weak form of associativity of m. For a brief history of the notion of Lie 2-groups, see the introduction of Chapter 3.

Given a Lie 2-group $\mathfrak{G}$, there is a natural way to define actions of $\mathfrak{G}$ on differentiable 1-stacks. One can also define a *principal* $\mathfrak{G}$*-bundle* $\pi : \mathfrak{P} \to M$ to be a fibration of differentiable 1-stacks carrying an action of $\mathfrak{G}$ that is principal in the sense that it induces an equivalence $\mathfrak{P} \times \mathfrak{G} \cong \mathfrak{P} \times_M \mathfrak{P}$. However, the notion of *connection* on these bundles is in general not well understood. We review in more detail the history of this concept in the introduction to Chapter 4. For now we will just mention that non-flat connections on $\mathfrak{G}$-bundles behaving as expected from physics have been defined for certain Lie 2-groups which by results of [238] admit an alternative description as *multiplicative gerbes* [235, 237, 273]. This notion of connection depends on the choice of some additional data on the multiplicative gerbe, called itself a *connection* in [271, 273].

The definition from [273] is inspired by early work on string theory and supergravity [38, 74, 81, 137, 165, 277], where the bosonic field content of these theories is described as a connection A on a G-bundle over spacetime $P \to M$ and some 2-forms B_i, defined only locally over M, and coupled to A in such a way that the combination

$$H := dB_i + \langle dA_i \wedge A_i \rangle + \frac{1}{3}\langle A_i \wedge [A_i \wedge A_i] \rangle \tag{1.7}$$

is a globally well-defined 3-form on M satisfying

$$dH - \langle F_A \wedge F_A \rangle = 0, \tag{1.8}$$

for $F_A \in \Omega^2(ad\, P)$ the curvature of A. Here $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ is an Ad-invariant, symmetric, bilinear form that must be chosen beforehand. If $\langle \cdot, \cdot \rangle$ satisfies a certain integrality condition, then [271] defines a multiplicative gerbe with connection $\mathcal{G}$ such that the notion of a connection on a $\mathcal{G}$-bundle, as defined in [273], formalizes the bosonic field content in [38, 81, 137]. The pair (F_A, H), with H defined by (1.7), is called the *curvature* of the connection.

In particular, the study of moduli spaces of connections on $\mathcal{G}$-bundles, where $\mathcal{G}$ is constructed as in [271], is related to the study of configuration spaces in supergravity and string theory, suggesting rich geometries on them, as in the case of moduli spaces of connections on G-bundles sketched in Section 1.3. This is precisely the content of our main results. In fact, these physical theories often include an additional parameter, called the *dilaton* and represented by a global function, and the spaces we construct need to accommodate this freedom in order to be equipped with non-degenerate shifted symplectic structures. The following theorem is presented in the body of the thesis as Example 6.3 and Theorem 6.7.

Theorem 1.1. *Let $\mathcal{G}$ be the multiplicative gerbe with connection associated to a Lie group G and an Ad-invariant, symmetric, integral, bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$, and* *let $\mathcal{P} \to M$ be a $\mathcal{G}$-bundle over a compact manifold of* $\dim_{\mathbb{R}} M = n$*. Then*

1. *There is a simplicial derived manifold $\mathcal{B}^{\flat,d}(\mathcal{P})$ parameterizing gauge equivalence classes of flat connections on $\mathfrak{P}$.*

2. *If $\langle\cdot,\cdot\rangle$ is non-degenerate and M is orientable, then there is a $(2-n)$-shifted symplectic structure on $\mathcal{B}^{\flat,d}(\mathcal{P}) \times \Omega_{dR}(M)^*$, where $\Omega_{dR}(M)^*$ is a derived manifold parameterizing locally constant positive functions on M and called the* dilaton moduli*.*

Lie 2-groups can also be defined in the holomorphic context. Such objects have not been studied in detail in the literature (see, however, the introduction to Chapter 5 for a summary of some precursors). A straightforward generalization of the results in [238] and [271, 273] enables us to construct a holomorphic multiplicative gerbe $\mathcal{G}$ with holomorphic connection from the data of a complex Lie group G and an Ad-invariant, symmetric $\mathbb{C}$-bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ satisfying an integrality condition.

We can also define holomorphic structures on $\mathcal{G}$-bundles, as well as holomorphic connections on holomorphic $\mathcal{G}$-bundles. Interestingly, one can also define the notion of a *holomorphic structure with holomorphic connective structure* on a $\mathcal{G}$-bundle, which is an intermediate structure between a holomorphic structure and a holomorphic structure with holomorphic connection. Then Examples 6.4 and 6.5 and Theorem 6.8 can be summarized as follows.

Theorem 1.2. *Let $\mathcal{G}$ be the holomorphic multiplicative gerbe with holomorphic connection associated to a complex Lie group G and an Ad-invariant, symmetric, integral, $\mathbb{C}$-bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$, and let $\mathcal{P} \to X$ be a smooth $\mathcal{G}$-bundle over a compact complex manifold of* $\dim_{\mathbb{C}} X = n$*. Then*

1. *There is a simplicial derived complex manifold $\mathcal{H}^d(\mathcal{P})$ parameterizing holomorphic structures on $\mathcal{P}$, up to isomorphism.*

2. *There is a simplicial derived complex manifold $\mathcal{H}'^{,d}(\mathcal{P})$ parameterizing holomorphic structures with holomorphic connective structures on $\mathcal{P}$, up to isomorphism.*

3. *If $\langle\cdot,\cdot\rangle$ is non-degenerate and X admits holomorphic volume forms, then there is a $(2-n)$-shifted holomorphic symplectic structure on $\mathcal{H}^d(\mathcal{P}) \times \Omega^{n,\bullet}_{\bar\partial}(X)^*$, where $\Omega^{n,\bullet}_{\bar\partial}(X)^*$ is a derived manifold parameterizing holomorphic volume forms on X and called the* axio-dilaton moduli*.*

The moduli spaces from Theorems 1.1 and 1.2 are inspired by work in mathematical physics [13, 14, 94, 174] and generalized geometry [84, 125, 127], but rigorous mathematical constructions reflecting all the symmetries of higher gauge theory have only been carried out until now for $G = \{*\}$ [63, 109, 110, 192, 256]. Even in this case, our shifted symplectic structures seem to be new. Theorem 1.2 is proved by applying the same techniques we develop for Theorem 1.1, after obtaining a gauge-theoretic description of holomorphic structures and holomorphic structures with holomorphic connective structures on $\mathcal{G}$-bundles similar to the description of holomorphic structures on a complex vector bundle E in terms of Dolbeault operators.

Recall that this classical description has a particularly nice expression when E is the complexification of a Hermitian vector bundle E_h. In this case, the so-called Chern correspondence establishes a bijection between holomorphic structures on E and unitary connections on E_h with curvature of type $(1,1)$ (see e.g. [233] for a generalization to principal bundles for a complex reductive Lie group). Moreover, this is the relation that is used to define a map between the moduli space of solutions to the Hermitian Yang-Mills equations on E_h and the moduli space of semistable holomorphic structures on E, which the Donaldson-Uhlenbeck-Yau theorem [100, 267] (extended to principal bundles for complex reductive Lie groups in [6]) proves to be a homeomorphism.

Our next result, which summarizes the content of Theorems 5.8 and 5.26, extends the Chern correspondence to the setting of Lie 2-groups, identifying the F-term equations that appear in supersymmetric heterotic string theory as the gauge-theoretic description of holomorphic structures with holomorphic connective structures on a principal 2-bundle. For this, we need to introduce the notion of *enhanced connection* on a principal 2-bundle, which generalizes the definition of connection from [273] by introducing an additional symmetric covariant tensor g on the base manifold.

Theorem 1.3. *Let $\mathcal{K}$ be the multiplicative gerbe with connection associated to a compact Lie group K and an Ad-invariant, symmetric, integral, bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{k}\otimes\mathfrak{k}\to\mathbb{R}$. Then*

1. *There exists a holomorphic multiplicative gerbe $\mathcal{K}^{\mathbb{C}}$ with holomorphic connection over the complexification $K^{\mathbb{C}}$ of K, such that there is a canonical faithful functor $\mathcal{P}_h \mapsto \mathcal{P}_h^{\mathbb{C}}$ from the bicategory of smooth $\mathcal{K}$-bundles to the bicategory of smooth $\mathcal{K}^{\mathbb{C}}$-bundles.*

2. *If $\mathcal{P}_h \to X$ is a $\mathcal{K}$-bundle over a complex manifold X, then holomorphic structures with holomorphic connective structure on $\mathcal{P}_h^{\mathbb{C}}$ are in bijection with enhanced connections $((A,B),g)$ on $\mathcal{P}_h$ such that*

$$g^{0,2} = 0, \quad F_A^{0,2} = 0, \quad H = i(\overline{\partial} - \partial)g(J\cdot,\cdot), \tag{1.9}$$

for J the complex structure on X and (F_A, H) the curvature of the connection (A,B).

Part 1 of Theorem 1.3 is to be regarded as a complexification theorem for Lie 2-groups. It generalizes a result from [268], which is equivalent to ours when $K^{\mathbb{C}}$ is a Stein group. Part 2 of Theorem 1.3 is proven for the case $G = \{*\}$ in [142], and is strongly related to a similar result in [127] relating equations (1.9) to holomorphic Courant algebroids. This relation is more than an analogy, as it follows from another original result (Theorems 4.23 and 4.26 and Proposition 5.32).

Theorem 1.4. *Let $\mathcal{G}$ be the multiplicative gerbe with connection associated to a Lie group G and a non-degenerate, Ad-invariant, symmetric, integral, bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$. Then*

1. *For a manifold M, there is a canonical functor $\mathcal{P}_\nabla \mapsto E_{\mathcal{P}_\nabla}$ from the bicategory of $\mathfrak{G}$-bundles with connective structure over M to the category of Courant algebroids over M.*
2. *(Enhanced) connections on $\mathcal{P}_\nabla \to M$ are in bijection with (possibly non-)isotropic splittings of the anchor $\pi : E_{\mathcal{P}_\nabla} \to TM$.*
3. *Let $ad\,\mathcal{P}_\nabla := Ker(\pi) \subset E_{\mathcal{P}_\nabla}$. There is a map from $\Gamma(ad\,\mathcal{P}_\nabla)$ to the 2-group $Gauge(\mathcal{P}_\nabla)$ of automorphisms of $\mathcal{P}_\nabla$ which induces a structure of Lie 2-group on $Gauge(\mathcal{P}_\nabla)$.*
4. *If $\mathcal{G}$ is a holomorphic multiplicative gerbe with holomorphic connection and X is a complex manifold, then there is a canonical functor $\mathcal{P}_\nabla \mapsto Q_{\mathcal{P}_\nabla}$ from the bicategory of holomorphic $\mathcal{G}$-bundles with holomorphic connective structure over X to the category of holomorphic Courant algebroids over X.*

The special case of Theorem 1.4 when $G = \{*\}$ is well-known in the literature as the construction of an exact Courant algebroid playing the role of the 'Atiyah algebroid' of a gerbe [85, 99, 142, 152, 221, 243]. Parts 1 and 2 of Theorem 1.4 are proven in [245], for a specific choice of multiplicative gerbe with connection called String(n). Part 3, which is an original contribution of this thesis, is crucial for our proof of Theorems 1.1 and 1.2, as the smooth structure on $Gauge(\mathcal{P}_\nabla)$ is responsible for the smooth structure on simplicial manifolds obtained as quotients by actions of $Gauge(\mathcal{P}_\nabla)$.

The following theorem is an original result on the general theory of multiplicative gerbes which can be found as Theorems 3.43 and 3.54 in the body of the thesis. Part 1 of Theorem 1.5 implies that the definition of connection on $\mathcal{G}$-bundles from [273] is valid for $\mathcal{G}$ any multiplicative gerbe with connection, and that so are all our main results. On the other hand, part 2 provides the main tool we use for proving part 3 of Theorem 1.4, where the role of the 1-form (1.10) is important because of its appearance in the transition functions (4.73) of the Courant algebroid $E_{\mathcal{P}_\nabla}$.

Theorem 1.5. *Let $\mathcal{G}$ be a multiplicative gerbe over a Lie group G with Lie algebra $\mathfrak{g}$. Then*

1. *$\mathcal{G}$ admits a connection if and only if it arises from an Ad-invariant, symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ as in [271].*

2. *$\mathcal{G}$ admits a connection if and only if $\exp^*\mathcal{G} \to \mathfrak{g}$ is trivial as an equivariant gerbe, for the adjoint action of G on $\mathfrak{g}$ and the equivariant structure on $\exp^*\mathcal{G}$ induced by the multiplicative structure. In this case, an equivariant trivialization can be chosen to have covariant derivative $\eta \in \Omega^1(G \times \mathfrak{g}, \mathbb{R})$ defined by*

$$\eta_{(g,v)}(v_g + \dot{v}) := 2\langle v, g^{-1} v_g \rangle, \tag{1.10}$$

   where $\langle \cdot, \cdot \rangle$ is the bilinear form from part 1.

Theorem 1.5 is also used to derive in a natural way the brackets on the Lie 2-algebra of the Lie 2-group associated to a multiplicative gerbe equipped with a connection (Proposition 3.51), while Corollary 3.55 interprets the equivariant trivializations from part 2 of Theorem 1.5 as a sort of *exponential map*. These constructions, along with the fact that part 1 can be used for defining connections on $\mathcal{G}$-bundles, raise the natural question of interpreting connections on multiplicative gerbes as special cases of some natural structure that can be defined for a general Lie 2-group $\mathfrak{G}$. We answer this by introducing the notion of *Maurer-Cartan forms* on Lie 2-groups (Definition 3.23).

We show that another example of Maurer-Cartan form on a Lie 2-group is provided by the notion from [220] of an *adjustment* on a Lie crossed module (see the introduction to Chapter 4 for a brief historical account). Notably, Lie crossed modules equipped with an adjustment are the only other family of Lie 2-groups for which there is a good notion of fully non-flat connection, apart from the multiplicative gerbes with connection from [271, 273]. We have checked in Propositions 3.66 and 4.21 that the approaches in [220] and [273] are equivalent, whenever it makes sense to compare them.

Theorem 1.6. *Let $\mathfrak{G}$ be a Lie 2-group that has a model as a multiplicative gerbe $\mathcal{G}$ and a model as a Lie crossed module $(\tilde{G}, H, f, \triangleright)$. Then*

1. *A connection on $\mathcal{G}$ determines an adjustment on $(\tilde{G}, H, f, \triangleright)$.*

2. *Choose a connection on $\mathcal{G}$. The corresponding category of connections on $\mathfrak{G}$-bundles defined as in [273] is equivalent to the category of connections on $\mathfrak{G}$-bundles defined as in [220] in terms of the corresponding adjustment on $(\tilde{G}, H, f, \triangleright)$.*

Theorem 1.6 suggests that, for a general Lie 2-group $\mathfrak{G}$, the notion of a Maurer-Cartan form on $\mathfrak{G}$ allows to define connections on $\mathfrak{G}$-bundles in a consistent, well-behaved way. This remains an open problem which we discuss in Sections 8.1.1 and 8.2.1. We furthermore conjecture that Maurer-Cartan forms can help solve open problems on the theory of general Lie 2-groups that involve in some way the Lie 2-algebra: a canonical construction of the L_∞-structure, the definition of an exponential map, a good notion of holonomy, etc. One problem that we have in fact solved using Maurer-Cartan forms is the definition of *moment maps* for actions of general Lie 2-groups on symplectic manifolds (see Proposition 6.11).

Theorem 1.7. *Let $\mathfrak{G}$ be a Lie 2-group equipped with a Maurer-Cartan form acting on a symplectic manifold (M, ω) with a moment map μ. Then there is a simplicial derived manifold $M//_{\mu}\mathfrak{G}$ with a 0-shifted symplectic structure. The moduli spaces from Theorems 1.1 and 1.2 when $n = 2$ are examples of this construction.*

To sum up, our results initiate the study of geometric structures on moduli spaces of connections and other associated structures in principal 2-bundles, boosting the interaction between generalized complex geometry, higher gauge theory and supersymmetric string theory. We believe that this area can be as fruitful as the interaction between complex geometry and gauge theory in the work of Narasimhan-Seshadri, Atiyah-Bott, Hitchin, Kobayashi, Donaldson, Uhlenbeck-Yau, Donaldson-Thomas, Witten and others [15, 100, 102, 151, 168, 202, 267, 276], especially through the relation between our main Theorems 1.1, 1.2 and their classical analogs discussed in Section 1.3.

1.5 Outline

The main results of this thesis can be summarized as follows.

1. On the theory of Lie 2-groups.
    - (a) Theorems 3.43 and 3.54 relate connections on a multiplicative gerbe $\mathcal{G} \to G$, Ad-invariant bilinear forms on $\mathfrak{g}$ and equivariant trivializations of $exp^*\mathcal{G} \to \mathfrak{g}$.
    - (b) Definition 3.23 of *Maurer-Cartan forms* on Lie 2-groups. Propositions 3.50 and 3.66 relate these with connections on multiplicative gerbes and with adjustments on crossed modules, respectively.
    - (c) Propositions 3.27 and 6.11 define symplectic reduction for actions of Lie 2-groups with Maurer-Cartan forms.
    - (d) Theorem 5.8 provides the complexification of a family of Lie 2-groups.

2. On the theory of connections on principal bundles for Lie 2-groups.

   (a) Proposition 4.21 relates adjusted connections to trivializations of Chern-Simons 2-gerbes.

   (b) Theorems 4.23 and 4.26 associate a Courant-Dorfman algebroid E to a principal 2-bundle $\mathcal{P}$ and prove that the gauge 2-group of $\mathcal{P}$ is a Lie 2-group modelled on the space of sections of a sub-bundle of E.

   (c) Definition 4.7 of *enhanced connection* on a principal 2-bundle. Theorem 5.26 uses these to relate holomorphic structures with holomorphic connective structures on principal 2-bundles to supersymmetric configurations in string theory.

3. On the theory of moduli spaces in higher gauge theory.

   (a) Examples 6.3, 6.4 and 6.5 present simplicial derived manifolds representing the moduli spaces of flat connections, holomorphic structures and holomorphic structures with holomorphic connective structure on principal 2-bundles.

   (b) Theorems 6.7 and 6.8 present shifted symplectic structures on these spaces.

The structure of this thesis is the following. Chapter 2 introduces the language of simplicial derived manifolds. Chapter 3 develops the theory of Lie 2-groups, while Chapter 4 develops the theory of connections on principal 2-bundles. Chapter 5 treats aspects of Lie 2-groups and principal 2-bundles which are special to the holomorphic context. Chapter 6 presents constructions of moduli spaces in higher gauge theory as simplicial derived manifolds. Chapter 7 contains a brief introduction to ∞-category theory and the ∞-category of higher derived differentiable stacks, as well as some original examples of higher Lie groups that are relevant in physics. Chapter 8 summarizes our conclusions and poses some open problems.

While Chapter 2 is necessary for understanding the moduli spaces from Chapter 6, the reader that is only interested in the theory of Lie 2-groups, principal 2-bundles and connections on them can skip it and read directly Chapters 3, 4 and 5. Section 7.2.2, with examples of higher Lie groups, might also be of interest in this case. The reader that is not concerned with complex geometry can omit Chapter 5. All the original results can be understood independently of Chapter 7, while the formalism presented there extends them to a better-behaved context.

Chapter 2

Simplicial derived manifolds

Simplicial manifolds are geometric objects that model topological spaces more general than ordinary smooth manifolds, but still using suitable smooth data. For example, given a Lie group G acting on a manifold M, then the topological space M/G might not have any manifold structure such that $M \to M/G$ is smooth if the action is not free. However, in order to have a model for M/G in terms of smooth objects, we can consider for each $n \in \mathbb{N}$ the space $(M//G)_n$ of n-simplices Δ^n such that

1. The vertices of Δ^n are labelled by points of M.
2. Edges of Δ^n from a vertex labelled by $x \in M$ to a vertex labelled by $y \in M$ are labelled by elements $g \in G$ such that $x \cdot g = y$.
3. The edges of each triangle in Δ^n form a commutative diagram of elements of G acting on M.

It is clear that $(M//G)_n = M \times G^n$ is a manifold, and the face maps $d_j^n : (M//G)_n \to (M//G)_{n-1}$, $j = 0, ..., n$ given by taking the j-th face of an n-simplex are smooth. Roughly, a general simplicial manifold $X_\bullet$ is a sequence of manifolds X_n, $n \in \mathbb{N}$ with smooth maps $d_j^n : X_n \to X_{n-1}$, $j = 0, ..., n$ satisfying the same identities as the face maps of n-simplices. Each manifold X_n is to be thought of as the space of n-simplices in some space, where arrows represent a notion of symmetry between points, 2-cells represent a notion of symmetry between arrows, etc., while the maps d_j^n are to be thought of as projecting an n-simplex to its j-th face. A simplicial manifold has an 'underlying topological space', called its *fat geometric realization*, and obtained precisely by considering a geometric n-simplex for each point of X_n and gluing these along the face maps.

An important feature of simplicial manifolds is that they can be studied with tools from differential geometry. For example, there is a notion of de Rham cohomology for simplicial manifolds which computes the singular cohomology of their geometric realizations [49], and the de Rham cohomology of $M//G$ coincides with the equivariant de Rham cohomology of M as originally defined in [79]. This way of thinking about simplicial manifolds is related to the fact that they can be used as models for differentiable ∞-stacks [213, 282], as we discuss in more detail in Chapter 7.

Derived manifolds are also geometric objects that model topological spaces more general than manifolds, but the nature of such spaces is different. Namely, while simplicial manifolds are well-suited for dealing with arbitrary smooth quotients, derived manifolds are well-suited for dealing with fibered products of manifolds along possibly non-transversal maps. While there are different approaches to derived manifolds [32, 45, 46, 75, 158, 250, 252], all shown to be equivalent [76], we follow [32] here and define a derived manifold to be a graded vector bundle $E \to M$ with a fiberwise structure of curved L_∞-algebra. Such a structure, defined on a $\mathbb{N}^{\geq 1}$-graded vector space V, is equivalent to a degree 2 element $\Phi \in V_2$ (called *curvature*) and a sequence of multilinear, graded skew-symmetric brackets $\{\cdot, ..., \cdot\} : V \otimes ... \otimes V \to V$ satisfying some identities that generalize the axioms of a differential graded Lie algebra. It was first defined in [251], based on the BRST complex from [283].

In order to understand how a curved L_∞-algebra models a space in derived geometry, we consider first a motivating example that yields a differential graded Lie algebra. Let $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ be a sequence of vector spaces with polynomial functions f, g of degree ≤ 2 with $f(0) = 0$, $g(0) = 0$ and $g \circ f = 0$. Since $f^{-1}(0) \subset V_1$ might not be a manifold around 0 if $Df_{|0}$ is not surjective, to treat this space with geometric tools we may proceed as in algebraic geometry and retain the information of the polynomial f itself. Equivalently, we keep track of the first and second derivatives $Df_{|0} : V_1 \to V_2$ and $D^2 f_{|0} : S^2 V_1 \to V_2$. Now we may also wish to encode the information that f takes values in $g^{-1}(0) \subset V_2$. If $Dg_{|0}$ is surjective, we can do this using the implicit function theorem by identifying a neighborhood of 0 inside $g^{-1}(0)$ with $Ker\,(Dg_{|0})$ and proceed as before, with f replaced by some $\tilde{f} : V_1 \to Ker\,(Dg_{|0})$. In the general case, we may just keep track of $Dg_{|0} : V_2 \to V_3$ and $D^2 g_{|0}(Df_{|0}(\cdot), \cdot) : V_1 \otimes V_2 \to V_3$, noting that $0 = g \circ f$ is equivalent to

$$\begin{aligned}
0 &= Dg_{|0}(Df_{|0}(v)), \\
0 &= D^2 g_{|0}(Df_{|0}(v), Df_{|0}(w)) + Dg_{|0}(D^2 f_{|0}(v, w)), \\
0 &= D^2 g_{|0}(Df_{|0}(u), Df^2_{|0}(v, w)) \\
&\quad + D^2 g_{|0}(Df_{|0}(v), Df^2_{|0}(u, w)) + D^2 g_{|0}(Df_{|0}(w), Df^2_{|0}(u, v)),
\end{aligned} \tag{2.1}$$

which we might recognize as the axioms of a differential graded Lie algebra structure on $V_1 \oplus V_2 \oplus V_3$.

A general curved L_∞-algebra on a $\mathbb{N}^{\geq 1}$-graded vector space V models an infinitesimal neighborhood of a point x in a space X with a possibly singular subspace $Z \subset X$, in the following way. Firstly, $\Phi = 0$ if and only if $x \in Z$. Then an element $e \in V$ of degree 1 represents an infinitesimal deformation of x within X which can be lifted to an actual curve with endpoint on Z if and only if it satisfies the *Maurer-Cartan equation*

$$\Phi + \{e\} + \frac{1}{2}\{e, e\} + \frac{1}{6}\{e, e, e\} + ... = 0. \tag{2.2}$$

Elements in degree 2 represent infinitesimal obstructions, in the sense that the left-hand side of (2.2) lies in V_2. Elements in degrees ≥ 3 represent higher infinitesimal obstructions, in the sense that the left-hand side of (2.2) lies in a subset of V_2 that is itself described as the zero set of a polynomial function to V_3, whose image is again in the zero set of a polynomial function to V_4, etc. This way of thinking about curved L_∞-algebras is related to the fact that they can be used as models for derived differential geometry [32, 75], as we discuss in more detail in Chapter 7.

One can also combine both theories and define simplicial derived manifolds, which are geometric objects modelling topological spaces such as those that can be obtained by (iterated) quotients and fibered products of manifolds. Simplicial derived manifolds turn out to be well-suited for modelling moduli spaces, as these can often be written as $\{x \in \mathcal{A} \mid \Phi(x) = 0\}/G$ for some manifold $\mathcal{A}$, smooth map Φ and Lie group G. This observation is related to the fact that simplicial derived manifolds can be used as models for higher derived differential geometry, as we discuss in more detail in Chapter 7.

Simplicial derived manifolds can be equipped with a certain type of geometric structure, called *shifted symplectic structures*, which generalize standard symplectic structures on manifolds. They were introduced in [210] within the setting of higher derived algebraic geometry (see [68, 232] for surveys). Although the quasi-symplectic Lie groupoids from [275, 278] and the symplectic *dg*-manifolds from [224, 242] (based on [1]) are precursors, a systematic treatment of shifted symplectic structures in differential geometry is only available for now in the context of higher (not derived) differential geometry [92].

One of the most important features of this theory is that many moduli spaces of interest carry canonical shifted symplectic structures. These generalize and unify classical symplectic structures, such as the Atiyah-Bott [15] construction, or Mukai's holomorphic symplectic structure on the smooth locus of the moduli space of G-bundles on an abelian or $K3$ surface [196]. Moreover, they are related to topological field theories through the

AKSZ construction [1, 69, 242] and thus they provide invariants of manifolds. In particular, Donaldson-Thomas invariants have been studied and generalized using this point of view in [34, 47, 159, 160, 164].

In this chapter we define and provide examples of simplicial derived manifolds and shifted symplectic structures on them. In Section 2.1.1 we introduce simplicial manifolds. In Section 2.1.2 we follow [96, 134] to provide a brief overview of sheaf cohomology on simplicial manifolds, which is a useful tool for stating and proving classification results for geometric structures. In Section 2.2.1 we define L_∞-algebras, and in Section 2.2.2 we recall the approach from [32] to derived manifolds. Then we define simplicial derived manifolds in Section 2.2.3. In Section 2.3.1 we present a notion of shifted symplectic structures on simplicial derived manifolds, based on [92, 210, 224]. In Section 2.3.2 we present some basic constructions of shifted symplectic structures, and in Section 2.3.3 we discuss examples of shifted symplectic structures on moduli spaces. The constructions and results in this chapter are adapted from the references and there is no claim of originality except for minor presentation aspects.

2.1 Simplicial manifolds

2.1.1 Simplicial manifolds

We use the category of simplicial manifolds as an approximate model for the ∞-category of differentiable ∞-stacks. While this ∞-category is defined in Section 7.2.3, all the relevant constructions in this thesis are described without losing rigor with the terminology of this section. We establish first a notational convention for the whole thesis. Namely, for a category C, we write C_0 for the class of objects and C_1 for the class of arrows. For x, $y \in C_0$, we write $C(x, y)$ for the class of arrows from x to y.

Definition 2.1 ([282]). A *simplicial manifold* $M_\bullet$ is the following data.

1. For $n \in \mathbb{N}$, a manifold M_n.

2. For $n \in \mathbb{N}$ and $j = 0, ..., n$, smooth maps $d_j^n : M_n \to M_{n-1}$ (*face maps*) and $s_j^n : M_n \to M_{n+1}$ (*degeneracy maps*) that satisfy

$$\begin{aligned} d_i^{n-1} d_j^n &= d_{j-1}^{n-1} d_i^n, && i < j, \\ s_i^{n+1} s_j^n &= s_{j+1}^{n+1} s_i^n, && i \leq j, \\ d_i^{n+1} s_j^n &= s_{j-1}^{n-1} d_i^n, && i < j, \\ d_i^{n+1} s_j^n &= id, && i = j,\, j+1, \\ d_i^{n+1} s_j^n &= s_j^{n-1} d_{i-1}^n, && i > j+1. \end{aligned} \tag{2.3}$$

A *semi-simplicial manifold* $M_\bullet$ is a sequence of manifolds M_n, $n \in \mathbb{N}$, with *face maps* $d_j^n : M_n \to M_{n-1}$ satisfying the first equation in (2.3). A *morphism of (semi)-simplicial manifolds* $f_\bullet : M_\bullet \to N_\bullet$ is a family of smooth maps $f_n : M_n \to N_n$ commuting with all face and degeneracy maps. We write $sMan$ for the category of simplicial manifolds and $s_{inj}Man$ for the category of semi-simplicial manifolds.

We will omit the superscripts in the face and degeneracy maps of a simplicial manifold when these are clear from context. There is an alternative characterization of simplicial manifolds which is sometimes useful to present examples. Define the sets

$$[n] := \{0, ..., n\}, \qquad n \in \mathbb{N}. \tag{2.4}$$

Definition 2.2 ([106])**.** The *simplex category* is the small category Δ whose set of objects is $\{[n] | n \in \mathbb{N}\}$ and such that $\Delta([n], [m])$ for $n, m \in \mathbb{N}$ is the set of non-decreasing functions $[n] \to [m]$. The *semi-simplex category* is the subcategory $\Delta_{inj} \subset \Delta$ with the same objects, but with only strictly increasing functions as arrows.

Any arrow in Δ can be written in a canonical way as a composition of certain canonical arrows of the form $\delta_j^n : [n-1] \to [n]$ and $\sigma_j^n : [n+1] \to [n]$, $n \in \mathbb{N}$, $j = 0, ..., n$, called *coface* and *codegeneracy* maps [106]. They are defined by

$$\delta_j^n(i) := \begin{cases} i & i < j \\ i+1 & i \geq j \end{cases}, \qquad \sigma_j^n(i) := \begin{cases} i & i \leq j \\ i-1 & i > j \end{cases}, \tag{2.5}$$

and satisfy the relations

$$
\begin{aligned}
\delta_i^{n+1}\delta_j^n &= \delta_{j+1}^{n+1}\delta_i^n, && i \le j,\\
\sigma_j^n\sigma_i^{n+1} &= \sigma_i^n\sigma_{j+1}^{n+1}, && i \le j,\\
\sigma_j^n\delta_i^{n+1} &= \delta_i^n\sigma_{j-1}^{n-1}, && i < j,\\
\sigma_j^n\delta_i^{n+1} &= id, && i = j,\, j+1,\\
\sigma_j^n\delta_i^{n+1} &= \delta_{i-1}^n\sigma_j^{n-1}, && i > j+1.
\end{aligned}
\tag{2.6}
$$

This implies the following.

Proposition 2.3. *The category sMan as in Definition 2.1 is equivalent to the category whose objects are functors $\Delta^{op} \to Man$ and whose arrows are natural transformations. The category $s_{inj}Man$ as in Definition 2.1 is equivalent to the category whose objects are functors $\Delta_{inj}^{op} \to Man$ and whose arrows are natural transformations.*

In particular, Proposition 2.3 is useful to define the *Cartesian product* of simplicial manifolds $M_\bullet$, $N_\bullet : \Delta^{op} \to \mathrm{Man}$ as the simplicial manifold $(M \times N)_\bullet : \Delta^{op} \to \mathrm{Man}$ given by

$$(M \times N)_\bullet([n]) := M([n]) \times N([n]). \tag{2.7}$$

For $n \in \mathbb{N}$, write

$$|\Delta^n| := \{(x_0, ..., x_n) \in [0,1]^{n+1} \mid \textstyle\sum_i x_i = 1\} \tag{2.8}$$

for the *geometric n-simplex*. For $0 \le j \le n$, write $d_j^\Delta : |\Delta^{n-1}| \to |\Delta^n|$ and $s_j^\Delta : |\Delta^{n+1}| \to |\Delta^n|$ for the continuous maps

$$d_j^\Delta(x_0, ..., x_{n-1}) = (x_0, ..., x_{j-1}, 0, x_j, ..., x_{n-1}), \tag{2.9}$$

$$s_j^\Delta(x_0, ..., x_{n+1}) = (x_0, ..., x_j + x_{j+1}, ..., x_{n+1}). \tag{2.10}$$

A simplicial manifold $M_\bullet$ is a geometric model for the topological space

$$\pi_0(M_\bullet) := M_0/\sim, \tag{2.11}$$

where $p_0 \sim p_1$ if $\exists f \in M_1$ with $d_0(f) = p_0$, $d_1(f) = p_1$. The simplicial manifold $M_\bullet$ contains additional information on how points in $\pi_0(M_\bullet)$ are identified. This information is also encoded by another associated topological space, called the *geometric realization* [215] and defined by

$$|M_\bullet| := \bigsqcup_{n\in\mathbb{N}} M_n \times |\Delta^n| \,/ \sim, \tag{2.12}$$

where the equivalence relation is

$$(p, d_j^{\Delta}(x)) \sim (d_j(p), x), \qquad p \in M_n,\ x \in |\Delta^{n-1}|,\ \ j = 0, \ldots, n, \tag{2.13}$$
$$(p, s_j^{\Delta}(x)) \sim (s_j(p), x), \qquad p \in M_n,\ x \in |\Delta^{n+1}|,\ \ j = 0, \ldots, n. \tag{2.14}$$

The *fat geometric realization* $||M_\bullet||$ of $M_\bullet$ is similarly defined, but quotienting only by relation (2.13). In particular, it can also be defined for semi-simplicial manifolds. Fat geometric realization is in general better behaved, as it commutes with Cartesian products, but all the examples of simplicial manifolds in this thesis satisfy that degeneracies are embeddings, which implies that their fat geometric realization is homotopy equivalent to their geometric realization [264].

We say $M_\bullet$ is a geometric model in the sense that it is described by smooth manifolds and smooth maps between them. In particular, a simplicial manifold $M_\bullet$ has a *Moore tangent complex*, which is the $\mathbb{Z}^{\leq 0}$-graded chain complex of vector bundles over M_0

$$\ldots \xrightarrow{\delta} s^*TM_n \xrightarrow{\delta} s^*TM_{n-1} \xrightarrow{\delta} \ldots \to s^*TM_1 \xrightarrow{\delta} TM_0. \tag{2.15}$$

Here $s : M_0 \to M_n$ is the map obtained by composing degeneracy maps (which does not depend on which degeneracies are chosen, by the simplicial identities), and $\delta := \sum_j (-1)^j d^n_{j,*}$. This complex is meant to be considered up to quasi-isomorphism. The Dold-Kan correspondence [162] implies that the Moore tangent complex is quasi-isomorphic in the derived category of vector bundles to the following complex, which we call the *normalized tangent complex.*

$$\ldots \xrightarrow{\partial} A_{-n} \xrightarrow{\partial} A_{-n+1} \xrightarrow{\partial} \ldots \to A_{-1} \xrightarrow{\partial} A_0 = TM_0, \tag{2.16}$$

where

$$A_{-n} := \frac{s^*TM_n}{\oplus_{j=0}^{n-1} s^{n-1}_{j,*}(s^*TM_{n-1})} = \cap_{j=0}^{n-1} \ker(d^n_{j,*} : s^*TM_n \to s^*TM_{n-1}), \tag{2.17}$$
$$\partial := \sum_{j=0}^{n} (-1)^j d^n_{j,*} = d^n_{n,*} \tag{2.18}$$

A caveat is that the normalized tangent complex is not in general a complex of vector bundles. For simplicial manifolds satisfying a property called *Kan conditions*, which is treated more carefully in Chapter 7, this is not a problem. All the simplicial manifolds in this thesis satisfy this condition and so we will only use the normalized tangent complex, to which we will simply refer as the *tangent complex.* The *cotangent complex* of $M_\bullet$ is the dual of the tangent complex; i.e., it is the $\mathbb{Z}^{\geq 0}$-graded chain complex of vector bundles over X_0

$$T^*M_0 = A_0^* \overset{\partial^*}{\to} A_{-1}^* \to ... \overset{\partial^*}{\to} A_{-n+1}^* \overset{\partial^*}{\to} A_{-n}^* \overset{\partial^*}{\to} .... \tag{2.19}$$

A morphism of simplicial manifolds $f_\bullet : M_\bullet \to N_\bullet$ induces a map f_* between the tangent complexes of $M_\bullet$ and $N_\bullet$, as it follows simply from the fact that $f_\bullet$ commutes with all face and degeneracy maps.

Remark 2.4. The tangent bundle TM of an ordinary manifold M is canonically equipped with the Lie bracket $[\cdot,\cdot] : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(TM)$. While one would perhaps expect that the tangent complex of a simplicial manifold is canonically equipped with some analog structure, it turns out that this is not true in general. However, upon choosing certain connection-like data, one can indeed define a structure of $\mathbb{Z}^{\geq 0}$-graded L_∞-algebroid on it (see [178] for details).

Example 2.5. If M is a manifold, then we may see it as the simplicial manifold $M_\bullet$ with $M_n = M$ and all face and degeneracy maps equal to the identity. More interestingly, if G is a Lie group acting smoothly on M, then the *quotient groupoid* $(M//G)_\bullet$ is the simplicial manifold with $(M//G)_n = M \times G^n$ and simplicial maps

$$\begin{aligned}
d_0(p, g_1, ..., g_n) &= (pg_1, g_2, ..., g_n), \\
d_j(p, g_1, ..., g_n) &= (p, g_1, ..., g_{j-1}, g_j g_{j+1}, g_{j+2}, ..., g_n), \quad j = 1, ..., n-1, \\
d_n(p, g_1, ..., g_n) &= (p, g_1, ..., g_{n-1}), \\
s_j(p, g_1, ..., g_n) &= (p, g_1, ..., g_j, 1, g_{j+1}, ..., g_n), \qquad j = 0, ..., n.
\end{aligned} \tag{2.20}$$

The use of the word *groupoid* will be justified in Chapter 7. The tangent complex of $(M//G)_\bullet$ is the chain complex of vector bundles over M

$$\underline{\mathfrak{g}} \overset{\rho_*}{\to} TM, \tag{2.21}$$

where $\underline{\mathfrak{g}}$ is the trivial bundle (placed in degree -1) with fiber $\mathfrak{g}$ and ρ_* denotes the infinitesimal action map. This is often called the *action Lie algebroid* for the action of G on M. An alternative description of $(M//G)_\bullet$ is

$$\begin{aligned}
(M//G)_n = \{(\{p_i\}_{i\in[n]}, \{g_{ij}\}_{i\leq j\in[n]}) \in M^{n+1} \times G^{\binom{n+1}{2}} |\\
\forall i, \qquad g_{ii} = 1,\\
\forall i < j, \qquad p_i g_{ij} = p_j,\\
\forall i < j < k, \; g_{ij} g_{jk} = g_{ik}\};
\end{aligned} \tag{2.22}$$

indeed, note that a point $(\{p_i\}_i, \{g_{ij}\}_{i,j}) \in (M//G)_n$ is completely determined by the point $(p_0, g_{01}, g_{12}, ..., g_{n-1,n}) \in M \times G^n$. This description is more suitable for seeing $(M//G)_\bullet$ as a functor $\Delta^{op} \to \text{Man}$, because then for an arbitrary non-decreasing function

$f : [n_1] \to [n_2]$ we can define $f^* : (M//G)_{n_2} \to (M//G)_{n_1}$ by

$$\begin{aligned}(\{p_i\}_{i\in[n_2]}, \{g_{ij}\}_{i\le j\in[n_2]}) \mapsto (\{f^* p_i\}_{i\in[n_1]}, \{f^* g_{ij}\}_{i\le j\in[n_1]}) \\ f^* p_i := p_{f(i)}, \quad f^* g_{ij} := g_{f(i)f(j)}.\end{aligned} \tag{2.23}$$

The geometric realization of $(M//G)_\bullet$ is the *homotopy quotient*

$$|(M//G)_\bullet| = (M \times EG)/G, \tag{2.24}$$

where $EG \to BG$ is the universal bundle of G. This has a canonical surjective map $|(M//G)_\bullet| \to M/G$ to the standard topological quotient $M/G = \pi_0((M//G)_\bullet)$, and its fiber over each $[p] \in M/G$ is weakly homotopy equivalent to $BIso(p)$, the classifying space of the isotropy topological sub-group $Iso(p) \subset G$ of p. In this sense, $(M//G)_\bullet$ is a geometric model for the quotient which exists even when the action is poorly behaved, and which retains information about the isotropy of the action. In particular, when the action is free, then $|(M//G)_\bullet|$ is weakly homotopy equivalent to M/G.

Example 2.6. For G a Lie group and its trivial action on a point $\{*\}$, the corresponding quotient groupoid as in Example 2.5 is denoted $BG_\bullet := (\{*\}//G)_\bullet$ and called the *delooping* of G. The quotient groupoid corresponding to the action of G on itself by right multiplication is denoted $EG_\bullet := (G//G)_\bullet$. There are maps of simplicial manifolds $G \to EG_\bullet \to BG_\bullet$ given at each level by $G \overset{id\times 1}{\to} G \times G^n \overset{\pi_2}{\to} \{*\} \times G^n$, and the geometric realization of this sequence is the universal bundle $G \to EG \to BG$ of G as a topological group [146]. Note the tangent complex of $BG_\bullet$ is just the Lie algebra of G, seen as a degree -1 vector bundle over a point, while the tangent complex of EG is the complex of vector bundles over G, concentrated in degrees -1 and 0, with fiber $\mathfrak{g} \overset{id}{\to} \mathfrak{g}$.

Example 2.7. If M is a manifold and $\mathcal{U} = \{U_a\}_{a\in A}$ is an open cover of M, then we write $U_{a_1\ldots a_p} := \cap_{i=1}^p U_{a_i}$ for p-fold intersections. The *Čech groupoid* of M with respect to $\mathcal{U}$ is the simplicial manifold $\check{C}(M,\mathcal{U})_\bullet$ defined by $\check{C}(M,\mathcal{U})_n := \sqcup_{a_0,\ldots,a_n\in A^{n+1}} U_{a_0\ldots a_n}$, with face and degeneracy maps given by

$$\begin{aligned}d_j(a_0, ..., a_n, x) &= (a_0, ..., a_{j-1}, a_{j+1}, ..., a_n, x), \quad j = 0, \ ..., \ n, \\ s_j(a_0, ..., a_n, x) &= (a_0, ..., a_j, a_j, a_{j+1}, ..., a_n, x), \quad j = 0, \ ..., \ n.\end{aligned} \tag{2.25}$$

The geometric realization of $\check{C}(M,\mathcal{U})_\bullet$ is weakly homotopy equivalent to M itself. For G a Lie group, a morphism of simplicial manifolds $g : \check{C}(M,\mathcal{U})_\bullet \to BG_\bullet$ is the same as a collection of smooth functions $g_{ab} : U_{ab} \to G$ satisfying $g_{ab}g_{bc} = g_{ac}$ on U_{abc}. Given one such morphism g, then we define the simplicial manifold $(g^*EG)_\bullet$ by

$$(g^*EG)_n = \{(\{(a_i, g_i)\}_{i\in[n]}, x) \in (A \times G)^{n+1} \times M | \\ x \in \cap_{i=0}^n U_{a_i},\ \forall i < j,\ g_i g_{a_i a_j}(x) = g_j\}. \tag{2.26}$$

There is an obvious map of simplicial manifolds $(g^*EG)_\bullet \to M$, whose geometric realization is weakly homotopy equivalent to the G-bundle $P \to M$ defined by $P := \sqcup_{a\in A} U_a \times G/\sim$, with $(a, x, g_{ab}(x)g) \sim (b, x, g)$.

Example 2.8 ([178])**.** For T an abelian topological group, the homotopy type BT admits a model as an abelian topological group, and so inductively one can define again $B^2T := B(BT)$, $B^3T := B(B^2T)$, etc. We present a 'smooth' analog of this construction. Let T be an abelian Lie group and let $l \geq 1$. We define a simplicial manifold $(B^lT)_\bullet$ by

$$(B^lT)_n := \{\{t_{i_0,\dots,i_l}\}_{i_0\leq i_1\leq\dots\leq i_l\in[n]} \mid \forall j \in [l-1],\ t_{i_0,\dots,i_j,i_j,\dots,i_{l-1}} = 0, \\ \forall i_0 \leq \dots \leq i_{l+1} \in [n],\ \textstyle\sum_j (-1)^j t_{i_0,\dots,\hat{i_j},\dots,i_{l+1}} = 0\}, \tag{2.27}$$

where we are using additive notation on T. For the simplicial maps, given a non-decreasing function $f : [n_1] \to [n_2]$ we define the pull-back map

$$\{t_{i_0,\dots,i_l}\}_{i_0\leq i_1\leq\dots\leq i_l\in[n_2]} \mapsto \{f^*t_{i_0,\dots,i_l}\}_{i_0\leq i_1\leq\dots\leq i_l\in[n_1]} \\ f^*t_{i_0,\dots,i_l} := t_{f(i_0),\dots,f(i_l)}, \tag{2.28}$$

Note that for $l = 1$ we recover the simplicial manifold BT from Example 2.6. For general l, the inclusion-exclusion principle implies that $(B^lT)_n = T^{c(n,l)}$, where

$$c(n, l) = \binom{n+1}{l+1} - \binom{n+1}{l+2} + \binom{n+1}{l+3} - \dots = \binom{n}{l}, \tag{2.29}$$

as $\binom{n+1}{l+1}$ is the number of non-degenerate l-faces of Δ^n. However, the simplicial maps are harder to describe if we write $B^lT = T^{\binom{n}{l}}$. The presentation from (2.27) is more natural, as it lets us picture each $t_{i_0,\dots,i_l}$ as labelling the l-face of Δ^n with vertices i_0, ..., i_l. We also define a simplicial manifold $(EB^{l-1}T)_\bullet$ by

$$(EB^{l-1}T)_n := \{(\{t_{i_0,\dots,i_{l-1}}\}_{i_0\leq i_1\leq\dots\leq i_{l-1}\in[n]}, \{t_{i_0,\dots,i_l}\}_{i_0\leq i_1\leq\dots\leq i_l\in[n]}) \mid \\ \forall j \in [l-1],\qquad t_{i_0,\dots,i_j,i_j,\dots,i_{l-2}} = 0, \\ \forall i_0 \leq \dots \leq i_l \in [n],\ \textstyle\sum_j (-1)^j t_{i_0,\dots,\hat{i_j},\dots,i_l} = t_{i_0\dots i_l}\}. \tag{2.30}$$

There is a sequence of simplicial manifolds $B^{l-1}T_\bullet \to EB^{l-1}T_\bullet \to B^lT_\bullet$, and its geometric realization is the universal bundle of $B^{l-1}T$. Now if M is a manifold and $\mathcal{U} = \{U_a\}_{a\in A}$ is an open cover of M, then a map of simplicial manifolds $t : \check{C}(M,\mathcal{U})_\bullet \to B^lT_\bullet$ is the same as a T-valued Čech l-cocycle; i.e., a family of functions $t_{a_0 \ldots a_l} : U_{a_0 \ldots a_l} \to T$ with $\sum_{j=0}^{l+1}(-1)^j t_{i_0,\ldots,\hat{i_j},\ldots,i_{l+1}} = 0$. Associated to one such function t we can define $(t^* EB^{l-1}T)_\bullet$ by

$$\begin{aligned}(t^* EB^{l-1}T)_n := \{(\{a_i\}_{i \in [n]}, x, \{t_{i_0,\ldots,i_{l-1}}\}_{i_0 \le i_1 \le \ldots \le i_{l-1} \in [n]}) \mid \\ x \in \cap_{i=0}^n U_{a_i}, \\ \forall j \in [l-1], \qquad t_{i_0,\ldots,i_j,i_j,\ldots,i_{l-2}} = 0, \\ \forall i_0 \le \ldots \le i_l \in [n], \sum_j (-1)^j t_{i_0,\ldots,\hat{i_j},\ldots,i_l} = t_{a_{i_0} \ldots a_{i_l}}(x)\}.\end{aligned} \tag{2.31}$$

The geometric realization of $(t^* EB^{l-1}T)_\bullet$ is weakly homotopy equivalent to a $B^{l-1}T$-bundle over M, but the advantage of working with $t^* EB^{l-1}T$ is that it is completely described by smooth data, and so, as we will see, one can define differential geometric notions such as connections, symplectic structures, etc. on it.

2.1.2 Sheaf cohomology on semi-simplicial manifolds

Definition 2.9 ([96, 134])**.** Let $M_\bullet$ be a semi-simplicial manifold. A *sheaf of abelian groups* $\mathcal{S}^\bullet$ on $M_\bullet$ is a collection of sheaves of abelian groups $\mathcal{S}^n$ on M_n with maps $\partial_j^n : (d_j^n)^* \mathcal{S}^{n-1} \to \mathcal{S}^n$ satisfying the condition

$$\partial_j^n \circ (d_j^n)^* \partial_i^{n-1} = \partial_i^n \circ (d_i^n)^* \partial_{j-1}^{n-1} \tag{2.32}$$

for $i < j$. For a sheaf $\mathcal{S}$ we define the operators $\delta_{n-1} : \mathcal{S}^{n-1}(M_{n-1}) \to \mathcal{S}^n(M_n)$ by

$$\delta_{n-1} := \sum_{j=0}^{n-1} (-1)^j \partial_j^{n-1} (d_j^{n-1})^*. \tag{2.33}$$

A *morphism of sheaves* $\mathcal{S}_0^\bullet \to \mathcal{S}_1^\bullet$ is a collection of morphisms $\mathcal{S}_0^n \to \mathcal{S}_1^n$ commuting with the maps $(\partial_j^n)_0$, $(\partial_j^n)_1$. The *global sections functor* $\Gamma : \mathrm{AbSh}(M_\bullet) \to \mathrm{Ab}$ is the functor from the category of sheaves of abelian groups on $M_\bullet$ to the category of abelian groups acting as $\mathcal{S}^\bullet \mapsto \ker(\delta_0 : \mathcal{S}^0(M_0) \to \mathcal{S}^1(M_1))$.

For $M_\bullet$ a semi-simplicial manifold, the category $\mathrm{AbSh}(M_\bullet)$ is abelian. In fact, it can be interpreted as the category of sheaves in a Grothendieck site and so it has enough injectives [257, 2.1.2]. For $p \in \mathbb{N}$, the *sheaf cohomology group* $H^p(M_\bullet, \mathcal{S}^\bullet)$ is then the image of $\mathcal{S}^\bullet$ by the p-th derived functor of Γ. *Sheaf hypercohomology* $\mathbb{H}(M_\bullet, \mathcal{S}_0^\bullet \to \ldots \to \mathcal{S}_l^\bullet)$ of a complex of sheaves is similarly defined as the derived functors of the functor

$$\mathcal{S}_0^\bullet \xrightarrow{t} \ldots \xrightarrow{t} \mathcal{S}_l^\bullet \mapsto \ker(\delta_0 \oplus t : \mathcal{S}_0^0(M_0) \to \mathcal{S}_0^1(M_1) \oplus \mathcal{S}_1^0(M_0)). \tag{2.34}$$

The cohomology of a sheaf $\mathcal{S}^\bullet$ can be computed by taking a resolution in AbSh$(M_\bullet)$ $\mathcal{S}^\bullet \xrightarrow{\epsilon} I^{\bullet,0} \xrightarrow{d} I^{\bullet,1} \xrightarrow{d} \ldots$ such that the standard sheaf cohomology groups $H^r(M_n, I^{n,m})$ vanish for $r > 0$ and n, $m \geq 0$. This can be done, for example, by taking functorial acyclic resolutions of each sheaf $\mathcal{S}^n$ over M_n. Then $H^*(M_\bullet, \mathcal{S}^\bullet)$ is obtained as the total cohomology of the double complex $(I^{n,m}(M_n), \delta, d)$. Similarly, the hypercohomology of a complex of sheaves $\mathcal{S}_0^\bullet \xrightarrow{t} \ldots \xrightarrow{t} \mathcal{S}_l^\bullet$ can be computed by taking acyclic resolutions $\mathcal{S}_l \to I_l^{\bullet,\bullet}$ of each $\mathcal{S}_l$ with maps $I_0^{\bullet,\bullet} \xrightarrow{t} I_1^{\bullet,\bullet} \xrightarrow{t} \ldots$ commuting with the maps from the complex. Then $\mathbb{H}(M_\bullet, \mathcal{S}_0^\bullet \to \ldots \to \mathcal{S}_l^\bullet)$ is the cohomology of the triple complex $(I_k^{n,m}(M_n), \delta, d, t)$.

All the examples of sheaves on a semi-simplicial manifold $M_\bullet$ appearing in this thesis arise from considering a sheaf $\mathcal{S}$ that is defined functorially over all manifolds and letting $\mathcal{S}^n$ be the sheaf $\mathcal{S}$ on M_n, with ∂_j^n the maps obtained by functoriality. For example, for a fixed finite-dimensional vector space V and for fixed $q \in \mathbb{N}$, we define the sheaf Ω_V^q of V-valued q-forms on a semi-simplicial manifold $M_\bullet$ to be the sheaf on $M_\bullet$ which at each level M_n is the sheaf of V-valued q-forms on M_n, with the maps $\partial_j^n = (d_j^n)^*$ between them. In this case, the maps $\delta : \Omega^q(M_{n-1}) \to \Omega^q(M_n)$ are defined by

$$\delta := \sum_{j=0}^{n} (-1)^j (d_j^n)^*. \tag{2.35}$$

Remark 2.10. If $\mathcal{S}^\bullet$ is a sheaf over a semi-simplicial manifold $M_\bullet$ such that each $\mathcal{S}^n$ is an acyclic sheaf over M_n, then

$$H^p(M_\bullet, \mathcal{S}_\bullet) = \frac{\ker(\delta : \mathcal{S}^p(M_p) \to \mathcal{S}^{p+1}(M_{p+1}))}{Im(\delta : \mathcal{S}^{p-1}(M_{p-1}) \to \mathcal{S}^p(M_p))}, \tag{2.36}$$

so it does not follow in general that $\mathcal{S}^\bullet$ is acyclic as a sheaf on $M_\bullet$. In particular, for any semi-simplicial manifold $M_\bullet$ and any p, $q \in \mathbb{N}$ we have

$$H^p(M_\bullet, \Omega_V^q) = \frac{\ker(\delta : \Omega^q(M_p, V) \to \Omega^q(M_{p+1}, V))}{Im(\delta : \Omega^q(M_{p-1}, V) \to \Omega^q(M_p, V))}. \tag{2.37}$$

We recall now two theorems that relate sheaf cohomology on semi-simplicial manifolds with other cohomology theories, and which we will use to prove some classification results in Chapters 3 and 4.

Theorem 2.11 ([49])**.** *Let Z be an abelian Lie group, let $M_\bullet$ be a semi-simplicial manifold and let $\underline{Z}$ be the sheaf of locally constant Z-valued functions on $M_\bullet$. Then*

$$H^*(M_\bullet, \underline{Z}) = H^*(||M_\bullet||, Z), \tag{2.38}$$

where the right-hand side of (2.38) denotes singular cohomology of the fat geometric realization (cf. Section 2.1.1) of $M_\bullet$.

Remark 2.12. In the situation of Theorem 2.11, and for $Z = V$ a vector space, $H^*(M_\bullet, \underline{V})$ can be computed by taking de Rham resolutions $\Omega^\bullet_V$ on each M_n

$$\underline{V} \to C^\infty_V \overset{d}{\to} \Omega^1_V \overset{d}{\to} \dots \tag{2.39}$$

Recall that the maps $\delta : \Omega^m_V(M_{n-1}) \to \Omega^m_V(M_n)$ are given by (2.35) . Then $H^*(M_\bullet, V)$ is the total cohomology of $(\Omega^m_V(M_n), \delta, d)$, which provides a useful tool for computing singular cohomology of topological spaces much more general than manifolds.

For the next result, we recall the notion of group cohomology [54]. For G a group, a *G-module* is an abelian group M with an action by automorphisms of G. Given a G-module M, write $C^r(G, M)$ for the space of (set-theoretical) functions $G^r \to M$ and define a differential $d : C^r(G, M) \to C^{r+1}(G, M)$ by

$$\begin{aligned} dm(g_0, \dots, g_r) =& g_0 \cdot m(g_1, \dots, g_r) + \sum_{j=1}^{r} (-1)^j m(g_0, \dots, g_{j-1}g_j, g_{j+1}, \dots, g_r) \\ &+ (-1)^{r+1} m(g_0, \dots, g_{r-1}). \end{aligned} \tag{2.40}$$

Group cohomology with coefficients on M is defined by

$$H^r_{gr}(G, M) := \frac{\{m \in C^r(G, M) \mid dm = 0\}}{dC^{r-1}(G, M)}. \tag{2.41}$$

If G is a topological group acting continuously by automorphisms on the topological abelian group M, then we write $H^r_{gr,cont}(G, M)$ for the space defined as in (2.41) but replacing $C^r(G, M)$ with the space of continuous maps $G^r \to M$.

Theorem 2.13 ([48]). *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let V be a finite-dimensional vector space. Then*

$$H^p(BG_\bullet, \Omega^q_V) = H^{p-q}_{gr,cont}(G, S^q\mathfrak{g}^* \otimes V), \tag{2.42}$$

where the G-module structure on $S^q\mathfrak{g}^* \otimes V$ is given by the coadjoint action. Moreover, if G is compact, then for $r > 0$ one has

$$H^r_{gr,cont}(G, S^q\mathfrak{g}^* \otimes V) = 0. \tag{2.43}$$

We also need the following explicit formulas for the isomorphism (2.42) in low degrees. In the sequel we will use the following notation: for manifolds X, Y and points $x \in X$,

$y \in Y$ we identify $T_{(x,y)}(X \times Y) = T_x X \oplus T_y Y$ and write $v_x + v_y \in T_{(x,y)}(X \times Y)$ for vectors tangent to the product manifold.

Lemma 2.14. *Let G be a Lie group and let V be a vector space. Then*

1. *The isomorphism $H^2(BG_\bullet, \Omega^1_V) = H^1_{gr,cont}(G, \mathfrak{g}^* \otimes V)$ is induced by the map*

$$\phi : \frac{\{\tau \in \Omega^1(G^2, V) \mid \delta\tau = 0\}}{\{\delta\sigma \mid \sigma \in \Omega^1(G, V)\}} \to \frac{\{\kappa : G \times \mathfrak{g} \to V \mid \kappa(g_1 g_2, v) = \kappa(g_1, v) + \kappa(g_2, g_1^{-1} v g_1)\}}{\{\kappa(g, v) = \chi(g^{-1} v g) - \chi(v) \mid \chi : \mathfrak{g} \to V\}}$$

with

$$\phi(\tau)(g, v) = \tau_{(g^{-1},1)}(0 + v) + \tau_{(g^{-1},g)}(g^{-1} v + 0), \tag{2.44}$$
$$\phi^{-1}(\kappa)_{(g_1,g_2)}(v_{g_1} + v_{g_2}) = \kappa(g_2, g_1^{-1} v_{g_1}). \tag{2.45}$$

2. *The isomorphism $H^2(BG_\bullet, \Omega^2_V) = H^0_{gr,cont}(G, S^2\mathfrak{g}^* \otimes V)$ is induced by the map:*

$$\psi : \frac{\{\nu \in \Omega^2(G^2, V) \mid \delta\nu = 0\}}{\{\delta\sigma \mid \sigma \in \Omega^2(G, V)\}} \to \{\langle\cdot,\cdot\rangle : S^2\mathfrak{g} \to V \mid \langle Ad(g)u, Ad(g)v\rangle = \langle u, v\rangle\}$$

with

$$\psi(\nu)(u, v) = \frac{1}{2}\nu_{(1,1)}(0 + u, v + 0) + \frac{1}{2}\nu_{(1,1)}(0 + v, u + 0), \tag{2.46}$$
$$\psi^{-1}(\langle\cdot,\cdot\rangle) = -\langle g_1^* \theta^L \wedge g_2^* \theta^R\rangle, \tag{2.47}$$

where θ^L, $\theta^R \in \Omega^1(G, \mathfrak{g})$ are the left- and right-invariant Maurer-Cartan 1-forms on G.

Proof. We show the proof of 2, as 1 follows from similar computations. First, let $\nu \in \Omega^1(G^2, V)$ satisfy $\delta\nu = 0$. It is easy to see that the expression (2.46) is invariant under changing ν by $\delta\sigma$, for $\sigma \in \Omega^2(G, V)$. Moreover, we claim that

$$\langle u, v\rangle = \frac{1}{2}\nu_{(g_1,g_2)}(0 + ug_2, g_1 v + 0) + \frac{1}{2}\nu_{(g_1,g_2)}(0 + vg_2, g_1 u + 0) \tag{2.48}$$

is actually independent of g_1, $g_2 \in G$. This can be seen by symmetrizing u and v in the following identities that follow from the cocycle condition for ν.

$$\nu_{(g_1,g_2)}(0 + ug_2, g_1 v + 0) + \nu_{(g_1^{-1}, g_1 g_2)}(0 + g_1 u g_2, 0 + g_1 v g_2) = \nu_{(1,g_2)}(0 + ug_2, v + 0), \tag{2.49}$$

$$\nu_{(g_1,g_2)}(0 + ug_2, g_1 v + 0) + \nu_{(g_1 g_2, g_2^{-1})}(g_1 u g_2 + 0, g_1 v g_2 + 0) = \nu_{(g_1,1)}(0 + u, g_1 v + 0). \tag{2.50}$$

Then, Ad-invariance of $\langle\cdot,\cdot\rangle$ follows from applying the cocycle condition on $(1,g,g^{-1})\in G^3$ and $0+0+ug^{-1}$, $gvg^{-1}+0+0\in T_{(1,g,g^{-1})}G^3$, which yields

$$\nu_{(1,1)}(0+gug^{-1},gvg^{-1}+0)=\nu_{(g,g^{-1})}(0+ug^{-1},gv+0). \tag{2.51}$$

This concludes that the map ψ is well-defined. To see that ψ^{-1} is well-defined, we note simply that if $\langle\cdot,\cdot\rangle : S^2\mathfrak{g}\to V$ is Ad-invariant then

$$\begin{aligned}\langle (g_1g_2)^*\theta^L\wedge g_3^*\theta^R\rangle &- \langle g_1^*\theta^L\wedge (g_2g_3)^*\theta^R\rangle\\ &= \langle Ad(g_2^{-1})g_1^*\theta^L+g_2^*\theta^L\wedge g_3^*\theta^R\rangle - \langle g_1^*\theta^L\wedge g_2^*\theta^R + Ad(g_2)g_3^*\theta^R\rangle\\ &= \langle g_2^*\theta^L\wedge g_3^*\theta^R\rangle - \langle g_1^*\theta^L\wedge g_2^*\theta^R\rangle.\end{aligned} \tag{2.52}$$

Checking that $\psi\circ\psi^{-1}=id$ is immediate, while $\psi^{-1}\circ\psi=id$ follows from noting that for $\nu\in\Omega^2(G^2,V)$ with $\delta\nu=0$ we have the following identities.

$$\begin{aligned}\nu_{(g_1,g_2)}(u_{g_1}+u_{g_2},v_{g_1}+v_{g_2}) &+ \nu_{(g_2^{-1}g_1^{-1},g_1g_2)}(0+u_{g_1}u_{g_2},v_{g_2}^{-1}v_{g_1}^{-1}+v_{g_1}v_{g_2})\\ &= \nu_{(g_2^{-1}g_1^{-1},g_1)}(0+u_{g_1},v_{g_2}^{-1}v_{g_1}^{-1}+v_{g_1}) + \nu_{(g_2^{-1},g_2)}(g_2^{-1}g_1^{-1}u_{g_1}+u_{g_2},v_{g_2}^{-1}+v_{g_2}),\end{aligned} \tag{2.53}$$

$$\begin{aligned}\nu_{(g_2^{-1}g_1^{-1},g_1)}&(0+u_{g_1},v_{g_2}^{-1}v_{g_1}^{-1}+v_{g_1})\\ &= \nu_{(g_1^{-1},g_1)}(0+u_{g_1},v_{g_1}^{-1}+v_{g_1}) + \nu_{(g_2^{-1},1)}(0+g_1^{-1}u_{g_1},v_{g_2}^{-1}+0),\end{aligned} \tag{2.54}$$

$$\begin{aligned}\nu_{(g_2^{-1},1)}(0+g_1^{-1}u_{g_1},v_{g_2}^{-1}+0) &+ \nu_{(g_2^{-1},g_2)}(g_2^{-1}g_1^{-1}u_{g_1}+0,v_{g_2}^{-1}+0)\\ &= \nu_{(g_2^{-1},g_2)}(0+g_1^{-1}u_{g_1}g_2,v_{g_2}^{-1}+0).\end{aligned} \tag{2.55}$$

Here we are writing, for $u_g\in T_gG$, $u_g^{-1}:=dinv_g(u_g)\in T_{g^{-1}}G$, where $inv:G\to G$ is the map $g\mapsto g^{-1}$. Adding these identities and skew-symmetrizing $u_{g_1}+u_{g_2}$ and $v_{g_1}+v_{g_2}$ yields

$$\nu = g_1^*\sigma-(g_1g_2)^*\sigma+g_2^*\sigma-\langle g_1^*\theta^L\wedge g_2^*\theta^R\rangle, \tag{2.56}$$

for $\langle\cdot,\cdot\rangle$ defined by (2.46) and $\sigma\in\Omega^2(G,V)$ defined by

$$\sigma_g(u_g,v_g):=\frac{1}{2}\nu_{(g^{-1},g)}(0+u_g,v_g^{-1}+v_g)+\frac{1}{2}\nu_{(g^{-1},g)}(u_g^{-1}+u_g,0+v_g), \tag{2.57}$$

which concludes the proof. □

2.2 Derived manifolds

2.2.1 Graded algebra and L_∞-algebras

We fix the following notations and conventions throughout the whole thesis. First, 'graded' objects will always be $\mathbb{Z}$-graded unless otherwise stated. For $V = \oplus_{n\in\mathbb{Z}} V_n$ a graded vector space, we write $|f| = d$ for the degree of an homogeneous element $f \in V_d$. Sometimes we will also write $(-1)^f := (-1)^{|f|}$. All equations that depend on the degree of vectors are stated for homogeneous elements and then extended to all vectors by linearity. The dual V^* is regarded as a graded vector space with $(V^*)_{-n} = (V_n)^*$. For $d \in \mathbb{Z}$, we write $V[d]$ for the graded vector space with V_n in degree $n - d$. A graded $\mathbb{R}$-algebra A is commutative if the product satisfies $f \cdot g = (-1)^{|f||g|} g \cdot f$. A graded left module M for A is a left A-module with a grading such that $|f \cdot m| = |f| + |m|$ for $f \in A$, $m \in M$. An A-multilinear map of degree l on M with values on the left A-module N is a map $\omega : M^p \to N$ such that

$$|\omega(m_1, ..., m_p)| = |m_1| + ... + |m_p| + l, \tag{2.58}$$

$$\omega(m_1, ..., f \cdot m_j, ..., m_p) = (-1)^{f(l+m_1+...+m_{j-1})} f \cdot \omega(m_1, ..., m_p). \tag{2.59}$$

It is graded symmetric when

$$\omega(m_1, ..., m_i, m_{i+1}, ..., m_p) = (-1)^{|m_i||m_{i+1}|} \omega(m_1, ..., m_{i+1}, m_i, ..., m_p) \tag{2.60}$$

and it is graded skew-symmetric when

$$\omega(m_1, ..., m_i, m_{i+1}, ..., m_p) = -(-1)^{|m_i||m_{i+1}|} \omega(m_1, ..., m_{i+1}, m_i, ..., m_p). \tag{2.61}$$

We write $S^\bullet M^* \otimes N = \oplus_p S^p M^* \otimes N$ for the graded A-module of graded symmetric multilinear maps and $\Lambda^\bullet M^* \otimes N = \oplus_p \Lambda^p M^* \otimes N$ for the graded A-module of graded skew-symmetric multilinear maps (and we supress N when $N = A$). The space $S^\bullet M^*$ is a commutative graded algebra under the product

$$\begin{aligned}(\omega_1 \odot \omega_2)(m_1, ..., m_{p_1+p_2}) := \sum_{\sigma \in S_{p_1,p_2}} & \omega_1(m_{\sigma(1)}, ..., m_{\sigma(p_1)}) \omega_2(m_{\sigma(p_1+1)}, ..., m_{\sigma(p_1+p_2)}) \\ & \times (-1)^{\omega_2(m_{\sigma(1)}+...+m_{\sigma(p_1)})} (-1)^{\gamma(\sigma)}.\end{aligned} \tag{2.62}$$

Here

$$\begin{aligned} S_{p_1,p_2,\ldots,p_j} := \{\sigma \in S_{p_1+\ldots+p_j} \mid\ & \sigma(1) < \ldots < \sigma(p_1), \\ & \sigma(p_1+1) < \ldots < \sigma(p_1+p_2), \\ & \ldots, \\ & \sigma(p_1+\ldots+p_{j-1}+1) < \ldots < \sigma(p_1+\ldots+p_{j-1}+p_j)\} \end{aligned} \tag{2.63}$$

and $(-1)^{\gamma(\sigma)}$ is the *Kozul sign*; that is, it is the result of writing the permutation

$$(m_1, \ldots, m_{p_1+p_2}) \mapsto (m_{\sigma(1)}, \ldots, m_{\sigma(p_1+p_2)}) \tag{2.64}$$

as a product of transpositions $(m_{i_0}, m_{j_0}) \cdot \ldots \cdot (m_{i_N}, m_{j_N})$, and defining

$$(-1)^{\gamma(\sigma)} := \prod_s (-1)^{m_{i_s} m_{j_s}}. \tag{2.65}$$

Note that, although this is not reflected in the notation, $(-1)^{\gamma(\sigma)}$ depends on the degrees of the permuted elements m_i. We will also write $(-1)^\sigma$ for the sign of a permutation σ. The *décalage isomorphism* is the isomorphism of graded A-modules

$$\begin{gathered} S^p((M[1])^*) \to (\Lambda^p M^*)[-p], \\ \omega \mapsto \omega_{sk}, \end{gathered} \tag{2.66}$$

where

$$\omega_{sk}(m_1, \ldots, m_p) := (-1)^{pm_1+(p-1)m_2+pm_3+\ldots+m_{p-1}} \omega(m_1, \ldots, m_p). \tag{2.67}$$

We also write $\omega \mapsto \omega_{sy}$ for its inverse. In particular, it induces structure of graded commutative algebra on $\oplus_p (\Lambda^p M^*)[-p]$, which as a vector space coincides with $\Lambda^\bullet M^*$. A derivation of A of degree l is an $\mathbb{R}$-linear map $X : A \to A[l]$ satisfying

$$X(fg) = X(f) \cdot g + (-1)^{l|f|} f X(g). \tag{2.68}$$

The space $Der(A)$ of derivations of A is a graded A-module. We equip it with the graded Lie bracket

$$[X, Y](f) := X(Y(f)) - (-1)^{|X||Y|} Y(X(f)). \tag{2.69}$$

This satisfies

$$|[X,Y]| = |X| + |Y|, \tag{2.70}$$

$$[X,Y] = -(-1)^{|X||Y|}[Y,X], \tag{2.71}$$

$$[X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z]], \tag{2.72}$$

$$[X,fY] = X(f)Y + (-1)^{|X||f|}f[X,Y]. \tag{2.73}$$

When $A = S^\bullet V^*$ for V a finite-dimensional graded vector space, then there is a canonical isomorphism $Der(A) = A \otimes V$, where each $v \in V$ induces a derivation defined on $V^* \subset A$ by $\alpha \mapsto \alpha(v)$[1] and then extended to all of A imposing Leibniz's rule.

Definition 2.15 ([175, 251])**.** A *curved L_∞-algebra* (V, Q) is a graded vector space V with a degree 1 derivation Q of $S^\bullet(V[1])^*$ (the *homological vector field*) such that $Q^2 = 0$. A *morphism of curved L_∞-algebras* $(W, Q_W) \to (V, Q_V)$ is a morphism of graded algebras $\varphi : S^\bullet(V[1])^* \to S^\bullet(W[1])^*$ such that $\varphi(Q_V(f)) = Q_W(\varphi(f))$.

Since $Der(S^\bullet(V[1])^*) = S^\bullet(V[1])^* \otimes (V[1])$, it follows from Definition 2.15 that the homological vector field of a curved L_∞-algebra determines a sequence indexed by $p \in \mathbb{N}$ of degree 1 graded symmetric p-linear maps $\{\cdot, ..., \cdot\} : S^p V[1] \to V[1]$. In particular, the 0-bracket is an element $\Phi \in V_2$, which we call *curvature*. The condition $Q^2 = 0$ is equivalent to the *higher Jacobi identities*: for $n \in \mathbb{N}$ and v_1, ..., $v_n \in V[1]$,

$$\sum_{p+q=n} \sum_{\sigma \in S_{p,q}} (-1)^{\gamma(\sigma)} \{\{v_{\sigma(1)}, ..., v_{\sigma(p)}\}, v_{\sigma(p+1)}, ..., v_{\sigma(n)}\} = 0, \tag{2.74}$$

where the Kozul sign $(-1)^{\gamma(\sigma)}$ is computed with the degrees of v_i as elements of $V[1]$. Alternatively, we can use the décalage isomorphism (2.66) to define for each $p \in \mathbb{N}$ the map $\{\cdot, ..., \cdot\}_{sk}$, which is a graded skew-symmetric p-linear map on V with values on V and of degree $2 - p$. The Jacobi identities for the skew-symmetric maps is

$$\sum_{p+q=n} \sum_{\sigma \in S_{p,q}} (-1)^{pq}(-1)^{\sigma}(-1)^{\gamma(\sigma)} \{\{v_{\sigma(1)}, ..., v_{\sigma(p)}\}_{sk}, v_{\sigma(p+1)}, ..., v_{\sigma(n)}\}_{sk} = 0, \tag{2.75}$$

where the Koszul sign is now computed with the degrees of v_i as elements of V. The homological vector field Q can be recovered from the brackets by the formula

$$Q(\xi) := \xi \left(\Phi + \{\cdot\} + \frac{1}{2}\{\cdot,\cdot\} + \frac{1}{6}\{\cdot,\cdot,\cdot\} + ... \right) \tag{2.76}$$

[1]One could also define it by $\alpha \mapsto (-1)^{|v||\alpha|}\alpha(v)$, but we keep the above notation, as it seems more natural in examples.

for $\xi \in V[1]^*$, which is then extended using Leibniz's rule. When $\Phi = 0$, one of the Jacobi identities implies that the 1-bracket squares to 0 and so it induces a structure of chain complex on V.

Given two curved L_∞-algebras (W, Q_W), (V, Q_V), morphisms of graded algebras $\varphi : S^\bullet(V[1])^* \to S^\bullet(W[1])^*$ are in bijection with degree-preserving linear maps $g : V[1]^* \to S^\bullet(W[1])^*$. We can decompose these as $g = g_1 + g_2 + ...$, with g_p the projection of g onto $S^p(W[1])^*$, and dualize to obtain degree-preserving linear maps $f_p : S^p(W[1]) \to V[1]$. One can then check that φ defines a morphism of curved L_∞-algebras if and only if the following identities are satisfied for $n \in \mathbb{N}$ and w_1, ..., $w_n \in W[1]$.

$$\begin{aligned}\sum_{j\geq 1} \sum_{p_1+...+p_j=n} \sum_{\sigma\in S_{p_1,...,p_j}} &(-1)^{\gamma(\sigma)} \{f_{p_1}(w_{\sigma(1)}, ..., w_{\sigma(p_1)}), ..., f_{p_j}(w_{\sigma(n-p_j+1)}, ..., w_{\sigma(n)})\}^V \\ &= \sum_{p+q=n} \sum_{\sigma\in S_{p,q}} (-1)^{\gamma(\sigma)} f_{q+1}(\{w_{\sigma(1)}, ..., w_{\sigma(p)}\}^W, w_{\sigma(p+1)}, ..., w_{\sigma(p+q)}).\end{aligned} \tag{2.77}$$

In particular, when $\Phi_V = 0$ and $\Phi_W = 0$, then $f_1 : W[1] \to V[1]$ commutes with the 1-brackets. In this case, we say φ is a *quasi-isomorphism* if f_1 induces an isomorphism on the cohomology of the corresponding chain complexes.

The description of curved L_∞-algebras in terms of homological vector fields is very convenient to prove abstract results, since it is more concise, but practical examples appear naturally in the bracket description. We dedicate the rest of this section to spell out in detail the axioms, in the skew-symmetric bracket formulation, of a certain family of curved L_∞-algebras and a certain family of morphisms between them which will cover all the examples in this thesis.

Definition 2.16. A *curved cubic L_∞-algebra* is a graded vector space $V = \oplus_n V_n$ equipped with the following data.

1. A degree 2 element $\Phi \in V$, called *curvature*,
2. A degree 1 linear map $d : V \to V$,
3. A graded skew-symmetric degree 0 bilinear map $[\cdot, \cdot] : V \otimes V \to V$,
4. A graded skew-symmetric degree -1 trilinear map $\{\cdot, \cdot, \cdot\} : V \otimes V \otimes V \to V$,

subject to

$$
\begin{aligned}
d\Phi &= 0, && (2.78)\\
d^2(e) &= -[\Phi, e], && (2.79)\\
d[e_1, e_2] &= [de_1, e_2] + (-1)^{e_1}[e_1, de_2] - \{\Phi, e_1, e_2\}, && (2.80)
\end{aligned}
$$

$$
\begin{aligned}
d\{e_1, e_2, e_3\} &+ \{de_1, e_2, e_3\} + (-1)^{e_1}\{e_1, de_2, e_3\} + (-1)^{e_1+e_2}\{e_1, e_2, de_3\} \\
&= [e_1, [e_2, e_3]] - [[e_1, e_2], e_3] - (-1)^{e_1 e_2}[e_2, [e_1, e_3]],
\end{aligned} \tag{2.81}
$$

$$
\begin{aligned}
\{[e_1, e_2], e_3, e_4\} &+ (-1)^{e_2 e_3 + 1}\{[e_1, e_3], e_2, e_4\} + (-1)^{e_4(e_2+e_3)}\{[e_1, e_4], e_2, e_3\} \\
&+ (-1)^{e_1(e_2+e_3)}\{[e_2, e_3], e_1, e_4\} + (-1)^{e_1(e_2+e_4)+e_3 e_4+1}\{[e_2, e_4], e_1, e_3\} \\
&+ (-1)^{(e_1+e_2)(e_3+e_4)}\{[e_3, e_4], e_1, e_2\} \\
&= [\{e_1, e_2, e_3\}, e_4] + (-1)^{e_3 e_4 + 1}[\{e_1, e_2, e_4\}, e_3] \\
&+ (-1)^{e_2(e_3+e_4)}[\{e_1, e_3, e_4\}, e_2] + (-1)^{e_1(e_2+e_3+e_4)}[\{e_2, e_3, e_4\}, e_1]
\end{aligned} \tag{2.82}
$$

for e_1, e_2, e_3, $e_4 \in V$. In particular, a curved cubic L_∞-algebra with $\Phi = 0$, $\{\cdot, \cdot, \cdot\} = 0$ is a *differential graded Lie algebra* (DGLA). For

$$
(W, \Phi^W, d^W, [\cdot, \cdot]^W, \{\cdot, \cdot, \cdot\}^W),\ (V, \Phi^V, d^V, [\cdot, \cdot]^V, \{\cdot, \cdot, \cdot\}^V)
$$

curved cubic L_∞-algebras, a *quadratic morphism* between them is a pair of linear maps $f_1 : W \to V$, $f_2 : W \otimes W \to V$ such that

1. f_1 has degree 0 and f_2 is graded skew-symmetric of degree -1.
2. The following identities are satisfied.

$$
\begin{aligned}
f_1(\Phi^W) - \Phi^V &= 0, && (2.83)\\
f_1(d^W e) - d^V(f_1(e)) &= f_2(\Phi^W, e), && (2.84)
\end{aligned}
$$

$$
\begin{aligned}
f_1([e_1, e_2]^W) - [f_1(e_1), f_1(e_2)]^V &= d^V f_2(e_1, e_2) \\
&+ f_2(d^W e_1, e_2) + (-1)^{e_1} f_2(e_1, d^W e_2),
\end{aligned} \tag{2.85}
$$

$$
\begin{aligned}
f_1(\{e_1, e_2, e_3\}^W) &- \{f_1(e_1), f_1(e_2), f_1(e_3)\}^V = \\
&= -[f_2(e_1, e_2), f_1(e_3)]^V + (-1)^{e_2 e_3}[f_2(e_1, e_3), f_1(e_2)]^V \\
&- (-1)^{e_1(e_2+e_3)}[f_2(e_2, e_3), f_1(e_1)]^V - f_2([e_1, e_2]^W, e_3) \\
&+ (-1)^{e_2 e_3} f_2([e_1, e_3]^W, e_2) - (-1)^{e_1(e_2+e_3)} f_2([e_2, e_3]^W, e_1)
\end{aligned} \tag{2.86}
$$

$$\begin{aligned}
&f_2(\{e_1,e_2,e_3\}^W,e_4)-(-1)^{e_3e_4}f_2(\{e_1,e_2,e_4\}^W,e_3)\\
&+(-1)^{e_2(e_3+e_4)}f_2(\{e_1,e_3,e_4\}^W,e_2)-(-1)^{e_1(e_2+e_3+e_4)}f_2(\{e_2,e_3,e_4\}^W,e_1)\\
&=-\{f_2(e_1,e_2),f_1(e_3),f_1(e_4)\}^V+(-1)^{e_2e_3}\{f_2(e_1,e_2),f_1(e_3),f_1(e_4)\}^V\\
&-(-1)^{e_4(e_2+e_3)}\{f_2(e_1,e_4),f_1(e_2),f_1(e_3)\}^V\\
&-(-1)^{e_1(e_2+e_3)}\{f_2(e_2,e_3),f_1(e_1),f_1(e_4)\}^V\\
&+(-1)^{e_1(e_2+e_4)+e_3e_4}\{f_2(e_2,e_4),f_1(e_1),f_1(e_3)\}^V\\
&-(-1)^{(e_1+e_2)(e_3+e_4)}\{f_2(e_3,e_4),e_1,e_2\}^V
\end{aligned}\tag{2.87}$$

If $\Phi^W=\Phi^V=0$, then we say that (f_1,f_2) is a *quasi-isomorphism* when f_1 induces a quasi-isomorphism between the chain complexes (W,d^W) and (V,d^V).

2.2.2 Derived manifolds

We give here a brief overview of the theory of derived manifolds following [32], see [80, 107, 269, 270] for detailed expositions in the equivalent language of differential graded manifolds. While the ∞-category of derived manifolds is defined in Section 7.2.3, all the relevant constructions in this thesis are described without losing rigor with the terminology of this section.

We start with some conventions and notations. A graded vector bundle $E\to M$ is simply a vector bundle equipped with a decomposition into sub-vector bundles $E=\bigoplus_{n\in\mathbb{Z}}E_n$. For $E\to M$ a graded vector bundle and $d\in\mathbb{Z}$ we write $E[d]$ for the graded vector bundle whose fibers are shifted by $-d$. The graded bundles E^*, $S^\bullet E^*$ and $\Lambda^\bullet E^*$ are defined similarly as in Section 2.2.1.

Definition 2.17 ([32])**.** A *derived manifold* $\mathcal{M}=(M,E,Q)$ is the folowing data.

1. A manifold M with a finite rank, $\mathbb{N}^{\geq 2}$-graded, vector bundle $E\to M$.
2. A degree 1 derivation (the *homological vector field*) $Q:\Gamma(S^\bullet E[1]^*)\to\Gamma(S^\bullet E[1]^*)$ such that $Q^2=0$.

A *morphism of derived manifolds* $\mathcal{M}^1\to\mathcal{M}^2$ is a pair (φ,ψ) consisting of a smooth map $\varphi:M^1\to M^2$ and a homomorphism of graded algebras $\psi:\Gamma(\varphi^*S^\bullet E^2[1]^*)\to\Gamma(S^\bullet E^1[1]^*)$ such that $\psi(\varphi^*Q_2(f)))=Q_1(\psi(f))$. The *Cartesian product* of derived manifolds (M,E,Q_E), (N,F,Q_F) is the derived manifold $(M\times N,p_M^*E\otimes p_N^*F,p_M^*Q_E\otimes p_N^*Q_F)$. We write *dMan* for the category of derived manifolds.

Remark 2.18. Definition 2.17 implies that Q is tensorial, in the sense that $Q(fe)=fQ(e)$ for $e\in\Gamma(S^\bullet E[1]^*)$ and $f\in C^\infty(M)$. This follows from the chain rule for derivations, and the fact that $Q(f) = 0$ for $f \in C^\infty(M)$, since $Q(f)$ must be of degree 1 but $\Gamma(S^\bullet E[1]^*)$ is concentrated in non-positive degrees.

A derived manifold $\mathcal{M} = (M, E, Q)$ such that $M = \{*\}$ is the same as a curved L_∞-algebra V concentrated in degrees ≥ 2 (cf. Definition 2.15). In general, a derived manifold is a bundle of $\mathbb{N}^{\geq 2}$-graded curved L_∞-algebras in that it determines (and is determined by) through the same procedure as in Section 2.2.1 a section $\Phi \in \Gamma(E_2)$, a degree 1 map $d : \Gamma(E[1]) \to \Gamma(E[1])$, and for $p \geq 2$, degree 1 graded symmetric p-linear maps $\{\cdot, ..., \cdot\} : \Gamma(S^p E[1]) \to \Gamma(E[1])$ satisfying the higher Jacobi identities (2.74). One can also consider $\mathbb{Z}^{\leq 0}$-*graded dg-manifolds*, which are objects that locally look like bundles of $\mathbb{N}^{\geq 2}$-graded curved L_∞-algebras, but which are glued along possibly polynomial isomorphisms of L_∞-algebras (hence, not defining a global vector bundle). Despite seeming more general, the category of *dg*-manifolds is actually equivalent to the category of derived manifolds. [32].

Remark 2.19. Given derived manifolds $\mathcal{M}^1 = (M^1, E^1, Q^1)$, $\mathcal{M}^2 = (M^2, E^2, Q^2)$, a morphism $\mathcal{M}^1 \to \mathcal{M}^2$ can be described in terms of the multilinear brackets as a smooth map $\varphi : M^1 \to M^2$, together with fiberwise morphisms of curved L_∞-algebras $E^1_{|x} \to E^2_{\varphi(x)}$ varying smoothly over $x \in M$. In particular, a linear map $E^1 \to \varphi^* E^2$ preserving the degrees, Φ and all the brackets is an example of a morphism of derived manifolds, but more general morphisms exist, such as those given by the quadratic morphisms from Definition 2.16.

We regard a derived manifold $\mathcal{M} = (M, E, Q)$ as a geometric model for the topological space $Z(\mathcal{M}) := \{x \in M \,|\, \Phi(x) = 0\} \subset M$ (the *zero locus* of $\mathcal{M}$), where $\Phi : M \to E_2$ is the section determined by Q. As for the case of L_∞-algebras motivated in the introduction to this chapter, $\mathcal{M}$ contains additional information about higher obstructions of the equation $\Phi(x) = 0$. The *tangent complex* of $\mathcal{M}$ is the $\mathbb{Z}^{\geq 0}$-graded chain complex of vector bundles $T\mathcal{M}$ over $Z(\mathcal{M})$ whose fiber at each point $x \in Z(\mathcal{M})$ is

$$T_x M \overset{d\Phi_x}{\to} E_{2|x} \overset{d(x)}{\to} E_{3|x} \overset{d(x)}{\to} \dots \overset{d(x)}{\to} E_{m|x} \to 0 \to \dots, \tag{2.88}$$

for d the 1-bracket induced by Q. Here we are slightly abusing notation by writing $d\Phi_x$ for the composition of $d\Phi_x : T_x M \to T_{(x,0)} E_2$ with the projection $T_{(x,0)} E_2 \to E_{2|x}$, $v \mapsto v - d0(d\pi(v))$, where $0 : M \to E_2$ is the zero section. The fact that (2.88) is indeed a complex follows from the axioms

$$\begin{aligned} &d\Phi(x) = 0, \\ &d^2(e_x) = [\Phi(x), e_x], \end{aligned} \tag{2.89}$$

for $x \in M$ and $e \in E_x$, which we may evaluate at $x \in Z(\mathcal{M})$. In (2.88), T_xM is regarded in degree 0 and the other vector spaces are graded accordingly. The *cotangent complex* is the dual of the above complex; i.e, it is the $\mathbb{Z}^{\leq 0}$-graded chain complex of vector bundles over $Z(\mathcal{M})$ whose fiber at each point $x \in Z(\mathcal{M})$ is

$$... \to 0 \to E^*_{m|x} \overset{d(x)^*}{\to} E^*_{m-1|x} \overset{d(x)^*}{\to} ... \overset{d(x)^*}{\to} E^*_{2|x} \overset{d\Phi(x)^*}{\to} T^*_xM, \tag{2.90}$$

where T^*_xM has degree 0.

If $(\varphi, \psi) : \mathcal{M}^1 \to \mathcal{M}^2$ is a morphism of derived manifolds, then ψ induces a linear map $\Gamma(\varphi^* E^2[1]^*) \to \Gamma(E^1[1]^*)$ by restricting and projecting in the obvious way. This determines a degree preserving map of vector bundles $\psi^1 : E^1 \to E^2$ covering the map $\varphi : M^1 \to M^2$. Since (φ, ψ) preserves the homological vector fields, one can check that $\varphi(Z(\mathcal{M}^1)) \subset Z(\mathcal{M}^2)$, and that ψ^1_* induces a chain map ψ_* between the tangent complexes of $\mathcal{M}_1$ and $\mathcal{M}_2$ covering φ; we call this the *differential* of (φ, ψ).

Example 2.20. Given a section $\Phi : M \to E$ of a vector bundle $\pi : E \to M$, we construct the derived manifold $\mathcal{M} = (M, E[-2], Q)$, with $Q : \Gamma(S^\bullet E[-1]^*) \to \Gamma(S^\bullet E[-1]^*)$ defined by contracting with $\Phi \in \Gamma(E)$. We regard $\mathcal{M}$ as a smooth model for the topological space $\Phi^{-1}(0)$, which we can construct without assuming that Φ is transversal to the zero section $0 : M \to E$. The tangent complex of $\mathcal{M}$ is the complex of vector bundles over $\Phi^{-1}(0) \subset M$ defined by

$$TM \overset{d\Phi}{\to} E \to 0 \to ... . \tag{2.91}$$

Asumme further that there is a map of vector bundles $d : E \to F$ such that $d\Phi = 0$. Then we may want to construct a derived manifold that keeps track of this 'higher obstruction'. For this we take the derived manifold defined by the graded vector bundle $E[-2] \oplus F[-3] \to M$ and the homological vector field Q on $\Gamma(S^\bullet E[-1]^* \otimes S^\bullet F[-2]^*)$ that acts as $\xi \mapsto \xi(\Phi)$ for $\xi \in \Gamma(E[-1]^*)$ and as $\xi \mapsto \xi(d\cdot)$ for $\xi \in \Gamma(F[-2]^*)$. The tangent complex of this derived manifold is the complex of vector bundles over $\Phi^{-1}(0) \subset M$ defined by

$$TM \overset{d\Phi}{\to} E \overset{d}{\to} F \to 0 \to ... . \tag{2.92}$$

Some of the derived manifolds in this thesis are actually constructed from vector bundles $E \to M$ with infinite-dimensional fibers over infinite-dimensional manifolds M, and we will omit treating the technical problems that this could produce. We will also refer to *complex* derived manifolds when $E \to M$ is a holomorphic vector bundle over a complex manifold and all the maps defining the curved L_∞-algebra structure are holomorphic.

Example 2.21. Let G be a Lie group and let $P \to M$ be a principal G-bundle. Write $\mathcal{A}(P)$ for the space of connections on P. For each $A \in \mathcal{A}(P)$, there is a structure of curved *DGLA* on $\Omega^{\geq 2}(ad\,P)$ (cf. Definition 2.16) given by $\Phi = F_A \in \Omega^2(ad\,P)$, the exterior covariant derivative $d^A : \Omega^j(ad\,P) \to \Omega^{j+1}(ad\,P)$ and the Lie bracket $[\cdot \wedge \cdot] : \Omega^{j_1}(ad\,P) \otimes \Omega^{j_2}(ad\,P) \to \Omega^{j_1+j_2}(ad\,P)$. The *derived space of flat connections on* P is the derived manifold associated to this structure, seen as a bundle of curved *DGLA*s over $\mathcal{A}(P)$. Note that it is a model for the space $\{A \in \mathcal{A}(P) \mid F_A = 0\}$ of flat connections, containing information about higher obstructions in a similar way to (but richer than) Example 2.20. Its tangent complex is the following chain complex of vector bundles over $\{A \in \mathcal{A}(P) \mid F_A = 0\}$.

$$\underline{\Omega^1(ad\,P)} \overset{d^A}{\to} \underline{\Omega^2(ad\,P)} \overset{d^A}{\to} \dots \overset{d^A}{\to} \underline{\Omega^n(ad\,P)} \to 0 \to \dots . \tag{2.93}$$

Similarly, if G is a complex Lie group and $P \to X$ is a smooth G-bundle over a complex manifold, then we can consider the space $\mathcal{D}(P) := \mathcal{A}(P)/\Omega^{1,0}(ad\,P)$ of semiconnections on P and define for each $A \in \mathcal{D}(P)$ a curved *DGLA* on $\Omega^{(0,\geq 2)}(ad\,P)$ with $\Phi = F_A^{0,2}$, $d = \overline{\partial}^A$ and the restriction of the Lie bracket from before. The result is a derived complex manifold, which we call the *derived space of holomorphic structures on* P. Its tangent complex is the following chain complex of vector bundles over $\{[A] \in \mathcal{D}(P) \mid F_A^{0,2} = 0\}$.

$$\underline{\Omega^{0,1}(ad\,P)} \overset{\overline{\partial}^A}{\to} \underline{\Omega^{0,2}(ad\,P)} \overset{\overline{\partial}^A}{\to} \dots \overset{\overline{\partial}^A}{\to} \underline{\Omega^{0,n}(ad\,P)} \to 0 \to \dots . \tag{2.94}$$

2.2.3 Simplicial derived manifolds

We use the category of simplicial derived manifolds as an approximate model for the ∞-category of derived differentiable ∞-stacks. While this ∞-category is defined in Section 7.2.3, all the relevant constructions in this thesis are described without losing rigor with the terminology of this section.

Definition 2.22. A *simplicial derived manifold* $\mathcal{M}_\bullet$ is the following data.

1. For $n \in \mathbb{N}$, a derived manifold $\mathcal{M}_n$.
2. For $n \in \mathbb{N}$ and $j = 0, \dots, n$, morphisms of derived manifolds $d_j^n : \mathcal{M}_n \to \mathcal{M}_{n-1}$ (*face maps*) and $s_j^n : \mathcal{M}_n \to \mathcal{M}_{n+1}$ (*degeneracy maps*) that satisfy (2.3).

A *morphism of simplicial derived manifolds* $f_\bullet : \mathcal{M}_\bullet \to \mathcal{N}_\bullet$ is a family of morphisms of derived manifolds $f_n : \mathcal{M}_n \to \mathcal{N}_n$ commuting with all the face and degeneracy maps. We write *sdMan* for the category of simplicial derived manifolds.

As in the case of simplicial manifolds, a simplicial derived manifold can be seen as a functor $\Delta^{op} \to \mathrm{dMan}$. Then we define the *Cartesian product* of simplicial derived manifolds $\mathcal{M}_\bullet, \mathcal{N}_\bullet : \Delta^{op} \to \mathrm{dMan}$ as the simplicial derived manifold $(\mathcal{M} \times \mathcal{N})_\bullet : \Delta^{op} \to \mathrm{dMan}$ defined by

$$(\mathcal{M} \times \mathcal{N})([n]) := \mathcal{M}([n]) \times \mathcal{N}([n]). \tag{2.95}$$

We regard a simplicial derived manifold $\mathcal{M}_\bullet$ as a geometric model for the topological space $Z(\mathcal{M}_0)/\sim$, where $p_0 \sim p_1$ if $\exists f \in Z(\mathcal{M}_1)$ with $d_0(f) = p_0$ and $d_1(f) = p_1$. In order to define the Moore and normalized tangent complexes of a simplicial derived manifold $\mathcal{M}_\bullet$, we proceed in the following steps.

1. For each $n \in \mathbb{N}$, write $\mathcal{M}_n = (M_n, E_{n,\bullet}, Q_n)$ and recall that its tangent complex is the following complex of vector bundles over $Z(\mathcal{M}_n)$.

$$TM_{n|Z(\mathcal{M}_n)} \xrightarrow{\Phi_*} E_{n,2|Z(\mathcal{M}_n)} \xrightarrow{d} E_{n,3|Z(\mathcal{M}_n)} \xrightarrow{d} \dots \xrightarrow{d} E_{n,m|Z(\mathcal{M}_n)} \to \dots . \tag{2.96}$$

2. Use the unique map $s : \mathcal{M}_0 \to \mathcal{M}_n$ obtained by composing degeneracies to pull-back the tangent complex of $\mathcal{M}_n$ to $Z(\mathcal{M}_0)$. Along with the maps $\delta := \sum_j (-1)^j d^n_{j,*}$ given by the alternating sum of the differentials of the simplicial maps $d^n_j : \mathcal{M}_n \to \mathcal{M}_{n-1}$, this yields a double complex of vector bundles over $Z(\mathcal{M}_0)$. We call this the *Moore tangent complex* of $\mathcal{M}_\bullet$ and denote it by

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] \arrow[d, "\Phi_*"] & TM_{3|Z(\mathcal{M}_0)} \arrow[r, "\partial"] \arrow[d, "\Phi_*"] & TM_{2|Z(\mathcal{M}_0)} \arrow[r, "\partial"] \arrow[d, "\Phi_*"] & TM_{1|Z(\mathcal{M}_0)} \arrow[r, "\partial"] \arrow[d, "\Phi_*"] & TM_{0|Z(\mathcal{M}_0)} \arrow[d, "\Phi_*"] \\
\dots \arrow[r, "\partial"] \arrow[d, "d"] & E_{3,2|Z(\mathcal{M}_0)} \arrow[r, "\partial"] \arrow[d, "d"] & E_{2,2|Z(\mathcal{M}_0)} \arrow[r, "\partial"] \arrow[d, "d"] & E_{1,2|Z(\mathcal{M}_0)} \arrow[r, "\partial"] \arrow[d, "d"] & E_{0,2|Z(\mathcal{M}_0)} \arrow[d, "d"] \\
\dots \arrow[r, "\partial"] \arrow[d, "d"] & E_{3,3|Z(\mathcal{M}_0)} \arrow[r, "\partial"] \arrow[d, "d"] & E_{2,3|Z(\mathcal{M}_0)} \arrow[r, "\partial"] \arrow[d, "d"] & E_{1,3|Z(\mathcal{M}_0)} \arrow[r, "\partial"] \arrow[d, "d"] & E_{0,3|Z(\mathcal{M}_0)} \arrow[d, "d"] \\
\dots \arrow[r, "\partial"] & \dots \arrow[r, "\partial"] & \dots \arrow[r, "\partial"] & \dots \arrow[r, "\partial"] & \dots,
\end{tikzcd} \tag{2.97}$$

where the pull-backs to $Z(\mathcal{M}_0)$ are omitted from the notation.

3. Let $A_{0,0} := TM_{0|Z(\mathcal{M}_0)}$ and define, for n, $m \geq 1$,

$$\begin{aligned}
A_{-n,0} &:= \frac{TM_{n|Z(\mathcal{M}_0)}}{\oplus_{j=0}^{n-1} s^{n-1}_{j,*}\big(TM_{n-1|Z(\mathcal{M}_0)}\big)}, \\
A_{-n,m} &:= \frac{E_{n,m+1|Z(\mathcal{M}_0)}}{\oplus_{j=0}^{n-1} s^{n-1}_{j,*}\big(E_{n-1,m+1|Z(\mathcal{M}_0)}\big)}.
\end{aligned} \tag{2.98}$$

Then ∂, Φ_* and d_* descend to give a double complex $(A_{\bullet,\bullet}, \partial, Q_*)$ (where Q_* denotes either Φ_* or d_* depending on the degree) of vector bundles[2] over $Z(\mathcal{M}_0)$.

4. The *(normalized) tangent complex* of $\mathcal{M}_\bullet$ is the $\mathbb{Z}$-graded chain complex of vector bundles over $Z(\mathcal{M}_0)$ obtained by taking the total complex of $(A_{\bullet,\bullet}, \partial, Q_*)$.

A morphism $f_\bullet : \mathcal{M}_\bullet \to \mathcal{N}_\bullet$ of simplicial derived manifolds induces at each level a morphism $f_{n,*}$ between the tangent complexes of $\mathcal{M}_n$ and $\mathcal{N}_n$ as in Section 2.2.2, and these descend to give a morphism between the tangent complexes of $\mathcal{M}_\bullet$ and $\mathcal{N}_\bullet$ since $f_\bullet$ commutes with the face and degeneracy maps.

Example 2.23. Let G be a Lie group acting smoothly on a derived manifold $\mathcal{M} = (M, E, Q)$. That is, we have a morphism of derived manifolds $\rho : \mathcal{M} \times G \to \mathcal{M}$ such that

1. $\rho \circ (\rho \times id) = \rho \circ (id \times m)$ as morphisms $\mathcal{M} \times G \times G \to \mathcal{M}$, where m is the product of G.

2. $\rho \circ (id \times 1) = id$ as morphisms $\mathcal{M} \to \mathcal{M}$, where $1 : \{*\} \to G$ is the inclusion of the unit.

As it follows from Remark 2.19, one such action could be given, for example, by a smooth action of G on M covered by a smooth, fiberwise linear, degree-preserving action of G on the total space of E such that $\Phi(x \cdot g) = \Phi(x) \cdot g$ and $\{e_1 g, ..., e_n g\} = \{e_1, ..., e_n\} g$ for $x \in M$, $g \in G$, e_1, ..., $e_n \in E$ and Φ, $\{\cdot, ..., \cdot\}$ the structure of curved L_∞ algebra on E induced by Q. In particular, the action of G on M preserves $Z(\mathcal{M})$. Then we can construct a simplicial derived manifold $(\mathcal{M}//G)_\bullet$, called the *quotient groupoid*, and serving as a geometric model for $Z(\mathcal{M})/G$, exactly as in Example 2.5. Namely, we let $(\mathcal{M}//G)_n = \mathcal{M} \times G^n$ and define simplicial and degeneracy maps by the same formulas as in Example 2.5, now understood as morphisms of graded manifolds in the obvious way. It is easy to see that the tangent complex of this simplicial derived manifold is the following chain complex of vector bundles over $Z(\mathcal{M})$.

$$... \to 0 \to \underline{\mathfrak{g}} \xrightarrow{\rho_*} TM \xrightarrow{d\Phi} E_2 \xrightarrow{d} E_3 \xrightarrow{d} ... \xrightarrow{d} E_m \to 0 \to ..., \tag{2.99}$$

with TM in degree 0, and where $d\Phi$ is defined as in Section 2.2.2.

[2]For an arbitrary simplicial derived manifold, it is not necessarily true that $A_{-n,m}$ are vector bundles, as there is no condition on the rank of $s_{j,*}$. This technical issue will not appear in the simplicial derived manifolds that we consider in this thesis, and will be treated more carefully in Chapter 7 by introducing the Kan conditions

Example 2.24. Let $\mathcal{M}$ be the derived manifold of flat connections on a G-bundle $P \to X$ from Example 2.21. Then $Gauge(P)$, the gauge group of P, is an infinite-dimensional Lie group with Lie algebra $\Omega^0(ad\, P)$. It acts on the space $\mathcal{A}(P)$ through its natural action of connections and on the space $\Omega^{\geq 2}(ad\, P)$ through the adjoint action. This induces an action on $\mathcal{M}$ because d^A and $[\cdot,\cdot]$ are equivariant with respect to these actions. The derived quotient stack $\mathcal{B}^{\flat,d}(P) := \mathcal{M}//Gauge(P)$ as in Example 2.23 is called the *derived moduli stack of flat connections on* P[3]. Its tangent complex is the chain complex of vector bundles over $\{A \in \mathcal{A}(P) \mid F_A = 0\}$ whose fiber at each flat connection A is the elliptic complex

$$\Omega^0(ad\, P) \xrightarrow{d^A} \Omega^1(ad\, P) \xrightarrow{d^A} \dots \xrightarrow{d^A} \Omega^n(ad\, P) \to 0 \to \dots, \tag{2.100}$$

with $\Omega^1(ad\, P)$ in degree 0. Similarly, if G is a complex Lie group, X is a complex manifold and $P \to X$ is a smooth G-bundle, then we may take $\mathcal{X}$ to be the (complex) derived manifold of holomorphic structures P as in Example 2.21. There is again an action of $Gauge(P)$ (which is now a complex Lie group) on $\mathcal{X}$ and the corresponding simplicial derived complex manifold $\mathcal{H}^d(P) := \mathcal{X}//Gauge(P)$ is called the *derived moduli stack of holomorphic structures on* P. Its tangent complex is the chain complex of vector bundles over $\{[A] \in \mathcal{D}(P) \mid F_A^{0,2} = 0\}$ whose fiber at each holomorphic structure $\overline{\partial}^A$ is

$$\Omega^0(ad\, P) \xrightarrow{\overline{\partial}^A} \Omega^{0,1}(ad\, P) \xrightarrow{\overline{\partial}^A} \dots \xrightarrow{\overline{\partial}^A} \Omega^{0,n}(ad\, P) \to 0 \to \dots. \tag{2.101}$$

2.3 Shifted symplectic structures

2.3.1 Shifted symplectic structures on simplicial derived manifolds

Let $\mathcal{M} = (M, E, Q)$ be a derived manifold. We write $C^\infty(\mathcal{M}) := \Gamma(S^\bullet E[1]^*)$, and refer to this as the *algebra of functions* on $\mathcal{M}$. A *vector field* on $\mathcal{M}$ is a derivation of $C^\infty(\mathcal{M})$. We write $\Gamma(T\mathcal{M})$ for the space of vector fields, which is a $C^\infty(\mathcal{M})$-module. It is also a differential graded Lie algebra under the graded commutator

$$[V, W] := VW - (-1)^{|V||W|}WV \tag{2.102}$$

and the differential $[Q,\cdot]$. A *differential p-form of degree l* on $\mathcal{M}$ is a graded skew-symmetric $C^\infty(\mathcal{M})$-linear map $\Gamma(T\mathcal{M})^{\otimes p} \to C^\infty(\mathcal{M})$ of degree l. We write $\Omega^p(\mathcal{M})_l$ for

[3]The use of the word *stack* is justified in Chapter 7, where we discuss in which sense simplicial manifolds are models for ∞-stacks.

the space of p-forms of degree l. There are operators

$$d : \Omega^p(\mathcal{M})_l \to \Omega^{p+1}(\mathcal{M})_l, \tag{2.103}$$

$$\iota_V : \Omega^p(\mathcal{M})_l \to \Omega^{p-1}(\mathcal{M})_{l+|V|}, \quad V \in \Gamma(T\mathcal{M}), \tag{2.104}$$

$$L_V : \Omega^p(\mathcal{M})_d \to \Omega^p(\mathcal{M})_{l+|V|}, \quad V \in \Gamma(T\mathcal{M}), \tag{2.105}$$

defined by

$$\begin{aligned} d\alpha(X_0, ..., X_p) &:= \sum_j (-1)^{X_j(\alpha+X_0+...+X_{j-1})+j} X_j(\alpha(X_0, ..., \hat{X}_j, ..., X_p)) \\ &+ \sum_{i<j} (-1)^{X_i(X_0+...+X_{i-1})+X_j(X_0+...+\hat{X}_i+...+X_{j-1})+i+j} \alpha([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p), \end{aligned} \tag{2.106}$$

$$\iota_V \alpha(X_1, ..., X_{p-1}) := (-1)^{\alpha V} \alpha(V, X_1, ..., X_{p-1}), \tag{2.107}$$

$$L_V := d\iota_V + \iota_V d. \tag{2.108}$$

They satisfy the Cartan relations

$$d^2 = 0, \tag{2.109}$$

$$dL_V - L_V d = 0, \tag{2.110}$$

$$\iota_V \iota_W + (-1)^{VW} \iota_W \iota_V = 0, \tag{2.111}$$

$$L_V \iota_W - (-1)^{VW} \iota_W L_V = \iota_{[V,W]}, \tag{2.112}$$

$$L_V L_W - (-1)^{VW} L_W L_V = L_{[V,W]}. \tag{2.113}$$

In particular, note that L_Q has degree 1 and satisfies $L_Q^2 = 0$. It can also be computed by

$$\begin{aligned} &L_Q \alpha(X_1, ..., X_p) = \\ &\quad Q(\alpha(X_1, ..., X_p)) - \sum_j (-1)^{\alpha+X_1+...+X_{j-1}} \alpha(X_1, ..., X_{j-1}, [Q, X_j], X_{j+1}, ..., X_p). \end{aligned} \tag{2.114}$$

Differential forms can be pulled-back along morphisms $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$; this operation commutes with d and it satisfies $\varphi^* L_{Q_2} \omega = L_{Q_1} \varphi^* \omega$. In particular, given a simplicial derived manifold $\mathcal{M}_\bullet$, there are degree-preserving operators $\delta : \Omega^\bullet(\mathcal{M}_{n-1}) \to \Omega^\bullet(\mathcal{M}_n)$ with $\delta^2 = 0$, defined by $\delta := \sum_{j=0}^n (d_j^n)^*$. They commute with the exterior differentials d and the Lie derivatives L_Q of each derived manifold $\mathcal{M}_n$.

Definition 2.25 ([68, 92, 210])**.** Let $\mathcal{M}_\bullet$ be a simplicial derived manifold. The *triple complex of differential forms* on $\mathcal{M}_\bullet$ is the triple complex

$$(\{\Omega^\bullet(\mathcal{M}_\bullet)_\bullet\}, d, \delta, L_Q). \tag{2.115}$$

A differential form $\alpha \in \Omega^r(\mathcal{M}_p)_q$ is *normalized* if $s_j^*\alpha = 0$, $j = 0, ..., p$. These form a sub-complex $\hat{\Omega}^\bullet(\mathcal{M}_\bullet)_\bullet$, called the *normalized triple complex of differential forms.* For $l \in \mathbb{Z}$, an *l-shifted presymplectic form* on $\mathcal{M}_\bullet$ is a closed element $\omega_\bullet$ of degree $2+l$ in the total complex associated to the triple complex

$$(\{\hat{\Omega}^{\bullet \geq 2}(\mathcal{M}_\bullet)_\bullet\}, d, \delta, L_Q); \tag{2.116}$$

i.e. it is a family of normalized forms

$$\omega^r_{p,q} \in \hat{\Omega}^r(\mathcal{M}_p)_q, \quad r \geq 2,\ \ p \geq 0,\ \ q \leq 0,\ \ p + r + q = 2 + l \tag{2.117}$$

such that

$$d\omega^{r-1}_{p,q} + \delta\omega^r_{p-1,q} + L_Q\omega^r_{p,q-1} = 0. \tag{2.118}$$

The *leading term* of an l-shifted presymplectic form $\omega_\bullet$ is its projection $\omega^2_\bullet$ to $\hat{\Omega}^2(\mathcal{M}_\bullet)_\bullet$. Given a morphism of simplicial derived manifolds $\mathcal{N}_\bullet \xrightarrow{f} \mathcal{M}_\bullet$ and an l-shifted presymplectic form $\omega_\bullet$ on $\mathcal{M}$, then an *isotropic structure* on f (relative to $\omega_\bullet$) is an element $\omega^L_\bullet$ of degree $1 + l$ in the total complex associated to the triple complex

$$(\{\hat{\Omega}^{\bullet \geq 2}(\mathcal{N}_\bullet)_\bullet\}, d, \delta, L_Q). \tag{2.119}$$

whose total derivative is $f^*\omega_\bullet$. That is, it is a family of normalized forms

$$\omega^{r,L}_{p,q} \in \hat{\Omega}^r(\mathcal{M}_p)_q, \quad r \geq 2,\ \ p \geq 0,\ \ q \leq 0,\ \ p + r + q = 1 + l \tag{2.120}$$

such that

$$d\omega^{r-1,L}_{p,q} + \delta\omega^{r,L}_{p-1,q} + L_Q\omega^{r,L}_{p,q-1} = f^*\omega^r_{p,q}. \tag{2.121}$$

Its *leading term* is its projection $\omega^{2,L}_\bullet$ to $\hat{\Omega}^2(\mathcal{N}_\bullet)_\bullet$.

Let $\mathcal{M} = (M, E, Q)$ be a derived manifold. Upon choosing a degree-preserving connection ∇ on E, we obtain an isomorphism

$$\Gamma(T\mathcal{M}) = \Gamma(S^\bullet(E[1])^* \otimes (TM \oplus E[1])), \tag{2.122}$$

where we identify $V \in \Gamma(TM)$ and $e \in \Gamma(E)$ with the derivations of $\Gamma(S^\bullet E[1]^*)$ that act on $\xi \in \Gamma(E[1]^*)$ as

$$\xi \mapsto V(\xi(\cdot)) - \xi(\nabla_V \cdot), \quad \xi \mapsto \xi(e), \tag{2.123}$$

respectively, and which are extended to all of $\Gamma(S^\bullet E[1]^*)$ imposing Leibniz's rule. Under this isomorphism, the DGLA structure on $\Gamma(T\mathcal{M})$ is completely described by the following formulas.

$$[X + e_1, Y + e_2] = [X, Y] + \nabla_X e_2 - \nabla_Y e_1 - F^\nabla(X, Y), \tag{2.124a}$$

$$[Q, X](\xi) = -\xi\bigg(\nabla_X \Phi + \nabla_X(\{\cdot\})(\cdot) + \frac{1}{2}\nabla_X(\{\cdot,\cdot\})(\cdot,\cdot) \tag{2.124b}$$

$$+ \frac{1}{6}\nabla_X(\{\cdot,\cdot,\cdot\})(\cdot,\cdot,\cdot) + ...\bigg), \tag{2.124c}$$

$$[Q, e] = (-1)^e \left(\{e\} + \{e,\cdot\} + \frac{1}{2}\{e,\cdot,\cdot\} + \frac{1}{6}\{e,\cdot,\cdot,\cdot\} + ...\right), \tag{2.124d}$$

where we see $F^\nabla(X,Y) \in \Gamma(E^* \otimes E) \subset \Gamma(S^\bullet E[1]^* \otimes (TM \oplus E))$ and we write $\nabla_X \Phi$, $\nabla_X(\{\cdot\})$, $\nabla_X(\{\cdot,\cdot\})$, etc. for the covariant derivatives of the tensors Φ, $\{\cdot\}$, $\{\cdot,\cdot\}$, etc. defining the fiberwise curved L_∞-algebra structure (with the convention that the brackets are graded symmetric in $E[1]$). In particular, differential p-forms on $\mathcal{M}$ can be identified with graded skew-symmetric $C^\infty(M)$-multilinear maps $\Gamma(TM \oplus E)^{\otimes p} \to \Gamma(S^\bullet E[1])$ and d, L_Q can be computed using (2.124).

Remark 2.26. Let $\mathcal{M} = (M, E, Q)$ be a derived manifold, and let $\omega \in \Omega^p(E)$ be an ordinary differential form in the total space of E such that $\omega(X_1, ..., X_p) \in \Gamma(E^*) \subset C^\infty(E)$ whenever X_1, ..., X_p are linear vector fields on E. Then ω induces canonically a differential form $\omega^{\mathcal{M}}$ on $\mathcal{M}$. This can be seen by choosing a connection ∇ on E and using the description above to define $\omega^{\mathcal{M}}(X_1 + e_1, ..., X_p + e_p) := \omega(X_1^h + X_{e_1}, ..., X_p^h + X_{e_p}) \in \Gamma(E^*) \subset \Gamma(S^\bullet E[1]^*)$, where X^h denotes horizontal lift of $X \in \Gamma(TM)$ with respect to ∇ and X_e is the vertical vector field associated to $e \in \Gamma(E)$. Changing the connection does not change $\omega^{\mathcal{M}}$, and so this is well-defined independently of ∇. It is also worth noting that in this case $(d\omega)^{\mathcal{M}} = d(\omega^{\mathcal{M}})$.

Now let $\mathcal{M}_\bullet$ be a simplicial derived manifold, write $\mathcal{M}_p = (M_p, E_{p,\bullet}, Q_p)$ and let $\omega \in \Omega^2(\mathcal{M}_p)_q$ be normalized. Recall the vector bundles $A_{i,j} \to Z(\mathcal{M}_0)$ defined in (2.98) for $(i,j) \in \mathbb{Z}^{\leq 0} \times \mathbb{Z}^{\geq 0}$. We use ω to define a map $\tilde{\omega} : A_{i,j} \otimes A_{p-i,q-j} \to \underline{\mathbb{R}}$, where $\underline{\mathbb{R}}$ is the trivial vector bundle with fiber $\mathbb{R}$. For $x \in Z(\mathcal{M}_0)$ and $[v_x] \in A_{i,j|x}$, we write $V \in \Gamma(T\mathcal{M}_p)$ for any vector field obtained as follows.

1. Choose a representative $v_x \in E_{p,j+1|x}$ (if $j \geq 1$) or $v_x \in TM_{p|x}$ (if $j = 0$).
2. Extend v_x to either a local section of $E_{p,j+1}$ (if $j \geq 1$) or a local vector field on M_p (if $j = 0$).

3. Use any connection to identify v_x with $V \in \Gamma(T\mathcal{M}_p)$ via (2.122).

Of course, V depends on all these choices, but the following formula does not.

$$\tilde{\omega}_x(v_x^1, v_x^2) := \sum_{\sigma \in S_{i,p-i}} (-1)^\sigma \omega(s_{\sigma(p)-1,*}\circ\ldots\circ s_{\sigma(i+1)-1,*}(V^1), s_{\sigma(i)-1,*}\circ\ldots\circ s_{\sigma(1)-1,*}(V^2))(s(x)) \tag{2.125}$$

The degrees are chosen so that each term on the sum in (2.125) is an element of $\Gamma(S^\bullet E_{p,\bullet}[1]^*)$ of degree 0; i.e., a function on M_p, and so it makes sense to evaluate it at $s(x)$, for $s : M_0 \to M_p$ the degeneracy map. To state the following lemma we recall that the *homotopy fiber* of a map $\psi : A_\bullet \to B_\bullet$ of chain complexes is the chain complex $\mathrm{Hofib}(\psi)$ defined by

$$\mathrm{Hofib}(\psi)_n = A_n \oplus B_{n-1}, \quad d(a+b) = d^A a + (d^B b - \psi(a)). \tag{2.126}$$

Lemma 2.27. *Let $\mathcal{M}_\bullet$ be a simplicial derived manifold and let $\omega \in \Omega^2(\mathcal{M}_p)_q$ be normalized. Then $\tilde{\omega} : A_{i,j} \otimes A_{p-i,q-j} \to \mathbb{R}$ is well defined independently of choices by (2.125) and it satisfies*

$$(-1)^p \widetilde{\delta\omega}(v^1, v^2) = \tilde{\omega}(\partial v^1, v^2) + (-1)^i \tilde{\omega}(v^1, \partial v^2), \tag{2.127}$$

$$(-1)^q \widetilde{L_Q\omega}(v^1, w^2) = \tilde{\omega}(Q_* v^1, w^2) + (-1)^j \tilde{\omega}(v^1, Q_* w^2) \tag{2.128}$$

for $v^1 \in A_{i,j}$, $v^2 \in A_{p+1-i,q-j}$, $w^2 \in A_{p-i,q+1-j}$. In particular, the leading term $\omega_\bullet^2$ of an l-shifted presymplectic form on $\mathcal{M}_\bullet$ determines by (2.125) a map of chain complexes of vector bundles over $Z(\mathcal{M}_0)$

$$T\mathcal{M} \to T^*[l]\mathcal{M}, \tag{2.129}$$

where $T^*[l]\mathcal{M}$ denotes the cotangent complex of $\mathcal{M}_\bullet$ shifted by l, and the leading term $\omega_\bullet^{2,L}$ of an isotropic structure on a morphism of simplicial derived manifolds $\mathcal{N}_\bullet \xrightarrow{f} \mathcal{M}_\bullet$ determines by (2.125) a map of chain complexes of vector bundles over $Z(\mathcal{N}_0)$

$$T\mathcal{N} \xrightarrow{f_* + \tilde{\omega}^L} \mathit{Hofib}(f^*T\mathcal{M} \xrightarrow{f^*\tilde{\omega}} f^*T^*[l]\mathcal{M} \xrightarrow{f^*_*} T^*[l]\mathcal{N}). \tag{2.130}$$

Proof. That $\tilde{\omega}$ does not depend on the choice of $v_x \in E_{p,j+1|x}$ follows from the fact that ω is normalized, which implies that the right-hand side of (2.125) vanishes for $V^1 \in Im\,(s_i^*)$ or $V^2 \in Im\,(s_i^*)$ by a similar argument to the one in [92, Lemma E.1.]. Independence on the choice of extension of v_x is clear since the right-hand side of (2.125) is evaluated at $s(x)$, while independence on the choice of connection follows from noting that the terms that appear upon changing the connection vanish for degree reasons. Then (2.127) also follows as in [92, Lemma E.1.], since the computations there are purely algebraic, while (2.128) follows from formulas (2.114) and (2.124), noting that

$Q(\omega(V_1, V_2)) = 0$ when $deg(V_1) + deg(V_2) = deg(\omega)$. Finally, maps (2.129) and (2.130) are defined in a straightforward way by formula (2.125); then equations (2.127), (2.128) and the closedness conditions from Definition 2.25 imply that these maps preserve the chain differentials. □

Definition 2.28. Let $(\mathcal{M}_\bullet, \omega_\bullet)$ be a simplicial derived manifold with an l-shifted presymplectic form. We say $(\mathcal{M}_\bullet, \omega_\bullet)$ is *l-shifted symplectic* if the map (2.129) is a quasi-isomorphism. Let $\mathcal{N}_\bullet \overset{(f_\bullet, \omega^L_\bullet)}{\to} (\mathcal{M}_\bullet, \omega_\bullet)$ be a morphism of simplicial derived manifolds with an isotropic structure and assume that $(\mathcal{M}_\bullet, \omega_\bullet)$ is l-shifted symplectic. We say $(f_\bullet, \omega^L_\bullet)$ is *Lagrangian* if the map (2.130) is a quasi-isomorphism.

The following result provides two methods for constructing shifted (pre)symplectic structures from previously known ones. At the time of writing of this thesis, it has only been proven rigorously in the algebraic setting, but it will serve us for inspiration to construct shifted (pre)symplectic structures on simplicial derived manifolds.

Theorem 2.29 ([210])**.**

1. *Let $(\mathfrak{X}, \omega)$ be an l-shifted presymplectic derived Artin stack and let $\mathfrak{Y}_i \overset{f_i}{\to} \mathfrak{X}$, $i = 1, 2$ be morphisms of derived Artin stacks equipped with isotropic structures λ_i. Then $\lambda_2 - \lambda_1$ defines an $(l-1)$-shifted presymplectic structure on $\mathfrak{Y}_1 \times^h_{\mathfrak{X}} \mathfrak{Y}_2$. If ω is symplectic and λ_1, λ_2 are Lagrangian, then $\lambda_2 - \lambda_1$ is symplectic.*

2. *Let $(\mathfrak{X}, \omega)$ be an l-shifted presymplectic derived Artin stack and let $\mathfrak{Y}$ be a compact, d-oriented derived Artin stack in the sense of [210, Def. 2.1 and 2.4]. If the internal hom $\underline{dSt}(\mathfrak{Y}, \mathfrak{X})$ is a derived Artin stack, then $\int_{\mathfrak{Y}} ev^*\omega$ is an $(l-d)$-shifted presymplectic structure on $\underline{dSt}(\mathfrak{Y}, \mathfrak{X})$, where $ev : \underline{dSt}(\mathfrak{Y}, \mathfrak{X}) \times \mathfrak{Y} \to \mathfrak{X}$ is the evaluation map and $\int_{\mathfrak{Y}}$ is defined by the d-orientation on $\mathfrak{Y}$. If ω is symplectic, then so is $\int_{\mathfrak{Y}} ev^*\omega$.*

In our examples, when $\mathfrak{X}$ is a simplicial derived complex manifold, we will also consider *shifted holomorphic symplectic* forms. By this we mean the analog of Definition 2.28, but where the triple complex of $\mathbb{R}$-valued differential forms is replaced by the triple complex of $\mathbb{C}$-valued, $\mathbb{C}$-multilinear differential forms.

2.3.2 First examples

All the examples in this section can be found at least within the context of derived algebraic geometry in [8, 36, 67, 68, 92, 210, 278].

Example 2.30 ([67, 88])**.** Let M be a manifold. For $l \in \mathbb{Z}^{<0}$ we define the *shifted cotangent* $T^*[l]M$ to be the derived manifold given by the vector bundle $T^*M \to M$, with T^*M in degree $1-l$, and zero homological vector field. The convention for the degree is chosen so that the tangent complex of $T^*[l]M$ is $TX \xrightarrow{0} ... \xrightarrow{0} T^*X$, with TX in degree 0 and T^*X in degree $-l$. The canonical symplectic form ω on T^*M can be seen as an l-shifted symplectic form on $T^*[l]M$ by Remark 2.26. For $l \in \mathbb{Z}^{>0}$, the *shifted cotangent* $T^*[l]M_\bullet$ is the simplicial manifold with

$$(T^*[l]M)_n = \{(x, \{\alpha_{i_0\dots i_l}\}_{i_0<\dots<i_l\in[n]}) \in M \times (T^*M)^{\binom{n+1}{l+1}} \mid \alpha_{i_0\dots i_l} \in T^*_xM, \\ \forall i_0 < \dots < i_{l+1},\ (\delta\alpha)_{i_0\dots i_{l+1}} = 0\}, \tag{2.131}$$

where we are writing

$$(\delta\alpha)_{i_0\dots i_{l+1}} := \sum_{j=0}^{l+1}(-1)^j \alpha_{i_0\dots i_{j-1}i_{j+1}\dots i_{l+1}}, \tag{2.132}$$

and, for non-decreasing $f : [n_1] \to [n_2]$, the map $f^* : (T^*[l]M)_{n_2} \to (T^*[l]M)_{n_1}$ is defined as in (2.23). In particular, note that $(T^*[l]M)_l = T^*M$. Then the canonical symplectic form ω on T^*M can be seen as an l-shifted symplectic form on $T^*[l]M$ because it is linear on the fibers of T^*M and this implies $\delta\omega = 0$. The tangent complex to $T^*[l]M$ in the case $l > 0$ is $T^*M \xrightarrow{0} ... \xrightarrow{0} TM$, with TM in degree 0 and T^*M in degree $-l$. It is then easy to see that, in both cases $l < 0$ and $l > 0$, the non-degeneracy condition for ω follows simply from the fact that it induces isomorphisms $(TM)^* = T^*M$ and $(T^*M)^* = TM$. This construction is generalized in [67], which proves in the algebraic setting that for any derived stack $\mathfrak{X}$ and for $l \in \mathbb{Z}$ one can define a derived stack $T^*[l]\mathfrak{X}$ with a canonical l-shifted symplectic structure.

Example 2.31 ([8, 36])**.** Let M be a manifold and let G be a Lie group acting smoothly on M with infinitesimal action map $\rho : \underline{\mathfrak{g}} \to TM$. Then we define $T^*(M//G)$ to be the following simplicial derived manifold. First, define the derived manifold $\mathcal{M} := (T^*M, \underline{\mathfrak{g}}^*[-2], Q)$, where Q is defined simply by the curvature map $\Phi : T^*M \to \underline{\mathfrak{g}}^*[-2]$, $\Phi = \rho^*$. The action of G on M induces an action on $\mathcal{M}$, where the action on T^*M is given by pull-back and the action on $\mathfrak{g}^*$ is the coadjoint action (this defines an action on $\mathcal{M}$ since ρ^* is G-equivariant). Then define $T^*(M//G) := \mathcal{M}//G$ in the sense of Example 2.23. Its tangent complex is the following chain complex of vector bundles over T^*M.

$$\underline{\mathfrak{g}}[1] \xrightarrow{\rho^{T^*M}} TT^*M \xrightarrow{(\rho^*)_*} \underline{\mathfrak{g}}^*[-1], \tag{2.133}$$

where ρ^{T^*M} is the infinitesimal action map for the action of G on T^*M, while the cotangent complex is its dual,

$$\underline{\mathfrak{g}}[1] \xrightarrow{((\rho^*)_*))^*} T^*T^*M \xrightarrow{(\rho^{T^*M})^*} \underline{\mathfrak{g}}^*[-1]. \tag{2.134}$$

The canonical isomorphism betwen these two complexes is induced by a 0-shifted symplectic structure on $T^*(M//G)$ which we can describe as follows. It is given by the canonical symplectic structure on T^*M, which can be seen as a degree 0, d-exact, 2-form $\omega^0 = d\lambda^0 \in \Omega^2(\mathcal{M})_0$, and the canonical symplectic structure on $T^*G = G \times \mathfrak{g}^*$, which can be seen as a degree -1, d-exact, 2-form $\omega^1 = d\lambda^1 \in \Omega^2(\mathcal{M} \times G)_{-1}$ by Remark 2.26. Let us check the closedness condition (2.118).

1. $L_Q\omega^0 = 0$ for degree reasons.
2. $L_Q\omega^1 = \delta\omega^0$ follows from

$$\begin{aligned}(L_Q\lambda^1)_{(\alpha_p,g,\xi)}(\dot{\alpha}_p + v_g + \dot{\xi}) &= Q((\alpha_p, g, \xi) \mapsto \xi(v_g g^{-1}))_{(\alpha_p,g,\xi)} \\ &= (\rho^*\alpha_p)(v_g g^{-1}) = \alpha_p(p v_g g^{-1}),\end{aligned} \tag{2.135}$$

$$(\delta\lambda^0)_{(\alpha_p,g,\xi)}(\dot{\alpha}_p + v_g + \dot{\xi}) = \alpha_{pg}(v_p g) + \alpha_{pg}(p v_g) - \alpha_p(v_p) = \alpha_p(p v_g g^{-1}) \tag{2.136}$$

   for $p \in M$, $\alpha_p \in T_p^*M$, $g \in G$, $\xi \in \mathfrak{g}^*$, $\dot{\alpha}_p \in T_{\alpha_p}T^*M$, $v_g \in T_gG$, $\dot{\xi} \in T_\xi\mathfrak{g}^*$.
3. $\delta\omega^1 = 0$ states the multiplicative property of the symplectic structure on T^*G.

Hence, (ω^0, ω^1) is indeed a 0-shifted symplectic structure on $T^*(M//G)$.

Example 2.32. Let M be a manifold and let G be a Lie group acting smoothly on M. Let $S : M \to \mathbb{R}$ be a G-invariant function. Then $dS : M \to T^*M$ satisfies $\rho^* dS = 0$. This is precisely the condition for the map of vector bundles $M \times \{0\} \overset{(dS,0)}{\to} T^*M \times \mathfrak{g}^*[-2]$ to define a morphism of derived manifolds $M \to (T^*M, \underline{\mathfrak{g}^*}[-2], \rho^*)$. Moreover, this map is G-equivariant and so it determines a morphism of simplicial derived manifolds $dS : M//G \to T^*(M//G)$, where $T^*(M//G)$ is defined as in Example 2.31. Since $\alpha^*\omega = d\alpha$ for ω the symplectic form on T^*M and $\alpha : M \to T^*M$ any section, it follows that $(dS)^*(\omega^0, \omega^1) = 0$ for (ω^0, ω^1) the 0-shifted symplectic structure on $T^*(M//G)$. That is, dS carries a canonical isotropic structure given by 0. It is in fact Lagrangian, because the map (2.130) is in this case the following quasi-isomorphism of (vertical) chain complexes of vector bundles over M

$$\begin{tikzcd} \underline{\mathfrak{g}} \arrow[r, "id"] \arrow[d, "\rho_*"] & \underline{\mathfrak{g}} \arrow[d, "(dS)^*\rho^{T^*M}"] \\ TM \arrow[r, "dS_*"] \arrow[d] & (dS)^*TT^*M \arrow[d, "(dS)^*(\rho^*)_* - (dS_*)^* \circ \omega"] \\ 0 \arrow[r] \arrow[d] & \underline{\mathfrak{g}^*} \oplus T^*M \arrow[d, "id - \rho^*"] \\ 0 \arrow[r] & \underline{\mathfrak{g}^*} \end{tikzcd} \; . \tag{2.137}$$

In particular, the function $0 : M \to \mathbb{R}$ also defines a canonical Lagrangian structure on $d0 : M//G \to T^*(M//G)$ and then Theorem 2.29 suggests the existence of a simplicial derived manifold $dCrit(S)$ with a (-1)-shifted symplectic structure modelling the space $\{x \in M \mid dS = 0\}/G$. Note that, in physics terminology, this is the configuration space of a system with G-symmetry defined by the action functional S. We present now a model for $dCrit(S)$, which is called the *derived critical locus of* S.

We proceed as in Example 2.20. Let $\mathcal{N} := (M, T^*[-2]M \oplus \underline{\mathfrak{g}}^*[-3], Q)$, where Q is defined by the fiberwise structure of curved L_∞-algebra given by the curvature $\Phi : M \to T^*[-2]M$, $\Phi = dS$ and the differential $d : T^*[-2]M \to \underline{\mathfrak{g}}^*[-3]$, $d = \rho^*$. This gives a derived manifold, since $d\Phi = 0$ follows from S being G-invariant. The action of G on M lifts to an action on $\mathcal{N}$, where the action on $T^*[-2]M$ is given by pull-back and the action on $\underline{\mathfrak{g}}^*[-3]$ is the coadjoint action (this defines an action on $\mathcal{N}$ since dS and ρ^* are G-equivariant). Then define $dCrit(S) := \mathcal{N}//G$ in the sense of Example 2.23. Its tangent complex is the following chain complex of vector bundles over M.

$$\underline{\mathfrak{g}}[1] \xrightarrow{\rho} TM \xrightarrow{dS} T^*[-1]M \xrightarrow{\rho^*} \underline{\mathfrak{g}}^*[-2], \tag{2.138}$$

while the cotangent complex is its dual,

$$\underline{\mathfrak{g}}[2] \xrightarrow{\rho} T[1]M \xrightarrow{dS^*} T^*M \xrightarrow{\rho^*} \underline{\mathfrak{g}}^*[-1]. \tag{2.139}$$

The canonical isomorphism $T(dCrit(S)) \cong T^*(dCrit(S))[-1]$ is induced by a (-1)-shifted symplectic structure on $dCrit(S)$ which we can describe as follows. It is given by the canonical symplectic structure on T^*M, which can be seen as a degree -1, d-exact, 2-form $\omega^0 = d\lambda^0 \in \Omega^2(\mathcal{N})_{-1}$ by Remark 2.26, and the canonical symplectic structure on $T^*G = G \times \mathfrak{g}^*$, which can be seen as a degree -2, d-exact, 2-form $\omega^1 = d\lambda^1 \in \Omega^2(\mathcal{M} \times G)_{-2}$ for the same reason. Let us check the closedness condition (2.118).

1. $L_Q\omega^0 = 0$ because $L_Q\lambda^0 = dS$:

$$(L_Q\lambda^0)_\alpha(\dot{\alpha}) = Q(\alpha \mapsto \alpha(d\pi(\dot{\alpha}))) = dS(d\pi(\dot{\alpha})). \tag{2.140}$$

2. $L_Q\omega^1 = \delta\omega^0$ as in Example 2.31.
3. $\delta\omega^1 = 0$ as in Example 2.31.

Hence, (ω^0, ω^1) is indeed a (-1)-shifted symplectic structure on $dCrit(S)$. Note the construction also works if dS is replaced by any closed G-invariant 1-form $\alpha \in \Omega^1(M)$. At least in the algebraic setting, this example can be generalized replacing $M//G$ by any derived stack $\mathfrak{X}$ and dS by any closed 1-form α of degree l (for any $l \in \mathbb{Z}$) on $\mathfrak{X}$ [8].

Then $T^*[l]\mathfrak{X}$ is l-shifted symplectic by Example 2.30, $\alpha : \mathfrak{X} \to T^*[l]\mathfrak{X}$ is Lagrangian by closedness and so its homotopy fibered product with the zero section is $(l-1)$-shifted symplectic by Theorem 2.29.

Example 2.33 ([8, 210, 278])**.** Let G be a Lie group. Then we form the quotient groupoid (cf. Example 2.5) $\mathfrak{g}^*//G$ for the coadjoint action of G on $\mathfrak{g}^*$. Its tangent space is the following complex of vector bundles over $\mathfrak{g}^*$

$$\underline{\mathfrak{g}}[1] \stackrel{ad^*}{\to} \underline{\mathfrak{g}}^*, \tag{2.141}$$

where we write $ad^*_{|\xi}(v)(\cdot) := \xi([v, \cdot])$ for $\xi \in \mathfrak{g}^*$ and $v \in \mathfrak{g}$. Its cotangent is

$$\underline{\mathfrak{g}} \stackrel{(ad^*)^*}{\to} \underline{\mathfrak{g}}^*[-1], \tag{2.142}$$

where

$$(ad^*)^*_\xi(v)(\cdot) = ad^*_\xi(\cdot)(v) = \xi([\cdot, v]) = -ad^*_\xi(v)(\cdot). \tag{2.143}$$

The canonical isomorphism $T(\mathfrak{g}^*//G) \cong T^*(\mathfrak{g}^*//G)[1]$ is induced by a canonical 1-shifted symplectic form on $\mathfrak{g}^*//G$. It is defined by the canonical symplectic form ω^G on $T^*G = G \times \mathfrak{g}^* = (\mathfrak{g}^*//G)_1$ (one must only check that $d\omega^G = 0$, $\delta\omega^G = 0$ as before). In fact, the existence of this symplectic form follows from the fact that $\mathfrak{g}^*//G$ can be thought of as $T^*[1](BG)$ (cf. Examples 2.30 and 2.31).

Example 2.34. Let M be a manifold with a smooth action of a Lie group G. Then it is easy to see from the definitions that a morphism of simplicial manifolds $M//G \to \mathfrak{g}^*//G$ is the same as a G-equivariant map $\mu : M \to \mathfrak{g}^*$. An isotropic structure on it, for the 1-shifted symplectic structure on $\mathfrak{g}^*//G$ from Example 2.33, is the data of $\omega \in \Omega^2(M, \mathbb{R})$ such that $\delta\omega = \mu^*\omega^G$. This is equivalent to

$$g^*\omega = 0, \qquad \forall g \in G, \tag{2.144}$$

$$\iota_{\rho(v)}\omega = d(\mu(\cdot)(v)) \quad \forall v \in \mathfrak{g} \tag{2.145}$$

In conclusion, isotropic structures on $\mu : M//G \to \mathfrak{g}^*//G$ are in bijection with presymplectic structures on M such that μ is a moment map for the G-action. The corresponding map (2.130) is the following map of chain complexes of vector bundles over M

$$\begin{tikzcd} \underline{\mathfrak{g}} \arrow[r, "id"] \arrow[d, "\rho"] & \underline{\mathfrak{g}} \arrow[d, "\mu^* ad^* - d\mu^*"] \\ TM \arrow[r, "d\mu+\omega"] \arrow[d] & \underline{\mathfrak{g}}^* \oplus T^*M \arrow[d, "id-\rho^*"] \\ 0 \arrow[r] & \underline{\mathfrak{g}}^* \end{tikzcd}, \tag{2.146}$$

which is a quasi-isomorphism if and only if ω is symplectic. In particular, $\{*\}$ regarded as a symplectic manifold with trivial G-action and trivial moment map defines a Lagrangian structure on $\{*\}//G \to \mathfrak{g}^*//G$. Then Theorem 2.29 suggests that, for any symplectic manifold with moment map (M, ω, μ), we may take $M//_\mu G := (M//G)\,{}_\mu\!\times_0^h(\{*\}//G)$ to obtain a simplicial derived manifold with a 0-shifted symplectic structure. Note that this is a model for the classical Marsden-Weinstein quotient which exists as a geometric object without regularity assumptions on the action. We proceed to present this explicitly.

As a simplicial derived manifold, the fibered product $M//_\mu G$ is simply the quotient groupoid as in Example 2.23 for the G-action on the derived manifold $(M, \underline{\mathfrak{g}^*}[-2], Q)$ where Q is just given by $\Phi = \mu : M \to \mathfrak{g}^*$ (cf. the derived zero set from Example 2.20; G acts here since μ is equivariant). The tangent complex of $M//_\mu G$ is the chain complex of vector bundles over $\mu^{-1}(0)$

$$\underline{\mathfrak{g}}[1] \xrightarrow{\rho} TM \xrightarrow{d\mu} \underline{\mathfrak{g}^*}[-1], \tag{2.147}$$

while the cotangent complex is

$$\underline{\mathfrak{g}}[1] \xrightarrow{d\mu^*} T^*M \xrightarrow{d\mu} \underline{\rho^{**}}[-1]. \tag{2.148}$$

A 0-shifted symplectic structure inducing an isomorphism between these two complexes is given by the symplectic form ω on M, seen as a d-closed 2-form $\omega^0 \in \Omega^2((M//_\mu G)_0)_0$, and the canonical symplectic form on T^*G, seen as a d-closed degree -1 2-form $\omega^1 \in \Omega^2((M//_\mu G)_1)_{-1}$. Then $L_Q\omega^0 = 0$ for degree reasons, $\delta\omega^0 = L_Q\omega^1$ because of the moment map condition and $\delta\omega^1 = 0$ because of multiplicativity of ω^G, which yields the closedness condition (2.118). This example can be generalized replacing (M, ω) by any l-shifted symplectic derived stack (i.e. simplicial derived manifold) $(\mathfrak{X}, \omega)$ with an action of G; then the notion of moment map can be defined directly as a map $\mu : \mathfrak{X}//G \to \mathfrak{g}^*[l]//G$ such that ω is a Lagrangian structure for $\mu^*\omega^G$ [8].

Example 2.35 ([68, 92, 210])**.** Let G be a Lie group. The tangent complex to BG (cf. Example 2.6) is the chain complex of vector bundles over $\{*\}$ (i.e. the chain complex of vector spaces)

$$\mathfrak{g}[1]. \tag{2.149}$$

The cotangent complex is thus

$$\mathfrak{g}^*[-1]. \tag{2.150}$$

Let $\langle\cdot,\cdot\rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ be an Ad-invariant, symmetric bilinear form. This induces a 2-shifted presymplectic structure on BG. It is defined by

$$\mu := \frac{1}{6}\langle\theta^L \wedge [\theta^L \wedge \theta^L]\rangle \in \Omega^3(G), \quad \nu := -\langle g_1^*\theta^L \wedge g_2^*\theta^R\rangle \in \Omega^2(G^2), \tag{2.151}$$

where θ^L and θ^R are the left- and right- invariant Maurer-Cartan forms on G. This is indeed a presymplectic structure because these differential forms satisfy

$$d\mu = 0, \quad d\nu = -\delta\mu, \quad \delta\nu = 0, \tag{2.152}$$

as it follows from straightforward computations (this is originally due to [246]). Of course, it induces the morphism $\mathfrak{g}[1] \to \mathfrak{g}^*[-1]$ given by $\langle \cdot, \cdot \rangle$ and so this is a 2-shifted symplectic structure precisely when $\langle \cdot, \cdot \rangle$ is non-degenerate.

2.3.3 Shifted symplectic structures on derived moduli stacks

Example 2.35, along with Theorem 2.29, is used in [210] to prove the following.

Theorem 2.36 ([210]). *1. Let G be a reductive affine group scheme over Spec$\mathbb{R}$ with an Ad-invariant, non-degenerate, symmetric form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ and let M be a compact, connected, oriented manifold of $\dim_{\mathbb{R}} M = n$. Then there is a derived Deligne-Mumford stack with a $(2-n)$-shifted symplectic structure parameterizing flat G-bundles on M.*

2. Let G be a reductive affine group scheme over Spec$\mathbb{C}$ with an Ad-invariant, non-degenerate, symmetric $\mathbb{C}$-linear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ and let X be a compact, connected complex manifold with a holomorphic volume form of $\dim_{\mathbb{C}} X = n$. Then there is a derived Deligne-Mumford stack with a $(2-n)$-shifted holomorphic symplectic structure parameterizing holomorphic G-bundles on X.

We proceed to present a gauge-theoretic description of the shifted symplectic structures from Theorem 2.36 within the language of (infinite-dimensional) simplicial derived manifolds. This presentation allows us to deal with an arbitrary Lie group G. The ideas that we use are known to experts, although we are not aware of any prior explicit presentation of such shifted symplectic structures in these terms.

Let G be a Lie group and let $P \to M$ be a G-bundle over a compact, oriented manifold with $\dim_{\mathbb{R}} M = n$. The derived moduli stack $\mathcal{B}^{\flat,d}(P)$ of flat connections on P is defined in Example 2.23. Its tangent complex is the following complex of vector bundles over $\{A \in \mathcal{A}(P) \mid F_A = 0\}$

$$\underline{\Omega^0(ad\,P)}[1] \xrightarrow{d^A} \underline{\Omega^1(ad\,P)} \xrightarrow{d^A} \underline{\Omega^2(ad\,P)}[-1] \xrightarrow{d^A} \dots \xrightarrow{d^A} \underline{\Omega^n(ad\,P)}[-n+1]. \tag{2.153}$$

If $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ is a non-degenerate, Ad-invariant, symmetric bilinear form then we define a $(2-n)$-shifted symplectic structure on $\mathcal{B}^{\flat,d}(P)$ as follows. We describe differential forms on $\mathcal{B}^{\flat,d}(P)_0$ and $\mathcal{B}^{\flat,d}(P)_1$ as explained in Section 2.3.1, using the trivial connection on $\underline{\Omega^{\geq 2}(ad\, P)} \to \mathcal{A}(P)$ and taking also into account that $T\mathcal{A}(P) = \mathcal{A}(P) \times \Omega^1(ad\, P)$ and $TGauge(P) = Gauge(P) \times \Omega^0(ad\, P)$ (for this we identify $\Omega^0(ad\, P)$ with right-invariant vector fields). The symplectic structure is then given by

$$\omega^0 \in \Omega^2(\mathcal{B}^{\flat,d}(P)_0)_{2-n}, \quad \omega^1 \in \Omega^2(\mathcal{B}^{\flat,d}(P)_1)_{1-n} \tag{2.154}$$

defined by

$$\begin{aligned}\omega^0_{|A}(\dot{a}_1^1 + \dot{a}_1^2 + ... + \dot{a}_1^n, \dot{a}_2^1 + \dot{a}_2^2 + ... + \dot{a}_2^n) = \int_M &((\langle \dot{a}_1^1 \wedge \dot{a}_2^{n-1}\rangle - \langle \dot{a}_2^1 \wedge \dot{a}_1^{n-1}\rangle)\\ &+ ((-1)^n \langle \dot{a}_1^2 \wedge \dot{a}_2^{n-2}\rangle + \langle \dot{a}_2^2 \wedge \dot{a}_1^{n-2}\rangle) + (\langle \dot{a}_1^3 \wedge \dot{a}_2^{n-3}\rangle - \langle \dot{a}_2^3 \wedge \dot{a}_1^{n-3}\rangle)\\ &+ ((-1)^n \langle \dot{a}_1^4 \wedge \dot{a}_2^{n-4}\rangle + \langle \dot{a}_2^4 \wedge \dot{a}_1^{n-4}\rangle) + ... + (-1)^n(\langle \dot{a}_1^{n-1} \wedge \dot{a}_2^1\rangle - \langle \dot{a}_2^{n-1} \wedge \dot{a}_1^1\rangle))\end{aligned} \tag{2.155}$$

$$\omega^1_{|(A,g)}(\dot{a}_1^0 + ... + \dot{a}_1^n, \dot{a}_2^0 + + ... + \dot{a}_2^n) = 2\int_M (\langle \dot{a}_1^0, \dot{a}_2^n\rangle - \langle \dot{a}_2^0, \dot{a}_1^n\rangle + (-1)^n \langle [\dot{a}_1^0, \dot{a}_2^0] \wedge a^n\rangle), \tag{2.156}$$

with a_i^p, $\dot{a}_i^p \in \Omega^p(ad\, P)$. Here, and similarly for the following examples, the right-hand side of (2.156) is to be read as function on the derived manifold

$$\mathcal{B}^{\flat,d}(P)_1 = (\mathcal{A}(P) \times Gauge(P), \underline{\Omega^{\geq 2}(ad\, P)}, (d^A, [\cdot,\cdot])),$$

defined through its action on $a^n \in \Gamma(\underline{\Omega^{\geq 2}(ad\, P)})$. It is clear that ω^1 is normalized, as it equals 0 when $\dot{a}_1^0 = \dot{a}_2^0 = 0$ (and this condition is vacuous for ω^0). Let us check the closedness condition (2.118). It is clear that $d\omega^0 = 0$ and $d\omega^1 = 0$, while $\delta\omega^1 = 0$ expresses the multiplicativity of the canonical symplectic form on $T^*Gauge(P)$. Then $L_Q\omega^0 = 0$ follows from

$$\int_M (\langle d^A \dot{a}^j \wedge \dot{a}^{n-j}\rangle + (-1)^j \langle \dot{a}^j \wedge d^A \dot{a}^{n-j}\rangle) = \int_M d\langle \dot{a}^j \wedge \dot{a}^{n-j}\rangle = 0, \tag{2.157}$$

$$\langle \dot{a}^i \wedge [\dot{a}^j \wedge a^{n-i-j}]\rangle + (-1)^{ij} \langle \dot{a}^j \wedge [\dot{a}^i \wedge a^{n-i-j}]\rangle = 0. \tag{2.158}$$

Finally, $\delta\omega^0 = L_Q\omega^1$ follows from the fact that all terms not containing $\dot{a}^1_1$, $\dot{a}^1_2$ are δ-closed by Ad-invariance of $\langle\cdot,\cdot\rangle$, while

$$\int_M \langle Ad(g^{-1})d^A\dot{a}^0 \wedge Ad(g^{-1})\dot{a}^{n-1}\rangle = \int_M \langle d^A\dot{a}^0 \wedge \dot{a}^{n-1}\rangle = -\int_M \langle \dot{a}^0 \wedge d^A\dot{a}^{n-1}\rangle, \tag{2.159}$$

$$\begin{aligned}\int_M (\langle Ad(g^{-1})d^A\dot{a}^0_1 \wedge Ad(g^{-1})[\dot{a}^0_2, a^{n-1}]\rangle - \langle Ad(g^{-1})d^A\dot{a}^0_2 \wedge Ad(g^{-1})[\dot{a}^0_1, a^{n-1}]\rangle)\\ = \int_M \langle d^A[\dot{a}^0_1,\dot{a}^0_2] \wedge a^{n-1}\rangle = -\int_M \langle [\dot{a}^0_1,\dot{a}^0_2] \wedge d^A a^{n-1}\rangle,\end{aligned} \tag{2.160}$$

$$\langle Ad(g^{-1})\dot{a}^1 \wedge Ad(g^{-1})[\dot{a}^0, a^{n-1}]\rangle = -\langle \dot{a}^0, [\dot{a}^1 \wedge a^{n-1}]\rangle. \tag{2.161}$$

The case $n = 2$ is an extension of the Atiyah-Bott symplectic form on the smooth locus of the moduli space of G-local systems on a Riemann surface [15] to the whole derived moduli stack $\mathcal{B}^{\flat,d}(P)$. It can also be constructed as the derived symplectic reduction (cf. Example 2.34) for the action of $Gauge(P)$ on $\mathcal{A}(P)$, with symplectic structure given by ω^0 as above and moment map $\mu : \mathcal{A}(P) \to \Omega^0(ad\,P)^* \cong \Omega^2(ad\,P)$ the curvature map; i.e., $\mu(A)(s) = \int_M \langle F_A, s\rangle$ for $A \in \mathcal{A}(P)$ and $s \in \Omega^0(ad\,P)$.

The case $n = 3$ can be constructed as the derived critical locus (cf. Example 2.32) of the Chern-Simons functional on the groupoid $\mathcal{A}(P)//Gauge(P)$ of connections modulo gauge. More precisely, it is the derived critical locus of the $Gauge(P)$-invariant closed 1-form $\alpha \in \Omega^1(\mathcal{A}(P))$ defined by

$$\alpha_A(\dot{a}) = \int_M \langle \dot{a} \wedge F_A\rangle \qquad A \in \mathcal{A}(P),\ \dot{a} \in \Omega^1(ad\,P) \cong T_A\mathcal{A}(P). \tag{2.162}$$

Note that for $n \geq 2$ we can write $\omega^1 = (-1)^{n+1}d\lambda^1$ and for $n \geq 3$ we can write $\omega^0 = (-1)^{n+1}d\lambda^0$, where

$$\lambda^0(\dot{a}^1 + \dot{a}^2 + ... + \dot{a}^n) = \int_M (2\langle a^{n-1} \wedge \dot{a}^1\rangle + \langle a^{n-2} \wedge \dot{a}^2\rangle + ... + \langle a^2 \wedge \dot{a}^{n-2}\rangle), \tag{2.163}$$

$$\lambda^1(\dot{a}^0 + \dot{a}^1 + \dot{a}^2 + ... + \dot{a}^n) = \int_M 2\langle a^n, \dot{a}^0\rangle. \tag{2.164}$$

These satisfy $\delta\lambda^0 = L_Q\lambda^1$, $\delta\lambda^1 = 0$ but $L_Q\lambda^0 \neq 0$. In fact,

$$\begin{aligned}L_Q\lambda^0(\dot{a}^1 + ... + \dot{a}^n) &= \int_M (\langle d^A a^{n-2} \wedge \dot{a}^1\rangle + \langle F_A \wedge \dot{a}^{n-2}\rangle)\\ &+ \frac{(-1)^n}{2} \sum_{\substack{2\leq i\leq n-4\\ 2\leq j\leq n-2-i}} \int_M \langle a^i \wedge [\dot{a}^j \wedge a^{n-i-j}]\rangle.\end{aligned} \tag{2.165}$$

Similarly, for G a complex Lie group with $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}$ non-degenerate, symmetric, Ad-invariant, $\mathbb{C}$-linear, and $P\to X$ a smooth G-bundle over a compact, complex manifold with $\dim_{\mathbb{C}} X = n$ and a holomorphic volume form $\Omega\in\Omega^{n,0}(X,\mathbb{C})$, the derived moduli stack $\mathcal{H}^d(P)_\bullet$ of holomorphic structures on P defined in Example 2.23 carries a $(2-n)$-shifted holomorphic symplectic structure which we describe as follows. The tangent complex of $\mathcal{H}^d(P)$ is the following chain complex of vector bundles over the space $\{A\in\mathcal{A}(P)/\Omega^{1,0}(ad\,P)\,|F_A^{0,2}=0\}$.

$$\underline{\Omega^0(ad\,P)}[1]\overset{\overline{\partial}^A}{\to}\underline{\Omega^{0,1}(ad\,P)}\overset{\overline{\partial}^A}{\to}\underline{\Omega^{0,2}(ad\,P)}[-1]\overset{\overline{\partial}^A}{\to}\dots\overset{\overline{\partial}^A}{\to}\underline{\Omega^{0,n}(ad\,P)}[-n+1], \tag{2.166}$$

The symplectic forms are defined exactly as in (2.155), (2.156), except that all the integrals are performed now against the volume form Ω. Note that the condition $d\Omega=0$, which allows to integrate by parts on such integrals, is necessary for the corresponding 2-forms ω_0, ω_1 to satisfy $L_Q\omega_0=0$ and $L_Q\omega_1=\delta\omega_0$ as before.

The case $n=2$ extends Mukai's holomorphic symplectic structure in the smooth locus of the moduli space of G-bundles on a $K3$ surface [196], and it can also be constructed as a holomorphic symplectic reduction with holomorphic symplectic form ω^0 and moment map $\mu:\mathcal{A}(P)/\Omega^{1,0}(ad\,P)\to\Omega^0(ad\,P)^*\cong\Omega^{0,2}(ad\,P)$, $A\mapsto F_A^{0,2}$. The case $n=3$ can be constructed as the derived critical locus of the holomorphic Chern-Simons functional and has been extensively studied in [34, 159, 160, 164] for its relation with Donaldson-Thomas invariants. The (-2)-shifted symplectic structure in the case $n=2$ has also been studied in search for invariants of Calabi-Yau fourfolds in [47, 208, 209].

Remark 2.37. Let (X,ω) be a compact, Hermitian manifold with $\dim_{\mathbb{C}} X = n$ and $d(\omega^{n-1})=0$. Let K be a compact Lie group with complexification G, let $P_h\to X$ be a K-bundle and write $P:=(P_h\times G)/K$ for its complexification (where we identify $(pk,g)\to(p,kg)$). A positive-definite, symmetric, Ad-invariant bilinear form $\langle\cdot,\cdot\rangle$: $\mathfrak{k}\otimes\mathfrak{k}\to\mathbb{R}$ determines a Kähler structure on the space $\mathcal{A}(P_h)$ of connections on P_h. Its complex structure $I^{\mathcal{A}}$ is given by identifying

$$T_A\mathcal{A}(P_h)=\Omega^1(ad\,P_h)=\Omega^{0,1}(ad\,P); \tag{2.167}$$

i.e.,

$$I_A^{\mathcal{A}}(\dot a)=(i\dot a^{0,1}-i\dot a^{1,0}). \tag{2.168}$$

Its symplectic structure $\omega^{\mathcal{A}}$ is given by

$$\omega_A^{\mathcal{A}}(\dot a_1,\dot a_2)=\int_X\langle\dot a_1\wedge\dot a_2\rangle\wedge\frac{\omega^{n-1}}{(n-1)!}. \tag{2.169}$$

This can also be written as

$$\omega_A^{\mathcal{A}} := \int_X \langle F_{\mathbb{A}} \wedge F_{\mathbb{A}} \rangle \wedge \frac{\omega^{n-1}}{(n-1)!}, \tag{2.170}$$

where $F_{\mathbb{A}} \in \Omega^2(\mathcal{A}(P_h) \times X, ad\, P_h)$ is the curvature of the connection $\mathbb{A}$ on the G-bundle $\mathcal{A}(P_h) \times P \to \mathcal{A}(P_h) \times X$ defined by

$$\mathbb{A}_{(A,p)}(\dot{a} + v_p) = A_p(v_p). \tag{2.171}$$

The space $\mathcal{A}^{1,1}(P_h) \subset \mathcal{A}(P_h)$ is formally a Kähler submanifold. The action of the gauge group $Gauge(P_h)$ on $\mathcal{A}(P_h)$ restricts to $\mathcal{A}^{1,1}(P_h)$ and is Hamiltonian, with moment map $\mu : \mathcal{A}^{1,1}(P_h) \to \Gamma(ad\, P_h)^*$ defined by

$$\mu(A)(s) = \int_X \langle F_A, s \rangle \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{2.172}$$

for $s \in \Gamma(ad\, P_h)$. Thus, ignoring again smoothness problems, $\mathcal{M} := \mu^{-1}(0)/Gauge(P_h)$ is a symplectic manifold. Note that there is a map $\mathcal{M} \to \mathcal{H}^d(P)$, where $\mathcal{H}^d(P)$ is as in Example 2.21. The Donaldson-Uhlenbeck-Yau theorem [100, 267] states that the restriction of this map to adequate (open) smooth locus of both moduli spaces is a diffeomorphism, and that the smooth locus of $\mathcal{M}$ is Kähler with respect to the complex structure induced by $\mathcal{H}^d(P)$. When X also admits a holomorphic volume form, then one may wonder what the relation is between the $(2-n)$-shifted holomorphic symplectic form on $\mathcal{H}^d(P)$ from above and the Kähler form on $\mathcal{M}$. We clarify this when $n = 2$.

Let (X, I, ω) be a Hermitian manifold with $\dim_{\mathbb{C}} X = 2$ and $d\omega = 0$, and let $\Omega \in \Omega^{2,0}(X, \mathbb{C})$ be a nowhere-vanishing $(2,0)$-form with $d\Omega = 0$. The Calabi-Yau theorem [279] implies that X is hyperkähler; that is, there are complex structures J, K such that

1. $IJK = -Id$,
2. $g := \omega(I\cdot, \cdot)$ is also Kähler with respect to J, K
3. $\Omega = \omega_J + i\omega_K$ for $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

Then for $P_h \to M$ a K-bundle with complexification P, it follows that each of the Hermitian structures (I, ω), (J, ω_J), (K, ω_K) determines as before a Kähler structure $(I^{\mathcal{A}}, \omega^{\mathcal{A}})$, $(J^{\mathcal{A}}, \omega_J^{\mathcal{A}})$, $(K^{\mathcal{A}}, \omega_K^{\mathcal{A}})$ on $\mathcal{A}(P_h)$. These also satisfy the hyperkähler relations, and in particular $\Omega^{\mathcal{A}} := \omega_J^{\mathcal{A}} + i\omega_K^{\mathcal{A}}$ is the holomorphic symplectic form on $\mathcal{A}(P)/\Omega^{1,0}(ad\, P) = \mathcal{A}(P_h)$ from above.

Chapter 3

Lie 2-groups and Maurer-Cartan forms

Lie groupoids are geometric objects that generalize differentiable manifolds to model possibly singular spaces, such as poorly behaved quotients or foliations [105, 187]. While manifolds and smooth maps between them form a category, Lie groupoids are more naturally thought of as the objects of a bicategory in which arrows are called *anafunctors* and 2-cells are called *transformations*. This bicategory is equivalent to the bicategory of differentiable stacks [33].

Lie 2-groups are geometric objects that model symmetries of Lie groupoids, in the same way that Lie groups model symmetries of manifolds. The study of 'set-theoretical' 2-groups dates back to [55, 190, 249]. The original definition of Lie 2-groups from [16] has been weakened since then in [19, 238] to capture important examples in geometry and physics. Roughly, a Lie 2-group is a Lie groupoid $\mathfrak{G}$ equipped with an anafunctor $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ which may not be strictly associative, but which is equipped with a transformation $\alpha : m \circ (m \times id) \Rightarrow m \circ (id \times m) : \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$, expressing a weak form of associativity of m. The product m is also required to have a distinguished element $1 \in \mathfrak{G}_0$ playing the role of the unit, and to admit inverses in a weak sense.

Two important families of Lie 2-groups in the literature are central extensions of G by BT (where G and T are Lie groups with T abelian) [238] and strict Lie 2-groups [19]. The former admit an alternative description as multiplicative gerbes, while the latter can be modelled by Lie crossed modules.

Gerbes were first defined by Giraud [133] as certain sheaves of groupoids and they were then studied in the work of Brylinski [56–58], who set up a working definition in terms of Čech cocycle data which has been extensively used since then, as well as the essentially equivalent definitions of Hitchin-Chatterjee gerbes [82, 152] and bundle gerbes [198]. This language was used in [58, 78] to study certain special gerbes over Lie groups which were called *multiplicative*, and Schommer-Pries proved in [238] that the classification of these coincides with the classification of Lie 2-group central extensions of G by BT. The notion of a *connection* on a multiplicative gerbe is introduced in [271] and used to prove classification results for Chern-Simons theories over arbitrary Lie groups.

As for strict Lie 2-groups, the fact that they can be presented by Lie crossed modules as in [16] is a straightforward generalization of the analogous result for set-theoretical 2-groups from [55]. Trying to axiomatize the observation from [235, 237] that the Lie 2-algebra of a Lie 2-group called $\mathrm{String}(n)$ has some additional structure that is useful for defining $\mathrm{String}(n)$-connections, [230] defined the notion of an *adjustment* on a Lie crossed module as a deformation of its Weil algebra. This structure was then presented in a finite way in [220].

Some aspects of Lie group theory can be generalized to Lie 2-groups, but this is not always done in a straightforward way. The main difference between (general) Lie 2-groups and ordinary Lie groups is that, by virtue of how anafunctors between Lie groupoids are defined, the product of two objects in a Lie 2-group $\mathfrak{G}$ is only well-defined up to isomorphism. This has many consequences, among which we highlight the following.

1. For an arbitrary $g \in \mathfrak{G}_0$, there is no canonical map $T_1\mathfrak{G}_0 \to T_1\mathfrak{G}_0$, $v \mapsto g \cdot v \cdot g^{-1}$ generalizing the adjoint action of a Lie group.
2. For an arbitrary $g \in \mathfrak{G}_0$, there is no canonical map $T_g(\mathfrak{G}_0) \to T_1(\mathfrak{G}_0)$ generalizing the map $dL_{g^{-1}} : T_gG \to T_1G$ of a Lie group. Thus, there is no canonical analog of the Maurer-Cartan form.
3. Since there is no canonical analog of the Maurer-Cartan form, there is no canonical way to extend some $v \in T_1(\mathfrak{G}_0)$ to a 'left-invariant' vector field; hence no analog of the Lie bracket on T_1G for G a Lie group.
4. Since there is no notion of left-invariant vector fields on $\mathfrak{G}$, we cannot take the flow of such vector fields to define an analog of the exponential map of a Lie group.

Problem 3 is related to the well-known problem in higher differential geometry that the tangent complex of a simplicial manifold is not equipped in general with a canonical structure generalizing the Lie bracket of vector fields on a manifold. It is shown in [178] that this can be solved by choosing connections on appropriate vector bundles, while [265] addresses problem 1 (in the more general context of Lie m-groups for $m \in \mathbb{N}$) by choosing other connection-like data.

In this chapter we introduce Lie 2-groups and we present some original definitions and results on this theory, relating them to previous constructions and studying the problems above. In Sections 3.1.1 and 3.1.2 we recall the definitions of the bicategories of Lie groupoids and Lie 2-groups, respectively, and we relate them to the simplicial manifolds from Chapter 2. In section 3.1.3 we define actions of Lie 2-groups. In Section 3.1.4 we introduce the original notion of Maurer-Cartan forms on a Lie 2-group $\mathfrak{G}$, which axiomatizes the structure from problems 1 and 2 as an additional datum on $\mathfrak{G}$, and we show that it allows to generalize Example 2.33 to the setting of arbitrary Lie 2-groups.

In Section 3.2.1 we set some conventions on gerbes, and in Section 3.2.2 we recall the definition of multiplicative gerbes, along with some classification results. In Section 3.2.3 we introduce the notion of *connective structure* on a multiplicative gerbe, closely related to the connections from [271], and we show that any multiplicative T-gerbe over G with a connective structure determines an Ad-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{t}$ that classifies it under some topological assumptions, providing a converse for a construction in [271]. In Section 3.2.4 we relate connective structures on multiplicative gerbes over G to prequantizations of the 2-shifted presymplectic structure on BG from Example 2.35, and to Maurer-Cartan forms on their corresponding Lie 2-groups. This is used to construct in a natural way the brackets of their Lie 2-algebras. In Section 3.2.5 we prove that a connective structure on a multiplicative gerbe $\mathcal{G} \to G$ determines equivariant trivializations of $exp^*\mathcal{G} \to \mathfrak{g}$, and we interpret these as the exponential map of the corresponding Lie 2-group.

In Section 3.3.1 we recall the relation between strict Lie 2-groups and Lie crossed modules, as well as the notion of adjustment from [220], and we provide a new interpretation for these by relating them to Maurer-Cartan forms. In Section 3.3.2 we show how adjustments relate to connective structures for Lie 2-groups that admit models both as a multiplicative gerbe and as a crossed module. Finally, in Section 3.3.3 we provide a more explicit construction of the equivariant trivializations of $exp^*\mathcal{G} \to \mathfrak{g}$ when the Lie 2-group associated to the multiplicative gerbe $\mathcal{G}$ also admits a model as a Lie crossed module.

3.1 Lie 2-groups

3.1.1 Lie groupoids and anafunctors

Definition 3.1 ([238]). A *Lie groupoid* is a small category $\mathfrak{X}$ such that

1. The space of objects $\mathfrak{X}_0$ and the space of arrows $\mathfrak{X}_1$ are manifolds.

2. The source and target maps $s, t : \mathfrak{X}_1 \to \mathfrak{X}_0$ are surjective submersions.

3. The identity $id : \mathfrak{X}_0 \to \mathfrak{X}_1$ and composition $\circ : \mathfrak{X}_1 {}_s\!\times_t \mathfrak{X}_1 \to \mathfrak{X}_1$ maps are smooth.

An *anafunctor* $(F, \pi_0, \rho_0, \pi_1, \rho_1) : \mathfrak{X} \to \mathfrak{Y}$ between Lie groupoids is a manifold (*total space*) F with smooth *anchor* and *action maps*

$$\begin{aligned} \pi_0 &: F \to \mathfrak{X}_0, & \rho_0 &: \mathfrak{X}_1 {}_s\!\times_{\pi_0} F \to F, \\ \pi_1 &: F \to \mathfrak{Y}_0, & \rho_1 &: F {}_{\pi_1}\!\times_t \mathfrak{Y}_1 \to F, \end{aligned} \tag{3.1}$$

satisfying the relations

$$\begin{aligned} &\pi_0(\rho_0(\gamma, x)) = t(\gamma), \quad \rho_0(\gamma' \circ \gamma, x) = \rho_0(\gamma', \rho_0(\gamma, x)), \quad \rho_0(id_{\pi_0(x)}, x) = x, \\ &\pi_1(\rho_1(x, \eta)) = s(\eta), \quad \rho_1(x, \eta' \circ \eta) = \rho_1(\rho_1(x, \eta'), \eta), \quad \rho_1(x, id_{\pi_1(x)}) = x, \\ &\pi_0(\rho_1(x, \eta)) = \pi_0(x), \quad \pi_1(\rho_0(\gamma, x)) = \pi_1(x), \quad \rho_1(\rho_0(\gamma, x), \eta) = \rho_0(\gamma, \rho_1(x, \eta)), \end{aligned} \tag{3.2}$$

for $x \in F$, γ, $\gamma' \in \mathfrak{X}_1$, η, $\eta' \in \mathfrak{Y}_1$ and such that

1. π_0 is a surjective submersion,

2. $F {}_{\pi_1}\!\times_t \mathfrak{Y}_1 \to F {}_{\pi_0}\!\times_{\pi_0} F$, $(x, \eta) \mapsto (x, \rho_1(x, \eta))$ is a diffeomorphism.

Given two anafunctors

$$(F, \pi_0, \rho_0, \pi_1, \rho_1), \, (F', \pi_0', \rho_0', \pi_1', \rho_1') : \mathfrak{X} \to \mathfrak{Y},$$

a *transformation* $\alpha : F \Rightarrow F'$ between them is a smooth map $\alpha : F \to F'$ with

$$\begin{aligned} \pi_0'(\alpha(x)) &= \pi_0(x), & \rho_0'(\gamma, \alpha(x)) &= \alpha(\rho_0(\gamma, x)), \\ \pi_1'(\alpha(x)) &= \pi_1(x), & \rho_1'(\alpha(x), \eta) &= \alpha(\rho_1(x, \eta)). \end{aligned} \tag{3.3}$$

The *composition* of anafunctors $\mathfrak{X} \xrightarrow{F} \mathfrak{Y} \xrightarrow{G} \mathfrak{Z}$ is the anafunctor $G \circ F : \mathfrak{X} \to \mathfrak{Z}$ with total space $(F {}_{\pi_1^F}\!\times_{\pi_0^G} G)/\sim$, where the equivalence relation is

$$(x, \rho_0^G(\eta, y)) \sim (\rho_1^F(x, \eta), y), \quad x \in F, \; y \in G, \; \eta \in \mathfrak{Y}_1, \tag{3.4}$$

and with

$$\begin{aligned} \pi_0^{G\circ F}([x, y]) &= \pi_0^F(x), & \rho_0^{G\circ F}(\gamma, [x, y]) &= [\rho_0^F(\gamma, x), y], \\ \pi_1^{G\circ F}([x, y]) &= \pi_1^G(y), & \rho_1^{G\circ F}([x, y], \zeta) &= [x, \rho_1^G(y, \zeta)] \end{aligned} \tag{3.5}$$

for $\gamma \in \mathfrak{X}_1$, $\zeta \in \mathfrak{Z}_1$.

Lie groupoids, anafunctors and transformations between them form a bicategory (in the sense of [35]) LieGpd. As discussed in [33], this bicategory is equivalent to the bicategory of differentiable 1-stacks that we define in Section 7.2.1.

Example 3.2. For a Lie group G, define the Lie groupoid BG with $BG_0 = \{*\}$, $(BG)_1 = G$ and composition defined by the group product. For Lie groups G_1, G_2, an anafunctor $F : BG_1 \to BG_2$ is the same as a right G_2-torsor with a left G_1-action commuting with the G_2-action. In particular, F is weakly invertible if and only if it is also a G_1-torsor, in which case an inverse F^{-1} is given simply by taking the same total space with inverted actions.

Example 3.3. A smooth functor $f : \mathfrak{X} \to \mathfrak{Y}$ between Lie groupoids gives an anafunctor with total space $F := \mathfrak{X}_0 \,{}_{f_0}\!\times_t \mathfrak{Y}_1$ and

$$\begin{aligned} \pi_0(x,\eta) &= x, & \rho_0(\gamma,(x,\eta)) &= (t(\gamma), f_1(\gamma)\circ\eta), \\ \pi_1(x,\eta) &= s(\eta), & \rho_1((x,\eta),\eta') &= (x, \eta\circ\eta'). \end{aligned} \tag{3.6}$$

For a general anafunctor, the condition that π_0 is a surjective submersion and that $F \,{}_{\pi_1}\!\times_t \mathfrak{Y}_1 \to F \,{}_{\pi_0}\!\times_{\pi_0} F$ is a diffeomorphism implies that there are local sections $\sigma : U \subset \mathfrak{X}_0 \to F$ of π_0 inducing isomorphisms $U \,{}_{\pi_1\circ\sigma}\!\times_t \mathfrak{Y}_1 = \pi_0^{-1}(U) \subset F$. In this sense, an anafunctor is to be regarded as an object constructed from gluing 'locally defined' smooth functors. The following proposition generalizes to our setting the classical result from category theory stating that a functor has a weak inverse if and only if it is essentially surjective and fully faithful.

Proposition 3.4 ([205])**.** *An anafunctor $(F, \pi_0, \rho_0, \pi_1, \rho_1) : \mathfrak{X} \to \mathfrak{Y}$ is weakly invertible if and only if the following two conditions are satisfied.*

1. *π_1 is a surjective submersion.*
2. *$\mathfrak{X}_1 \,{}_s\!\times_{\pi_0} F \to F \,{}_{\pi_1}\!\times_{\pi_1} F$, $(\gamma, x) \mapsto (\rho_0(\gamma, x), x)$ is a diffeomorphism.*

A Lie groupoid $\mathfrak{X}$ has an associated simplicial manifold (cf. Definition 2.1), called its *nerve* and denoted $N(\mathfrak{X})_\bullet$. It is defined by

$$\begin{aligned} N(\mathfrak{X})_n = \{(\{x_i\}_{i\in[n]}, \{f_{ij}\}_{i\leq j\in[n]}) \in \mathfrak{X}_0^{n+1} \times \mathfrak{X}_1^{\binom{n+2}{2}} |& \\ \forall i, \qquad & f_{ii} = id_{x_i}, \\ \forall i<j, \qquad & x_j \xrightarrow{f_{ij}} x_i, \\ \forall i<j<k,\ & f_{ij}\circ f_{jk} = f_{ik}\}, \end{aligned} \tag{3.7}$$

with simplicial maps defined similarly as in (2.23). Each $N(\mathfrak{X})_n$ is indeed a manifold, as it can also be described as the n-fold fibered product $N(\mathfrak{X})_n = \mathfrak{X}_1 {}_s\times_t \cdots {}_s\times_t \mathfrak{X}_1$, and s, t are submersions.

Example 3.5. Let G be a Lie group acting on a manifold M, then the *quotient groupoid* $M//G$ is the Lie groupoid $(M//G)_0 = M$ and $(M//G)_1 = M$, where $(x, g) \in M \times G$ is seen as an arrow $x \mapsto xg$ and composition is defined by $(xg_1, g_2) \circ (x, g_1) := (x, g_1 g_2)$. In particular, when $M = \{*\}$ with trivial G-action we obtain $\{*\}//G = BG$, for BG defined as in Example 3.2. Note that the nerve of $M//G$ is the simplicial manifold from Example 2.5, hence the notation.

Example 3.6. Let M be a manifold with an open cover $\{M_a\}_{a\in A}$, let T be an abelian Lie group T and let $\lambda_{abc} : M_{abc} \to T$ be a T-valued Čech 2-cocycle. We construct with this data a Lie groupoid $\mathcal{L}$ defined by $\mathcal{L}_0 := \sqcup_{a\in A} M_a$ and $\mathcal{L}_1 := \sqcup_{a,b\in A} M_{ab} \times T$, where $(a, b, x, t_{ab}) \in \mathcal{L}_1$ is seen as an arrow $(a, x) \to (b, x)$ and composition is given by

$$\begin{tikzcd} (a,x) \arrow[r, bend left, "{(a,b,x,t_{ab})}"] & (b,x) \arrow[r, bend left, "{(b,c,x,t_{bc})}"] & (c,x) \end{tikzcd} = \begin{tikzcd} (a,x) \arrow[rr, bend left, "{(a,c,x,t_{ab}t_{bc}\lambda_{abc}(x))}"] & & (c,x) \end{tikzcd} \tag{3.8}$$

The cocycle condition on λ_{abc} ensures that this composition map is associative. The nerve of $\mathcal{L}$ is the simplicial manifold $\lambda^* EBT$ from Example 2.8.

Definition 3.7 ([188])**.** Let $\mathfrak{X}$ be a Lie groupoid.

1. The *tangent Lie groupoid* of $\mathfrak{X}$ is the Lie groupoid $T\mathfrak{X}$ defined by $(T\mathfrak{X})_0 := T\mathfrak{X}_0$ and $(T\mathfrak{X})_1 := T\mathfrak{X}_1$, with source, target and composition maps given by push-forward along the source, target and composition maps of $\mathfrak{X}$.

2. A *multiplicative vector field* on $\mathfrak{X}$ is a pair $(X_0, X_1) \in \Gamma(T\mathfrak{X}_0) \oplus \Gamma(T\mathfrak{X}_1)$ defining a smooth functor $X : \mathfrak{X} \to T\mathfrak{X}$. We write $\Gamma(T\mathfrak{X})^{\mathfrak{X}}$ for the space of multiplicative vector fields.

3. The *Lie algebroid* of $\mathfrak{X}$ is the vector bundle $A_{\mathfrak{X}} \to \mathfrak{X}_0$ defined by

$$A_{\mathfrak{X}} := id^*(Ker(ds : T\mathfrak{X}_1 \to s^* T\mathfrak{X}_0)). \tag{3.9}$$

4. Given $\alpha \in \Gamma(A_{\mathfrak{X}})$, we define its *right-invariant extension* $\alpha^R \in \Gamma(T\mathfrak{X}_1)$ and its *left-invariant extension* $\alpha^L \in \Gamma(T\mathfrak{X}_1)$ by

$$\begin{aligned} \alpha^R(\gamma) &:= dR_\gamma(\alpha(t(\gamma))), \\ \alpha^L(\gamma) &:= d(L_\gamma \circ inv)(\alpha(t(\gamma))), \end{aligned} \tag{3.10}$$

for $\gamma \in \mathfrak{X}_1$. Here R_γ and L_γ denote right- and left- composition with γ, respectively, while $inv : \mathfrak{X}_1 \to \mathfrak{X}_1$ is the inversion map.

Note that, given $(X_0, X_1) \in \Gamma(T\mathfrak{X})^{\mathfrak{X}}$, then X_0 is determined by $X_0(x) = ds(X_1(id_x))$. Thus, we may see $\Gamma(T\mathfrak{X})^{\mathfrak{X}}$ as a subspace of $\Gamma(T\mathfrak{X}_1)$.

Proposition 3.8 ([40]). *Let $\mathfrak{X}$ be a Lie groupoid.*

1. *The space $\Gamma(T\mathfrak{X})^{\mathfrak{X}}$ is closed under the Lie bracket of vector fields on $\mathfrak{X}_1$.*
2. *There is a Lie algebra action of $\Gamma(T\mathfrak{X})^{\mathfrak{X}}$ on $\Gamma(A_{\mathfrak{X}})$, defined by*
$$[X, \alpha](x) := [X, \alpha^L](id_x) \tag{3.11}$$
for $X \in \Gamma(T\mathfrak{X})^{\mathfrak{X}}$, $\alpha \in \Gamma(A_{\mathfrak{X}})$ and $x \in \mathfrak{X}_0$.
3. *The map $\partial : \Gamma(A_{\mathfrak{X}}) \to \Gamma(T\mathfrak{X}_1)$, $\alpha \mapsto \alpha^L + \alpha^R$ takes values in $\Gamma(T\mathfrak{X})^{\mathfrak{X}}$ and satisfies*
$$\partial[X, \alpha] = [X, \partial\alpha] \tag{3.12}$$
for $X \in \Gamma(T\mathfrak{X})^{\mathfrak{X}}$ and $\alpha \in \Gamma(A_{\mathfrak{X}})$.

In other words, $[\cdot,\cdot]$ and ∂ define a structure of L_∞-algebra (cf. Definition 2.16) on $\Gamma(T\mathfrak{X}) := \Gamma(T\mathfrak{X})^{\mathfrak{X}} \oplus \Gamma(A_{\mathfrak{X}})[1]$. We call this the Lie 2-algebra of vector fields *on $\mathfrak{X}$.*

3.1.2 Lie 2-groups

Definition 3.9 ([238]). A *Lie 2-group* is a Lie groupoid $\mathfrak{G}$ with

1. a functor $1 : \{*\} \to \mathfrak{G}$ (the *unit*) and an anafunctor $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ (the *product*),
2. transformations $r : m \circ (id \times 1) \Rightarrow id : \mathfrak{G} \to \mathfrak{G}$ (the *right unitor*), $l : m \circ (1 \times id) \Rightarrow id : \mathfrak{G} \to \mathfrak{G}$ (the *left unitor*) and $\alpha : m \circ (m \times id) \Rightarrow m \circ (id \times m) : \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ (the *associator*)

satisfying

1. The *pentagon identity*:

$$
\begin{tikzcd}[column sep=huge]
\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \arrow[r, bend left=80, "((g_1g_2)g_3)g_4", ""{name=A, below}] \arrow[r, bend left=25, "(g_1(g_2g_3))g_4"{description}, ""{name=B}] \arrow[r, bend right=25, "g_1((g_2g_3)g_4)"{description}, ""{name=C}] \arrow[r, bend right=80, "g_1(g_2(g_3g_4))"', ""{name=D, above}] & \mathfrak{G}
\arrow[Rightarrow, from=A, to=B, "\alpha"'] \arrow[Rightarrow, from=B, to=C, "\alpha"'] \arrow[Rightarrow, from=C, to=D, "\alpha"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \arrow[r, bend left=70, "((g_1g_2)g_3)g_4", ""{name=A, below}] \arrow[r, "(g_1g_2)(g_3g_4)"{description}, ""{name=B}] \arrow[r, bend right=70, "g_1(g_2(g_3g_4))"', ""{name=C, above}] & \mathfrak{G}
\arrow[Rightarrow, from=A, to=B, "\alpha"'] \arrow[Rightarrow, from=B, to=C, "\alpha"']
\end{tikzcd}
. \tag{3.13}
$$

2. The *triangle identity*:

$$
\begin{tikzcd}[column sep=huge]
\mathfrak{G}\times\mathfrak{G} \arrow[r, bend left=60, "(g_11)g_2", ""{name=A, below}] \arrow[r, "g_1(1g_2)"{description}, ""{name=B}] \arrow[r, bend right=60, "g_1g_2"', ""{name=C, above}] & \mathfrak{G}\times\mathfrak{G}
\arrow[Rightarrow, from=A, to=B, "\alpha"'] \arrow[Rightarrow, from=B, to=C, "l"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=large]
\mathfrak{G}\times\mathfrak{G} \arrow[r, bend left=50, "(g_11)g_2", ""{name=A, below}] \arrow[r, bend right=50, "g_1g_2"', ""{name=B, above}] & \mathfrak{G}\times\mathfrak{G}
\arrow[Rightarrow, from=A, to=B, "r"']
\end{tikzcd}
. \tag{3.14}
$$

3. *Existence of inverses*: The anafunctor $p_1 \times m : \mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}\times\mathfrak{G}$ is weakly invertible, where p_1 denotes projection of the first factor.

A *homomorphism of Lie 2-groups* $(F, \alpha^F) : \mathfrak{G} \to \mathfrak{H}$ is an anafunctor $F : \mathfrak{G} \to \mathfrak{H}$ with a transformation $\alpha^F : F \circ m_{\mathfrak{G}} \Rightarrow m_{\mathfrak{H}} \circ F : \mathfrak{G}\times\mathfrak{G} \to \mathfrak{H}$ such that

$$
\begin{tikzcd}[column sep=huge]
\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \arrow[r, bend left=80, "F((g_1g_2)g_3)", ""{name=A, below}] \arrow[r, bend left=25, "F(g_1g_2)F(g_3)"{description}, ""{name=B}] \arrow[r, bend right=25, "(F(g_1)F(g_2))F(g_3)"{description}, ""{name=C}] \arrow[r, bend right=80, "F(g_1)(F(g_2)F(g_3))"', ""{name=D, above}] & \mathfrak{H}
\arrow[Rightarrow, from=A, to=B, "\alpha^F"'] \arrow[Rightarrow, from=B, to=C, "\alpha^F"'] \arrow[Rightarrow, from=C, to=D, "\alpha^{\mathfrak{H}}"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \arrow[r, bend left=80, "F((g_1g_2)g_3)", ""{name=A, below}] \arrow[r, bend left=25, "F(g_1(g_2g_3))"{description}, ""{name=B}] \arrow[r, bend right=25, "F(g_1)F(g_2g_3)"{description}, ""{name=C}] \arrow[r, bend right=80, "F(g_1)(F(g_2)F(g_3))"', ""{name=D, above}] & \mathfrak{H}
\arrow[Rightarrow, from=A, to=B, "\alpha^{\mathfrak{G}}"'] \arrow[Rightarrow, from=B, to=C, "\alpha^F"'] \arrow[Rightarrow, from=C, to=D, "\alpha^F"']
\end{tikzcd}
. \tag{3.15}
$$

Given homomorphisms (F_1, α^{F_1}), $(F_2, \alpha^{F_2}) : \mathfrak{G} \to \mathfrak{H}$, then a *transformation* between them is a transformation of anafunctors $\psi : F_1 \Rightarrow F_2 : \mathfrak{G} \to \mathfrak{H}$ such that

$$
\begin{tikzcd}[column sep=huge]
\mathfrak{G}\times\mathfrak{G} \arrow[r, bend left=70, "F_1(g_1g_2)", ""{name=A, below}] \arrow[r, "F_1(g_1)F_1(g_2)"{description}, ""{name=B}] \arrow[r, bend right=70, "F_2(g_1)F_2(g_2)"', ""{name=C, above}] & \mathfrak{H}
\arrow[Rightarrow, from=A, to=B, "\alpha^{F_1}"'] \arrow[Rightarrow, from=B, to=C, "\psi"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathfrak{G}\times\mathfrak{G} \arrow[r, bend left=70, "F_1(g_1g_2)", ""{name=A, below}] \arrow[r, "F_2(g_1g_2)"{description}, ""{name=B}] \arrow[r, bend right=70, "F_2(g_1)F_2(g_2)"', ""{name=C, above}] & \mathfrak{H}
\arrow[Rightarrow, from=A, to=B, "\psi"'] \arrow[Rightarrow, from=B, to=C, "\alpha^{F_2}"']
\end{tikzcd}
. \tag{3.16}
$$

Lie 2-groups, their homomorphisms and transformations form the *bicategory of Lie 2-groups*.

Remark 3.10. The diagrams in Definition 3.9 are equalities between transformations of anafunctors. For example, the pentagon identity is an equality between two transformations $m \circ (m \times id) \circ (m \times id \times id) \Rightarrow m \circ (id \times m) \circ (id \times id \times m) : \mathfrak{G}^4 \to \mathfrak{G}$ that are defined by composing α in the different ways that the diagram depicts: each black arrow represents an anafunctor (for example, we write $((g_1g_2)g_3)g_4$ for the anafunctor $m \circ (m \times id) \circ (m \times id \times id))$ and each 2-cell represents a smooth transformation that is defined in terms of α in the only possible way. We will make frequent use of these diagrams throughout the whole thesis.

Remark 3.11. We will also deal with (not necessarily Lie) 2-groups; i.e. groupoids with the structure from Definition 3.9 but without the smoothness assumptions. For example, if C is a bicategory and $x \in C_0$ is an object, then the *automorphism 2-group* of x is the groupoid $Aut(x)$ with $Aut(x)_0$ the set of invertible arrows $f : x \to x$ in C and with $Aut(x)_1(f,g)$ the set of invertible 2-cells $\alpha : f \Rightarrow : x \to y$ in C. Composition in $Aut(x)$ is given by vertical composition in C, while the product $m : Aut(x) \times Aut(x) \to Aut(x)$ is given by horizontal composition in C.

Let $(\mathfrak{G}, 1, m, r, l, \alpha)$ be a Lie 2-group. We construct an associated simplicial manifold, denoted $B\mathfrak{G}_\bullet$ and called the *delooping* of $\mathfrak{G}$. Let M be the total space of $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$. We write $d_2 \times d_0 : M \to \mathfrak{G}_0 \times \mathfrak{G}_0$ and $d_1 : M \to \mathfrak{G}_0$ for the corresponding anchor maps π_0, π_1 defining the anafunctor. Note that the unit 1 is just a point $\{*\} \to \mathfrak{G}_0$, while r, l are completely described by smooth functions $r, l : \mathfrak{G}_0 \to M$ with $d_2(r(g)) = d_1(r(g)) = d_0(l(g)) = d_1(l(g)) = g$, $d_0(r(g)) = d_2(l(g)) = 1$. Then we let

$$B\mathfrak{G}_0 := \{*\}, \qquad B\mathfrak{G}_1 := \mathfrak{G}_0, \qquad B\mathfrak{G}_2 := M, \tag{3.17}$$

with face maps $M \to \mathfrak{G}_0$ given precisely by d_0, d_1, d_2, degeneracy $s_0 : \{*\} \to \mathfrak{G}_0$ given by the unit of $\mathfrak{G}$ and degeneracies s_0, $s_1 : \mathfrak{G}_0 \to M$ given by l, r. To construct $B\mathfrak{G}_3$ we note first that α is a diffeomorphism between the manifolds

$$\begin{aligned} &\{(m_{1,2}, m_{12,3}) \in M^2 \mid d_1(m_{1,2}) = d_2(m_{12,3})\}/\sim, \\ &\{(m_{2,3}, m_{1,23}) \in M^2 \mid d_1(m_{2,3}) = d_0(m_{1,23})\}/\sim, \end{aligned} \tag{3.18}$$

where the equivalence relations are

$$(\rho(m_{1,2}, \gamma), m_{12,3}) \sim (m_{1,2}, \rho((\gamma, id), m_{12,3})), \tag{3.19}$$

$$(\rho(m_{2,3}, \gamma), m_{1,23}) \sim (m_{2,3}, \rho((id, \gamma), m_{1,23})) \tag{3.20}$$

for $\gamma \in \mathfrak{G}_1$. We define then

$$B\mathfrak{G}_3 := \{(m_{2,3}, m_{12,3}, m_{1,23}, m_{1,2}) \in M^4 \mid d_1(m_{1,2}) = d_2(m_{12,3}),\ d_1(m_{2,3}) = d_0(m_{1,23}),\\ d_0(m_{12,3}) = d_0(m_{2,3}),\ \alpha([m_{1,2}, m_{12,3}]) = [m_{2,3}, m_{1,23}]\}, \tag{3.21}$$

and let the face maps $B\mathfrak{G}_3 \to M$ be $d_0 = m_{2,3}$, $d_1 = m_{12,3}$, $d_2 = m_{1,23}$, $d_3 = m_{1,2}$. The pentagon and triangle identities ensure that $B\mathfrak{G}_3 \rightrightarrows M \rightrightarrows \mathfrak{G}_0 \rightrightarrows \{*\}$ can be extended to a simplicial manifold by the coskeleton construction [281]. Explicitly, we can describe all levels with the following formula.

$$\begin{aligned} B\mathfrak{G}_n := \{(\{g_{ij}\}_{i<j\in[n]}, \{m_{ijk}\}_{i<j<k\in[n]}) \in \mathfrak{G}_0^{\binom{n+1}{2}} \times M^{\binom{n+1}{3}} \mid \\ \forall i<j<k \in [n],\ d_2(m_{ijk}) = g_{ij},\ d_1(m_{ijk}) = g_{ik},\ d_0(m_{ijk}) = g_{jk}, \\ \forall\, i<j<k<l \in [n],\ \alpha([m_{jkl}, m_{ijl}]) = [m_{ijk}, m_{ikl}]\} \end{aligned} \tag{3.22}$$

Example 3.12. If G is a Lie group, then we see it as a Lie 2-group by associating to it the Lie groupoid with G as set of objects, only identity arrows, and multiplication functor given by the product of G.

Example 3.13. If T is an abelian Lie group, then we let BT be the groupoid with set of objects $\{*\}$ and manifold of arrows T, with composition given by the product of T. Then, since T is abelian, the product of T also defines an associative functor $m : BT \times BT \to BT$, endowing BT with structure of Lie 2-group. In this case we write $B^2T := B(BT)$ for its delooping.

A general Lie 2-group $(\mathfrak{G}, 1, m, l, r, \alpha)$ determines the following three invariants.

1. The topological group $\pi_0(\mathfrak{G})$ of isomorphism classes of objects with group product induced by m.
2. The Lie group $\pi_1(\mathfrak{G})$ of automorphisms in $\mathfrak{G}_1$ of $1 \in \mathfrak{G}_0$.
3. A continuous action $\triangleright$ of $\pi_0(\mathfrak{G})$ on $\pi_1(\mathfrak{G})$, defined by $[g] \triangleright f := id_g \cdot f \cdot id_{g^{-1}}$[1] for $[g] \in \pi_0(\mathfrak{G})$ and $f \in \pi_1(\mathfrak{G})$.

Using that the group structure on $\pi_1(\mathfrak{G})$ can be described by either m or the composition of $\mathfrak{G}$, an Eckmann-Hilton argument shows that $\pi_1(\mathfrak{G})$ is abelian. Moreover, there is an exact sequence of topological 2-groups (defined in an analogous way as for groups)

$$1 \to B\pi_1(\mathfrak{G}) \to \mathfrak{G} \to \pi_0(\mathfrak{G}) \to 1, \tag{3.23}$$

[1]More precisely: let A be the total space of the anafunctor $m \circ (m \times id) : \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$, then for $[g] \in \pi_0(\mathfrak{G})$ and $f \in \pi_1(\mathfrak{G})$ choose $a \in A$ with $\pi_1(a) = 1$, $\pi_0(a) = (g, 1, g^{-1})$ for some g, $g^{-1} \in \mathfrak{G}_0$ representing $[g]$, $[g]^{-1}$ and define $[g] \triangleright f \in \mathfrak{G}_1$ by $\rho_0((id_g, f, id_{g^{-1}}), a) = \rho_1(a, [g] \triangleright f)$.

where the functor $B\pi_1(\mathfrak{G}) \to \mathfrak{G}$ maps $\{*\} = (B\pi_1(\mathfrak{G}))_0$ to $1 \in \mathfrak{G}_0$ and equals the identity of $\pi_1(\mathfrak{G})$ on arrows, while the functor $\mathfrak{G} \to \pi_0(\mathfrak{G})$ is just the projection of each object of $\mathfrak{G}$ to its isomorphism class.

Example 3.14 ([119])**.** Given vector spaces V_0, V_1 with lattices $\Lambda_0 \subset V_0$, $\Lambda_1 \subset V_1$ and a bilinear form $\langle\cdot,\cdot\rangle : \Lambda_0 \otimes \Lambda_0 \to \Lambda_1$ (which we extend to $V_0 \otimes V_0 \to V_1$ by linearity), [119] constructs a Lie 2-group $\mathcal{T}$ which is an extension of V_0/Λ_0 by $B(V_1/\Lambda_1)$. Define $\mathcal{T}$ by $\mathcal{T}_0 = V_0$, $\mathcal{T}_1 = V_0 \times \Lambda_0 \times V_1/\Lambda_1$, where $(v^0, \lambda^0, [v^1]) \in \mathcal{T}_1$ is seen as an arrow $v^0 \to v^0 + \lambda^0$, and composition is defined as

$$(v^0 + \lambda^0_{01}, \lambda^0_{12}, [v^1_{12}]) \circ (v^0, \lambda^0_{01}, [v^1_{01}]) := (v_0, \lambda^0_{01} + \lambda^0_{12}, [v^1_{01} + v^1_{12}]). \tag{3.24}$$

We equip it with the smooth functor $m : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ acting on arrows as

$$m((u^0, \lambda^0, [u^1]), (v^0, \mu^0, [v^1])) := (u^0 + v^0, \lambda^0 + \mu^0, [u^1 + v^1 + \langle u^0, \mu^0\rangle]). \tag{3.25}$$

The unit is simply $0 \in \mathcal{T}_0$ and the associator, as well as the left and right unitors, are identities. It is shown in [119] that two 2-groups $\mathcal{T}^i$, $i = 1, 2$ constructed in this way from bilinear forms $\langle\cdot,\cdot\rangle_i : \Lambda_0 \times \Lambda_0 \to \Lambda_1$, $i = 1, 2$ are isomorphic if and only if the symmetric parts of $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ coincide. An isomorphism $\mathcal{T}^1 \to \mathcal{T}^2$ is given by choosing a bilinear form $B : \Lambda_0 \otimes \Lambda_0 \to \Lambda_1$ such that

$$B(u^0, v^0) - B(v^0, u^0) = \langle u^0, v^0\rangle_2 - \langle u^0, v^0\rangle_1, \tag{3.26}$$

from which we let $F : \mathcal{T}^1 \to \mathcal{T}^2$ be the functor defined on arrows by

$$F(u^0, \lambda^0, [u^1]) := (u^0, \lambda^0, [u^1 + B(u^0, \lambda^0)]), \tag{3.27}$$

and we let $\alpha^F : F \circ m_1 \Rightarrow m_2 \circ F : \mathcal{T}_1 \times \mathcal{T}_1 \to \mathcal{T}_2$ be the natural transformation defined by

$$\alpha^F(u_0, v_0) = (u_0 + v_0, 0, [B(v_0, u_0)]). \tag{3.28}$$

Example 3.15. Let M be a smooth manifold and let T be an abelian Lie group. We write $BT(M)$ for the groupoid with objects the class of all T-bundles on M and with isomorphisms of T-bundles as arrows. Similarly, we write $BT_\nabla(M)$ for the groupoid with objects the class of all T-bundles with connection on M and with flat isomorphisms of T-bundles as arrows. Given two T-bundles P_1, $P_2 \in BT(M)$, we define their tensor product by $P_1 \otimes P_2 := (P_1 \times_M P_2)/T$, where $(p_1 t, p_2) \sim (p_1, p_2 t)$. This can easily be enhanced to give functors $m : BT(M) \times BT(M) \to BT(M)$ and $m : BT_\nabla(M) \times BT_\nabla(M) \to BT_\nabla(M)$. Then, the isomorphisms $(P_1 \otimes P_2) \otimes P_3 \to P_1 \otimes (P_2 \otimes P_3)$, $[[p_1, p_2], p_3] \mapsto [p_1, [p_2, p_3]]$ and $P \otimes (M \times T) \to P$, $[p, (x, t)] \mapsto p$ define an associator and unitors, giving structure of strict 2-group to $BT(M)$ and $BT_\nabla(M)$. However, since the class of all T-bundles (resp.

the class of all T-bundles with connection) over M is not a set, we wish to have smaller descriptions of these 2-groups. We show how to obtain these in two different ways.

The first way is to choose a good open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M, so that every T-bundle over M can be trivialized over each open set U_i. Then we define the groupoid $BT(M)^{\mathcal{U}}$, where an object is a collection $\{t_{ij}\}_{i,j\in I}$ of functions $t_{ij} : U_{ij} \to T$ such that $t_{ij}t_{jk} = t_{ik}$ and $t_{ij} = t_{ji}^{-1}$, and where an arrow $\{t^1_{ij}\}_{i,j} \to \{t^2_{ij}\}_{i,j}$ is a collection $\{t_i\}_{i\in I}$ of functions $t_i : U_i \to T$ such that $t^1_{ij}t_j = t_i t^2_{ij}$. We equip $BT(M)^{\mathcal{U}}$ with the strictly associative functor $m : BT(M)^{\mathcal{U}} \times BT(M)^{\mathcal{U}} \to BT(M)^{\mathcal{U}}$ that acts as $(\{t^1_{ij}\}, \{t^2_{ij}\}) \mapsto \{t^1_{ij}t^2_{ij}\}$ on objects, and similarly on arrows. Then $BT(M)^{\mathcal{U}}$ is a small groupoid, with an obvious functor $BT(M)^{\mathcal{U}} \to BT(M)$ which is an equivalence of categories and which can be enhanced to a homomorphism of 2-groups. One can proceed similarly for the 2-group $BT_{\nabla}(M)$.

The second way does not depend on the choice of a cover, but it yields a weak 2-group. Let $C = \pi_0(BT(M))$ be the discrete abelian group of isomorphism classes of T-bundles over M. For each $c \in C$ we choose a representative T-bundle $L^c \in BT(M)_0$, with $L^0 = T \times M$ the trivial bundle, and for each pair $(c_1, c_2) \in C^2$ we choose an isomorphism $\phi(c_1, c_2) : L^{c_1+c_2} \to L^{c_1} \otimes L^{c_2}$, with $\phi(c, 0)$ the canonical isomorphism $L^c \otimes (T \times M) \to L^c$, $[p, t, m] \mapsto p$ and similarly for $\phi(0, c)$. Then for $(c_1, c_2, c_3) \in C^3$ we let $\alpha(c_1, c_2, c_3) \in C^\infty(M, T)$ be the unique function such that the following diagram commutes

$$\begin{tikzcd}
L^{c_1+c_2+c_3} \arrow[r, "\phi(c_1+c_2,c_3)"] \arrow[d, "\phi(c_1,c_2+c_3)"'] & L^{c_1+c_2} \otimes L^{c_3} \arrow[d, "\phi(c_1,c_2)\alpha(c_1,c_2,c_3)"] \\
L^{c_1} \otimes L^{c_2+c_3} \arrow[r, "\phi(c_2,c_3)"] & L^{c_1} \otimes L^{c_2} \otimes L^{c_3}.
\end{tikzcd} \tag{3.29}$$

In particular, note $\alpha(c_1, c_2, c_3) \equiv 1$ if $c_i = 0$ for some $i = 1, 2, 3$. It is clear that the inclusion $\sqcup_{c\in C}\mathcal{A}(L^c)//C^\infty(M, T) \to BT_{\nabla}(M)$ is an equivalence of groupoids, where $\mathcal{A}(L^c)$ is the space of connections on L^c and $C^\infty(M, T)$ acts on it by the gauge action. Now we equip the groupoid $\sqcup_{c\in C}\mathcal{A}(L^c)//C^\infty(M, T)$ with 2-group structure. The multiplication functor is defined on objects by

$$\begin{aligned} m : \sqcup_{c\in C}\mathcal{A}(L^c)//C^\infty(M,T) \times \sqcup_{c\in C}\mathcal{A}(L^c)//C^\infty(M,T) &\to \sqcup_{c\in C}\mathcal{A}(L^c)//C^\infty(M,T) \\ ((c_1, \nabla_1), (c_2, \nabla_2)) &\mapsto (c_1 + c_2, \phi(c_1, c_2)^*(\nabla_1 \otimes \nabla_2)), \end{aligned} \tag{3.30}$$

while acting as $(f_1, f_2) \mapsto f_1 f_2$ on arrows. The associator is the natural transformation that sends the triple (c_1, ∇_1), (c_2, ∇_2), (c_3, ∇_3) to the arrow

$$
\begin{aligned}
(c_1 + c_2 + c_3, &\phi(c_1 + c_2, c_3)^*(\phi(c_1, c_2)^*(\nabla_1 \otimes \nabla_2) \otimes \nabla_3)) \\
&\overset{\alpha(c_1,c_2,c_3)}{\to} (c_1 + c_2 + c_3, \phi(c_1, c_2 + c_3)^*(\nabla_1 \otimes \phi(c_2, c_3)^*(\nabla_2 \otimes \nabla_3))).
\end{aligned}
\tag{3.31}
$$

The left and right unitors are identities. The inclusion $\sqcup_{c\in C}\mathcal{A}(L^c)//C^\infty(M,T) \to BT_\nabla(M)$, along with the isomorphisms $\phi(c_1, c_2)$, defines a homomorphism of 2-groups which is an equivalence at the level of groupoids, hence we regard these as two equivalent presentations of the 2-group $BT_\nabla(M)$. By forgetting the connections we obtain a similar presentation of $BT(M)$.

3.1.3 Actions of Lie 2-groups

Definition 3.16. For $\mathfrak{G}$ a Lie 2-group and $\mathfrak{P}$ a Lie groupoid, a *smooth (right) action* of $\mathfrak{G}$ on $\mathfrak{P}$ is an anafunctor $\rho : \mathfrak{P} \times \mathfrak{G} \to \mathfrak{P}$ with smooth transformations $r^\rho : \rho \circ (id \times 1) \Rightarrow id : \mathfrak{P} \to \mathfrak{P}$ and $\alpha^\rho : \rho \circ (\rho \times id) \Rightarrow \rho \circ (id \times m) : \mathfrak{P} \times \mathfrak{G} \times \mathfrak{G} \to \mathfrak{P}$ such that

$$
\begin{tikzcd}[column sep=huge]
\mathfrak{P}\times\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}
\arrow[r, bend left=70, "((pg_1)g_2)g_3"{name=A}]
\arrow[r, bend left=20, "(p(g_1g_2))g_3"{name=B, description}]
\arrow[r, bend right=20, "p((g_1g_2)g_3)"{name=C, description}]
\arrow[r, bend right=70, "p(g_1(g_2g_3))"'{name=D}]
& \mathfrak{P}
\arrow[from=A, to=B, Rightarrow, "\alpha^\rho"']
\arrow[from=B, to=C, Rightarrow, "\alpha^\rho"']
\arrow[from=C, to=D, Rightarrow, "\alpha"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathfrak{P}\times\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}
\arrow[r, bend left=60, "((pg_1)g_2)g_3"{name=A}]
\arrow[r, "(pg_1)(g_2g_3)"{name=B, description}]
\arrow[r, bend right=60, "p(g_1(g_2g_3))"'{name=D}]
& \mathfrak{P}
\arrow[from=A, to=B, Rightarrow, "\alpha^\rho"']
\arrow[from=B, to=D, Rightarrow, "\alpha^\rho"']
\end{tikzcd}
, \tag{3.32}
$$

$$
\begin{tikzcd}[column sep=huge]
\mathfrak{P}\times\mathfrak{G}
\arrow[r, bend left=60, "(p1)g"{name=A}]
\arrow[r, "p(1g)"{name=B, description}]
\arrow[r, bend right=60, "pg"'{name=D}]
& \mathfrak{G}\times\mathfrak{G}
\arrow[from=A, to=B, Rightarrow, "\alpha^\rho"']
\arrow[from=B, to=D, Rightarrow, "l"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathfrak{G}\times\mathfrak{G}
\arrow[r, bend left=50, "(p1)g"{name=A}]
\arrow[r, bend right=50, "pg"'{name=D}]
& \mathfrak{G}\times\mathfrak{G}
\arrow[from=A, to=D, Rightarrow, "r^\rho"']
\end{tikzcd}
, \tag{3.33}
$$

$$
\begin{tikzcd}[column sep=huge]
\mathfrak{P}\times\mathfrak{G}
\arrow[r, bend left=60, "(pg)1"{name=A}]
\arrow[r, "p(g1)"{name=B, description}]
\arrow[r, bend right=60, "pg"'{name=D}]
& \mathfrak{G}\times\mathfrak{G}
\arrow[from=A, to=B, Rightarrow, "\alpha^\rho"']
\arrow[from=B, to=D, Rightarrow, "r"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathfrak{G}\times\mathfrak{G}
\arrow[r, bend left=50, "(pg)1"{name=A}]
\arrow[r, bend right=50, "pg"'{name=D}]
& \mathfrak{G}\times\mathfrak{G}
\arrow[from=A, to=D, Rightarrow, "r^\rho"']
\end{tikzcd}
. \tag{3.34}
$$

Left actions are similarly defined. Given an anafunctor $F : \mathfrak{P}_1 \to \mathfrak{P}_2$ and actions $(\rho_i, \alpha^{\rho_i})$ of a Lie 2-group $\mathfrak{G}$ on $\mathfrak{P}_i$ for $i = 1, 2$, an *equivariant structure* on F is a transformation $\alpha^F : F \circ \rho_1 \Rightarrow \rho_2 \circ (F \times id) : \mathfrak{P}_1 \times \mathfrak{G} \to \mathfrak{P}_2$ such that

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathfrak{P}_1 \times \mathfrak{G} \times \mathfrak{G} \arrow[r, bend left=80, "F((pg_1)g_2)"{name=A}] \arrow[r, bend left=25, "F(pg_1)g_2"{name=B, description}] \arrow[r, bend right=25, "(F(p)g_1)g_2"{name=C, description}] \arrow[r, bend right=80, "F(p)(g_1g_2)"'{name=D}] \arrow[from=A, to=B, phantom, "\Downarrow \alpha^F"] \arrow[from=B, to=C, phantom, "\Downarrow \alpha^F"] \arrow[from=C, to=D, phantom, "\Downarrow \alpha^{\rho_2}"] & \mathfrak{P}_2
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=large]
\mathfrak{P}_1 \times \mathfrak{G} \times \mathfrak{G} \arrow[r, bend left=70, "F((pg_1)g_2)"{name=A}] \arrow[r, "F(p(g_1g_2))"{name=B, description}] \arrow[r, bend right=70, "F(p)(g_1g_2)"'{name=C}] \arrow[from=A, to=B, phantom, "\Downarrow \alpha^{\rho_1}"] \arrow[from=B, to=C, phantom, "\Downarrow \alpha^F"] & \mathfrak{P}_2
\end{tikzcd}. \tag{3.35}$$

A *smooth action by a functor* of $\mathfrak{G}$ on $\mathfrak{P}$ is a smooth action such that ρ is a smooth functor. A *strict smooth action* is a smooth action by a functor such that $r^\rho = id$ and $\alpha^\rho = id$.

Remark 3.17. Let $\mathfrak{G}$ be a Lie 2-group and let P be a manifold, regarded as a Lie groupoid with only identity arrows. Then, any smooth action of $\mathfrak{G}$ on P factorizes by a continuous action of the topological group $\pi_0(\mathfrak{G})$ of isomorphism classes of objects of $\mathfrak{G}$ on P. However, since $\pi_0(\mathfrak{G})$ is in general not a Lie group, the smoothness condition cannot be stated in terms of this action. This remark follows from noting that, since there are only trivial arrows on P, an anafunctor $\rho : P \times \mathfrak{G} \to P$ is just a smooth map $\rho_0 : P \times \mathfrak{G}_0 \to P$, which we denote $(x, g) \mapsto xg$, such that $xg = xg'$ whenever there exists an arrow $g \to g'$ in $\mathfrak{G}_1$. Moreover, smooth transformations r^ρ, α^ρ as in Definition 3.16 exist if and only if $x1 = x$ and for $g_1, g_2, g_{12} \in \mathfrak{G}_0$ such that there exists $m \in M$ (the total space of the anafunctor $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$) with $d_2(m) = g_1$, $d_0(m) = g_2$, $d_1(m) = g_{12}$ we have $(xg_1)g_2 = xg_{12}$; in this case, r^ρ and α^ρ are unique.

Let $\mathfrak{G}$ be a Lie 2-group acting through ρ, α^ρ on a Lie groupoid $\mathfrak{P}$ in the sense of Definition 3.16. We define a simplicial manifold $\mathfrak{P}//\mathfrak{G}$, called the *quotient 2-groupoid*, as follows. Let

$$\mathfrak{P}_0 \times \mathfrak{G}_0 \overset{d_2 \times d_0}{\leftarrow} R \overset{d_1}{\to} \mathfrak{P}_0 \tag{3.36}$$

be the total space and anchors of the anafunctor ρ. Then α^ρ is a diffeomorphism between the manifolds

$$\begin{aligned} \{(r_{01}, r_{12}) \in R^2 \mid d_1(r_{01}) = d_2(r_{12})\}/\sim, \\ \{(m_{012}, r_{02}) \in R \times M \mid d_1(m_{012}) = d_0(r_{02})\}/\sim, \end{aligned} \tag{3.37}$$

where the equivalence relations are

$$(\rho(r_{01}, f), r_{1,2}) \sim (r_{01}, \rho((f, id), r_{12})), \tag{3.38}$$

$$(\rho(m_{012}, \gamma), r_{02}) \sim (m_{012}, \rho((id, \gamma), r_{0,2})) \tag{3.39}$$

for $f \in \mathfrak{X}_1$ and $\gamma \in \mathfrak{G}_1$. We define $\mathfrak{P}//\mathfrak{G}$ by

$$\begin{aligned}(\mathfrak{P}//\mathfrak{G})_n = \{(\{p_i\}_{i\in[n]}, \{r_{ij}\}_{i<j\in[n]}, \{m_{ijk}\}_{i<j<k}) \in \mathfrak{P}_0^{\binom{n+1}{1}} \times R^{\binom{n+1}{2}} \times M^{\binom{n+1}{3}} \mid \\ \forall i<j\in[n], \quad d_2(r_{ij}) = p_i,\ d_1(r_{ij}) = p_j \\ \forall i<j<k\in[n],\ d_2(m_{ijk}) = d_0(r_{ij}),\ d_1(m_{ijk}) = d_0(r_{ik}), \\ d_0(m_{ijk}) = d_0(r_{jk}),\ \alpha^\rho([r_{ij}, r_{jk}]) = [r_{ik}, m_{ijk}], \\ \forall\, i<j<k<l\in[n],\ \alpha([m_{jkl}, m_{ijl}]) = [m_{ijk}, m_{ikl}]\},\end{aligned} \tag{3.40}$$

with simplicial maps defined similarly as in (2.23). In particular,

$$\begin{aligned}(\mathfrak{P}//\mathfrak{G})_0 &= \mathfrak{P}_0, \\ (\mathfrak{P}//\mathfrak{G})_1 &= \{(p_0, p_1, r_{01}) \in P^2 \times R \,|\, d_2(r_{01}) = p_0,\ d_1(r_{01}) = p_1\} \cong R, \\ (\mathfrak{P}//\mathfrak{G})_2 &= \{(p_0, p_1, p_2, r_{01}, r_{12}, r_{02}, m_{012}) \in P^3 \times R^3 \times M \,|\, d_2(r_{01}) = d_2(r_{02}) = p_0, \\ &\qquad d_1(r_{01}) = d_2(r_{12}) = p_1,\ d_1(r_{02}) = d_1(r_{12}) = p_2, \\ &\qquad \alpha^\rho([r_{01}, r_{12}]) = [r_{02}, m_{012}]\} \cong (R\, {}_{d_1}\!\times_{d_2} R)\, {}_{d_0\times d_0}\!\times_{d_0\times d_2} M.\end{aligned} \tag{3.41}$$

Remark 3.18. In the following we shall also need to perform the construction of the quotient 2-groupoid $\mathfrak{P}//\mathfrak{G}$ when $\mathfrak{P}$ is a groupoid internal to the category of derived manifolds (i.e., the analog of Definition 3.1 replacing manifolds by the derived manifolds from Section 2.2.2). This is done in a completely analogous way, similarly as in Example 2.23.

For example, if $\mathfrak{P}$ is just a derived manifold (M, E, Q) whose corresponding fiberwise structure of curved L_∞-algebra on E is denoted by Φ, $\{\cdot, ..., \cdot\}$, then an action of $\mathfrak{G}$ on $\mathfrak{P}$ could be given by a smooth action of $\mathfrak{G}$ on the manifold M, $(x, g) \mapsto xg$ (cf. Remark 3.17) and a smooth action of $\mathfrak{G}$ on the total space of E, $(e, g) \mapsto eg$, fitting in a commutative diagram with the projection $\pi : E \to M$, and such that for each $g \in \mathfrak{G}_0$ we have that $e \mapsto eg$ is degree-preserving, fiberwise linear, and satisfies $\Phi(xg) = \Phi(x)g$, $\{e_1 g, ..., e_n g\} = \{e_1, ..., e_n\}g$ for $x \in M$, e_1, ..., $e_n \in E$. Although more general actions exist, these will be sufficient for the purposes of this thesis. In this case, the quotient 2-groupoid $\mathfrak{P}//\mathfrak{G}$ is given by $(\mathfrak{P}//\mathfrak{G})_n = \mathfrak{P} \times B\mathfrak{G}_n$, with simplicial maps defined similarly as in Example 2.5, using Remark 2.19 to see the action map of $\mathfrak{G}$ on E as a map of derived manifolds $\mathfrak{P} \times \mathfrak{G}_0 \to \mathfrak{P}$, with the arrows of $\mathfrak{G}$ acting trivially.

Our next goal is to present the conjugation action of a Lie 2-group on itself. For this we need to introduce first a new definition.

Definition 3.19 ([19]). Let $(\mathfrak{G}, 1, m, r, l, \alpha)$ be a Lie 2-group. A *coherent inversor* is an anafunctor $inv : \mathfrak{G} \to \mathfrak{G}$ with transformations $e : m \circ (inv \times id) :\Rightarrow 1 : \mathfrak{G} \to \mathfrak{G}$ and

$i : 1 \Rightarrow m \circ (id \times inv) : \mathfrak{G} \to \mathfrak{G}$ satisfying the *zig-zag identities*

$$\begin{tikzcd}[column sep=huge]
\mathfrak{G} \arrow[r, bend left=70, "1\cdot g"{name=A}] \arrow[r, bend left=20, "(g\cdot g^{-1})\cdot g"{name=B, description}] \arrow[r, bend right=20, "g\cdot(g^{-1}\cdot g)"{name=C, description}] \arrow[r, bend right=70, "g\cdot 1"'{name=D}] & \mathfrak{G}
\arrow[Rightarrow, from=A, to=B, "i"'] \arrow[Rightarrow, from=B, to=C, "\alpha"'] \arrow[Rightarrow, from=C, to=D, "e"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathfrak{G} \arrow[r, bend left=50, "1\cdot g"{name=A}] \arrow[r, "g"{name=B, description}] \arrow[r, bend right=50, "g\cdot 1"'{name=C}] & \mathfrak{G}
\arrow[Rightarrow, from=A, to=B, "l"'] \arrow[Rightarrow, from=B, to=C, "r"']
\end{tikzcd}, \tag{3.42}$$

$$\begin{tikzcd}[column sep=huge]
\mathfrak{G} \arrow[r, bend left=70, "g^{-1}\cdot 1"{name=A}] \arrow[r, bend left=20, "g^{-1}\cdot(g\cdot g^{-1})"{name=B, description}] \arrow[r, bend right=20, "(g^{-1}\cdot g)\cdot g^{-1}"{name=C, description}] \arrow[r, bend right=70, "1\cdot g^{-1}"'{name=D}] & \mathfrak{G}
\arrow[Rightarrow, from=A, to=B, "i"'] \arrow[Rightarrow, from=B, to=C, "\alpha"'] \arrow[Rightarrow, from=C, to=D, "e"']
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
\mathfrak{G} \arrow[r, bend left=50, "g^{-1}\cdot 1"{name=A}] \arrow[r, "g^{-1}"{name=B, description}] \arrow[r, bend right=50, "1\cdot g^{-1}"'{name=C}] & \mathfrak{G}
\arrow[Rightarrow, from=A, to=B, "r"'] \arrow[Rightarrow, from=B, to=C, "l"']
\end{tikzcd}. \tag{3.43}$$

As proven in [19], any set-theoretical 2-group in the sense of Definition 3.9 can be equipped with a coherent inversor, and any two choices of coherent inversor are essentially equivalent. Thanks to the use of anafunctors, the proof can probably be adapted to yield an equivalent result for Lie 2-groups, although we will not pursue this here. We do note that, for $\mathfrak{G}$ a Lie 2-group with a coherent inversor (inv, i, e), there are canonical transformations

$$\begin{aligned}
1^{inv} &: inv \circ 1 \Rightarrow 1 : \{*\} \to \mathfrak{G}, \\
m^{inv} &: m \circ (inv \times inv) \circ Flip \Rightarrow inv \circ m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G},
\end{aligned} \tag{3.44}$$

where $Flip : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G} \times \mathfrak{G}$ is the functor that switches the order of the two factors. They are defined by composing α, i, e, l, r as follows.

$$1^{-1} \xrightarrow{r} 1^{-1} \cdot 1 \xrightarrow{e} 1, \tag{3.45}$$

$$\begin{aligned}
g_2^{-1} g_1^{-1} &\xrightarrow{i} (g_2^{-1} g_1^{-1})((g_1 g_2)(g_1 g_2)^{-1}) \xrightarrow{\alpha} ((g_2^{-1} g_1^{-1})(g_1 g_2))(g_1 g_2)^{-1} \\
&\xrightarrow{\alpha} (g_2^{-1}(g_1^{-1}(g_1 g_2))(g_1 g_2)^{-1} \xrightarrow{\alpha} (g_2^{-1}((g_1^{-1} g_1) g_2))(g_1 g_2)^{-1} \\
&\xrightarrow{e} (g_2^{-1}(1 g_2))(g_1 g_2)^{-1} \xrightarrow{l} (g_2^{-1} g_2)(g_1 g_2)^{-1} \xrightarrow{e} 1 \cdot (g_1 g_2)^{-1} \xrightarrow{l} (g_1 g_2)^{-1}.
\end{aligned} \tag{3.46}$$

Here each arrow is a natural transformation between anafunctors that are defined by composing m, inv and 1 in a hopefully self-explanatory way.

Lemma 3.20. *Let $\mathfrak{G}$ be a Lie 2-group equipped with a coherent inversor (inv, e, i). Then the anafunctor $Ad^{-1} : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ defined by*

$$Ad^{-1} := \mathfrak{G} \times \mathfrak{G} \overset{(inv \circ p_2) \times m}{\longrightarrow} \mathfrak{G} \times \mathfrak{G} \overset{m}{\to} \mathfrak{G}, \tag{3.47}$$

where p_2 denotes projection of the second factor, together with the transformation $r^{Ad^{-1}}$: $Ad^{-1} \circ (id \times 1) \Rightarrow id : \mathfrak{G} \to \mathfrak{G}$ defined by

$$1^{-1}(h \cdot 1) \xrightarrow{r} 1^{-1} \cdot h \xrightarrow{1^{inv}} 1 \cdot h \xrightarrow{l} h \tag{3.48}$$

and the transformation

$$\alpha^{Ad^{-1}} : Ad^{-1} \circ (Ad^{-1} \times id) \Rightarrow Ad^{-1} \circ (id \times m) : \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G} \tag{3.49}$$

defined by using α, m^{inv} in the following way

$$\begin{aligned} g_2^{-1}((g_1^{-1}(hg_1))g_2) &\xrightarrow{\alpha} (g_2^{-1}(g_1^{-1}(hg_1)))g_2 \xrightarrow{\alpha} ((g_2^{-1}g_1^{-1})(hg_1))g_2 \\ &\xrightarrow{\alpha} (g_2^{-1}g_1^{-1})((hg_1)g_2) \xrightarrow{\alpha} (g_2^{-1}g_1^{-1})(h(g_1g_2)) \xrightarrow{m^{inv}} (g_1g_2)^{-1}(h(g_1g_2)) \end{aligned} \tag{3.50}$$

determine a right action of $\mathfrak{G}$ on itself, called the conjugation action.

Proof. Checking that (3.32), (3.33) and (3.34) are satisfied directly would involve very tedious computations. However, we can invoke the coherence theorem for bicategories [212], which implies that any two natural transformations with the same source and target obtained by composing α, r, l, i, e in any way coincide. □

Example 3.21. Let M be a manifold, let T be an abelian Lie group and let $BT_\nabla(M)$ be the 2-group of T-bundles with connection on M. To each T-bundle $P \to M$ we associate the T-bundle $P^* \to M$ defined by $P^* := \{(x, \phi) \mid x \in M,\ \phi : P_x \to T,\ \phi(pt) = t\phi(p)\}$. This can easily be enhanced to give a functor $inv : BT_\nabla(M) \to BT_\nabla(M)$, which together with the isomorphisms $e_P : P^* \otimes P \to M \times T$, $[(x, \phi), p] \mapsto (x, \phi(p))$ provides a coherent inversor for $BT_\nabla(M)$. Its corresponding conjugation action is the functor $((P_1, \nabla_1), (P_2, \nabla_2)) \mapsto (P_2^*, \nabla_2^*) \otimes ((P_1, \nabla_1) \otimes (P_2, \nabla_2))$, equipped with the corresponding transformations from Lemma 3.20.

As in Example 3.15, we can present $BT_\nabla(M)$ as a small 2-group by letting C be the abelian group of isomorphism classes of T-bundles over M and choosing representatives L^c for each $c \in C$ with isomorphisms $\phi(c_1, c_2) : L^{c_1+c_2} \to L^{c_1} \otimes L^{c_2}$. In this presentation, there is a coherent inversor for $BT_\nabla(M)$ defined by $(c, \nabla) \mapsto (-c, -\nabla)$, where $-\nabla$ denotes the unique connection on L^{-c} such that $\phi(-c, c) : M \times T \to L^{-c} \otimes L^c$ is flat with respect to the trivial connection on $M \times T$ and the connection $(-\nabla) \otimes \nabla$ on $L^{-c} \otimes L^c$. The corresponding conjugation functor Ad^{-1} acts on objects as $((c_1, \nabla_1), (c_2, \nabla_2)) \mapsto (c_1, \nabla_1 + \chi(c_1, c_2)^*\theta^T)$, where $\chi(c_1, c_2) : M \to T$ is the unique function such that the following is a commutative diagram.

$$\begin{tikzcd} L^{c_1} \arrow[r, "\phi(-c_2,c_1+c_2)"] \arrow[d, "\phi(-c_2,c_2)"'] & L^{-c_2} \otimes L^{c_1+c_2} \arrow[d, "\phi(c_1,c_2)"] \\ L^{-c_2} \otimes L^{c_2} \otimes L^{c_1} \arrow[r, "\chi(c_1,c_2)"'] & L^{-c_2} \otimes L^{c_1} \otimes L^{c_2} \end{tikzcd} \tag{3.51}$$

In particular, note $\chi(0,c) \equiv 1$. The natural transformations $r^{Ad^{-1}}$ and $\alpha^{Ad^{-1}}$ can be computed as in Lemma 3.20; it turns out that $r^{Ad^{-1}}$ is trivial, while $\alpha^{Ad^{-1}}$ is the natural transformation that sends $((c,\nabla),(c_1,\nabla_1),(c_2,\nabla_2))$ to the isomorphism

$$(c, \nabla + \chi(c,c_1)^*\theta_T + \chi(c,c_2)^*\theta^T) \overset{\alpha^\chi(c,c_1,c_2)}{\to} (c, \nabla + \chi(c, c_1+c_2)^*\theta^T), \tag{3.52}$$

for $\alpha^\chi(c,c_1,c_2) := \chi(c,c_1+c_2)^{-1}\chi(c,c_1)\chi(c,c_2)$. In particular, note $\alpha^\chi(0,c_1,c_2) \equiv 1$.

3.1.4 Maurer-Cartan forms: relating Lie 2-groups and Lie 2-algebras

Definition 3.22 ([19])**.** Let $\mathfrak{G}$ be a Lie 2-group. Its *Lie 2-algebra* is the 2-step complex of vector spaces $\mathfrak{h} \overset{t_*}{\to} \mathfrak{g}$, where $\mathfrak{g} := T_1\mathfrak{G}_0$, $\mathfrak{h} := Ker(s_*) \subset T_{id(1)}\mathfrak{G}_1$ and we write $1 \in \mathfrak{G}_0$ for the image of $1 : \{*\} \to \mathfrak{G}_0$.

Note we have not defined any brackets defining an L_∞-algebra structure on $\mathfrak{h} \overset{t_*}{\to} \mathfrak{g}$. This is because such brackets are not canonical in general, for the same reasons as in Remark 2.4. Other familiar notions from the theory of Lie groups such as the adjoint representation, the exponential map or the notion of connection on principal bundles have not been defined in the literature for arbitrary Lie 2-groups without additional structure. We will try to solve some of these problems by introducing a new notion of Maurer-Cartan forms.

If $\mathfrak{G}$ is a Lie 2-group with delooping $B\mathfrak{G}_\bullet$, then we note the following.

1. $T_1(B\mathfrak{G}_1) = \mathfrak{g}$
2. Since $\mathfrak{h} = Ker(s_*) \subset T_{id(1)}\mathfrak{G}_1$, the map $\rho_1 : B\mathfrak{G}_2 \,{}_{\pi_1}\!\times_t \mathfrak{G}_1 \to B\mathfrak{G}_2$ determines a subspace $\rho_{1*} : \mathfrak{h} \to T_{s_0(1)}B\mathfrak{G}_2$. There is a canonical projection

$$\begin{aligned} T_{s_0(1)}(B\mathfrak{G}_2) &\to \mathfrak{h} \\ v &\mapsto v - s_{0*}d_{0*}(v) - s_{1*}d_{2*}v. \end{aligned} \tag{3.53}$$

These maps induce an isomorphism between the Lie 2-algebra of $\mathfrak{G}$, as in Definition 3.22, and the (shifted by 1) tangent complex of the simplicial manifold $B\mathfrak{G}$, as defined in Section 2.1.1.

Definition 3.23. A *(left) adjoint action* of a Lie 2-group $\mathfrak{G}$ with Lie 2-algebra $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$ is a pair of smooth, linear, left actions by functors of $\mathfrak{G}$ on $\mathfrak{g}$ and on $\mathfrak{h}$, denoted as in Remark 3.17 by $Ad : \mathfrak{G}_0 \times \mathfrak{g} \to \mathfrak{g}$ and $Ad : \mathfrak{G}_0 \times \mathfrak{h} \to \mathfrak{h}$ such that $t_* : \mathfrak{h} \to \mathfrak{g}$ is equivariant for the induced $\pi_0(\mathfrak{G})$-action. For such a left adjoint action, a *(right-invariant) Maurer-Cartan form* on $\mathfrak{G}$ is a pair

$$\theta^0 \in \Omega^1(B\mathfrak{G}_1, \mathfrak{g}), \quad \theta^1 \in \Omega^1(B\mathfrak{G}_2, \mathfrak{h}) \tag{3.54}$$

such that

$$t_*\theta^1 = d_2^*\theta^0 - d_1^*\theta^0 + Ad(d_2(\cdot))d_0^*\theta^0 \in \Omega^1(B\mathfrak{G}_2, \mathfrak{g}), \tag{3.55}$$

$$0 = d_3^*\theta^1 - d_2^*\theta^1 + d_1^*\theta^1 + Ad(d_2 \circ d_3(\cdot))d_0^*\theta^1 \in \Omega^1(B\mathfrak{G}_3, \mathfrak{g}). \tag{3.56}$$

and such that θ_1^0 is the identity and $\theta^1_{s_0(1)}$ is (3.53). A *right adjoint action* Ad^{-1} is defined similarly, but with $\mathfrak{G}$ acting on the right, and for such a right adjoint action a *(left-invariant) Maurer-Cartan form* is defined similarly, replacing (3.55) and (3.56) by

$$t_*\theta^1 = Ad^{-1}(d_0(\cdot))d_2^*\theta^0 - d_1^*\theta^0 + d_0^*\theta^0 \in \Omega^1(B\mathfrak{G}_2, \mathfrak{g}), \tag{3.57}$$

$$0 = Ad^{-1}(d_0 \circ d_0(\cdot))d_3^*\theta^1 - d_2^*\theta^1 + d_1^*\theta^1 + d_0^*\theta^1 \in \Omega^1(B\mathfrak{G}_3, \mathfrak{g}). \tag{3.58}$$

Note that an adjoint action of a Lie 2-group $\mathfrak{G}$ with Lie 2-algebra $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$ determines a (strict) action of $\mathfrak{G}$ on $\mathfrak{g}//\mathfrak{h}$ in the sense of Definition 3.16, where $\mathfrak{g}//\mathfrak{h}$ is the quotient groupoid associated to the action of $\mathfrak{h}$ on $\mathfrak{g}$ given by t_*. This is the most natural way of thinking about an adjoint action, and it is possible that Maurer-Cartan forms can also be defined for weak actions. The standard adjoint action of a Lie group and its Maurer-Cartan forms fit into Definition 3.23. In Sections 3.2.4 and 3.3.1 we will see more examples of Maurer-Cartan forms, relating them to *connections* on multiplicative gerbes [271, 273] and to *adjustments* on crossed modules [220]. These are structures that have been used to define connections on principal bundles for certain families of Lie 2-groups.

Definition 3.24. Let $\mathfrak{G}$ be a Lie 2-group with Lie 2-algebra $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$. Then

1. A *differentiation* of $\mathfrak{G}$ is a structure of L_∞-algebra on $\mathfrak{g} \oplus \mathfrak{h}[1]$ whose differential is t_*, together with a morphism of L_∞-algebras $\mathfrak{g} \oplus \mathfrak{h}[1] \to \Gamma(T\mathfrak{G})$. Here $\Gamma(T\mathfrak{G})$ is the Lie 2-algebra of vector fields on $\mathfrak{G}$ as in Proposition 3.8.

2. Assume $\mathfrak{G}$ has a coherent inversor (Definition 3.19) and an adjoint action (Definition 3.23). An *exponential map* on $\mathfrak{G}$ is an anafunctor $(E, \pi_0, \rho_0, \pi_1, \rho_1) : \mathfrak{g}//\mathfrak{h} \to \mathfrak{G}$, equipped with an equivariant structure α^{exp} for the adjoint action of $\mathfrak{G}$ on $\mathfrak{g}//\mathfrak{h}$ and the conjugation action of $\mathfrak{G}$ on itself (Lemma 3.20), and such that

(a) There exist a neighborhood $U \subset \mathfrak{G}_0$ of 1 and a neighborhood $V \subset \mathfrak{g}$ of 0 such that the map $\pi_1 : E \to \mathfrak{G}_0$ restricts to a surjective submersion $\pi_0^{-1}(V) \to U$.

(b) There exists a neighborhood $W \subset \mathfrak{g} \oplus \mathfrak{h} = (\mathfrak{g}//\mathfrak{h})_1$ of $(0,0)$ such that the map $(\mathfrak{g}//\mathfrak{h})_1 \,{}_s\!\times_{\pi_0} E \to E \,{}_{\pi_1}\!\times_{\pi_1} E$, $(\gamma, x) \mapsto (\rho_0(\gamma, x), x)$ restricts to a diffeomorphism from $W \,{}_s\!\times_{\pi_0} \pi^{-1}(V)$ onto its image.

In light of Proposition 3.4, conditions 2a and 2b are to be thought of as imposing that the exponential map is a 'local equivalence' of Lie groupoids. In Sections 3.2.4 and 3.2.5 we will show that a Maurer-Cartan form on a Lie 2-group $\mathfrak{G}$ associated to a multiplicative gerbe determines a differentiation of $\mathfrak{G}$ and an exponential map on it. We conjecture this is also the case for Maurer-Cartan forms on general Lie 2-groups (see Section 8.2.1 for some speculations on this). Proposition 3.27 below uses Maurer-Cartan forms to generalize another construction for Lie groups (namely, Example 2.33) to the setting of general Lie 2-groups. It will let us define Hamiltonian actions of Lie 2-groups in Proposition 6.11. First we need two lemmas.

Lemma 3.25. *Let $\mathfrak{G}$ be a Lie 2-group with a left adjoint action Ad and a right-invariant Maurer-Cartan form (θ^0, θ^1). Then*

$$\theta^0_g(d_{1,*}(v) - d_{0,*}(v)) - \theta^0_1(d_{2,*}(v)) = -t_*\theta^1_{s_0(g)}(v), \tag{3.59}$$
$$\theta^0_g(d_{1,*}(w) - d_{2,*}(w)) - Ad(g)\theta^0_1(d_{0,*}(w)) = -t_*\theta^1_{s_1(g)}(w), \tag{3.60}$$

for $g \in \mathfrak{G}_0$, $v \in T_{s_0(g)}B\mathfrak{G}_2$, $w \in T_{s_1(g)}B\mathfrak{G}_2$. Moreover, for v_1, $v_2 \in T_1B\mathfrak{G}_1$ we have

$$d\theta^0_1(v_1, v_2) + ad(v_1)(v_2) = -t_*d\theta^1_{s_0(1)}(s_{0,*}(v_1), s_{1,*}(v_2)), \tag{3.61}$$

where $ad(v_1)(v_2)$ denotes the differential of the map $\mathfrak{G} \to \mathfrak{g}$, $g \mapsto Ad(g)(v_2)$ evaluated at $1 \in \mathfrak{G}$, $v_1 \in T_1\mathfrak{G}$. In particular,

$$ad(v_1)(v_2) + ad(v_2)(v_1) = -t_*\left(d\theta^1_{s_0(1)}(s_{0,*}(v_1), s_{1,*}(v_2)) + d\theta^1_{s_0(1)}(s_{1,*}(v_2), s_{0,*}(v_1)\right). \tag{3.62}$$

Proof. (3.59) and (3.60) follow from evaluating (3.55) at $s_0(g)$, $s_1(g) \in B\mathfrak{G}_2$ for $g \in B\mathfrak{G}_1$ and s_0, $s_1 : B\mathfrak{G}_1 \to B\mathfrak{G}_2$ the degeneracy maps. On the other hand, (3.61) follows from taking the exterior derivative of (3.55) and evaluating at $s_0(1) \in B\mathfrak{G}_2$, $s_{0,*}(v_1) \in T_{s_0(1)}B\mathfrak{G}_2$, $s_{1,*}(v_2) \in T_{s_0(1)}B\mathfrak{G}_2$ for v_1, $v_2 \in T_1B\mathfrak{G}_1$. □

Lemma 3.26. *Let $\mathfrak{G}$ be a Lie 2-group acting on a derived manifold $\mathcal{M} = (M, E, Q)$ with a smooth functor $\rho : \mathcal{M} \times \mathfrak{G} \to \mathcal{M}$. Then the tangent complex to the quotient 2-groupoid*

$\mathcal{M}//\mathfrak{G}$ (cf. Remark 3.18) is the following chain complex of vector bundles over $Z(\mathcal{M})$

$$\underline{\mathfrak{h}} \xrightarrow{t_*} \underline{\mathfrak{g}} \xrightarrow{\rho_*} T\mathcal{M}, \tag{3.63}$$

where ρ_* is the partial differential at $M \times \{1\}$ of the underlying map $\rho_0 : M \times \mathfrak{G}_0 \to M$ of ρ, and $T\mathcal{M}$ is the tangent complex of $\mathcal{M}$.

Proof. Recall from Remark 3.18 that $(\mathcal{M}//\mathfrak{G})_n = \mathcal{M} \times B\mathfrak{G}_n$. From here it is easy to see that diagram (2.97) is in this case

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] \arrow[d, "\Phi_*"] & TM \times TB\mathfrak{G}_3 \arrow[r, "\partial"] \arrow[d, "\Phi_*"] & TM \times TB\mathfrak{G}_2 \arrow[r, "\partial"] \arrow[d, "\Phi_*"] & TM \times TB\mathfrak{G}_1 \arrow[r, "\partial"] \arrow[d, "\Phi_*"] & TM \arrow[d, "\Phi_*"] \\
\dots \arrow[r, "\partial"] \arrow[d, "d_*"] & TE_2 \times TB\mathfrak{G}_3 \arrow[r, "\partial"] \arrow[d, "d_*"] & TE_2 \times TB\mathfrak{G}_2 \arrow[r, "\partial"] \arrow[d, "d_*"] & TE_2 \times TB\mathfrak{G}_1 \arrow[r, "\partial"] \arrow[d, "d_*"] & TE_2 \arrow[d, "d_*"] \\
\dots \arrow[r, "\partial"] \arrow[d, "d_*"] & TE_3 \times TB\mathfrak{G}_3 \arrow[r, "\partial"] \arrow[d, "d_*"] & TE_2 \times TB\mathfrak{G}_2 \arrow[r, "\partial"] \arrow[d, "d_*"] & TE_2 \times TB\mathfrak{G}_1 \arrow[r, "\partial"] \arrow[d, "d_*"] & TE_3 \arrow[d, "d_*"] \\
\dots \arrow[r, "\partial"] & \dots \arrow[r, "\partial"] & \dots \arrow[r, "\partial"] & \dots \arrow[r, "\partial"] & \dots
\end{tikzcd} . \tag{3.64}$$

Then, after pulling-back to $Z(\mathcal{M})$ and performing the quotients (2.98), we obtain

$$A_{0,0} = TM_{0|Z(\mathcal{M}_0)}, \;\; A_{0,m} = E_{m+1|Z(\mathcal{M}_0)}, \;\; A_{-1,0} = \mathfrak{g}, \;\; A_{-2,0} = \mathfrak{h} \tag{3.65}$$

and $A_{-n,m} = 0$ otherwise, which yields the desired result after taking the associated total complex. □

Let $\mathfrak{G}$ be a Lie 2-group with Lie 2-algebra $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$. In the following proposition we regard the dual $\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*$ of this chain complex as the derived manifold (cf. Section 2.2.2) $(\mathfrak{g}^*, \underline{\mathfrak{h}}^*[-2], Q)$, where Q is given simply by the 'curvature' $\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*$.

Proposition 3.27. *Let $\mathfrak{G}$ be a Lie 2-group with Lie 2-algebra $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$. Then, a left (resp. right) adjoint action determines an action of $\mathfrak{G}$ on the derived manifold $\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*$ and a right (resp. left)-invariant Maurer-Cartan form for this action defines a 1-shifted symplectic structure on the quotient 2-groupoid $(\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*)//\mathfrak{G}$ (cf. Remark 3.18).*

Proof. The fact that an adjoint action determines an action of $\mathfrak{G}$ on $\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*$ follows from Remark 3.18, as well as the fact that $(\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*)//\mathfrak{G}$ is described simply by

$$((\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*)//\mathfrak{G})_n = (\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*) \times B\mathfrak{G}_n, \tag{3.66}$$

with simplicial maps defined analogously as in Example 2.5. Then define the forms

$$\lambda^0 \in \Omega^1(\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^* \times B\mathfrak{G}_1, \mathbb{R}), \quad \lambda^1 \in \Omega^1(\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^* \times B\mathfrak{G}_2, \mathbb{R}) \tag{3.67}$$

by

$$\lambda^0_{(\xi,\eta,g)}(\dot{\xi}+\dot{\eta}+v_g)=\xi(\theta^0_g(v_g)), \tag{3.68}$$
$$\lambda^1_{(\xi,\eta,\gamma)}(\dot{\xi}+\dot{\eta}+v_\gamma)=\eta(\theta^1_\gamma(v_\gamma)), \tag{3.69}$$

where $\xi \in \mathfrak{g}^*$, $\eta \in \mathfrak{h}^*$, $g \in B\mathfrak{G}_1$, $\gamma \in B\mathfrak{G}_2$, $\dot{\xi} \in T_\xi \mathfrak{g}^*$, $\dot{\eta} \in T_\eta \mathfrak{h}^*$, $v_g \in T_g B\mathfrak{G}_1$, $v_\gamma \in T_\gamma B\mathfrak{G}_2$. Then $L_Q\lambda^0 = 0$ for degree reasons, while equations (3.55) and (3.56) are equivalent to $\delta\lambda^0 = L_Q\lambda^1$, $\delta\lambda^1 = 0$. It follows that $\omega^0 := d\lambda^0$, $\omega^1 := d\lambda^1$ is a 1-shifted presymplectic form. Explicitly,

$$\omega^0_{(\xi,\eta,g)}(\dot{\xi}^1+\dot{\eta}^1+v^1_g,\dot{\xi}^2+\dot{\eta}^2+v^2_g)=\dot{\xi}^1(\theta^0_g(v^2_g))-\dot{\xi}^2(\theta^0_g(v^1_g))-\xi(d\theta^0_g(v^1_g,v^2_g)), \tag{3.70}$$
$$\omega^1_{(\xi,\eta,\gamma)}(\dot{\xi}^1+\dot{\eta}^1+v^1_\gamma,\dot{\xi}^2+\dot{\eta}^2+v^2_\gamma)=\dot{\eta}^1(\theta^1_\gamma(v^2_\gamma))-\dot{\eta}^2(\theta^1_\gamma(v^1_\gamma))-\eta(d\theta^1_\gamma(v^1_\gamma,v^2_\gamma)) \tag{3.71}$$

In order to check the non-degeneracy condition, we note that the tangent complex of $(\mathfrak{g}^* \xrightarrow{t^*_*} \mathfrak{h}^*)//\mathfrak{G}$ is the chain complex of vector bundles over $\xi \in Ker(t^*_*) \subset \mathfrak{g}^*$

$$\underline{\mathfrak{h}}[2] \xrightarrow{t_*} \underline{\mathfrak{g}}[1] \xrightarrow{ad^*} \underline{\mathfrak{g}}^* \xrightarrow{t^*_*} \underline{\mathfrak{h}}^*[-1], \tag{3.72}$$

where we write $ad^*_\xi(v_1)(\cdot) = \xi(ad(v_1)(\cdot))$ for ad defined as in Lemma 3.25. The cotangent complex is

$$\underline{\mathfrak{h}}[1] \xrightarrow{t_*} \underline{\mathfrak{g}} \xrightarrow{(ad^*)^*} \underline{\mathfrak{g}}^*[-1] \xrightarrow{t^*_*} \underline{\mathfrak{h}}^*[-2], \tag{3.73}$$

where $ad^*_\xi(v_1)(\cdot) = \xi(ad(\cdot)(v_1))$. Using (3.70), (3.71) it is easy to see that the map (2.129) induced by (ω^0, ω^1) is just the identity (with some signs) on each degree, hence it is clearly an isomorphism. Note that this is indeed a chain map for $\xi \in Ker(t^*_*) \subset \mathfrak{g}^*$ by (3.62). $\square$

3.2 Multiplicative gerbes

3.2.1 Gerbes

We dedicate this section to fix some notation and conventions regarding gerbes that are extensively used throughout the thesis. We start by fixing a manifold M and an abelian Lie group T with Lie algebra $\mathfrak{t}$.

Definition 3.28 ([56, 82, 133])**.** A T-*gerbe* over M is the data of an open cover $\{U_i\}_{i\in I}$ of M and a T-valued Čech 2-cocycle in this cover; i.e., functions $\lambda_{ijk} : U_{ijk} \to T$ satisfying

$$\lambda^{-1}_{jkl}\lambda_{ikl}\lambda^{-1}_{ijl}\lambda_{ijk} = 1. \tag{3.74}$$

A *connective structure* on it is the data of 1-forms $\Lambda_{ij} \in \Omega^1(U_{ij}, \mathfrak{t})$ such that

$$\Lambda_{ij} - \Lambda_{ik} + \Lambda_{jk} = \lambda^*_{ijk}\theta, \tag{3.75}$$

for $\theta \in \Omega^1(T, \mathfrak{t})$ the Maurer-Cartan form on T. A *curving* for it is a collection of 2-forms $B_i \in \Omega^2(U_i, \mathfrak{t})$ such that

$$B_i - B_j = d\Lambda_{ij}. \tag{3.76}$$

An *enhanced curving* is a collection $\{B_i^{en}\}_i$ of $B_i^{en} \in \Gamma(T^*U_i \otimes T^*U_i \otimes \mathfrak{t})$ such that

$$B_i^{en} - B_j^{en} = d\Lambda_{ij}; \tag{3.77}$$

equivalently, it is a pair $(\{B_i\}_i, h)$ of a curving $\{B_i\}_i$ and a $h \in \Gamma(S^2T^*M \otimes \mathfrak{t})$. A *connection* $(\Lambda, B) := (\{\Lambda_{ij}\}_{i,j}, \{B_i\}_i)$ for a gerbe is a connective structure with a curving and an *enhanced connection* is a connective structure with an enhanced curving. In any case, the *curvature* is $H \in \Omega^3_{cl}(M, \mathfrak{t})$ given locally by $H_{|U_i} = dB_i$.

Given two gerbes $(\{U_i^1\}_{i\in I_1}, \{\lambda^1_{ijk}\}_{i,j,k\in I_1})$, $(\{U_i^2\}_{i\in I_2}, \{\lambda^2_{ijk}\}_{i,j,k\in I_2})$, an *isomorphism* between them is the data of $\{V_a\}_{a\in A}$ a common refinement of $\{U_i^1\}_{i\in I_1}$ and $\{U_i^2\}_{i\in I_2}$ (with refinement maps $i_1 : A \to I_1$ and $i_2 : A \to I_2$) and functions $s_{ab} : V_{ab} \to T$ satisfying

$$s_{ac}\lambda^1_{i_1(a)i_1(b)i_1(c)} = \lambda^2_{i_2(a)i_2(b)i_2(c)} s_{bc} s_{ab} \tag{3.78}$$

If the gerbes have connective structures $\Lambda^1_{i_1j_1}$, $\Lambda^2_{i_2j_2}$, then we define a *connection* on the isomorphism to be a collection of 1-forms $\Lambda_a \in \Omega^1(V_a, \mathfrak{t})$ satisfying

$$\Lambda_a - \Lambda_b = \Lambda^1_{i_1(a)i_1(b)} - \Lambda^2_{i_2(a)i_2(b)} - s^*_{ab}\theta^T \tag{3.79}$$

An isomorphism of gerbes with a connection is also called an *isomorphism of gerbes with connective structure*, while a connection on the identity isomorphism is also called an *isomorphism of connective structures.* If the gerbes also have (enhanced) curvings $B^{en,1}_{i_1}$, $B^{en,2}_{i_2}$, then the *curvature* with respect to $B^{en,1}_{i_1}$, $B^{en,2}_{i_2}$ is $F \in \Gamma(T^*M \otimes T^*M \otimes \mathfrak{t})$ defined by

$$F = d\Lambda_a - B^{en,1}_{i_1(a)} + B^{en,2}_{i_2(a)} = d\Lambda_a - B^1_{i_1(a)} + B^2_{i_2(a)} - h_1 + h_2 \tag{3.80}$$

Its skew-symmetric part F^{sk} satisfies $dF^{sk} = H_2 - H_1$. A *trivialization* of a gerbe $(\{U_i^1\}_i, \{\lambda_{ijk}\}_{i,j,k})$ is an isomorphism $1 \to (\{U_i^1\}_i, \{\lambda_{ijk}\}_{i,j,k})$, where 1 denotes the trivial gerbe, and a *connection* on it is a connection on the isomorphism, where 1 is regarded with the trivial connective structure and curving.

Given two isomorphisms $(\{V_a\}, \{s_{ab}\})$, $(\{V'_a\}, \{s'_{ab}\})$ between the same gerbes, a *2-isomorphism* between them is the data of a common refinement $\{W_r\}_{r\in R}$ of $\{V_a\}_{a\in A}$

and $\{V'_a\}_{a\in A'}$ such that the refinement maps $R \to A \to I_n$, $R \to A' \to I_n$ coincide for $n = 1, 2$ and functions $t_r : W_r \to T$ such that

$$t_r s_{a(r)a(s)} = s'_{a'(r)a'(s)} t_s \tag{3.81}$$

If the gerbes have connective structures Λ^1_{ij}, Λ^2_{ij} and the isomorphisms have connections Λ_a, Λ'_a then the *covariant derivative* of the 2-isomorphism is $\eta \in \Omega^1(M, \mathfrak{t})$ given locally by $\eta = \Lambda'_{a'(r)} - \Lambda_{a(r)} + t_r^*\theta^T$. The 2-isomorphism is a *2-isomorphism of gerbes with connective structure* or a *flat 2-isomorphism* when $\eta = 0$.

For $\Phi := (\{V_a\}_a, \{s_{ab}\}_{a,b})$ an isomorphism of gerbes from $\mathcal{L}_1 = (\{U_{i_1}\}, \{\lambda^1_{i_1j_1k_1}\})$ to $\mathcal{L}_2 = (\{U_{i_2}\}, \{\lambda^2_{i_2j_2k_2}\})$ and $(\{\Lambda^1_{i_1j_1}\}_{i_1j_1}, \{B^1_{i_1}\}_{i_1})$ a connection on $\mathcal{L}_1$, we write $(\Phi^{-1})^*(\Lambda^1_{ij}, B^1_i) = (\Lambda^2_{ab}, B^2_a)$ for the connection on $\mathcal{L}_2$ defined by

$$\Lambda^2_{ab} := \Lambda_{i_1(a)i_1(b)} - s^*_{ab}\theta, \tag{3.82}$$

$$B^2_a := B_{i_1(a)}. \tag{3.83}$$

If $\psi : (\{W_r\}_r, \{t_r\}_r)$ is a 2-isomorphism $\Phi \Rightarrow \Phi'$ and Λ_{i_1} is an isomorphism of connections $(\{\Lambda^1_{i_1j_1}\}_{i_1j_1}, \{B^1_{i_1}\}_{i_1}) \to (\{\tilde{\Lambda}^1_{i_1j_1},\}_{i_1j_1}, \{\tilde{B}^1_{i_1}\}_{i_1})$, then we write

$$(\psi^{-1})^*\Lambda_{i_1} : (\Phi^{-1})^*(\{\Lambda^1_{i_1j_1}\}_{i_1j_1}, \{B^1_{i_1}\}_{i_1}) \to (\Phi')^{-1}(\{\tilde{\Lambda}^1_{i_1j_1},\}_{i_1j_1}, \{\tilde{B}^1_{i_1}\}_{i_1}) \tag{3.84}$$

for the isomorphism of connections defined by the 1-forms $\Lambda_{i_1(r)} + t_r^*\theta$. When $\lambda^1_{i_1j_1k_1} = \lambda^2_{i_2j_2k_2}$, this is called the *gauge action* of the gerbe. For

$$\Phi_\nabla = (\{V_a\}_a, \{s_{ab}\}_{a,b}, \{\Lambda_a\}) : (\{U_{i_1}\}, \{\lambda^1_{i_1j_1k_1}\}, \{\Lambda^1_{i_1j_1}\}) \to (\{U_{i_2}\}, \{\lambda^2_{i_2j_2k_2}\}, \{\Lambda^2_{i_2j_2}\}) \tag{3.85}$$

an isomorphism of gerbes with connective structure and B^1_i a curving on $\Lambda^1_{i_1j_1}$, we write $(\Phi_\nabla^{-1})^*B^1_i$ for the curving on $\Lambda^2_{i_2j_2}$ defined by the two-forms $B^1_{i_1(a)} - d\Lambda_2$. Note that when two isomorphisms of gerbes with connective structure are related by a flat 2-isomorphism then their action on curvings coincides. When $\lambda^1_{i_1j_1k_1} = \lambda^2_{i_2j_2k_2}$ and $\Lambda_{i_1j_1} = \Lambda_{i_2j_2}$, this is called the *gauge action* of the gerbe with connective structure.

The *tensor product* of two gerbes (with connective structure and curving)

$$\mathcal{L}^1 = (\{U_i\}_i, \{\lambda^1_{ijk}\}_{i,j,k}, \{\Lambda^1_{ij}\}_{i,j}, \{B^1_i\}_i), \quad \mathcal{L}^2 = (\{U_i\}_i, \{\lambda^2_{ijk}\}_{i,j,k}, \{\Lambda^2_{ij}\}_{i,j}, \{B^2_i\}_i)$$

is the gerbe $\mathcal{L}^1 \otimes \mathcal{L}^2$ (with connective structure and curving) described by the cocycle data $(\{U_i\}_i, \{\lambda^1_{ijk}\lambda^2_{ijk}\}_{i,j,k}, \{\Lambda^1_{ij} + \Lambda^2_{ij}\}_{i,j}, \{B^1_i + B^2_i\}_i)$. The *dual* of a gerbe (with connective structure and curving) $\mathcal{L} = (\{U_i\}_i, \{\lambda^1_{ijk}\}_{i,j,k}, \{\Lambda^1_{ij}\}_{i,j}, \{B^1_i\}_i)$ is the gerbe (with connective structure and curving) $\mathcal{L}^{-1} := (\{U_i\}_i, \{(\lambda^1_{ijk})^{-1}\}_{i,j,k}, \{-\Lambda^1_{ij}\}_{i,j}, \{-B^1_i\}_i)$.

We also recall Murray's notion of *bundle gerbes* [198], which is completely analogous to Definition 3.28 but replacing the use of a cover of M by a general surjective submersion $\pi : Y \to M$. For such a surjective submersion, write $Y^{[n]} := Y \times_M Y \times_M ... \times_M Y$ for the nth fibered product over M and $d_j : Y^{[n]} \to Y^{[n-1]}$, $j = 0, ..., n-1$ for the map that forgets the j-th point. Then a T-gerbe over M can be defined as a principal T-bundle $L \to Y^{[2]}$ with an isomorphism $\lambda : d_2^* L \otimes d_0^* L \to d_1^* L$ of T-bundles over $Y^{[3]}$ satisfying $d_1^*\lambda \circ d_3^*\lambda = d_2^*\lambda \circ d_0^*\lambda$ over $Y^{[4]}$; a connective structure on it is a T-connection ∇ on $L \to Y^{[2]}$ such that λ is flat and a curving for it is a 2-form $B \in \Omega^2(Y, \mathfrak{t})$ such that $d_1^* B - d_0^* B = F_\nabla$, for F_∇ the curvature of ∇.

For two gerbes described by (L_1, λ_1) and (L_2, λ_2), an isomorphism between them is described by a T-bundle $M \to Y$ with an isomorphism $s : d_0^* M \otimes L_1 \to L_2 \otimes d_1^* M$ of T-bundles over $Y^{[2]}$ inducing a commutative diagram with λ_1 and λ_2 over $Y^{[3]}$. If (L_1, λ_1) and (L_2, λ_2) have connective structures given by connections ∇_1, ∇_2, then a connection on the isomorphism (M, s) is a connection on M such that s is flat. Given two isomorphisms (M, s) and (M', s'), a 2-isomorphism between them is an isomorphism $M \to M'$ inducing a commutative diagram with s and s' over $Y^{[2]}$.

Both Definition 3.28 and the approach with bundle gerbes are working definitions. More rigorously, T-gerbes are principal 2-bundles (cf. Definition 4.1) for the 2-group BT. The relation between this and Definition 3.28 is given by the construction in Example 3.6, which can be seen as the 'total space' of the gerbe that can be constructed with a 2-cocycle.

For the following proposition we let T be a connected abelian Lie group and we let $Z \subset \mathfrak{t}$ be the kernel of the exponential map $exp : \mathfrak{t} \to T$.

Proposition 3.29 ([82]). *1. Every T-gerbe admits a connection.*

2. *T-gerbes over M are classifed by $H^3(M, Z)$ and the class of a gerbe is represented in de Rham cohomology by taking the curvature of any curving.*

3. *A gerbe admits a flat connection if and only if it admits locally constant cocycle data. An isomorphism of flat gerbes admits a flat connection if and only if it admits locally constant cocycle data in the same frame in which the gerbes are described by locally constant cocycle data. A 2-isomorphism between isomorphisms with flat connections is flat if and only if it is described by locally constant functions in the same frame in which the gerbes and the isomorphisms are described by locally constant cocycle data.*

4. *The automorphism 2-group of a gerbe $\mathcal{L}$ (cf. Remark 3.11) is equivalent to the 2-group $BT(M)$ of T-bundles over M from Example 3.15. The gauge action is*

an action of $BT(M)$ on the groupoid $\mathcal{A}(\mathcal{L})$ of connections (with isomorphisms of connections as arrows), in the sense of Definition 3.16.

5. *The automorphism 2-group of a gerbe with connective structure $\mathcal{L}_\nabla$ is equivalent to the 2-group $BT_\nabla(M)$ of T-bundles with connection over M from Example 3.15. The gauge action is an action of $BT_\nabla(M)$ on the set $\mathcal{A}(\mathcal{L}_\nabla)$ of curvings for the given connective structure, in the sense of Definition 3.16.*

Given a gerbe, we can regard the groupoid $\mathcal{A}(\mathcal{L})$ as a *topological* groupoid (defined similarly as in Definition 3.1) taking the Fréchet topology on the spaces $\Omega^1(U_{ij}, \mathfrak{t})$, $\Omega^2(U_i, \mathfrak{t})$ and $\Omega^1(U_i, \mathfrak{t})$. We also think of the 2-group $BT(M)$ as a *topological* 2-group (defined similarly as in Definition 3.9), describing T-bundles in terms of transitions functions as in Example 3.15 and using Fréchet topologies on $C^\infty(U_{ij}, T)$ and $C^\infty(U_i, T)$. The gauge action from Proposition 3.29 is an action by a continuous functor and so we can form the simplicial *topological* space (defined similarly as in Definition 2.1) $\mathcal{A}(\mathcal{L})//BT(M)$ as in Section 3.1.3.

Given a gerbe with connective structure $\mathcal{L}_\nabla$, we can regard the set $\mathcal{A}(\mathcal{L}_\nabla)$, which is a torsor for $\Omega^2(M, \mathfrak{t})$, as a topological space with Fréchet topology. The 2-group $BT_\nabla(M)$ can also be thought of as a topological 2-group by describing it in the second presentation from Example 3.15 and taking Fréchet topologies on the spaces $C^\infty(M, T)$ and $\mathcal{A}(L^c) \cong \Omega^1(M, \mathfrak{t})$. With this topology, the gauge action from Proposition 3.29 is continuous and so we can form the simplicial topological space $\mathcal{A}(\mathcal{L}_\nabla)//BT_\nabla(M)$.

Note $\mathcal{A}(\mathcal{L})//BT(M)$ is a model for the space of connections modulo gauge on $\mathcal{L}$. On the other hand, $\mathcal{A}(\mathcal{L}_\nabla)//BT_\nabla(M)$ might seem a priori a less natural object, as it requires fixing a connective structure on $\mathcal{L}$. However, the following proposition states that both simplicial topological spaces are essentially equivalent. As it will become more evident in the non-abelian generalization from Section 4.2.2 and in the constructions of moduli spaces from Section 6.1.2, we note this because it is easier to treat $\mathcal{A}(\mathcal{L}_\nabla)//BT_\nabla(M)$ as a geometric object, as the topology of each space $(\mathcal{A}(\mathcal{L}_\nabla)//BT_\nabla(M))_n$, $n \in \mathbb{N}$ is modelled on the space of *global* sections of some vector bundle over M independently of any choice of cover on M.

Proposition 3.30. *Let $\mathcal{L}_\nabla$ be a gerbe with connective structure. There is a canonical morphism of simplicial topological spaces $\mathcal{A}(\mathcal{L}_\nabla)//BT_\nabla(M) \to \mathcal{A}(\mathcal{L})//BT(M)$ inducing a weak homotopy equivalence on their geometric realizations.*

Proof. This follows from noting the following.

1. An object in $\mathcal{A}(\mathcal{L})//BT(M)$ is a connection on $\mathcal{L}$. Given a fixed connective structure Λ on $\mathcal{L}$ and two curvings B, B' for Λ, an arrow $(\Lambda, B) \to (\Lambda, B')$ in

$\mathcal{A}(\mathcal{L})//BT(M)$ is a pair (L, ϕ) where $L \in BT(M)_0$ is an automorphism of $\mathcal{L}$ and $\phi : (\Lambda, B) \to (L^*\Lambda, B')$ is an isomorphism of connections. A 2-cell between (L, ϕ) and (L', ϕ') is an isomorphism of T-bundles $\psi : L \to L'$ inducing a commutative diagram with ϕ and ϕ'.

2. By looking at the cocycle data that defines each structure, one sees that a pair (L, ϕ) as above is exactly the same as a T-bundle with connection $L_\nabla \in BT_\nabla(M)_0$ whose curvature F_∇ satisfies $B' - B = F_\nabla$, and that an isomorphism of T-bundles $\psi : L \to L'$ that are equipped with connections ∇, ∇' induces a commutative diagram with the corresponding ϕ, ϕ' if and only if it preserves the connections. This means that for any fixed Λ there is a map of simplicial topological spaces $\mathcal{A}(\mathcal{L}_\nabla)//BT_\nabla(M) \to \mathcal{A}(\mathcal{L})//BT(M)$ whose geometric realization induces an isomorphism on π_1 and π_2.

3. Any two connective structures are always isomorphic. This means that the map $\mathcal{A}(\mathcal{L}_\nabla)//BT_\nabla(M) \to \mathcal{A}(\mathcal{L})//BT(M)$ from before induces an isomorphism on π_0.

□

3.2.2 Multiplicative gerbes

Let G be a Lie group. Recall the simplicial manifold $BG_\bullet$ from Example 2.6 with its maps $d_j^n : G^n \to G^{n-1}$. In order to simplify notation, we denote in what follows any possible composition of these maps by its image; for example, $g_1g_2g_3 : G^3 \to G$ is the map $d_1^2 \circ d_1^3$, while $(g_1g_2, g_3g_4) : G^4 \to G^2$ is the map $d_2^3 \circ d_1^4$.

Definition 3.31 ([58, 78, 195])**.** For T an abelian Lie group, a *multiplicative T-gerbe* over a Lie group G is the following data:

1. A T-gerbe $\mathcal{G} \to G$,
2. An isomorphism m of T-gerbes over $G \times G$ (the *product*) $m : g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \to (g_1g_2)^*\mathcal{G}$,
3. A 2-isomorphism α of T-gerbes over $G \times G \times G$ (the *associator*)

$$\begin{tikzcd}
g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \otimes g_3^*\mathcal{G} \arrow[r, "(g_1,g_2)^*m"] \arrow[d, "(g_2,g_3)^*m"'] & (g_1g_2)^*\mathcal{G} \otimes g_3^*\mathcal{G} \arrow[d, "(g_1g_2,g_3)^*m"] \arrow[dl, Rightarrow, "\alpha"] \\
g_1^*\mathcal{G} \otimes (g_2g_3)^*\mathcal{G} \arrow[r, "(g_1,g_2g_3)^*m"'] & (g_1g_2g_3)^*\mathcal{G}
\end{tikzcd} \tag{3.86}$$

such that, over $G \times G \times G \times G$,

$$\begin{tikzcd}
g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \otimes g_3^*\mathcal{G} \otimes g_4^*\mathcal{G} \arrow[rr, bend left=60, "m(((g_1g_2)g_3)g_4)"] \arrow[rr, bend left=20, "m((g_1(g_2g_3))g_4)" description] \arrow[rr, bend right=20, "m(g_1((g_2g_3)g_4))" description] \arrow[rr, bend right=60, "m(g_1(g_2(g_3g_4)))"'] \arrow[rr, phantom, shift left=5, "\Downarrow\alpha"] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[rr, phantom, shift right=5, "\Downarrow\alpha"] & & (g_1g_2g_3g_4)^*\mathcal{G}
\end{tikzcd} = \begin{tikzcd}
g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \otimes g_3^*\mathcal{G} \otimes g_4^*\mathcal{G} \arrow[rr, bend left=50, "m(((g_1g_2)g_3)g_4)"] \arrow[rr, bend left=15, "m((g_1g_2)(g_3g_4))" description] \arrow[rr, bend right=30, "m(g_1(g_2(g_3g_4)))"'] \arrow[rr, phantom, shift left=4, "\Downarrow\alpha"] \arrow[rr, phantom, shift right=2, "\Downarrow\alpha"] & & (g_1g_2g_3g_4)^*\mathcal{G}
\end{tikzcd}. \tag{3.87}$$

Isomorphisms and *2-isomorphisms* of multiplicative gerbes are defined similarly as in Definition 3.9, replacing anafunctors by isomorphisms of gerbes and transformations by 2-isomorphisms of gerbes. This yields the *bicategory of multiplicative T-gerbes over G*.

Remark 3.32. The last diagram of Definition 3.31 is an equality between 2-isomorphisms of gerbes: each black arrow represents an isomorphism (for example, we are writing $m(((g_1g_2)g_3)g_4) := (g_1g_2g_3, g_4)^*m \circ (g_1g_2, g_3)^*m \otimes id \circ (g_1, g_2)^*m \otimes id \otimes id)$ and each 2-cell is a 2-isomorphism constructed from α.

Remark 3.33. The data of Definition 3.31 is sufficient to construct other canonical structures that might be useful. For example, we write $1_{\mathcal{G}} : 1 \to 1^*\mathcal{G}$ for the following trivialization of the gerbe $1^*\mathcal{G} \to \{*\}$, where we write $1 : \{*\} \to G$ for the inclusion of the unit element. First, there is an isomorphism

$$1^*\mathcal{G} \overset{(1,1)^*m^{-1}}{\to} 1^*\mathcal{G} \otimes 1^*\mathcal{G}, \tag{3.88}$$

and so we can define $1_{\mathcal{G}}$ as the following composition

$$1 \overset{e}{\to} 1^*\mathcal{G} \otimes 1^*\mathcal{G}^{-1} \overset{(1,1)^*m^{-1} \otimes id_{1^*\mathcal{G}^{-1}}}{\to} 1^*\mathcal{G} \otimes 1^*\mathcal{G} \otimes 1^*\mathcal{G}^{-1} \overset{id_{1^*\mathcal{G}} \otimes e^{-1}}{\to} 1^*\mathcal{G}, \tag{3.89}$$

where e is the canonical trivialization of $1^*\mathcal{G} \otimes 1^*\mathcal{G}^{-1}$. Similarly, we can define a 2-isomorphism of gerbes over G, called the *right unitor*,

$$\begin{tikzcd}
g^*\mathcal{G} \otimes 1^*\mathcal{G} \arrow[r, bend left=50, "(g,1)^*m"] \arrow[r, bend right=50, "id_{\mathcal{G}} \otimes 1_{\mathcal{G}}^{-1}"'] \arrow[r, phantom, "\Downarrow r"] & g^*\mathcal{G}.
\end{tikzcd} \tag{3.90}$$

It is defined by considering the 2-isomorphism

$$\begin{tikzcd} g^*\mathcal{G}\otimes 1^*\mathcal{G}\otimes 1^*\mathcal{G} \arrow[r, "(g,1)^*m"] \arrow[d, "(1,1)^*m"'] & g^*\mathcal{G}\otimes 1^*\mathcal{G} \arrow[d, "(g,1)^*m"] \arrow[dl, Rightarrow, "(g,1,1)^*\alpha"] \\ g^*\mathcal{G}\otimes 1^*\mathcal{G} \arrow[r, "(g,1)^*m"'] & g^*\mathcal{G} \end{tikzcd}; \tag{3.91}$$

then note that $(g,1,1)^*\alpha \circ id_{(g,1)^*m^{-1}}$ is a 2-isomorphism of the form

$$\begin{tikzcd} g^*\mathcal{G}\otimes 1^*\mathcal{G}\otimes 1^*\mathcal{G} \arrow[r, bend left, "(g,1)^*m"] \arrow[r, bend right, "(1,1)^*m"'] \arrow[r, phantom, "\Downarrow"] & g^*\mathcal{G}\otimes 1^*\mathcal{G}, \end{tikzcd} \tag{3.92}$$

and then construct r by tensoring everything with $1^*\mathcal{G}^{-1}$. We will also use the notation

$$inv : 1 \to g^*\mathcal{G}\otimes (g^{-1})^*\mathcal{G} \tag{3.93}$$

for the isomorphism $(g,g^{-1})^*m^{-1}\circ 1_{\mathcal{G}}$. Since all these maps are defined canonically from m, α, they satisfy good properties with respect to them (a precise statement in this respect is the coherence theorem for bicategories [212]).

In order to describe multiplicative gerbes in terms of cocycle data one must take a *good semi-simplicial cover* of $BG_\bullet$. This is a collection $\{\mathcal{U}_n\}_{n\geq 1}$, where each $\mathcal{U}_n = \{U^n_{i_n}\}_{i_n\in I_n}$ is a good cover of G^n indexed by a set I_n, together with maps $\tilde{d}^n_j : I_n \to I_{n-1}$ such that $d^n_j(U^n_{i_n}) \subset U^{n-1}_{\tilde{d}^n_j(i_n)}$ and that $\{I_n, \tilde{d}^n_j\}_{n,j}$ is a semi-simplicial set. In what follows we abuse notation by writing simply $\tilde{d}^n_j = d^n_j$; furthermore, we denote $U^n_{i^1_n i^2_n \dots i^k_n} := \bigcap_{s=1}^k U^n_{i^s_n}$. There are constructions of good semi-simplicial covers of $BG_\bullet$ in [58, 195].

Given a good semi-simplicial cover $\{\mathcal{U}_n\}_n$ of $BG_\bullet$, it follows directly from the definitions that a multiplicative T-gerbe over G is given by

$$\lambda_{i_1j_1k_1} : U^1_{i_1j_1k_1} \to T, \quad m_{i_2j_2} : U^2_{i_2j_2} \to T, \quad \alpha_{i_3} : U^3_{i_3} \to T \tag{3.94}$$

satisfying

$$
\begin{aligned}
&\lambda_{i_1j_1k_1}(g)\lambda_{i_1k_1l_1}(g) = \lambda_{i_1j_1l_1}(g)\lambda_{j_1k_1l_1}(g),\\
&\qquad m_{i_2j_2}(g_1,g_2)m_{j_2k_2}(g_1,g_2)\lambda_{d_1(i_2)d_1(j_2)d_1(k_2)}\\
&\qquad\qquad\qquad = m_{i_2k_2}(g_1,g_2)\lambda_{d_0(i_2)d_0(j_2)d_0(k_2)}(g_2)(g_1g_2)\lambda_{d_2(i_2)d_2(j_2)d_2(k_2)}(g_1),\\
&\qquad \alpha_{i_3}(g_1,g_2,g_3)m_{d_3(i_3)d_3(j_3)}(g_1,g_2)m_{d_1(i_3)d_1(j_3)}(g_1g_2,g_3)\\
&\qquad\qquad\qquad = \alpha_{j_3}(g_1,g_2,g_3)m_{d_2(i_3)d_2(j_3)}(g_1,g_2g_3)m_{d_0(i_3)d_0(j_3)}(g_2,g_3),\\
&\quad \alpha_{d_4(i_4)}(g_1,g_2,g_3)\alpha_{d_2(i_4)}(g_1,g_2g_3,g_4)\alpha_{d_0(i_4)}(g_2,g_3,g_4)\\
&\qquad\qquad\qquad = \alpha_{d_3(i_4)}(g_1,g_2,g_3g_4)\alpha_{d_1(i_4)}(g_1g_2,g_3,g_4).
\end{aligned}
\tag{3.95}
$$

For G, T any Lie groups with T abelian, we let $Ext(G,BT)$ be the set of multiplicative T-gerbes over G up to isomorphism. We also write for the rest of the thesis $\mathfrak{g}$ and $\mathfrak{t}$ for the Lie algebras of G and T, respectively. The following is a classification result that is well-known in the literature at least when G is compact (e.g. [238]). To state it we recall the theory of sheaf cohomology on semi-simplicial manifolds from Section 2.1.2, and group cohomology defined by (2.41).

Proposition 3.34. *Let G, T be Lie groups with T abelian, and let C^∞_T be the sheaf of smooth T-valued functions. Then*

1. *$Ext(G,BT) = H^3(BG_\bullet, C^\infty_T)$*
2. *If T is connected, then there is an exact sequence*

$$H^3_{gr,cont}(G,\mathfrak{t}) \to Ext(G,BT) \to H^4(BG,Z) \to H^4_{gr,cont}(G,\mathfrak{t}), \tag{3.96}$$

where $H^*(BG,Z)$ denotes singular cohomology of the classifying space of G and $Z := \ker exp_T \subset \mathfrak{t}$. In particular, $Ext(G,BT) = H^4(BG,Z)$ when G is compact.

Proof. Consider over $BG_\bullet$ the sheaf C^∞_T of smooth T-valued functions. A good semi-simplicial cover of $BG_\bullet$ gives an injective resolution of this sheaf by taking the Čech resolutions $(\check{C}^\bullet(C^\infty_{T,G^n},\mathcal{U}_n),\check{\delta})$ of C^∞_T with respect to each cover $\mathcal{U}_n$, and using the maps $\tilde{d}^n_j : I_n \to I_{n-1}$ to define the sheaf morphisms $\partial^n_j : (d^n_j)^*\check{C}^p(C^\infty_{T,G^{n-1}},\mathcal{U}_{n-1}) \to \check{C}^p(C^\infty_{T,G^n},\mathcal{U}_n)$. Thus the total cohomology of the double complex $(\check{C}^\bullet(C^\infty_{T,G^\bullet},\mathcal{U}_n),\check{\delta},\delta)$ computes $H^*(BG_\bullet,C^\infty_T)$, and the cocycle data for multiplicative T-gerbes over G gives precisely an element in $H^3(BG_\bullet,C^\infty_T)$ which classifies them completely. In other words, $H^3(BG_\bullet,C^\infty_T) = Ext(G,BT)$. This is valid for any Lie groups G, T but when T is connected then $1 \to Z \to \mathfrak{t} \overset{exp}{\to} T \to 1$ is exact and so there is an exact sequence

$$H^3(BG_\bullet, C^\infty_{\mathfrak{t}}) \to H^3(BG_\bullet, C^\infty_T) \to H^4(BG_\bullet, \underline{Z}) \to H^4(BG_\bullet, C^\infty_{\mathfrak{t}}), \tag{3.97}$$

where $C^\infty_{\mathfrak{t}}$ is the sheaf of smooth $\mathfrak{t}$-valued functions on $BG_\bullet$. The result follows then from Theorems 2.11 and 2.13. □

Definition 3.35. Let G, T be Lie groups with T abelian and let $(\mathcal{G}, m, \alpha)$ be a multiplicative T-gerbe over G. We say $(\mathcal{G}, m, \alpha)$ is *flat* if its class in $Ext(G, BT) = H^3(BG_\bullet, C^\infty_T)$ lies in the image of $H^3(BG_\bullet, \underline{T}) \to H^3(BG_\bullet, C^\infty_T)$, for $\underline{T}$ the sheaf of locally constant T-valued functions.

It follows from Proposition 3.34 that, for connected T, a multiplicative T-gerbe $\mathcal{G}$ over G has a class $c(\mathcal{G}) \in H^4(BG, Z)$. This has an image

$$c_{\mathfrak{t}}(\mathcal{G}) \in H^4(BG, \mathfrak{t}), \tag{3.98}$$

which we call the *de Rham class* of the multiplicative gerbe.

Lemma 3.36. *The group $H^4(BG, \mathfrak{t})$ is isomorphic to the following quotient.*

$$\frac{\{(\tau_3, \tau_2, \tau_1, \tau_0) \mid \tau_i \in \Omega^i(G^{4-i}, \mathfrak{t}),\ d\tau_3 = 0,\ d\tau_2 = -\delta\tau_3,\ d\tau_1 = \delta\tau_2,\ d\tau_0 = -\delta\tau_1,\ \ 0 = \delta\tau_0\}}{\{(d\beta_2, \delta\beta_2 + d\beta_1, -\delta\beta_1 + d\beta_0, \delta\beta_0) \mid \beta_i \in \Omega^i(G^{3-i}, \mathfrak{t})\}}.$$

Moreover, if T is connected, then

1. *The de Rham class $c_{\mathfrak{t}}(\mathcal{G}) \in H^4(BG, \mathfrak{t})$ of a multiplicative T-gerbe $\mathcal{G}$ over G admits a representative $[(\tau_3, \tau_2, \tau_1, \tau_0)]$ as above with $\tau_0 = 0$.*

2. *The de Rham class $c_{\mathfrak{t}}(\mathcal{G}) \in H^4(BG, \mathfrak{t})$ of a multiplicative T-gerbe $\mathcal{G}$ over G is 0 if and only if $\mathcal{G}$ is flat.*

Proof. The description of $H^4(BG, \mathfrak{t})$ follows from Remark 2.12. Now note that the exact sequence of sheaves $0 \to Z \to C^\infty_{\mathfrak{t}} \to C^\infty_T \to 0$ induces the exact sequence

$$H^3(BG_\bullet, C^\infty_T) \to H^4(BG_\bullet, Z) \to H^4(BG_\bullet, C^\infty_{\mathfrak{t}}), \tag{3.99}$$

which yields 1. Similarly, the exact sequence of sheaves $Z \to \underline{\mathfrak{t}} \to \underline{T}$ gives the exact sequence

$$H^3(BG, \underline{T}) \to H^4(BG, Z) \to H^4(BG, \mathfrak{t}), \tag{3.100}$$

which implies 2. □

Given a multiplicative T-gerbe over G, one representative $c_t(\mathcal{G}) = [(\tau_3, \tau_2, \tau_1, 0)]$ as in part 1 of Lemma 3.36 can be obtained by choosing any connective structure and curving on $\mathcal{G}$ and any connection on m. This yields the curvature 3-form τ_3 of $\mathcal{G}$ on G, the curvature 2-form $-\tau_2$ of m on G^2 and the covariant derivative 1-form τ_1 of α on G^3, which satisfy the equations from Lemma 3.36.

Example 3.37 ([58, 195])**.** For G a compact, simple, simply connected Lie group, Proposition 3.34 and the fact that $H^4(BG, \mathbb{Z}) = H^3(G, \mathbb{Z}) = \mathbb{Z}$ in this case imply that multiplicative $U(1)$-gerbes over G are classified by $\mathbb{Z}$. The multiplicative gerbe corresponding to a choice of generator of $H^4(BG, \mathbb{Z})$ is called $\mathrm{String}(G)$. The image of such generator in $H^4(BG_\bullet, \mathbb{R})$ can be described in de Rham cohomology by the forms

$$\begin{aligned} \tau_3 &:= \frac{1}{6}\langle \theta^L, [\theta^L \wedge \theta^L]\rangle \in \Omega^3(G, \mathbb{R}), \\ \tau_2 &:= \langle g_1^* \theta^L \wedge g_2^* \theta^R \rangle \in \Omega^2(G \times G, \mathbb{R}), \\ \tau_1 &:= 0, \\ \tau_0 &:= 0, \end{aligned} \tag{3.101}$$

(which satisfy $d\tau_3 = 0$, $d\tau_2 = \delta\tau_3$, $\delta\tau_2 = 0$), for θ^L, $\theta^R \in \Omega^1(G, \mathfrak{g})$ the left- and right-invariant Maurer-Cartan forms on G, respectively, and $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ the Killing form, normalized so that $[\tau_3] \in H^3(G, \mathbb{Z}) = \mathbb{Z}$ is a generator. It follows that a finite-dimensional model for $\mathrm{String}(G)$ can be obtained by choosing potentials for the forms τ_3, τ_2 in a semi-simplicial cover of G, which give a cocycle presentation of the multiplicative gerbe to which Theorem 3.48 below can be applied to construct a Lie 2-group. However, as this is not a canonical procedure, it does not yield an explicit description of the Lie 2-group $\mathrm{String}(G)$. The gerbe $\mathrm{String}(G) \to G$ is described explicitly in a cover of G in [195], where an *equivariant structure* on it is also given, but there is no known explicit cocycle data for the product m and the associator α.

Example 3.38. Example 3.14 can also be presented as a multiplicative V_1/Λ_1-gerbe over V_0/Λ_0. Take $\mathcal{G} \to G$ to be the trivial gerbe. Then the product m is just a V_1/Λ_1-bundle M over $V_0/\Lambda_0 \times V_0/\Lambda_0$. It is defined by

$$M := (V_0/\Lambda_0 \times V_0 \times V_1/\Lambda_1)/\Lambda_0, \tag{3.102}$$

where the action of Λ_0 is

$$([u^0], v^0, [u^1]) \cdot \mu^0 := ([u^0], v^0 + \lambda^0, [u^1 + \langle u^0, \mu^0 \rangle]). \tag{3.103}$$

The associator α is the following canonical isomorphism of V_1/Λ_1-bundles over $(V_0/\Lambda_0)^3$.

$$\begin{aligned} \alpha : d_3^*M \otimes d_1^*M &\to d_0^*M \otimes d_2^*M \\ [u^0, v^0, u^1] \otimes [u^0 + v^0, w^0, v^1] &\mapsto [v^0, w^0, u^1] \otimes [u^0, v^0 + w^0, v^1]. \end{aligned} \tag{3.104}$$

It is straightforward to check that α satisfies the pentagon identity. Note also that M carries a canonical connection $\theta \in \Omega^1(M, V_1)$ such that α is flat. It is defined by $\theta = du^1 - \langle du^0, v^0\rangle$ and its curvature is $\langle du^0 \wedge dv^0\rangle \in \Omega^2(V_0/\Lambda_0 \times V_0/\Lambda_0, V_1)$.

Example 3.39 ([271])**.** Generalizing Examples 3.37 and 3.38, let G, T be Lie groups with T abelian and connected and let $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{t}$ be a symmetric, *Ad*-invariant bilinear form. Then the three forms τ_3, τ_2 defined as in (3.101) satisfy $d\tau_3 = 0$, $d\tau_2 = \delta\tau_3$ and $\delta\tau_2 = 0$. By Lemma 3.36, they define a class in $H^4(BG_\bullet, \mathfrak{t})$. If $\langle\cdot,\cdot\rangle$ is such that this class lies in the image of $H^4(BG_\bullet, Z)$, and $H^3(BG, \underline{T}) = 0$ (i.e., there are no non-trivial flat multiplicative T-gerbes over G), then Proposition 3.34 implies that this data determines uniquely a multiplicative T-gerbe over G.

3.2.3 Connective structures on multiplicative gerbes

Definition 3.40. Let $(\mathcal{G}, m, \alpha)$ be a multiplicative T-gerbe over G. A *connective structure* on it is the following data:

1. A connective structure ∇ on the gerbe $\mathcal{G} \to G$,
2. A connection ∇_m on the isomorphism of gerbes m such that α is a flat 2-isomorphism.

We often write $(\mathcal{G}_\nabla, m_\nabla, \alpha)$ or simply $\mathcal{G}_\nabla$ for a multiplicative gerbe equipped with a connective structure. An *isomorphism of connective structures on a multiplicative T-gerbe* $(\nabla_1, \nabla_{m_1}) \to (\nabla_2, \nabla_{m_2})$ is an isomorphism of connective structures on gerbes $\phi : \nabla_1 \to \nabla_2$ such that the following is a commutative diagram of isomorphisms of gerbes with connective structures

$$\begin{tikzcd} g_1^*\mathcal{G}_{\nabla_1} \otimes g_2^*\mathcal{G}_{\nabla_1} \arrow[r, "(m,\nabla_{m_1})"] \arrow[d, "(id, g_1^*\phi\otimes g_2^*\phi)"'] & (g_1g_2)^*\mathcal{G}_{\nabla_1} \arrow[d, "(id,(g_1g_2)^*\phi)"] \\ g_1^*\mathcal{G}_{\nabla_2} \otimes g_2^*\mathcal{G}_{\nabla_2} \arrow[r, "(m,\nabla_{m_2})"'] & (g_1g_2)^*\mathcal{G}_{\nabla_2}. \end{tikzcd} \tag{3.105}$$

Remark 3.41. In [271], a *connection* on a multiplicative gerbe is defined as the same piece of structure as in Definition 3.40 but with an additional curving on $\mathcal{G}$ (not necessarily preserved by m). However, as we will see in Theorem 3.43 below, the data of a connective structure already determines a canonical curving.

Given a cocycle description $\lambda_{i_1j_1k_1}$, $m_{i_2j_2}$, α_{i_3} of the multiplicative gerbe in a good semi-simplicial cover of $BG_\bullet$ as in (3.94), a connective structure on it is then described by

$$A_{i_1j_1} \in \Omega^1(U^1_{i_1j_1}, \mathfrak{t}), \quad M_{i_2} \in \Omega^1(U^2_{i_2}, \mathfrak{t}) \tag{3.106}$$

satisfying

$$\begin{aligned} A_{i_1j_1} - A_{i_1k_1} + A_{j_1k_1} &= \lambda^*_{i_1j_1k_1}\theta^T, \\ M_{i_2} + d_1^* A_{d_1(i_2)d_1(j_2)} + m^*_{i_2j_2}\theta^T &= d_0^* A_{d_0(i_2)d_0(j_2)} + d_2^* A_{d_2(i_2)d_2(j_2)} + M_{j_2}, \\ \alpha^*_{i_3}\theta^T + d_0^* M_{d_0(i_3)} + d_2^* M_{d_2(i_3)} &= d_1^* M_{d_1(i_3)} + d_3^* M_{d_3(i_3)}, \end{aligned} \tag{3.107}$$

where $\theta^T \in \Omega^1(T, \mathfrak{t})$ is the Maurer-Cartan form on T.

The following is an existence/classification result for connective structures on multiplicative gerbes (see [271] for similar results, with the difference that an additional curving on the multiplicative gerbe is considered as part of the structure to classify). For fixed T, G, let $Ext(G, BT)$ be the space of multiplicative T-gerbes over G up to isomorphism and $Ext(G, BT_\nabla)$ the space of multiplicative T-gerbes with connective structure over G up to isomorphism.

Proposition 3.42. *A multiplicative gerbe $\mathcal{G}$ admits a connective structure if and only if its de Rham class (3.98) admits a representative $(\tau_3, \tau_2, \tau_1, \tau_0)$ with $\tau_1 = 0$, $\tau_0 = 0$. Moreover, there is an exact sequence*

$$H^1_{gr,cont}(G, \mathfrak{g}^* \otimes \mathfrak{t}) \to Ext(G, BT_\nabla) \to Ext(G, BT) \to H^2_{gr,cont}(G, \mathfrak{g}^* \otimes \mathfrak{t}). \tag{3.108}$$

In particular, $Ext(G, BT_\nabla) = Ext(G, BT)$ for compact G.

Proof. Representatives $(\tau_3, \tau_2, \tau_1, 0)$ of the de Rham class of $\mathcal{G}$ are obtained from taking a connective structure and curving on $\mathcal{G}$ and a connection on m; since τ_1 measures the failure of the associator α to preserve the connective structure it is clear that the multiplicative gerbe admits a connective structure if and only if the choices can be made so that $\tau_1 = 0$. Now it follows from the cocycle data above, as in the proof of Proposition 3.34, that $Ext(G, BT_\nabla) = \mathbb{H}^3(BG_\bullet, C^\infty_T \to \Omega^1_\mathfrak{t})$, for $C^\infty_T \to \Omega^1_\mathfrak{t}$ the complex of sheaves on $BG_\bullet$ of smooth T-valued functions and smooth $\mathfrak{t}$-valued 1-forms, respectively, with the map $f \mapsto f^*\theta^T$ between them. Then the above sequence follows from the exact sequence of complexes $0 \to (0 \to \Omega^1_\mathfrak{t}) \to (C^\infty_T \to \Omega^1_\mathfrak{t}) \to (C^\infty_T \to 0) \to 0$ and Theorem 2.13. □

Recall from Example 3.39 the construction of a multiplicative T-gerbe over G from the data of an Ad-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{t}$. By Proposition 3.42, it admits a connective structure, as it was first proven by Waldorf in [271]. We recap this in Theorem 3.43 below, along with a converse to this result which seems to be new. From now on we write θ^L, $\theta^R \in \Omega^1(G, \mathfrak{g})$ for the left- and right-invariant Maurer-Cartan forms on G, respectively. We recall the notion of *enhanced curving* on a gerbe that was defined in Section 3.2.1 and we write $H^4(BG_\bullet, \underline{\mathfrak{t}}) \xrightarrow{exp} H^4(BG_\bullet, \underline{T})$ for the map induced by $exp : \mathfrak{t} \to T$.

Theorem 3.43. *Let G, T be Lie groups with T abelian and connected. Given an Ad-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{t}$, the differential forms*

$$\mu := \frac{1}{6}\langle \theta^L \wedge [\theta^L \wedge \theta^L]\rangle \in \Omega^3(G, \mathfrak{t}), \quad \nu := -\langle g_1^*\theta^L \wedge g_2^*\theta^R \rangle \in \Omega^2(G^2, \mathfrak{t}) \tag{3.109}$$

define a class $[\mu, -\nu, 0, 0] \in H^4(BG, \mathfrak{t})$. If $exp([\mu, -\nu, 0, 0]) \in H^4(BG_\bullet, \underline{T})$ vanishes then there is a multiplicative T-gerbe with connective structure $(\mathcal{G}_\nabla, m_\nabla, \alpha)$ over G whose de Rham class (3.98) is $c_\mathfrak{t}(\mathcal{G}) = [\mu, -\nu, 0, 0]$. Furthermore, such $(\mathcal{G}_\nabla, m_\nabla, \alpha)$ is unique up to tensor product with flat multiplicative gerbes.

Conversely, a multiplicative T-gerbe with connective structure $(\mathcal{G}_\nabla, m_\nabla, \alpha)$ over G has a unique enhanced curving $\Theta^{L,en} = (\Theta^L, h)$ such that the curvature $F \in \Gamma(T^*G^2 \otimes T^*G^2 \otimes \mathfrak{t})$ of ∇_m with respect to $\Theta^{L,en}$ is determined by $h \in \Gamma(S^2T^*G \otimes \mathfrak{t})$ as

$$F = 2h_1(g_1^*\theta^L \otimes g_2^*\theta^R), \tag{3.110}$$

where $h_1 \in S^2\mathfrak{g}^* \otimes \mathfrak{t}$ is h evaluated at $1 \in G$. Its curvature is $\frac{1}{6}h_1(\theta^L \wedge [\theta^L \wedge \theta^L]\rangle$. In other words, $(\mathcal{G}_\nabla, m_\nabla, \alpha)$ determines the following data.

1. *An Ad-invariant symmetric bilinear form $h_1 = \langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{t}$ such that, for μ, ν defined as in (3.109), $exp([\mu, -\nu, 0, 0]) \in H^4(BG_\bullet, \underline{T})$ vanishes.*
2. *A curving Θ^L on $\mathcal{G}_\nabla$ with curvature μ and such that ∇_m has curvature ν with respect to it.*

We call $\Theta^{L,en}$ the Maurer-Cartan enhanced curving *on $\mathcal{G}_\nabla$ and we call Θ^L the* Maurer-Cartan curving *on $\mathcal{G}_\nabla$. If T is connected, then both constructions are inverse to each other up to tensor product with flat multiplicative gerbes.*

Proof. The first part, which is due to Waldorf [271], follows from Example 3.39 and Proposition 3.42. For the second part, we note first that the equivalence between both descriptions of the data determined by $(\mathcal{G}_\nabla, m_\nabla, \alpha)$ follow from taking the skew-symmetric and symmetric part of (3.110). Then we consider the following exact triangle in the derived category of sheaves of abelian groups over $BG_\bullet$.

$$\begin{tikzcd} \underline{T} \arrow[r] \arrow[d] & C^\infty_T \arrow[r] \arrow[d] & 0 \arrow[r, "{[1]}"] \arrow[d] & {} \\ 0 \arrow[r] & \Omega^1_{\mathfrak{t}} \arrow[r, "d"] & \Omega^2_{\mathfrak{t},d-cl} \arrow[r, "{[1]}"] & {} \end{tikzcd},$$

where $\Omega^2_{\mathfrak{t},d-cl}$ is the sheaf of closed $\mathfrak{t}$-valued two-forms. This yields the sequence

$$H^1(BG_\bullet, \Omega^2_{\mathfrak{t},d-cl}) \to H^3(BG, \underline{T}) \to Ext(G, BT_\nabla) \to H^2(BG_\bullet, \Omega^2_{\mathfrak{t},d-cl}) \to H^4(BG, \underline{T}).$$

Consider now the exact sequence $0 \to \Omega^2_{\mathfrak{t},d-cl} \to \Omega^2_{\mathfrak{t}} \xrightarrow{d} \Omega^3_{\mathfrak{t}} \to 0$ and apply Theorem 2.13 to obtain $H^1(BG_\bullet, \Omega^2_{\mathfrak{t},d-cl}) = 0$ and

$$H^2(BG_\bullet, \Omega^2_{\mathfrak{t},d-cl}) = H^2(BG_\bullet, \Omega^2_{\mathfrak{t}}) = H^0_{gr,cont}(G, S^2\mathfrak{g}^* \otimes \mathfrak{t}); \tag{3.111}$$

hence,

$$0 \to H^3(BG, \underline{T}) \to Ext(G, BT_\nabla) \to H^0_{gr,cont}(G, S^2\mathfrak{g}^* \otimes \mathfrak{t}) \to H^4(BG, \underline{T}). \tag{3.112}$$

The theorem follows from chasing how the maps in this sequence are defined. From the triangle above it is clear that the map $Ext(G, BT_\nabla) \to H^2(BG_\bullet, \Omega^2_{\mathfrak{t}})$ sends a multiplicative gerbe with connective structure $(\mathcal{G}_\nabla, m_\nabla, \alpha)$ to the class of the curvature 2-form $-\tau_2 \in \Omega^2(G^2, \mathfrak{t})$ of ∇_m with respect to any choice of curving on $\mathcal{G}_\nabla$ (it satisfies $\delta\tau_2 = 0$ as α preserves the connection on m). By part (2) of Lemma 2.14 this determines an Ad-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{t}$, characterized by the condition that the curving on $\mathcal{G}_\nabla$ can be chosen so that $\tau_2 = \nu$. Let Θ^L be one such curving.

The curvature of Θ^L is some $H \in \Omega^3(G, \mathfrak{t})$ with $\delta H = d\nu$; since μ satisfies this, we obtain $H = \mu + h$ with $\delta h = 0$, but we see from Theorem 2.13 and Remark 2.10 that $H^1(BG_\bullet, \Omega^3_{\mathfrak{t}}) = \ker(\delta : \Omega^3(G, \mathfrak{t}) \to \Omega^3(G^2, \mathfrak{t})) = 0$ and so $H = \mu$. Then take $\Theta^{L,en}$ to be given by Θ^L and the symmetric tensor $h = -\frac{1}{2}\langle \theta^L \odot \theta^L \rangle$; this enhanced curving satisfies the condition above. It is moreover unique with such property, as any other enhanced curving differs from this one by $b \in \Omega^2(G, \mathfrak{t})$ and $h' \in \Gamma(S^2T^*G \otimes \mathfrak{t})$ but then the curvature condition imposes $h' = \frac{1}{2}\langle \theta^L \odot \theta^L \rangle'$ for some Ad-invariant $\langle \cdot, \cdot \rangle'$ and $\delta b = \langle g_1^*\theta^L \wedge g_2^*\theta^R \rangle'$; hence, $\langle \cdot, \cdot \rangle' = 0$ and $b = 0$ by part (2) of Lemma 2.14 and by Theorem 2.13, which yields $H^1(BG_\bullet, \Omega^2_{\mathfrak{t}}) = \ker(\delta : \Omega^2(G, \mathfrak{t}) \to \Omega^2(G^2, \mathfrak{t})) = 0$. This concludes the proof of the theorem. □

Remark 3.44. Given a multiplicative T-gerbe with connective structure over G described by cocycle data (3.94), (3.106), one can use formula (2.46) from Lemma 2.14 to compute the pairing from Theorem 3.43 (1) as

$$\langle u,v\rangle = \frac{1}{2}dM_{i_2|(g_1,g_2)}(0+ug_2, g_1v+0) + \frac{1}{2}dM_{i_2|(g_1,g_2)}(0+vg_2, g_1u+0), \tag{3.113}$$

for any choice of $(g_1,g_2)\in G^2$ and any choice of $i_2\in I_2$ with $(g_1,g_2)\in U^2_{i_2}$. Alternatively, one can use the cocycle equations (3.107) to check directly that this formula gives a well-defined Ad-invariant, symmetric pairing. Furthermore, one can prove that the Maurer-Cartan curving is given by the two-forms $\Theta^L_{i_1}\in\Omega^2(U^1_{i_1},\mathfrak{t})$ defined by

$$\begin{aligned}\Theta^L_{i_1|g}(u_g,v_g) &= dA_{i_1 d_0(i_2)|g}(u_g,v_g) + \frac{1}{2}dM_{i_2|(g^{-1},g)}(0+u_g, v_g^{-1}+v_g)\\ &\quad + \frac{1}{2}dM_{i_2|(g^{-1},g)}(u_g^{-1}+u_g, 0+v_g),\end{aligned} \tag{3.114}$$

as it follows from similar computations to those in the proof of Lemma 2.14 that they satisfy the required properties. Here $i_2\in I_2$ is any choice of index such that $(g^{-1},g)\in U^2_{i_2}$ and by u_g^{-1} we mean $dinv_g(u_g)$ for $inv: G\to G$, $g\mapsto g^{-1}$. In particular, these explicit computations yield another proof of the existence of $\langle\cdot,\cdot\rangle$ and Θ^L in Theorem 3.43 which is also valid when T is not connected. Note also that we can add the two formulas to obtain a formula for $\Theta^{L,en}$:

$$\Theta^{L,en}_{i_1|g}(u_g,v_g) = dA_{i_1 d_0(i_2)|g}(u_g,v_g) + dM_{i_2|(g^{-1},g)}(u_g^{-1}+u_g, 0+v_g), \tag{3.115}$$

Corollary 3.45. *Let G, T be Lie groups with G compact and T abelian. Any multiplicative T-gerbe $(\mathcal{G},m,\alpha)$ over G determines an Ad-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{t}$, a connective structure (∇,∇_m) on $(\mathcal{G},m,\alpha)$ well-defined up to isomorphism and a curving Θ^L on $\mathcal{G}_\nabla$ with curvature μ and such that ∇_m has curvature ν, where μ, ν are as in (3.109).*

Proof. Straightforward from Proposition 3.42 and Theorem 3.43. □

Example 3.46. Let $\mathcal{T}$ be the multiplicative V_1/Λ_1-gerbe over V_0/Λ_0 constructed from a bilinear form $\langle\cdot,\cdot\rangle:\Lambda_0\otimes\Lambda_0\to\Lambda_1$ as in Example 3.38 and write $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{sy}+\langle\cdot,\cdot\rangle_{sk}$ for its decomposition in symmetric and skew-symmetric forms. The connection θ described in Example 3.38 is a connective structure on $\mathcal{T}$. Its corresponding pairing by Theorem 3.43 is $\langle\cdot,\cdot\rangle_{sy}$, while the Maurer-Cartan curving is $\Theta=\frac{1}{2}\langle dv\wedge dv\rangle_{sk}\in\Omega^2(V_0,V_1)$; thus, the Maurer-Cartan enhanced curving is $\Theta^{en}=\frac{1}{2}\langle dv\otimes dv\rangle$.

Example 3.47. For the multiplicative $U(1)$-gerbe String(G) over a compact simple Lie group G from Example 3.37, it follows by construction and by Theorem 3.43 that there is a connective structure on it inducing as pairing the multiple of the Killing form such that the corresponding $[\mu]\in H^3(G,\mathbb{Z})=\mathbb{Z}$ is a generator of $H^3(G,\mathbb{Z})$. [195] provides an explicit description of cocycle data for a connective structure and a curving on the gerbe $\mathrm{String}(G) \to G$ with curvature μ (without a connection on m, as there is no known explicit description of m itself).

3.2.4 Connective structures as Maurer-Cartan forms and as prequantizations of BG

The main result from [238] states that, for Lie groups G, T with T abelian, the classification of multiplicative T-gerbes over G coincides with the classification of *central extensions* of G by BT. To state this result we recall from section 3.1.2 that a Lie 2-group $\mathfrak{G}$ determines topological groups G, T with T abelian such that $\mathfrak{G}$ fits in an extension of topological 2-groups of the form

$$1 \to BT \to \mathfrak{G} \to G \to 1, \tag{3.116}$$

as well as a continuous action $\triangleright$ of G on T. We say $\mathfrak{G}$ is a *central extension* of G by T if $\triangleright$ is trivial. We recall here the proof of the main result from [238].

Theorem 3.48 ([238])**.** *Let G, T be Lie groups with T abelian. There is an equivalence of bicategories between the bicategory of central extensions of G by BT as Lie 2-groups and the bicategory of multiplicative T-gerbes over G.*

Proof. Given a multiplicative T-gerbe over G, we describe it with cocycle data $\lambda_{i_1j_1k_1}$, $m_{i_2j_2}$, α_{i_3} (3.94), assuming for simplicity that $\lambda_{i_1j_1k_1}$ is normalized (i.e., it equals 1 whenever there are two coinciding indices). Then we construct the Lie groupoid $\mathfrak{G}$ as in Example 3.6; i.e. we let $\mathfrak{G}_0 := \sqcup_{i_1\in I_1} U^1_{i_1}$, $\mathfrak{G}_1 := \sqcup_{i_1j_1\in I_1} U^1_{i_1j_1} \times T$, where $(i_1, j_1, g, t) \in \mathfrak{G}_1$ is seen as an arrow $(i_1, g) \to (j_1, g)$ and composition is defined as

$$(j_1, k_1, g, t_2) \circ (i_1, j_1, g, t_1) := (i_1, k_1, g, t_1t_2\lambda_{i_1j_1k_1}(g)); \tag{3.117}$$

the cocycle condition for $\lambda_{i_1j_1k_1}$ ensures that this is associative. Then the anafunctor $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ is defined by the total space

$$\begin{aligned} M := \{(i^1_1, i^2_1, i^{12}_1, i_2, g_1, g_2, t) \in I^3_1 \times I_2 \times G^2 \times T \mid \\ g_1 \in U^1_{i^1_1},\ g_2 \in U^1_{i^2_1},\ g_1g_2 \in U^1_{i^{12}_1},\ (g_1, g_2) \in U^1_{i_2}\}/\sim, \end{aligned} \tag{3.118}$$

where the equivalence relation is $(i^1_1, i^2_1, i^{12}_1, i_2, g_1, g_2, t) \sim (i^1_1, i^2_1, i^{12}_1, j_2, g_1, g_2, t')$ for

$$t' := t \cdot \lambda^{-1}_{i^1_1 d_2(i_2) d_2(j_2)}(g_1) \cdot \lambda^{-1}_{i^2_1 d_0(i_2) d_0(j_2)}(g_2) \cdot \lambda_{i^{12}_1 d_1(i_2) d_1(j_2)}(g_1g_2) \cdot m^{-1}_{i_2j_2}(g_1, g_2). \tag{3.119}$$

The cocycle condition for $m_{i_2j_2}$ ensures that this is indeed an equivalence relation. The anchor maps of the anafunctor are

$$\begin{aligned}\pi_0([i_1^1, i_1^2, i_1^{12}, i_2, g_1, g_2, t]) &= ((i_1^1, g_1), (i_1^2, g_2)),\\ \pi_1([i_1^1, i_1^2, i_1^{12}, i_2, g_1, g_2, t]) &= (i_1^{12}, g_1g_2)\end{aligned} \tag{3.120}$$

and the actions are

$$\begin{aligned}\rho_0(((i_1^1, j_1^1, g_1, t_1),&(i_1^2, j_1^2, g_2, t_2)), [i_1^1, i_1^2, i_1^{12}, i_2, g_1, g_2, t])\\ &= [j_1^1, j_1^2, i_1^{12}, i_2, g_1, g_2, t \cdot t_1t_2 \cdot \lambda_{i_1^1j_1^1d_2(i_2)}(g_1) \cdot \lambda_{i_1^2j_1^2d_0(i_2)}(g_2)],\end{aligned} \tag{3.121}$$

$$\begin{aligned}\rho_1([i_1^1, i_1^2, i_1^{12}, i_2, g_1, g_2, t],&(j_1^{12}, i_1^{12}, g_1g_2, t_{12}))\\ &= [i_1^1, i_1^2, j_1^{12}, i_2, g_1, g_2, t \cdot t_{12} \cdot \lambda^{-1}_{i_1^{12}j_1^{12}d_1(i_2)}(g_1g_2)].\end{aligned} \tag{3.122}$$

The cocycle condition for $m_{i_2j_2}$ ensures that these are well-defined. The transformation α is defined as follows. First, the total spaces of the anafunctors $F := m \circ (m \times id)$ and $F' := m \circ (id \times m)$ are

$$\begin{aligned}F := \{(i_1^1, i_1^2, i_1^3, i_1^{12}, i_1^{123}, i_2^{1,2}, i_2^{12,3}, g_1, g_2, g_3, t) \in I_1^5 \times I_2^2 \times G^3 \times T \mid\\ g_1 \in U^1_{i_1^1},\ g_2 \in U^1_{i_1^2},\ g_3 \in U^1_{i_1^3},\ g_1g_2 \in U^1_{i_1^{12}},\\ g_1g_2g_3 \in U^1_{i_1^{123}},\ (g_1, g_2) \in U^1_{i_2^{1,2}},\ (g_1g_2, g_3) \in U^2_{i_2^{12,3}}\}/\sim,\end{aligned} \tag{3.123}$$

$$\begin{aligned}F' := \{(i_1^1, i_1^2, i_1^3, i_1^{23}, i_1^{123}, i_2^{2,3}, i_2^{1,23}, g_1, g_2, g_3, t) \in I_1^5 \times I_2^2 \times G^3 \times T \mid\\ g_1 \in U^1_{i_1^1},\ g_2 \in U^1_{i_1^2},\ g_3 \in U^1_{i_1^3},\ g_2g_3 \in U^1_{i_1^{23}},\\ g_1g_2g_3 \in U^1_{i_1^{123}},\ (g_2, g_3) \in U^1_{i_2^{2,3}},\ (g_1, g_2g_3) \in U^2_{i_2^{1,23}}\}/\sim,\end{aligned} \tag{3.124}$$

where the equivalence relations are

$$\begin{aligned}&(i_1^1, i_1^2, i_1^3, i_1^{12}, i_1^{123}, i_2^{1,2}, i_2^{12,3}, g_1, g_2, g_3, t) \sim (i_1^1, i_1^2, i_1^3, j_1^{12}, i_1^{123}, j_2^{1,2}, j_2^{12,3}, g_1, g_2, g_3, t')\\ &t' := t \cdot \lambda^{-1}_{i_1^1d_2(i_2^{1,2})d_2(j_2^{1,2})}(g_1) \cdot \lambda^{-1}_{i_1^2d_0(i_2^{1,2})d_0(j_2^{1,2})}(g_2) \cdot \lambda^{-1}_{i_1^3d_0(i_2^{12,3})d_0(j_2^{12,3})}(g_3)\\ &\lambda_{i_1^{12}d_1(i_2^{1,2})d_1(j_2^{1,2})}(g_1g_2) \cdot \lambda^{-1}_{i_1^{12}d_2(i_2^{12,3})d_2(j_2^{12,3})}(g_1g_2) \cdot \lambda_{i_1^{123}d_1(i_2^{12,3})d_1(j_2^{12,3})}(g_1g_2g_3)\\ &\lambda_{i_1^{12}j_1^{12}d_1(i_2^{1,2})}(g_1g_2) \cdot \lambda^{-1}_{i_1^{12}j_1^{12}d_2(i_2^{12,3})}(g_1g_2) \cdot m^{-1}_{i_2^{1,2}j_2^{1,2}}(g_1, g_2) \cdot m^{-1}_{i_2^{12,3}j_2^{12,3}}(g_1g_2, g_3).\end{aligned} \tag{3.125}$$

and

$$
\begin{aligned}
&(i_1^1, i_1^2, i_1^3, i_1^{23}, i_1^{123}, i_2^{2,3}, i_2^{1,23}, g_1, g_2, g_3, t) \sim (i_1^1, i_1^2, i_1^3, j_1^{23}, i_1^{123}, j_2^{2,3}, j_2^{1,23}, g_1, g_2, g_3, t') \\
&t' := t \cdot \lambda^{-1}_{i_1^1 d_2(i_2^{1,23}) d_2(j_2^{1,23})}(g_1) \cdot \lambda^{-1}_{i_1^2 d_2(i_2^{2,3}) d_2(j_2^{2,3})}(g_2) \cdot \lambda^{-1}_{i_1^3 d_0(i_2^{2,3}) d_0(j_2^{2,3})}(g_3) \\
&\lambda_{i_1^{23} d_1(i_2^{2,3}) d_1(j_2^{2,3})}(g_2 g_3) \cdot \lambda^{-1}_{i_1^{23} d_0(i_2^{1,23}) d_0(j_2^{1,23})}(g_2 g_3) \cdot \lambda_{i_1^{123} d_1(i_2^{1,23}) d_1(j_2^{1,23})}(g_1 g_2 g_3) \\
&\lambda_{i_1^{23} j_1^{23} d_1(i_2^{2,3})}(g_2 g_3) \cdot \lambda^{-1}_{i_1^{23} j_1^{23} d_0(i_2^{1,23})}(g_2 g_3) \cdot m^{-1}_{i_2^{2,3} j_2^{2,3}}(g_2, g_3) \cdot m^{-1}_{i_2^{1,23} j_2^{1,23}}(g_1, g_2 g_3).
\end{aligned}
\tag{3.126}
$$

The anchors and actions are defined similarly as in (3.121), (3.122). Then we define a map $\alpha : F \to F'$ by

$$
\begin{aligned}
&[(i_1^1, i_1^2, i_1^3, i_1^{12}, i_1^{123}, d_3(i_3), d_1(i_3), g_1, g_2, g_3, t)] \\
&\qquad \mapsto [(i_1^1, i_1^2, i_1^3, i_1^{23}, i_1^{123}, d_0(i_3), d_2(i_3), g_1, g_2, g_3, t\alpha_{i_3}(g_1, g_2, g_3))],
\end{aligned}
\tag{3.127}
$$

where $i_3 \in I^3$ is any choice of index such that $(g_1, g_2, g_3) \in I^3$. The simplicial identities for the index sets of the simplicial cover, together with the cocycle equations for α_{i_3}, imply that α is a well-defined transformation of anafunctors satisfying the pentagon identity. The unit and the unitors can be constructed similarly, from the canonical data of Remark 3.33. This construction can be enhanced to an equivalence of bicategories. □

Remark 3.49. In the proof of Theorem 3.48 we have presented all the explicit computations for completeness, but a more straightforward way to prove this is by noting the following observations that we have used in the proof of Theorem 3.48.

1. A gerbe has an associated Lie groupoid as in Example 3.6.
2. An isomorphism of gerbes determines an anafunctor between the corresponding Lie groupoids.
3. A 2-isomorphism of gerbes determines a transformation between the corresponding anafunctors.

Then it is clear that multiplicative gerbes as in Definition 3.31 are equivalent to Lie 2-groups as in Definition 3.9.

Let $\mathfrak{G}$ be the Lie 2-group corresponding to a multiplicative T-gerbe $\mathcal{G} \to G$ by Theorem 3.48. From the description of $\mathfrak{G}_0$ and $\mathfrak{G}_1$ in the proof, it is easy to see that its Lie 2-algebra is the 2-step complex of vector spaces $\mathfrak{t} \xrightarrow{0} \mathfrak{g}$. At least when G is compact, simple and simply connected, and $\mathcal{G}$ is the String(G) group from Example 3.37, it is customary to regard $\mathbb{R} \xrightarrow{0} \mathfrak{g}$ as equipped with the L_∞-structure defined by the Lie bracket of $\mathfrak{g}$

and the 3-bracket $\{v_1, v_2, v_3\} := \langle v_1, [v_2, v_3]\rangle$, where $\langle\cdot,\cdot\rangle$ is the pairing associated to String(G) by Corollary 3.45. We proceed to derive this structure in a natural way using the notion of Maurer-Cartan forms from Section 3.1.4.

Proposition 3.50. *Let $\mathfrak{G}$ be the Lie 2-group corresponding to a multiplicative T-gerbe $\mathcal{G} \to G$ by Theorem 3.48. Then, a connective structure on $\mathcal{G}$ determines*

$$\theta^0 \in \Omega^1(B\mathfrak{G}_1, \mathfrak{g}), \quad \theta^1 \in \Omega^1(B\mathfrak{G}_2, \mathfrak{t}), \quad \Theta^0 \in \Omega^2(B\mathfrak{G}_1, \mathfrak{t}) \tag{3.128}$$

such that

$$0 = d_2^*\theta^0 - d_1^*\theta^0 + Ad(d_2(\cdot))d_0^*\theta^0, \tag{3.129}$$
$$0 = \delta\theta^1, \tag{3.130}$$
$$d\theta^0 = -\frac{1}{2}[\theta^0 \wedge \theta^0], \tag{3.131}$$
$$d\Theta^0 = \frac{1}{6}\langle\theta^0 \wedge [\theta^0 \wedge \theta^0]\rangle, \tag{3.132}$$
$$d\theta^1 - \delta\Theta^0 = \langle d_2^*\theta^0 \wedge Ad(d_2(\cdot))d_0^*\theta^0\rangle. \tag{3.133}$$

In particular, (θ^0, θ^1) is a right-invariant Maurer-Cartan form on $\mathfrak{G}$ for the action of $\mathfrak{G}$ on $\mathfrak{t} \xrightarrow{0} \mathfrak{g}$ given by the adjoint action of G on $\mathfrak{g}$ and the trivial action on $\mathfrak{t}$.

Proof. In this case $\pi : B\mathfrak{G}_1 \to G$ is the surjective submersion on which the gerbe $\mathcal{G}$ is described and $B\mathfrak{G}_2$ is the total space of the isomorphism of gerbes $m : g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \to (g_1g_2)^*\mathcal{G}$. Then taking $\theta^0 := \pi^*\theta^R$, θ^1 the connection 1-form of the connection on m and Θ^0 the Maurer-Cartan curving yields the equations above by Theorem 3.43. □

Proposition 3.51. *Let $\mathfrak{G}$ be the Lie 2-group corresponding to a multiplicative T-gerbe $\mathcal{G} \to G$ by Theorem 3.48. Then, a connective structure on $\mathcal{G}$ determines a differentiation of $\mathfrak{G}$ in the sense of Definition 3.24, where $\mathfrak{t} \xrightarrow{0} \mathfrak{g}$ is equipped with the cubic L_∞-structure defined by*

$$\begin{aligned}
[v_1, v_2] &:= \begin{cases} [v_1, v_2]_{\mathfrak{g}} & v_1,\ v_2 \in \mathfrak{g}, \\ 0 & \textit{otherwise} \end{cases}, \\
\{v_1, v_2, v_3\} &:= \begin{cases} \langle v_1, [v_2, v_3]\rangle & v_1,\ v_2,\ v_3 \in \mathfrak{g}, \\ 0 & \textit{otherwise} \end{cases},
\end{aligned} \tag{3.134}$$

where $[\cdot,\cdot]_{\mathfrak{g}}$ denotes the Lie bracket of $\mathfrak{g}$.

Proof. Choose cocycle data $\lambda_{i_1j_1k_1}$, $m_{i_2j_2}$, α_{i_3} (3.94) for $\mathcal{G}$ and use the construction of $\mathfrak{G}$ in the proof of Theorem 3.48. Let $A_{i_1j_1}$, M_{i_2} be cocycle data for a connective structure as in (3.106) and let Θ_{i_1} be the corresponding Maurer-Cartan curving defined by (3.114). For v, v_1, $v_2 \in \mathfrak{g}$ and $u \in \mathfrak{t}$ we define $X^v \in \Gamma(T\mathfrak{G})^{\mathfrak{G}}$ and α^u, $\alpha^{(v_1,v_2)} \in \Gamma(A_{\mathfrak{G}})$ by

$$\begin{aligned} X^v(i_1, j_1, g, t) &:= (i_1, j_1, v \cdot g, A_{i_1 j_1|g}(v \cdot g) \cdot t), \\ \alpha^u(i, g) &:= (i, i, 0_g, u), \\ \alpha^{(v_1,v_2)}(i, g) &:= (i, i, 0_g, \Theta_{i|g}(v_1 \cdot g, v_2 \cdot g)). \end{aligned} \tag{3.135}$$

From Definition 2.16 we see that this defines a morphism of L_∞-algebras $\mathfrak{g}\oplus\mathfrak{t}[1] \to \Gamma(T\mathfrak{G})$ if and only if

$$\partial\alpha^u = 0, \tag{3.136}$$

$$[X^{v_1}, X^{v_2}] - X^{[v_1,v_2]} = \partial\alpha^{(v_1,v_2)}, \tag{3.137}$$

$$[X^v, \alpha^u] = 0, \tag{3.138}$$

$$\begin{aligned} \alpha^{\langle v_1,[v_2,v_3]\rangle} &= X^{v_1}(\alpha^{(v_2,v_3)}) - X^{v_2}(\alpha^{(v_1,v_3)}) + X^{v_3}(\alpha^{(v_1,v_2)}) \\ &\quad - \alpha^{[v_1,v_2],v_3} + \alpha^{([v_1,v_3],v_2)} - \alpha^{([v_2,v_3],v_1}). \end{aligned} \tag{3.139}$$

for v, v_1, v_2, $v_3 \in \mathfrak{g}$ and $u \in \mathfrak{t}$. Now (3.136) and (3.138) follow easily from the definition of ∂ and $[\cdot,\cdot]$, while (3.137) is equivalent to $dA_{i_1 j_1} = \Theta_{i_1} - \Theta_{j_1}$ and (3.139) is equivalent to $\frac{1}{6}\langle\theta^R \wedge [\theta^R \wedge \theta^R]\rangle = d\Theta_{i_1}$, which concludes the proof. $\square$

In Section 3.2.5 we will also show how to use connective structures on multiplicative gerbes to define an exponential map on their associated Lie 2-groups. There is yet another interesting construction associated to the choice of a connective structure on a multiplicative gerbe. Namely, recall from Example 2.35 that, for any Lie group G, the data of a symmetric, Ad-invariant (possibly not) non-degenerate bilinear form $\langle\cdot,\cdot\rangle$: $\mathfrak{g}\otimes\mathfrak{g} \to \mathbb{R}$ induces a 2-shifted (pre)symplectic structure $\omega = (\mu, \nu)$ on $BG_\bullet$ given by the differential forms

$$\mu = \frac{1}{6}\langle\theta^L \wedge [\theta^L \wedge \theta^L]\rangle \in \Omega^3(BG_1, \mathbb{R}), \quad \nu = \langle g_1^*\theta^L \wedge g_2^*\theta^R\rangle \in \Omega^2(BG_2, \mathbb{R}) \tag{3.140}$$

When $\langle\cdot,\cdot\rangle$ is the pairing associated to a multiplicative $U(1)$-gerbe with connective structure over G by Theorem 3.43, and $\mathfrak{G}$ is the corresponding Lie 2-group from Theorem 3.48, then there is a sequence of simplicial manifolds

$$B^2U(1) \to B\mathfrak{G} \xrightarrow{\pi} BG \tag{3.141}$$

(where $B^2U(1) := B(BU(1))$) and equations (3.132) and (3.133) from Proposition 3.50 imply that (θ^0, θ^1) is a 2-shifted 1-form on $B\mathfrak{G}$ with total derivative $\pi^*(\mu,\nu)$. In the language of [231], $(B\mathfrak{G}, \theta^1, \Theta^0)$ is a *prequantization* of (BG, μ, ν). We can also see (θ^1, Θ^0) as an example of a 2-*shifted (pre)contact structure* on $B\mathfrak{G}$, in the sense of [39] (the author thanks Miquel Cueca and Chenchang Zhu for this observation). In particular, we can associate to it the analog of the (pre)symplectic cone of a standard contact structure.

Proposition 3.52. *Let $\mathfrak{G}$ be the Lie 2-group corresponding to a multiplicative $U(1)$-gerbe $\mathcal{G} \to G$ by Theorem 3.48. Then, a connective structure on $\mathcal{G}$ determines a 2-shifted presymplectic structure on $B\mathfrak{G} \times \mathbb{R}^*$ defined by*

$$\frac{t}{6}\pi^*\langle \theta^L \wedge [\theta^L \wedge \theta^L] \rangle + dt \wedge \Theta^0 \in \Omega^3(B\mathfrak{G}_1 \times \mathbb{R}^*, \mathbb{R}), \tag{3.142}$$

$$t\pi^*\langle g_1^*\theta^L \wedge g_2^*\theta^R \rangle + dt \wedge \theta^1 \in \Omega^2(B\mathfrak{G}_2 \times \mathbb{R}^*, \mathbb{R}), \tag{3.143}$$

where t is the coordinate in $\mathbb{R}^*$, $\pi : B\mathfrak{G}_n \to BG_n$ is the projection map and $\langle \cdot, \cdot \rangle$ is the pairing associated to the connective structure by Theorem 3.43. This is in fact 2-shifted symplectic if and only if $\langle \cdot, \cdot \rangle$ is non-degenerate.

Proof. That this is a 2-shifted presymplectic structure is straightforward by Proposition 3.50, since δ does not act on the t coordinate. The tangent complex of $B\mathfrak{G} \times \mathbb{R}^*$ is the following chain complex of vector bundles over $\mathbb{R}^*$

$$\underline{\mathbb{R}}[2] \xrightarrow{0} \underline{\mathfrak{g}}[1] \xrightarrow{0} \underline{\mathbb{R}}; \tag{3.144}$$

the cotangent complex is

$$\underline{\mathbb{R}}^* \xrightarrow{0} \underline{\mathfrak{g}}^*[-1] \xrightarrow{0} \underline{\mathbb{R}}^*[-2]. \tag{3.145}$$

Then $t\pi^*\langle g_1^*\theta^L \wedge g_2^*\theta^R \rangle + dt \wedge \theta^1$ induces the map $\underline{\mathfrak{g}} \to \underline{\mathfrak{g}}^*$ given at each $t \in \mathbb{R}^*$ by $v \mapsto t\langle v, \cdot \rangle$ and the canonical isomorphisms $\underline{\mathbb{R}} \to \underline{\mathbb{R}}^*$. Thus, this structure is 2-shifted symplectic precisely when $\langle \cdot, \cdot \rangle$ is non-degenerate. $\square$

3.2.5 The exponential map

Let $\mathcal{G}$ be a multiplicative T-gerbe over a Lie group G and let $exp^*\mathcal{G} \to \mathfrak{g}$ be the pull-back gerbe by the exponential map $exp : \mathfrak{g} \to G$. The multiplicative structure on $\mathcal{G}$, and the fact that the exponential map is Ad-equivariant, implies that $exp^*\mathcal{G} \to \mathfrak{g}$ is an *equivariant gerbe* in the sense of [199]. Since $\mathfrak{g}$ is contractible, it is clear that $exp^*\mathcal{G} \to \mathfrak{g}$ is trivial as a gerbe, but one might wonder whether it is trivial as an equivariant gerbe.

Definition 3.53. Let $\mathcal{G}$ be a multiplicative T-gerbe over G and let ϵ be a trivialization of $exp^*\mathcal{G} \to \mathfrak{g}$ as a gerbe, where $exp : \mathfrak{g} \to G$ is the exponential map. An *equivariant* *structure* on ϵ is a 2-isomorphism α^ϵ of gerbes over $G \times \mathfrak{g}$

$$\begin{tikzcd}
(\text{Trivial}) \arrow[r, "v^*\epsilon"] \arrow[d, "(Ad(g)v)^*\epsilon"'] \arrow[dr, phantom, "\Leftarrow\alpha^\epsilon"] & exp(v)^*\mathcal{G} \arrow[d, "g^*inv"] \\
(exp(Ad(g)v))^*\mathcal{G} & g^*\mathcal{G} \otimes exp(v)^*\mathcal{G} \otimes (g^{-1})^*\mathcal{G} \arrow[l, "Ad"]
\end{tikzcd}, \tag{3.146}$$

where $Ad := (gexp(v), g^{-1})^*m \circ (g, exp(v))^*m$ and inv is defined by (3.93), such that over $G \times G \times \mathfrak{g}$ we have

$$\begin{tikzcd}
exp(Ad(g_1g_2)v)^*\mathcal{G} & (Trivial) \arrow[l, "(Ad(g_1g_2)v)^*\epsilon"'] \arrow[r, "v^*\epsilon"] \arrow[d, phantom, "\Leftarrow (g_1g_2,v)^*\alpha^\epsilon"] & exp(v)^*\mathcal{G} \arrow[dl, bend left, "(g_1g_2)^*inv"] \\
& (g_1g_2)^*\mathcal{G} \otimes exp(v)^*\mathcal{G} \otimes (g_2^{-1}g_1^{-1})^*\mathcal{G} \arrow[ul, bend left, "(g_1g_2,v)^*Ad"] &
\end{tikzcd} =$$

$$\begin{tikzcd}
exp(Ad(g_1g_2)v)^*\mathcal{G} \arrow[dr, phantom, "\Leftarrow (g_1,Ad(g_2)v)^*\alpha^\epsilon"] & (Trivial) \arrow[l, "(Ad(g_1g_2)v)^*\epsilon"'] \arrow[r, "v^*\epsilon"] \arrow[d, "(Ad(g_2)v)^*\epsilon"'] \arrow[dr, phantom, "\Leftarrow (g_2,v)^*\alpha^\epsilon"] & exp(v)^*\mathcal{G} \arrow[d, "g_2^*inv"] \arrow[ll, bend right=60, "(g_1g_2,v)^*Ad \circ (g_1g_2)^*inv"', "\Uparrow \alpha^{Ad}"] \\
g_1^*\mathcal{G} \otimes exp(Ad(g_2)v)^*\mathcal{G} \otimes (g_1^{-1})^*\mathcal{G} \arrow[u, "(g_1,Ad(g_2)v)^*Ad"] & exp(Ad(g_2)v)^*\mathcal{G} \arrow[l, "g_1^*inv"] & g_2^*\mathcal{G} \otimes exp(v)^*\mathcal{G} \otimes (g_2^{-1})^*\mathcal{G} \arrow[l, "(g_2,v)^*Ad"]
\end{tikzcd}. \tag{3.147}$$

In (3.147) we write α^{Ad} for the following 2-isomorphism of gerbes obtained from the associator of $\mathcal{G}$.

$$\begin{aligned}(g_1, g_2, exp(v)g_2^{-1}g_1^{-1})^*\alpha^{-1} \circ (g_2, exp(v)g_2^{-1}, g_1^{-1})^*\alpha \circ (exp(v), g_2^{-1}, g_1^{-1})^*\alpha \\ \circ (g_2, g_2^{-1}, g_1^{-1})^*\alpha^{-1} \circ (g_1, g_2, g_2^{-1}g_1^{-1})^*\alpha.\end{aligned} \tag{3.148}$$

The following is a new result that will allow us to define the exponential map of a multiplicative gerbe with connective structure and to endow the gauge 2-group of a principal 2-bundle with a smooth structure. The 1-form (3.149) is crucial for the latter purpose, as it appears in the transition functions of heterotic Courant algebroids (4.73).

Theorem 3.54. *Let $\mathcal{G}$ be a multiplicative T-gerbe over G. Then,*

1. *$\mathcal{G}$ admits a connective structure if and only if every trivialization ϵ of $exp^*\mathcal{G} \to \mathfrak{g}$ admits an equivariant structure.*

2. *If $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{t}$ corresponds to a connective structure on $\mathcal{G}$, then any trivialization ϵ can be equipped with a connection such that there exists an equivariant structure α^ϵ whose covariant derivative is $\eta^\epsilon \in \Omega^1(G\times\mathfrak{g},\mathfrak{t})$ defined by*

$$\eta^\epsilon_{|(g,v)}(v_g + \dot{v}) := 2\langle v, g^{-1}v_g\rangle. \tag{3.149}$$

Proof. Assume first that ϵ is a trivialization of $exp^*\mathcal{G}$ with equivariant structure α^ϵ. Recall from Proposition 3.42 that by choosing any connections on $\mathcal{G}$ and m we obtain $\tau \in \Omega^1(G^3,\mathfrak{t})$ such that $\delta\tau = 0$, and that a connective structure on $\mathcal{G}$ exists if and only if there is $\sigma \in \Omega^1(G^2,\mathfrak{t})$ with $\delta\sigma = \tau$. By Bott's Theorem 2.13, we may assume without loss of generality that

$$\tau_{(g_1,g_2,g_3)}(v_{g_1} + v_{g_2} + v_{g_3}) = \kappa(g_2, g_3, g_1^{-1}v_{g_1}) \tag{3.150}$$

for $\kappa : G\times G\times\mathfrak{g} \to \mathfrak{t}$, linear on $\mathfrak{g}$, satisfying

$$\kappa(g_1,g_2,v) - \kappa(g_1,g_2g_3,v) + \kappa(g_1g_2,g_3,v) - \kappa(g_2,g_3,g_1^{-1}v_{g_1}) = 0. \tag{3.151}$$

The existence of σ as above is then equivalent to the existence of $\chi : G\times\mathfrak{g} \to \mathfrak{t}$ with

$$\kappa(g_1,g_2,v) = \chi(g_1,v) - \chi(g_1g_2,v) + \chi(g_2, Ad(g_1^{-1})v). \tag{3.152}$$

Now choose a connection on ϵ and let $\eta \in \Omega^1(G\times\mathfrak{g},\mathfrak{t})$ be the covariant derivative of α^ϵ. Then cocycle condition (3.147) implies the following identity of 1-forms over $G\times G\times\mathfrak{g}$

$$\begin{aligned}(g_1g_2,v)^*\eta - (g_1,Ad(g_2)v)^*\eta - (g_2,v)^*\eta =& \\ = -(g_1,g_2,exp(v)g_2^{-1}g_1^{-1})^*\tau + (g_2,exp(v)g_2^{-1},g_1^{-1})^*\tau + (exp(v),g_2^{-1},g_1^{-1})^*\tau& \\ - (g_2,g_2^{-1},g_1^{-1})^*\tau + (g_1,g_2,g_2^{-1}g_1^{-1})^*\tau.&\end{aligned} \tag{3.153}$$

Evaluating at $(g_1,g_2,0) \in G\times G\times\mathfrak{g}$, $(0,0,v) \in T_{g_1}G_1\times T_{g_2}G\times T_0\mathfrak{g}$ and using formula (3.150) for τ we see that $\chi(g,v) := -\eta_{(g^{-1},0)}(0+v)$ satisfies (3.152) and so $\mathcal{G}$ admits a connective structure, as we wanted to show.

Conversely, assume that $\mathcal{G}$ has a connective structure and let ϵ be any trivialization of $exp^*\mathcal{G} \to \mathfrak{g}$. The two corresponding trivializations of $exp(Ad(g)v)^*\mathcal{G}$ in diagram (3.146) differ by a T-bundle $P^\epsilon \to G\times\mathfrak{g}$ and a 2-isomorphism α^ϵ is equivalent to a section of P^ϵ. Since any two trivializations of $exp^*\mathcal{G} \to \mathfrak{g}$ are isomorphic, we can assume without loss of generality that $0^*\epsilon = 1_{\mathcal{G}}$. Then it is easy to see that there is a canonical 2-isomorphism α^ϵ over $G\times\{0\}$, given by the right unitor of $\mathcal{G}$. This defines a section s of $P^\epsilon_{|G\times\{0\}}$, which we proceed to extend to a global section on $G\times\mathfrak{g}$. For this, choose an arbitrary connection $\nabla^{\epsilon,0}$ on ϵ. Since *inv* and *Ad* are also equipped with connections, this defines a connection $\nabla^{P,0}$ on P^ϵ. We define then $s(g,v) \in P_{(g,v)}$ to be the parallel transport at time 1 of $s(g,0) \in P_{(g,0)}$ along the curve $\gamma^{g,v} : \mathbb{R} \to G \times \mathfrak{g}$, $t \mapsto (g,tv)$. For the corresponding 2-isomorphism α^ϵ, the cocycle condition (3.147) is equivalent to $\alpha(s(g_2,v) \otimes s(g_1, Ad(g_2)v)) = s(g_1g_2,v)$, where $\alpha : P_{(g_2,v)} \otimes P_{(g_1,Ad(g_2v)} \to P_{(g_1g_2,v)}$ is an isomorphism defined by the associator of $\mathcal{G}$. Since the associator behaves well with respect to the right unitor, it follows that α preserves the values at $v = 0$ of the sections. Since the associator preserves the connective structure of $\mathcal{G}$, it follows that α preserves the connections on the T-bundles. Hence, it preserves the sections, as they are defined by parallel transport.

To prove 2, we note first that the covariant derivative of the 2-isomorphism α^ϵ constructed from $\nabla^{\epsilon,0}$ but measured with respect to an arbitrary connection $\nabla^{\epsilon,\sigma} := \nabla^{\epsilon,0}+\sigma$, $\sigma \in \Omega^1(\mathfrak{g},\mathfrak{t})$, is precisely

$$s^*\nabla^{P,0} + (gvg^{-1})^*\sigma - v^*\sigma \in \Omega^1(G \times \mathfrak{g},\mathfrak{t}). \tag{3.154}$$

From the standard formula for parallel transport and Stokes theorem one can deduce that

$$s^*\nabla^{P,0}_{(g,v)}(v_g + \dot{v}) = \int_0^1 F^{P,0}_{(g,tv)}(0+v, v_g + t\cdot v)dt, \tag{3.155}$$

where $F^{P,0} \in \Omega^2(G \times \mathfrak{g},\mathfrak{t})$ is the curvature of $\nabla^{P,0}$. If $F^{\epsilon,0} \in \Omega^2(\mathfrak{g},\mathfrak{t})$ is the curvature of the trivialization ϵ with connection $\nabla^{\epsilon,0}$, then

$$F^{P,0} = (gvg^{-1})^*F^{\epsilon,0} - v^*F^{\epsilon,0} - \langle g^*\theta^L \wedge exp(v)^*\theta^R\rangle - \langle (gexp(v))^*\theta^L \wedge (g^{-1})^*\theta^R\rangle. \tag{3.156}$$

One can then check that the following choice of σ yields the desired covariant derivative.

$$\sigma_v(\dot{v}) = -\int_0^1 tF^{\epsilon,0}_{tv}(v,\dot{v})dt. \tag{3.157}$$

□

Now let $\mathfrak{G}$ be the Lie 2-group corresponding to a multiplicative T-gerbe $\mathcal{G} \to G$ by Theorem 3.48. Then it follows from Remarks 3.33 and 3.49 that $\mathfrak{G}$ has a coherent inversor (Definition 3.19). We also consider the adjoint action of $\mathfrak{G}$ on $\mathfrak{t} \xrightarrow{0} \mathfrak{g}$ given by the adjoint action of G on $\mathfrak{g}$ and the trivial action on $\mathfrak{t}$.

Corollary 3.55. *Let $\mathfrak{G}$ be the Lie 2-group corresponding to a multiplicative T-gerbe $\mathcal{G} \to G$ by Theorem 3.48. Then, a connective structure on $\mathcal{G}$ determines an exponential map, in the sense of Definition 3.24, for $\mathfrak{G}$ equipped with the coherent inversor and the adjoint action above.*

Proof. Choose a trivialization ϵ of $exp^*\mathcal{G} \to \mathfrak{g}$. If $\mathcal{G}$ is given by cocycle data $\lambda_{i_1j_1k_1}$, $m_{i_2j_2}$, α_{i_3} (3.94), then ϵ is given by functions $\epsilon_{i_1j_1} : exp^*U_{i_1j_1} \to T$, where

$$exp^*U_{i_1j_1} := \{v \in \mathfrak{g} \mid exp(v) \in U_{i_1j_1}\}, \tag{3.158}$$

such that $\epsilon_{i_1j_1}\epsilon_{j_1k_1} = \epsilon_{i_1k_1}\lambda_{i_1j_1k_1}(exp(v))$. This determines an anafunctor $exp : \mathfrak{g}//\mathfrak{t} \to \mathfrak{G}$ with total space

$$\begin{aligned} E := \{(i_1, j_1, v, t) \in I_1 \times I_1 \times \mathfrak{g} \times T \mid exp(v) \in U_{i_1j_1}\}/\sim \\ (i_1, j_1, v, t) \sim (i_1', j_1, v, t \cdot \lambda_{i_1'i_1j_1}(exp(v))\epsilon_{i_1'i_1}(v)^{-1}). \end{aligned} \tag{3.159}$$

The anchor maps are $\pi_0([i_1, j_1, v, t]) = v$ and $\pi_1([i_1, j_1, v, t]) = (j_1, exp(v))$, while the action maps are

$$\begin{aligned} \rho_0(u, [i_1, j_1, v, t]) = [i_1, j_1, v, texp(u)], \\ \rho_1([i_1, j_1, v, t], (j_1, k_1, exp(v), t')) = [i_1, k_1, v, t \cdot t' \cdot \lambda_{i_1j_1k_1}(exp(v)]. \end{aligned} \tag{3.160}$$

Since the exponential maps of G and T are local diffeomorphisms, it follows that conditions 2a and 2b from Definition 3.24 are satisfied for the anafunctor exp. Then Remark 3.49 implies that an equivariant structure on ϵ in the sense of Definition 3.53 determines an equivariant structure on exp in the sense of Definition 3.24. Hence, the result follows from Theorem 3.54. □

3.3 Strict Lie 2-groups

3.3.1 Lie crossed modules and adjustments

Definition 3.56 ([19, 55])**.** A Lie 2-group $(\mathfrak{G}, m, \alpha)$ is *strict* if $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ is a smooth functor, the quasi-inverse of $p_1 \times m$ is also a smooth functor and $\alpha = id$. A *strict homomorphism* of Lie 2-groups is a strict smooth functor preserving the product functors. A *Lie crossed module* is a quadruple $(\tilde{G}, H, f, \triangleright)$, where $\tilde{G}$, H are Lie groups, $f : H \to \tilde{G}$ is a smooth homomorphism and $g \triangleright h$ denotes a smooth left action of $\tilde{G}$ on H by automorphisms that satisfies

$$f(g \triangleright h) = gf(h)g^{-1}, \tag{3.161}$$

$$f(h_1) \triangleright h_2 = h_1h_2h_1^{-1}, \tag{3.162}$$

for $g \in \tilde{G}$, h, h_1, $h_2 \in H$. A *strict homomorphism of Lie crossed modules* $(\tilde{G}_1, H_1, f_1, \triangleright_1) \to (\tilde{G}_2, H_2, f_2, \triangleright_2)$ is a pair of group homomorphisms $\tilde{G}_1 \to \tilde{G}_2$, $H_1 \to H_2$ preserving f, $\triangleright$.

Proposition 3.57 ([19, 55]). *The category of strict Lie 2-groups and strict homomorphisms is equivalent to the category of Lie crossed modules and strict homomorphisms.*

Proof. Given a Lie crossed module $(\tilde{G}, H, f, \triangleright)$, construct a strict Lie 2-group $(\mathfrak{G}, m)$ as follows. As a groupoid, $\mathfrak{G} := H\backslash\backslash\tilde{G}$ (cf. Example 3.5), with H acting on $\tilde{G}$ on the left as $h \cdot g = f(h)g$. The product $m : \mathfrak{G} \times \mathfrak{G} \to \mathfrak{G}$ is defined on arrows as the semi-direct product

$$m((g_1, h_1), (g_2, h_2)) = (g_1 g_2, h_1 \cdot g_1 \triangleright h_2). \tag{3.163}$$

The axioms imply that this is a well-defined associative functor. Conversely, given a strict Lie 2-group $(\mathfrak{G}, m)$, the since m is strictly associative it follows that it induces Lie group structures on $\mathfrak{G}_0$ and $\mathfrak{G}_1$, with $s,\, t : \mathfrak{G}_1 \to \mathfrak{G}_0$ smooth group homomorphisms. Let $\tilde{G} := \mathfrak{G}_0$, $H := Ker(s : \mathfrak{G}_1 \to \mathfrak{G}_0)$ and $f : H \to \tilde{G}$ be the restriction of $t : \mathfrak{G}_1 \to \mathfrak{G}_0$. Then $\tilde{G}$ acts on H as $g \triangleright h := m(m(id_g, h), id_{g^{-1}})$ making $(\tilde{G}, G, f, \triangleright)$. This construction can be enhanced to an equivalence of categories, see [19]. $\square$

Let $(\tilde{G}, H, f, \triangleright)$ be a Lie crossed module and let $\mathfrak{G}$ be its corresponding Lie 2-group by Proposition 3.57. We can present its delooping $B\mathfrak{G}_\bullet$ (cf. Section 3.1.2) in terms of the Lie crossed module structure. It is the simplicial manifold with

$$\begin{aligned} B\mathfrak{G}_n := \{(\{g_{ij}\}_{i<j\in[n]}, &\{h_{ijk}\}_{i<j<k\in[n]}) \in \tilde{G}^{\binom{n}{2}} \times H^{\binom{n}{3}} \mid \\ &\forall\, i<j<k \in [n],\; g_{ik} = f(h_{ijk}) g_{ij} g_{jk}, \\ &\forall\, i<j<k<l \in [n],\; h_{ikl} h_{ijk} = h_{ijl} \cdot g_{ij} \triangleright h_{jkl}\} \end{aligned} \tag{3.164}$$

and simplicial maps given by sending a non-decreasing function $f : [n_1] \to [n_2]$ to the function

$$\begin{aligned} f^* : B\mathfrak{G}_{n_2} &\to B\mathfrak{G}_{n_1} \\ (\{g_{ij}\}_{i<j\in[n_2]}, \{h_{ijk}\}_{i<j<k\in[n_2]}) &\mapsto (\{f^* g_{ij}\}_{i<j\in[n_1]}, \{f^* h_{ijk}\}_{i<j<k\in[n_1]}), \end{aligned} \tag{3.165}$$

where

$$f^* g_{ij} := \begin{cases} g_{f(i)f(j)} & \text{if } f(i) < f(j) \\ 1 & \text{if } f(i) = f(j) \end{cases}, \quad f^* h_{ijk} := \begin{cases} h_{f(i)f(j)f(k)} & \text{if } f(i) < f(j) < f(k) \\ 1 & \text{otherwise} \end{cases}. \tag{3.166}$$

The first levels can also be identified with $B\mathfrak{G}_1 = \tilde{G}$, $B\mathfrak{G}_2 = \tilde{G}^2 \times H$, $B\mathfrak{G}_3 = \tilde{G}^3 \times H^3$ with face maps

$$
\begin{aligned}
d_0(g_1, g_2, h) &= g_2, \\
d_1(g_1, g_2, h) &= f(h) g_1 g_2, \\
d_2(g_1, g_2, h) &= g_1, \\
d_0(g_1, g_2, g_3, h_{1,2}, h_{12,3}, h_{2,3}) &= (g_2, g_3, h_{2,3}), \\
d_1(g_1, g_2, g_3, h_{1,2}, h_{12,3}, h_{2,3}) &= (f(h_{1,2}) g_1 g_2, g_3, h_{12,3}), \\
d_2(g_1, g_2, g_3, h_{1,2}, h_{12,3}, h_{2,3}) &= (g_1, f(h_{2,3}) g_2 g_3, h_{12,3} \cdot h_{1,2} \cdot g_1 \triangleright h_{2,3}^{-1}), \\
d_3(g_1, g_2, g_3, h_{1,2}, h_{12,3}, h_{2,3}) &= (g_1, g_2, h_{1,2}).
\end{aligned}
\tag{3.167}
$$

For a Lie crossed module $(\tilde{G}, H, f, \triangleright)$, we will write $\tilde{\mathfrak{g}}$, $\mathfrak{h}$ for the Lie algebras of $\tilde{G}$ and H, respectively. It is clear that the Lie 2-algebra of the Lie 2-group $\mathfrak{G}$ associated to $(\tilde{G}, H, f, \triangleright)$ is the complex $\mathfrak{h} \xrightarrow{f} \tilde{\mathfrak{g}}$, where we abuse notation by writing f for the linearization of $f : H \to \tilde{G}$.

Definition 3.58 ([220])**.** Let $(\tilde{G}, H, f, \triangleright)$ be a crossed module. An *adjustment* on it is a map $\tilde{\kappa} : \tilde{G} \times \tilde{\mathfrak{g}} \to \mathfrak{h}$, linear in $\tilde{\mathfrak{g}}$, such that

$$\tilde{\kappa}(g_1 g_2, v) = g_1 \triangleright \tilde{\kappa}(g_2, v) - \tilde{\kappa}(g_1, f(\kappa(g_2, v))) + \tilde{\kappa}(g_1, Ad(g_2)v), \tag{3.168}$$

$$\tilde{\kappa}(f(h), v) = h \cdot v \triangleright h^{-1}. \tag{3.169}$$

Proposition 3.59. *Let $\mathfrak{G}$ be the Lie 2-group corresponding to a Lie crossed module $(\tilde{G}, H, f, \triangleright)$. Then, an adjustment $\tilde{\kappa} : \tilde{G} \times \tilde{\mathfrak{g}} \to \mathfrak{h}$ defines a left adjoint action of $\mathfrak{G}$ in the sense of Definition 3.23 by*

$$\tilde{g} \cdot v := Ad(\tilde{g})v - f\tilde{\kappa}(\tilde{g}, v), \qquad \tilde{g} \in \tilde{G},\ v \in \tilde{\mathfrak{g}}, \tag{3.170}$$

$$\tilde{g} \cdot u := \tilde{g} \triangleright u - \tilde{\kappa}(\tilde{g}, fu), \qquad \tilde{g} \in \tilde{G},\ u \in \mathfrak{h}, \tag{3.171}$$

and a right-invariant Maurer-Cartan form, where

$$\theta^0 \in \Omega^1(\tilde{G}, \tilde{\mathfrak{g}}),\ \theta^1 \in \Omega^1(\tilde{G}^2 \times H, \mathfrak{h})$$

are defined by

$$\theta^0_g(v_g) := v_g g^{-1}, \tag{3.172}$$

$$\theta^1_{(g_1, g_2, h)}(v_{g_1} + v_{g_2} + v_h) := v_h \cdot (v_{g_1} v_{g_2} g_2^{-1} g_1^{-1}) \triangleright h^{-1} + \tilde{\kappa}(g_1, v_{g_2} g_2^{-1}). \tag{3.173}$$

Proof. It follows from a straightforward computation using axioms (3.168), (3.169) and the description in (3.167) of the face maps of $B\mathfrak{G}_\bullet$. □

Remark 3.60. In the setting of Proposition 3.59, we can also define a right adjoint action by $v \cdot \tilde{g} = \tilde{g}^{-1} \cdot v$. In this case, the following 1-forms define a left-invariant Maurer-Cartan form.

$$\theta^0_g(v_g) := g^{-1} v_g, \tag{3.174}$$

$$\theta^1_{(g_1,g_2,h)}(v_{g_1} + v_{g_2} + v_h) := g_2^{-1} g_1^{-1} \triangleright h^{-1} v_h + \tilde{\kappa}(g_2^{-1}, g_1^{-1} v_{g_1}). \tag{3.175}$$

Example 3.61. The categorical tori from Example 3.14 can be described as Lie crossed modules by letting $\tilde{G} := V_0$, $H := \Lambda_0 \times V_1/\Lambda_1$ and

$$f(\lambda_0, [v_1]) := \lambda_0, \tag{3.176}$$

$$u_0 \triangleright (\lambda_0, [u_1]) := (\lambda_0, [u_1 + \langle u_0, \lambda_0 \rangle]). \tag{3.177}$$

A canonical adjustment is given by $\kappa(u_0, v_0) := -\langle u_0, v_0 \rangle$.

Example 3.62. For G a compact, simple, simply connected Lie group, the Lie 2-group associated to the multiplicative gerbe String(G) from Example 3.37 admits models as an infinite-dimensional Lie crossed module. The first such model was constructed in [18], and is equipped with an adjustment in [220]. A simplified model is presented in [180].

3.3.2 Central Lie crossed modules

Definition 3.63 ([206])**.** A *central Lie crossed module* is a Lie crossed module $(\tilde{G}, H, f, \triangleright)$ such that the induced action of $G := \tilde{G}/Im(f)$ on $T := Ker(f)$ by $\triangleright$ is trivial.

If $\mathfrak{G}$ is the Lie 2-group associated to a central Lie crossed module $(\tilde{G}, H, f, \triangleright)$, then $\mathfrak{G}$ is a central extension of G by BT in the sense of Section 3.2.4. Thus, if G is a Lie group, then Theorem 3.48 implies that there is also a model for $\mathfrak{G}$ as a multiplicative T-gerbe over G. As discussed in [206], such model can be explicitly presented (with the language of *bundle gerbes*, see Section 3.2.1) in terms of the crossed module structure as follows.

First, in order to give a T-gerbe over G we use the surjective submersion $\tilde{G} \to G$ and we define a T-bundle $L \to \tilde{G} \times_G \tilde{G}$ by $L := \tilde{G} \times H$, with projection $(g, h) \mapsto (g, f(h)g)$ and T acting on H through the group multiplication; then there is a canonical isomorphism $p_{12}^* L \otimes p_{23}^* L \to p_{13}^* L$ over $\tilde{G} \times_G \tilde{G} \times_G \tilde{G}$ because

$$p_{13}^* L = \{(g, g', g'', h^{0,''}) \in \tilde{G}^3 \times H \mid g'' = f(h^{0,''})g\}$$
$$p_{12}^* L \otimes p_{23}^* L = \{(g, g', g'', h^{0,'}, h^{','' }) \in \tilde{G}^3 \times H^2 \mid g' = f(h^{','' })g,\ g'' = f(h^{0,'})g\}/\sim$$

with $(h^{0,'}, h^{','' }) \sim (th^{0,'}, t^{-1}h^{','' })$ for $t \in T$ and so we may define $[h^{0,'}, h^{','' }] \mapsto h^{','' }h^{0,'}$. This completes the construction of the bundle gerbe $\mathcal{G} \to G$. Now, in order to give an isomorphism $g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \to (g_1 g_2)^*\mathcal{G}$, we cover $G \times G$ by $\tilde{G} \times \tilde{G}$ and give an isomorphism between the T-bundles over $(\tilde{G} \times \tilde{G}) \times_{G\times G} (\tilde{G} \times \tilde{G})$ that describe the gerbes $g_1^*\mathcal{G} \otimes g_2^*\mathcal{G}$ and $(g_1 g_2)^*\mathcal{G}$; these are

$$\begin{aligned} \{(g_1, g_2, g_1', g_2', h_1, h_2) \in \tilde{G}^4 \times H^2 \mid g_1' = f(h_1)g_1,\ g_2' = f(h_2)g_2\}/\sim, \\ \{(g_1, g_2, g_1', g_2', h_{12}) \in \tilde{G}^4 \times H \mid g_1'g_2' = f(h_{12})g_1g_2\}, \end{aligned}$$

respectively, with the equivalence relation $(h_1, h_2) \sim (th_1, t^{-1}h_2)$ for $t \in T$. The isomorphism is then $(g_1, g_2, g_1', g_2', [h_1, h_2]) \mapsto (g_1, g_2, g_1', g_2', h_1 \cdot g_1 \triangleright h_2)$. For this to be well-defined we use that $g \triangleright t = t$ and for it to be an isomorphism of gerbes we have to check that it is compatible over $(\tilde{G} \times \tilde{G}) \times_{G\times G} (\tilde{G} \times \tilde{G}) \times_{G\times G} (\tilde{G} \times \tilde{G})$ with the gerbe product; this reduces to proving

$$h_1^{','' } h_1^{0,'} g_1 \triangleright (h_2^{','' }, h_2^{0,'}) = h_1^{','' } g_1' \triangleright h_2^{','' } \cdot h_1^{0,'} \cdot g_1 \triangleright h_2^{0,'} \tag{3.178}$$

when $g_1' = f(h_1^{0,'})g_1$, which follows from the axioms. To conclude the construction of a multiplicative T-gerbe over G, it only remains to give the 2-isomorphism α but in this case we can simply take the identity, which follows essentially from

$$h_1 \cdot g_1 \triangleright h_2 \cdot (g_1 g_2) \triangleright h_3 = h_1 \cdot g_1 \triangleright (h_2 \cdot g_2 \triangleright h_3). \tag{3.179}$$

We proceed to characterize the category of connective structures on this multiplicative gerbe in terms of the adjustments from Section 3.3.1.

Definition 3.64. Let $(\tilde{G}, H, f, \triangleright)$ be a central Lie crossed module and let $G := \tilde{G}/Im(f)$, $T := Ker(f)$. A *strong adjustment* on $(\tilde{G}, H, f, \triangleright)$ is a pair (s, κ), where

1. $s : \mathfrak{h} \to \mathfrak{t}$ is a linear splitting of the inclusion map $\mathfrak{t} \to \mathfrak{h}$.

2. $\kappa : \tilde{G} \times \tilde{\mathfrak{g}} \to \mathfrak{t}$ is linear on $\tilde{\mathfrak{g}}$ and satisfies

$$\kappa(g_1 g_2, v) = \kappa(g_2, Ad(g_1^{-1})v) + \kappa(g_1, v), \tag{3.180}$$

$$\kappa(f(h), v) = s(h^{-1} \cdot v \triangleright h), \tag{3.181}$$

$$\kappa(g, f(u)) = s(g^{-1} \triangleright u - u). \tag{3.182}$$

Given two strong adjustments (s_1, κ_1), (s_2, κ_2), an *isomorphism* between them is a linear map $\phi : \tilde{\mathfrak{g}} \to \mathfrak{t}$ such that $s_2(u) - s_1(u) = \phi f(u)$ and $\kappa_2(g, v) - \kappa_1(g, v) = \phi(Ad(g^{-1})v - v)$.

Remark 3.65. Given a strong adjustment (s, κ) in a central Lie crossed module $(\tilde{G}, H, f, \triangleright)$, then we may obtain an adjustment $\tilde{\kappa}$ in the sense of Definition 3.58 as follows. Consider the exact sequence of Lie algebras

$$0 \to \mathfrak{t} \to \mathfrak{h} \xrightarrow{f} \tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0 \tag{3.183}$$

and the induced short exact sequences

$$0 \to \mathfrak{t} \to \mathfrak{h} \xrightarrow{f} Im(f) \to 0, \tag{3.184}$$

$$0 \to Im(f) \to \tilde{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0. \tag{3.185}$$

Choose a linear splitting $l : \mathfrak{g} \to \tilde{\mathfrak{g}}$ of (3.185) and note that there is a unique linear map $r : \tilde{\mathfrak{g}} \to \mathfrak{h}$ such that

$$0 \leftarrow \mathfrak{t} \xleftarrow{s} \mathfrak{h} \xleftarrow{r} \tilde{\mathfrak{g}} \xleftarrow{l} \mathfrak{g} \leftarrow 0. \tag{3.186}$$

is exact. Then one can easily check that $\tilde{\kappa}(g, v) := \kappa(g^{-1}, v) + r(Ad(g^{-1})v - v)$ is an adjustment. In fact, any adjustment $\tilde{\kappa}$ such that $\tilde{\kappa}(g, f(u)) = g^{-1} \triangleright u - u$ and such that there exist splittings s, l of (3.184), (3.185) with $rf\tilde{\kappa}(g, v) = r(Ad(g^{-1})v - v)$ arises from a strong adjustment. All the explicit adjustments in [166] and [220] satisfy these conditions.

Proposition 3.66. *Let $(\tilde{G}, H, f, \triangleright)$ be a central Lie crossed module such that $G := \tilde{G}/Im(f)$ is a Lie group, and let $\mathcal{G}$ be the corresponding multiplicative T-gerbe over G. Then the category of connective structures on $\mathcal{G}$ is equivalent to the category of strong adjustments on $(\tilde{G}, H, f, \triangleright)$.*

Proof. By unwinding the definitions in this case we see that a connective structure on $\mathcal{G}$ is precisely the data of $\nabla \in \Omega^1(\tilde{G} \times H, \mathfrak{t})$ and $\tau \in \Omega^1(\tilde{G} \times \tilde{G}, \mathfrak{t})$ satisfying the following relations (here $\theta^T \in \Omega^1(T, \mathfrak{t})$ is the Maurer-Cartan form on T):

$$\begin{gathered}
(g, ht)^*\nabla - (g, h)^*\nabla = t^*\theta^T, \\
(g, h)^*\nabla + (f(h)g, h')^*\nabla = (g, h'h)^*\nabla, \\
(f(h_1)g_1, f(h_2)g_2)^*\tau - (g_1, g_2)^*\tau = (g_1g_2, h_1g_1 \triangleright h_2)^*\nabla - (g_1, h_1)^*\nabla - (g_2, h_2)^*\nabla, \\
(g_1, g_2)^*\tau + (g_1g_2, g_3)^*\tau = (g_1, g_2g_3)^*\tau + (g_2, g_3)^*\tau
\end{gathered} \tag{3.187}$$

Moreover, two such connective structures (∇, τ) and (∇', τ') are isomorphic whenever there exists $\sigma \in \Omega^1(\tilde{G}, \mathfrak{t})$ such that

$$\begin{aligned} \nabla' - \nabla &= (f(h)g)^*\sigma - g^*\sigma, \\ \tau' - \tau &= (g_1 g_2)^*\sigma - g_1^*\sigma - g_2^*\sigma. \end{aligned} \tag{3.188}$$

Then let (s, κ) be a strong adjustment. We claim that $\nabla^s_{(g,h)}(v_g + v_h) := s(g^{-1} \triangleright h^{-1} v_h)$, $\tau^{s,\kappa}_{(g_1,g_2)}(v_{g_1} + v_{g_2}) := \kappa(g_2, g_1^{-1} v_{g_1})$ defines a connective structure on the multiplicative gerbe, which boils down to straightforward computations. Similarly, one checks that if ϕ is an isomorphism $(s_1, \kappa_1) \to (s_2, \kappa_2)$ then $\sigma^\phi_g(v_g) := \phi(g^{-1} v_g)$ is an isomorphism $(\nabla^{s_1}, \tau^{s_1,\kappa_1}) \to (\nabla^{s_2}, \tau^{s_2,\kappa_2})$. In fact, if $\sigma : (\nabla^{s_1}, \tau^{s_1,\kappa_1}) \to (\nabla^{s_2}, \tau^{s_2,\kappa_2})$ is an arbitrary isomorphism, then $\phi^\sigma(v) := \sigma_1(v)$ is an isomorphism $(s_1, \kappa_1) \to (s_2, \kappa_2)$ and $\sigma = \sigma^{\phi^\sigma}$, so we have defined a fully faithful functor from the category of strong adjustments to the category of connective structures. It is also essentially surjective: since any two connective structures on a gerbe are always isomorphic, we may restrict our attention to those (∇, τ) such that $\nabla = \nabla^s$ for a given splitting $s : \mathfrak{h} \to \mathfrak{t}$. Then one can check that for $\kappa(g, v) := \tau_{(g^{-1},1)}(0+v) + \tau_{(g^{-1},g)}(g^{-1}v+0)$ we have that (s, κ) is a strong adjustment with (∇^s, τ^κ) isomorphic to (∇^s, τ). □

Remark 3.67. Proposition 3.66 restricted to the case of the crossed module $(G, T, f, \triangleright)$ with trivial f and trivial $\triangleright$ coincides precisely with the isomorphism $H^2(BG_\bullet, \Omega^1_{\mathfrak{t}}) \to H^1_{gr,cont}(G, \mathfrak{g}^* \otimes \mathfrak{t})$ from Lemma 2.14.

In particular, Proposition 3.66 and Theorem 3.43 imply that a strong adjustment (s, κ) on a central Lie crossed module $(\tilde{G}, H, f, \alpha)$ gives an Ad-invariant pairing $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{t}$ and a Maurer-Cartan curving $\Theta^L \in \Omega^2(\tilde{G}, \mathfrak{t})$, which we can compute using the formulas in Remark 3.44. For this we write first $\partial_g \kappa : T_g\tilde{G} \times \tilde{\mathfrak{g}} \to \mathfrak{t}$ for the partial derivative of κ at $g \in \tilde{G}$, whose main properties are

$$\begin{aligned} \partial_g \kappa(v_g, v) &= \partial_1 \kappa(g^{-1} v_g, Ad(g^{-1})v), \\ \partial_1 \kappa(Ad(g^{-1})u, Ad(g^{-1})v) &= \partial_1 \kappa(u, v) - \kappa(g, [u, v]), \\ \partial_1 \kappa(f(u), v) &= s(v \triangleright u) = -\partial_1 \kappa(v, f(u)). \end{aligned} \tag{3.189}$$

Here we are writing $v \triangleright u := \frac{d}{dt}_{|t=0} exp(tv) \triangleright u$, $v \in \tilde{\mathfrak{g}}$, $u \in \mathfrak{h}$ for the Lie algebra action of $\tilde{\mathfrak{g}}$ on $\mathfrak{h}$, which satisfies $f(v \triangleright u) = [v, f(u)]$ and $f(u_1) \triangleright u_2 = [u_1, u_2]$. Then,

$$\langle u, v \rangle = \frac{1}{2}(\partial_1 \kappa(u, v) + \partial_1 \kappa(v, u)) \tag{3.190}$$

$$\Theta^L_g(u_g, v_g) = -\frac{1}{2}(\partial_1 \kappa(g^{-1} u_g, g^{-1} v_g) - \partial_1 \kappa(g^{-1} v_g, g^{-1} u_g)). \tag{3.191}$$

Although $\partial_1\kappa$ is in principle defined over $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$, (3.190) is well-defined over $\mathfrak{g} \otimes \mathfrak{g}$ by (3.189). Note that $\langle\cdot,\cdot\rangle$ and $-\Theta_1^L$ are the symmetric and skew-symmetric parts of the tensor $\partial_1\kappa$. In terms of the Maurer-Cartan *enhanced* curving we obtain the simple formula

$$\Theta_g^{L,en}(u_g, v_g) = -\partial_1\kappa(g^{-1}u_g, g^{-1}v_g) = -\partial_g\kappa(u_g, v_g g^{-1}). \tag{3.192}$$

3.3.3 The exponential map of a central Lie crossed module

The proof of Theorem 3.54 can be made more explicit for a multiplicative gerbe arising from a central Lie crossed module.

Theorem 3.68. *Let $(\tilde{G}, H, f, \triangleright)$ be a central Lie crossed module such that $G := \tilde{G}/Im(f)$ is a Lie group, let (s,κ) be a strong adjustment on it, and let $l : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be a section of $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$. Then there exists a function $\chi : \tilde{G} \times \mathfrak{g} \to H$ satisfying*

$$\chi(g_1g_2, v) = g_2^{-1} \triangleright \chi(g_1, v) \cdot \chi(g_2, Ad(g_1^{-1})v), \tag{3.193}$$

$$\chi(f(h), v) = h^{-1} \cdot exp(l(v)) \triangleright h, \tag{3.194}$$

$$f\chi(g, v) = exp(Ad(g^{-1})l(v))exp(lAd(g^{-1})v))^{-1}, \tag{3.195}$$

for g, g_1, $g_2 \in \tilde{G}$, $h \in H$, $v \in \mathfrak{g}$. There also exists $\sigma \in \Omega^1(\mathfrak{g}, \mathfrak{t})$ such that

$$\begin{aligned}
&\kappa\left(g, exp^*\theta^L_{l(v)}(l(\dot{v}))\right) - \kappa\left(g^{-1}exp(l(v))g, l(g^{-1}v_g)\right) + 2\langle v_g g^{-1}, v\rangle \\
&= s\left((g^{-1}exp(-l(v))g) \triangleright \frac{d}{dt}_{t=0}\chi(exp(tl(v_g g^{-1}))g, v + t\dot{v})\chi(g, v)^{-1}\right) \\
&\quad + \sigma_{Ad(g^{-1})v}\left(Ad(g^{-1})\dot{v} - [g^{-1}v_g, Ad(g^{-1})v]\right) - \sigma_v(\dot{v})
\end{aligned} \tag{3.196}$$

for $g \in \tilde{G}$, $v_g \in T_g\tilde{G}$, v, $\dot{v} \in \mathfrak{g}$, and where $\langle\cdot,\cdot\rangle$ is defined by (3.190).

Proof. For $g \in \tilde{G}$ and $v \in \mathfrak{g}$ define the curve

$$\begin{aligned}
\gamma_{g,v} : [0,\infty) &\to \tilde{G} \times_G \tilde{G} \\
t &\mapsto \left(exp(tl(g^{-1}vg)), exp(tg^{-1}l(v)g)\right).
\end{aligned} \tag{3.197}$$

Then let $\gamma^s_{g,v} : [0,\infty) \to H$ be defined by parallel transport of $\gamma_{g,v}$ with respect to the connection ∇^s from the proof of Proposition 3.66 on the T-bundle $\tilde{G} \times H \to \tilde{G} \times_G \tilde{G}$.

That is, $\gamma^s_{g,v}$ is the unique solution to the following ODE.

$$s\left(g^{-1}exp(-tl(v))g \triangleright \frac{d}{d\epsilon}_{\epsilon=0}\gamma^s_{g,v}(t+\epsilon)\gamma^s_{g,v}(t)^{-1}\right) = 0, \tag{3.198}$$

$$exp(tg^{-1}l(v)g)exp(tl(g^{-1}vg))^{-1} = f\gamma^s_{g,v}(t), \tag{3.199}$$

$$1 = \gamma^s_{g,v}(0). \tag{3.200}$$

Then we claim that $\chi(g,v) := \gamma^s_{g,v}(1)\cdot exp(\kappa(g,l(v)))$ satisfies the desired properties. This is because

$$\gamma^s_{g_1g_2,v}(t) = g_2^{-1}\triangleright\chi(g_1,tv)\cdot\chi(g_2,Ad(g_1^{-1})tv)\cdot exp(-t\kappa(g_1g_2,l(v))), \tag{3.201}$$

$$\gamma^s_{f(h),v}(t) = h^{-1}\cdot exp(tl(v))\triangleright h\cdot exp(-t\kappa(f(h),l(v))), \tag{3.202}$$

as it can be shown by checking that the right-hand sides of (3.201) and (3.202) satisfy their corresponding ODEs. We proceed to construct σ. Using (3.190) and our expression for χ, we see that (3.196) at $g=1$, $v_g=u$ is equivalent to

$$\begin{aligned}\sigma_v([u,v]) &= \kappa(exp(l(v)),l(u)) - \partial_1\kappa(l(v),l(u))\\ &+ s\left(exp(-l(v))\triangleright\frac{d}{d\epsilon}_{|\epsilon=0}\gamma^s_{exp(\epsilon u),v}(1)\right).\end{aligned} \tag{3.203}$$

Define, for $u\in\tilde{\mathfrak{g}}$ and $v\in\mathfrak{g}$,

$$\begin{aligned}h_{u,v}(t) &:= \kappa(exp(tl(v)),l(u)),\\ \rho_{u,v}(t) &:= \frac{d}{d\epsilon}_{\epsilon=0}\gamma^s_{exp(\epsilon l(u)),v}(t).\end{aligned} \tag{3.204}$$

Using (3.189) we obtain

$$\dot{h}_{u,v}(t) = \partial_1\kappa(l(v),Ad^{-1}(exp(tl(v)))l(u)), \tag{3.205}$$

while differentiating the ODE for $\gamma^s_{g,v}$ yields the following ODE for $\rho_{u,v}$.

$$s\left(exp(-tl(v))\triangleright\dot{\rho}_{u,v}\right) = 0, \tag{3.206}$$

$$Ad(exp(tl(v)))l(u) - l(u) + exp^*\theta^R_{tl(v)}(tl[u,v]) = f\rho_{u,v}(t), \tag{3.207}$$

$$0 = \rho_{u,v}(0). \tag{3.208}$$

Using the fundamental theorem of calculus we rewrite (3.203) as

$$\begin{aligned}\sigma_v([u,v]) &= \int_0^1 \dot{h}_{u,v}(t)dt - \partial_1\kappa(l(v),l(u))\\ &+ s\left(\int_0^1 (exp(-tl(v))\triangleright\dot{\rho}_{u,v}(t) - l(v)\triangleright exp(-tl(v))\triangleright\rho_{u,v}(t))\right)\\ &= \int_0^1 \partial_1\kappa(l(v), Ad^{-1}(exp(tl(v)))l(u) - l(u))\\ &+ \int_0^1 \partial_1\kappa(l(v), l(u) - Ad^{-1}(exp(tl(v)))l(u) + exp^*\theta^L_{tl(v)}(tl[u,v])),\end{aligned} \tag{3.209}$$

where in the last step we have used (3.205), (3.206) and (3.189). Thus, if we define σ

$$\sigma_v(\dot{v}) := \int_0^1 \partial_1\kappa\left(l(v), exp^*\theta^L_{\xi l(v)}(\xi l(\dot{v}))\right)d\xi, \tag{3.210}$$

then (3.196) is satisfied at $g=1$. Using the properties of κ, $\langle\cdot,\cdot\rangle$ and χ one can show that it is now sufficient to prove that (3.196) is also satisified for arbitrary g and $v_g = 0$ to conclude the proof. That is, we only need to show that σ defined by (3.210) satisfies

$$\sigma_{Ad(g^{-1})v}(Ad(g^{-1})\dot{v}) - \sigma_v(\dot{v}) = \kappa\left(g, exp^*\theta^L_{l(v)}(l(\dot{v})) - l(\dot{v})\right) - s\left(g^{-1}exp(-l(v))g\triangleright\nu(1)\right), \tag{3.211}$$

where ν is defined by

$$\nu(t) := \frac{d}{d\epsilon}_{\epsilon=0}\gamma^s_{g,v+\epsilon\dot{v}}(t)\gamma^s_{g,v}(t)^{-1}. \tag{3.212}$$

Differentiating the ODE for $\gamma^{g,v}$, and using (3.189) and $d\theta^L = -\frac{1}{2}[\theta^L\wedge\theta^L]$, we obtain

$$\begin{aligned}&s\left(g^{-1}exp(-tl(v))g\triangleright\dot{\nu}(t)\right) = s[(\gamma^s_{g,\cdot}(\cdot))_*(\dot{v}), (\gamma^s_{g,\cdot}(\cdot))_*(\partial_t)]_{(t,v)}\\ &-\partial_1\kappa\big(Ad(g^{-1})exp^*\theta^L_{tl(v)}(tl(\dot{v})), Ad(g^{-1})l(v) - l(Ad(g^{-1})v)\big)\end{aligned} \tag{3.213}$$

$$\begin{aligned}&f\nu(t) =\\ &Ad(g^{-1})exp^*\theta^R_{tl(v)}(tl(\dot{v})) - Ad(g^{-1}exp(tl(v))g)exp^*\theta^L_{tl(Ad(g^{-1})v)}(tl(Ad(g^{-1})\dot{v})),\end{aligned} \tag{3.214}$$

where $(\gamma^s_{g,\cdot}(\cdot))_*(\dot{v})$ and $(\gamma^s_{g,\cdot}(\cdot))_*(\partial_t)$ stand for the vector fields on H obtained through push-forward by $\gamma^s_{g,\cdot}(\cdot): \mathfrak{g}\times[0,\infty)\to H$ of $\dot{v}\in\Gamma(T\mathfrak{g})$ and $\partial_t\in\Gamma(T[0,\infty))$. We can rewrite this ODE by noting that, for u_1, $u_2\in\mathfrak{h}$, we have $s[u_1,u_2] = s[rf(u_1), rf(u_2)] = \partial_1\kappa(f(u_1), f(u_2))$ and so

$$\begin{aligned}&s\left(g^{-1}exp(-tl(v))g\triangleright\dot{\nu}(t)\right) =\\ &\quad-\partial_1\kappa\big(exp^*\theta^L_{tl(Ad(g^{-1}v)}(tl(Ad(g^{-1}\dot{v})), Ad(g^{-1})l(v) - l(Ad(g^{-1})v)\big).\end{aligned} \tag{3.215}$$

Then we use the fundamental theorem of calculus to write

$$\begin{aligned}
& s\Big(g^{-1}exp(-l(v))g \triangleright \nu(1)\Big) \\
&\qquad = \int_0^1 s\left(g^{-1}exp(-tl(v))g \triangleright \dot{\nu}(t) - g^{-1}l(v)g \triangleright g^{-1}exp(tl(v))^{-1}g \triangleright \nu(t)\right) dt \\
&\qquad = \int_0^1 -\partial_1\kappa\big(exp^*\theta^L_{tl(Ad(g^{-1})v)}(tl(Ad(g^{-1})\dot{v})), Ad(g^{-1})l(v) - l(Ad(g^{-1})v)\big) dt \\
&\qquad + \int_0^1 \partial_1\kappa(Ad(g^{-1})l(v), Ad(g^{-1})exp^*\theta^L_{tl(v)}(tl(\dot{v})) - exp^*\theta^L_{tlAd(g^{-1})v}(tlAd(g^{-1})\dot{v}))dt \\
&\qquad = \int_0^1 \partial_1\kappa(l(Ad(g^{-1})v), -exp^*\theta^L_{tlAd(g^{-1})v}(tlAd(g^{-1})\dot{v}))dt \\
&\qquad + \int_0^1 \partial_1\kappa(l(v), exp^*\theta^L_{tl(v)}(tl(\dot{v})))dt - \int_0^1 \kappa(g, [l(v), exp^*\theta^L_{tl(v)}(tl(\dot{v}))])dt.
\end{aligned} \tag{3.216}$$

Finally, we use the standard formula

$$exp^*\theta^L_v(\dot{v}) = \int_0^1 Ad(exp(-\xi v))\dot{v}d\xi \tag{3.217}$$

to show that

$$\begin{aligned}
-\int_0^1 [l(v), exp^*\theta^L_{tl(v)}(tl(\dot{v}))]dt &= \int_0^1\int_0^1 [-tl(v), Ad(exp(-t\xi l(v)))l(\dot{v})]d\xi dt \\
&= \int_0^1 [Ad(exp(-t\xi l(v)))l(\dot{v})]_{\xi=0}^{\xi=1}\, dt \\
&= \int_0^1 Ad(exp(-tl(v)))l(\dot{v}) - l(\dot{v}) \\
&= exp^*\theta^L_{l(v)}(l(\dot{v})) - l(\dot{v}),
\end{aligned} \tag{3.218}$$

which concludes the proof. □

Let $(\tilde{G}, H, f, \triangleright)$ be a central Lie crossed module such that $G := \tilde{G}/Im(f)$ is a Lie group, let (s, κ) be a strong adjustment on it and let $\mathcal{G}_\nabla \to G$ be the corresponding multiplicative gerbe with connective structure by Proposition 3.66. We use the data of Theorem 3.68 to construct a trivialization ϵ of $exp^*\mathcal{G}_\nabla \to \mathfrak{g}$ with an equivariant structure α^ϵ whose covariant derivative is as in Theorem 3.54.

The gerbe $exp^*\mathcal{G}_\nabla \to \mathfrak{g}$ is described by covering $\mathfrak{g}$ with

$$exp^*\tilde{G} := \{(g_v, v) \in \tilde{G} \times \mathfrak{g} \mid \pi(g_v) = exp(v)\} \tag{3.219}$$

and taking the T-bundle

$$\begin{aligned} exp^*L := \{(g_v, h, v) \in \tilde{G} \times G \times \mathfrak{g} \,|\pi(g_v) = exp(v)\} \to exp^*\tilde{G} \times_{\mathfrak{g}} exp^*\tilde{G} \\ (g_v, h, v) \mapsto ((g_v, v), (f(h)g_v, v)), \end{aligned} \tag{3.220}$$

with the obvious isomorphisms $p_{12}^*L \otimes p_{23}^*L \to p_{13}^*L$ defined as in Section 3.3.2, and the connection $exp^*\nabla^s_{(g_v,h,v)}(v_g + v_h + \dot{v}) := s(g_v^{-1} \triangleright h^{-1}v_h)$ from the proof of Proposition 3.66. We trivialize this by defining the following T-bundle with connection over $exp^*\tilde{G}$

$$\begin{aligned} E := H \times \mathfrak{g} \to exp^*\tilde{G}, \quad (h, v) \mapsto (f(h)exp(l(v)), v), \\ \nabla_{(h,v)}(v_h + \dot{v}) := s(exp(l(v))^{-1} \triangleright h^{-1}v_h) + \sigma_v(\dot{v}), \end{aligned} \tag{3.221}$$

where σ is the 1-form in Theorem 3.68, and the following isomorphism of T-bundles with connection over $exp^*\tilde{G} \times_{\mathfrak{g}} exp^*\tilde{G}$.

$$\begin{aligned} p_1^*E \otimes L \xrightarrow{\epsilon} p_2^*E \\ (h_E, v) \otimes (g_v, h_L, v) \mapsto (h_L h_E, v). \end{aligned} \tag{3.222}$$

It is easy to check that this behaves well with respect to $p_{12}^*L \otimes p_{23}^*L \to p_{13}^*L$. Then over $G \times \mathfrak{g}$ we have two trivializations of $exp^*\mathcal{G}_\nabla$:

1. One is given simply by pulling-back (E, ϵ) through the projection $G \times \mathfrak{g} \to \mathfrak{g}$.
2. The other one is given by the T-bundle $E^{Ad} \to \tilde{G} \times exp^*\tilde{G}$

   $$\begin{aligned} E^{Ad} := \{(g_v, h, g, v) \in & \tilde{G} \times H \times \tilde{G} \times \mathfrak{g} \,| \\ & \pi(g_v) = exp(v),\; f(h)exp(l(Ad(g)v)) = Ad(g)g_v\} \end{aligned} \tag{3.223}$$

   with projection map $(g_v, h, g, v) \mapsto (g, g_v, v)$, equipped with the connection

   $$\begin{aligned} \nabla_{(g_v,h,g,v)} = {} & (exp(l(Ad(g)v)), h)^*\nabla^s - (g, g_v)^*\tau^{s,\kappa} - (gg_v, g^{-1})^*\tau^{s,\kappa} \\ & + (g, g^{-1})^*\tau^{s,\kappa} + (Ad(g)v)^*\sigma \end{aligned} \tag{3.224}$$

   for $\tau^{s,\kappa}$ as in the proof of Proposition 3.66, and the following isomorphism of T-bundles with connection over $\tilde{G} \times_G \tilde{G} \times exp^*\tilde{G} \times_{\mathfrak{g}} exp^*\tilde{G}$.

   $$\begin{aligned} p_1^*E^{Ad} \otimes L \xrightarrow{\epsilon^{Ad}} p_2^*E^{Ad} \\ (g_v, h_{E^{Ad}}, g, g', v) \otimes (g_v, h_L, v) \mapsto (f(h_L)g_v, h', g, g', v), \\ h' := h_0 \cdot g \triangleright h_L \cdot h_{E^{Ad}} \cdot exp(l(Ad(g)v)) \triangleright h_0^{-1}, \end{aligned} \tag{3.225}$$

   where $h_0 \in H$ is any element with $g' = f(h_0)g$.

Then we define an equivariant structure α^ϵ on the trivialization ϵ by the following isomorphism of T-bundles over $\tilde{G} \times exp^*\tilde{G}$.

$$\begin{aligned} E &\to E^{Ad} \\ (g, h_E, g_v, v) &\mapsto (g_v, g \triangleright h_E \cdot \chi(g^{-1}, v), g, v). \end{aligned} \tag{3.226}$$

The properties of χ in Theorem 3.68 ensure that this is indeed an isomorphism of trivializations of $exp^*\mathcal{G}$, and the property of σ means that its covariant derivative is $\eta \in \Omega^1(G \times \mathfrak{g}, \mathfrak{t})$, $\eta_{(g,v)}(v_g + \dot{v}) = -2\langle v_g g^{-1}, v\rangle$.

Chapter 4

Principal 2-bundles and Courant algebroids

Some fields in string theory and supergravity [38, 81], or more generally in two-dimensional sigma-models, are described by local potential 2-forms with values in the Lie algebra of an abelian Lie group T, their symmetries are given by 1-forms, and these have themselves 'higher' symmetries given by functions to T. Moreover, these fields may interact non-trivially with classical gauge fields for a non abelian Lie group G. Based on observations from [243], it was proposed in [52] that splittings of certain transitive Courant algeboids could provide a mathematical model for these fields. This approach, later expanded in [24, 120, 121], has been very fruitful, yielding models for T-Duality and for the construction of moduli spaces of solutions to string-theoretic equations both in the mathematical [84, 125, 127] and the physical [13, 14] literature.

However, using Courant algebroids for modelling these fields yields a problem: the natural symmetries of these fields that are dictated by physics form a 2-group, and there is no way to construct such 2-group just from the data of the Courant algebroid. This is similar to how the gauge group of a G-bundle P cannot be recovered from just the data of its Atiyah algebroid TP/G. In this sense, splittings of Courant algebroids are only a shadow of the actual physical fields, which physics suggests should be seen as some kind of 'connections' on bundles whose fibers are isomorphic to some Lie 2-group $\mathfrak{G}$ of the form $1 \to BT \to \mathfrak{G} \to G \to 1$. There are now many models for the type of Lie 2-groups that appear in physics, among which we can mention [18, 62, 149, 180, 238, 272].

When $G = \{*\}$, mathematical formalizations of this idea can be traced back to [131], which uses Deligne cohomology as a model, and its reinterpretation in terms of gerbes from [77, 118, 132]. The next step was to develop an analogous theory for strict Lie 2-groups. For these, the original notion of connection that was proposed in [12, 16, 20] turned out to be problematic, as it needs to impose a condition called *fake flatness* for consistency of higher gauge transformations, and this condition renders the theory essentially abelian [129].

Based on the original physics literature, it was noticed in [235, 237] that this problem can be solved for the String(G) 2-groups, as in this case the notion of connection can be modified by using the existence of an additional structure in their Lie 2-algebras, which they call *Chern-Simons terms*. At around the same time, [273] developed an equivalent notion of connection on String(G)-bundles, also based on the relevance of Chern-Simons forms in the physics literature and in the work of Stolz-Teichner [253], but using the notion of multiplicative gerbes from [78] as its mathematical formalism. Some years later, [245] related these approaches to the one based on Courant algebroids, generalizing a well-known relation between gerbes and exact Courant algebroids [152, 243].

It is also clear in [273] that its approach can be extended to define connections on $\mathfrak{G}$-bundles, whenever $\mathfrak{G}$ is the Lie 2-group corresponding to the multiplicative gerbe constructed from an Ad-invariant, symmetric bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{t}$ as in Example 3.39. Our Theorem 3.43 characterizes such multiplicative gerbes as precisely those that admit a connective structure. Similarly, but for a different family of Lie 2-groups, the notion of an adjustment from [167, 220, 230] (cf. Section 3.3.1) abstracts the work of [235, 237] to characterize the data that a strict Lie 2-group $\mathfrak{G}$ must have to yield a good notion of connection on $\mathfrak{G}$-bundles. Recall that in Section 3.3.2 we proved that connective structures and adjustments are essentially equivalent for central Lie crossed modules.

In this chapter we present principal 2-bundles and we unify all the above approaches to the modelling of these fields. In Section 4.1.1 we define principal 2-bundles for multiplicative gerbes and provide cocycle data for them which is equivalent to but a bit simpler than others in the literature [98, 235, 245]. In Section 4.1.2 we extend the work of [273] to define connections on bundles for multiplicative gerbes with connective structure, and we enrich the definition by allowing for *enhanced* connections. In Section 4.1.3 we provide cocycle data for gauge transformations and their action on connections, and in Section 4.1.4 we compare this notion of connection with the one based on adjustments. In Section 4.2.1 we generalize the work of [245] to construct a Courant algebroid out of a principal 2-bundle with with structure 2-group determined by a multiplicative gerbe with connective structure. In Section 4.2.2, we prove an original theorem that lets us model the gauge 2-group of a principal 2-bundle as an infinite-dimensional Lie 2-group modelled on the space of sections of its associated Courant algebroid. Finally, we use this in Section 4.2.3 to prove a slice theorem for the space of connections modulo gauge on a principal 2-bundle.

4.1 Principal 2-bundles

4.1.1 Definition and cocycle data

For $\mathfrak{G}$ a Lie 2-group and $\mathfrak{P}$ a Lie groupoid, recall from Definition 3.16 the notion of an action of $\mathfrak{G}$ on $\mathfrak{P}$.

Definition 4.1 ([205])**.** Let $\mathfrak{G}$ be a Lie 2-group. A *principal 2-bundle* with structure 2-group $\mathfrak{G}$ over a manifold M is a Lie groupoid $\mathfrak{P}$ with a smooth functor $\pi : \mathfrak{P} \to M$ that is a surjective submersion on objects and an action (ρ, α^ρ) of $\mathfrak{G}$ on $\mathfrak{P}$ such that

1. $\pi \circ \rho = \pi \circ p_1 : \mathfrak{P} \times \mathfrak{G} \to M$,
2. The anafunctor $p_1 \times \rho : \mathfrak{P} \times \mathfrak{G} \to \mathfrak{P} \times_M \mathfrak{P}$ is weakly invertible, where p_1 denotes projection of the first factor.

Isomorphisms and 2-isomorphisms of principal 2-bundles are defined similarly as in Definition 3.9, yielding a bicategory.

If $\mathfrak{G}$ is the Lie 2-group arising from a multiplicative gerbe as in Theorem 3.48, then the bicategory of $\mathfrak{G}$-bundles admits an equivalent description which has been studied for the 2-groups of the form $\mathrm{String}(G)$ (cf. Example 3.37) in [64, 78, 273].

Definition 4.2 ([64, 78, 273])**.** Let $(\mathcal{G}, m, \alpha)$ be a multiplicative T-gerbe over G. A *principal $\mathcal{G}$-bundle* $(P, \mathcal{P}, \rho, \alpha^\rho)$ over a manifold M is the following data.

1. A principal G-bundle $P \to M$.
2. A T-gerbe $\mathcal{P} \to P$.
3. An isomorphism of T-gerbes $\rho : p^*\mathcal{P} \otimes g^*\mathcal{G} \to (pg)^*\mathcal{P}$ over $P \times G$.
4. A 2-isomorphism of T-gerbes over $P \times G \times G$

$$\begin{tikzcd} p^*\mathcal{P} \otimes g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \arrow[r, "(p,g_1)^*\rho"] \arrow[d, "(g_1,g_2)^*m"'] & (pg_1)^*\mathcal{P} \otimes g_2^*\mathcal{G} \arrow[d, "(pg_1,g_2)^*\rho"] \arrow[dl, Rightarrow, "\alpha^\rho"] \\ p^*\mathcal{P} \otimes (g_1g_2)^*\mathcal{G} \arrow[r, "(p,g_1g_2)^*\rho"'] & (pg_1g_2)^*\mathcal{P} \end{tikzcd} \tag{4.1}$$

such that, over $P \times G \times G \times G$, we have

$$
\begin{tikzcd}[column sep=huge]
p^*\mathcal{P} \otimes g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \otimes g_3^*\mathcal{G} \arrow[r, bend left=70, "\rho(((pg_1)g_2)g_3)"] \arrow[r, bend left=25, "\rho((p(g_1g_2))g_3)" description] \arrow[r, bend right=25, "\rho(p((g_1g_2)g_3))" description] \arrow[r, bend right=70, "\rho(p(g_1(g_2g_3)))"'] \arrow[r, phantom, shift left=6, "\Downarrow\alpha^\rho"] \arrow[r, phantom, "\Downarrow\alpha^\rho"] \arrow[r, phantom, shift right=6, "\Downarrow\alpha"] & (pg_1g_2g_3)^*\mathcal{P}
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
p^*\mathcal{P} \otimes g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \otimes g_3^*\mathcal{G} \arrow[r, bend left=70, "\rho(((pg_1)g_2)g_3)"] \arrow[r, bend left=25, "\rho((pg_1)(g_2g_3))" description] \arrow[r, bend right=25, "\rho(p(g_1(g_2g_3)))"'] \arrow[r, phantom, shift left=6, "\Downarrow\alpha^\rho"] \arrow[r, phantom, "\Downarrow\alpha^\rho"] & (pg_1g_2g_3)^*\mathcal{P}
\end{tikzcd}
.
\tag{4.2}
$$

Given $(P^i, \mathcal{P}^i, \rho^i, \alpha^{\rho_i})$, $i = 1, 2$, then an *isomorphism of $\mathcal{G}$-bundles* is the following data.

1. An equivariant map $u : P^1 \to P^2$ covering the identity on M.
2. An isomorphism of T-gerbes $\varphi : \mathcal{P}^1 \to u^*\mathcal{P}_2$ over P_1.
3. A 2-isomorphism of T-gerbes over $P^1 \times G$

$$
\begin{tikzcd}
(p^1)^*\mathcal{P}^1 \otimes g^*\mathcal{G} \arrow[r, "(p^1,g)^*\rho^1"] \arrow[d, "(p^1)^*\varphi"'] & (p^1g)^*\mathcal{P}^1 \arrow[d, "(p^1g)^*\varphi"] \arrow[dl, Rightarrow, "\alpha^\varphi"] \\
u^*\mathcal{P}^2 \otimes g^*\mathcal{G} \arrow[r, "(u,g)^*\rho^2"'] & (u(p^1)g)^*\mathcal{P}^2
\end{tikzcd}
\tag{4.3}
$$

such that, over $P^1 \times G \times G$,

$$
\begin{tikzcd}[column sep=huge]
p_1^*\mathcal{P}^1 \otimes g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \arrow[r, bend left=70, "\varphi((p_1g_1)g_2)"] \arrow[r, bend left=25, "\varphi(p_1g_1)g_2" description] \arrow[r, bend right=25, "(\varphi(p_1)g_1)g_2" description] \arrow[r, bend right=70, "\varphi(p_1)(g_1g_2)"'] \arrow[r, phantom, shift left=6, "\Downarrow\alpha^\varphi"] \arrow[r, phantom, "\Downarrow\alpha^\varphi"] \arrow[r, phantom, shift right=6, "\Downarrow\alpha^{\rho_2}"] & (u(p_1)g_1g_2)^*\mathcal{P}^2
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
p_1^*\mathcal{P}^1 \otimes g_1^*\mathcal{G} \otimes g_2^*\mathcal{G} \arrow[r, bend left=70, "\varphi((p_1g_1)g_2)"] \arrow[r, bend left=25, "\varphi(p_1(g_1g_2))" description] \arrow[r, bend right=25, "\varphi(p_1)(g_1g_2)"'] \arrow[r, phantom, shift left=6, "\Downarrow\alpha^{\rho_1}"] \arrow[r, phantom, "\Downarrow\alpha^\varphi"] & (u(p_1)g_1g_2)^*\mathcal{P}^2
\end{tikzcd}
.
\tag{4.4}
$$

Given $(u, \varphi, \alpha^\varphi), (u', \varphi', \alpha^{\varphi'}) : (P^1, \mathcal{P}^1, \rho^1, \alpha^{\rho^1}) \to (P^2, \mathcal{P}^2, \rho^2, \alpha^{\rho^2})$, then a 2-*isomorphism* between them can only exist if $u = u'$ and is then given by a 2-isomorphism $\psi : \varphi \Rightarrow \varphi'$ such that, over $P_1 \times G$,

$$
\begin{tikzcd}[column sep=huge]
p_1^*\mathcal{P}^1 \otimes g^*\mathcal{G} \arrow[r, bend left=70, "\varphi(p_1g)"] \arrow[r, bend left=20, "\varphi(p_1)g" description] \arrow[r, bend right=30, "\varphi'(p_1)g"'] \arrow[r, phantom, shift left=6, "\Downarrow\alpha^\varphi"] \arrow[r, phantom, "\Downarrow\psi"] & (u(p_1)g)^*\mathcal{P}^2
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
p_1^*\mathcal{P}^1 \otimes g^*\mathcal{G} \arrow[r, bend left=70, "\varphi(p_1g)"] \arrow[r, bend left=20, "\varphi'(p_1g)" description] \arrow[r, bend right=30, "\varphi'(p_1)g"'] \arrow[r, phantom, shift left=6, "\Downarrow\psi"] \arrow[r, phantom, "\Downarrow\alpha^{\varphi'}"] & (u(p_1)g_1g_2)^*\mathcal{P}^2
\end{tikzcd}
.
\tag{4.5}
$$

We often abbreviate all the data $(P, \mathcal{P}, \rho, \alpha^\rho)$ of a $\mathcal{G}$-bundle by $(P, \mathcal{P})$ or $\mathcal{P}$.

Proposition 4.3. *Let $\mathcal{G}$ be a multiplicative T-gerbe over G with associated Lie 2-group $\mathfrak{G}$. There is a canonical equivalence of bicategories between $\mathcal{G}$-bundles in the sense of Definition 4.2 and $\mathfrak{G}$-bundles in the sense of Definition 4.1.*

Proof. Analogous to Theorem 3.48. □

Proposition 4.4. *Let $\{M_a\}_{a\in\Lambda}$ be a good open cover of M. Then $\mathcal{G}$-bundles on M are described by the following cocycle data.*

1. *$g_{ab} : M_{ab} \to G$ with $g_{ab}g_{bc} = g_{ac}$.*
2. *σ^{ab} trivializations of $g_{ab}^*\mathcal{G}$.*
3. *$\tau^{abc} : m \circ (\sigma^{ab} \otimes \sigma^{bc}) \to \sigma^{ac}$ isomorphisms of trivializations of $g_{ac}^*\mathcal{G}_\nabla$ such that*

$$\begin{tikzcd}[column sep=large]
1 \arrow[rr, bend left=70, "(\sigma_{ab}\sigma_{bc})\sigma_{cd}"] \arrow[rr, bend left=15, "\sigma_{ab}(\sigma_{bc}\sigma_{cd})" description] \arrow[rr, bend right=15, "\sigma_{ab}\sigma_{bd}" description] \arrow[rr, bend right=70, "\sigma_{ad}"'] \arrow[rr, phantom, shift left=6ex, "\alpha\Downarrow"] \arrow[rr, phantom, "\tau^{bcd}\Downarrow"] \arrow[rr, phantom, shift right=6ex, "\tau^{abd}\Downarrow"] & & (g_{ad})^*\mathcal{G}
\end{tikzcd}
=
\begin{tikzcd}[column sep=large]
1 \arrow[rr, bend left=50, "(\sigma_{ab}\sigma_{bc})\sigma_{cd}"] \arrow[rr, "\sigma_{ac}\sigma_{cd}" description] \arrow[rr, bend right=50, "\sigma_{ad}"'] \arrow[rr, phantom, shift left=3ex, "\tau^{abc}\Downarrow"] \arrow[rr, phantom, shift right=3ex, "\tau^{acd}\Downarrow"] & & (g_{ad})^*\mathcal{G}
\end{tikzcd}
\tag{4.6}$$

Proof. Given a $\mathcal{G}$-bundle $(P, \mathcal{P}, \rho, \alpha^\rho)$, one can obtain this data by taking local sections $s_a : M_a \to P$ and trivializations η_a of $s_a^*\mathcal{P} \to M_a$. Then g_{ab} are defined as the unique functions such that $s_a g_{ab} = s_b$, while σ^{ab} are defined as the composition

$$1 \xrightarrow{\eta_b} s_b^*\mathcal{P} \xrightarrow{(s_a, g_{ab})^*\rho^{-1}} s_a^*\mathcal{P} \otimes g_{ab}^*\mathcal{G} \xrightarrow{\eta_a^{-1}} g_{ab}^*\mathcal{G}$$

and similarly $\tau^{abc} = (s_a, g_{ab}, g_{bc})^*(\alpha^\rho)^{-1}$. Conversely, given such data one constructs

$$P := \sqcup_a M_a \times G/\sim$$

with $(a, x, g_{ab}(x)g) \sim (b, x, g)$ and defines $r_a : \pi^{-1}(M_a) \subset P \to G$ to be $[a, x, g] \mapsto g$; these satisfy $g_{ab}r_b = r_a$. Then $\mathcal{P} \to P$ is constructed by gluing the gerbes $r_a^*\mathcal{G} \to \pi^{-1}(M_a)$ with the isomorphisms

$$r_a^*\mathcal{G} \xrightarrow{(g_{ab}, r_b)^* m^{-1}} g_{ab}^*\mathcal{G} \otimes r_b^*\mathcal{G} \xrightarrow{\sigma_{ab}^{-1}} r_b^*\mathcal{G}$$

and the 2-isomorphisms $(g_{ab}, g_{bc}, r_c)^*\alpha^{-1} \otimes \tau^{abc}$. □

Proposition 4.5 ([239, 253]). *Let G be a Lie group and let $P \to M$ be a G-bundle. For T a connected abelian Lie group, a multiplicative T-gerbe $\mathcal{G} \to G$ determines a characteristic class in $c(P) \in H^4(M, Z)$ such that $c(P) = 0$ if and only if P can be lifted to a $\mathcal{G}$-bundle, where $Z := ker(exp) \subset \mathfrak{t}$. We call this the* Pontryagin class *of P with respect to $\mathcal{G}$.*

Proof. Let $c \in H^4(BG, Z)$ be the class corresponding to $\mathcal{G}$ by Proposition 3.34. Letting $f : M \to BG$ be any map such that $f^*EG \cong P$ we define $c(P) := f^*c$ and the result follows by abstract non-sense since $B\mathcal{G}$ is the homotopy fiber of the map $BG \to K(4, Z)$ determined by c. Alternatively, we can define c from cocycle data $g_{ab} : M_{ab} \to G$ for P as follows. Let σ_{ab} be any trivializations of $g^*_{ab}\mathcal{G}$ and let τ_{abc} be any isomorphisms of trivializations $m \circ \sigma_{ab} \otimes \sigma_{bc} \to \sigma_{ac}$. For these τ_{abc}, the failure of equality 4.6 to hold is measured by functions $l_{abcd} : M_{abcd} \to T$ which define a Čech cocycle by the pentagon identity for α. This yields an element in $H^3(BG, T) = H^4(BG, Z)$ which clearly vanishes precisely when P lifts to a $\mathcal{G}$-bundle. □

4.1.2 Connections

For G a Lie group and $P \to M$ a G-bundle, recall that a *connection* $P \to M$ is a $A \in \Omega^1(P, \mathfrak{g})$ such that, over $P \times G$,

$$(pg)^*A = Ad(g(\cdot)^{-1})p^*A + g^*\theta^L. \tag{4.7}$$

By taking vectors tangent to either P or G in this equation, we see that this is equivalent to the two conditions

$$A_{pg}(v_p g) = g^{-1}A_p(v_p)g, \quad A_{p\cdot 1}(pu) = u, \qquad \text{for} \qquad p \in P,\ v_p \in T_pP,\ g \in G,\ u \in T_1G.$$

For the following lemma, recall the simplicial manifold $P//G_\bullet$ (cf. Example 2.5) and its corresponding simplicial differential $\delta : \Omega^*(P \times G^r) \to \Omega^*(P \times G^{r+1})$ defined by (2.35). In low simplicial degrees, we can write this explicitly as follows. For $\alpha \in \Omega^*(P)$ and $\beta \in \Omega^*(P \times G)$,

$$\begin{aligned} \delta\alpha &:= (pg)^*\alpha - p^*\alpha \in \Omega^*(P \times G), \\ \delta\beta &:= -(p, g_1)^*\beta + (p, g_1g_2)^*\beta - (pg_1, g_2)^*\beta \in \Omega^*(P \times G^2). \end{aligned} \tag{4.8}$$

Lemma 4.6 ([246])**.** *Let P be a G-bundle, let $A \in \Omega^1(P,\mathfrak{g})$ be a connection on it and let $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{t}$ be Ad-invariant and symmetric. Then the forms*

$$CS(A) := \langle dA \wedge A\rangle + \frac{1}{3}\langle A\wedge[A\wedge A]\rangle \in \Omega^3(P,\mathfrak{t}), \quad R(A) := \langle p^*A \wedge g^*\theta^R\rangle \in \Omega^2(P\times G,\mathfrak{t})$$

satisfy

$$dCS(A) = \langle F_A \wedge F_A\rangle, \quad dR(A) - \delta CS(A) = g^*\mu, \quad \delta R(A) = g^*\nu, \tag{4.9}$$

where μ, ν are defined by (3.109), $F_A := dA + \frac{1}{2}[A\wedge A]$ and δ is the simplicial differential of the simplicial manifold $P//G_\bullet$.

In particular, when T is connected and $\langle\cdot,\cdot\rangle$ is the pairing associated to a multiplicative T-gerbe over G, then the image of the Pontryagin class $c(P) \in H^4(M,Z)$ from Proposition 4.5 in $H^4(M,\mathfrak{t})$ is represented in de Rham cohomology by $\langle F_A \wedge F_A\rangle \in \Omega^4(M,\mathfrak{t})$.

Proof. Equation (4.9) follows from straightforward computations. First,

$$\begin{aligned} dCS(A) &= \langle dA \wedge dA\rangle + \langle dA \wedge [A\wedge A]\rangle \\ &= \langle dA\wedge dA\rangle + \frac{1}{2}\langle dA\wedge[A\wedge A]\rangle + \frac{1}{2}\langle [A\wedge A]\wedge dA\rangle + \frac{1}{4}\langle [A\wedge A]\wedge[A\wedge A]\rangle \\ &= \langle F_A\wedge F_A\rangle. \end{aligned} \tag{4.10}$$

Then, use $(pg)^*F_A = Ad(g(\cdot)^{-1})p^*F_A$, $d\theta^R - \frac{1}{2}[\theta^R\wedge\theta^R] = 0$ and (4.7) to see

$$\begin{aligned} &- p^*CS(A) + (pg)^*CS(A) + g^*\mu = \\ &= -\langle p^*F_A \wedge p^*A\rangle + \frac{1}{6}\langle p^*A\wedge[p^*A\wedge p^*A]\rangle + \langle (pg)^*F_A\wedge(pg)^*A\rangle \\ &\quad - \frac{1}{6}\langle (pg)^*A \wedge [(pg)^*A\wedge(pg)^*A]\rangle + \frac{1}{6}\langle g^*\theta^L\wedge[g^*\theta^L\wedge g^*\theta^L]\rangle \\ &= \langle p^*F_A \wedge (-p^*A + Ad(g(\cdot))(pg)^*A)\rangle + \frac{1}{6}\langle p^*A\wedge[p^*A\wedge p^*A]\rangle \\ &\quad - \frac{1}{6}\langle (p^*A + g^*\theta^R)\wedge[(p^*A+g^*\theta^R)\wedge(p^*A+g^*\theta^R)]\rangle + \frac{1}{6}\langle g^*\theta^L\wedge[g^*\theta^L\wedge g^*\theta^L]\rangle \\ &= \langle p^*F_A\wedge g^*\theta^R\rangle - \frac{1}{2}\langle g^*\theta^R\wedge[p^*A\wedge p^*A]\rangle - \frac{1}{2}\langle p^*A\wedge[g^*\theta^R, g^*\theta^R]\rangle \\ &= \langle p^*dA\wedge g^*\theta^R\rangle - \frac{1}{2}\langle p^*A\wedge[g^*\theta^R\wedge g^*\theta^R]\rangle = dR(A). \end{aligned} \tag{4.11}$$

Finally,

$$\begin{aligned}
&(p,g_1)^*R(A) - (p,g_1g_2)^*R(A) + (pg_1,g_2)^*R(A) + (g_1,g_2)^*\nu = \\
&= \langle p^*A, g_1^*\theta^R\rangle - \langle p^*A, (g_1g_2)^*\theta^R\rangle + \langle (pg_1)^*A, g_2^*\theta^R\rangle - \langle g_1^*\theta^L, g_2^*\theta^R\rangle \\
&= \langle p^*A, -Ad(g_1(\cdot))g_2^*\theta^R\rangle + \langle Ad((g_1(\cdot)^{-1})p^*A, g_2^*\theta^R\rangle = 0.
\end{aligned} \tag{4.12}$$

For the second part, recall from Theorem 3.43 that the image of the class $c \in H^4(BG, Z)$ in $H^4(BG, \mathfrak{t})$ classifying the multiplicative gerbe is represented in simplicial de Rham cohomology of $BG_\bullet$ (cf. Theorem 2.12) by the differential forms $(\mu, \nu, 0, 0)$. Now note that the simplicial manifold $P//G_\bullet$ has a canonical map $g : P//G_\bullet \to BG_\bullet$ whose geometric realization is a classifying map $M \to BG$ for P, which implies by the above that the image of $c(P) \in H^4(M, \mathfrak{t}) = H^4(|P//G_\bullet|, \mathfrak{t})$ is represented in simplicial de Rham cohomology by $(g^*\mu, g^*\nu, 0, 0)$. Finally, (4.9) states precisely that $(g^*\mu, g^*\nu, 0, 0)$ and $\langle F_A \wedge F_A\rangle$ define the same class in simplicial de Rham cohomology, as they differ by the total derivative of $(CS(A), R(A))$. □

Connections on String(G)-principal bundles are defined in [273] as trivializations of an associated Chern-Simons 2-gerbe with connection. The existence of this 2-gerbe with connection relies essentially on the fact that String(G) has a canonical enhanced curving. Hence, Theorem 3.43 allows us to generalize loc. cit. to define connections for $\mathcal{G}$-bundles, where $\mathcal{G}$ is any multiplicative gerbe equipped with a connective structure. We also expand the definition by introducing *enhanced connections*. From now on we fix $\mathcal{G}_\nabla$ a multiplicative T-gerbe over G with connective structure and write $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{t}$, Θ^L for the associated pairing and curving from Theorem 3.43.

Definition 4.7. Let $(P, \mathcal{P}, \rho, \alpha^\rho)$ be a principal $\mathcal{G}$-bundle. A *connective structure* on it is the following data.

1. A connective structure ∇ on the gerbe $\mathcal{P} \to P$.
2. A connection ∇_ρ on the isomorphism of gerbes $\rho : p^*\mathcal{P}_\nabla \otimes g^*\mathcal{G}_\nabla \to (pg)^*\mathcal{P}_\nabla$ such that α^ρ is a flat 2-isomorphism of gerbes.

We write $(P, \mathcal{P}_\nabla, \rho_\nabla, \alpha^\rho)$ or simply $\mathcal{P}_\nabla$ for a principal $\mathcal{G}$-bundle with connective structure, also called a *principal $\mathcal{G}_\nabla$-bundle*. An *isomorphism of connective structures on a $\mathcal{G}$-bundle* $(\nabla_1, \nabla_{\rho_1}) \to (\nabla_2, \nabla_{\rho_2})$ is an isomorphism of connective structures on gerbes $\phi : \nabla_1 \to \nabla_2$ such that the following is a commutative diagram of isomorphisms of gerbes with connective structures

$$\begin{tikzcd} p^*\mathcal{P}_{\nabla_1} \otimes g^*\mathcal{G}_\nabla \arrow[r, "(\rho,\nabla_{\rho_1})"] \arrow[d, "(id, p^*\phi\otimes id)"'] & (pg)^*\mathcal{P}_{\nabla_1} \arrow[d, "(id,(pg)^*\phi)"] \\ p^*\mathcal{P}_{\nabla_2} \otimes g^*\mathcal{G}_\nabla \arrow[r, "(\rho,\nabla_{\rho_2})"'] & (pg)^*\mathcal{P}_{\nabla_2}. \end{tikzcd} \tag{4.13}$$

A *connection* (resp. *enhanced connection*) on $(P, \mathcal{P}, \rho, \alpha^\rho)$ is

1. A connective structure (∇, ∇_ρ) on $(P, \mathcal{P}, \rho, \alpha^\rho)$.
2. A G-connection $A \in \Omega^1(P, \mathfrak{g})$ on P.
3. A curving (resp. *enhanced curving*) B on $\mathcal{P}_\nabla \to P$ such that ∇_ρ has curvature

$$-\langle p^*A \wedge g^*\theta^R\rangle \in \Omega^2(P \times G, \mathfrak{t}) \tag{4.14}$$

(resp. $-\langle p^*A \otimes g^*\theta^R\rangle \in \Gamma(T^*(P\times G)\otimes T^*(P\times G)\otimes \mathfrak{t})$) with respect to $p^*B \otimes g^*\Theta^L$ (resp. $p^*B \otimes g^*\Theta^{L,en}$) and $(pg)^*B$.

An *isomorphism of (enhanced) connections* $(\nabla_1, \nabla_{\rho_1}, A_1, B_1) \to (\nabla_2, \nabla_{\rho_2}, A_2, B_2)$ can only exist if $A_1 = A_2$ and is then given by an isomorphism $\phi : (\nabla_1, \nabla_{\rho_1}) \to (\nabla_2, \nabla_{\rho_2})$, flat with respect to B_1, B_2. We write $\mathcal{A}(\mathcal{P})$ for the groupoid of connections on $\mathcal{P}$ and $\mathcal{A}^{en}(\mathcal{P})$ for the groupoid of enhanced connections on $\mathcal{P}$. Whenever a connective structure (∇, ∇_ρ) on $\mathcal{P}$ is fixed, we write $\mathcal{A}(\mathcal{P}_\nabla)$ for the set of connections with such connective structure and $\mathcal{A}^{en}(\mathcal{P}_\nabla)$ for the set of enhanced connections with such connective structure.

Remark 4.8. An enhanced connection can also be defined as a pair $((\nabla, \nabla_\rho, A, B), h)$ of a connection $(\nabla, \nabla_\rho, A, B)$ and a symmetric tensor $h \in \Gamma(S^2T^*M \otimes \mathfrak{t})$. This is because condition 3 in Definition 4.7 states that an $h^P \in \Gamma(S^2T^*P \otimes \mathfrak{t})$ is the symmetric part of an enhanced connection with underlying G-connection A if and only if $h := h^P + \frac{1}{2}\langle A \odot A\rangle \in \Gamma(S^2T^*P \otimes \mathfrak{t})$ is basic, which can thus be identified with a symmetric tensor on M. Note the relation with the Kaluza-Klein mechanism [166].

Lemma 4.9. *Let $(P, \mathcal{P}_\nabla, \rho_\nabla, \alpha^\rho, A, B)$ be a principal $\mathcal{G}$-bundle with connection. Then the curvature $\hat{H} \in \Omega^3(P, \mathfrak{t})$ of B as a curving on $\mathcal{P}_\nabla \to P$ is of the form $\hat{H} := \pi^*H - CS(A)$ for some $H \in \Omega^3(M, \mathfrak{t})$, where $CS(A)$ is as in Lemma 4.6.*

Proof. It follows from Theorem 3.43 and the general properties of curvings on gerbes that the curvature of B is some $\hat{H} \in \Omega^3(P, \mathfrak{t})$ with $\delta\hat{H} - g^*\mu = -d\langle p^*A \wedge g^*\theta^R\rangle$, so $H := \hat{H} + CS(A)$ satisfies $\delta H = 0$ by Lemma 4.6, which means that it is a basic 3-form on P, as we wanted to show. □

Definition 4.10. The *curvature* of a connection $(\nabla, \nabla_\rho, A, B)$ on a $\mathcal{G}$-bundle $(P, \mathcal{P}, \rho, \alpha^\rho)$ is the pair $(F_A, H) \in \Omega^2(M, ad\, P) \oplus \Omega^3(M, \mathfrak{t})$, where $F_A = dA + \frac{1}{2}[A \wedge A]$ and H is the three-form in Lemma 4.9. A $\mathcal{G}$-bundle with connection is *flat* if $F_A = 0$ and $H = 0$.

The following proposition shows that the curvature of a connection on a $\mathcal{G}$-bundle satisfies the *Green-Schwartz anomally cancellation equation* [137] which is expected in string theory from the field strength of a Kalb-Ramond field coupled to an ordinary gauge field.

Proposition 4.11 (Bianchi Identity [273])**.** *Let $(P, \mathcal{P}_\nabla, \rho_\nabla, \alpha^\rho, A, B)$ be a principal $\mathcal{G}$-bundle with connection. Then its curvature $(F_A, H) \in \Omega^2(M, ad\, P) \oplus \Omega^3(M, \mathfrak{t})$ satisfies*

$$d_A F_A = 0, \tag{4.15}$$

$$dH - \langle F_A \wedge F_A \rangle = 0. \tag{4.16}$$

Proof. The equation $d_A F_A = 0$ is the classical Bianchi identity for connections on G-bundles. On the other hand, the curvature $\hat{H} \in \Omega^3(P, \mathfrak{t})$ of B as a curving on the gerbe $\mathcal{P}_\nabla \to P$ satisfies $d\hat{H} = 0$ and so Lemma 4.6 implies $dH - \langle F_A \wedge F_A \rangle = 0$. □

We can give cocycle data for connections that generalizes and simplifies the descriptions of connections on String(n)-bundles in [98, 237, 245].

Proposition 4.12. *Let $\mathcal{P} \to M$ be a $\mathcal{G}$-bundle described in a good open cover $\{M_a\}_a$ of M by cocycle data g_{ab}, σ_{ab}, τ_{abc} as in Proposition 4.4. Then a connection on $\mathcal{P}$ is described by the following cocycle data.*

1. *Connections ∇_{ab} on σ_{ab} such that τ_{abc} are flat.*

2. *$A_a \in \Omega^1(M_a, \mathfrak{g})$ such that*

$$A_b - Ad(g_{ab}^{-1})A_a = g_{ab}^* \theta^L \tag{4.17}$$

3. *$B_a \in \Omega^2(M_a, \mathfrak{t})$ such that*

$$B_b - B_a = F_{ab} - \langle A_a \wedge g_{ab}^* \theta^R \rangle, \tag{4.18}$$

where F_{ab} is the curvature of ∇_{ab} with respect to $g_{ab}^* \Theta^L$.

Its curvature (F, H) is described locally by

$$F = dA_a + \frac{1}{2}[A_a \wedge A_a], \tag{4.19}$$

$$H = dB_a + \langle dA_a \wedge A_a \rangle + \frac{1}{3}\langle A_a \wedge [A_a \wedge A_a] \rangle. \tag{4.20}$$

An isomorphism of connections $(\nabla^1_{ab}, A_a, B_a) \to (\nabla^2_{ab}, A_a, B^2_a)$ *is described by* $\Lambda_a \in \Omega^1(M_a, \mathfrak{t})$ *such that*

$$\nabla^2_{ab} + \Lambda_a = \nabla^1_{ab} + \Lambda_b, \tag{4.21}$$

$$B^2_a - B^1_a = d\Lambda_a. \tag{4.22}$$

Proof. Assume first that $\mathcal{P}$ carries a connection. If $s_a : M_a \to P$, $\eta_a : 1 \to s_a^*\mathcal{P}$ define the cocycle data for $\mathcal{P}$ as in Proposition 4.4, then equipping η_a with any connection defines a connection on $\sigma_{ab} = \eta_a^{-1} \circ (s_a, g_{ab})^*\rho^{-1} \circ \eta_b$ such that τ_{abc} is flat. Then we let $A_a := s_a^* A$ and we let B_a be the curvature of η_a; it is easy to see that the definition of connection implies (4.17), (4.18). Conversely, from this data recall that P is constructed as $P := \sqcup_a M_a \times G/\sim$ with $(a, x, g_{ab}(x)g) \sim (b, x, g)$, and that we define $r_a : \pi^{-1}(M_a) \subset P \to G$ to be $[a, x, g] \mapsto g$. Then the connection A is constructed as $A_{|\pi^{-1}(M_a)} := r_a^*\theta^L + Ad(r_a^{-1})A_a$, while $(\mathcal{P}_\nabla, B)$ is obtained by gluing the gerbes with connective structure and curving

$$(r_a^*\mathcal{G}_\nabla, r_a^*\Theta + B_a - \langle A_a \wedge r_a^*\theta^R\rangle) \to \pi^{-1}(M_a)$$

with the isomorphisms

$$r_a^*\mathcal{G}_\nabla \overset{(g_{ab}, r_b)^* m^{-1}}{\to} g_{ab}^*\mathcal{G} \otimes r_b^*\mathcal{G} \overset{\sigma_{ab}^{-1}}{\to} r_b^*\mathcal{G}$$

and the 2-isomorphisms $(g_{ab}, g_{bc}, r_c)^*\alpha^{-1} \otimes \tau^{abc}$. The result for isomorphisms of connections follows similarly. □

Definition 4.13. A $\mathcal{G}$-bundle $\mathcal{P} \to M$ *admits locally constant cocycle data* if it admits a connective structure such that there is a cocycle description as in Proposition 4.12 with g_{ab} locally constant and σ_{ab} flat.

We introduce some notation for the following proposition. For $P \to M$ a G-bundle and $ad\,P \to M$ its associated $\mathfrak{g}$-bundle we write $\Omega^1(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^2(M, \mathfrak{t})$ for the group with underlying set $\Omega^1(ad\,P) \times \Omega^2(M, \mathfrak{t})$ but with product

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 + a_2, b_1 + b_2 + \langle a_1 \wedge a_2\rangle) \tag{4.23}$$

i.e., it is a non-trivial central extension of $\Omega^1(ad\,P)$ by $\Omega^2(M, \mathfrak{t})$. Recall also the groupoids $\mathcal{A}(\mathcal{P})$, $\mathcal{A}^{en}(\mathcal{P})$ and sets $\mathcal{A}(\mathcal{P}_\nabla)$, $\mathcal{A}^{en}(\mathcal{P}_\nabla)$ introduced in Definition 4.7.

Proposition 4.14. *Let* $(P, \mathcal{P}, \rho, \alpha^\rho) \to M$ *be a* $\mathcal{G}$*-bundle.*

1. *Connective structures on* $\mathcal{P}$ *always exist and any two are isomorphic; the set of isomorphisms between them is a torsor for* $\Omega^1(M, \mathfrak{t})$.

2. *For any fixed connective structure (∇, ∇_ρ) on $\mathcal{P}$, the set $\mathcal{A}(\mathcal{P}_\nabla)$ is a right torsor for $\Omega^1(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^2(M, \mathfrak{t})$. In particular, it is an affine bundle over the space of G-connections on P with fiber $\Omega^2(M, \mathfrak{t})$.*

3. *For any fixed connective structure (∇, ∇_ρ) on $\mathcal{P}$, the set $\mathcal{A}^{en}(\mathcal{P}_\nabla)$ is a right torsor for $\Omega^1(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^2(M, \mathfrak{t}) \times \Gamma(S^2T^*M \otimes \mathfrak{t})$. In particular, it is an affine bundle over the space of G-connections on P with fiber $\Gamma(T^*M \otimes T^*M \otimes \mathfrak{t})$.*

4. *$\mathcal{P}$ admits locally constant cocycle data (cf. Definition 4.13) if and only if it admits a flat connection.*

Proof. Let g_{ab}, σ_{ab}, τ_{abc} be cocycle data for $\mathcal{P}$. Choose any connections $\tilde{\nabla}_{ab}$ on σ_{ab} and let $\eta_{abc} \in \Omega^1(M_{abc}, \mathfrak{t})$ be the covariant derivative of τ_{abc}. The cocycle condition for τ_{abc} implies that η_{abc} is a Čech cocycle of 1-forms, so we may write $\eta_{abc} = \eta_{ab} - \eta_{ac} + \eta_{bc}$ and then $\nabla_{ab} := \tilde{\nabla}_{ab} - \eta_{ab}$ is a connective structure on $\mathcal{P}$. Note any two connections ∇^1_{ab}, ∇^2_{ab} on σ_{ab} differ by a $\gamma_{ab} \in \Omega^1(M_{ab}, \mathfrak{t})$; if ∇^1_{ab}, ∇^2_{ab} define connective structures on $\mathcal{P}$ then γ_{ab} is a Čech cocycle and the set of its trivializations, which is a torsor for $\Omega^1(M, \mathfrak{t})$, is the set of isomorphisms $\nabla^1_{ab} \to \nabla^2_{ab}$, proving 1.

Once a connective structure ∇_{ab} is chosen, the existence of the 1-forms A_a is simply the classical existence of connections on G-bundles and then the existence of B_a follows from the fact that Theorem 3.43 and the existence of τ_{abc} implies

$$F_{ab} - F_{ac} + F_{bc} = \langle g^*_{ab}\theta^L \wedge g^*_{bc}\theta^R \rangle, \tag{4.24}$$

so $-\langle A_a \wedge g^*_{ab}\theta^R \rangle + F_{ab}$ is a cocycle of 2-forms in M for any choice of A. The action of $(a, b) \in \Omega^1(M, ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^2(M, \mathfrak{t})$ on $\mathcal{A}(\mathcal{P}_\nabla)$ is defined by

$$(A_a, B_a) \cdot (a_a, b) := (A_a + a_a, B + \langle A_a \wedge a_a \rangle + b), \tag{4.25}$$

where $a_a \in \Omega^1(M_a, \mathfrak{g})$ satisfies $a_b = Ad(g^{-1}_{ab})a_a$ and $b \in \Omega^2(M, \mathfrak{t})$; it is easy to see that $\mathcal{A}(\mathcal{P}_\nabla)$ is a right torsor for this action, proving 2. Then 3 follows directly from Remark 4.8.

If $\mathcal{P}$ admits locally constant cocycle data g_{ab}, σ_{ab}, τ^{abc} then letting ∇_{ab} be a flat connection on σ_{ab} such that τ^{abc} is flat we see that $A_a = 0$, $B_a = 0$ is clearly a flat connection. Conversely, if $\mathcal{P}$ has a flat connection then a classical result states that there are local sections $s_a : M_a \to P$ such that $s^*_a A = 0$ with corresponding transition functions $g_{ab} : M_{ab} \to G$ locally constant. We obtain the rest of the cocycle data σ^{ab}, τ^{abc}, ∇_{ab}, B_a by the method above and we see that $0 = H = dB_a$ implies $B_a = d\Lambda_a$ for some Λ_a; thus, $\nabla_{ab} + \Lambda_a - \Lambda_b$ is a flat connection on σ_{ab} such that τ_{abc} is flat, which concludes the proof. □

Remark 4.15. Parts 1 and 2 from Proposition 4.14 can be summarized in a single, more abstract statement by saying that the groupoid $\mathcal{A}(\mathcal{P})$ is a torsor (i.e., a principal 2-bundle over a point) for the Lie 2-group associated to the crossed module

$$(\Omega^1(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^2(M,\mathfrak{t}), \Omega^1(M,\mathfrak{t}), f, \triangleright)$$

with $\triangleright$ trivial and $f : \Omega^1(M,\mathfrak{t}) \to \Omega^1(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^2(M,\mathfrak{t})$, $\eta \mapsto (0, d\eta)$. Similarly, $\mathcal{A}^{en}(\mathcal{P})$ is a torsor for $(\Omega^1(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^2(M,\mathfrak{t}) \times \Gamma(S^2T^*M \otimes \mathfrak{t}), \Omega^1(M,\mathfrak{t}), f, \triangleright)$. These results are analogous to the classical fact that the space of connections on a G-bundle is a torsor for $\Omega^1(M, adP)$. This analogy will become even more clear in Section 4.2.1, when we introduce the Courant algebroid associated to a $\mathcal{G}$-bundle.

4.1.3 The gauge 2-group

Recall from Definition 3.28 that isomorphisms and 2-isomorphisms of gerbes act on connections and their isomorphisms. Similarly, there is an obvious way to define the action of isomorphisms and 2-isomorphisms of $\mathcal{G}$-bundles on connections and isomorphisms of connections using Definitions 4.2, 4.7. Our next goal is to generalize Proposition 3.30 to the case of $\mathcal{G}$-bundles, for which we need the following definition.

Definition 4.16. Let $\mathcal{P}^1_\nabla$, $\mathcal{P}^2_\nabla$ be $\mathcal{G}$-bundles with connective structure (i.e., $\mathcal{G}_\nabla$-bundles) and let $(u, \varphi, \alpha^\varphi)$ be an isomorphism $\mathcal{P}^1 \to \mathcal{P}^2$. A *connection* on it is a connection on φ such that α^φ is flat. An isomorphism of $\mathcal{G}$-bundles with a connection is also called an *isomorphism of $\mathcal{G}_\nabla$-bundles*. Given two isomorphisms with connection $(u, \varphi_\nabla, \alpha^\varphi)$, $(u, \varphi'_\nabla, \alpha^{\varphi,'})$ and a 2-isomorphism $\psi : (u, \varphi, \alpha^\varphi) \to (u, \varphi', \alpha^{\varphi,'})$, then we say ψ is flat if it is flat as a 2-isomorphism of gerbes. We write $Gauge(\mathcal{P})$ for the automorphism 2-group (cf. Remark 3.11) of a given $\mathcal{G}$-bundle, and $Gauge(\mathcal{P}_\nabla)$ for the automorphism 2-group of a given $\mathcal{G}_\nabla$-bundle, with only flat 2-isomorphisms as arrows.

While the 2-group $Gauge(\mathcal{P})$ acts on the groupoid of all connections $\mathcal{A}(\mathcal{P})$, the 2-group $Gauge(\mathcal{P}_\nabla)$ acts on the set $\mathcal{A}(\mathcal{P}_\nabla)$ of connections whose connective structure is prescribed. In particular, the arrows of $Gauge(\mathcal{P}_\nabla)$ act trivially on $\mathcal{A}(\mathcal{P}_\nabla)$, as this is a set (cf. Remark 3.17). As in Section 3.2.1, we regard $Gauge(\mathcal{P})$, $Gauge(\mathcal{P}_\nabla)$, $\mathcal{A}(\mathcal{P})$ and $\mathcal{A}(\mathcal{P}_\nabla)$ endowed with Fréchet topologies and we construct the simplicial topological spaces $\mathcal{A}(\mathcal{P}_\nabla)//Gauge(\mathcal{P}_\nabla)$, $\mathcal{A}(\mathcal{P})//Gauge(\mathcal{P})$ by the quotient 2-groupoid construction from Section 3.1.2. We do this because Theorem 4.26 below will let us treat $\mathcal{A}(\mathcal{P}_\nabla)//Gauge(\mathcal{P}_\nabla)$ as a simplicial manifold modelled in spaces of global sections of familiar vector bundles, which is crucial for our constructions in Chapter 6 and does not seem to be so easily done for $\mathcal{A}(\mathcal{P})//Gauge(\mathcal{P})$, and because the following proposition shows that $\mathcal{A}(\mathcal{P})//Gauge(\mathcal{P})$ and $\mathcal{A}(\mathcal{P}_\nabla)//Gauge(\mathcal{P}_\nabla)$ are essentially equivalent. It also provides a description of $Gauge(\mathcal{P})$ and $Gauge(\mathcal{P}_\nabla)$ closely related to the description of automorphisms of string Courant algebroids in [126].

Proposition 4.17. *Let $\mathcal{P}$ be a $\mathcal{G}$-bundle. Then there is an exact sequence of 2-groups*

$$1 \to BT(M) \to Gauge(\mathcal{P}) \to Gauge(P) \xrightarrow{r} H^3(X, Z) \to 1, \tag{4.26}$$

where $BT(M)$ is the 2-group whose objects are T-bundles on M (cf. Section 3.2.1) and the image of r in $H^3(X, \mathfrak{t})$ can be represented by choosing a connection $A \in \Omega^1(P, \mathfrak{g})$ as

$$u \mapsto [CS(u^*A) - CS(A) - d\langle u^*A \wedge A\rangle] \in H^3(X, \mathfrak{t}). \tag{4.27}$$

Let $\mathcal{P}_\nabla$ be a $\mathcal{G}$-bundle with connective structure. Then there is an exact sequence of 2-groups

$$1 \to BT_\nabla(M) \to Gauge(\mathcal{P}_\nabla) \to Gauge(P) \xrightarrow{r} H^3(X, Z) \to 1, \tag{4.28}$$

where $BT_\nabla(M)$ is the 2-group of T-bundles with connection on M and the map r is the same as in (4.26). Moreover, there is a canonical morphism of simplicial topological spaces $\mathcal{A}(\mathcal{P}_\nabla)//Gauge(\mathcal{P}_\nabla) \to \mathcal{A}(\mathcal{P})//Gauge(\mathcal{P})$ inducing weak homotopy equivalence on geometric realizations.

Proof. The existence of an exact sequence

$$1 \to BT(M) \to Gauge(\mathcal{P}) \to Gauge(P) \tag{4.29}$$

follows directly from Definition 4.2. Now the condition for $u \in Gauge(P)$ to lift to a gauge transformation of $\mathcal{P}$ can be described as follows. First, note that the gerbe $u^*\mathcal{P} \otimes \mathcal{P}^{-1} \to P$ is equipped with an equivariant structure [199] given by $u^*\rho \otimes \rho^{-1}$ and $u^*\alpha^\rho \otimes \alpha^{\rho,-1}$. Thus it defines a gerbe over $P/G = M$, which is trivial precisely when u lifts to $Gauge(\mathcal{P})$. The map r assigns to each u the class in $H^3(M, Z)$ of the corresponding gerbe. It can be represented as in (4.27) because a connection on the $\mathcal{G}$-bundle $\mathcal{P}$ (with A the underlying connection on P) determines a curving on $u^*\mathcal{P} \otimes \mathcal{P}^{-1} \to P$ that descends to the quotient and has curvature $CS(u^*A) - CS(A) - d\langle u^*A \wedge A\rangle$. The exact sequence 4.28 is obtained similarly. Then the equivalence $\mathcal{A}(\mathcal{P})//Gauge(\mathcal{P}) \cong \mathcal{A}(\mathcal{P}_\nabla)//Gauge(\mathcal{P}_\nabla)$ follows from the exact sequences (4.26), (4.28), Proposition 3.30 and the fact that any two connective structures on $\mathcal{P}$ are isomorphic (cf. Proposition 4.14). □

Remark 4.18. The equivalence of quotient 2-groupoids from Proposition 4.17 is useful because the action of $Gauge(\mathcal{P}_\nabla)$ on $\mathcal{A}(\mathcal{P}_\nabla)$ is easier to describe than that of $Gauge(\mathcal{P})$

on $\mathcal{A}(\mathcal{P})$. In fact, using the presentation $\mathcal{A}(\mathcal{P}_\nabla)//Gauge(\mathcal{P}_\nabla)$ will let us define geometric structures on moduli spaces of connections on principal 2-bundles in Chapter 6.

Proposition 4.19. *Let $\mathcal{P}^1_\nabla$, $\mathcal{P}^2_\nabla$ be $\mathcal{G}_\nabla$-bundles over M described on a good open cover $\{M_a\}_a$ by cocycle data $(g^i_{ab}, \sigma^i_{ab,\nabla}, \tau^i_{abc})$, $i = 1, 2$. Then an isomorphism with connection $\mathcal{P}^1_\nabla \to \mathcal{P}^2_\nabla$ is given by the following data.*

1. *Functions $\varphi_a : M_a \to G$ with $\varphi_a g^1_{ab} = g^2_{ab}\varphi_b$.*
2. *Trivializations with connections $\Phi_{a,\nabla}$ of $\varphi_a^*\mathcal{G}_\nabla$.*
3. *Flat isomorphisms of trivializations $\psi_{ab} : m\circ(\Phi_a \otimes \sigma^1_{ab}) \to m\circ(\sigma^2_{ab}\otimes\Phi_b)$ satisfying*

$$
\begin{tikzcd}
& m((\Phi_a\sigma^1_{ab})\sigma^1_{bc}) \arrow[d, Rightarrow, "\alpha"'] \arrow[dr, bend left] & \\
& m(\Phi_a(\sigma^1_{ab}\sigma^1_{bc})) \arrow[d, Rightarrow, "\tau_1^{abc}"'] \arrow[dr] & \\
1 \arrow[uur, bend left, dash] \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] \arrow[ddr, bend right, dash] & m(\Phi_a\sigma^1_{ac}) \arrow[r] \arrow[d, Rightarrow, "\psi_{ac}"'] & (g^2_{ac}\varphi_c)^*\mathcal{G} \\
& m(\sigma^2_{ac}\Phi_c) \arrow[ur] \arrow[d, Rightarrow, "(\tau_2^{abc})^{-1}"'] & \\
& m((\sigma^2_{ab}\sigma^2_{bc})\Phi_c) \arrow[uur, bend right] &
\end{tikzcd}
=
\begin{tikzcd}
& m((\Phi_a\sigma^1_{ab})\sigma^1_{bc}) \arrow[d, Rightarrow, "\psi_{ab}"'] \arrow[dr, bend left] & \\
& m((\sigma^2_{ab}\Phi_b)\sigma^1_{bc}) \arrow[d, Rightarrow, "\alpha"'] \arrow[dr] & \\
1 \arrow[uur, bend left, dash] \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] \arrow[ddr, bend right, dash] & m(\sigma^2_{ab}(\Phi_b\sigma^1_{bc})) \arrow[r] \arrow[d, Rightarrow, "\psi_{bc}"'] & (g^2_{ac}\varphi_c)^*\mathcal{G}. \\
& m(\sigma^2_{ab}(\sigma^2_{bc}\Phi_c)) \arrow[ur] \arrow[d, Rightarrow, "\alpha"'] & \\
& m((\sigma^2_{ab}\sigma^2_{bc})\Phi_c) \arrow[uur, bend right] &
\end{tikzcd}
\tag{4.30}
$$

If $(\varphi_a, \Phi_{a,\nabla}, \psi_{ab})$, $(\varphi'_a, \Phi'_{a,\nabla}, \psi'_{ab})$ are two isomorphisms $\mathcal{P}^1_\nabla \to \mathcal{P}^2_\nabla$, then a flat 2-isomorphism between them can only exist if $\varphi_a = \varphi'_a$ and in that case it is given by flat 2-isomorphisms $\chi_a : \Phi_a \Rightarrow \Phi'_a$ such that

$$
\begin{tikzcd}
& m(\Phi_a\sigma^1_{ab}) \arrow[d, Rightarrow, "\chi_a"'] \arrow[dr, bend left] & \\
1 \arrow[ur, bend left, dash] \arrow[r, dash] \arrow[dr, bend right, dash] & m(\Phi'_a\sigma^1_{ab}) \arrow[r] \arrow[d, Rightarrow, "\psi'_{ab}"'] & (g^2_{ab}\varphi_b)^*\mathcal{G} \\
& m(\sigma^2_{ab}\Phi'_b) \arrow[ur, bend right] &
\end{tikzcd}
=
\begin{tikzcd}
& m(\Phi_a\sigma^1_{ab}) \arrow[d, Rightarrow, "\psi_{ab}"'] \arrow[dr, bend left] & \\
1 \arrow[ur, bend left, dash] \arrow[r, dash] \arrow[dr, bend right, dash] & m(\sigma^2_{ab}\Phi_b) \arrow[r] \arrow[d, Rightarrow, "\chi_b"'] & (g^2_{ab}\varphi_b)^*\mathcal{G} \\
& m(\sigma^2_{ab}\Phi'_b) \arrow[ur, bend right] &
\end{tikzcd}
. \tag{4.31}
$$

Composition of isomorphisms of $\mathcal{G}_\nabla$-bundles is defined by

$$(\varphi^{13}_a, \Phi^{13}_{a,\nabla}, \psi^{13}_{ab}) = \mathcal{P}^1_\nabla \xrightarrow{(\varphi^{12}_a, \Phi^{12}_{a,\nabla}, \psi^{12}_{ab})} \mathcal{P}^2_\nabla \xrightarrow{(\varphi^{23}_a, \Phi^{23}_{a,\nabla}, \psi^{23}_{ab})} \mathcal{P}^3_\nabla,$$

where $\varphi_a^{13} := \varphi_a^{23}\varphi_a^{12}$, $\Phi_{a,\nabla}^{13} := m \circ (\Phi_{a,\nabla}^{23} \otimes \Phi_{a,\nabla}^{12})$ *and* ψ_{ab}^{13} *is*

$$\begin{tikzcd}
 & (\varphi_a^{23}\varphi_a^{12})^*\mathcal{G} \otimes (g_{ab}^1)^*\mathcal{G} \arrow[dr] \arrow[d, Rightarrow, "\alpha"] & & \\
(\varphi_a^{23})^*\mathcal{G} \otimes (\varphi_a^{12})^*\mathcal{G} \otimes (g_{ab}^1)^*\mathcal{G} \arrow[ur] \arrow[r] \arrow[dr, Rightarrow, "\psi_{ab}^{12}"] & (\varphi_a^{23})^*\mathcal{G} \otimes (\varphi_a^{12}g_{ab}^1)^*\mathcal{G} \arrow[r] \arrow[dr, Rightarrow, "\alpha^{-1}"] & (\varphi_a^{23}\varphi_a^{12}g_{ab}^1)^*\mathcal{G} & \\
 & (\varphi_a^{23})^*\mathcal{G} \otimes (g_{ab}^2)^*\mathcal{G} \otimes (\varphi_b^{12})^*\mathcal{G} \arrow[u] \arrow[r] \arrow[dr, Rightarrow, "\psi_{ab}^{23}"'] & (g_{ab}^3\varphi_b^{23})^*\mathcal{G} \otimes (\varphi_b^{12})^*\mathcal{G} \arrow[u] \arrow[r, Rightarrow, "\alpha"] & (g_{ab}^3)^*\mathcal{G} \otimes (\varphi_b^{23}\varphi_b^{12})^*\mathcal{G} \arrow[ul] \\
1 \arrow[uu] \arrow[ur] \arrow[rr] & & (g_{ab}^3)^*\mathcal{G} \otimes (\varphi_b^{23})^*\mathcal{G} \otimes (\varphi_b^{12})^*\mathcal{G}. \arrow[u] \arrow[ur] &
\end{tikzcd} \tag{4.32}$$

The associator of this composition is given by $(\varphi_a^{34}, \varphi_a^{23}, \varphi_a^{12})^*\alpha$ *for a sequence of isomorphisms of the form*

$$\mathcal{P}_\nabla^1 \overset{(\varphi_a^{12}, \Phi_{a,\nabla}^{12}, \psi_{ab}^{12})}{\to} \mathcal{P}_\nabla^2 \overset{(\varphi_a^{23}, \Phi_{a,\nabla}^{23}, \psi_{ab}^{23})}{\to} \mathcal{P}_\nabla^3 \overset{(\varphi_a^{34}, \Phi_{a,\nabla}^{34}, \psi_{ab}^{34})}{\to} \mathcal{P}_\nabla^4. \tag{4.33}$$

Proof. Let $s_a^i : M_a \to P^i$, $\eta_{a,\nabla}^i : 1 \to s_a^*\mathcal{P}_\nabla^i$ define the cocycle data by the procedure in Proposition 4.12. Then for an isomorphism $(u, \varphi, \alpha^\varphi)$ we define $\varphi_a : M_a \to G$ by $u(s_a^1) = s_a^2\varphi_a$ and $\Phi_{a,\nabla}$ as the composition

$$1 \overset{\eta_{a,\nabla}^1}{\to} s_a^*\mathcal{P}^1 \overset{s_a^*\varphi}{\to} u(s_a^1)^*\mathcal{P}^1 \overset{(s_a^2,\varphi_a)^*\rho_2^{-1}}{\to} s_a^*\mathcal{P}^2 \otimes \varphi_a^*\mathcal{G} \overset{(\eta_{a,\nabla}^2)^{-1}}{\to} \varphi_a^*\mathcal{G}. \tag{4.34}$$

Then letting $\psi_{ab} := (s_a^1, g_{ab}^1)^*\alpha^\rho \otimes (s_a^2, \varphi_a, g_{ab}^1)^*\alpha^{\rho_2} \otimes (s_a^2, g_{ab}^2, \varphi_b)^*(\alpha^{\rho_2})^{-1}$ yields the desired cocycle condition. The rest follows similarly. □

Remark 4.20. Since isomorphisms of G-bundles and isomorphisms and 2-isomorphisms of gerbes can be inverted in a canonical, functorial way, it follows that $Gauge(\mathcal{P}_\nabla)$ has a canonical coherent inversor in the sense of Definition 3.19. In terms of the cocycle description from Proposition 4.19, the inverse of a gauge transformation described by $(\varphi_a, \Phi_{a,\nabla}, \psi_{ab})$ is the gauge transformation described by $(\varphi_a^{-1}, \tilde{\Phi}_{a,\nabla}, \tilde{\psi}_{ab})$, where $\tilde{\Phi}_{a,\nabla}$ is the trivialization with connection of $(\varphi_a^{-1})^*\mathcal{G}_\nabla$ constructed as

$$1 \overset{\varphi_a^* inv}{\to} \varphi_a^*\mathcal{G}_\nabla \otimes (\varphi_a^{-1})^*\mathcal{G}_\nabla \overset{(\Phi_{a,\nabla})^{-1}}{\to} (\varphi_a^{-1})^*\mathcal{G}_\nabla \tag{4.35}$$

(for inv defined as in Remark 3.33) and $\tilde{\psi}_{ab}$ is constructed in a similar way.

If $(g_{ab}^i, \sigma_{ab,\nabla}^i, \psi_{ab}^i)$, $i = 1, 2$, is cocycle data for $\mathcal{P}_\nabla^i$, while $(\varphi_a, \Phi_{a,\nabla}, \psi_{ab})$ is cocycle data for an isomorphism with connection $\Phi_\nabla : \mathcal{P}_\nabla^1 \to \mathcal{P}_\nabla^2$ and $(A_a^1, B_a^1) \in \mathcal{A}(\mathcal{P}_\nabla^1)$, then we define the connection $(\Phi_{\nabla}^{-1})^*(A,B) \in \mathcal{A}(\mathcal{P}^2_{\nabla})$ by

$$A^2_a := Ad(\varphi_a)A^1_a - \varphi_a^*\theta^R, \tag{4.36}$$
$$B^2_a := B_a - \langle\varphi_a^*\theta^L \wedge A^1_a\rangle - F_a, \tag{4.37}$$

where $F_a \in \Omega^2(M_a, \mathfrak{k})$ is the curvature of $\Phi_{a,\nabla}$ with respect to the Maurer-Cartan curving. One can check that this is well-defined by noting that

$$dF_a = \frac{1}{6}\langle\varphi_a^*\theta^L \wedge [\varphi_a^*\theta^L \wedge \varphi_a^*\theta^L]\rangle, \tag{4.38}$$
$$F_a - F_b = F^2_{ab} - \langle(g^2_{ab})^*\theta^L \wedge \varphi_b^*\theta^R\rangle - F^1_{ab} + \langle\varphi_a^*\theta^L \wedge (g^1_{ab})^*\theta^R\rangle, \tag{4.39}$$

as it follows from the existence of ψ_{ab}. In particular, note that isomorphisms with connection that are related by a flat 2-isomorphism act in the same way.

4.1.4 Comparison with adjusted connections

Let $(\tilde{G}, H, f, \triangleright)$ be a Lie crossed module and let $\mathfrak{G}$ be its corresponding Lie 2-group. As proven in [205], a $\mathfrak{G}$-bundle over a manifold M is described in a cover $\{M_a\}_a$ of M by maps $\tilde{g}_{ab} : M_{ab} \to \tilde{G}$, $h_{abc} : M_{abc} \to H$ such that

$$\begin{aligned} t(h_{abc})\tilde{g}_{ab}\tilde{g}_{bc} &= \tilde{g}_{ac}, \\ h_{acd}h_{abc} &= h_{abd}\tilde{g}_{ab} \triangleright h_{bcd}. \end{aligned} \tag{4.40}$$

Now let $\tilde{\kappa}$ be an adjustment on $(\tilde{G}, H, f, \triangleright)$ (cf. Definition 3.58). Then [220] provides the following cocycle data for *adjusted connections* on it. They are given by $\Lambda_{ab} \in \Omega^1(M_{ab}, \mathfrak{h})$, $\tilde{A}_a \in \Omega^1(M_a, \tilde{\mathfrak{g}})$ and $\tilde{B}_a \in \Omega^2(M_a, \mathfrak{h})$ satisfying

$$\begin{aligned} \Lambda_{bc} + g_{bc}^{-1} \triangleright \Lambda_{ab} &= \Lambda_{ac} - \tilde{g}_{ac}^{-1} \triangleright (h^*_{abc}\theta^R + \tilde{A}_a \triangleright h_{abc} \cdot h^{-1}_{abc}), \\ \tilde{A}_b - Ad(\tilde{g}_{ab}^{-1})A_a &= \tilde{g}^*_{ab}\theta^L - f(\Lambda_{ab}), \\ \tilde{g}_{ab}^{-1} \triangleright \tilde{B}_a - \tilde{B}_b &= -d\Lambda_{ab} - \frac{1}{2}[\Lambda_{ab} \wedge \Lambda_{ab}] - \tilde{A}_b \triangleright \Lambda_{ab} \\ &\quad + \tilde{\kappa}\left(\tilde{g}_{ab}^{-1}, d\tilde{A}_a + \frac{1}{2}[\tilde{A}_a \wedge \tilde{A}_a] + f(\tilde{B}_a)\right). \end{aligned} \tag{4.41}$$

An *isomorphism of adjusted connections* $(\Lambda_{ab}, \tilde{A}_a, \tilde{B}_a) \to (\Lambda'_{ab}, \tilde{A}'_a, \tilde{B}'_a)$ is a collection of 1-forms $\lambda_a \in \Omega^1(M_a, \mathfrak{h})$ such that

$$\begin{aligned} \Lambda'_{ab} - \Lambda_{ab} &= \lambda_b - \tilde{g}_{ab}^{-1} \triangleright \lambda_a, \\ \tilde{A}'_a - \tilde{A}_a &= -f(\lambda_a), \\ \tilde{B}'_a - \tilde{B}_a &= d\lambda_a + \frac{1}{2}[\lambda_a \wedge \lambda_a] + \tilde{A}'_a \triangleright \lambda_a. \end{aligned} \tag{4.42}$$

An important property of adjusted connections that can be derived directly from the above cocycle data is the following equivariance relation:

$$d\tilde{A}_b + \frac{1}{2}[\tilde{A}_b \wedge \tilde{A}_b] = Ad(\tilde{g}_{ab}^{-1})\left(d\tilde{A}_a + \frac{1}{2}[\tilde{A}_a \wedge \tilde{A}_a]\right) - f\left(d\Lambda_{ab} + \frac{1}{2}[\Lambda_{ab} \wedge \Lambda_{ab}] + \tilde{A}_b \triangleright \Lambda_{ab}\right) \tag{4.43}$$

If $r : \tilde{\mathfrak{g}} \to \mathfrak{h}$ is the map corresponding to splittings (s, l) as in (3.186), then an adjusted connection $(\Lambda_{ab}, \tilde{A}_a, \tilde{B}_a)$ is said to be *fake-flat with respect to* r if

$$r\left(d\tilde{A}_a + \frac{1}{2}[\tilde{A}_a \wedge \tilde{A}_a] + f\tilde{B}_a\right) = 0. \tag{4.44}$$

Proposition 4.21. *Let $(\tilde{G}, H, f, \triangleright, \tilde{\kappa})$ be a central Lie crossed module with an adjustment $\tilde{\kappa}$ obtained from a strong adjustment (s, κ) and $l : \mathfrak{g} \to \tilde{\mathfrak{g}}$ as in Remark 3.65. Let $\mathfrak{G}$ be the corresponding Lie 2-group and let $\mathcal{G}_\nabla$ be the corresponding multiplicative gerbe with connection by Proposition 3.66. There is an equivalence of bicategories between the bicategory of $\mathcal{G}$-bundles with connection in the sense of Definition 4.7 and the bicategory of $\mathfrak{G}$-bundles with adjusted connections that are fake-flat with respect to r.*

Proof. The equivalence between the bicategory of $\mathcal{G}$-bundles and the bicategory of $\mathfrak{G}$-bundles is Proposition 4.3, so we only need to prove that the categories of connections on a fixed bundle are equivalent in both approaches. For this we use first Proposition 4.12 adapted to the multiplicative gerbe with connective structure from Proposition 3.66. Unwinding the definitions and using the formulas for $\langle \cdot, \cdot \rangle$ and Θ^L, this is: $\sigma_{ab} \in \Omega^1(M_{ab}, \mathfrak{t})$, $A_a \in \Omega^1(M_a, \mathfrak{g})$, $B_a \in \Omega^2(M_a, \mathfrak{t})$ such that

$$\sigma_{ab} - \sigma_{ac} + \sigma_{bc} = -\kappa(\tilde{g}_{bc}, \tilde{g}_{ab}^*\theta^L) - s(\tilde{g}_{ac}^{-1} \triangleright h_{abc}^*\theta^R), \tag{4.45a}$$

$$A_b = Ad(g_{ab}^{-1})A_a + g_{ab}^*\theta^L, \tag{4.45b}$$

$$B_b - B_a = d\sigma_{ab} - \frac{1}{2}(\partial_1\kappa(\tilde{g}_{ab}^*\theta^L \wedge \tilde{g}_{ab}^*\theta^L) + \partial_1\kappa(A_a \wedge \tilde{g}_{ab}^*\theta^R) - \partial_1\kappa(\tilde{g}_{ab}^*\theta^R \wedge A_a)), \tag{4.45c}$$

where $g_{ab} := \pi(\tilde{g}_{ab}) : M_{ab} \to G$. An isomorphism of connections $(\sigma_{ab}, A_a, B_a) \to (\sigma'_{ab}, A'_a, B'_a)$ can only exist if $A_a = A'_a$ and is then given by $\lambda_a \in \Omega^1(M_a, \mathfrak{t})$ such that

$$\sigma'_{ab} - \sigma_{ab} = \lambda_b - \lambda_a, \tag{4.46}$$

$$B'_a - B_a = d\lambda_a. \tag{4.47}$$

Now letting $\xi_{ab} := (\kappa(\tilde{g}_{ab}, l(A_a)) + r(Ad(\tilde{g}_{ab}^{-1})l(A_a) + \tilde{g}_{ab}^*\theta^L))$, a quick computation reveals that

$$\kappa(\tilde{g}_{bc}, \tilde{g}_{ab}^*\theta^L) = \tilde{g}_{bc}^{-1} \triangleright \xi_{ab} - \xi_{ac} + \xi_{bc} + \tilde{g}_{ac}^{-1} \triangleright (l(A_a) \triangleright h_{abc} \cdot h_{abc}^{-1}) + rf(\tilde{g}_{ac}^{-1} \triangleright h_{abc}^*\theta^R), \tag{4.48}$$

$$s(d\xi_{ab}) = \partial_1\kappa(\tilde{g}_{ab}^*\theta^L \wedge Ad(\tilde{g}_{ab}^{-1})l(A_a)) + \kappa(\tilde{g}_{ab}, dl(A_a)), \tag{4.49}$$

which can be used to see that

$$\Lambda_{ab} := \xi_{ab} + \sigma_{ab}, \tag{4.50}$$

$$\tilde{A}_a := l(A_a), \tag{4.51}$$

$$\tilde{B}_a := B_a + \frac{1}{2}\partial_1\kappa(l(A_a) \wedge l(A_a)) - \frac{1}{2}r[l(A_a) \wedge l(A_a)] \tag{4.52}$$

is an adjusted connection which is fake-flat with respect to r. It is then clear that this map can be extended to a fully faithful functor. It is also essentially surjective, since for $(\Lambda_{ab}, \tilde{A}_a, \tilde{B}_a)$ a r-fake-flat adjusted connection, one has that $\sigma_{ab} := s(\Lambda_{ab} - \kappa(\tilde{g}_{ab}, \tilde{A}_a))$, $A_a := \pi(\tilde{A}_a)$, $B_a := s(\tilde{B}_a - \frac{1}{2}\partial_1\kappa(\tilde{A}_a \wedge \tilde{A}_a))$ is a $\mathcal{G}$-connection that maps under the above functor to an adjusted connection isomorphic to the original one. An isomorphism is given by the one-forms $\lambda_a := r(\tilde{A}_a)$ (here the fake-flatness condition is used to prove that this is indeed an isomorphism). □

Proposition 4.21 leads us to the following two observations. Firstly, we recall that the models for the String(G) 2-groups that are currently known are either explicit models as infinite-dimensional (adjusted, central) Lie crossed module, or non-explicit models as finite-dimensional multiplicative gerbe. Our cocycle equations (4.41) combine advantages from both approaches, since they can be defined over a fixed String(G)-principal bundle defined by transition functions with values in an explicit crossed module, but model connections in terms of differential forms taking values in the finite dimensional vector spaces $\mathfrak{g}$, $\mathfrak{t}$, allowing for a good theory of moduli spaces.

Secondly, we note that in [220] there is no definition of a *connective structure* on a bundle for an adjusted crossed module. In fact, in the cocycle equations (4.41), Λ_{ab} and $\tilde{A}_a$ are coupled, so in principle it is unclear how one could write such definition. Our Proposition 4.21 decouples the cocycle data, so in particular it gives the first notion of *connective structure* on a bundle for an adjusted, central Lie crossed module. This is particularly important in the holomorphic category, since Theorem 5.26 below states that supersymmetric equations in heterotic string theory are related to the existence of holomorphic connective structures, which cannot be defined with the approach in [220].

4.2 Courant algebroids

4.2.1 The Atiyah algebroid of a principal 2-bundle

The following definition is a mild generalization of the notion of Courant algebroid based on [52, 140, 227].

Definition 4.22. Let M be a manifold and let V be a vector space. A *V-Courant-Dorfman algebroid* over M is a quadruple $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], d_E)$, where

1. $E \to M$ is a smooth real vector bundle,
2. $\langle\cdot,\cdot\rangle : \Gamma(E) \otimes_{C^\infty(M)} \Gamma(E) \to C^\infty(M, V)$ is a symmetric $C^\infty(M,\mathbb{R})$-bilinear map,
3. $[\cdot,\cdot] : \Gamma(E) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E)$ is a $\mathbb{R}$-bilinear map,
4. $d_E : C^\infty(M, V) \to \Gamma(E)$ is a $\mathbb{R}$-linear map

such that, for e_1, e_2, $e_3 \in \Gamma(E)$, s, $t \in C^\infty(M, V)$ and $f \in C^\infty(M)$,

$$\langle e_1, d_E\langle e_2, e_3\rangle\rangle = \langle [e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle, \tag{4.53}$$

$$[e_1, e_2] + [e_2, e_1] = d_E\langle e_1, e_2\rangle, \tag{4.54}$$

$$[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]], \tag{4.55}$$

$$[e_1, f e_2] = f[e_1, e_2] + \pi(e_1)(f)e_2, \tag{4.56}$$

$$[d_E s, e] = 0, \tag{4.57}$$

$$\langle d_E s, d_E t\rangle = 0, \tag{4.58}$$

$$d_E(fs)(p) = d_E(f \cdot s(p))(p) + f(p) \cdot (d_E s)(p), \tag{4.59}$$

where $\pi : \Gamma(E) \to \Gamma(TM)$ is the *anchor map* defined as $\pi(e)(f)v = \langle e, d_E(fv)\rangle$ for any $v \in V$. The *coanchor map* is the unique vector bundle map $\pi^* : T^*M \otimes V \to E$ determined by $d_E : C^\infty(M, V) \to \Gamma(E)$ as $\pi^*(ds) = d_E s$ for $s \in C^\infty(M, V)$ and d the exterior derivative. A V-Courant-Dorfman algebroid $E \to M$ is called *transitive* if π is surjective and it is called *exact* if the sequence

$$0 \to T^*M \otimes V \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \to 0$$

is exact. We say E is in fact a *V-Courant algebroid* if the pairing $\langle\cdot,\cdot\rangle$ is non-degenerate.

Let T be an abelian Lie group and let $\mathcal{L}_\nabla \to M$ be a T-gerbe with connective structure over M described on a cover $\{M_a\}_a$ of M by $\lambda_{abc} : M_{abc} \to T$ and $\sigma_{ab} \in \Omega^1(M_{ab}, \mathfrak{t})$. As described in [243], one may construct an exact $\mathfrak{t}$-Courant algebroid $E_{\mathcal{L}_\nabla}$ from $\mathcal{L}_\nabla$. As a vector bundle, $E_{\mathcal{L}_\nabla} = \sqcup_{a\in A} TM_a \oplus (T^*M_a \otimes V)/\sim$ with the equivalence relation

$$(a, u_x + \xi_x) \sim (b, u_x + \xi_x + \iota_{u_x} d\sigma_{ab|x}) \tag{4.60}$$

for a, $b \in A$, $(x, u_x + \xi_x) \in TM_{ab} \oplus T^*M_{ab} \otimes V$. Thus a section of $E_{\mathcal{L}_\nabla}$ is an object of the form $X + \{\xi_a\}_a$ with $X \in \Gamma(TM)$ and $\xi_a \in \Gamma(T^*M_a \otimes V)$ satisfying $\xi_b = \xi_a + \iota_X d\sigma_{ab}$. The $\mathfrak{t}$-Courant structure is given by

$$\langle X + \{\xi_a\}_a, Y + \{\eta_a\}_a\rangle := \frac{1}{2}(\xi_a(Y) + \eta_a(X)), \tag{4.61}$$

$$[X + \{\xi_a\}_a, Y + \{\eta_a\}_a] := [X, Y] + \{L_X\eta_a - \iota_Y d\xi_a\}_a, \tag{4.62}$$

$$d_E(f) := 0 + \{df\}_{a\in A}. \tag{4.63}$$

An enhanced curving on $\mathcal{L}_\nabla$ described by (B_a, g) with $B_a \in \Omega^2(M_a, \mathfrak{t})$ such that $B_a - B_b = d\sigma_{ab}$ and $g \in \Gamma(S^2T^*M \otimes \mathfrak{t})$ induces a splitting of $\pi : E_{\mathcal{L}_\nabla} \to TM$, defined by sending $X \in \Gamma(TM)$ to the section $X + \{\iota_X B_a + g(X, \cdot)\}_a$. This yields an isomorphism $E = TM \oplus T^*M \otimes \mathfrak{t}$ and the $\mathfrak{t}$-Courant structure is given in this form by

$$\langle X + \xi, Y + \eta\rangle := \frac{1}{2}(\xi(Y) + \eta(X)) + g(X, Y), \tag{4.64}$$

$$[X + \xi, Y + \eta] := [X, Y] + L_X\eta - \iota_Y d\xi + 2\nabla^{*,g,H}X(Y, \cdot), \tag{4.65}$$

$$d_E(f) := 0 + df, \tag{4.66}$$

where $\nabla^{*,g,H}X \in \Gamma(T^*M \otimes T^*M \otimes \mathfrak{t})$ is defined as

$$\begin{aligned} 2\nabla^{*,g,H}X(Y, Z) :=& H(X, Y, Z) + X(g(Y, Z)) - Y(g(X, Z)) \\ &+ Z(g(X, Y)) - g([X, Y], Z) - g(Y, [X, Z]) + g(X, [Y, Z]), \end{aligned} \tag{4.67}$$

for $H := dB_a \in \Omega^3(M, \mathfrak{t})$ the curvature of the curving. Formula (4.65) can be obtained by using first the isotropic splitting induced by just $\{B_a\}$, which induces an isomorphism with the standard H-twisted exact Courant algebroid, and then computing the bracket $[X + g(X, \cdot), Y + g(Y, \cdot)]$. The notation is chosen because, when g is non-degenerate, then $\nabla^{*,g,H}X(Y, Z) = g(\nabla^{g,H}_Z X, Y)$ for $\nabla^{g,H}$ the unique g-metric connection with torsion

$$g(\nabla^{g,H}_X Y - \nabla^{g,H}_Y X - [X, Y], Z) = H(X, Y, Z). \tag{4.68}$$

That is,

$$g(\nabla^{g,H}_Z X, Y) = g(\nabla^g_Z X, Y) + \frac{1}{2}H(X, Y, Z), \tag{4.69}$$

for ∇^g the Levi-Civita connection of g.

The following result generalizes this construction for an arbitrary $\mathcal{G}_\nabla$-bundle $\mathcal{P}_\nabla$. One way to prove it is to show that the exact Courant algebroid $\hat{E} \to P$ associated to the gerbe with connective structure $\mathcal{P}_\nabla \to P$ has the necessary structure to descend to a $\mathfrak{t}$-Courant-Dorfman algebroid $E \to M$ by the procedure in [66], and this follows from Lemma 4.9 as in [120]. However, for the purposes of this thesis it is better to have a cocycle description of E, which we proceed to present. This generalizes a construction for String(n)-bundles in [245].

Theorem 4.23. *Let $\mathcal{G}_\nabla$ be a multiplicative T-gerbe with connective structure over G and let M be a manifold. There is a functor $\mathcal{P}_\nabla \mapsto E_{\mathcal{P}_\nabla}$ from the bicategory of $\mathcal{G}_\nabla$-bundles over M to the category of transitive $\mathfrak{t}$-Courant-Dorfman algebroids over M with the following properties.*

1. *For each $\mathcal{P}_\nabla$ there is a canonical exact sequence of vector bundles*
$$0 \to T^*M \otimes \mathfrak{t} \to E_{\mathcal{P}_\nabla} \to TP/G \to 0,$$
where the map $E_{\mathcal{P}_\nabla} \to TP/G$ preserves the anchor and the bracket.

2. *There is a canonical bijection between the set of enhanced connections on $\mathcal{P}_\nabla$ and the set of splittings of $\pi : E_{\mathcal{P}_\nabla} \to TM$.*

3. *Any enhanced connection $(A, B, g) \in \mathcal{A}_{en}(\mathcal{P}_\nabla)$ with curvature $(F_A, H) \in \Omega^2(M, ad\, P) \oplus \Omega^3(M, \mathfrak{t})$ induces an isomorphism $E_{\mathcal{P}_\nabla} \cong TM \oplus ad\, P \oplus T^*M \otimes \mathfrak{t}$ on which the $\mathfrak{t}$-Courant-Dorfman structure is given by*
$$\langle X + u + \xi, Y + v + \eta \rangle = \frac{1}{2}\left(\eta(X) + \xi(Y)\right) + \langle u, v\rangle + g(X, Y), \tag{4.70}$$
$$\begin{aligned}[X + u + \xi, Y + v + \eta] &= [X, Y] + (-[u, v] + \nabla^A_X v - \nabla^A_Y u - F_A(X, Y)) \\ &\quad + (L_X\eta - \iota_Y d\xi + 2\nabla^{*,g,H} X(Y, \cdot) + 2\langle \nabla^A u, v\rangle + 2\langle \iota_X F_A, v\rangle - 2\langle \iota_Y F_A, u\rangle),\end{aligned} \tag{4.71}$$
$$d(f) = 0 + 0 + df. \tag{4.72}$$

4. *$E_{\mathcal{P}_\nabla}$ is a $\mathfrak{t}$-Courant algebroid if and only if the pairing $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{t}$ is non-degenerate.*

We call $E_{\mathcal{P}_\nabla}$ the Atiyah algebroid *of $\mathcal{P}_\nabla$.*

Proof. If $\mathcal{P}_\nabla$ is described by $(\{g_{ab}\}, \{\sigma_{ab,\nabla}\}, \{\tau_{abc}\})$ in a cover $\{M_a\}_{a\in A}$ of M, write $F_{ab} \in \Omega^2(M_{ab}, \mathfrak{t})$ for the curvature of $\sigma_{ab,\nabla}$. Then define $E_{\mathcal{P}_\nabla} := \sqcup_a TM_a \oplus \mathfrak{g} \oplus T^*M_a \otimes \mathfrak{t}/\sim$, where the equivalence relation is

$$(a, x, v_x, e, \xi_x) \sim (b, x, v_x, Ad(g_{ab}^{-1})e - \iota_{v_x} g_{ab}^* \theta^L, \xi_x + \iota_{v_x} F_{ab} - \langle \iota_{v_x} g_{ab}^* \theta^R - 2e, g_{ab}^* \theta^R \rangle), \tag{4.73}$$

for $a, b \in A$, $x \in M_{ab}$, $v_x \in T_x M$, $e \in \mathfrak{g}$, $\xi_x \in T_x^* M \otimes \mathfrak{t}$. This is a well-defined equivalence relation by (4.24). In other words, sections of E can be described as $X + \{f_a + \xi_a\}_a$ with $X \in \Gamma(TX)$, $f_a \in C^\infty(M_a, \mathfrak{g})$, $\xi_a \in \Omega^1(M_a, \mathfrak{t})$ satisfying

$$f_b = Ad(g_{ab}^{-1}) f_a - \iota_X g_{ab}^* \theta^L, \tag{4.74}$$

$$\xi_b = \xi_a + \iota_X F_{ab} - \langle \iota_X g_{ab}^* \theta^R - 2 f_a, g_{ab}^* \theta^R \rangle. \tag{4.75}$$

The $\mathfrak{t}$-Courant-Dorfman structure is given by

$$\langle X + \{f_a + \xi_a\}, Y + \{g_a + \eta_a\} \rangle = \frac{1}{2} (\eta_a(X) + \xi_a(Y)) + \langle f_a, g_a \rangle, \tag{4.76}$$

$$\begin{aligned}[X + \{f_a + \xi_a\}, Y + \{g_a + \eta_a\}] = [X, Y] &+ (-[f_a, g_a] + X(g_a) - Y(f_a)) \\ &+ (L_X \eta_a - \iota_Y d\xi_a + 2\langle df_a, g_a \rangle),\end{aligned} \tag{4.77}$$

$$d(f) = 0 + (0 + df). \tag{4.78}$$

(4.24) and the fact that $dF_{ab} = \frac{1}{6} \langle g_{ab}^* \theta^L \wedge [g_{ab}^* \theta^L \wedge g_{ab}^* \theta^L] \rangle$ imply that these operations are well-defined, and then it is straightforward to check that they satisfy all the required axioms. An enhanced connection on the $\mathcal{G}$-bundle given by $A_a \in \Omega^1(M_a, \mathfrak{t})$, $B_a \in \Omega^2(M_a, \mathfrak{t})$, $g \in \Gamma(S^2 T^* M \otimes \mathfrak{t})$ gives a splitting of E defined by

$$X \mapsto X + \{-A_a(X) + \iota_X B_a - \langle A_a(X), A_a(\cdot) \rangle + g(X, \cdot)\}. \tag{4.79}$$

Such an enhanced connection gives an isomorphism $TM \oplus ad\, P \oplus T^* M \otimes \mathfrak{t} \to E$ by

$$X + u + \xi \mapsto X + \{-A_a(X) + u_a + \iota_X B_a - \langle A_a(X) - 2u_a, A_a(\cdot) \rangle + \xi + g(X, \cdot)\}. \tag{4.80}$$

A straightforward computation shows that the $\mathfrak{t}$-Courant-Dorfman structure is pulled-back under this isomorphism to the one in part 3 of the theorem. Part 2 is then immediate, since both spaces are torsors for the same group and we have defined an equivariant map between them.

At the level of morphisms, the functor is defined as follows. Let E^1, E^2 be the $\mathfrak{t}$-Courant-Dorfman algebroids corresponding to two $\mathcal{G}_\nabla$-bundles described by $(\{g_{ab}^1\}_{a,b}, \{F_{ab}^1\}_{a,b})$, $(\{g_{ab}^2\}_{a,b}, \{F_{ab}^2\}_{a,b})$. Then an isomorphism between the bundles given by cocycle data $(\varphi_a, \Phi_{a,\nabla}, \psi_{ab})$ yields in particular the two-forms $F_a \in \Omega^2(M_a, \mathfrak{t})$ satisfying (4.38), (4.38).

This can be used to check that

$$\Gamma(E^1) \to \Gamma(E^2) \tag{4.81}$$

$$X + \{f_a^1 + \xi_a^1\}_a \mapsto X + \{f_a^2 + \xi_a^2\}_a, \tag{4.82}$$

where

$$f_a^2 := Ad(\varphi_a)f_a^1 + \iota_X \varphi_a^* \theta^R, \tag{4.83}$$

$$\xi_a^2 := \xi_a^1 - \iota_X F_a - \langle \iota_X \varphi_a^* \theta^L + 2f_a^1, \varphi_a^* \theta^L \rangle \tag{4.84}$$

is well-defined and preserves all the structure maps. Note also that 2-isomorphic isomorphisms of $\mathcal{G}_\nabla$-bundles give the same morphism of $\mathfrak{t}$-Courant-Dorfman algebroids. □

In particular, Theorem 4.23 implies that for a fixed $\mathcal{G}_\nabla$-bundle $\mathcal{P}_\nabla \to M$ with Atiyah algebroid $E \to M$, there is a map $Gauge(\mathcal{P}_\nabla) \to Aut(E)$. Write $Aut_{\mathcal{P}}(E) \subset Aut(E)$ for the image of this map. Motivated by the observation in [245] that $Aut_{\mathcal{P}}(E) \neq Aut(E)$ for the case of String(n)-bundles, [127] defines two classes $Ham(E) \subset Aut_{\text{string}}(E) \subset Aut(E)$ of restricted automorphisms of string algebroids. We claim that $Ham(E) \subset Aut_{\mathcal{P}}(E) \subset Aut_{\text{string}}(E)$, but none of these is in general an equality, as we proceed to show.

In terms of the cocycle data above, $Aut_{\text{string}}(E)$ amounts to taking functions $\varphi_a : M_a \to G$ for an isomorphism $u : P^1 \to P^2$ of the underlying G-bundles and two-forms $F_a \in \Omega^2(M_a, \mathfrak{t})$ satisfying (4.38, 4.39). In particular, $Aut_{\mathcal{P}}(E) \subset Aut_{\text{string}}(E)$ is not an equality, as the 2-forms F_a of the automorphisms in $Aut_{\mathcal{P}}(E)$ must arise as the curvature of a trivialization of $\varphi_a^* \mathcal{G}_\nabla$ (so they form a torsor for $d\Lambda^1(X, \mathfrak{t})$ instead of $\Lambda^2_{cl}(X, \mathfrak{t})$). On the other hand, $Ham(E)$ is defined by integrating formally inside $Aut(E)$ the Lie algebra of $Aut_{\mathcal{P}}(E)$, so $Ham(E) \subset Aut_{\mathcal{P}}(E)$ but this integration fails to capture automorphisms in $Aut_{\mathcal{P}}(E)$ not connected to the identity, such as those given by topologically non-trivial line bundles with connection.

4.2.2 The gauge 2-group as a Lie 2-group

If P is a G-bundle with Atiyah algebroid $E_P = TP/G$, then $ad\, P = Ker\, \pi \subset E_P$ is a sub-Lie algebroid and there is an exponential map $\Gamma(ad\, P) \overset{exp}{\to} Gauge(P)$ which can be used to model $Gauge(P)$ as an infinite-dimensional Lie group with Lie algebra $\Gamma(ad\, P)$; in particular, $Gauge(P)$ acts through the adjoint action on $\Gamma(ad\, P)$. In this section we perform a similar construction for a $\mathcal{G}_\nabla$-bundle $\mathcal{P}_\nabla$.

Recall from Section 3.2.1 and Example 3.15 that already in the case of a T-gerbe $\mathcal{L}$ it is a good idea to fix a connective structure ∇, so that we can model the space of curvings $\mathcal{A}(\mathcal{L}_\nabla)$ as a torsor for the space $\Omega^2(M, \mathfrak{t})$ and the 2-group $BT_\nabla(M)$ as $\sqcup_{c\in C}\mathcal{A}(L^c)//C^\infty(M,T)$, where C is the discrete abelian group of isomorphism classes of T-bundles on M, each class c represented by the T-bundle L^c, and $\mathcal{A}(L^c) \cong \Omega^1(M, \mathfrak{t})$ is the space of connections on L^c. In this description, $\mathcal{A}(\mathcal{L}_\nabla)$ can be considered as a Fréchet manifold and $BT_\nabla(M)$ can be considered as a Fréchet Lie 2-group acting smoothly on $\mathcal{A}(\mathcal{L}_\nabla)$ and with Lie 2-algebra $C^\infty(M,\mathfrak{t}) \xrightarrow{d} \Gamma(T^*M \otimes \mathfrak{t})$.

The Lie 2-group $BT_\nabla(M)$ was also equipped with a coherent inversor in Example 3.21. As explained there, the corresponding conjugation action from Lemma 3.20 is trivial when restricted to the groupoid of connections on the trivial T-bundle. Thus, if we equip $BT_\nabla(M)$ with the trivial adjoint action on its Lie 2-algebra, then the following functor, equipped with the trivial equivariant structure, is an exponential map in the sense of Definition 3.24.

$$\begin{aligned} exp : \Gamma(T^*M \otimes \mathfrak{t})//C^\infty(M,\mathfrak{t}) &\to BT_\nabla(M) \\ \xi &\mapsto (M \times T, \theta^T + \xi), \\ (f : \xi \to \xi + df) &\mapsto \big(exp(f) : (M \times T, \theta^T + \xi) \to (M \times T, \theta^T + \xi + df)\big), \end{aligned} \tag{4.85}$$

Indeed, conditions 2a and 2b from Definition 3.24 follow from the fact that a connection on the trivial T-bundle is just a $\mathfrak{t}$-valued 1-form and from the fact that $C^\infty(M,\mathfrak{t}) \overset{exp}{\to} C^\infty(M,T)$ is a local diffeomorphism.

Now let $\mathcal{P}_\nabla$ be a $\mathcal{G}_\nabla$-bundle with Atiyah algebroid E. We write from now on $ad\,\mathcal{P}_\nabla := Ker\,\pi \subset E$. This is a sub-Courant-Dorfman algebroid of E fitting in exact sequences

$$0 \to T^*M \otimes \mathfrak{t} \to ad\,\mathcal{P}_\nabla \to ad\,P \to 0, \tag{4.86}$$
$$0 \to ad\,\mathcal{P}_\nabla \to E \to TM \to 0. \tag{4.87}$$

Recall from Theorem 4.23 that isomorphisms of $\mathcal{G}_\nabla$-bundles induce isomorphisms of their corresponding Atiyah algebroids. In particular, $Gauge(\mathcal{P}_\nabla)$ acts on the space of sections of the Atiyah algebroid of $\mathcal{P}_\nabla$. One can check that this action preserves $\Gamma(ad\,\mathcal{P}_\nabla)$ and is trivial on all of $\Gamma(T^*M \otimes \mathfrak{t})$. We call this the *adjoint action* of $Gauge(\mathcal{P}_\nabla)$ because, as it will follow from Theorem 4.26, there is a structure of Lie 2-group on $Gauge(\mathcal{P}_\nabla)$ such that this is an adjoint action in the sense of Definition 3.23.

Remark 4.24. It follows from the proof of Theorem 4.23 that the adjoint action of $Gauge(\mathcal{P}_\nabla)$ can be described as follows. Given $s + \xi \in \Gamma(ad\,P) \oplus \Gamma(T^*M \otimes \mathfrak{t})$, write $(s + \xi)_{(A,B)} \in \Gamma(ad\,\mathcal{P}_\nabla)$ for the section corresponding to $s + \xi$ through the isomorphism

$\Gamma(ad\,\mathcal{P}_\nabla) = \Gamma(T^*M \otimes \mathfrak{t}) \oplus \Gamma(ad\,P)$ induced by a connection $(A, B) \in \mathcal{A}(\mathcal{P}_\nabla)$. Then the action can be written as

$$u \cdot (s+\xi)_{(A,B)} = (Ad(g_u)s + \xi + 2\langle u^*A - A, Ad(g_u)s\rangle)_{(A,B)} = (Ad(g_u)s+\xi)_{u\cdot(A,B)} \quad (4.88)$$

for $u \in Gauge(\mathcal{P}_\nabla)_0$, where $g_u : M \to Ad\,P$ denotes the corresponding gauge transformation of P and $u \cdot (A, B)$ denotes the action of u on the connection (A, B).

Write $\Gamma(ad\,\mathcal{P}_\nabla)//C^\infty(M, \mathfrak{t})$ for the quotient groupoid associated to the 2-step complex of vector spaces $C^\infty(M, \mathfrak{t}) \overset{d_E}{\to} \Gamma(ad\,\mathcal{P}_\nabla)$. Instead of defining a smooth structure on $Gauge(\mathcal{P}_\nabla)$ and then trying to define an exponential map in the sense of Definition 3.24, we will construct first a continuous functor $exp : \Gamma(ad\,\mathcal{P}_\nabla)//C^\infty(M, \mathfrak{t}) \to Gauge(\mathcal{P}_\nabla)$ and then use it to define a smooth structure on $Gauge(\mathcal{P}_\nabla)$. This is analogous to how a topological group G can be given a smooth structure by giving a local homeomorphism from a neighborhood of 0 on a vector space to a neighborhood of $1 \in G$, and then translating this chart with the group product of G.

Proposition 4.25. *Let $\mathcal{P}_\nabla$ be a $\mathcal{G}_\nabla$-bundle with Atiyah algebroid E. Then there is a continuous functor $exp : \Gamma(ad\,\mathcal{P}_\nabla)//C^\infty(M, \mathfrak{t}) \to Gauge(\mathcal{P}_\nabla)$ fitting in a sequence*

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(T^*M\otimes\mathfrak{t})//C^\infty(M,\mathfrak{t}) \arrow[r] \arrow[d, "exp"] & \Gamma(ad\,\mathcal{P}_\nabla)//C^\infty(M,\mathfrak{t}) \arrow[r] \arrow[d, "exp"] & \Gamma(ad\,P) \arrow[r] \arrow[d, "exp"] & 0 \\
1 \arrow[r] & BT_\nabla(M) \arrow[r] & Gauge(\mathcal{P}_\nabla) \arrow[r] & Gauge_{\mathcal{P}}(P) \arrow[r] & 1,
\end{tikzcd} \quad (4.89)$$

where $Gauge_{\mathcal{P}}(P) \subset Gauge(P)$ is the subgroup of gauge transformations of P lifting to $\mathcal{P}$. Moreover, for $u \in Gauge(\mathcal{P}_\nabla)_0$ and $e \in \Gamma(ad\,\mathcal{P}_\nabla)$, there are canonical isomorphisms

$$(u \cdot exp(e)) \cdot u^{-1} \overset{\alpha^{Ad}(u,e)}{\to} exp(Ad(u)e) \quad (4.90)$$

such that the diagram

$$\begin{tikzcd}
((u_1\cdot u_2)\cdot exp(v))(u_2^{-1}\cdot u_1^{-1}) \arrow[r, "\alpha"] \arrow[d, "\alpha^{Ad}(u_1u_2,v)"'] & (u_1\cdot((u_2\cdot exp(v))\cdot u_2^{-1}))\cdot u_1^{-1} \arrow[d, "(id_{u_1}\cdot\alpha^{Ad}(g_2,v))\cdot id_{u_1^{-1}}"] \\
exp(u_1u_2vu_2^{-1}u_1^{-1}) & (u_1\cdot exp(u_2vu_2^{-1}))\cdot u_1^{-1} \arrow[l, "\alpha^{Ad}(u_1,u_2vu_2^{-1})"']
\end{tikzcd} \quad (4.91)$$

commutes, for α the associator of $Gauge(\mathcal{P}_\nabla)$.

Proof. Let ϵ be a trivialization of $exp^*\mathcal{G}_\nabla$, let α^ϵ be an equivariant structure on it and equip ϵ with a connection as in Theorem 3.54. We use this to define the functor exp as follows. Let $(g_{ab}, \sigma_{ab,\nabla}, \tau_{abc})$ be cocycle data for $\mathcal{P}_\nabla$. A section of $ad\,\mathcal{P}_\nabla$ is given by $f_a : M_a \to \mathfrak{g}$ and $\xi_a \in \Omega^1(M_a, \mathfrak{t})$ such that $f_b = Ad(g_{ab}^{-1})f_a$ and $\xi_b = \xi_a + 2\langle f_a, g_{ab}^*\theta^R\rangle$. Define then $\varphi_a : M_a \to G$ by $\varphi_a = exp(f_a)$ and trivializations $\Phi_{a,\nabla}$ of $\varphi_a^*\mathcal{G}_\nabla$ as $\Phi_{a,\nabla} := f_a^*\epsilon_\nabla$. There are 2-isomorphisms

$$\begin{tikzcd} (\text{Trivial}) \arrow[r, "\Phi_{a,\nabla}"] \arrow[d, "\Phi_{b,\nabla}"'] & \varphi_a^*\mathcal{G} \arrow[d, "(g_{ab}^{-1})^*inv"] \arrow[dl, phantom, "\xLeftarrow{(g_{ab}^{-1},f_a)^*\alpha^\epsilon}"] \\ \varphi_b^*\mathcal{G} & (g_{ab}^{-1})^*\mathcal{G}\otimes\varphi_a^*\mathcal{G}\otimes g_{ab}^*\mathcal{G} \arrow[l, "Ad"] \end{tikzcd} \tag{4.92}$$

with covariant derivative $(g_{ab}^{-1}, f_a)^*\eta^\epsilon = -2\langle f_a, g_{ab}^*\theta^R\rangle$. Thus these 2-isomorphisms are flat if we change the connection on $\Phi_{a,\nabla}$ by adding the 1-form ξ_a. Then equation (3.147) implies the necessary cocycle condition for $(\varphi_a, \Phi_{a,\nabla} + \xi_a, (g_{ab}^{-1}, f_a)^*\eta^\epsilon)$ to give a well-defined gauge transformation of $\mathcal{P}_\nabla$. At the level of arrows, the map exp is defined similarly as in the abelian case. Finally, a gauge transformation $u = (\tilde\varphi_a, \tilde\Phi_{a,\nabla}, \tilde\psi_{ab})$ acts on $e = f_a + \xi_a$ sending it to $u \cdot e = Ad(\tilde\varphi_a)f_a + \xi_a - 2\langle f_a, \tilde\varphi_a^*\theta^L\rangle$ and it follows from the formula in Proposition 4.19 for the product in $Gauge(\mathcal{P}_\nabla)$ and from cocycle condition (3.147) that $\alpha^{Ad}(u, e) := (\varphi_a, f_a)^*\alpha^\epsilon$ defines the desired isomorphisms $(u\circ exp(e))\circ u^{-1} \to exp(Ad(u)e)$. □

Theorem 4.26. *Let $\mathcal{P}_\nabla$ be a $\mathcal{G}_\nabla$-bundle. The 2-group $Gauge(\mathcal{P}_\nabla)$ admits a model as a Fréchet Lie 2-group with Lie 2-algebra $C^\infty(X, \mathfrak{t}) \xrightarrow{d_E} \Gamma(ad\,\mathcal{P}_\nabla)$. There is a right-invariant Maurer-Cartan form on $Gauge(\mathcal{P}_\nabla)$ for the adjoint action on $\Gamma(ad\,\mathcal{P}_\nabla)$ and the trivial action on $C^\infty(X, \mathfrak{t})$. The functor exp from Proposition 4.25 is an exponential map in the sense of Definition 3.24 for the coherent inversor from Remark 4.20.*

Proof. We proceed to construct a smooth structure for the 2-group $Gauge(\mathcal{P}_\nabla)$. First, choose a neighborhood $\mathcal{U}$ of $0 \in \Gamma(ad\,\mathcal{P}_\nabla)$ such that $exp : \mathcal{U} \to Gauge(\mathcal{P}_\nabla)_0$ is injective, which exists because it exists for the left and right vertical arrrows of (4.89), and define for each $u \in Gauge(\mathcal{P}_\nabla)_0$ the set $\mathcal{U}_u := \{exp(e) \cdot u \mid e \in \mathcal{U}\}$. Then choose a set $\Lambda \subset Gauge(\mathcal{P}_\nabla)_0$ such that every object of $Gauge(\mathcal{P}_\nabla)$ is isomorphic to some element of $\mathcal{U}_u$, for some $u \in \Lambda$. It follows that $Gauge(\mathcal{P}_\nabla)$ is equivalent to a groupoid $\mathfrak{X}$ with

$$\mathfrak{X}_0 = \bigsqcup_{u\in\Lambda} \mathcal{U}_u, \tag{4.93}$$

$$\mathfrak{X}_1 = \bigsqcup_{u_0,u_1\in\Lambda} (\mathcal{U}_{u_0} \times \mathcal{U}_{u_1})\,{}_{id\times id}\!\times_{s\times t} Gauge(\mathcal{P}_\nabla)_1. \tag{4.94}$$

Since $\mathcal{U}_u \cong \mathcal{U} \subset \Gamma(ad\,\mathcal{P}_\nabla)$, it is clear that $\mathfrak{X}_0$ is a Fréchet manifold. As for $\mathfrak{X}_1$, we model a neighborhood of any point $(exp(e_0)u_0, exp(e_1)u_1, \phi_{01}) \in \mathfrak{X}_1$ on a neighborhood $\mathcal{V} \times \mathcal{W}$ of $(0, 0)$ inside $\Gamma(ad\,\mathcal{P}_\nabla) \oplus C^\infty(X, t)$ as follows. First, choose $\mathcal{V} \subset \mathcal{U} \subset \Gamma(ad\,\mathcal{P}_\nabla)$ such that

$$(exp(e)(exp(e_0)u_0), exp(e)(exp(e_1)u_1), id_{exp(e)} \cdot \phi_{01}) \in \mathfrak{X}_1 \tag{4.95}$$

for $e \in \mathcal{V}$. Then choose $\mathcal{W} \subset C^\infty(X, \mathfrak{t})$ such that $exp : \mathcal{W} \to C^\infty(X, T)$ is injective. It follows that

$$\{(exp(e)(exp(e_0)u_0) - df, exp(e)(exp(e_1)u_1), exp(f) \cdot (id_{exp(e)} \cdot \phi_{01})) \mid (e, f) \in \mathcal{V} \times \mathcal{W}\} \tag{4.96}$$

is a neighborhood of $(exp(e_0)u_0, exp(e_1)u_1, \phi_{01}) \in \mathfrak{X}_1$ which is isomorphic to $\mathcal{V} \times \mathcal{W} \subset \Gamma(ad\,\mathcal{P}_\nabla) \oplus C^\infty(X, \mathfrak{t})$. Here we write $u + a$ when $u \in Gauge(\mathcal{P}_\nabla)_0$, $a \in \Omega^1(X, \mathfrak{t})$ for the gauge transformation of $\mathcal{P}_\nabla$ which coincides with u with connection shifted by a. It is easy to check that this gives a smooth atlas on $\mathfrak{X}_1$. Next we construct a manifold M which can serve as the total space of the product anafunctor $m : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$. We let

$$\begin{aligned} M := \{(u_1, u_2, u_{12}, e_1, e_2, e_{12}, \phi) \in & \Lambda^3 \times \mathcal{U}^3 \times Gauge(\mathcal{P}_\nabla)_1 \\ & \mid \phi : exp(e_{12})u_{12} \to (exp(e_1)u_1)(exp(e_2)u_2)\}. \end{aligned} \tag{4.97}$$

Provided that M is a manifold, there is an obvious structure making it into the total space of the desired anafunctor. To show that it is indeed a manifold, we construct local sections of $\pi_0 : M \to \mathfrak{X}_0 \times \mathfrak{X}_0$. Given $(u_1, u_2) \in \Lambda^2$, choose $u_{12} \in \Lambda$, $e_{12} \in \mathcal{U}$ and an isomorphism $\phi_{12} : exp(e_{12})u_{12} \to u_1u_2$ in $Gauge(\mathcal{P}_\nabla)$. Then for e_1, $e_2 \in \Gamma(ad\,\mathcal{P}_\nabla)$ we note that there is an isomorphism

$$\begin{aligned} (exp(e_1)exp(g_1e_2g_1^{-1}))(exp(e_{12})u_{12}) & \overset{id_{exp(e_1)exp(g_1e_2g_1^{-1})} \cdot \phi_{12}}{\to} (exp(e_1)exp(g_1e_2g_1^{-1}))(u_1u_2) \\ & \overset{id_{exp(e_1)} \cdot \alpha^{Ad^{-1}}(g_1, e_2) \cdot id_{u_2}}{\to} (exp(e_1)((u_1exp(e_2))u_1^{-1}))(u_1u_2) \\ & \overset{\alpha}{\to} (exp(e_1)u_1)(exp(e_2)u_2), \end{aligned} \tag{4.98}$$

where $g_1 \in \Gamma(Ad\,P)$ is the gauge transformation of P underlying u_1 and α is a shorthand for

$$\begin{aligned} \alpha^{-1}(exp(e_1), u_1, exp(e_2)u_2) \circ (id_{exp(e_1)} \cdot \alpha(u_1, exp(e_2), u_2))) \\ \circ (id_{exp(e_1)} \cdot \alpha(u_1exp(e_2), u_1^{-1}, u_1u_2)) \circ \alpha(exp(e_1), (u_1exp(e_1))u_1^{-1}, u_1u_2). \end{aligned} \tag{4.99}$$

Then for small e_1, e_2 the source of (4.98) remains in $\mathcal{U}_{u_{12}}$, which lets us give a local section $\sigma : U^{\phi_{12}, e_{12}}_{u_1, u_2} \subset \mathfrak{X}_0 \times \mathfrak{X}_0 \to M$. Then we can write $\pi_0^{-1}(U^{\phi_{12}, e_{12}}_{u_1, u_2}) = U^{\phi_{12}, e_{12}}_{u_1, u_2} {}_{\pi_{12}}\!\times_t \mathfrak{X}_1$, where $\pi_{12} := \pi_1 \circ \sigma$, which is a smooth manifold. For the same u_1, u_2, changing the choice of $(u_{12}, e_{12}, \phi_{12})$ to $(u'_{12}, e'_{12}, \phi'_{12})$ induces the diffeomorphism

$$\begin{aligned} & U^{\phi_{12}, e_{12}}_{u_1, u_2} {}_{\pi_{12}}\!\times_t \mathfrak{X}_1 \to U^{\phi'_{12}, e'_{12}}_{u_1, u_2} {}_{\pi'_{12}}\!\times_t \mathfrak{X}_1 \\ & (u_1, e_1, u_2, e_2, \phi) \mapsto (u_1, e_1, u_2, e_2, (id_{exp(e_1)exp(g_1e_2g_1^{-1})} \cdot (\phi'^{-1}_{12} \circ \phi_{12})) \circ \phi) \end{aligned} \tag{4.100}$$

and so this is indeed an atlas for M. One can check that the associator α and the adjoint action are smooth, for the unique way of defining them in this atlas. The functor exp from Proposition 4.25 is an exponential map in the sense of Definition 3.24 by construction of the smooth structure.

To show that $Gauge(\mathcal{P}_\nabla)$ has a Maurer-Cartan form we have to construct 1-forms

$$\theta^0 \in \Omega^1(BGauge(\mathcal{P}_\nabla)_1, \Gamma(ad\,\mathcal{P}_\nabla)), \quad \theta^1 \in \Omega^1(BGauge(\mathcal{P}_\nabla)_2, C^\infty(X, \mathfrak{t})) \tag{4.101}$$

satisfying

$$\delta\theta^1 = 0, \tag{4.102}$$

$$d_E(\theta^1) = d_2^*\theta^0 + Ad(d_2(\cdot))d_0^*\theta^0 - d_1^*\theta^0, \tag{4.103}$$

where $d_E : C^\infty(X, \mathfrak{t}) \to \Gamma(ad\,\mathcal{P}_\nabla)$ and we write $Ad(\cdot)$ for the adjoint action of $Gauge(\mathcal{P}_\nabla)$ on $\Gamma(ad\,\mathcal{P}_\nabla)$. First, θ^0 is defined simply by the inverse of the isomorphism $\Gamma(ad\,\mathcal{P}_\nabla) \to T_{(u,exp(e)u)}\mathcal{U}_u$, $\dot{e} \mapsto \frac{d}{dt}_{|t=0}(u, exp(t\dot{e})(exp(e)u))$. Then θ^1 is constructed by defining 1-forms $\theta^{u_1,u_2,\phi_{12},e_{12}} \in \Omega^1(U^{\phi_{12},e_{12}}_{u_1,u_2}\ {}_{\pi_{12}}\!\times_t \mathfrak{X}_1, C^\infty(X, \mathfrak{t})$ that are preserved by the diffeomorphisms (4.100). These are given by

$$\begin{aligned} \theta^{u_1,u_2,\phi_{12},e_{12}} &: TU^{\phi_{12},e_{12}}_{u_1,u_2}\ {}_{\pi_{12}}\!\times_t \mathfrak{X}_1 \to C^\infty(X, \mathfrak{t}) \\ &(u_1, u_2, e_1, e_2, \dot{e}_1, \dot{e}_2, \dot{\phi}) \mapsto f, \end{aligned} \tag{4.104}$$

where we are using θ^0 to see $\dot{e}_i \in T_{(u_i,exp(e_i)u_i)}U_{u_i} = \Gamma(ad\,\mathcal{P}_\nabla)$ and f is defined as follows. Recall that ϕ, $\dot{\phi}$ are of the form

$$\begin{aligned} &\phi : s(\phi) \to (exp(e_1)exp(g_1e_2g_1^{-1}))(exp(e_{12})u_{12}), \\ &\dot{\phi} : s(\dot{\phi}) \to \frac{d}{dt}_{t=0}((exp(t\dot{e}_1)exp(e_1))exp(g_1log(exp(t\dot{e}_2)exp(e_2))g_1^{-1}))(exp(e_{12})u_{12}), \end{aligned} \tag{4.105}$$

where $log : U \subset Gauge(\mathcal{P}_\nabla)_0 \to \Gamma(ad\,\mathcal{P}_\nabla)$ is the local inverse for exp. Write

$$\begin{aligned} \dot{v} := &\frac{d}{dt}_{t=0}((exp(t\dot{e}_1)exp(e_1))exp(g_1log(exp(t\dot{e}_2)exp(e_2))g_1^{-1}))(exp(g_1e_2g_1^{-1})^{-1}exp(e_1)^{-1}) \\ = &\frac{d}{dt}_{t=0}((exp(t\dot{e}_1)exp(e_1))exp(g_1e_2g_1^{-1}))(exp(g_1e_2g_1^{-1})^{-1}exp(e_1)^{-1}) \\ + &\frac{d}{dt}_{t=0}(exp(e_1)exp(g_1log(exp(t\dot{e}_2)exp(e_2))g_1^{-1}))(exp(g_1e_2g_1^{-1})^{-1}exp(e_1)^{-1}) \end{aligned} \tag{4.106}$$

and note that there are canonical vectors in the space of arrows

$$\dot{v} \xrightarrow{\dot{\psi}} \dot{e}_1 + Ad(exp(e_1)u_1)\dot{e}_2, \tag{4.107}$$

$$\dot{v} \cdot s(\phi) \xrightarrow{\dot{\chi}} t(\dot{\phi}), \tag{4.108}$$

where $\dot{\psi}$ is constructed from α and α^{Ad}, while $\dot{\chi}$ is constructed from α and ϕ. Then it follows from how the smooth structure of $\mathfrak{X}_1$ is defined that there exists a unique $f \in C^\infty(X, \mathfrak{t})$ such that $\dot{\phi}$ coincides with the arrow

$$(\dot{e}_1 + Ad(exp(e_1)u_1)\dot{e}_2) \cdot s(\phi) - \frac{d}{dt}_{t=0} t d_E f \xrightarrow{\dot{\psi} \cdot \frac{d}{dt}_{t=0} exp(tf)} \dot{v} \cdot s(\phi) \xrightarrow{\dot{\chi}} t(\dot{\phi}), \tag{4.109}$$

and this is the f in (4.104). A direct computation shows that the 1-forms defined by (4.104) are preserved by (4.100). Now (4.103) follows directly from the definition of f, while (4.102) is obtained by a direct computation, using the cocycle property of α^{Ad}. The additional conditions in Definition 3.23 for θ^0, θ^1 to be a Maurer-Cartan form are also straightforward to check. □

Theorem 4.26 suggests that there is a differentiation of $Gauge(\mathcal{P}_\nabla)$ in the sense of Definition 3.24, relating the brackets from the Lie 2-algebra of vector fields on $Gauge(\mathcal{P}_\nabla)$ with the brackets from the Lie 2-algebra associated to the Courant-Dorfman algebroid $ad\,\mathcal{P}_\nabla$ by the procedure in [224], as in the finite-dimensional case from Proposition 3.51. We leave this for future work, as our main results from Chapter 6 do not require such construction.

4.2.3 Slice theorem

Let G, T be Lie groups with T abelian. Let $\mathcal{G}_\nabla$ be a multiplicative T-gerbe with connective structure over G and let $\mathcal{P}_\nabla \to M$ be a $\mathcal{G}_\nabla$-bundle over a manifold M with Atiyah algebroid $E \to M$ (cf. Theorem 4.23). We write $ad\,\mathcal{P}_\nabla := Ker(\pi) \subset E$.

Recall from Proposition 4.14 that $\mathcal{A}(\mathcal{P}_\nabla)$ is a right torsor for $\Omega^1(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^2(M, \mathfrak{t})$. This means that we can regard it as an infinite-dimensional manifold in which vector fields are described by functions $f : \mathcal{A}(\mathcal{P}_\nabla) \to \Omega^1(ad\,P) \oplus \Omega^2(M, \mathfrak{t})$. For each such function f, write X_f for the corresponding vector field and $f = (f^a, f^b)$ for its decomposition in terms of the projections onto $\Omega^1(ad\,P)$ and $\Omega^2(M, \mathfrak{t})$. In this description, the Lie bracket of vector fields corresponds to

$$[f, g] = L_{X_f} g - L_{X_g} f - (0, 2\langle f^a \wedge g^a \rangle), \tag{4.110}$$

which follows from computing the Lie bracket of $\Omega^1(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^2(M,\mathfrak{t})$. The gauge 2-group $Gauge(\mathcal{P}_\nabla)$ acts on $\mathcal{A}(\mathcal{P}_\nabla)$ and using Proposition 4.25 we can define an infinitesimal action of $\Gamma(ad\,\mathcal{P}_\nabla)$ on $\mathcal{A}(\mathcal{P}_\nabla)$ by

$$e \cdot (A,B) = \frac{d}{dt}_{|t=0} exp(-te)^*(A,B) \tag{4.111}$$

for $e \in \Gamma(ad\,\mathcal{P}_\nabla)$, $(A,B) \in \mathcal{A}(\mathcal{P}_\nabla)$.

Proposition 4.27. *The infinitesimal action (4.111) is given by the homomorphism of Lie algebras $\rho : \Gamma(ad\,\mathcal{P}_\nabla)/dC^\infty(M) \to \Gamma(T\mathcal{A}(\mathcal{P}_\nabla))$, $\rho = (\rho^a, \rho^b)$ defined by*

$$\rho(e)^a(A,B)(X) := -\pi_{ad\,P}[e, s^{(A,B)}(X)], \tag{4.112}$$

$$\rho(e)^b(A,B)(X,Y) := -2\langle [e, s^{(A,B)}(X)], s^{(A,B)}(Y)\rangle, \tag{4.113}$$

where $X, Y \in \Gamma(TM)$, $\pi_{ad\,P} : ad\,\mathcal{P}_\nabla \to ad\,P$ denotes the canonical projection and $s^{(A,B)} : TM \to E$ is the splitting corresponding to the connection (A,B). Equivalently, using (A,B) to identify $ad\,\mathcal{P}_\nabla = ad\,P \oplus T^*M \otimes \mathfrak{t}$, $e \mapsto v + \xi$, then

$$\rho(e)^a(A,B) = d^A v, \tag{4.114}$$

$$\rho(e)^b(A,B) = d\xi + 2\langle F_A, v\rangle. \tag{4.115}$$

Proof. Let $(g_{ab}, \sigma_{ab,\nabla}, \tau_{abc})$ be cocycle data for $\mathcal{P}_\nabla$ as in Proposition 4.4 so that $(A,B) \in \mathcal{P}_\nabla$ is given by $A_a \in \Omega^1(M_a,\mathfrak{g})$, $B_a \in \Omega^2(M_a,\mathfrak{t})$ as in Proposition 4.12 and $e \in \Gamma(ad\,\mathcal{P}_\nabla)$ is given by $f_a : M_a \to \mathfrak{g}$, $\xi_a \in \Omega^1(M_a,\mathfrak{t})$ satisfying (4.74), (4.75) (with $X = 0$). Then using formulas (4.36), (4.37) for the gauge action we can see that the infinitesimal gauge action is given upon identifying $T_{(A,B)}\mathcal{A}(\mathcal{P}_\nabla) = \Omega^1(ad\,P) \oplus \Omega^2(M,\mathfrak{t})$ as above by

$$e \cdot (A,B) = (df_a + [A_a, f_a], d\xi_a + \langle f_a, [A_a \wedge A_a]\rangle - 2\langle df_a \wedge A_a\rangle). \tag{4.116}$$

Use then (4.77) and (4.80) to show that this is equivalent to the formulas above. □

Proposition 4.27 lets us prove a slice theorem for the topological space

$$\mathcal{B}(\mathcal{P}_\nabla) := \mathcal{A}(\mathcal{P}_\nabla)/Gauge(\mathcal{P}_\nabla) \tag{4.117}$$

of equivalence classes of connections on a $\mathcal{G}_\nabla$-bundle $\mathcal{P}_\nabla$, where we identify two connections if there exists some $(u, \varphi, \alpha^\varphi) \in Gauge(\mathcal{P}_\nabla)_0$ relating them. Given a connection $(A,B) \in \mathcal{A}(\mathcal{P}_\nabla)$, consider the following elliptic operator

$$\begin{aligned} d^{(A,B)} : \Omega^0(ad\,P) \oplus \Omega^1(M,\mathfrak{t}) &\to \Omega^1(ad\,P) \oplus \Omega^2(M,\mathfrak{t}) \\ v + \xi &\mapsto d^A v + d\xi + 2\langle F_A, v\rangle. \end{aligned} \tag{4.118}$$

We choose a Riemmanian metric on M and a positive-definite, Ad-invariant pairing on $\mathfrak{g}$ (not necessarily related to the bilinear form arising from the 2-group), and we let $d^{(A,B)*}$ be the adjoint of the above operator with respect to these metrics.

Theorem 4.28. *Let G, T be Lie groups with T abelian. Let $\mathcal{G}_\nabla$ be a multiplicative T-gerbe with connective structure over G and let $\mathcal{P}_\nabla \to M$ be a $\mathcal{G}_\nabla$-bundle. Then, for each $(A,B) \in \mathcal{A}(\mathcal{P}_\nabla)$, the map*

$$\begin{aligned} \{a+b \in \Omega^1(ad\,P) \oplus \Omega^2(M,\mathfrak{t}) \mid d^{(A,B)*}(a+b) = 0\}/\Gamma_A \to \mathcal{B}(\mathcal{P}_\nabla) \\ [(a,b)] \mapsto [(A,B)+(a,b)] \end{aligned} \tag{4.119}$$

is a local homeomorphism around 0, where $\Gamma_A \subset Gauge(P)$ is the isotropy subgroup of A, acting on $\Omega^1(ad\,P)$ through the adjoint action.

Proof. Fix $(A,B) \in \mathcal{A}(\mathcal{P}_\nabla)$ and consider the map

$$\begin{aligned} Im(d^{(A,B)*}) \times \Omega^1(ad\,P) \oplus \Omega^2(X,\mathfrak{t}) \to Im(d^{(A,B)*}) \\ (s+\xi,(a,b)) \mapsto d^{(A,B)*}((exp(s+\xi)\cdot((A,B)+(a,b))) - (A,B)). \end{aligned} \tag{4.120}$$

Its partial differential at 0 is

$$d^{(A,B)^*}d^{(A,B)} : Im(d^{(A,B)*}) \to Im(d^{(A,B)*}), \tag{4.121}$$

which is an isomorphism, and so by the implicit function theorem there exist neighborhoods of zero $U \subset \Omega^1(ad\,P)\oplus\Omega^2(X,\mathfrak{t})$, $V \subset Im(d^{(A,B)*})$ and a map $h : U \to V$ inducing a homeomorphism

$$U \to \{(s+\xi,(a,b)) \in V\times U \mid d^{(A,B)*}(exp(s+\xi)\cdot((A,B)+(a,b))-(A,B)) = 0\}. \tag{4.122}$$

This means that (4.119) is locally surjective around 0, and that in a neighborhood of $[(A,B)]$ the conditions

$$[(A,B)+(a_1,b_1)] = [(A,B)+(a_2,b_2)], \tag{4.123}$$
$$d^{(A,B)*}(a_1,b_1) = d^{(A,B)*}(a_2,b_2) = 0 \tag{4.124}$$

imply that the gauge transformation relating the two connections either is the identity or it does not lie in V. Thus, to prove local injectivity of (4.119), it suffices to show that, under the conditions above, and if (a_i,b_i) are sufficiently close to 0, then one can find gauge transformations $(u_i, \varphi_i, \alpha^{\varphi_i}) \in Gauge(\mathcal{P}_\nabla)$, $i = 1, 2$, such that

$$(u_1, \varphi_1, \alpha^{\varphi_1}) \in V, \tag{4.125}$$

$$(u_2, \varphi_2, \alpha^{\varphi_2}) \cdot (A, B) = (A, B), \tag{4.126}$$

$$\begin{aligned}(u_1, \varphi_1, \alpha^{\varphi_1}) \cdot ((A, B) + (a_1, b_1)) &= (u_2, \varphi_2, \alpha^{\varphi_2}) \cdot ((A, B) + (a_2, b_2)) \\ &= (A, B) + (Ad(u_2)^{-1} a_2, b_2).\end{aligned} \tag{4.127}$$

This can be shown by applying standard estimates for the Green operator (see [169, Th 3.17] for details) of the Laplacian

$$d^{(A,B)^*} d^{(A,B)} + d^{(A,B)} d^{(A,B)^*} : \Omega^1(ad\, P) \oplus \Omega^2(M, \mathfrak{t}) \to \Omega^1(ad\, P) \oplus \Omega^2(M, \mathfrak{t}), \tag{4.128}$$

where

$$\begin{aligned} d^{(A,B)} : \Omega^1(ad\, P) \oplus \Omega^2(M, \mathfrak{t}) &\to \Omega^2(ad\, P) \oplus \Omega^3(M, \mathfrak{t}), \\ a + b &\mapsto d^A a + db - 2\langle F_A \wedge a \rangle. \end{aligned} \tag{4.129}$$

□

We conclude by noting that results similar to Theorem 4.28 have been proved in related contexts with tools from generalized geometry [84, 125, 127, 229]. The main difference is that in those papers the symmetries are given by a group whose Lie algebra is a quotient of $\Gamma(ad\, \mathcal{P}_\nabla)$, while in our approach the symmetries are given by a 2-group with Lie 2-algebra $C^\infty(X, \mathfrak{t}) \xrightarrow{d} \Gamma(ad\, \mathcal{P}_\nabla)$. Since these are spaces of sections of vector bundles, while $\Gamma(ad\, \mathcal{P}_\nabla)/\sim$ is not, elliptic operator theory applies in a much more straightforward way in our approach.

Chapter 5

Complex Lie 2-groups and holomorphic principal 2-bundles

A complex Lie 2-group $\mathfrak{G}$ is a Lie 2-group as in Definition 3.9 such that $\mathfrak{G}_0$, $\mathfrak{G}_1$ are complex manifolds and such that all structure maps are holomorphic. By a straightforward generalization of Theorem 3.48, a family of complex Lie 2-groups is given by holomorphic multiplicative gerbes, which have been studied in [59, 268] based on previous work on holomorphic gerbes [57, 82, 193]. In particular, the main result in [268] can be thought of as the construction of a complexification for the Lie 2-group $\mathrm{String}(n)$. It is also noted there that equipping holomorphic multiplicative gerbes with holomorphic connective structures is interesting for applications to complex geometry.

Given a complex Lie 2-group $\mathfrak{G}$ and a complex manifold X, one can define *holomorphic* principal $\mathfrak{G}$-bundles over X similarly as in Section 4.1. A special feature of higher gauge theory is that, at least for complex Lie 2-groups $\mathfrak{G}$ that arise from holomorphic multiplicative gerbes with holomorphic connective structure, one can also define *holomorphic $\mathfrak{G}$-bundles with holomorphic connective structure* over X. These are intermediate objects between holomorphic $\mathfrak{G}$-bundles and holomorphic $\mathfrak{G}$-bundles with holomorphic connection. The abelian case (i.e. holomorphic gerbes with holomorphic connective structure) is studied in all the literature on holomorphic gerbes [57, 82, 193], as well as in generalized complex geometry [142].

A natural problem in this context is the construction of well-behaved moduli spaces of holomorphic $\mathfrak{G}$-bundles (or holomorphic $\mathfrak{G}$-bundles with holomorphic connective structure) over X, and geometric structures on them. The analog of this problem in classical gauge theory can be approached through gauge-theoretic methods thanks to two fundamental results. Firstly, for G a complex Lie group and P a smooth G-bundle, holomorphic structures on P are in bijection with integrable semiconnections on P. Secondly, when G is the complexification of a compact Lie group K and P is the complexification of a K-bundle P_h, then integrable semiconnections on P are in bijection with connections on P_h whose curvature is of type $(1,1)$ [233]. This theorem is called the *Chern correspondence*. There are analogs of these results for holomorphic gerbes [2, 82, 142], while a similar result in the context of Courant algebroids of string type is presented in [127], relating holomorphic Courant algebroids to supersymmetric configurations in heterotic string theory.

In this chapter we study complex Lie 2-groups, and we generalize the description of holomorphic structures on principal bundles in gauge-theoretic terms to this setting. In Section 5.1.1 we review fundamental aspects of holomorphic gerbes. In Section 5.1.2 we discuss the subtleties associated to connective structures on holomorphic multiplicative gerbes, and we prove an original theorem that generalizes the construction in [268] to complexify multiplicative gerbes over arbitrary compact groups. In Section 5.1.3 we discuss how Maurer-Cartan forms interact with the complex structure of a complex Lie 2-group, and the shifted holomorphic symplectic structures that they give rise to. In Section 5.2.1 we provide a gauge-theoretic description of holomorphic structures, and holomorphic structures with holomorphic connective structure, on $\mathfrak{G}$-bundles. In Section 5.2.2 we generalize the Chern correspondence to this context, using the complexification construction and our notion of enhanced connections from Section 4.1.2. Finally, in Section 5.2.3 we relate holomorphic $\mathfrak{G}$-bundles to holomorphic Courant algebroids, which lets us establish a link between our work and [127], as well as define structure of complex Lie 2-group on the gauge 2-group of a $\mathfrak{G}$-bundle, for $\mathfrak{G}$ a complex Lie 2-group.

5.1 Complex Lie 2-groups

5.1.1 Holomorphic gerbes and holomorphic connective structures

We establish first a notational convention that will be used for the rest of the thesis. For real vector spaces V, W with complex structures J_V, J_W, a skew-symmetric $\mathbb{R}$-multilinear map $\phi : \Lambda^k V \to W$ is of *type* $(p, k-p)$ if the map $\phi^{\mathbb{C}} : \Lambda^k(V \otimes \mathbb{C}) \to W$ defined by $\phi(iv, \cdot) = J_W \phi(v, \cdot)$ is zero outside $\Lambda^p(V^{1,0}) \otimes \Lambda^{k-p}(V^{0,1}) \subset \Lambda^k(V \otimes \mathbb{C})$, where $V^{1,0}$ and $V^{0,1}$ are the i and $-i$ eigenspaces of $J_V^{\mathbb{C}} : V \otimes \mathbb{C} \to V \otimes \mathbb{C}$, respectively. The type of a symmetric map is similarly defined.

Let X be a complex manifold, let T be a complex abelian Lie group with Lie algebra $\mathfrak{t}$. Recall the definition of smooth gerbes from Section 3.2.1.

Definition 5.1 ([57, 82]). A *holomorphic T-gerbe* over X is a T-gerbe $(\{U_i\}, \{\lambda_{ijk}\})$ over X such that λ_{ijk} are holomorphic functions. In this case, a connective structure

$\{\Lambda_{ij}\}$ is *compatible with the holomorphic structure* if $\Lambda_{ij} \in \Omega^{1,0}(U_{ij}, \mathfrak{t})$. A curving $\{B_i\}$ for a compatible connective structure is itself *compatible with the holomorphic structure* if $B_i \in \Omega^{2,0+1,1}(U_i, \mathfrak{t})$. A *holomorphic isomorphism* $(\{V_a\}, \{s_{ab}\})$ between holomorphic T-gerbes is an isomorphism of T-gerbes such that s_{ab} are holomorphic. If, moreover, the holomorphic gerbes have compatible connective structures, then a connection Λ_a on a holomorphic isomorphism is *compatible with the holomorphic structure* if $\Lambda_a \in \Omega^{1,0}(V_a, \mathfrak{t})$. A *holomorphic 2-isomorphism* $(\{W_r\}, \{t_r\})$ between holomorphic isomorphisms is a 2-isomorphism such that the functions t_r are holomorphic.

A *holomorphic T-gerbe with holomorphic connective structure* is a holomorphic T-gerbe $(\{U_i\}, \{\lambda_{ijk}\})$ with a connective structure $\{\Lambda_{ij}\}$ such that $\Lambda_{ij} \in \Omega^{1,0}(U_{ij}, \mathfrak{t})$ and $\bar{\partial} A_{ij} = 0$. In this case, a curving $\{B_i\}$ is *compatible with the holomorphic connective structure* if $B_i \in \Omega^{2,0}(U_i, \mathfrak{t})$. It is a *holomorphic curving* if, moreover, $\overline{\partial} B_i = 0$. Given a holomorphic isomorphism $(\{V_a\}, \{s_{ab}\})$ between holomorphic T-gerbes with holomorphic connective structures, a *holomorphic connection* on it is a connection $\{\Lambda_a\}$ such that $\Lambda_a \in \Omega^{1,0}(V_a, \mathfrak{t})$ and $\bar{\partial}\Lambda_a = 0$.

Let $\mathcal{L} = (\{U_i\}, \{\lambda_{ijk}\})$ be a smooth T-gerbe. A *1-semiconnection* on $\mathcal{L}$ is the data of $D_{ij} \in \Omega^{0,1}(U_{ij}, \mathfrak{t})$, $D_i \in \Omega^{0,2}(U_i, \mathfrak{t})$ such that

$$D_{ij} - D_{ik} + D_{jk} = \lambda_{ijk}^* \theta^{0,1}, \tag{5.1}$$

$$D_i - D_j = (dD_{ij})^{0,2}. \tag{5.2}$$

A 1-semiconnection $(\{D_{ij}\}, \{D_i\})$ is *integrable* if

$$(dD_i)^{0,3} = 0. \tag{5.3}$$

Let $\phi = (\{U_i\}, \{s_{ij}\}) : \mathcal{L}^1 \to \mathcal{L}^2$ be an isomorphism of smooth T-gerbes. If $\mathcal{L}^1$, $\mathcal{L}^2$ have 1-semiconnections D^1, D^2, then a *1-semiconnection* on ϕ is the data of $D_i^\phi \in \Omega^{0,1}(U_i, \mathfrak{t})$ with

$$D_i^\phi - D_j^\phi = D_{ij}^1 - D_{ij}^2 - s_{ij}^* \theta^{0,1} \tag{5.4}$$

and it is *integrable* if

$$(dD_i^\phi)^{0,2} = D_i^1 - D_i^2. \tag{5.5}$$

Let $\psi = (\{U_i\}, \{t_i\}) : \phi \Rightarrow \phi' : \mathcal{L}^1 \to \mathcal{L}^2$ be a 2-isomorphism of smooth T-gerbes. If $\mathcal{L}^1$, $\mathcal{L}^2$, ϕ, ϕ' have 1-semiconnections, then we say that ψ *preserves the 1-semiconnections* if

$$D_i^{\phi'} - D_i^\phi + t_i^* \theta^{0,1} = 0. \tag{5.6}$$

Let $\mathcal{L} = (\{U_i\}, \{\lambda_{ijk}\})$ be a smooth T-gerbe. A *2-semiconnection* on $\mathcal{L}$ is the data of $D_{ij} \in \Omega^1(U_{ij}, \mathfrak{t})$, $D_i \in \Omega^{1,1+0,2}(U_i, \mathfrak{t})$ such that

$$D_{ij} - D_{ik} + D_{jk} = \lambda^*_{ijk}\theta, \tag{5.7}$$

$$D_i - D_j = (dD_{ij})^{1,1+0,2}. \tag{5.8}$$

A 2-semiconnection $(\{D_{ij}\}, \{D_i\})$ is *integrable* if

$$(dD_i)^{1,2+0,3} = 0. \tag{5.9}$$

Let $\phi = (\{U_i\}, \{s_{ij}\}) : \mathcal{L}^1 \to \mathcal{L}^2$ be an isomorphism of smooth T-gerbes. If $\mathcal{L}^1$, $\mathcal{L}^2$ have 2-semiconnections D^1, D^2, then a *2-semiconnection* on ϕ is the data of $D^\phi_i \in \Omega^1(U_i, \mathfrak{t})$ such that

$$D^\phi_i - D^\phi_j = D^1_{ij} - D^2_{ij} - s^*_{ij}\theta \tag{5.10}$$

and it is *integrable* if

$$(dD^\phi_i)^{1,1+0,2} = D^1_i - D^2_i. \tag{5.11}$$

Let $\psi = (\{U_i\}, \{t_i\}) : \phi \Rightarrow \phi' : \mathcal{L}^1 \to \mathcal{L}^2$ be a 2-isomorphism of smooth T-gerbes. If $\mathcal{L}^1$, $\mathcal{L}^2$, ϕ, ϕ' have 2-semiconnections, then we say that ψ *preserves the 2-semiconnections* if

$$D^{\phi'}_i - D^\phi_i + t^*_i\theta = 0. \tag{5.12}$$

Let $T_\mathbb{R}$ be a compact abelian Lie group with Lie algebra $\mathfrak{t}_\mathbb{R}$ and let $j_T : T_\mathbb{R} \to T$ be its complexification (inducing an inclusion $dj_T : \mathfrak{t}_\mathbb{R} \to \mathfrak{t}$). For a $T_\mathbb{R}$-gerbe with connection $(\mathcal{L}, \Lambda, B) = (\{U_i\}, \{\lambda_{ijk}\}, \{\Lambda_{ij}\}, \{B_i\})$, its *fibrewise complexification* is the smooth T-gerbe with connection $(\mathcal{L}^\mathbb{C}, \nabla^\mathbb{C}, B^\mathbb{C})$ given by $(\{U_i\}, \{j_T(\lambda_{ijk})\}, \{dj_T \circ \Lambda_{ij}\}, \{dj_T \circ B_i\})$.

Write $\mathcal{O}_T$ for the sheaf of holomorphic T-valued functions and $\Omega^1_{\bar\partial-cl,\mathfrak{t}}$ for the sheaf of holomorphic $\mathfrak{t}$-valued 1-forms.

Proposition 5.2 ([2, 57, 82, 268]). *1. Holomorphic T-gerbes over X are classified by $H^2(X, \mathcal{O}_T)$, their automorphisms are classified by $H^1(X, \mathcal{O}_T)$ and there are $H^0(X, \mathcal{O}_T)$ 2-automorphisms of a given isomorphism.*

2. A smooth gerbe is isomorphic to a holomorphic gerbe if and only if it admits a connection with curvature H satisfying $H^{0,3} = 0$. A smooth isomorphism between holomorphic gerbes with compatible connective structures is 2-isomorphic to a holomorphic isomorphism if and only if it admits a connection with curvature F satisfying $F^{0,2} = 0$. A smooth 2-isomorphism between holomorphic isomorphisms with compatible connections is holomorphic if and only if its covariant derivative η satisfies $\eta^{0,1} = 0$.

3. *The bicategory of holomorphic T-gerbes, holomorphic isomorphisms and holomorphic 2-isomorphisms is equivalent to the bicategory of smooth T-gerbes with integrable 1-semiconnections, smooth isomorphisms with integrable 1-semiconnections and smooth 2-isomorphisms preserving the* 1*-semiconnections.*

4. *Holomorphic gerbes with holomorphic connective structure over X are classified by $\mathbb{H}^2(X, \mathcal{O}_{X,T} \xrightarrow{d} \Omega^1_{\bar{\partial}-cl,\mathfrak{t}})$, their automorphisms with holomorphic connection by $\mathbb{H}^1(X, \mathcal{O}_{X,T} \xrightarrow{d} \Omega^1_{\bar{\partial}-cl,\mathfrak{t}})$ and there are $\mathbb{H}^0(X, \mathcal{O}_{X,T} \xrightarrow{d} \Omega^1_{\bar{\partial}-cl,\mathfrak{t}})$ flat 2-automorphisms of a given automorphism .*

5. *A smooth gerbe is isomorphic to a holomorphic gerbe with holomorphic connective structure if and only if it admits a connection with curvature H satisfying $H^{1,2+0,3} = 0$. A smooth isomorphism between holomorphic gerbes with compatible connective structures is* 2*-isomorphic to a holomorphic isomorphism if and only if it admits a connection with curvature F satisfying $F^{1,1+0,2} = 0$.*

6. *The bicategory of holomorphic T-gerbes with holomorphic connective structure, holomorphic isomorphisms with holomorphic connection and flat 2-isomorphisms is equivalent to the bicategory of smooth T-gerbes with integrable 2-semiconnections, smooth isomorphisms with integrable* 2*-semiconnections and smooth 2-isomorphisms preserving the* 2*-semiconnections.*

Proof. We give a brief sketch of the proof of 3. If $\mathcal{L}$ is given by holomorphic data λ_{ijk}, then we define a 1-semiconnection by $D_{ij} = 0$, $D_i = 0$. Conversely, if $(\mathcal{L}, D)$ is a smooth gerbe with 1-semiconnection given by λ_{ijk}, D_{ij}, D_i then we choose $c_i \in \Omega^{0,1}(U_i, \mathfrak{t})$, $f_{ij} \in C^\infty(U_{ij}, \mathfrak{t})$ with $\bar{\partial} c_i = D_i$ and $\bar{\partial} f_{ij} = c_i - c_j - D_{ij}$; then $\lambda_{ijk} exp(f_{ij}) exp(f_{ik})^{-1} exp(f_{jk})$ is data for a holomorphic gerbe. Different choices of c_i, f_{ij} yield canonically isomorphic holomorphic gerbes. These maps can be extended to an equivalence of bicategories in a similar way. □

For a fixed smooth gerbe $\mathcal{L}$ we note that we can form a groupoid $\mathcal{D}_{int}(\mathcal{L})$ (resp. $\mathcal{D}'_{int}(\mathcal{L})$) with objects integrable 1-semiconnections (resp. 2-semiconnections) on $\mathcal{L}$ and with arrows integrable 1-semiconnections (resp. 2-semiconnections) on the identity automorphism of $\mathcal{L}$. The 2-group $BT(M)$ acts on these in a similar way to the gauge action from Definition 3.28. Then we define the quotient 2-groupoids (cf. Section 3.1.2)

1. $\mathcal{H}(\mathcal{L}) := \mathcal{D}_{int}(\mathcal{L}) // BT(M)$.

2. $\mathcal{H}'(\mathcal{L}) := \mathcal{D}'_{int}(\mathcal{L}) // BT(M)$.

Proposition 5.20 implies that $\mathcal{H}(\mathcal{L})$ is the space of holomorphic structures on $\mathcal{L}$ and that $\mathcal{H}'(\mathcal{L})$ is the space of holomorphic structures with holomorphic connective structure on $\mathcal{L}$. For a fixed smooth gerbe with connective structure $\mathcal{L}_\nabla$ we recall that $\mathcal{A}(\mathcal{L}_\nabla)$ is the set of curvings on $\mathcal{L}_\nabla$ and we write the following.

1. $\mathcal{D}(\mathcal{L}_\nabla) = \mathcal{A}(\mathcal{L}_\nabla)/\Omega^{2,0+1,1}(X, \mathfrak{t})$.
2. $\mathcal{D}'(\mathcal{L}_\nabla) = \mathcal{A}(\mathcal{L}_\nabla)/\Omega^{2,0}(X, \mathfrak{t})$.

Finally, we write $BT_{\nabla^{0,1}}(X)$ for the groupoid whose objects are T-bundles over X equipped with an equivalence class of connections, up to addition of 1-forms in $\Omega^{1,0}(X, \mathfrak{t})$, and whose arrows are isomorphisms of T-bundles whose covariant derivative with respect to any choice of representing connection is of type $(1, 0)$. This is a 2-group, with product given by tensor product of T-bundles with connection. All these objects are considered with Fréchet topology.

Proposition 5.3. *Let $\mathcal{L}_\nabla$ be a smooth gerbe with connective structure. Then*

1. *$BT_{\nabla^{0,1}}(X)$ acts on $\mathcal{D}(\mathcal{L}_\nabla)$ and there is a canonical map of simplicial topological spaces $\{[B] \in \mathcal{D}(\mathcal{L}_\nabla) \mid H^{0,3} = 0\}//BT_{\nabla^{0,1}}(X) \to \mathcal{H}(\mathcal{L})$ inducing weak homotopy equivalence on geometric realizations.*
2. *$BT_\nabla(X)$ acts on $\mathcal{D}'(\mathcal{L}_\nabla)$ and there is a canonical map of simplicial topological spaces $\{[B] \in \mathcal{D}'(\mathcal{L}_\nabla) \mid H^{(1,2)+(0,3)} = 0\}//BT_\nabla(X) \to \mathcal{H}'(\mathcal{L})$ inducing weak homotopy equivalence on geometric realizations.*

Proof. Analogous to the proof of Proposition 3.30. □

5.1.2 Holomorphic multiplicative gerbes

Definition 5.4. Let G, T be complex Lie groups with T abelian. A *holomorphic multiplicative T-gerbe* over G is a multiplicative T-gerbe $(\mathcal{G}, m, \alpha)$ over G as in Definition 3.31 such that $\mathcal{G}$, m and α are holomorphic. A *compatible (resp. holomorphic) connective structure* on it is a compatible (resp. holomorphic) connective structure on the gerbe $\mathcal{G}$ with a compatible (resp. holomorphic) connection on the isomorphism of gerbes m such that α is a flat 2-isomorphism of gerbes.

In terms of cocycle data (3.94) in a good semi-simplicial cover of $BG_\bullet$, a holomorphic multiplicative T-gerbe over G is a multiplicative T-gerbe for which $\lambda_{i_1j_1k_1}$, $m_{i_2j_2}$, α_{i_3} can be chosen to be holomorphic. In terms of cocycle data (3.106), a compatible connective structure is one for which $A_{i_1j_1}$, M_{i_2} can be chosen to be of type $(1,0)$ and a holomorphic connective structure is one for which they can be chosen to be of type $(1,0)$ and to satisfy $\bar{\partial}A_{i_1j_1}=0$, $\bar{\partial}M_{i_2}=0$.

For a smooth multiplicative gerbe $\mathcal{G}$, we say that it *admits* a holomorphic structure (with compatible or holomorphic connective structure) if it is isomorphic as a smooth multiplicative gerbe to the underlying smooth multiplicative gerbe of a holomorphic multiplicative gerbe (with compatible or holomorphic connective structure).

Proposition 5.5 ([268])**.** *Let G, T be complex Lie groups with T abelian and let $\mathcal{G}$ be a smooth multiplicative T-gerbe over G. Then*

1. *$\mathcal{G}$ admits a holomorphic structure if and only if its de Rham class* (3.98) *admits a representative $(\tau_3,\tau_2,\tau_1,0)$ with $\tau_3^{0,3}=0$, $\tau_2^{0,2}=0$, $\tau_1^{0,1}=0$.*

2. *$\mathcal{G}$ admits a holomorphic structure with compatible connective structure if and only if its de Rham class* (3.98) *admits a representative $(\tau_3,\tau_2,\tau_1,0)$ with $\tau_3^{0,3}=0$, $\tau_2^{0,2}=0$, $\tau_1=0$.*

3. *$\mathcal{G}$ admits a holomorphic structure with holomorphic connective structure if and only if its de Rham class* (3.98) *admits a representative $(\tau_3,\tau_2,\tau_1,0)$ with $\tau_3^{1,2+0,3}=0$, $\tau_2^{1,1+0,2}=0$, $\tau_1=0$.*

Proof. Straightforward by Proposition 5.2. □

For a smooth multiplicative gerbe with connective structure $\mathcal{G}_\nabla$, we say that it *admits* a holomorphic structure with compatible (resp. holomorphic) connective structure if it is isomorphic as a smooth multiplicative gerbe with connective structure to the underlying smooth multiplicative gerbe with connective structure of a holomorphic multiplicative gerbe with compatible (resp. holomorphic) connective structure.

Proposition 5.6. *Let G, T be complex Lie groups with T abelian and let $\mathcal{G}_\nabla$ be a smooth multiplicative T-gerbe with connective structure over G. Then*

1. *$\mathcal{G}_\nabla$ admits a holomorphic structure with compatible connective structure if and only if the pairing $\langle\cdot,\cdot\rangle:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{t}$ from Theorem 3.43 satisfies $\langle\cdot,\cdot\rangle^{0,2}=0$. In this case, there is a unique such holomorphic structure with compatible connective structure up to holomorphic isomorphism with compatible connection, and the Maurer-Cartan curving is compatible with it.*

2. *$\mathcal{G}_\nabla$ admits a holomorphic structure with holomorphic connective structure if and only if the pairing $\langle\cdot,\cdot\rangle:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{t}$ from Theorem 3.43 satisfies $\langle\cdot,\cdot\rangle^{1,1+0,2}=0$. In*

this case, there is a unique such holomorphic structure with holomorphic connective structure up to holomorphic isomorphism with holomorphic connection and the Maurer-Cartan curving is holomorphic.

Proof. We only prove 1, as 2 follows similarly. If $\mathcal{G}_\nabla$ is holomorphic with compatible connective structure, then take cocycle data (3.106) such that $A_{i_1j_1}$, M_{i_2} are of type $(1,0)$ and compute Θ^L, $\langle\cdot,\cdot\rangle$ as in Remark 3.44. Since $dA^{0,2}_{i_1j_1}=0$, $dM^{0,2}_{i_2}=0$, it follows that $\langle\cdot,\cdot\rangle^{0,2}=0$, as we wanted to show. But we also see that $(\Theta^L_{i_1})^{0,2}=0$, which means that the $(0,2)$-part of the Maurer-Cartan curving is an integrable 1-semiconnection for $\mathcal{G}_\nabla$, and that the $(0,1)$-part of the connection on m is an integrable 1-semiconnection on m. Since the $(0,2)$-part of the Maurer-Cartan curving and the $(0,1)$-part of the connection on m are preserved by isomorphisms of smooth $\mathcal{G}_\nabla$-bundles, this means that a smooth $\mathcal{G}_\nabla$-bundle admits at most 1 structure of holomorphic multiplicative gerbe with compatible connective structure, and it does so whenever the $(0,2)$-part of the Maurer-Cartan curving and the $(0,1)$-part of the connection on m are integrable as 1-semiconnections. That is, when the corresponding μ, ν from (3.109) satisfy $\mu^{0,3}=0$ and $\nu^{0,2}=0$, which happens precisely when $\langle\cdot,\cdot\rangle^{0,2}=0$. □

For $\mathcal{G}$ a smooth multiplicative T-gerbe over G, recall from Definition 3.53 the notion of an equivariant structure on a trivialization of $exp^*\mathcal{G}\to\mathfrak{g}$. We can analogously define a *holomorphic equivariant structure* on a holomorphic trivialization of $exp^*\mathcal{G}\to\mathfrak{g}$. Then the proof of Theorem 3.54 can be adapted to yield the following result.

Corollary 5.7. *Let G, T be complex Lie groups with T abelian and let $\mathcal{G}$ be a holomorphic multiplicative T-gerbe over G. Then,*

1. *$\mathcal{G}$ admits a holomorphic connective structure if and only if every holomorphic trivialization ϵ of $exp^*\mathcal{G}\to\mathfrak{g}$ admits a holomorphic equivariant structure.*

2. *If $\langle\cdot,\cdot\rangle:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{t}$ corresponds to a holomorphic connective structure on $\mathcal{G}$, then any holomorphic trivialization ϵ can be equipped with a holomorphic connection such that the covariant derivative of a holomorphic equivariant structure α^ϵ is $\eta^\epsilon\in\Omega^{1,0}(G\times\mathfrak{g},\mathfrak{t})$ defined by*

$$\eta^\epsilon_{|(g,v)}(v_g+\dot{v}):=2\langle v,g^{-1}v_g\rangle. \tag{5.13}$$

Recall the fibrewise complexification $\mathcal{L}^{\mathbb{C}}$ of a gerbe $\mathcal{L}$ from Definition 5.1. Upmeier [268] used the theory of Stein manifolds to construct a holomorphic $\mathbb{C}^*$-gerbe with holomorphic connective structure over $GL(n,\mathbb{C})$ that restricts over $U(n)$ to the fibrewise complexification of the $U(1)$-gerbe $\mathrm{String}(U(n))$. The following theorem generalizes that construction.

Theorem 5.8. *Let K, $T_{\mathbb{R}}$ be compact, connected Lie groups with $T_{\mathbb{R}}$ abelian and let $j_{T_{\mathbb{R}}} : T_{\mathbb{R}} \to T$, $j_K : K \to G$ be their complexifications. For $\mathcal{K}$ any $T_{\mathbb{R}}$-multiplicative gerbe over K there is a unique holomorphic multiplicative T-gerbe with holomorphic connective structure $\mathcal{G}_{\nabla}$ over G such that $j_K^*\mathcal{G} = \mathcal{K}^{\mathbb{C}}$ as smooth multiplicative T-gerbes over K. We call $\mathcal{G}_{\nabla}$ the* complexification *of $\mathcal{K}$.*

Proof. Let $\langle\cdot,\cdot\rangle : \mathfrak{k}\otimes\mathfrak{k} \to \mathfrak{t}_{\mathbb{R}}$ be the pairing associated to $\mathcal{K}$ via Corollary 3.45, and let $\langle\cdot,\cdot\rangle_{\mathbb{C}} : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{t}$ be its complexification. If $\mathcal{G}_{\nabla}$ as in the theorem exists, then by Proposition 5.6 its pairing $\mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{t}$ must be $\langle\cdot,\cdot\rangle_{\mathbb{C}}$, as it must be $\mathbb{C}$-linear and restrict to $\langle\cdot,\cdot\rangle$ on $\mathfrak{k}$, so let us show that there exists a $\mathcal{G}_{\nabla}$ with pairing $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. By Theorem 3.43, this happens if and only if the forms $\mu_{\mathbb{C}} := \frac{1}{6}\langle\theta^L\wedge[\theta^L\wedge\theta^L]\rangle_{\mathbb{C}}$, $\nu_{\mathbb{C}} := -\langle g_1^*\theta^L\wedge g_2^*\theta^R\rangle_{\mathbb{C}}$ determine $[\mu_{\mathbb{C}}, -\nu_{\mathbb{C}}, 0, 0] \in H^4(BG, \mathfrak{t})$ such that $exp([\mu_{\mathbb{C}}, -\nu_{\mathbb{C}}, 0, 0]) = 0 \in H^4(BG, \underline{T})$. Now the inclusion $j_K : K \to G$ is a homotopy equivalence, therefore there is a commutative diagram

$$\begin{array}{ccc} H^4(BG,\mathfrak{t}) & \xrightarrow{exp} & H^4(BG,\underline{T}) \\ {\scriptstyle j_K^*}\big\downarrow & & {\scriptstyle j_K^*}\big\downarrow \\ H^4(BK,\mathfrak{t}) & \xrightarrow{exp} & H^4(BK,\underline{T}) \end{array}$$

where the vertical arrows are isomorphisms, and it is clear that $j_K^*[\mu_{\mathbb{C}}, -\nu_{\mathbb{C}}, 0, 0] = [\mu, -\nu, 0, 0]$ for μ, ν given by (3.109). Now we have $exp([\mu, -\nu, 0, 0]) = 0$ because of the existence of $\mathcal{K}$, and so $exp([\mu_{\mathbb{C}}, \nu_{\mathbb{C}}, 0, 0]) = 0$, implying that there is a smooth multiplicative T-gerbe with connective structure $\mathcal{G}_{\nabla}$ over G whose associated pairing is $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. Moreover, it follows from Proposition 5.6 that $\mathcal{G}_{\nabla}$ has one and only one holomorphic structure with holomorphic connective structure. In principle, Theorem 3.43 implies that $\mathcal{G}_{\nabla}$ as a smooth gerbe is only determined by $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ up to flat gerbes, i.e., classes in $H^3(BG, \underline{T})$, but again since $j_K^* : H^3(BG, \underline{T}) \to H^3(BK, \underline{T})$ is an isomorphism, this dependence is fixed by imposing $j^*\mathcal{G} = \mathcal{K}^{\mathbb{C}}$. □

Remark 5.9. Theorem 5.8 is also true for K non compact and not connected and T non compact, as long as K and $T_{\mathbb{R}}$ are Lie groups admitting complexifications G and T with $j_K : K \to G$ a homotopy equivalence and $\mathcal{K}$ is a multiplicative $T_{\mathbb{R}}$-gerbe with connective structure over K.

Example 5.10. Let $\Lambda_0 \subset V_0$, $\Lambda_1 \subset V_1$ be real lattices inside complex vector spaces, let $\langle\cdot,\cdot\rangle : \Lambda_0 \otimes \Lambda_0 \to \Lambda_1$ be a bilinear form and write $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{sy} + \langle\cdot,\cdot\rangle_{sk}$ for its decomposition in symmetric and skew-symmetric forms. Let $\mathcal{T}$ be the corresponding multiplicative V_1/Λ_1-gerbe with connective structure over V_0/Λ_0 from examples 3.38, 3.46. Then we note the following.

1. We see by construction that $\mathcal{T}$ is a holomorphic multiplicative gerbe $\Leftrightarrow$ $\langle\cdot,\cdot\rangle$ is $\mathbb{C}$-linear in the first entry. In this case, the connection θ is compatible with the holomorphic structure.

2. The connection θ is holomorphic $\Leftrightarrow$ $\langle\cdot,\cdot\rangle$ is $\mathbb{C}$-linear in both entries.

Moreover, recall from Example 3.14 that these multiplicative gerbes are actually classified by $\langle\cdot,\cdot\rangle_{sy}$. We obtain the following.

1. $\mathcal{T}$ is isomorphic to a holomorphic multiplicative gerbe $\Leftrightarrow$ $\langle\cdot,\cdot\rangle_{sy}^{0,2} = 0$. In this case, it carries a compatible connective structure.

2. $\mathcal{T}$ is isomorphic to a holomorphic multiplicative gerbe with holomorphic connective structure $\Leftrightarrow$ $\langle\cdot,\cdot\rangle_{sy}^{1,1+0,2} = 0$.

This gives a direct proof of Proposition 5.6 for this family of multiplicative gerbes with connective structure.

Example 5.11. For G a complex reductive Lie group with compact form K, Brylinski [59] constructs a holomorphic $\mathbb{C}^*$-gerbe $\mathcal{G}$ over G that restricts over K to the fibrewise complexification of the gerbe $\mathrm{String}(K)$ from Example 3.37 (see also an alternative construction when $G = GL(n,\mathbb{C})$ in [268], based on [195, 200]). It follows from Theorem 5.8 that $\mathcal{G}$ admits a unique holomorphic multiplicative structure with holomorphic connective structure, which defines the complexification of $\mathrm{String}(K)$ as a multiplicative gerbe. As in the smooth case (cf. Example 3.37), there is no known explicit cocycle description of the multiplicative structure on $\mathcal{G}$ but there is an explicit equivariant structure in the original work of Brylinski [59].

5.1.3 Maurer-Cartan forms and shifted holomorphic symplectic structures

Definition 5.12. A *complex Lie groupoid* $\mathfrak{X}$ is a Lie groupoid as in Definition 3.1 such that $\mathfrak{X}_0$, $\mathfrak{X}_1$ are complex manifolds and such that all structure maps are holomorphic. A *holomorphic anafunctor* between complex Lie groupoids is an anafunctor of Lie groupoids whose total space is a complex manifold, and such that all structure maps are holomorphic. A *holomorphic transformation* between holomorphic anafunctors is a transformation of anafunctors which is holomorphic as a map between complex manifolds.

A *complex Lie 2-group* $\mathfrak{G}$ is a Lie 2-group $(\mathfrak{G}, 1, m, \alpha, r, l)$ as in Definition 3.9 such that $\mathfrak{G}$ is a complex Lie groupoid and 1, m, α, r, l are holomorphic. A *holomorphic action* of a complex Lie 2-group $(\mathfrak{G}, 1, m, \alpha, r, l)$ on a complex Lie groupoid $\mathfrak{P}$ is an action as in Definition 3.16 such that ρ, r^ρ, α^ρ are holomorphic.

A (left) adjoint action (cf. Definition 3.23) of a complex Lie 2-group $\mathfrak{G}$ with Lie 2-algebra $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$ is *compatible with the holomorphic structure* if, for every $g \in \mathfrak{G}_0$, $Ad(g) : \mathfrak{g} \to \mathfrak{g}$ and $Ad(g) : \mathfrak{h} \to \mathfrak{h}$ are $\mathbb{C}$-linear. For such a left adjoint action, a (right-invariant) Maurer-Cartan form (θ^0, θ^1) on $\mathfrak{G}$ is *compatible with the holomorphic structure* if θ^0, θ^1 are of type $(1,0)$. A (left) adjoint action is *holomorphic* if the actions of $\mathfrak{G}$ on both $\mathfrak{g}$ and $\mathfrak{h}$ are holomorphic, and in this case a (right-invariant) Maurer-Cartan form (θ^0, θ^1) on $\mathfrak{G}$ is *holomorphic* if θ^0, θ^1 are of type $(1,0)$ and satisfy $\overline{\partial}\theta^0 = 0$, $\overline{\partial}\theta^1 = 0$.

Example 5.13. Let $(\tilde{G}, H, f, \triangleright, \tilde{\kappa})$ be a Lie crossed module with $\tilde{\kappa} : \tilde{G} \times \tilde{\mathfrak{g}} \to \mathfrak{h}$ an adjustment, as in Definitions 3.56, 3.58. By Propositions 3.57 and 3.59, this determines a Lie 2-group $\mathfrak{G}$ with an adjoint action Ad and a Maurer-Cartan form (θ^0, θ^1).

1. $\mathfrak{G}$ is a complex Lie 2-group if and only if $\tilde{G}$, H are complex Lie groups and f, $\triangleright$ are holomorphic.
2. In that case, Ad and (θ^0, θ^1) are compatible with the holomorphic structure if and only if $\tilde{\kappa}$ is $\mathbb{C}$-linear on $\tilde{\mathfrak{g}}$.
3. In that case, Ad and (θ^0, θ^1) are holomorphic if and only if $\tilde{\kappa}$ is a holomorphic function.

Let $\mathfrak{G}$ be a complex Lie 2-group with Lie 2-algebra $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$. In the following proposition we regard the dual $\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*$ of this chain complex as the complex derived manifold (cf. Section 2.2.2) $(\mathfrak{g}^*, \underline{\mathfrak{h}}^*[-2], Q)$, where Q is given simply by the 'curvature' $\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*$.

Proposition 5.14. *Let $\mathfrak{G}$ be a complex Lie 2-group with Lie 2-algebra $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$. Then, a holomorphic left adjoint action determines a holomorphic action of $\mathfrak{G}$ on the derived manifold $\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*$ and a holomorphic right-invariant Maurer-Cartan form for this action defines a 1-shifted holomorphic symplectic structure on the derived quotient 2-groupoid $(\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*)//\mathfrak{G}$ (cf. Remark 3.18).*

Proof. Analogous to Proposition 3.27. We just emphasize that the 1-forms λ^0, λ^1 defined as in (3.68), (3.69) are of type $(1,0)$ as long as the Maurer-Cartan form is compatible with the holomorphic structure, while the 2-forms $\omega^0 = d\lambda^0$, $\omega^1 = d\lambda^1$ are of type $(2,0)$ as long as the Maurer-Cartan form is holomorphic. $\square$

We proceed to summarize the holomorphic analogs of the results in Section 3.2.4 that relate multiplicative gerbes and connective structures on them to Lie 2-groups and Maurer-Cartan forms on them, and to shifted symplectic structures on associated derived stacks. The proofs of these results are completely analogous to their smooth counterparts.

Corollary 5.15. *Let G, T be complex Lie groups with T abelian. There is an equivalence of bicategories between the bicategory of central extensions of G by BT as complex Lie 2-groups and the bicategory of holomorphic multiplicative T-gerbes over G.*

Proposition 5.16. *Let $\mathfrak{G}$ be the complex Lie 2-group corresponding to a holomorphic multiplicative T-gerbe $\mathcal{G} \to G$ by Corollary 5.15.*

1. *There is a holomorphic adjoint action of $\mathfrak{G}$ on $\mathfrak{t} \xrightarrow{0} \mathfrak{g}$ given by the adjoint action of G on $\mathfrak{g}$ and the trivial action on $\mathfrak{t}$.*

2. *A compatible connective structure on $\mathcal{G}$ determines a compatible right-invariant Maurer-Cartan form on $\mathfrak{G}$ for this adjoint action.*

3. *A holomorphic connective structure on $\mathcal{G}$ determines a holomorphic right-invariant Maurer-Cartan form on $\mathfrak{G}$ for this adjoint action. In particular, it induces a 1-shifted holomorphic symplectic structure on $(\mathfrak{g}^* \xrightarrow{0} \mathfrak{t}^*)//\mathfrak{G}$.*

Proposition 5.17. *Let $\mathfrak{G}$ be the complex Lie 2-group corresponding to a holomorphic multiplicative $\mathbb{C}^*$-gerbe $\mathcal{G} \to G$ by Corollary 5.15. Then, a holomorphic connective structure on $\mathcal{G}$ determines a 2-shifted holomorphic presymplectic structure on $B\mathfrak{G} \times \mathbb{C}^*$ defined by*

$$\frac{t}{6}\pi^*\langle\theta^L \wedge [\theta^L \wedge \theta^L]\rangle + dt \wedge \Theta^0 \in \Omega^{3,0}(B\mathfrak{G}_1 \times \mathbb{C}^*, \mathbb{C}), \tag{5.14}$$

$$t\pi^*\langle g_1^*\theta^L \wedge g_2^*\theta^R\rangle + dt \wedge \theta^1 \in \Omega^{2,0}(B\mathfrak{G}_2 \times \mathbb{C}^*, \mathbb{C}), \tag{5.15}$$

where t is the coordinate in $\mathbb{C}^*$, $\pi : B\mathfrak{G}_n \to BG_n$ is the projection map and $\langle\cdot,\cdot\rangle$ is the pairing associated to the connective structure by Theorem 3.43. This is in fact 2-shifted holomorphic symplectic if and only if $\langle\cdot,\cdot\rangle$ is non-degenerate.

5.2 Holomorphic principal 2-bundles

5.2.1 Semiconnections on principal 2-bundles

Fix X a complex manifold, G, T complex Lie groups with T abelian and $(\mathcal{G}_\nabla, m_\nabla, \alpha)$ a holomorphic multiplicative T-gerbe over G with holomorphic connective structure as in Definition 5.4.

Definition 5.18. A *holomorphic principal $\mathcal{G}$-bundle* over X is a principal $\mathcal{G}$-bundle $(P, \mathcal{P}, \rho, \alpha^\rho)$ over X (cf. Definition 4.2) such that P is a holomorphic manifold with the action of G on P holomorphic, $\mathcal{P} \to P$ is a holomorphic gerbe and ρ, α^ρ are an isomorphism and a 2-isomorphism of holomorphic gerbes. A connection $(\nabla, \nabla_\rho, A, B)$

(cf. Definition 4.7) on it is *compatible with the holomorphic structure* if (∇, B) is compatible with the holomorphic structure of the gerbe $\mathcal{P} \to P$, ∇_ρ is compatible with the holomorphic isomorphism ρ and $A \in \Omega^1(P, \mathfrak{g})$ is of type $(1,0)$. Note $F_A^{0,2} = 0$, $H^{0,3} = 0$ in this case.

A *holomorphic principal $\mathcal{G}_\nabla$-bundle* over X is a holomorphic $\mathcal{G}$-bundle $(P, \mathcal{P}, \rho, \alpha^\rho)$ over X with a connective structure (∇, ∇_ρ) as in Definition 4.7 such that ∇ is a holomorphic connective structure on $\mathcal{P} \to P$ and ∇_ρ is a holomorphic connection on ρ. A connection (A, B) for this connective structure is *compatible with the holomorphic connective structure* if B is compatible with the holomorphic connective structure ∇ and $A \in \Omega^1(P, \mathfrak{g})$ is of type $(1,0)$. Note $F_A^{0,2} = 0$, $H^{1,2+0,3} = 0$ in this case.

In terms of the cocycle data from Propositions 4.4 and 4.12, a holomorphic $\mathcal{G}$-bundle is one for which g_{ab}, σ_{ab} and τ_{abc} can be chosen to be holomorphic. In that case, a compatible connection is one for which ∇_{ab} can be chosen to be compatible with the holomorphic structure on σ_{ab}, A_a can be chosen to be of type $(1,0)$ and B_a can be chosen to be of type $(2,0) + (1,1)$. On the other hand, a holomorphic $\mathcal{G}_\nabla$-bundle is one for which g_{ab}, σ_{ab}, τ_{abc}, ∇_{ab} can be chosen to be holomorphic and in that case a compatible connection is one for which A_a can be chosen to be of type $(1,0)$ and B_a can be chosen to be of type $(2,0)$. It can be proven similarly as in Proposition 4.14 that compatible connections on holomorphic $\mathcal{G}$-bundles and on holomorphic $\mathcal{G}_\nabla$-bundles always exist.

Recall that, for any complex Lie group G, the category of holomorphic G-bundles is equivalent to the category of smooth G-bundles with *integrable semiconnections* [101]. Here a semiconnection on a smooth G-bundle P is defined as an equivalence class of smooth G-connections $A \in \Omega^1(P, \mathfrak{g})$, where we identify $A_1 \sim A_2$ if $A_1 - A_2 \in \Omega^{1,0}(X, ad\, P)$, and we say that it is integrable if $F_A^{0,2} = 0$ for any choice of representing connection. Recall also that we proved the equivalent result for gerbes in Proposition 5.2, based on Definition 5.1.

Definition 5.19. The bicategory of smooth $\mathcal{G}$-bundles with integrable 1-semiconnections is defined in the following way.

1. A *smooth $\mathcal{G}$-bundle with integrable 1-semiconnection* $(P, \mathcal{P}, \rho, \alpha^\rho, D_A, D_B, D^\rho)$ is a smooth $\mathcal{G}$-bundle $(P, \mathcal{P}, \rho, \alpha^\rho)$ with:
   - An integrable semiconnection D_A on the G-bundle $P \to X$.
   - An integrable 1-semiconnection D_B on the gerbe $\mathcal{P} \to P$, where we see P as a complex manifold with the complex structure induced by D_A.

- An integrable 1-semiconnection D^ρ on the isomorphism of gerbes with 1-semiconnection

$$\rho : (p^*\mathcal{P} \otimes g^*\mathcal{G}, p^*D_B \otimes g^*(\Theta^L)) \to ((pg)^*\mathcal{P}, (pg)^*D_B)$$

that is preserved by α^ρ. Here we see Θ^L as a 1-semiconnection by Proposition 5.6.

2. An *isomorphism of smooth $\mathcal{G}$-bundles with integrable 1-semiconnections*

$$(u, \phi, \alpha^\phi, D^\phi) : (P^1, \mathcal{P}^1, \rho^1, \alpha^{\rho,1}, D^1_A, D^1_B, D^{\rho^1}) \to (P^2, \mathcal{P}^2, \rho^2, \alpha^{\rho,2}, D^2_A, D^2_B, D^{\rho^2})$$

is an isomorphism of smooth $\mathcal{G}$-bundles (u, ϕ, α^ϕ) such that $u^*D^2_A = D^1_A$ with an integrable 1-semiconnection D^ϕ on the isomorphism of gerbes with 1-semiconnections $\phi : (\mathcal{P}^1, D_1) \to (u^*\mathcal{P}^2, u^*D_2)$ that is preserved by α^ϕ.

3. A *2-isomorphism of smooth $\mathcal{G}$-bundles with 1-semiconnections* $\psi : (u, \phi, \alpha^\phi, D^\phi) \Rightarrow (u', \phi', \alpha^{\phi'}, D^{\phi'})$ is a 2-isomorphism ψ of smooth $\mathcal{G}$-bundles such that $\psi : \phi \Rightarrow \phi'$ preserves the 1-semiconnections D^ϕ, $D^{\phi'}$.

We define in an analogous way the bicategory of smooth $\mathcal{G}$-bundles with integrable 2-semiconnections.

The following proposition follows directly from Definition 5.19 and Proposition 5.2.

Proposition 5.20. *The bicategory of holomorphic $\mathcal{G}$-bundles is equivalent to the bicategory of smooth $\mathcal{G}$-bundles with integrable 1-semiconnections. The bicategory of holomorphic $\mathcal{G}_\nabla$-bundles is equivalent to the bicategory of smooth $\mathcal{G}$-bundles with integrable 2-semiconnections.*

For a fixed smooth gerbe $\mathcal{L}$ Proposition 5.20 implies that $\mathcal{H}(\mathcal{L})$ is the space of holomorphic structures on $\mathcal{L}$ and that $\mathcal{H}'(\mathcal{L})$ is the space of holomorphic structures with holomorphic connective structure on $\mathcal{L}$. For a fixed smooth gerbe with connective structure $\mathcal{L}_\nabla$ we recall that $\mathcal{A}(\mathcal{L}_\nabla)$ is the set of curvings on $\mathcal{L}_\nabla$ and we write the following.

1. $\mathcal{D}(\mathcal{L}_\nabla) = \mathcal{A}(\mathcal{L}_\nabla)/\Omega^{2,0+1,1}(X, \mathfrak{t})$.

2. $\mathcal{D}'(\mathcal{L}_\nabla) = \mathcal{A}(\mathcal{L}_\nabla)/\Omega^{2,0}(X, \mathfrak{t})$.

Finally, we write $BT_{\nabla^{0,1}}(X)$ for the groupoid whose objects are T-bundles over X equipped with an equivalence class of connections, up to addition of 1-forms in $\Omega^{1,0}(X, \mathfrak{t})$, and whose arrows are isomorphisms of T-bundles whose covariant derivative with respect to any choice of representing connection is of type $(1,0)$. This is a 2-group, with product given by tensor product of T-bundles with connection. All these objects are considered with Fréchet topology.

For a fixed smooth $\mathcal{G}$-bundle $\mathcal{P}$ we note that we can form a groupoid $\mathcal{D}_{int}(\mathcal{P})$ (resp. $\mathcal{D}'_{int}(\mathcal{L})$) with objects integrable 1-semiconnections (resp. 2-semiconnections) on $\mathcal{P}$ and with arrows integrable 1-semiconnections (resp. 2-semiconnections) on the identity automorphism of $\mathcal{P}$. The 2-group $Gauge(\mathcal{P})$ acts on these in a similar way to the gauge action on connections. Then we define the quotient 2-groupoids (cf. Section 3.1.2)

1. $\mathcal{H}(\mathcal{P}) := \mathcal{D}_{int}(\mathcal{P})//Gauge(\mathcal{P})$.
2. $\mathcal{H}'(\mathcal{P}) := \mathcal{D}'_{int}(\mathcal{P})//Gauge(\mathcal{P})$.

Proposition 5.20 implies that $\mathcal{H}(\mathcal{P})$ is the space of holomorphic structures on $\mathcal{P}$ and that $\mathcal{H}'(\mathcal{P})$ is the space of holomorphic structures with holomorphic connective structure on $\mathcal{P}$.

For a smooth $\mathcal{G}_\nabla$-bundle $\mathcal{P}_\nabla$, write $\mathcal{D}_{int}(\mathcal{P}_\nabla)$ for the set of integrable 1-semiconnections (D_A, D_B, D^ρ) on $\mathcal{P}$ such that $\{(D_B)_{ij},\ D_i^\rho\}$ is the $(0,1)$-part of the given connective structure. We write $\mathcal{D}'_{int}(\mathcal{P}_\nabla)$ for the set of integrable 2-semiconnections whose underlying connective structure is the given one.

It is convenient to describe 1-semiconnections and 2-semiconnections on $\mathcal{G}$-bundles in a more straightforward way than the one in Definition 5.19. For this, we recall from Proposition 4.14 that the space $\mathcal{A}(\mathcal{P}_\nabla)$ of connections on a $\mathcal{G}_\nabla$-bundle over X is a torsor for the group $\Omega^1(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^2(X,\mathfrak{t})$ with product given by (4.23) and define similarly subgroups $\Omega^{1,0}(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^{2,0+1,1}(X,\mathfrak{t})$ and $\Omega^{1,0}(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^{2,0}(X,\mathfrak{t})$.

Proposition 5.21. *Let $\mathcal{P}_\nabla$ be a smooth $\mathcal{G}_\nabla$-bundle. There are canonical bijections*

$$\mathcal{D}_{int}(\mathcal{P}_\nabla) \to \{(A,B) \in \mathcal{A}(\mathcal{P}_\nabla) \mid F_A^{0,2} = 0,\ H^{0,3} = 0\}/\Omega^{1,0}(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^{2,0+1,1}(M,\mathfrak{t}),$$
$$\mathcal{D}'_{int}(\mathcal{P}_\nabla) \to \{(A,B) \in \mathcal{A}(\mathcal{P}_\nabla) \mid F_A^{0,2} = 0,\ H^{1,2+0,3} = 0\}/\Omega^{1,0}(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^{2,0}(M,\mathfrak{t}).$$

These maps send an integrable 1-semiconnection (resp. an integrable 2-semiconnection) (D_A, D_B, D^ρ) to the set of all connections on $\mathcal{P}_\nabla$ that are compatible with the holomorphic structure (resp. holomorphic structure with holomorphic connective structure) induced by (D_A, D_B, D^ρ).

Proof. We prove the case of integrable 2-semiconnections, as the other one is similar. It follows from Definition 5.19 and Proposition 5.2 that an element of $\mathcal{D}_{int}(\mathcal{P}_\nabla)$ is a pair $([A],[B])$ of an equivalence class of connections A on the G-bundle P such that $F_A^{0,2} = 0$ up to addition of $\Omega^{1,0}(ad\,P)$ (yielding a holomorphic structure on P such that $A \in \Omega^{1,0}(P,\mathfrak{g})$ and which we use in the following), and an equivalence class of curvings B on the gerbe with connective structure $\mathcal{P}_\nabla \to P$ whose curvature $\hat{H}$ is of type $(3,0)+(2,1)$ and such that the curvature of $\rho_\nabla : p^*\mathcal{P}_\nabla \otimes g^*\mathcal{G}_\nabla \to (pg)^*\mathcal{P}_\nabla$ with respect to $p^*B \otimes g^*\Theta^L$, $(pg)^*B$ is some $\tau_B \in \Omega^2(P\times G,\mathfrak{t})$ of type $(2,0)$, up to addition of $\Omega^{2,0}(P,\mathfrak{t})$. For such pair $([A],[B])$ choose a representing connection $A \in [A]$. We claim that we may always choose a representing curving $B \in [B]$ such that $\tau_B = -R(A)$, for $R(A)$ as in Lemma 4.6. This is because for any representing $B \in [B]$ we have that $\tau_B + R(A) \in \Omega^{2,0}(P\times G,\mathfrak{t})$ satisfies $\delta(\tau_B + R(A)) = 0$ for δ the simplicial differential of $P//G_\bullet$, hence there is $b \in \Omega^{2,0}(P,\mathfrak{t})$ with $\delta b = \tau_B + R(A)$. Then for such a choice of B we have $(A,B) \in \mathcal{A}(\mathcal{P}_\nabla)$ with $F_A^{0,2} = 0$ and $H^{1,2+0,3} = \hat{H}^{1,2+0,3} + CS(A)^{1,2+0,3} = 0$; it is easy to check that this gives a bijection as above. □

In light of Proposition 5.21, we define

$$\begin{aligned} \mathcal{D}(\mathcal{P}_\nabla) &:= \mathcal{A}(\mathcal{P}_\nabla)/\Omega^{1,0}(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^{2,0+1,1}(M,\mathfrak{t}), \\ \mathcal{D}'(\mathcal{P}_\nabla) &:= \mathcal{A}(\mathcal{P}_\nabla)/\Omega^{1,0}(ad\,P) \times_{\langle\cdot,\cdot\rangle} \Omega^{2,0}(M,\mathfrak{t}), \end{aligned} \tag{5.16}$$

and refer to these as the spaces of *1-semiconnections* and *2-semiconnections*, respectively. Finally, for $\mathcal{P}_\nabla$ a smooth $\mathcal{G}_\nabla$-bundle we define the 2-group $Gauge(\mathcal{P}_{\nabla^{0,1}})$ whose objects are equivalence classes of $(u,\varphi_\nabla,\psi) \in Gauge(\mathcal{P}_\nabla)$, where we identify $(u,\varphi_\nabla,\psi) \to (u,\varphi_\nabla+\Lambda,\psi)$ for $\Lambda \in \Omega^{1,0}(X,\mathfrak{t})$, and whose arrows $[(u,\varphi_\nabla,\psi)] \to [(u,\varphi'_\nabla,\psi')]$ are 2-isomorphisms $\psi : (u,\varphi,\psi) \to (u,\varphi',\psi')$ whose covariant derivative with respect to any choice of representing connection on φ, φ' is of type $(1,0)$.

Proposition 5.22. *Let $\mathcal{P}_\nabla$ be a smooth $\mathcal{G}$-bundle with connective structure. Then*

1. *$Gauge(\mathcal{P}_{\nabla^{0,1}})$ acts on $\mathcal{D}(\mathcal{P}_\nabla)$ preserving $\mathcal{D}_{int}(\mathcal{P}_\nabla)$ and there is a canonical map of simplicial topological spaces $\mathcal{D}_{int}(\mathcal{P}_\nabla)//Gauge(\mathcal{P}_{\nabla^{0,1}}) \to \mathcal{H}(\mathcal{P})$ inducing weak homotopy equivalence on geometric realizations.*

2. *$Gauge(\mathcal{P}_\nabla)$ acts on $\mathcal{D}(\mathcal{P}_\nabla)$ preserving $\mathcal{D}'_{int}(\mathcal{P}_\nabla)$ and there is a canonical map of simplicial topological spaces $\mathcal{D}'_{int}(\mathcal{P}_\nabla)//Gauge(\mathcal{P}_\nabla) \to \mathcal{H}'(\mathcal{P})$ inducing weak homotopy equivalence on geometric realizations.*

Proof. It follows directly from Propositions 4.17, 5.3 and 5.21. □

Proposition 5.23. *Let $\mathcal{P}_\nabla$ be a smooth $\mathcal{G}$-bundle with connective structure over X described in a cover $\{X_a\}_{a\in A}$ by cocycle data g_{ab}, σ_{ab}, τ_{abc}, ∇_{ab} as in Propositions 4.4,* *4.12. Then an integrable 1-semiconnection on $\mathcal{P}_\nabla$ is given by*

$$A_a \in \Omega^{0,1}(X_a, \mathfrak{g}), \quad D_a \in \Omega^{0,2}(X_a, \mathfrak{t}) \tag{5.17}$$

such that

$$\begin{aligned} A_b - Ad(g_{ab}^{-1})A_a &= (g_{ab}^*\theta^L)^{0,1}, \\ D_b - D_a &= F_{ab}^{0,2} - \langle A_a \wedge (g_{ab}^*\theta^R)^{0,1} \rangle, \\ \bar\partial A_a &= -\frac{1}{2}[A_a \wedge A_a], \\ \bar\partial D_a &= -\langle \bar\partial A_a \wedge A_a \rangle - \frac{1}{3}\langle A_a \wedge [A_a \wedge A_a] \rangle, \end{aligned} \tag{5.18}$$

where F_{ab} is the curvature of ∇_{ab}. An integrable 2-semiconnection on $\mathcal{P}_\nabla$ is given by

$$A_a \in \Omega^{0,1}(X_a, \mathfrak{g}), \quad D_a \in \Omega^{1,1+0,2}(X_a, \mathfrak{t}) \tag{5.19}$$

such that

$$\begin{aligned} A_b - Ad(g_{ab}^{-1})A_a &= (g_{ab}^*\theta^L)^{0,1}, \\ D_b - D_a &= F_{ab}^{1,1+0,2} - \langle A_a \wedge g_{ab}^*\theta^R \rangle + \langle (g_{ab}^*\theta^L)^{1,0} \wedge A_b \rangle, \\ \bar\partial A_a &= -\frac{1}{2}[A_a \wedge A_a], \\ (dD_a)^{1,2+0,3} &= -\langle dA_a \wedge A_a \rangle - \frac{1}{3}\langle A_a \wedge [A_a \wedge A_a] \rangle = 0. \end{aligned} \tag{5.20}$$

An isomorphism $\mathcal{P}_\nabla \to \mathcal{P}^2_\nabla$ described by $(\varphi_a, \Phi_{a,\nabla}, \psi_{ab})$ as in Proposition 4.19 acts on a 1-semiconnection (A_a, D_a) for $\mathcal{P}_\nabla$ by sending it to the 1-semiconnection (A_a^2, D_a^2) for $\mathcal{P}^2_\nabla$ with

$$A_a^2 = Ad(\varphi_a)A_a - (\varphi_a^*\theta^R)^{0,1}, \tag{5.21}$$
$$D_a^2 = D_a - \langle (\varphi_a^*\theta^L)^{0,1} \wedge A_a \rangle - F_a^{0,2}, \tag{5.22}$$

where F_a is the curvature of $\Phi_{a,\nabla}$, and it acts on a 2-semiconnection (A_a, D_a) for $\mathcal{P}_\nabla$ by sending it to the 2-semiconnection (A_a^2, D_a^2) for $\mathcal{P}^2_\nabla$ with

$$A_a^2 = Ad(\varphi_a)A_a^1 - (\varphi_a^*\theta^R)^{0,1}, \tag{5.23}$$
$$D_a^2 = D_a^1 - \langle (\varphi_a^*\theta^L)^{0,1} \wedge A_a^1 \rangle + \langle A_a^2 \wedge (\varphi_a^*\theta^R)^{1,0} \rangle - \langle (\varphi_a^*\theta^L)^{1,0} \wedge A_a^1 \rangle - F_a^{1,1+0,2}. \tag{5.24}$$

Proof. We prove the result for 2-semiconnections, as the other follows similarly. Straightforward computations show the following.

1. If $\{A_a\}$, $\{B_a\}$ is a connection with $F_A^{0,2} = 0$, $H^{1,2+0,3} = 0$, then $\{A_a^{0,1}\}$, $\{D_a := B_a^{1,1+0,2} + \langle A_a^{1,0} \wedge A_a^{0,1}\rangle\}$ satisfies (5.20)

2. Given $A_a \in \Omega^{0,1}(X_a, \mathfrak{g})$, $\quad D_a \in \Omega^{1,1+0,2}(X_a, \mathfrak{t})$ satisfying (5.20), one can always find a connection (A_a, B_a) such that $D_a = B_a^{1,1+0,2} + \langle A_a^{1,0} \wedge A_a^{0,1}\rangle$.

3. Two connections differ by $a \in \Omega^{1,0}(ad\, P)$, $b \in \Omega^{2,0}(X, \mathfrak{t})$ if and only if the corresponding data $(A_a^{0,1}, B_a^{1,1+0,2} + \langle A_a^{1,0} \wedge A_a^{0,1}\rangle)$ coincides.

4. Under the map $(A_a, B_a) \mapsto (A_a^{0,1}, B_a^{1,1+0,2} + \langle A_a^{1,0} \wedge A_a^{0,1}\rangle)$, equations (4.36), (4.37) become (5.23), (5.24).

Hence, the result follows from Proposition 5.21. $\square$

5.2.2 The Chern correspondence

We fix compact connected Lie groups K, $T_{\mathbb{R}}$ with $T_{\mathbb{R}}$ abelian, a multiplicative $T_{\mathbb{R}}$-gerbe $\mathcal{K}$ over K and a complex manifold X. We write G, T and $\mathcal{G}_\nabla$ for the complexifications of K, $T_{\mathbb{R}}$, $\mathcal{K}$, respectively, as in Theorem 5.8, and we write $j : K \to G$ for the complexification map.

If $P_h \to X$ is a K-bundle, then its *fibrewise complexification* is the smooth G-bundle $P_h^{\mathbb{C}} := (P_h \times G)/K$, where $k \in K$ acts as $(p, g) \cdot k = (pk, k^{-1}g)$. Note there is a canonical K-equivariant map $l : P_h \to P_h^{\mathbb{C}}$. This defines a functor from K-bundles to G-bundles. It can also be promoted to a functor from K-bundles with connection to G-bundles with connection which we denote by $(P_h, A_h) \mapsto (P_h^{\mathbb{C}}, A_h^{\mathbb{C}})$. Recall also the fibrewise complexification of gerbes from definition 5.1.

Proposition 5.24. *Let $(P_h, \mathcal{P}_h, \rho_h, \alpha^{\rho_h}) \to X$ be a $\mathcal{K}$-bundle. Then, there is a unique smooth $\mathcal{G}$-bundle $(P, \mathcal{P}, \rho, \alpha^\rho) \to X$ such that $P = P_h^{\mathbb{C}}$ and*

1. *There is an isomorphism of T-gerbes $\phi : \mathcal{P}_h^{\mathbb{C}} \to l^*\mathcal{P}$ over P_h.*

2. *There is a 2-isomorphism of T-gerbes over $P_h \times K$*

$$\begin{tikzcd}
p^*\mathcal{P}_h^{\mathbb{C}} \otimes k^*\mathcal{K}^{\mathbb{C}} \arrow[r, "\rho_h^{\mathbb{C}}"] \arrow[d, "p^*\phi\otimes id"'] & (pk)^*\mathcal{P}_h^{\mathbb{C}} \arrow[d, "(pk)^*\phi"] \arrow[dl, Rightarrow, "\psi"] \\
l(p)^*\mathcal{P} \otimes j(k)^*\mathcal{G} \arrow[r, "(l(p),j(k))^*\rho"'] & (l(p)j(k))^*\mathcal{P}
\end{tikzcd}$$

commuting with α^{ρ_h} and α^ρ.

We call $(P, \mathcal{P})$ the fibrewise complexification *of $(P_h, \mathcal{P}_h)$ and denote it by $(P_h^{\mathbb{C}}, \mathcal{P}_h^{\mathbb{C}})$.*

Proof. If such $\mathcal{G}$-bundle exists then it is unique up to isomorphism because if $\mathcal{P}^1$, $\mathcal{P}^2 \to P$ are any two such bundles then $\rho_1 \otimes \rho_2^{-1}$, $\alpha^{\rho_1} \otimes \alpha^{\rho_2^{-1}}$ equips the T-gerbe $\mathcal{P}^1 \otimes (\mathcal{P}^2)^* \to P$ with the necessary descent data to give a T-gerbe over X which is trivial precisely when $\mathcal{P}^1$ is isomorphic to $\mathcal{P}^2$ as $\mathcal{G}$-bundles [199]. But since $K \to G$ is injective this T-gerbe can also be obtained by descending $l^*\mathcal{P}^1 \otimes l^*\mathcal{P}_2^*$ and the maps ϕ, ψ give precisely a trivialization of the descent of this T-gerbe. In order to construct the $\mathcal{G}$-bundle, choose cocycle data $\{g_{ab,h}, \sigma_{ab,h}, \tau_{abc,h}\}$ for $(P_h, \mathcal{P}_{\nabla,h})$ over a cover $\{X_a\}_{a\in A}$ of X as in Proposition 4.4. Then define $g_{ab} := j \circ g_{ab}^h : X_{ab} \to G$ and we note that $g_{ab}^*\mathcal{G} = (g_{ab}^h)^* j^*\mathcal{G} = (g_{ab}^h)^*\mathcal{K}^{\mathbb{C}}$; hence $(g_{ab}, \sigma_{ab,h}^{\mathbb{C}}, \tau_{abc,h}^{\mathbb{C}})$ is cocycle data for a $\mathcal{G}$-bundle with ϕ, ψ as above. □

Remark 5.25. Fibrewise complexification can also be promoted to a functor of bicategories $C : \langle \text{Smooth } \mathcal{K} - \text{bundles}\rangle \to \langle \text{Smooth } \mathcal{G} - \text{bundles}\rangle$. Similarly, there are complexification functors between the bicategories of bundles with connective structures and between the bicategories of bundles with connections which we denote by $(P_h, \mathcal{P}_{h,\nabla_h}, A_h, B_h) \mapsto (P_h^{\mathbb{C}}, \mathcal{P}_{h,\nabla_h}^{\mathbb{C}}, A_h^{\mathbb{C}}, B_h^{\mathbb{C}})$.

The Chern correspondence from ordinary gauge theory relates connections on K-bundles with holomorphic structures on their complexifications [233]. Analogous results for gerbes appear in [82, 142]. Our next theorem generalizes all these. Recall enhanced connections from Definition 4.7 and use Remark 4.8 to write $((A, B), g) \in \mathcal{A}^{en}(\mathcal{P}_\nabla)$ for an enhanced connection thought of as a pair of a connection $(A, B) \in \mathcal{A}(\mathcal{P}_\nabla)$ and a $g \in \Gamma(S^2T^*X \otimes \mathfrak{t})$. Write $d^c : \Omega^p(X, \mathfrak{t}) \to \Omega^{p+1}(X, \mathfrak{t})$ for the operator $d^c := J_{\mathfrak{t}}(\overline{\partial} - \partial)$ and note that it preserves $\mathfrak{t}_{\mathbb{R}}$-valued forms.

Theorem 5.26. *Let $\mathcal{P}_{h,\nabla_h}$ be a $\mathcal{K}$-bundle with connective structure over X and let $\mathcal{P}_\nabla$ be its fibrewise complexification. Then*

1. *There is a canonical bijection*

$$\mathcal{D}_{int}(\mathcal{P}_\nabla) \to \{(A_h, B_h) \in \mathcal{A}(\mathcal{P}_{h,\nabla_h}) \mid F_{A_h}^{0,2} = 0,\ H_h^{0,3} = 0\}/\Omega^{1,1}(X, \mathfrak{t}_{\mathbb{R}}).$$

   This map sends a 1-semiconnection (D_A, D_B, D^ρ) to the set of connections on $\mathcal{P}_{h,\nabla_h}$ whose complexification is compatible with the holomorphic structure on $\mathcal{P}_\nabla$ determined by (D_A, D_B, D^ρ).

2. *There is a canonical bijection*

$$\mathcal{D}'_{int}(\mathcal{P}_\nabla) \to \{((A_h, B_h), g) \in \mathcal{A}^{en}(\mathcal{P}_{h,\nabla_h}) \mid g^{0,2} = 0,\ F_{A_h}^{0,2} = 0,\ H_h = d^c(g(J\cdot, \cdot))\}.$$

   This map sends a 2-semiconnection (D_A, D_B, D^ρ) to the unique enhanced connection $((A_h, B_h), g) \in \mathcal{A}^{en}(\mathcal{P}_{h,\nabla_h})$ such that $g^{0,2} = 0$ and that the connection

$(A_h^{\mathbb{C}}, B_h^{\mathbb{C}} - J_{\mathfrak{t}} g(J\cdot,\cdot))$ on $\mathcal{P}_\nabla$ is compatible with the holomorphic structure with holomorphic connective structure determined by (D_A, D_B, D^ρ).

Proof. We only show the proof of part 2, as the other one is similar. Recall (5.16). We shall prove that the map

$$\begin{aligned}\phi : \{((A_h, B_h), g) \in \mathcal{A}^{en}(\mathcal{P}_{h,\nabla_h}) \,|\, g^{0,2} = 0\} &\to \mathcal{D}'(\mathcal{P}_\nabla)\\ ((A_h, B_h), g) &\mapsto [A_h^{\mathbb{C}}, B_h^{\mathbb{C}} - J_{\mathfrak{t}} g(J\cdot,\cdot)]\end{aligned}$$

is a bijection. The theorem will follow then from Proposition 5.21 and the fact that the curvature (F_{A_h}, H_h) of (A_h, B_h) satisfies $F_{A_h}^{0,2} = 0$, $H_h = d^c g(J\cdot,\cdot)$ if and only if the curvature (F_A, H) of $(A_h^{\mathbb{C}}, B_h^{\mathbb{C}} - J_{\mathfrak{t}} g(J\cdot,\cdot))$ satisfies $F_A^{0,2} = 0$, $H^{1,2+0,3} = 0$. To see that ϕ is a bijection, recall from part 3 of Proposition 4.14 that the space

$$\{((A_h, B_h), g) \in \mathcal{A}^{en}(\mathcal{P}_{h,\nabla_h}) \,|\, g^{0,2} = 0\}$$

is a $\Omega^2(X, \mathfrak{t}_{\mathbb{R}}) \times \Gamma(S^{1,1}T^*X \otimes \mathfrak{t}_{\mathbb{R}})$-bundle over $\mathcal{A}(P_h)$. Here $\Gamma(S^{1,1}T^*X \otimes \mathfrak{t}_{\mathbb{R}})$ stands for the intersection of the symmetric $\mathfrak{t}$-valued tensors of type $(1,1)$ on X with the $\mathfrak{t}_{\mathbb{R}}$-valued tensors. On the other hand, the space

$$\mathcal{D}'(\mathcal{P}_\nabla) = \mathcal{A}(\mathcal{P}_\nabla)/\Omega^{1,0}(ad\, P) \times_{\langle\cdot,\cdot\rangle} \Omega^{2,0}(X, \mathfrak{t})$$

is a $\Omega^2(X, \mathfrak{t})/\Omega^{2,0}(X, \mathfrak{t})$-bundle over $\mathcal{A}(P)/\Omega^{1,0}(ad\, P)$. Now the map

$$\begin{aligned}\mathcal{A}(P_h) &\to \mathcal{A}(P)/\Omega^{1,0}(ad\, P)\\ A_h &\mapsto [A_h^{\mathbb{C}}]\end{aligned}$$

is a bijection, while the map

$$\begin{aligned}\Omega^2(X, \mathfrak{t}_{\mathbb{R}}) \times \Gamma(S^{1,1}T^*X \otimes \mathfrak{t}_{\mathbb{R}}) &\to \Omega^2(X, \mathfrak{t})/\Omega^{2,0}(X, \mathfrak{t})\\ (b, g) &\mapsto [b - J_{\mathfrak{t}} g(J\cdot,\cdot)]\end{aligned}$$

is an isomorphism of groups. Thus ϕ may be regarded as an equivariant map between affine bundles with the same fiber over the same space and so it is a bijection. □

We obtain the following corollary, which can be interpreted as saying that holomorphic $\mathcal{G}_\nabla$-bundles with a reduction to a $\mathcal{K}_\nabla$-bundle are the geometric objects prequantizing the Hermitian metrics proposed by Yau [280] as a generalization of Kähler metrics with the potential to fulfill Reid's fantasy [217].

Corollary 5.27. *Let $(\mathcal{P}_h, \nabla_h)$ be a $\mathcal{K}$-bundle with connective structure over X. A holomorphic structure with holomorphic connective structure on its complexification determines an $\omega \in \Omega^{1,1}(X, \mathfrak{t}_{\mathbb{R}})$ such that*

$$dd^c\omega - \langle F_h \wedge F_h \rangle = 0, \tag{5.25}$$

where $F_h \in \Omega^2(ad\, P_h)$ is the curvature of the Chern connection on P_h.

Definition 5.28. For $(\mathcal{P}_h, \nabla_h)$ a $\mathcal{K}$-bundle with connective structure and a choice of holomorphic structure with holomorphic connective structure on its complexification $(\mathcal{P}, \nabla)$, the corresponding enhanced connection $((A_h, B_h), g)) \in \mathcal{A}^{en}(\mathcal{P}_{h,\nabla_h})$ from Theorem 5.26 is called the *unitary Chern enhanced connection*, while the connection $(A_h^{\mathbb{C}}, B_h^{\mathbb{C}} - J_{\mathfrak{t}}\omega) \in \mathcal{A}(\mathcal{P}_\nabla)$ is called the *complex Chern connection*, for $\omega = g(J\cdot, \cdot)$.

We write now $\bar{\cdot} : \mathfrak{g} \to \mathfrak{g}$, $\bar{\cdot} : \mathfrak{t} \to \mathfrak{t}$ for the $\mathbb{C}$-antilinear involutions that leave $J_{\mathfrak{g}}\mathfrak{k}$ and $J_{\mathfrak{t}}\mathfrak{t}_{\mathbb{R}}$ invariant, repectively, and we write $Im : \mathfrak{t} \to \mathfrak{t}_{\mathbb{R}}$, $Re : \mathfrak{t} \to J_{\mathfrak{t}}\mathfrak{t}_{\mathbb{R}}$ for the two projections, by analogy with the case $\mathfrak{t}_{\mathbb{R}} = i\mathbb{R}$, $\mathfrak{t} = \mathbb{C}$.

Proposition 5.29. *Let $(\{g_{ab}^h\}, \{\sigma_\nabla^{ab,h}\}, \{\tau_\nabla^{abc,h}\})$ be cocycle data in a cover $\{X_a\}_a$ of X for a $\mathcal{K}_\nabla$-bundle $(P_h, \mathcal{P}_{\nabla,h})$, as in Proposition 4.4. Let $(\{g_{ab}\}, \{\sigma_\nabla^{ab}\}, \{\tau_\nabla^{abc}\})$ be holomorphic cocycle data for a holomorphic $\mathcal{G}_\nabla$-bundle $(P, \mathcal{P}_\nabla)$ and let $(\{\varphi_a\}, \{\Phi_{a,\nabla}\}, \{\psi_{ab}\})$ be cocyle data for an isomorphism $(P, \mathcal{P}_\nabla) \to (P_h^{\mathbb{C}}, \mathcal{P}_{\nabla,h}^{\mathbb{C}})$ as in Proposition 4.19. Then the unitary Chern enhanced connection on $(P_h, \mathcal{P}_{\nabla,h})$ is given by*

$$\begin{aligned}
A_a^h &= -(\varphi_a^*\theta^R)^{0,1} + \overline{(\varphi_a^*\theta^R)^{0,1}},\\
B_a^h &= -J_{\mathfrak{t}} Im\left(F_a^{1,1} + 2F_a^{0,2} - \langle(\varphi_a^*\theta^L)^{1,0} \wedge (\varphi_a^*\theta^L)^{0,1}\rangle - \langle\overline{(\varphi_a^*\theta^L)^{1,0}} \wedge (\varphi_a^*\theta^L)^{0,1}\rangle\right),\\
\omega &= -J_{\mathfrak{t}} Re\left(F_a^{1,1} - \langle(\varphi_a^*\theta^L)^{1,0} \wedge (\varphi_a^*\theta^L)^{0,1}\rangle\right),
\end{aligned} \tag{5.26}$$

while the complex Chern connection on $(P, \mathcal{P}_\nabla)$ is given by

$$\begin{aligned}
A_a &= (\varphi_a^*\theta^L)^{1,0} + \overline{(\varphi_a^*\theta^L)^{0,1}},\\
B_a &= F_a^{2,0} + \overline{F_a^{0,2}} + \langle(\varphi_a^*\theta^L)^{1,0} \wedge \overline{(\varphi_a^*\theta^L)^{0,1}}\rangle,
\end{aligned} \tag{5.27}$$

where F_a is the curvature of $\Phi_{a,\nabla}$.

Proof. Let F_{ab}^h be the curvature of $\sigma_\nabla^{ab,h}$ and let F_{ab} be the curvature of σ_∇^{ab}. Recall from (4.39) that

$$F_{ab} - \langle A_a \wedge g_{ab}^*\theta^R\rangle) - (F_{ab}^h - \langle A_a^h \wedge (g_{ab}^h)^*\theta^R\rangle) = (F_b + \langle\varphi_b^*\theta^L \wedge A_b\rangle) - (F_a + \langle\varphi_a^*\theta^L \wedge A_a\rangle). \tag{5.28}$$

Then take $-J_{\mathfrak{t}}Im((\cdot)^{1,1}+2(\cdot)^{0,2})$, $(\cdot)^{2,0}+\overline{(\cdot)^{0,2}}$ and $-J_{\mathfrak{t}}Re((\cdot)^{1,1})$ on both sides to see that (5.26) and (5.27) are well-defined connections on $(P_h, \mathcal{P}_{\nabla,h})$, $(P, \mathcal{P}_\nabla)$, respectively. Use (4.36), (4.37) to prove that $(\{\varphi_a\}, \{\Phi_{a,\nabla}\}, \{\psi_{ab}\})$ sends (A_a, B_a) to $(A_a^h, B_a^h - J_{\mathfrak{t}}\omega)$. $\square$

The classical Chern correspondence also states that, if P_h^1, P_h^2 are K-bundles with holomorphic structures on their complexifications determining Chern connections A_h^1, A_h^2, then an isomorphism of K-bundles $P_h^1 \to P_h^2$ is flat with respect to A_h^1, A_h^2 precisely when its complexification is holomorphic with respect to the corresponding holomorphic structures. Similarly, our Chern correspondence can be improved to an equivalence of bicategories of which Theorem 5.26 is the result at the level of objects. To state this equivalence, consider the forgetful functors of bicategories

$$F : \langle \text{Holomorphic } \mathcal{G}\text{-bundles} \rangle \to \langle \text{Smooth } \mathcal{G}\text{-bundles} \rangle,$$
$$F_\nabla : \langle \text{Holomorphic } \mathcal{G}_\nabla\text{-bundles} \rangle \to \langle \text{Smooth } \mathcal{G}_\nabla\text{-bundles} \rangle$$

and the complexification functors of bicategories

$$C : \langle \text{Smooth } \mathcal{K}\text{-bundles} \rangle \to \langle \text{Smooth } \mathcal{G}\text{-bundles} \rangle,$$
$$C_\nabla : \langle \text{Smooth } \mathcal{K}_\nabla\text{-bundles} \rangle \to \langle \text{Smooth } \mathcal{G}_\nabla\text{-bundles} \rangle.$$

We define the following bicategories as categorical fibered products.

$$\langle \text{Holomorphic } \mathcal{K}\text{-bundles} \rangle := \langle \text{Holomorphic } \mathcal{G}\text{-bundles} \rangle \;{}_F\!\times_C \langle \text{Smooth } \mathcal{K}\text{-bundles} \rangle,$$
$$\langle \text{Holomorphic } \mathcal{K}_\nabla\text{-bundles} \rangle := \langle \text{Holomorphic } \mathcal{G}_\nabla\text{-bundles} \rangle \;{}_{F_\nabla}\!\times_{C_\nabla} \langle \text{Smooth } \mathcal{K}_\nabla\text{-bundles} \rangle.$$

Corollary 5.30. *1. The bicategory of holomorphic $\mathcal{K}$-bundles is equivalent to the bicategory D with:*

- *An object in D is an equivalence class of $\mathcal{K}$-bundles with connection $(\mathcal{P}_h, \nabla_h, A_h, B_h)$ whose curvature satisfies $F_{A_h}^{0,2} = 0$, $H_h^{0,3} = 0$ and we identify $(\mathcal{P}_h, \nabla_h, A_h, B_h) \sim (\mathcal{P}_h, \nabla_h, A_h, B_h + b)$ for any $b \in \Omega^{1,1}(X, \mathfrak{t}_{\mathbb{R}})$.*
- *Isomorphisms $(\mathcal{P}_h, \nabla_h, A_h, [B_h]) \to (\mathcal{P}_{h,2}, \nabla_{h,2}, A_{h,2}, [B_{h,2}])$ in D are isomorphisms of $\mathcal{K}$-bundles with connection $(\mathcal{P}_h, \nabla_h, A_h, B_h) \to (\mathcal{P}_{h,2}, \nabla_{h,2}, A_{h,2}, B_{h,2})$ that are flat up to forms in $\Omega^{1,1}(X, \mathfrak{t}_{\mathbb{R}})$.*
- *2-isomorphisms in D are flat 2-isomorphisms between the corresponding isomorphisms of $\mathcal{K}$-bundles with connection.*

2. *The bicategory of holomorphic $\mathcal{K}_\nabla$-bundles is equivalent to the full sub-bicategory D' of the bicategory of smooth $\mathcal{K}$-bundles with enhanced connections $((A_h, B_h), g))$ spanned by objects such that $g^{0,2} = 0$, $F_{A_h}^{0,2} = 0$, $H_h = d^c g(J\cdot, \cdot)$.*

Proof. We show the proof of part 2, as the other one follows similarly. First, it follows from Proposition 5.20 that the bicategory of holomorphic $\mathcal{K}_\nabla$-bundles is equivalent to the bicategory whose objects are $\mathcal{K}_\nabla$-bundles with 2-semiconnections in their complexifications, whose isomorphisms are isomorphisms of $\mathcal{K}_\nabla$-bundles that complexify to isomorphisms of 2-semiconnections, and whose 2-isomorphisms are 2-isomorphisms of $\mathcal{K}$-bundles preserving the 2-semiconnections.

Theorem 5.26 implies then that every object in $\langle$Holomorphic $\mathcal{K}_\nabla$-bundles$\rangle$ can be described by an object in D', and conversely. An isomorphism in $\langle$Holomorphic $\mathcal{K}_\nabla$-bundles$\rangle$ between the $\mathcal{K}_\nabla$-bundles corresponding to $(\mathcal{P}_h, \nabla_h, A_h, B_h, g)$ and $(\mathcal{P}_{h,2}, \nabla_{h,2}, A_{h,2}, B_{h,2}, g_2)$ is then an isomorphism of $\mathcal{K}_\nabla$-bundles $(u, \varphi^u_\nabla, \alpha^u) : (\mathcal{P}_h, \nabla_h) \to (\mathcal{P}^2_h, \nabla^2_h)$ such that u is flat (by the classical Chern correspondence) and that $(\varphi^u_\nabla)^{\mathbb{C}}$ has curvature of type $(2,0)$ with respect to the curvings $B_{h,1} - J_{\mathfrak{t}} g_1(J\cdot, \cdot)$ and $u^*(B_{h,2} - J_{\mathfrak{t}} g_2(J\cdot, \cdot))$ (by Theorem 5.26 and Proposition 5.2). But its curvature with respect to these curvings equals its curvature with respect to $B_{h,1}$, $u^* B_{h,2}$ (which is $\mathfrak{t}_{\mathbb{R}}$-valued because so are $B_{h,1}$, $B_{h,2}$ and φ^u_∇) plus the $J_{\mathfrak{t}}\mathfrak{t}_{\mathbb{R}}$-valued form $J_{\mathfrak{t}}(g_1(J\cdot, \cdot) - g_2(J\cdot, \cdot))$; such a sum can only be of type $(2,0)$ when both terms are zero. This means that all isomorphisms in $\langle$Holomorphic $\mathcal{K}_\nabla$-bundles$\rangle$ are described by isomorphisms in D', and conversely. The same holds trivially for 2-isomorphisms, which concludes the proof. $\square$

5.2.3 Holomorphic Atiyah algebroids

Definition 5.31 ([126])**.** Let X be a complex manifold and let V be a complex vector space. A *complex V-Courant-Dorfman algebroid* over X is a quadruple $(E', \langle\cdot,\cdot\rangle, [\cdot,\cdot], d)$, where

1. $E' \to X$ is a smooth complex vector bundle,
2. $\langle\cdot,\cdot\rangle : \Gamma(E') \otimes_{C^\infty(M,\mathbb{C})} \Gamma(E') \to C^\infty(X, V)$ is a symmetric $C^\infty(X,\mathbb{C})$-bilinear map,
3. $[\cdot,\cdot] : \Gamma(E') \otimes_{\mathbb{C}} \Gamma(E') \to \Gamma(E')$ is a $\mathbb{C}$-bilinear map
4. $d : C^\infty(X, V) \to \Gamma(E')$ is a $\mathbb{C}$-linear map

satisfying the same axioms as in Definition 4.22. In particular, for $\underline{V}$ a real vector space and E a $\underline{V}$-Courant-Dorfman algebroid, the *complexification* of E is the complex $\underline{V} \otimes_{\mathbb{R}} \mathbb{C}$-Courant-Dorfman algebroid $E \otimes_{\mathbb{R}} \mathbb{C}$ with its obvious bracket, pairing and differential. A *holomorphic V-Courant-Dorfman algebroid* over X is a quadruple $(Q, \langle\cdot,\cdot\rangle, [\cdot,\cdot], d)$, where

1. $Q \to X$ is a holomorphic vector bundle with sheaf of sections also denoted by Q,

2. $\langle\cdot,\cdot\rangle : Q \otimes_{\mathcal{O}_X} Q \to \mathcal{O}_{X,V}$ is a symmetric morphism of sheaves,
3. $[\cdot,\cdot] : Q \otimes_{\underline{\mathbb{C}}} Q \to Q$ is a morphism of sheaves,
4. $d : \mathcal{O}_{X,V} \to Q$ is a morphisms of sheaves

such that the same axioms as in Definition 4.22 are satisfied.

If E' is a complex V-Courant-Dorfman algebroid, then there is a $\mathbb{C}$-linear anchor map $\pi : E' \to T_{\mathbb{C}}X$ with $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$ as in Definition 4.22. We note that $\pi^{-1}(T^{0,1}X) \subset E'$ is always a V-Courant-Dorfman subalgebroid. We say E' is *transitive* if π is surjective. In this case, or more generally if $\pi : E' \to T^{0,1}X$ is surjective, we define an *(involutive) lifting of* $T^{0,1}X$ to E' to be an isotropic (involutive) splitting $s : T^{0,1}X \to E'$ of $\pi : \pi^{-1}(T^{0,1}X) \subset E' \to T^{0,1}X$. An involutive lifting of $T^{0,1}X$ to E' determines a holomorphic V-Courant-Dorfman algebroid $Q_{E',s}$ as in [127]. Namely, write $L := s(T^{0,1}X) \subset E'$ and define $Q_{E',s} := L^{\perp}/L$, with holomorphic structure given by

$$\overline{\partial}_X e = [s(X), \tilde{e}] \mod L,$$

for $X \in \Gamma(T^{0,1}X)$, $e \in \Gamma(L^{\perp}/L)$ and $\tilde{e} \in \Gamma(L^{\perp})$ any representative of e. Then $\langle\cdot,\cdot\rangle$, $[\cdot,\cdot]$ and d_E are well-defined on holomorphic sections of $Q_{E',s}$ by restricting them from $L^{\perp}$. This construction is related to the description of holomorphic $\mathcal{G}$-bundles and holomorphic $\mathcal{G}_{\nabla}$-bundles in terms of semiconnections from Theorem 5.26.

Proposition 5.32. *Let X be a complex manifold and let $\mathcal{G}_{\nabla}$ be a holomorphic multiplicative T-gerbe with holomorphic connective structure.*

1. *A smooth $\mathcal{G}_{\nabla}$-bundle $(P, \mathcal{P}_{\nabla})$ over a complex manifold X gives rise to a complex $\mathfrak{t}$-Courant-Dorfman algebroid $E'_{\mathcal{P}_{\nabla}}$ fitting in a sequence*

$$0 \to T^*X \otimes \mathfrak{t} \to E'_{\mathcal{P}_{\nabla}} \to At_{\mathbb{C}}P \to 0, \tag{5.29}$$

where $At_{\mathbb{C}}P := (TP/G \otimes \mathbb{C})/(\mathrm{ad}\, P)^{0,1}$ is the complex Atiyah algebroid of P. The quotient $\pi^{-1}(T^{0,1}X)/(T^*X \otimes \mathfrak{t})^{1,0}$ is also a complex $\mathfrak{t}$-Courant-Dorfman algebroid.

2. *If $\mathcal{G}_{\nabla}$ is the complexification of $\mathcal{K}_{\nabla}$ and $\mathcal{P}_{\nabla} = \mathcal{P}^{\mathbb{C}}_{h,\nabla_h}$, then $E'_{\mathcal{P}_{\nabla}} = E^{\mathbb{C}}_{\mathcal{P}_{h,\nabla_h}}$.*
3. *(Integrable) 1-semiconnections on $\mathcal{P}_{\nabla}$ are in bijection with (involutive) liftings of $T^{0,1}X$ to $\pi^{-1}(T^{0,1}X)/(T^*X \otimes \mathfrak{t})^{1,0}$. (Integrable) 2-semiconnections on $\mathcal{P}_{\nabla}$ are in bijection with (involutive) liftings of $T^{0,1}X$ to $E'_{\mathcal{P}_{\nabla}}$.*

4. *A holomorphic $\mathcal{G}_\nabla$-bundle $(P, \mathcal{P}_\nabla)$ over X gives rise to a holomorphic $\mathfrak{t}$-Courant-Dorfman algebroid $Q_{\mathcal{P}_\nabla}$ fitting in a sequence*

$$0 \to (T^*X \otimes \mathfrak{t})^{1,0} \to Q_{\mathcal{P}_\nabla} \to At_{hol}P \to 0, \tag{5.30}$$

where $At_{hol}P := T^{1,0}P/G$ is the holomorphic Atiyah algebroid of P. Compatible connections on $(P, \mathcal{P}_\nabla)$ are in bijection with isotropic splittings of $Q_{\mathcal{P}_\nabla} \to T^{1,0}X$.

5. *Given an integrable 2-semiconnection on a smooth $\mathcal{G}_\nabla$-bundle $\mathcal{P}_\nabla$, the holomorphic Courant algebroid constructed from $E'_{\mathcal{P}_\nabla}$ and the involutive lifting of $T^{0,1}X$ of 3 coincides with the holomorphic Courant algebroid constructed in 4 from the corresponding holomorphic $\mathcal{G}_\nabla$-bundle.*

Proof. For 1, let $(\{g_{ab}\}, \{\sigma_{ab,\nabla}\}, \{\tau_{abc}\})$ be cocycle data for $(P, \mathcal{P}_\nabla)$ in a cover $\{X_a\}_{a\in A}$ of X and construct $E'_{\mathcal{P}_\nabla}$ by gluing $TX_a \otimes \mathbb{C} \oplus \mathfrak{g} \oplus T^*X_a \otimes \mathfrak{t}$ as in the proof of Theorem 4.23. Equivalently, $E' = E_{\mathcal{P}_\nabla} \otimes \mathbb{C}/(Ker\,\pi)^{0,1}$. With this description, 2 is clear and the formula for the bracket and pairing in Theorem 4.23 implies that $(T^*X \otimes \mathfrak{t})^{1,0}$ is an isotropic ideal inside $\pi^{-1}(T^{0,1}X)$, so that the bracket and pairing descend to $\pi^{-1}(T^{0,1}X)/(T^*X \otimes \mathfrak{t})^{1,0}$. For 3, note that connections on $\mathcal{P}_\nabla$ induce complex linear splittings of $E'_{\mathcal{P}_\nabla} \to TX \otimes \mathbb{C}$ as in Theorem 4.23, and check how these splittings behave when moving the connection by $a \in \Omega^{1,0}(ad\,P)$, $b \in \Omega^{(2,0)+(1,1)}(X, \mathfrak{t})$. The relation between being integrable and involutive follows from the formula for the bracket in Theorem 4.23. For 4, let $(\{g_{ab}\}, \{\sigma_{ab,\nabla}\}, \{\tau_{abc}\}))$ be holomorphic cocycle data for $(P, \mathcal{P}_\nabla)$ (so in particular $g_{ab}^*\theta \in \Omega^{1,0}(X, \mathfrak{g})$ and $F_{ab} \in \Omega^{2,0}(X, \mathfrak{t})$) and construct $Q_{\mathcal{P}_\nabla}$ by gluing $T^{1,0}X_a \oplus \mathfrak{g} \oplus (T^*X \otimes \mathfrak{t})^{1,0}$ as in Theorem 4.23. Then 5 follows directly by construction, as we can also use this cocycle data to construct $E'_{\mathcal{P}_\nabla}$ as before and then the lifting of $T^{0,1}X$ is obtained in this gauge by gluing $T^{0,1}X_a \oplus \{0\} \oplus \{0\}$. □

Remark 5.33. Given an integrable 1-semiconnection on a smooth $\mathcal{G}_\nabla$-bundle, the holomorphic $\mathfrak{t}$-Courant-Dorfman algebroid constructed from the involutive splitting of $T^{0,1}X$ to $\pi^{-1}(T^{0,1}X)/(T^*X \otimes \mathfrak{t})^{1,0}$ of part 3 of Proposition 5.32 is simply $T^{1,0}P/G$, the holomorphic Atiyah algebroid of P.

Let $\mathcal{G}_\nabla$ be a holomorphic multiplicative T-gerbe with holomorphic connective structure. As in Theorem 4.23, Proposition 5.32 can be enhanced to give a functor from the bicategory of smooth $\mathcal{G}_\nabla$-bundles to the category of complex $\mathfrak{t}$-Courant-Dorfman algebroids. In particular, for a fixed $\mathcal{G}_\nabla$-bundle $\mathcal{P}_\nabla$ with complex Atiyah algebroid E' we obtain an action of $Gauge(\mathcal{P}_\nabla)$ on E', which we call the *adjoint action*. This can be described as in the proof of Theorem 4.23; in particular, it preserves $ad\,\mathcal{P}'_\nabla := Ker(\pi) \subset E'$. Recall also the 2-group $Gauge(\mathcal{P}^{0,1}_\nabla)$ from Section 5.2.1 and note that it acts similarly on $ad\,\mathcal{P}'_\nabla/(T^*X \otimes \mathfrak{t})^{1,0}$. The proof of Theorem 4.26 can be adapted in a straightforward way to yield the following.

Corollary 5.34. *Let $\mathcal{G}_\nabla$ be a holomorphic multiplicative T-gerbe with holomorphic connective structure and let $\mathcal{P}_\nabla$ be a smooth $\mathcal{G}_\nabla$-bundle. Then*

1. *The 2-group $Gauge(\mathcal{P}_{\nabla^{0,1}})$ admits a model as a complex Lie 2-group with Lie 2-algebra $C^\infty(X,\mathfrak{t}) \xrightarrow{d_E} \Gamma(ad\,\mathcal{P}'_\nabla/(T^*X \otimes \mathfrak{t})^{1,0})$.*

2. *The 2-group $Gauge(\mathcal{P}_\nabla)$ admits a model as a complex Lie 2-group with Lie 2-algebra $C^\infty(X,\mathfrak{t}) \xrightarrow{d_E} \Gamma(ad\,\mathcal{P}'_\nabla)$.*

In both cases, the adjoint action is holomorphic and it admits a holomorphic right-invariant Maurer-Cartan form in the sense of Definition 5.12.

Remark 5.35. Let $\mathcal{P}_\nabla$ be a holomorphic $\mathcal{G}_\nabla$-bundle and take a compatible connection $(A,B) \in \mathcal{A}(\mathcal{P}_\nabla)$ to split $ad\,\mathcal{P}'_\nabla = T^*X \otimes \mathfrak{t} \oplus ad\,P$. Define the operator

$$\begin{aligned} \overline{\partial} : \Gamma(ad\,P) \oplus \Omega^1(X,\mathfrak{t}) &\to \Omega^{0,1}(ad\,P) \oplus \Omega^{1,1+0,2}(X,\mathfrak{t}) \\ v + \xi &\mapsto \overline{\partial}^A s + (d\xi)^{1,1+0,2} + 2\langle F_A^{1,1}, s\rangle. \end{aligned} \tag{5.31}$$

As it follows from Proposition 4.27, the Lie 2-algebra of the 2-group of holomorphic automorphisms of $\mathcal{P}_\nabla$ can be described as $H^0(X,\mathfrak{t}) \xrightarrow{d} Ker(\overline{\partial})$. In particular, the Lie 2-algebra $H^0(X,\mathfrak{t}) \xrightarrow{d} Ker(\pi : H^0(Q) \to H^0(T^{1,0}X))$ embeds there but in a non-surjective way, as one might have expected a priori from Corollary 5.34.

Chapter 6

Geometry of moduli spaces

In Section 2.3.3 we presented the derived moduli stack of flat connections on a G-bundle $P \to M$, for G a Lie group and M a smooth manifold, and the derived moduli stack of holomorphic structures on a G-bundle $P \to X$, for G a complex Lie group and X a complex manifold, and we constructed shifted symplectic structures on them. In this chapter we show that similar constructions can be performed in the context of higher gauge theory.

Recall that the 'derived' structure of the moduli spaces from Section 2.3.3 arises from the curved DGLA $(\Omega^{\geq 2}(ad\, P), d^A, [\cdot,\cdot])$ associated to a connection A on a G-bundle P. This curved DGLA is obtained from a general procedure which takes a Lie algebroid $E \to M$ with a sub-bundle $D \subset E$ and an ideal $L \subset E$ such that $D' \oplus L = E$, and produces a curved DGLA structure on $\Gamma(\Lambda^\bullet D^* \otimes L)$ whose Maurer-Cartan elements are in bijection with involutive sub-bundles $D' \subset E$ such that $D \oplus L = E$ [156, Section 6.2]. Indeed, applying this construction to $E = TP/G$, $D = s^A(TM)$ for $s^A : TM \to E$ the splitting induced by A and $L = ad\, P$ yields the desired curved DGLA.

In light of Theorem 4.23, we can mimic this construction within the context of higher gauge theory as follows. Given a multiplicative gerbe with connective structure $\mathcal{G}_\nabla$ and a $\mathcal{G}_\nabla$-bundle $\mathcal{P}_\nabla \to M$ with Atiyah algebroid $E \to M$, each enhanced connection $((A, B), g)$ on $\mathcal{P}_\nabla$ induces a splitting $s : TM \to E$. Deforming $((A, B), g)$ to a flat connection is equivalent to deforming s to an isotropic, involutive splitting, and so we can recast the problem of finding the 'derived' structure on the moduli space of flat connections on $\mathcal{P}_\nabla$ as the problem of finding a curved L_∞-algebra controlling deformations of a sub-bundle of a Courant-Dorfman algebroid.

The deformation theory of Dirac structures on Courant algebroids was one of the first problems to be addressed in generalized geometry [179]. Since then, many generalizations and alternative constructions have appeared [91, 115, 143, 163, 228]. They all use techniques from generalized and graded geometry to define curved L_∞-algebras controlling the problem of deforming a given Lagrangian subbundle inside a Courant algebroid $D \subset E$ to an involutive Lagrangian subbundle. The L_∞-algebra is presented upon choosing a Lagrangian complement to L. Note that, for the applications we have in mind, D and L are coisotropic and isotropic, respectively, but not Lagrangian, so the results in these papers cannot be applied directly. We propose a more general construction that yields the previously studied L_∞-algebras when D and L are Lagrangian, and which is also related to L_∞-algebras in the literature on mathematical physics [13].

Once these L_∞-algebras are constructed, defining simplicial derived manifolds representing our moduli spaces of interest is immediate from Theorem 4.26, which states that the gauge 2-group of a principal 2-bundle is smooth. In order to construct shifted symplectic structures associated to these, we recall that the shifted symplectic structures from Section 2.3.3 were motivated by Theorem 2.29 and the 2-shifted symplectic structure from Example 2.35. The analog of this in higher gauge theory is the 2-shifted symplectic structure on $B\mathfrak{G} \times \mathbb{R}^{>0}$ from Proposition 3.52, suggesting the existence of shifted symplectic structures on the moduli spaces of pairs $((A, B), \phi)$, where (A, B) is a flat connection on a principal 2-bundle over an oriented manifold M and ϕ is a non-vanishing constant function on M. As we will see, we can indeed construct such a shifted symplectic structure if we interpret ϕ as a rescaling of the volume form on M, relating it with the dilaton from string theory as suggested by [174].

An interesting observation from Section 2.3.3 is that some of the moduli spaces studied there can be constructed as symplectic reductions or derived critical loci. Recall also from Examples 2.34 and 2.32 that the shifted symplectic structures on these spaces are related to the shifted symplectic structure on $\mathfrak{g}^*//G$ from Example 2.33, while in Proposition 3.27 we constructed an analogous structure for Lie 2-groups with a Maurer-Cartan form. We can use this result to define Hamiltonian actions of Lie 2-groups and their corresponding symplectic reductions, as well as derived critical loci for Lie 2-group-invariant functions, illustrating the constructions with some of our moduli spaces.

In Section 6.1.1 we present our approach to the deformation theory of isotropic, involutive sub-bundles on a Courant-Dorfman algebroid. In Section 6.1.2 we apply this to construct derived moduli stacks of flat connections, holomorphic stuctures and holomorphic structures with holomorphic connective structures on principal 2-bundles. In Section 6.2.1 we construct shifted symplectic structures on these moduli spaces, and in Section 6.2.2 we provide a theory of Hamiltonian reduction for actions of 2-groups which we can apply to some of our examples of moduli spaces. Finally, in Section 6.2.3 we relate these constructions to the study of the Hull-Strominger system from [127].

6.1 Derived moduli stacks

6.1.1 Deformations of isotropic, involutive sub-bundles of a Courant-Dorfman algebroid

Let $E \to M$ be a V-Courant-Dorfman algebroid, let $D \subset E$ be an arbitrary sub-bundle and let $L \subset E$ be an ideal with $D \cap L = \{0\}$ and $D \oplus L = E$. We write $\Pi_L : E \to L$ and $\Pi_D : E \to D$ for the corresponding projections. Then

$$\begin{aligned} \Gamma(D^* \otimes L) &\to \{D' \subset E \mid D' \cap L = 0\} \\ \alpha &\mapsto \{v + \alpha(v) \mid v \in D\} \end{aligned} \tag{6.1}$$

induces a bijection between isotropic, involutive sub-bundles $D' \subset E$ with $D' \cap L = \{0\}$, $D' \oplus L = E$ and $\alpha \in \Gamma(D^* \otimes L)$ satisfying

$$\Pi_L([X,Y]) + [X, \alpha(Y)] + [\alpha(X), Y] - \alpha(\Pi_D[X,Y]) + [\alpha(X), \alpha(Y)] = 0, \tag{6.2}$$

$$\langle X, Y\rangle + \langle \alpha(X), Y\rangle + \langle X, \alpha(Y)\rangle + \langle \alpha(X), \alpha(Y)\rangle = 0. \tag{6.3}$$

Conditions (6.2) and (6.3) can be modelled as the Maurer-Cartan equation of a curved DGLA as follows. First, define for $p \geq 2$ the vector space $K^p(D, L)$ consisting on pairs (ω, τ), where

1. $\tau \in \Gamma(\Lambda^{p-2} D^* \otimes S^2 D^* \otimes V)$ satisfying

$$\begin{aligned} \tau(X_1, ..., X_{p-3}, X_{p-2}, X_{p-1}, X_p) + \tau(X_1, ..., X_{p-3}, X_{p-1}, X_p, X_{p-2}) \\ + \tau(X_1, ..., X_{p-3}, X_p, X_{p-2}, X_{p-1}) = 0, \end{aligned} \tag{6.4}$$

2. $\omega : \Gamma(D) \otimes \overset{p}{\cdots} \otimes \Gamma(D) \to \Gamma(L)$ $\mathbb{R}$-linear satisfying

$$\omega(X_1, ..., fX_p) - f\omega(X_1, ..., X_p) = 0, \qquad f \in C^\infty(X, \mathbb{R}), \tag{6.5}$$

$$\omega(X_1, ..., X_i, X_{i+1}, ..., X_p) + \omega(X_1, ..., X_{i+1}, X_i, ..., X_p) = 0, \qquad 1 \leq i \leq p-2, \tag{6.6}$$

$$\omega(X_1, ..., X_{p-1}, X_p) + \omega(X_1, ..., X_p, X_{p-1}) = d_E(\tau(X_1, ..., X_p)). \tag{6.7}$$

We also define

$$K^1(D,L) := \Gamma(D^*\otimes L), \quad K^0(D,L) := \Gamma(L), \quad K^{-1}(D,L) := C^\infty(X,V), \tag{6.8}$$

and $K^i = 0$ for $i \leq -2$.

The bracket $[\cdot,\cdot] : K^{p_1}(D,L)\otimes K^{p_2}(D,L)\to K^{p_1+p_2}(D,L)$ is defined as

$$[(\omega_1,\tau_1),(\omega_2,\tau_2)] = ([\omega_1,\omega_2],\tau_{[\omega_1,\omega_2]}), \tag{6.9}$$

where

$$\begin{aligned}
&[\omega_1,\omega_2](X_1,...,X_{p_1+p_2}) := \\
&\qquad \sum_{\sigma\in S_{p_1,p_2-1}} (-1)^\sigma [\omega_1(X_{\sigma(1)},...,X_{\sigma(p_1)}),\omega_2(X_{\sigma(p_1+1)},...,X_{\sigma(p_1+p_2-1)},X_{p_1+p_2})] \\
&+(-1)^{p_2+1}\sum_{\sigma\in S_{p_1-1,p_2}} (-1)^\sigma [\omega_2(X_{\sigma(p_1)},...,X_{\sigma(p_1+p_2-1)}),\omega_2(X_{\sigma(1)},...,X_{\sigma(p_1-1)},X_{p_1+p_2})],
\end{aligned} \tag{6.10}$$

$$\begin{aligned}
&\tau_{[\omega_1,\omega_2]}(X_1,...,X_{p_1+p_2}) := \\
&(-1)^{p_2+1}\sum_{\sigma\in S_{p_1-1,p_2-1}} (-1)^\sigma \langle\omega_1(X_{\sigma(1)},...,X_{\sigma(p_1-1)},X_{p_1+p_2-1}),\omega_2(X_{\sigma(p_1)},...,X_{\sigma(p_1+p_2-2)},X_{p_1+p_2})\rangle \\
&\qquad +\langle\omega_1(X_{\sigma(1)},...,X_{\sigma(p_1-1)},X_{p_1+p_2}),\omega_2(X_{\sigma(p_1)},...,X_{\sigma(p_1+p_2-2)},X_{p_1+p_2-1})\rangle
\end{aligned} \tag{6.11}$$

The differential $d : K^p(D,L)\to K^{p+1}(D,L)$ is

$$d(\omega,\tau) = (d\omega, d\tau + (-1)^{p-1}\omega^s), \tag{6.12}$$

where

$$\begin{aligned}
d\omega(X_1,...,X_{p+1}) := &\sum_{\sigma\in S_{1,p-1}} (-1)^\sigma [X_{\sigma(1)},\omega(X_{\sigma(2)},...,X_{\sigma(p)},X_{p+1})] \\
&-(-1)^p[\omega(X_1,...,X_p),X_{p+1}] \\
&+\sum_{\sigma\in S_{2,p-2}} (-1)^\sigma \omega(\pi_M[X_{\sigma(1)},X_{\sigma(2)}],X_{\sigma(3)},...,X_{\sigma(p)},X_{p+1}) \\
&-\sum_{\sigma\in S_{1,p-1}} (-1)^\sigma \omega(X_{\sigma(2)},...,X_{\sigma(p)},\pi_M[X_{\sigma(1)},X_{p+1}]),
\end{aligned} \tag{6.13}$$

$$\begin{aligned} d\tau(X_1,...,X_{p+1}) := &\sum_{\sigma\in S_{1,p-2}} (-1)^\sigma \pi(X_{\sigma(1)})(\tau(X_{\sigma(2)},...,X_{\sigma(p-1)},X_p,X_{p+1})) \\ &+ \sum_{\sigma\in S_{2,p-3}} (-1)^\sigma \tau(\pi_M[X_{\sigma(1)},X_{\sigma(2)}],X_{\sigma(3)},...,X_{\sigma(p-1)},X_p,X_{p+1}) \\ &- \sum_{\sigma\in S_{1,p-2}} (-1)^\sigma \tau(X_{\sigma(2)},X_{\sigma(3)},...,X_{\sigma(p-1)},\Pi_D[X_{\sigma(1)},X_p],X_{p+1}) \\ &- \sum_{\sigma\in S_{1,p-2}} (-1)^\sigma \tau(X_{\sigma(2)},X_{\sigma(3)},...,X_{\sigma(p-1)},X_p,\Pi_D[X_{\sigma(1)},X_{p+1}]), \end{aligned} \tag{6.14}$$

$$\omega^s(X_1,...,X_{p+1}) := \langle\omega(X_1,...X_{p-1},X_p),X_{p+1}\rangle + \langle\omega(X_1,...,X_{p-1},X_{p+1}),X_p\rangle \tag{6.15}$$

The curvature is $\Phi \in K^2(D,L)$, $\Phi = (\omega_\Phi,\tau_\Phi)$, where

$$\omega_\Phi(X_1,X_2) = -\Pi_L[X_1,X_2], \tag{6.16}$$

$$\tau_\Phi(X_1,X_2) = -\langle X_1,X_2\rangle. \tag{6.17}$$

Note that all these formulas are analogous to the invariant formulas for the DGLA controlling deformations of an involutive sub-bundle of a Lie algebroid [156, Cor. 6.2.17].

Proposition 6.1. $(K^\bullet(D,L),\Phi,d,[\cdot,\cdot])$ *is a curved differential graded Lie algebra whose Maurer-Cartan elements are in bijection with isotropic, involutive subbundles* $D' \subset E$ *such that* $D'\cap L = 0$ *and* $D'\oplus L = E$*.*

Proof. It follows from tedious but straightforward computations analogous to the corresponding statement for Lie algebroids [156, Section 6.2.], using the axioms from Definition 4.22 and the fact that the bijections of shuffle permutations (see (2.63))

$$\begin{matrix} S_{p_1,p_2}\times S_{p_1+p_2,p_3}\to S_{p_1,p_2,p_3} & S_{p_1,p_2+p_3}\times S_{p_2,p_3}\to S_{p_1,p_2,p_3} & S_{p_1,p_2}\to S_{p_2,p_1} \\ (\sigma_{1,2},\sigma_{12,3})\mapsto\sigma_{1,2,3}, & (\sigma_{1,23},\sigma_{2,3})\mapsto\sigma'_{1,2,3}, & \sigma_{1,2}\mapsto\sigma_{2,1} \end{matrix} \tag{6.18}$$

defined by

$$\sigma_{1,2,3} := \begin{cases} \sigma_{12,3}(\sigma_{1,2}(i)) & 1\le i\le p_1+p_2 \\ \sigma_{12,3}(i) & p_1+p_2 < i \le p_1+p_2+p_3, \end{cases} \tag{6.19}$$

$$\sigma'_{1,2,3} := \begin{cases} \sigma_{1,23}(i) & 1\le i\le p_1 \\ \sigma_{1,23}(p_1+\sigma_{2,3}(i-p_1)) & p_1 < i \le p_2+p_3, \end{cases} \tag{6.20}$$

$$\sigma_{2,1} := \begin{cases} \sigma_{1,2}(p_1+i) & 1\le i\le p_2 \\ \sigma_{1,2}(p_2-i) & p_2 < i \le p_1+p_2, \end{cases} \tag{6.21}$$

satisfy

$$\begin{aligned}(-1)^{\sigma_{1,2,3}} &= (-1)^{\sigma_{1,2}}(-1)^{\sigma_{12,3}},\\ (-1)^{\sigma'_{1,2,3}} &= (-1)^{\sigma_{1,23}}(-1)^{\sigma_{2,3}},\\ (-1)^{\sigma_{2,1}} &= (-1)^{p_1p_2}(-1)^{\sigma_{1,2}}.\end{aligned} \tag{6.22}$$

□

It is often desirable to present the DGLA structure on $K^\bullet(D,L)$ by a smaller — albeit algebraically more complicated —, quasi-isomorphic L_∞-algebra. For this, define

$$\begin{aligned}&W^p(D,L) := \{(\omega,\tau)\in K^p(D,L)\mid \tau=0,\ \omega^s=0\}\subset\Gamma(\Lambda^pD^*\otimes L) && p\geq 2,\\ &W^p(D,L) := K^p(D,L) && p\leq 1.\end{aligned} \tag{6.23}$$

Lemma 6.2. *Assume that $\tau_\Phi = 0$ and that there exists a map $l : D^*\otimes V\to L$ such that*

$$\alpha(X) = \langle l(\alpha), X\rangle, \tag{6.24}$$

$$\langle l(\alpha), l'\rangle = 0, \tag{6.25}$$

$$[l(\alpha), l'] = 0 \tag{6.26}$$

for $\alpha\in D^*\otimes V$, $X\in D$, $l'\in L$. Then there is a structure of curved cubic L_∞-algebra on $\bigoplus_p W^p(D,L)$ and a quadratic morphism $W^\bullet(D,L)\to K^\bullet(D,L)$ which is a quasi-isomorphism when $\Phi=0$.

Proof. Note first that we can use l to define for each $\tau\in\Lambda^{p-2}D^*\otimes S^2D^*\otimes V$ satisfying (6.4) an $\alpha_\tau\in\Lambda^{p-1}D^*\otimes L\subset K^{p-1}(D,L)$ with $\alpha_\tau^s = (-1)^{p-1}\tau$; namely,

$$\begin{aligned}\alpha_\tau(X_1,...,X_{p-1}) = \frac{(-1)^{p-1}}{p}\big(&l(\tau(X_1,...,X_{p-2},X_{p-1},\cdot) - l(\tau(X_1,...,X_{p-1},X_{p-2},\cdot)) - ...\\ &-l(\tau(X_1,X_{p-1},...,X_{p-2},X_2,\cdot)) - l(\tau(X_{p-1},X_2,...,X_{p-2},X_1,\cdot))\big).\end{aligned} \tag{6.27}$$

Then define $f_1 : W^\bullet(D,L)\to K^\bullet(D,L)$ as the inclusion and

$$f_2 : W^\bullet(D,L)\otimes W^\bullet(D,L)\to K^\bullet(D,L)$$

by $f_2(\omega_1,\omega_2) := -\alpha_{\tau_{[\omega_1,\omega_2]}}$. Note that $[f_2(\omega_1,\omega_2),\cdot]=0$. We will show that this gives a quasi-isomorphism between a curved cubic L_∞-algebra structure on $W^\bullet(D,L)$ and $K^\bullet(D,L)$ by applying the axioms of the DGLA structure of $K^\bullet(D,L)$ to elements of $W^\bullet(D,L)$.

1. For $\omega \in W^p(D,L) \subset K^p(D,L)$,

$$-([\Phi,\omega],\tau_{[\Phi,\omega]}) = d^2(\omega,0) = (d^2\omega,(-1)^p(d\omega)^s) \quad \Rightarrow \quad (d\omega)^s = -f_2(\Phi,\omega)^s; \tag{6.28}$$

thus we may define

$$d^W\omega := d\omega + f_2(\Phi,\omega) \tag{6.29}$$

2. For $\omega_i \in W^{p_i}(D,L) \subset K^{p_i}(D,L)$, $i = 1$, 2, note first that

$$\begin{aligned} 0 &= -([\Phi, f_2(\omega_1,\omega_2)], \tau_{[\Phi,f_2(\omega_1,\omega_2)]}) \\ &= (d^2 f_2(\omega_1,\omega_2), d\tau_{[\omega_1,\omega_2]} + (-1)^{p_1+p_2+1}(df_2(\omega_1,\omega_2))^s) \\ &\Rightarrow \quad (df_2(\omega_1,\omega_2))^s = (-1)^{p_1+p_2} d\tau_{[\omega_1,\omega_2]}. \end{aligned} \tag{6.30}$$

Then we see

$$\begin{aligned} &([d\omega_1,\omega_2]+(-1)^{p_1}[\omega_1,d\omega_2], \tau_{[d\omega_1,\omega_2]} + (-1)^{p_1}\tau_{[\omega_1,d\omega_2]}) \\ &\qquad = (d[\omega_1,\omega_2], d\tau_{[\omega_1,\omega_2]} + (-1)^{p_1+p_2-1}[\omega_1,\omega_2]^s) \\ &\qquad \Rightarrow [\omega_1,\omega_2]^s + df_2(\omega_1,\omega_2)^s + f_2(d\omega_1,\omega_2)^s + (-1)^{p_1} f_2(\omega_1,d\omega_2)^s = 0, \end{aligned} \tag{6.31}$$

and so we may define

$$[\omega_1,\omega_2]^W := [\omega_1,\omega_2] + df_2(\omega_1,\omega_2) + f_2(d\omega_1,\omega_2) + (-1)^{p_1} f_2(\omega_1,d\omega_2). \tag{6.32}$$

3. For $\omega_i \in W^{p_i}(D,L) \subset K^{p_i}(D,L)$, $i = 1$, 2, 3, we note

$$\begin{aligned} &f_2([\omega_1,\omega_2]^W,\omega_3)^s - (-1)^{e_2e_3} f_2([\omega_1,\omega_3]^W,\omega_2)^s + (-1)^{e_1(e_2+e_3)} f_2([\omega_2,\omega_3]^W,\omega_1)^s \\ &= (-1)^{p_1+p_2+p_3}\big(\tau_{[[\omega_1,\omega_2]^V,\omega_3]} - (-1)^{e_2e_3}\tau_{[[\omega_1,\omega_3]^V,\omega_2]} + (-1)^{e_1(e_2+e_3)}\tau_{[[\omega_2,\omega_3]^V,\omega_1]}\big) \\ &= 0, \end{aligned} \tag{6.33}$$

and so we can define

$$\begin{aligned} \{\omega_1,\omega_2,\omega_3\}^W := &- f_2([\omega_1,\omega_2]^W,\omega_3) + (-1)^{e_2e_3} f_2([\omega_1,\omega_3]^W,\omega_2) \\ &- (-1)^{e_1(e_2+e_3)} f_2([\omega_2,\omega_3]^W,\omega_1). \end{aligned} \tag{6.34}$$

By construction, this defines a structure of curved cubic L_∞-algebra on $W^\bullet(D,L)$ with f_1, f_2 defining a quadratic morphism to $K^\bullet(D,L)$. To see that this is in fact a quasi-isomorphism when $\Phi = 0$, let $(\omega,\tau) \in K^p(D,L)$ satisfy $d(\omega,\tau) = 0$; i.e. $d\omega = 0$ and $d\tau = (-1)^p\omega^s$. Then we claim that $(\omega,\tau) + d(\alpha_\tau,0) \in W^p$, which implies that

$H^p(W,d) \to H^p(K,d)$ is surjective. Indeed,

$$d(\alpha_\tau, 0) = (d\alpha_\tau, (-1)^p \alpha_\tau^s) = (d\alpha_\tau, -\tau), \tag{6.35}$$

$$0 = d^2(\alpha_\tau, 0) = (d^2\alpha_\tau, -d\tau + (-1)^{p-1}(d\alpha_\tau)^s) \quad \Rightarrow (d\alpha_\tau)^s = -\omega^s, \tag{6.36}$$

which shows the claim. To show that $H^p(W,d) \to H^p(K,d)$ is injective, we let $\omega \in W^p(D,L)$ and $(\mu,\sigma) \in K^{p-1}(D,L)$ satisfy $d(\mu,\sigma) = \omega$; i.e., $d\mu = \omega$ and $d\sigma + (-1)^p \mu^s = 0$. Then, by a similar reasoning as before, $(\mu,\sigma) + d(\alpha_\sigma, 0) \in W^{p-1}$, which concludes the proof. □

6.1.2 Derived stacks of flat connections, holomorphic structures and holomorphic connective structures

We proceed to apply the general procedure of Section 6.1.1 to obtain curved L_∞-algebras controlling deformation problems in higher gauge theory.

Example 6.3 (Deformations of flat connections)**.** Let G, T be Lie groups with T abelian, let $\mathcal{G}_\nabla$ be a multiplicative T-gerbe with connective structure over G, and let $\mathcal{P}_\nabla \to M$ be a $\mathcal{G}_\nabla$-bundle with Atiyah algebroid E (cf. Theorem 4.23). Let $(A,B) \in \mathcal{A}(\mathcal{P}_\nabla)$ be a connection on $\mathcal{P}_\nabla$ and let $s^{(A,B)} : TM \to E$ be its corresponding splitting. We wish to construct a curved L_∞-algebra controlling the problem of deforming (A,B) to a flat connection.

We note that deforming (A,B) to a flat connection is equivalent to deforming the sub-bundle $s^{(A,B)}(TM) \subset E$ to an isotropic, involutive sub-bundle complementary to $Ker\,\pi \subset E$. Hence, we perform the construction from Section 6.1.1. We define the vector spaces (6.23) and apply Lemma 6.2 with l the inclusion $T^*M \otimes \mathfrak{t} \to Ker\,\pi$. We also use (A,B) to identify $Ker\,\pi \cong T^*M \otimes \mathfrak{t} \oplus ad\,P$, obtaining the following curved cubic L_∞-algebra.

The graded vector space is

$$W = \bigoplus_{p=-1}^{n} \Omega^p(ad\,P) \oplus \Omega^{p+1}(M,\mathfrak{t})$$

with $W^p := \Omega^p(ad\,P) \oplus \Omega^{p+1}(M,\mathfrak{t})$ in degree p. The curved L_∞-structure is given by

$$\Phi^{(A,B)} := F_A - H, \tag{6.37}$$

$$d^{(A,B)}(a+b) := d^A a + (db + (-1)^a \langle F_A \wedge a \rangle), \tag{6.38}$$

$$[a_1 + b_1, a_2 + b_2] := -[a_1 \wedge a_2] + 0, \tag{6.39}$$

$$\{a_1 + b_1, a_2 + b_2, a_3 + b_3\} := 0 + (-1)^{a_1+a_2+a_3} \langle a_1 \wedge [a_2 \wedge a_3] \rangle. \tag{6.40}$$

for $a_i \in \Omega^\bullet(ad\, P)$ and $b_i \in \Omega^{\bullet+1}(M, \mathfrak{t})$.

Now consider the derived manifold $\mathcal{N} = (\mathcal{A}(\mathcal{P}_\nabla), \underline{W}^{\geq 2}, Q)$, where Q is defined by the fiberwise structure of curved cubic L_∞-algebra on $\underline{W}^{\geq 2} \to \mathcal{A}(\mathcal{P}_\nabla)$. The gauge 2-group $Gauge(\mathcal{P}_\nabla)$, which is a Lie group by Theorem 4.26, acts smoothly on $\mathcal{A}(\mathcal{P}_\nabla)$ and we can lift this to an action on $\mathcal{N}$ by letting $(u, \varphi_\nabla, \alpha^\varphi) \in Gauge(\mathcal{P}_\nabla)_0$ act on $a + b \in \Omega^p(ad\, P) \oplus \Omega^{p+1}(M, \mathfrak{t})$ as

$$u \cdot (a + b) = Ad(g_u)a + b,$$

where $g_u : M \to Ad\, P$ is the underlying gauge transformation of P. It is easy to check that the curved L_∞-algebra structure is equivariant for this action; hence, this is a well-defined action on $\mathcal{N}$. We write $\mathcal{B}^{\flat,d}(\mathcal{P}_\nabla)_\bullet := \mathcal{N}//Gauge(\mathcal{P}_\nabla)$ for the quotient 2-groupoid (cf. Remark 3.18) and call this the *derived moduli stack of flat connections* on $\mathcal{P}_\nabla$. Note that, since 2-isomorphic gauge transformations act in the same way, $\mathcal{B}^{\flat,d}(\mathcal{P}_\nabla)_n = \mathcal{N} \times BGauge(\mathcal{P}_\nabla)_n$, with simplicial maps defined in a similar way to Example 2.5.

Example 6.4 (Deformations of holomorphic structures)**.** Let G, T be complex Lie groups with T abelian, let $\mathcal{G}_\nabla$ be a holomorphic multiplicative T-gerbe with holomorphic connective structure over G, and let $\mathcal{P}_\nabla \to X$ be a smooth $\mathcal{G}_\nabla$-bundle over a complex manifold X with complex Atiyah algebroid E' (cf. Proposition 5.32). Let $D \in \mathcal{D}(\mathcal{P}_\nabla)$ be a 1-semiconnection and let $s^D : T^{0,1}X \to \pi^{-1}(T^{0,1}X)/(T^*X \otimes \mathfrak{t})^{1,0}$ be its corresponding splitting. We wish to construct a curved L_∞-algebra controlling the problem of deforming D to an integrable 1-semiconnection.

We note that deforming D to an integrable 1-semiconnection is equivalent to deforming the sub-bundle $s^D(T^{0,1}X) \subset \pi^{-1}(T^{0,1}X)/(T^*X \otimes \mathfrak{t})^{1,0}$ to an isotropic, involutive sub-bundle complementary to $Ker(\pi) \subset \pi^{-1}(T^{0,1}X)/(T^*X \otimes \mathfrak{t})^{1,0}$. Hence, we perform the construction from Section 6.1.1. We define the vector spaces (6.23) and apply Lemma 6.2 with l the inclusion $(T^*X \otimes \mathfrak{t})^{0,1} \to Ker(\pi)$, using also D to obtain a splitting $Ker(\pi) = (T^*X \otimes \mathfrak{t})^{0,1} \oplus ad\, P$. The result is the following curved cubic L_∞-algebra.

The graded vector space is

$$W = \bigoplus_{p=-1}^{n} \Omega^{(0,p)}(ad\, P) \oplus \Omega^{(0,p+1)}(X, \mathfrak{t})$$

with $W^p := \Omega^{(0,p)}(ad\,P) \oplus \Omega^{(0,p+1)}(X,\mathfrak{t})$ in degree p. The curved L_∞-structure is given by

$$\Phi^D := F_D^{0,2} - H^{0,3}, \tag{6.41}$$

$$d^{(A,B)}(a+b) := \overline{\partial}^D a + (\overline{\partial} b + (-1)^a \langle F_D^{0,2} \wedge a \rangle), \tag{6.42}$$

$$[a_1 + b_1, a_2 + b_2] := -[a_1 \wedge a_2] + 0, \tag{6.43}$$

$$\{a_1 + b_1, a_2 + b_2, a_3 + b_3\} := 0 + (-1)^{a_1+a_2+a_3} \langle a_1 \wedge [a_2 \wedge a_3] \rangle. \tag{6.44}$$

for $a_i \in \Omega^{(0,\bullet)}(ad\,P)$ and $b_i \in \Omega^{(0,\bullet+1)}(X,\mathfrak{t})$.

Now consider the derived manifold $\mathcal{N} = (\mathcal{D}(\mathcal{P}_\nabla), \underline{W}^{\geq 2}, Q)$, where Q is defined by the fiberwise structure of curved cubic L_∞-algebra on $\underline{W}^{\geq 2} \to \mathcal{D}(\mathcal{P}_\nabla)$. The complex Lie 2-group $Gauge(\mathcal{P}_{\nabla^{0,1}})$ acts on $\mathcal{D}(\mathcal{P}_\nabla)$ and we can lift this to an action on $\mathcal{N}$ by letting $(u, [\varphi_\nabla], \alpha^\varphi) \in Gauge(\mathcal{P}_{\nabla^{0,1}})_0$ act on $a + b \in \Omega^{0,p}(ad\,P) \oplus \Omega^{0,p+1}(M,\mathfrak{t})$ as

$$u \cdot (a+b) = Ad(g_u)a + b,$$

where $g_u : M \to Ad\,P$ is the underlying gauge transformation of P. It is easy to check that the curved L_∞-algebra structure is equivariant for this action; hence, this is a well-defined action on $\mathcal{N}$. We write $\mathcal{H}^d(\mathcal{P}_\nabla)_\bullet := \mathcal{N}//Gauge(\mathcal{P}_{\nabla^{0,1}})$ for the quotient 2-groupoid (cf. Remark 3.18) and, in light of Proposition 5.22, we call this the *derived moduli stack of holomorphic structures* on $\mathcal{P}_\nabla$. Note that, since 2-isomorphic gauge transformations act in the same way, $\mathcal{H}^d(\mathcal{P}_\nabla)_n = \mathcal{N} \times BGauge(\mathcal{P}_{\nabla^{0,1}})_n$, with simplicial maps defined in a similar way to Example 2.5.

Example 6.5 (Deformations of holomorphic structures with holomorphic connective structure)**.** Let G, T be complex Lie groups with T abelian, let $\mathcal{G}_\nabla$ be a holomorphic multiplicative T-gerbe with holomorphic connective structure over G, and let $\mathcal{P}_\nabla \to X$ be a smooth $\mathcal{G}_\nabla$-bundle over a complex manifold X with complex Atiyah algebroid E' (cf. Proposition 5.32). Let $D \in \mathcal{D}'(\mathcal{P}_\nabla)$ be a 2-semiconnection and let $s^D : T^{0,1}X \to E'$ be its corresponding lifting of $T^{0,1}X$. We wish to construct a curved L_∞-algebra controlling the problem of deforming D to an integrable 2-semiconnection.

We note that deforming D to an integrable 2-semiconnection is equivalent to deforming the sub-bundle $s^D(T^{0,1}X) \subset \pi^{-1}(T^{0,1}X)$ to an isotropic, involutive sub-bundle complementary to $Ker(\pi) \subset \pi^{-1}(T^{0,1}X)$. Hence, we perform the construction from Section 6.1.1. We define the vector spaces (6.23) and apply Lemma 6.2 with l the inclusion $(T^*X \otimes \mathfrak{t})^{0,1} \to Ker(\pi)$. The result is a structure of curved cubic L_∞-algebra on the graded vector space

$$W = \bigoplus_{p=2}^{n} W^{p,D},$$

where $W^{p,D}$ is defined by

$$\begin{aligned} W^{p,D} = \{\omega \in & \Lambda^{0,p}T^*X \otimes (Ker\,\pi) \mid \\ & \langle \omega(X_1, ..., X_p), s^D(X_{p+1})\rangle + \langle \omega(X_1, ..., X_{p+1}), s^D(X_p)\rangle = 0\}. \end{aligned} \tag{6.45}$$

In order to write the curved L_∞-structure, we choose a connection (A, B) compatible with the holomorphic structure induced by D. This gives an isomorphism $Ker(\pi) \cong T^*X \otimes \mathfrak{t} \oplus ad\,P$ and thus an isomorphism $W^{p,D} \cong \Omega^{0,p}(ad\,P) \oplus \Omega^{(1,p)+(0,p+1)}(X, \mathfrak{t})$. In this presentation, the structure is described as follows.

$$\Phi^D := F_A^{0,2} - H^{(1,2)+(0,3)}, \tag{6.46}$$

$$d^D(a + b) := \overline{\partial}^A a + (db)^{(1,p+1)+(0,p+2)} + (-1)^a \langle F_A^{0,2} + 2F_A^{1,1} \wedge a\rangle, \tag{6.47}$$

$$[a_1 + b_1, a_2 + b_2]^D := -[a_1 \wedge a_2] + (-1)^{a_1+a_2}(\langle \partial^A a_1 \wedge a_2\rangle - (-1)^{a_1}\langle a_1 \wedge \partial^A a_2\rangle), \tag{6.48}$$

$$\{a_1 + b_1, a_2 + b_2, a_3 + b_3\} := 0 + (-1)^{a_1+a_2+a_3}\langle a_1 \wedge [a_2 \wedge a_3]\rangle, \tag{6.49}$$

for $a_i \in \Omega^{(0,\bullet)}(ad\,P)$ and $b_i \in \Omega^{(1,\bullet)+(0,\bullet+1)}(X, \mathfrak{t})$. Note that when $\mathcal{G}_\nabla$ is the complexification of $\mathcal{K}_\nabla$ and $\mathcal{P}_\nabla$ is the complexification of a $\mathcal{K}_\nabla$-bundle $\mathcal{P}_{h,\nabla_h}$ then each $D \in \mathcal{D}'(\mathcal{P}_\nabla)$ determines by Theorem 5.26 a compatible connection $(A_h, B_h - J_{\mathfrak{t}}\omega)$, so the formulas above can be written with respect to it. If we do not want to choose a compatible connection in the general case, we can simply check how the isomorphisms $W^{p,D} \cong^{(A,B)} \Omega^{0,p}(ad\,P) \oplus \Omega^{(1,p)+(0,p+1)}(X, \mathfrak{t})$ behave under changing the connection by $a \in \Omega^{1,0}(ad\,P)$ and $b \in \Omega^{2,0}(X, \mathfrak{t})$. This gives a canonical isomorphism

$$W^{p,D} := \Big(\mathcal{A}_D(\mathcal{P}_\nabla) \times \Omega^{0,p}(ad\,P) \oplus \Omega^{(1,p)+(0,p+1)}(X, \mathfrak{t})\Big) / \sim, \tag{6.50}$$

where $\mathcal{A}_D(\mathcal{P}_\nabla)$ is the set of connections that are compatible with D, and the equivalence relation is

$$(A, B, a, b) \sim (A', B', a, b - 2\langle a \wedge (A' - A)\rangle) \tag{6.51}$$

for $a \in \Omega^{0,p}(ad\,P)$, $b \in \Omega^{(1,p)+(0,p+1)}(X, \mathfrak{t})$. The curved L_∞-algebra above is well-defined over the spaces (6.50) independently of any choices.

Now consider the derived manifold $\mathcal{N} = (\mathcal{D}'(\mathcal{P}_\nabla), W^{\geq 2}, Q)$, where $W^{\geq 2} \to \mathcal{D}'(\mathcal{P}_\nabla)$ is the graded vector bundle with fiber $W^{\geq 2,D}$ at $D \in \mathcal{D}'(\mathcal{P}_\nabla)$, and Q is defined by the preceding fiberwise structure of curved cubic L_∞-algebra on W. The complex Lie 2-group $Gauge(\mathcal{P}_\nabla)$ acts on $\mathcal{D}(\mathcal{P}_\nabla)$ and we can lift this to an action on $\mathcal{N}$ by letting $(u, \varphi_\nabla, \alpha^\varphi) \in Gauge(\mathcal{P}_\nabla)_0$ act on $a + b \in \Omega^{0,p}(ad\,P) \oplus \Omega^{(1,p)+(0,p+1)}(M, \mathfrak{t})$ as

$$u \cdot (a + b) = Ad(g_u)a + b,$$

where $g_u : M \to Ad\, P$ is the underlying gauge transformation of P. This defines an action on the graded vector bundle $W^{\geq 2} \to \mathcal{D}'(\mathcal{P}_\nabla)$ since (6.51) is equivariant. It is also easy to check that the curved L_∞-algebra structure is equivariant for this action; hence, this is a well-defined action on $\mathcal{N}$. We write $\mathcal{H}'^{,d}(\mathcal{P}_\nabla) := \mathcal{N}//Gauge(\mathcal{P}_\nabla)$ for the quotient 2-groupoid (cf. Remark 3.18) and, in light of Proposition 5.22, we call this the *derived moduli stack of holomorphic structures with holomorphic connective structures* on $\mathcal{P}_\nabla$. Note that, since 2-isomorphic gauge transformations act in the same way, $\mathcal{H}'^{,d}(\mathcal{P}_\nabla)_n = \mathcal{N} \times BGauge(\mathcal{P}_\nabla)_n$, with simplicial maps defined in a similar way to Example 2.5.

Remark 6.6. The L_∞-algebra in Example 6.5 is closely related to constructions in the literature on mathematical physics [13] and generalized geometry [126].

1. The L_∞-algebra in [13] is defined for an integrable 2-semiconnection $D \in \mathcal{D}'(\mathcal{P}_\nabla)$ (i.e., $F_A^{0,2} = 0$, $H^{1,2+0,3} = 0$) over a vector space of the form

$$\bigoplus_p \Omega^{0,p}(ad\, P) \oplus \Omega^{(1,p)+(0,p+1)}(X, \mathfrak{t}) \oplus \Omega^{0,p}(T^{1,0}X).$$

   The subspace $\bigoplus_p \Omega^{0,p}(ad\, P) \oplus \Omega^{(1,p)+(0,p+1)}(X, \mathfrak{t})$ is a subalgebra, and its bracket coincides with ours in the uncurved case.

2. The differential graded Lie algebra in [126] is defined for an integrable 2-semiconnection $D \in \mathcal{D}'(\mathcal{P}_\nabla)$ (i.e., $F_A^{0,2} = 0$, $H^{1,2+0,3} = 0$) over a vector space of the form

$$\bigoplus_p \Omega^{0,p}(ad\, P) \oplus \Omega^{(2,p)+\ldots+(p+2,0)}(X, \mathfrak{t}).$$

   The map $a + b \mapsto a + (-1)^a \partial b^{1,p} + 2\langle F_A^{2,0} \wedge a\rangle$ gives a morphism between our L_∞-algebra (in the uncurved case) and the one in [126].

6.2 Shifted symplectic structures

6.2.1 Shifted symplectic moduli spaces

Let $\mathcal{G}_\nabla$ be a multiplicative $U(1)$-gerbe with connective structure over a Lie group G such that the induced bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ from Theorem 3.43 is non-degenerate. Then Proposition 3.27 and Theorem 2.29 suggest that, for M an oriented, compact manifold with $\dim_{\mathbb{R}} M = n$, there is a $(2-n)$-shifted symplectic structure on the moduli space of pairs $((A, B), \phi)$, where (A, B) is a flat connection on a $\mathcal{G}$-bundle and $\phi : M \to \mathbb{R}^*$ is a constant function. We proceed to construct such structure.

First we introduce the following notation. For any manifold M we define the following derived manifold, called the *dilaton moduli*.

$$\Omega_{dR}(M)^* := (C^\infty(M,\mathbb{R}^*), \underline{\Omega^{\geq 1}(M,\mathbb{R})}[-1], Q), \tag{6.52}$$

The notation is such that we regard the trivial vector bundle with fiber $\Omega^j(M,\mathbb{R})$ in degree $j+1$. Here Q is defined by the curvature map $d : C^\infty(M,\mathbb{R}^*) \to \Omega^1(M,\mathbb{R})$ and the differential $d : \Omega^\bullet(M,\mathbb{R}) \to \Omega^{\bullet+1}(M,\mathbb{R})$, both given by the exterior derivative. Note that $\Omega_{dR}(M)^*$ is a model in derived geometry for the space of non-vanishing constant functions on M. Its role in the following result seems to be related to the role of dilatons in heterotic string theory.

Theorem 6.7. *Let $\mathcal{G}_\nabla$ be a multiplicative $U(1)$-gerbe with connective structure over a Lie group G such that the induced bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{R}$ from Theorem 3.43 is non-degenerate, let M be an oriented, compact manifold with $\dim_{\mathbb{R}} M = n$ and let $\mathcal{P}_\nabla \to M$ be a $\mathcal{G}_\nabla$-bundle. Then there is a $(2-n)$-shifted symplectic structure on the Cartesian product (see (2.95))*

$$\mathfrak{M}_\bullet := \mathcal{B}^{\flat,d}(\mathcal{P}_\nabla)_\bullet \times \Omega_{dR}(M)^*, \tag{6.53}$$

where $\mathcal{B}^{\flat,d}(\mathcal{P}_\nabla)_\bullet$ is as in Example 6.3 and $\Omega_{dR}(M)^*$ is as in (6.52).

Proof. Note that $\mathfrak{M}_j = \mathcal{N} \times BGauge(\mathcal{P}_\nabla)_j \times \Omega_{dR}(M)^*$, where

$$\mathcal{N} = (\mathcal{A}(\mathcal{P}_\nabla), \underline{\Omega^{\geq 2}(ad\, P)} \oplus \underline{\Omega^{\geq 3}(M,\mathbb{R})}[1], Q)$$

is as in Example 6.3. The simplicial maps of $\mathfrak{M}_\bullet$ are defined by the action of $Gauge(\mathcal{P}_\nabla)$ on $\mathcal{N}$ as in Example 2.5. We define the $(2-n)$-shifted symplectic structure on $\mathfrak{M}_\bullet$ first for $n \geq 3$. Recall the Maurer-Cartan 1-form on $Gauge(\mathcal{P}_\nabla)$ constructed in Theorem 4.26. We use this to send vectors tangent to $BGauge(\mathcal{P}_\nabla)_1$ and $BGauge(\mathcal{P}_\nabla)_2$ to elements $\dot{a}^0 + \dot{b}^1 \in \Gamma(ad\, P) \oplus \Omega^1(X,\mathbb{R}) \cong^{(A,B)} \Gamma(ad\,\mathcal{P})$ and $\dot{b}^0 \in C^\infty(X,\mathfrak{t})$, respectively. The differential forms defining the shifted symplectic structure only depend on these images and so we will abuse notation by writing just $\dot{a}^0 + \dot{b}^1$ and $\dot{b}^0$ in their entries. The identification $\Gamma(ad\, P) \oplus \Omega^1(X,\mathbb{R}) \cong^{(A,B)} \Gamma(ad\,\mathcal{P})$ is done at each point of $\mathfrak{M}_j$ using the corresponding connection $(A,B) \in \mathcal{A}(\mathcal{P}_\nabla)$. We let

$$\omega^0 \in \Omega^2(\mathfrak{M}_0)_{2-n}, \quad \omega^1 \in \Omega^2(\mathfrak{M}_1)_{1-n}, \quad \omega^2 \in \Omega^2(\mathfrak{M}_2)_{-n} \tag{6.54}$$

be given by $\omega_i = (-1)^{n+1} d\lambda_i$ for

$$\begin{aligned}
&\lambda^0(\dot a^1+\dot b^2+\dot\phi^0+...+\dot a^n+\dot b^n+\dot\phi^n)=\\
&\int_M(2\langle a^{n-1}\wedge\dot a^1\rangle+\langle a^{n-2}\wedge\dot a^2\rangle+\langle a^{n-3}\wedge\dot a^3\rangle+...+\langle a^2\wedge\dot a^{n-2}\rangle+(-1)^n\dot b^n)\phi^0\\
&+\int_M(2\langle a^{n-2}\wedge\dot a^1\rangle-\langle a^{n-3}\wedge\dot a^2\rangle+\langle a^{n-4}\wedge\dot a^3\rangle-...+(-1)^n\langle a^2\wedge\dot a^{n-3}\rangle+\dot b^{n-1})\wedge\phi^1\\
&+\int_M(2\langle a^{n-3}\wedge\dot a^1\rangle+\langle a^{n-4}\wedge\dot a^2\rangle+\langle a^{n-5}\wedge\dot a^3\rangle+...+\langle a^2\wedge\dot a^{n-4}\rangle+(-1)^n\dot b^{n-2})\wedge\phi^2\\
&+...+\int_M(2\langle a^2\wedge\dot a^1\rangle+(-1)^n\dot b^3)\wedge\phi^{n-3}+\int_M\dot b^2\wedge\phi^{n-2},
\end{aligned}\tag{6.55}$$

$$\begin{aligned}
&\lambda^1(\dot a^0+\dot b^1+\dot a^1+\dot b^2+\dot\phi^0+...+\dot a^n+\dot b^n+\dot\phi^n)=\\
&\int_M((-1)^n2\langle a^n,\dot a^0\rangle\phi^0-(-1)^n2\langle a^{n-1},\dot a^0\rangle\wedge\phi^1+...+2\langle a^2,\dot a^0\rangle\wedge\phi^{n-2}+\dot b^1\wedge\phi^{n-1}).
\end{aligned}\tag{6.56}$$

$$\lambda^2(\dot b^0+\dot a^1+\dot b^2+\dot\phi^0+...+\dot a^n+\dot b^n+\dot\phi^n)=-\int_M\dot b^0\phi^n,\tag{6.57}$$

where $a_i^p\in\Omega^p(ad\,P)$, $b_i^p\in\Omega^p(M,\mathbb{R})$, $\phi_i^p\in\Omega^p(M,\mathbb{R})$ stand for the parameters of a function on $\mathfrak{M}_j$ and $\dot a_i^p\in\Omega^p(ad\,P)$, $\dot b_i^p\in\Omega^p(M,\mathbb{R})$, $\dot\phi_i^p\in\Omega^p(M,\mathbb{R})$ determine vector fields on $\mathfrak{M}_j$ as discussed in Section 2.3.1. Computations similar to the ones from Section 2.3.3, and in particular (2.165), show that the L_Q-derivative of the first line in (6.55) is

$$\begin{aligned}
&\int_M(d\langle a^{n-2}\wedge\dot a^1\rangle+...+d\langle a^2\wedge\dot a^{n-3}\rangle+(-1)^{n+1}d\dot b^{n-1})\phi^0\\
&+\int_M(\langle d^Aa^{n-2}\wedge\dot a^1\rangle+\langle F_A\wedge\dot a^{n-2}\rangle)\phi^0+\frac{(-1)^n}{2}\sum_{\substack{2\le i\le n-4\\2\le j\le n-2-i}}\int_M\langle a^i\wedge[\dot a^j\wedge a^{n-i-j}]\rangle\phi^0\\
&-\int_M((-1)^{n+1}\langle d^A\dot a^1\wedge a^{n-2}\rangle+\langle F_A\wedge\dot a^{n-2}\rangle)\phi^0+\frac12\sum_{\substack{2\le i\le n-4\\2\le j\le n-2-i}}(-1)^{ij+n}\int_M\langle\dot a^i\wedge[a^j\wedge a^{n-i-j}]\rangle\phi^0\\
&=\int_M d(2\langle a^{n-2}\wedge\dot a^1\rangle+\langle a^{n-3}\wedge\dot a^2\rangle+...+\langle a^2\wedge\dot a^{n-3}\rangle+(-1)^{n+1}\dot b^{n-1})\phi^0.
\end{aligned}\tag{6.58}$$

The terms in the third line of (6.58) arise from applying L_Q to the term $\int_M\dot b^n\phi^0$. Then, similarly, the L_Q-derivative of the second line in (6.55) is

$$\begin{aligned}
&\int_M d(2\langle a^{n-3}\wedge\dot a^1\rangle-\langle a^{n-4}\wedge\dot a^2\rangle+...+(-1)^n\langle a^2\wedge\dot a^{n-4}\rangle+\dot b^{n-2})\wedge\phi^1\\
&+\int_M((-1)^{n+1}(2\langle a^{n-2}\wedge\dot a^1\rangle+\langle a^{n-3}\wedge\dot a^2\rangle+...+\langle a^2\wedge\dot a^{n-3}\rangle)+\dot b^{n-1})\wedge d\phi^0,
\end{aligned}\tag{6.59}$$

and thus iterating and integrating by parts we obtain $L_Q\lambda^0=0$. Then $L_Q\lambda^1=\delta\lambda^0$,

$L_Q\lambda^2 = \delta\lambda^1$ and $\delta\lambda^2 = 0$ follow similarly, using relations (4.102), (4.103) for computing the δ-differentials. This implies that $(\omega^0, \omega^1, \omega^2)$ defines a $(2-n)$-shifted presymplectic structure on $\mathfrak{M}_\bullet$ for $n \geq 3$. When $n = 2$ the formulas for ω^1 and ω^2 still work, but in this case we define ω^0 as follows.

$$\omega^0(\dot{a}_1^1 + \ldots + \dot{\phi}_1^2, \dot{a}_2^1 + \ldots + \dot{\phi}_2^2) = \int_M 2\langle \dot{a}_1^1 \wedge \dot{a}_2^1 \rangle \phi^0 - \int_M (\dot{b}_1^2 \dot{\phi}_2^0 - \dot{b}_2^2 \dot{\phi}_1^0). \tag{6.60}$$

Note that $d\omega^0 = 0$, as it follows from formula (4.110) for the Lie bracket of vector fields on $\mathcal{A}(\mathcal{P}_\nabla)$. To check the non-degeneracy condition for $(\omega^0, \omega^1, \omega^2)$, we note that the tangent complex of $\mathfrak{M}$ is the following chain complex of vector bundles over the space $\{((A, B), \phi) \in \mathcal{A}(\mathcal{P}_\nabla) \times C^\infty(M, \mathbb{R}^*) \mid (F_A, H, d\phi) = (0, 0, 0)\}$.

$$\begin{aligned} C^\infty(M, \mathbb{R}) &\xrightarrow{d} \Omega^0(ad\,P) \oplus \Omega^1(M, \mathbb{R}) \xrightarrow{d^A + d} \Omega^1(ad\,P) \oplus \Omega^2(M, \mathbb{R}) \oplus C^\infty(M, \mathbb{R}) \to \\ &\xrightarrow{d^A + d + d} \Omega^2(ad\,P) \oplus \Omega^3(M, \mathbb{R}) \oplus \Omega^1(M, \mathbb{R}) \xrightarrow{d^A + d + d} \ldots \\ &\ldots \xrightarrow{d^A + d + d} \Omega^{n-2}(ad\,P) \oplus \Omega^{n-1}(M, \mathbb{R}) \oplus \Omega^{n-3}(M, \mathbb{R}) \xrightarrow{d^A + d + d} \\ &\to \Omega^{n-1}(ad\,P) \oplus \Omega^n(M, \mathbb{R}) \oplus \Omega^{n-2}(M, \mathbb{R}) \xrightarrow{d^A + d} \Omega^n(ad\,P) \oplus \Omega^{n-1}(M, \mathbb{R}) \xrightarrow{d} \Omega^n(M, \mathbb{R}), \end{aligned} \tag{6.61}$$

with $\Omega^1(ad\,P) \oplus \Omega^2(M, \mathfrak{t}) \oplus C^\infty(M, \mathbb{R})$ in degree 0 and all other vector bundles graded accordingly. Here we are slightly abusing notation by writing V for the trivial vector bundle $\underline{V}$ with fiber the vector space V. One can check that $(\omega^0, \omega^1, \omega^2)$ induces the pairing

$$\begin{aligned} \tilde{\omega}_\phi(\dot{a}_1^j + \dot{b}_1^{j+1} + \dot{\phi}_1^{j-1}, \dot{a}_2^{n-j} + \dot{b}_2^{n-j+1} + \dot{\phi}_2^{n-j-1}) &= \int_M (2\langle \dot{a}_1^j \wedge \dot{a}_2^{n-j} \rangle \phi \\ &\pm \int_M (\dot{b}_1^{j+1} \wedge \dot{\phi}_2^{n-j-1} \pm \dot{b}_2^{n-j+1} \wedge \dot{\phi}_1^{j-1}), \end{aligned} \tag{6.62}$$

which is non-degenerate, as we wanted to show. □

There is a holomorphic analog of Theorem 6.7. To state the result, we define for any complex manifold X with $\dim_{\mathbb{C}} X = n$ the following complex derived manifold, called the *axio-dilaton moduli*.

$$\Omega^{n,\bullet}_{\overline{\partial}}(X)^* := (\Omega^{n,0}(X, \mathbb{C})^*, \underline{\Omega^{(n,\geq 1)}(X, \mathbb{C})}[-1], Q), \tag{6.63}$$

The notation is such that we regard the trivial vector bundle $\underline{\Omega^{(n,j)}(M, \mathbb{C})}$ in degree $j+1$. Here we write $\Omega^{n,0}(X, \mathbb{C})^*$ for the space of nowhere-vanishing $(n, 0)$-forms on X, and Q is defined by the curvature map $d : \Omega^{n,0}(X,\mathbb{C})^* \to \Omega^{n,1}(X,\mathbb{C})$ and the differential $d : \Omega^{(n,\bullet)}(X,\mathbb{C}) \to \Omega^{(n,\bullet+1)}(X,\mathbb{C})$, both given by the exterior derivative.

Theorem 6.8. *Let $\mathcal{G}_\nabla$ be a holomorphic multiplicative $\mathbb{C}^*$-gerbe with holomorphic connective structure over a complex Lie group G such that the induced bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{C}$ from Theorem 3.43 is non-degenerate, let X be a complex, compact manifold with $\dim_{\mathbb{C}} X = n$ admitting holomorphic volume forms, and let $\mathcal{P}_\nabla \to X$ be a smooth $\mathcal{G}_\nabla$-bundle. Then there is a $(2-n)$-shifted holomorphic symplectic structure on*

$$\mathfrak{X} := \mathcal{H}^d(\mathcal{P}_\nabla) \times \Omega^{n,\bullet}_{\bar\partial}(X)^*, \tag{6.64}$$

where $\mathcal{H}^d(\mathcal{P}_\nabla)$ is as in Example 6.4 and $\Omega^{n,\bullet}_{\bar\partial}(X)^*$ is as in (6.63).

Proof. Analogous to Theorem 6.7. Note that $\mathfrak{X}_j = \mathcal{N} \times BGauge(\mathcal{P}_{\nabla^{0,1}})_j \times \Omega^{n,\bullet}_{\bar\partial}(X)^*$, where

$$\mathcal{N} = (\mathcal{D}(\mathcal{P}_\nabla), \underline{\Omega^{(0,\geq 2)}(ad\,P)} \oplus \underline{\Omega^{(0,\geq 3)}(M,\mathbb{C})}[1], Q)$$

is as in Example 6.4. Recall the holomorphic Maurer-Cartan 1-form on $Gauge(\mathcal{P}_{\nabla^{0,1}})$ from Corollary 5.34 and use it to send vectors tangent to $BGauge(\mathcal{P}_{\nabla^{0,1}})_1$ and to $BGauge(\mathcal{P}_{\nabla^{0,1}})_2$ to elements $\dot a^0 + \dot b^{0,1} \in \Gamma(ad\,P) \oplus \Omega^{0,1}(X,\mathbb{C}) \cong^D \Gamma(ad\,\mathcal{P}'/T^*_{1,0}X)$ and $\dot b^0 \in C^\infty(X,\mathbb{C})$, respectively. The identification $\Gamma(ad\,P) \oplus \Omega^{0,1}(X,\mathbb{C}) \cong^D \Gamma(ad\,\mathcal{P}'/T^*_{1,0}X)$ is done at each point of $\mathfrak{X}_j$ using the corresponding 1-semiconnection $D \in \mathcal{D}(\mathcal{P}_\nabla)$. The $(2-n)$-shifted holomorphic symplectic structure is defined for $n \geq 3$ by

$$\omega^0 \in \Omega^{2,0}(\mathfrak{X}_0)_{2-n}, \quad \omega^1 \in \Omega^{2,0}(\mathfrak{X}_1)_{1-n}, \quad \omega^2 \in \Omega^{2,0}(\mathfrak{X}_2)_{-n}, \tag{6.65}$$

$\omega_i = (-1)^{n+1} d\lambda_i$, where

$$\begin{aligned}
&\lambda^0(\dot a^{0,1} + \dot b^{0,2} + \dot\Omega^{n,0} + \ldots + \dot a^{0,n} + \dot b^{0,n} + \dot\Omega^{n,n}) = \\
&\int_X (2\langle a^{0,n-1} \wedge \dot a^{0,1}\rangle + \langle a^{0,n-2} \wedge \dot a^{0,2}\rangle + \ldots + \langle a^{0,2} \wedge \dot a^{0,n-2}\rangle + (-1)^n \dot b^{0,n})\Omega^{n,0} \\
&+ \int_X (2\langle a^{0,n-2} \wedge \dot a^{0,1}\rangle - \langle a^{0,n-3} \wedge \dot a^{0,2}\rangle + \ldots + (-1)^n \langle a^{0,2} \wedge \dot a^{0,n-3}\rangle + \dot b^{0,n-1}) \wedge \Omega^{n,1} \\
&+ \int_X (2\langle a^{0,n-3} \wedge \dot a^{0,1}\rangle + \langle a^{0,n-4} \wedge \dot a^{0,2}\rangle + \ldots + \langle a^{0,2} \wedge \dot a^{0,n-4}\rangle + (-1)^n \dot b^{0,n-2}) \wedge \Omega^{n,2} \\
&+ \ldots + \int_X (2\langle a^{0,2} \wedge \dot a^{0,1}\rangle + (-1)^n \dot b^{0,3}) \wedge \Omega^{n,n-3} + \int_M \dot b^{0,2} \wedge \Omega^{n,n-2},
\end{aligned} \tag{6.66}$$

$$\lambda^1(\dot{a}^0 + \dot{b}^{0,1} + \dot{a}^{0,1} + \dot{b}^{0,2} + \dot{\Omega}^{n,0} + ... + \dot{a}^{0,n}) + \dot{b}^{0,n} + \dot{\Omega}^{n,n}) =$$
$$\int_X ((-1)^n 2\langle a^{0,n}, \dot{a}^0\rangle \Omega^{n,0} - (-1)^n 2\langle a^{0,n-1}, \dot{a}^0\rangle \wedge \Omega^{n,1} + ... + 2\langle a^{0,2}, \dot{a}^0\rangle \wedge \Omega^{n,n-2})$$
$$+ \int_X \dot{b}^{0,1} \wedge \Omega^{n,n-1}. \tag{6.67}$$

$$\lambda^2(\dot{b}^0 + \dot{a}^{0,1} + \dot{b}^{0,2} + \dot{\Omega}^{n,0} + ... + \dot{a}^{0,n} + \dot{b}^{0,n} + \dot{\Omega}^{n,n}) = -\int_X \dot{b}^0 \Omega^{n,n}, \tag{6.68}$$

where $a^{0,p} \in \Omega^{0,p}(ad\, P)$, $b^p \in \Omega^{0,p}(M, \mathbb{C})$, $\Omega^{n,p} \in \Omega^{n,p}(M, \mathbb{C})$. When $n = 2$, the formula for ω^0 is

$$\omega^0(\dot{a}_1^{0,1} + ... + \dot{\Omega}_1^{2,2}, \dot{a}_2^{0,1} + ... + \dot{\Omega}_2^{2,2}) = \int_M 2\langle \dot{a}_1^{0,1} \wedge \dot{a}_2^{0,1}\rangle \Omega^{2,0} - \int_M (\dot{b}_1^{0,2} \wedge \dot{\Omega}_2^{2,0} - \dot{b}_2^{0,2} \wedge \dot{\Omega}_1^{2,0}). \tag{6.69}$$

The tangent complex of $\mathfrak{X}$ is the following chain complex of vector bundles over the space $\{([(A,B)], \Omega) \in \mathcal{D}(\mathcal{P}_\nabla) \times \Omega^{n,0}(X, \mathbb{C})^* \mid (F_A^{0,2}, H^{0,3}, d\Omega) = (0,0,0)\}$.

$$\begin{aligned} C^\infty(M,\mathbb{C}) &\xrightarrow{\overline{\partial}} \Omega^0(ad\, P) \oplus \Omega^{0,1}(M,\mathbb{C}) \xrightarrow{\overline{\partial}^A + \overline{\partial}} \Omega^{0,1}(ad\, P) \oplus \Omega^{0,2}(M,\mathbb{C}) \oplus C^\infty(M,\mathbb{C}) \to \\ &\xrightarrow{\overline{\partial}^A + \overline{\partial} + \overline{\partial}} \Omega^{0,2}(ad\, P) \oplus \Omega^{0,3}(M,\mathbb{C}) \oplus \Omega^{0,1}(M,\mathbb{C}) \xrightarrow{\overline{\partial}^A + \overline{\partial} + \overline{\partial}} ... \\ &... \xrightarrow{\overline{\partial}^A + \overline{\partial} + \overline{\partial}} \Omega^{0,n-2}(ad\, P) \oplus \Omega^{0,n-1}(M,\mathbb{C}) \oplus \Omega^{0,n-3}(M,\mathbb{C}) \xrightarrow{\overline{\partial}^A + \overline{\partial} + \overline{\partial}} \\ \Omega^{0,n-1}(ad\, P) &\oplus \Omega^{0,n}(M,\mathbb{C}) \oplus \Omega^{0,n-2}(M,\mathbb{C}) \xrightarrow{\overline{\partial}^A + \overline{\partial}} \Omega^{0,n}(ad\, P) \oplus \Omega^{0,n-1}(M,\mathbb{R}) \xrightarrow{\overline{\partial}} \Omega^{0,n}(M,\mathbb{C}), \end{aligned} \tag{6.70}$$

and then non-degeneracy of $(\omega^0, \omega^1, \omega^2)$ follows as in Theorem 6.7. □

Remark 6.9. Let

$$\mathfrak{X}' := \mathcal{H}'^{,d}(\mathcal{P}_\nabla) \times \Omega^{n,\bullet}_{\overline{\partial}}(X)^*, \tag{6.71}$$

where $\mathcal{H}'^{,d}(\mathcal{P}_\nabla)$ is as in Example 6.5 and $\Omega^{n,\bullet}_{\overline{\partial}}(X)^*$ is as in (6.63). There is an obvious map $\mathfrak{X}' \to \mathfrak{X}$, where $\mathfrak{X}$ is as in Theorem 6.8, and the pull-back of the $(2-n)$-shifted holomorphic symplectic form on $\mathfrak{X}$ is a $(2-n)$-shifted holomorphic presymplectic form on $\mathfrak{X}'$. In Section 8.2.2 we comment on some conjectural ideas to make this form non-degenerate by introducing the complex structure on X as a parameter on the moduli space.

6.2.2 Action functionals and moment maps for Lie 2-group actions

We proceed to present alternative constructions for some of the moduli spaces in Section 6.2.1. They are based on general constructions of derived critical loci for Lie 2-group invariant functionals and symplectic reduction for actions of Lie 2-groups. While inspired by Examples 2.32 and 2.34, which we can generalize to the context of Lie 2-groups thanks to Proposition 3.27, these are original constructions that have not appeared before in the literature on algebraic derived geometry.

Proposition 6.10. *Let $\mathfrak{G}$ be a Lie 2-group with a Maurer-Cartan form (θ^0, θ^1) acting by a smooth functor $\rho : M \times \mathfrak{G} \to M$ on a manifold M, and let $S : M \to \mathbb{R}$ be a smooth, $\mathfrak{G}$-invariant function. Then there is a model $dCrit(S)$ for the space $\{dS = 0\}/\mathfrak{G}$ as a simplicial derived manifold with a (-1)-shifted symplectic structure.*

If $(\mathfrak{G}, \theta^0, \theta^1)$ is a complex Lie 2-group with a holomorphic Maurer-Cartan form acting holomorphically on a complex manifold X and $S : X \to \mathbb{C}$ is holomorphic and $\mathfrak{G}$-invariant, then $dCrit(S)$ is (-1)-shifted holomorphic symplectic.

Proof. In the algebraic setting, this follows from [67, 210], as discussed in Example 2.32. We present a model as a simplicial derived manifold. Define first the derived manifold $\mathcal{N} := (M, T^*[-2]M \oplus \underline{\mathfrak{g}}^*[-3] \oplus \underline{\mathfrak{h}}^*[-4], Q)$, where Q is defined by $\Phi : M \to T^*[-2]M$, $\Phi = dS$ and the differential $d : T^*[-2]M \to \underline{\mathfrak{g}}^*[-3]$, $d = \rho^*$, $d : \underline{\mathfrak{g}}^*[-3] \to \underline{\mathfrak{h}}^*[-4]$, $d = t_*^*$. This gives a derived manifold, since $d\Phi = 0$ follows from S being $\mathfrak{G}$-invariant and $t_*\rho_* = 0$ because isomorphic elements in $\mathfrak{G}$ must act in the same way on M.

The action of $\mathfrak{G}$ on M lifts to an action on $\mathcal{N}$, where the action on $\mathfrak{g}^*[-3]$, $\mathfrak{h}^*[-4]$ is the (dual of the) adjoint action defining the Maurer-Cartan form on $\mathfrak{G}$ and the action on $T^*[-2]M$ is given by pull-back. More precisely, $g \in \mathfrak{G}_0$ acts on $\alpha \in T_p^*M$ sending it to $g^*\alpha \in T^*_{\rho(p,g)}M$ defined by $(g^*\alpha)(v) = \alpha(\rho_*(v, g^{-1}))$, where $g^{-1} \in \mathfrak{G}_0$ is any point such that there exists $m \in B\mathfrak{G}_2$ with $d_2(m) = g$, $d_0(m) = g^{-1}$, $d_1(m) = 1$. It is clear that dS is equivariant, while the fact that ρ^* is equivariant follows from Lemma 3.25. Then define $dCrit(S) := \mathcal{N}//\mathfrak{G}$ in the sense of Remark 3.18. Its tangent complex is the following chain complex of vector bundles over M.

$$\underline{\mathfrak{h}}[2] \overset{t_*}{\to} \underline{\mathfrak{g}}[1] \overset{\rho}{\to} TM \overset{dS}{\to} T^*[-1]M \overset{\rho^*}{\to} \underline{\mathfrak{g}}^*[-2] \overset{t_*^*}{\to} \underline{\mathfrak{h}}^*[-3], \tag{6.72}$$

The canonical isomorphism $T(dCrit(S)) \cong T^*(dCrit(S))[-1]$ is induced by a (-1)-shifted symplectic structure on $dCrit(S)$ which we describe as follows. It is given by the canonical symplectic structure on T^*M, seen as a degree -1, d-exact, 2-form $\omega^0 = d\lambda^0 \in \Omega^2(\mathcal{N})_{-1}$, and the symplectic structure induced by the Maurer-Cartan form as in Proposition 3.27, seen now as a pair (ω^1, ω^2) of a degree -2, d-exact, 2-form $\omega^1 = d\lambda^1 \in \Omega^2(\mathcal{N} \times B\mathfrak{G}_1)_{-2}$ and a degree -3, d-exact, 2-form $\omega^2 = d\lambda^1 \in \Omega^2(\mathcal{N} \times B\mathfrak{G}_2)_{-3}$. Then $L_Q\omega^0 = 0$ follows as in Example 2.32, $L_Q\omega^2 = \delta\omega^1 = 0$ and $\delta\omega^2 = 0$ are implied by Proposition 3.27, and $L_Q\omega^1 = \delta\omega^0$ follows from the fact that θ_1^0 is the identity. Hence,

$(\omega^0, \omega^1, \omega^2)$ is indeed a (-1)-shifted symplectic structure on $dCrit(S)$. The holomorphic case follows analogously, using Proposition 5.14. □

Proposition 6.11. *Let $\mathfrak{G}$ be a Lie 2-group with Lie 2-algebra $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$ and Maurer-Cartan form (θ^0, θ^1) acting by a smooth functor $\rho : M \times \mathfrak{G} \to M$ on a symplectic manifold (M, ω). Let $\mu : M \to \mathfrak{g}^*$ be a map such that*

$$d(\mu(\cdot)(v)) = \iota_{X_v}\omega, \qquad v \in \mathfrak{g} \tag{6.73}$$

$$\mu(g \cdot x)(v) = \mu(x)(g \cdot v), \quad g \in \mathfrak{G}_0,\ v \in \mathfrak{g},\ x \in M \tag{6.74}$$

$$\mu(x)(t_* u) = 0, \qquad u \in \mathfrak{h},\ \ x \in M, \tag{6.75}$$

where $X_v(p) := \rho_{*|(p,1)}(0 + v) \in T_pM$. Then there is a model $M//_\mu\mathfrak{G}$ for the space $\mu^{-1}(0)/\mathfrak{G}$ as a simplicial derived manifold with a 0-shifted symplectic structure.

If $(\mathfrak{G}, \theta^0, \theta^1)$ is a complex Lie 2-group with a holomorphic Maurer-Cartan form acting holomorphically on a holomorphic symplectic manifold (X, ω) and $\mu : X \to \mathfrak{g}^*$ is holomorphic, then $M//_\mu\mathfrak{G}$ is 0-shifted holomorphic symplectic.

Proof. Define first the derived manifold $\mathcal{N} = (M, \underline{\mathfrak{h}^*}[-3] \oplus \underline{\mathfrak{g}^*}[-2], Q)$, where Q is defined simply by the curvature $\mu : M \to \underline{\mathfrak{g}^*}$ and the differential $\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*$. Note that (6.75) implies that the differential of the curvature is zero. Then the action of $\mathfrak{G}$ on M, together with (the dual of) the adjoint action defining the Maurer-Cartan form, determines an action of $\mathfrak{G}$ on $\mathcal{N}$ (note that (6.74) and $t : \mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$ being Ad-equivariant ensures that the action preserves Q). Then we define $M//_\mu\mathfrak{G} := \mathcal{N}//\mathfrak{G}$ (cf. Remark 3.18), which is a simplicial derived manifold with $(M//_\mu\mathfrak{G})_n = \mathcal{N} \times B\mathfrak{G}_n$. We define a 0-shifted symplectic structure on $M//_\mu\mathfrak{G}$ as follows. Consider the symplectic form ω on M, seen as a d-closed 2-form $\omega^0 \in \Omega^2(M) \subset \Omega^2(\mathcal{N})_0$, and the 2-forms $\omega^1 \in \Omega^1(B\mathfrak{G}_1 \times \mathfrak{g}^*, \mathbb{R})$, $\omega^2 \in \Omega^1(B\mathfrak{G}_2 \times \mathfrak{h}^*, \mathbb{R})$ defined by $\omega^i = d\lambda^i$, where

$$\lambda^1_{g,\xi}(v_g + \dot{\xi}) = \xi(\theta^0_g(v_g)), \tag{6.76}$$

$$\lambda^2_{\gamma,\eta}(v_\gamma + \dot{\eta}) = \eta(\theta^1_\gamma(v_\gamma)). \tag{6.77}$$

As in Proposition 3.27, ω^1 and ω^2 can be seen as d-closed 2-forms in $(M//_\mu\mathfrak{G})_1$, $(M//_\mu\mathfrak{G})_2$ by linearity on the $\mathfrak{g}^*$, $\mathfrak{h}^*$ components, and they satisfy $\delta\omega^1 = L_Q\omega^2$, $\delta\omega^2 = 0$. Then $L_Q\omega^0 = 0$ follows for degree reasons and $\delta\omega^0 = L_Q\omega^1$ is precisely condition (6.73). Thus $(\omega^0, \omega^1, \omega^2)$ is a 0-shifted presymplectic form on $M//_\mu\mathfrak{G}$. Note that the tangent complex of $M//_\mu\mathfrak{G}$ is the following complex of vector bundles over $\{x \in M \mid \mu(x) = 0\}$.

$$\underline{\mathfrak{h}} \xrightarrow{t_*} \underline{\mathfrak{g}} \xrightarrow{\rho} T_xM \xrightarrow{d\mu_{|x}} \underline{\mathfrak{g}^*} \xrightarrow{t_*^*} \underline{\mathfrak{h}^*}. \tag{6.78}$$

Then non-degeneracy of ω and of the symplectic form in Proposition 3.27 implies non-degeneracy of $(\omega^0, \omega^1, \omega^2)$. The holomorphic case follows analogously, using Proposition 5.14. $\square$

Example 6.12. Let $\mathcal{G}_\nabla$ be a multiplicative $U(1)$-gerbe with connective structure over a Lie group G such that the induced bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ from Theorem 3.43 is non-degenerate, let M be an oriented, compact manifold with $\dim_\mathbb{R} M = 3$ and let $\mathcal{P}_\nabla \to M$ be a $\mathcal{G}_\nabla$-bundle. Consider the infinite-dimensional manifold $\mathcal{M} = \mathcal{A}(\mathcal{P}_\nabla) \times C^\infty(M, \mathbb{R}^*)$ and the function

$$
\begin{aligned}
S : \mathcal{M} &\to \mathbb{R} \\
((A, B), \phi) &\mapsto \int_M H\phi.
\end{aligned}
\tag{6.79}
$$

Its differential is

$$
\begin{aligned}
dS_{((A,B),\phi)}(\dot{a} + \dot{b} + \dot{\phi}) &= \int_M (d\dot{b}\phi - 2\langle F_A \wedge \dot{a} \rangle \phi + H\dot{\phi}) \\
&= \int_M (-\dot{b} \wedge d\phi - 2\langle F_A \wedge \dot{a} \rangle \phi + H\dot{\phi}).
\end{aligned}
\tag{6.80}
$$

Ignoring foundational questions about infinite-dimensional manifolds, we may identify

$$
\begin{aligned}
&\mathcal{M} \times \Omega^2(ad\, P) \oplus \Omega^1(M, \mathbb{R}) \oplus \Omega^3(M, \mathbb{R}) \xrightarrow{\cong} T^*\mathcal{M} \\
&\qquad (((A, B), \phi), \dot{a}, \dot{\phi}, \dot{b}) \mapsto \left(((A, B), \phi), \int_M \langle \dot{a} \wedge \cdot \rangle \phi, \int_M \dot{\phi} \wedge \cdot, \int_M \dot{b} \cdot \right),
\end{aligned}
\tag{6.81}
$$

$$
\begin{aligned}
\mathcal{M} \times \Omega^3(ad\, P) \oplus \Omega^2(M, \mathbb{R}) &\xrightarrow{\cong} \mathcal{M} \times \Omega^0(ad\, P)^* \oplus \Omega^1(M, \mathbb{R})^* \\
(((A, B), \phi), \dot{a}, \dot{\phi}) &\mapsto \left(((A, B), \phi), \int_M \langle \dot{a}, \cdot \rangle \phi, \int_M \dot{\phi} \wedge \cdot \right),
\end{aligned}
\tag{6.82}
$$

$$
\begin{aligned}
\mathcal{M} \times \Omega^3(M, \mathbb{R}) &\xrightarrow{\cong} \mathcal{M} \times C^\infty(X, \mathbb{R})^* \\
(((A, B), \phi), \dot{\phi}) &\mapsto \left(((A, B), \phi), \int_M \dot{\phi} \cdot \right),
\end{aligned}
\tag{6.83}
$$

then we see that $dCrit(S)$ as in Proposition 6.10 coincides with the simplicial derived manifold $\mathfrak{M}$ from Theorem 6.7.

Similarly, when $\mathcal{G}_\nabla$ is a holomorphic multiplicative $\mathbb{C}^*$-gerbe with holomorphic connective structure over a complex Lie group G and X is a complex, compact manifold with $\dim_\mathbb{C} X = 3$ admitting holomorphic volume forms, then the simplicial derived manifold $\mathfrak{X}$ from Theorem 6.8 can be constructed as in Proposition 6.10 from the following

$Gauge(\mathcal{P}_{\nabla^{0,1}})$-invariant holomorphic functional on $\mathcal{D}(\mathcal{P}_\nabla) \times \Omega^{3,0}(X,\mathbb{C})^*$

$$\begin{aligned} \mathcal{D}(\mathcal{P}_\nabla) \times \Omega^{3,0}(X,\mathbb{C})^* &\to \mathbb{C} \\ ([(A,B)],\Omega) &\mapsto \int_X H \wedge \Omega. \end{aligned} \tag{6.84}$$

Note both (6.79) and (6.84) are analogs of the heterotic superpotential [13], adapted to our setting. In Section 8.2.2 we mention a conjecture regarding the moduli space that could be built as the derived critical locus of the full heterotic superpotential from string theory.

Example 6.13. Let $\mathcal{G}_\nabla$ be a multiplicative $U(1)$-gerbe with connective structure over a Lie group G such that the induced bilinear form $\langle \cdot,\cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ from Theorem 3.43 is non-degenerate, let M be an oriented, compact manifold with $\dim_\mathbb{R} M = 2$ and let $\mathcal{P}_\nabla \to M$ be a $\mathcal{G}_\nabla$-bundle. Consider the infinite-dimensional manifold $\mathcal{M} = \mathcal{A}(\mathcal{P}_\nabla) \times C^\infty(M,\mathbb{R}^*)$ and define a symplectic form ω on it by

$$\omega_{((A,B),\phi)}(\dot a_1^1 + \dot b_1^2 + \dot\phi_1^0, \dot a_2^1 + \dot b_2^2 + \dot\phi_2^0) = \int_M 2\langle \dot a_1^1 \wedge \dot a_2^1\rangle \phi - \int_M (\dot b_1^2 \dot\phi_2^0 - \dot b_2^2 \dot\phi_1^0), \tag{6.85}$$

where $\dot a_i^1 \in \Omega^1(ad\, P)$, $\dot b_i^2 \in \Omega^2(M,\mathbb{R})$, $\dot\phi_i^0 \in C^\infty(X,\mathbb{R})$, $i = 1, 2$. This is d-closed, as it follows from formula (4.110) for the Lie bracket on $\mathcal{A}(\mathcal{P}_\nabla)$. The 2-group $Gauge(\mathcal{P}_\nabla)$ acts on $\mathcal{M}$ and the map

$$\begin{aligned} \mu : \mathcal{M} &\to \Gamma(ad\, \mathcal{P}_\nabla)^* \cong \Omega^2(ad\, P) \oplus \Omega^1(X,\mathbb{R}) \\ ((A,B),\phi) &\mapsto F_A + d\phi; \end{aligned} \tag{6.86}$$

that is,

$$\mu(A,B,\phi)(s+\xi) := \int_M (2\langle F_A, s\rangle \phi + \xi \wedge d\phi). \tag{6.87}$$

for $s + \xi \in \Omega^0(ad\, P) \oplus \Omega^1(M,\mathbb{R}) \overset{(A,B)}{\cong} \Gamma(ad\, \mathcal{P}_\nabla)$ satisfies the conditions of Proposition 6.11. It is easy to check that $\mathcal{M}/\!/_\mu \mathfrak{G} = \mathfrak{M}$, for $\mathfrak{M}$ the simplicial derived manifold from Theorem 6.7.

Similarly, when $\mathcal{G}_\nabla$ is a holomorphic multiplicative $\mathbb{C}^*$-gerbe with holomorphic connective structure over a complex Lie group G and X is a complex, compact manifold with $\dim_\mathbb{C} X = 2$ admitting holomorphic volume forms, then the simplicial derived complex manifold $\mathfrak{X}$ from Theorem 6.8 can be constructed as in Proposition 6.11 from the infinite-dimensional complex manifold $\mathcal{M} = \mathcal{D}(\mathcal{P}_\nabla) \times \Omega^{2,0}(X,\mathbb{C})^*$ with holomorphic symplectic form

$$\begin{aligned}\omega_{([(A,B)],\Omega)}(\dot{a}_1^{0,1} + \dot{b}_1^{0,2} + \dot{\Omega}_1^{2,0}, \dot{a}_2^{0,1} + \dot{b}_2^{0,2} + \dot{\Omega}_2^{2,0}) \\ = \int_X 2\langle \dot{a}_1^{0,1} \wedge \dot{a}_2^{0,1}\rangle \Omega - \int_M (\dot{b}_1^{0,2} \wedge \dot{\Omega}_2^{2,0} - \dot{b}_2^{0,2} \wedge \dot{\Omega}_1^{2,0}),\end{aligned} \tag{6.88}$$

on which $Gauge(\mathcal{P}_{\nabla^{0,1}})$ acts holomorphically with holomorphic moment map

$$\begin{aligned}\mu : \mathcal{M} &\to \Gamma(ad\,\mathcal{P}_\nabla/(T^*X \otimes \mathfrak{t})^{1,0})^* \cong \Omega^{0,2}(ad\,P) \oplus \Omega^{2,1}(X,\mathbb{C}) \\ ([(A,B)],\Omega) &\mapsto F_A^{0,2} + d\Omega.\end{aligned} \tag{6.89}$$

6.2.3 Pre-Kähler and universal geometry of the Hull-Strominger system

Let K be a compact Lie group and let $\mathcal{K}$ be a multiplicative $U(1)$-gerbe over K whose induced pairing $\langle\cdot,\cdot\rangle : \mathfrak{k} \otimes \mathfrak{k} \to \mathbb{R}$ by Corollary 3.45 is non-degenerate. Let G, $\mathcal{G}_\nabla$ be the complexifications of K, $\mathcal{K}$, respectively, as in Theorem 5.8. Let $\mathcal{P}_{h,\nabla_h} \to X$ be a $\mathcal{K}_\nabla$-bundle over a complex manifold with $\dim_{\mathbb{C}} X = n$ and recall from Remark 4.8 that enhanced connections on $\mathcal{P}_{h,\nabla_h}$ can be identified with pairs $((A_h, B_h), g)$ with $(A_h, B_h) \in \mathcal{A}(\mathcal{P}_{h,\nabla_h})$ and $g \in \Gamma(S^2T^*X)$.

We write $\mathcal{A}^{en}_+(\mathcal{P}_{h,\nabla_h})$ for the open subset of $\mathcal{A}^{en}(\mathcal{P}_{h,\nabla_h})$ consisting of $((A_h, B_h), g)$ with $g^{1,1}$ positive definite.

Definition 6.14. Let $\mathcal{P}_{h,\nabla_h} \to X$ be a $\mathcal{K}$-bundle over a compact, complex manifold X of $\dim_{\mathbb{C}} X = n$ with a holomorphic volume form $\Omega \in \Omega^{n,0}(X,\mathbb{C})$. A solution to the *Hull-Strominger system* is $((A_h, B_h), g) \in \mathcal{A}^{en}_+(\mathcal{P}_{h,\nabla_h})$ such that

$$g^{0,2} = 0, \quad F_{A_h}^{0,2} = 0, \quad H_h = d^c\omega, \tag{6.90}$$

$$F_{A_h} \wedge \omega^{n-1} = 0, \quad d(e^{-f}\omega^{n-1}) = 0, \tag{6.91}$$

where $\omega := g(I\cdot,\cdot)$ for I the complex structure on X and $f : X \to \mathbb{R}$ is defined by

$$\frac{\omega^n}{n!} = e^{2f}(-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \overline{\Omega}. \tag{6.92}$$

Note that Theorem 5.26 implies that equations (6.90) state precisely that $((A_h, B_h), g)$ induces a holomorphic structure with holomorphic connective structure on $\mathcal{P}_\nabla := \mathcal{P}^{\mathbb{C}}_{h,\nabla_h}$. These are called *F-term* equations. The remaining equations (6.91), called *D-term* equations, are interpreted as moment map conditions in [127]. We recall (a slightly adapted version of) their construction.

Let $\mathcal{M} = \mathcal{A}_+^{en}(\mathcal{P}_{h,\nabla_h})$, considered as an infinite-dimensional complex manifold with complex structure given by identifying

$$\begin{aligned} T_{((A_h,B_h),g)}\mathcal{M} &= \Omega^1(ad\, P_h) \oplus \Gamma(T^*_{\mathbb{R}}X \otimes_{\mathbb{R}} T^*_{\mathbb{R}}X) \\ &= \Omega^{0,1}(ad\, P_h^{\mathbb{C}}) \oplus \Gamma(T^*_{0,1}X \otimes_{\mathbb{C}} T^*_{\mathbb{C}}X) \\ &= \Omega^{0,1}(ad\, \mathcal{P}'_{\nabla}), \end{aligned} \tag{6.93}$$

where $ad\, \mathcal{P}'_{\nabla} \subset E'$ is the kernel of the anchor of the complex Courant algebroid associated to $\mathcal{P}_{\nabla}$. In other words,

$$I^{\mathcal{M}}_{((A_h,B_h),g)}(\dot{a} + \dot{b} + \dot{g}) = (i\dot{a}^{0,1} - i\dot{a}^{1,0}, \dot{g}^{1,1}(I\cdot,\cdot), i\dot{b}^{1,1}(I\cdot,\cdot) + i\dot{g}^{0,2} - i\dot{g}^{2,0}). \tag{6.94}$$

We define a presymplectic structure $\omega^{\mathcal{M}}$ of type $(1,1)$ on $\mathcal{M}$ as follows. Consider the 1-form $\lambda \in \Omega^1(\mathcal{M}, \mathbb{R})$ given by

$$\lambda_{((A_h,B_h),g)}(\dot{a} + \dot{b} + \dot{g}) = \frac{1}{2}\int_X -\dot{b} \wedge e^{-f}\frac{\omega^{n-1}}{(n-1)!} \tag{6.95}$$

where $\omega = g(I\cdot,\cdot)$ and f is defined by (6.92). This can also be written as $\lambda = d^c M$ for $M : \mathcal{M} \to \mathbb{R}$ the *dilaton functional*

$$M((A_h,B_h),g) = \int_X e^{-f}\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} i^n \int_X e^f \Omega \wedge \overline{\Omega}; \tag{6.96}$$

in particular, it is clear that $\omega^{\mathcal{M}} = d\lambda = dd^c M$ is a presymplectic form of type $(1,1)$. Now $Gauge(\mathcal{P}_{h,\nabla_h})$ acts on $\mathcal{M}$ through its action on $\mathcal{A}(\mathcal{P}_{\nabla})$, and this action preserves λ by (4.88). This implies that it admits a moment map in the sense of Proposition 6.11, which is given by

$$\begin{aligned} \mu : \mathcal{M} &\to \Gamma(ad\, \mathcal{P}_{h,\nabla_h})^*, \\ ((A_h,B_h),g) &\mapsto \mu((A_h,B_h),g) \end{aligned} \tag{6.97}$$

defined over $s + \xi \in \Gamma(ad\, \mathcal{P}) \cong_{(A_h,B_h)} \Gamma(ad\, \mathcal{P}_{h,\nabla_h})$ as

$$\mu((A_h,B_h),g)(s+\xi) = \frac{1}{2}\int_X (d\xi + 2\langle F_{A_h}, s\rangle) \wedge e^{-f}\frac{\omega^{n-1}}{(n-1)!}. \tag{6.98}$$

Equations (6.91) are equivalent to $\mu = 0$. In particular, Proposition 6.11 implies that $\mathcal{M}/\!/_{\mu}\, Gauge(\mathcal{P}_{h,\nabla_h})$ has a natural 0-shifted presymplectic structure. In fact, if we define $\mathcal{M}^0 \subset \mathcal{M}$ to be the subspace of points satisfying the F-term equations (6.90), then (ignoring possible smoothness problems) one can check that $\mathcal{M}^0$ is a complex submanifold invariant by $Gauge(\mathcal{P}_{\nabla})$ and so one can also obtain a presymplectic structure on the moduli space of solutions to the Hull-Strominger system $\mathcal{M}^0/\!/_{\mu}\, Gauge(\mathcal{P}_{h,\nabla_h})$.

We conclude with some speculative comments on this construction based on Remark 2.37 and previous work in mathematical physics [72, 73]. First we note that there is an analog of formula (2.170) in this setting. Namely, consider the $\mathcal{K}_\nabla$-bundle $\mathcal{M}\times\mathcal{P}_{h,\nabla_h}\to\mathcal{M}\times X$, which carries a canonical connection $(\mathbb{A}_h,\mathbb{B}_h)$. If $\{X_a\}_a$ is a cover of X on which $\mathcal{P}_{h,\nabla_h}$ has cocycle data as in Proposition 4.4, then $\mathcal{M}\times\mathcal{P}_{h,\nabla_h}$ is described by the same cocycle data over the cover $\{\mathcal{M}\times X_a\}_a$ and we can define $(\mathbb{A}_h,\mathbb{B}_h)$ by the local forms

$$\mathbb{A}^a_{h|(((A_h,B_h),g),x)}(\dot a+\dot b+\dot g+v_x)=A^a_{h|x}(v_x), \tag{6.99}$$
$$\mathbb{B}^a_{h|(((A_h,B_h),g),x)}(\dot a+\dot b+\dot g+v_x)=B^a_{h|x}(v_x), \tag{6.100}$$

where $(A^a_h,\,B^a_h)$ are the local forms defining (A_h,B_h). The curvature of $(\mathbb{A}_h,\mathbb{B}_h)$ is $(F_{\mathbb{A}_h},\mathbb{H}_h)$ defined by

$$\begin{aligned}&F_{\mathbb{A}_{h|(((A_h,B_h),g),x)}}(\dot a_1+\dot b_1+\dot g_1+v^1_x,\dot a_2+\dot b_2+\dot g_2+v^2_x)\\&\qquad=F_{A_h}(v^1_x,v^2_x)+\dot a_1(v^2_x)-\dot a_2(v^1_x),\end{aligned} \tag{6.101}$$
$$\begin{aligned}&\mathbb{H}_{h_{|(((A_h,B_h),g),x)}}(\dot a_1+\dot b_1+\dot g_1+v^1_x,\dot a_2+\dot b_2+\dot g_2+v^2_x,\dot a_3+\dot b_3+\dot g_3+v^3_x)\\&\qquad=H_h(v^1_x,v^2_x,v^3_x)+\dot b_1(v^2_x,v^3_x)-\dot b_2(v^1_x,v^3_x)+\dot b_3(v^1_x,v^2_x)\end{aligned} \tag{6.102}$$

We also define $\mathbb{w}\in\Omega^{1,1}(\mathcal{M}\times X,\mathbb{R})$ and $\mathbb{f}:\mathcal{M}\times X\to\mathbb{R}$ by

$$\mathbb{w}_{|(((A_h,B_h),g),x)}(\dot a_1+\ldots+v^1_x,\dot a_2+\ldots+v^2_x)=\omega(v^1_x,v^2_x), \tag{6.103}$$
$$\mathbb{f}(((A_h,B_h),g),x)=f(x). \tag{6.104}$$

Then it is straightforward to check that the 1-form λ from (6.95) can be written as

$$\lambda=\frac{1}{2}\int_X\mathbb{H}_h\wedge e^{-\mathbb{f}}\frac{\mathbb{w}^{n-1}}{(n-1)!}. \tag{6.105}$$

This shows that the presymplectic form $\omega^{\mathcal{M}}=d\lambda$ on $\mathcal{M}$ is very natural: while the term $\langle F_{\mathbb{A}}\wedge F_{\mathbb{A}}\rangle\in\Omega^4(\mathcal{M}\times X,\mathbb{R})$ appearing in Donaldson's formula (2.170) is the pull-back of the 2-shifted symplectic form on $BK_\bullet$ by the map $\mathcal{M}\times X\to BK$ given by the universal K-bundle (as it follows from Lemma 4.6), the term $d(\mathbb{H}_h\wedge e^{-\mathbb{f}})$ appearing in $\omega^{\mathcal{M}}$ is the pull-back of the 2-shifted symplectic form on $B\mathcal{K}_\bullet\times\mathbb{R}^*$ (cf. Proposition 3.52) by the map $\mathcal{M}\times X\to B\mathcal{K}\times\mathbb{R}^*$ given by the universal $\mathcal{K}$-bundle and the functional $\mathbb{f}$.

Secondly, we comment on the relation between the presymplectic form $\omega^{\mathcal{M}}$ defined here and the $(2-n)$-shifted holomorphic symplectic form from Remark 6.9. As in the case of ordinary gauge theory (cf. Remark 2.37), we can make sense of this relation when $n=2$. In this case, note that a point $((A_h,B_h),g)\in\mu^{-1}(0)\subset\mathcal{M}$ determines by the Calabi-Yau theorem complex structures J, K such that $(I,e^{-f(\omega)}\omega)$, $(J,e^{-f(\omega)}\omega_J)$, $(K,e^{-f(\omega)}\omega_K)$ is a hyperkähler structure on X with $\Omega = e^{-f(\omega)}\omega_J + ie^{-f(\omega)}\omega_K$, where $\omega_J := g(J\cdot,\cdot)$, $\omega_K := g(K\cdot,\cdot)$.

This means that we can also define presymplectic forms $\omega_J^{\mathcal{M}}$ and $\omega_K^{\mathcal{M}}$ on $\mu^{-1}(0) \subset \mathcal{M}$ by a similar formula to (6.105). Each of these has again a moment map for the action of $Gauge(\mathcal{P}_{h,\nabla_h})$ and so they descend to the quotient $\mathcal{M}^0//_{\mu}Gauge(\mathcal{P}_{\nabla})$. Now there is a map $\psi : \mathcal{M}^0//_{\mu}Gauge(\mathcal{P}_{\nabla}) \to \mathcal{H}^{\prime,d}(\mathcal{P}_{\nabla}) \times \Omega^{n,\bullet}_{\overline{\partial}}(X)^*$ and the pull-back of the 0-shifted holomorphic presymplectic form from Remark 6.9 coincides with the reduction of $\omega_J^{\mathcal{M}} + i\omega_K^{\mathcal{M}}$. Note that other interpretations of the Hull-Strominger equations in terms of hyper-Kähler moment maps have been studied in [122].

The conclusion of this observation is the following. If ψ was proven to be a diffeomorphism between appropriate open smooth locus of both spaces, then such moduli space would be hyper-(pre-pseudo)-Kähler. One can probably get rid of the degeneracy of such structure by considering larger moduli spaces in which the complex structure on the base manifold X is also included as a parameter, as we discuss in more detail in Section 8.2.2.

Chapter 7

Higher derived differential geometry

In Chapter 6 we have constructed moduli spaces parameterizing geometric structures in higher gauge theory, and we have modelled them as simplicial derived manifolds using the formalism of Chapter 2. While there is a well-defined category of simplicial derived manifolds, the fundamental nature of these moduli spaces is more properly captured by regarding them as objects in the $(\infty,1)$-category of derived differentiable ∞-stacks, as explained in Sections 1.2 and 1.3. We expect our results to have applications in the construction of invariants with good functorial and combinatorial properties, but such applications will necessary require understanding the formalism of ∞-categories.

Informally, an (∞,∞)-*category* is

- a collection of *objects* or *0-cells*, for which we use letters x, y, z, ...,
- a collection of *arrows* or *1-cells* between objects $f : x \to y$,
- a collection of 2-*cells* between arrows $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$,
- a collection of 3-*cells* between 2-cells $\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd} \overset{\psi}{\Rrightarrow} \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\beta"] & y \end{tikzcd}$
- ...

in which all cells can be composed in many different ways. For $r \in \mathbb{N}$, an (∞, r)-*category* is the same data, but in which all m-cells for $m \geq r+1$ are invertible. For $n \in \mathbb{N}$ and

$r \leq n$, an (n,r)-*category* is the same data, but in which all m-cells for $m \geq r+1$ are invertible and all m-cells for $m \geq n+1$ are identities.

The study of higher categories has at least two different origins. The first one is the observation that sets form a category, while categories form a bicategory or $(2,2)$-category [35, 106], and bicategories form a tricategory or $(3,3)$-category [135], which suggests the natural problem of providing a rigorous axiomatization of all these structures. The second one arises from noting that many results about derived categories are proven by chasing morphisms and homotopies between them in the original abelian category, yielding the question of finding an algebraic structure that is well-suited for dealing with these homotopies. Quillen's answer [214] is to consider *simplicial model categories*, which provide examples of what we now call $(\infty,1)$-categories. These ideas motivated Grothendieck to envision models for ∞-categories and ∞-stacks in his manuscript *Pursuing Stacks* [138], which already included many of the fundamental ideas of ∞-category theory. This has now become a well-developed theory thanks to the work of many authors, such as [17, 26, 29, 104, 157, 181, 218, 255] and others.

We will focus our attention on $(\infty,1)$-categories and $(\infty,0)$-categories. An example of $(\infty,1)$-category is the $(\infty,1)$-category with objects topological spaces, arrows continuous maps, 2-cells homotopies, 3-cells homotopies between homotopies, etc. [214]. An example of an $(\infty,0)$-category is the *fundamental ∞-groupoid* $\Pi_\infty(X)$ [138, 203] of a topological space X: an object in $\Pi_\infty(X)$ is a point in X, an arrow $x \to y$ is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, a 2-cell is a homotopy of paths inside X, etc. Note that the composition of two paths $x \xrightarrow{\gamma} y \xrightarrow{\eta} z$ is only well-defined up to homotopy, in the sense that any two of the following paths could equally well be considered a composite of γ and η.

$$(\eta \circ \gamma)(t) := \begin{cases} \gamma(2t) & 0 \leq t \leq 1/2 \\ \eta(2t-1) & 1/2 \leq t \leq 1 \end{cases}, \quad (\eta \tilde{\circ} \gamma)(t) := \begin{cases} \gamma(3t) & 0 \leq t \leq 1/3 \\ \eta((3t-1)/2) & 1/3 \leq t \leq 1 \end{cases}. \tag{7.1}$$

This highlights a feature of $(\infty,1)$-categories that one should try to model: given k-cells in an $(\infty,1)$-category with adequate source and target, composites must exist but they may be non-unique; instead, any two composite k-cells must be related by a $(k+1)$-cell. In particular, note that this implies that composition of n-cells in an $(n,1)$-category is uniquely well-defined.

Most models for (∞,∞)-categories can be classified as either *algebraic* or *geometric*. Algebraic models axiomatize all the ways in which composites of cells can be assigned to composable cells in an $(n,1)$-category, and the conditions that these compositions must satisfy. For $n \leq 4$, this is covered by the theory of [135, 144, 266], but for general

$n \in \mathbb{N} \cup \{\infty\}$ these sort of definitions require too much combinatorial effort and are replaced by operadic approaches [29]. Geometric models axiomatize only the existence of composites of composable cells, without prescribing preferred composites. Most models are based on some category of 'prescribed shapes' for the k-cells such as the globe [254], the simplex [26, 157, 218, 255] or the opetope [17] categories. Comparisons and reviews of all these models can be found in [27, 83, 176, 219, 248].

The idea of seeing moduli spaces as objects in a higher category can be traced back to [10, 97, 197], where the notion of *algebraic stack* was introduced based on ideas from [9, 133] to deal with some aspects of the moduli space of curves of fixed genus over an algebraically closed field. Algebraic stacks model spaces with an internal notion of symmetry. It was suggested in [138] that a notion of higher algebraic stack could be useful for studying moduli problems in which the symmetries have themselves some notion of 'homotopy' between them, but it was not until [248] that a rigorous notion of higher algebraic stack was presented.

A parallel story is the development of derived geometry for the study of moduli spaces. Derived geometry started with the observation that the deformation theory of schemes can be studied by resolving them with smooth *dg*-algebras [7, 139, 155, 216], setting the foundations for what are now known as derived schemes. These model spaces with an internal notion of smooth deformation, which makes them suitable for dealing with iterated fibered products of schemes. The theory of derived algebraic geometry was developed and combined with the theory of higher algebraic stacks in [182, 183, 262, 263], observing in particular that many moduli spaces of interest in algebraic geometry have a natural structure of higher derived algebraic stack.

The use of higher derived geometry for the study of moduli spaces has also been immensely influenced by the literature on mathematical physics. The BV-BRST [28, 30] approach for quantizing gauge-theories is based on the idea of adding extra coordinates (*ghosts* and *antifields*) in the configuration space of a field theory. These extra coordinates were interpreted in [240, 241] as the variables of non-zero degree on a *differential graded supermanifold*, as defined in [37, 171], which is an infinitesimal approximation to the kind of geometric object that is studied in higher derived geometry.

In the differential geometric context, the full theory is less developed. Regarding higher differential geometry (i.e., the extension of differential geometry in which quotients can always be taken but intersections can still be problematic), differentiable ∞-stacks can be defined by a straightforward generalization of the algebraic setting from [247], as done for example in [239]. A more explicit approach is proposed in [282], defining Lie ∞-groupoids as simplicial manifolds satisfying certain conditions, and the results of [213] imply that both approaches are equivalent. In particular, all the simplicial manifolds that appear in this thesis aim to model objects in higher differential geometry and should therefore be thought of as objects in the $(\infty, 1)$-category of differentiable stacks.

On the other hand, the $(\infty, 1)$-category $dMan_\infty$ of derived manifolds is an extension of the category of manifolds in which fibered products always exist and satisfy good combinatorial properties, but in which quotients can still be problematic. The first attempt at constructing $dMan_\infty$ traces back to [250]. More recent approaches [46, 158, 252] mimic the definitions in the algebraic context, replacing commutative rings by C^∞-rings in order to capture the subtleties of the smooth setting. As proven in [76], all these approaches yield the same $(\infty, 1)$-category $dMan_\infty$. Based on work from [32], an alternative construction of $dMan_\infty$ is presented in [75] using the theory from Section 2.2.2, as it had originally been proposed by Kontsevich [170]. In particular, all the derived manifolds in this thesis aim to model objects in derived differential geometry and should therefore by thought of as objects of the $(\infty, 1)$-category $dMan_\infty$.

Given a model for $dMan_\infty$, constructing the ∞-category of derived differentiable stacks is immediate by mimicking the definitions from algebraic geometry [261]. Moreover, the results in [213] imply that simplicial objects in the category $dMan$ from Section 2.2.2 satisfying certain horn-filling conditions provide examples of derived differentiable stacks, and all the simplicial derived manifolds in this thesis should be regarded as such. However, at the time of writing of this thesis it is still an open problem to give an explicit model, along the lines of the Lie ∞-groupoids from [282], that suffices to construct all derived differentiable stacks [89].

In Section 7.1.1 we present the first definitions and examples of ∞-categories. In Section 7.1.2 we discuss the method of localization of categories of fibrant objects to construct ∞-categories, and in Section 7.1.3 we present the basics of ∞-sheaf theory. In Section 7.2.1 we define the $(\infty, 1)$-category of differentiable stacks, and we discuss how to present its objects by Lie ∞-groupoids. In Section 7.2.2 we define and present examples of Lie m-groups for high $m \in \mathbb{N}$, relating them to the theory from Chapters 3 and 4. In 7.2.3 we construct $dMan_\infty$ in terms of the category $dMan$ from Section 2.2.2, and we combine this theory with the one from Section 7.1.3 to define the $(\infty, 1)$-category ∞dManSt of derived differentiable ∞-stacks. The content of this chapter is mostly adapted from [32, 75, 181, 204, 213, 261, 282] and there is no claim of originality, except for Example 7.31 and some aspects of the presentation of Example 7.30.

7.1 ∞-categories

7.1.1 Kan conditions

Recall the categories Δ and Δ_{inj} from Definition 2.2.

Definition 7.1 ([106]). Let C be a category. A *simplicial object in* C is a functor $X : \Delta^{op} \to C$. A *semi-simplicial object in* C is a functor $X : \Delta^{op}_{inj} \to C$. A *morphism of (semi-)simplicial objects* $X \to Y$ is a natural transformation of functors $X \to Y$. If C is a monoidal category, the *Cartesian product* of two (semi-)simplicial objects X, Y is the (semi-)simplicial object $X \times Y$ defined by point-wise product in C. We write C_Δ for the category of simplicial objects in C and $C_{\Delta_{inj}}$ for the category of semi-simplicial objects in C.

A simplicial object X in C can equivalently be described by a sequence of objects X_n in C, $n \in \mathbb{N}$, with face arrows $d^n_j : X_n \to X_{n-1}$ and degeneracy arrows $s^n_j : X_n \to X_{n+1}$, $j = 0, ..., n$ satisfying the simplicial identities (2.3). Similarly, a semi-simplicial object in C is described by objects X_n and face maps $d^n_j : X_n \to X_{n-1}$ satisfying the first equation in (2.3). We will often use the notation $X_\bullet$ when we present a (semi-)simplicial object in this way, and we will omit the superscript in the face and degeneracy maps when it is clear from context. When C is the category of sets, groups, manifolds, etc. we refer to a simplicial object in X as a *simplicial set*, *simplicial group*, *simplicial manifold*, etc.

Example 7.2. For each $n \in \mathbb{N}$ we define the *combinatorial n-simplex* Δ^n as the simplicial set associated to the object $[n] \in \Delta$ by the Yoneda embedding. That is, $\Delta^n : \Delta \to \mathrm{Set}$ is defined as $\Delta^n([m]) = \Delta([m], [n])$ on objects, and it sends an arrow $f : [m_1] \to [m_2]$ to the corresponding pull-back map $f^* : \Delta([m_2], [n]) \to \Delta([m_1], [n])$. In particular, note that fully faithfulness of the Yoneda embedding implies that for any simplicial set $X_\bullet$ the corresponding set X_n can also be written as $X_n = \mathrm{Set}_\Delta(\Delta^n, X_\bullet)$.

Example 7.3. For $n \in \mathbb{N}^{\geq 1}$ and $j = 0, ..., n$, the (n, j) *horn* is the simplicial set Λ^n_j obtained from Δ^n by 'removing the jth face'. More explicitly, this means that

$$\Lambda^n_j([m]) := \{f \in \Delta([m], [n]) \mid f([m]) \cup \{j\} \neq [n]\}. \tag{7.2}$$

In particular, there is a canonical inclusion morphism $\Lambda^n_j \to \Delta^n$. The *inner horns* of Δ^n are the horns Λ^n_j with $j = 1, ..., n-1$ and the *outer horns* of Δ^n are the horns Λ^n_j with $j = 0, n$.

Example 7.4. If A is a small category, then we can build a simplicial set $N(A)_\bullet$, called the *nerve* of A, as follows. Let $N(A)_0$ be the set of objects of A, $N(A)_1$ be the set of arrows of A and for $n \geq 2$ let

$$N(A)_n := N(A)_1 \, {}_s{\times}_t N(A)_1 \, {}_s{\times}_t \cdots \, {}_s{\times}_t N(A)_1; \tag{7.3}$$

i.e., an element of $N(A)_n$ is a string $(g_1, ..., g_n)$ of n composable arrows of A. The face and degeneracy maps are then defined as

$$\begin{aligned}
d_0(g) &= s(g), \\
d_1(g) &= t(g), \\
d_0(g_1, ..., g_n) &= (g_2, ..., g_n), \\
d_j(g_1, ..., g_n) &= (g_1, ..., g_{j-1}, g_j \circ g_{j+1}, g_{j+2}, ..., g_n), \quad j = 1, \, ..., \, n-1, \\
d_n(g_1, ..., g_n) &= (g_1, ..., g_{n-1}), \\
s_0(x) &= id_x, \\
s_j(g_1, ..., g_n) &= (g_1, ..., g_j, id_{s(g_j)}, g_{j+1}, ..., g_n), \qquad j = 0, \, ..., \, n.
\end{aligned} \tag{7.4}$$

The usual axioms of a category, such as associativity of composition, imply the simplicial identities.

It is not hard to prove that, for X a simplicial set, there exists a small category A and an isomorphism $X \cong N(A)_\bullet$ if and only if the following condition is satisfied.

Condition 1. For $n \geq 2$ and $1 \leq j \leq n-1$, the map

$$\mathrm{Set}_\Delta(\Delta^n, X) \to \mathrm{Set}_\Delta(\Lambda^n_j, X) \tag{7.5}$$

induced by the inclusion $\Lambda^n_j \to \Delta^n$ is a bijection.

In this case, the category A is completely determined by X and so one can *define* small categories as simplicial sets satisfying Condition 1. Note also that A is a groupoid (i.e., every arrow is invertible) if and only if the map (7.5) is also a bijection for $j = 0$, n and $n \geq 2$

The previous observation motivates the following approach to define ∞-categories. Given a (small) $(\infty, 1)$-category A in the informal description from the introduction to Chapter 7, we should be able to define a simplicial set $N(A)_\bullet$ such that

- $N(A)_0$ is the set of objects of A,
- $N(A)_1$ is the set of arrows of A,
- $N(A)_2$ is the set of quadruples $(f_1, f_2, f_{12}, \alpha)$ such that f_1, f_2 are composable arrows in C and $\alpha : f_1 \circ f_2 \Rightarrow f_{12}$ is a 2-cell in A,

- $N(C)_3$ is the set of tuples

$$(f_1, f_2, f_3, f_{12}, f_{23}, f_{123}, \alpha_{1,2}, \alpha_{12,3}, \alpha_{2,3}, \alpha_{1,23}, \psi)$$

  such that f_1, f_2, f_3 are composable arrows in A, $\alpha_{1,2} : f_1 \circ f_2 \Rightarrow f_{12}$, $\alpha_{12,3} : f_{12} \circ f_3 \Rightarrow f_{123}$, $\alpha_{2,3} : f_2 \circ f_3 \Rightarrow f_{23}$, $\alpha_{1,23} : f_1 \circ f_{23} \Rightarrow f_{123}$ are 2-cells in A and $\psi : \alpha_{12,3} \circ \alpha_{1,2} \Rightarrow \alpha_{1,23} \circ \alpha_{2,3}$ is a 3-cell in A,

- ...

Since $N(A)_\bullet$ contains all the information about compositions of cells in A and the properties that these satisfy, anything that we want to do with A should be possible to be done with $N(A)_\bullet$, as it is the case with standard categories. Thus, instead of trying to define small $(\infty, 1)$-categories by axiomatizing all the ways in which cells can be composed, we may define $(\infty, 1)$-categories as simplicial sets satisfying the analog of Condition 1 that we would expect from the nerve of an 'informal' $(\infty, 1)$-category. This idea is due to [157], based on [44]. From now on we will ignore size issues, which can always be dealt with by working in appropriate universes.

Definition 7.5 ([44, 157, 181])**.** An $(\infty, 1)$-*category* A is a simplicial set $A_\bullet$ such that, for $n \geq 2$ and $1 \leq j \leq n-1$, the map

$$\mathrm{Set}_\Delta(\Delta^n, A) \to \mathrm{Set}_\Delta(\Lambda^n_j, A) \tag{7.6}$$

induced by the inclusion $\Lambda^n_j \to \Delta^n$ is a surjection. An ∞-*groupoid*, $(\infty, 0)$-*category* or *Kan complex* is an $(\infty, 1)$-category such that the maps (7.6) are also surjections for $j = 0, n$ and $n \geq 2$. For $m \in \mathbb{N}$, an $(m, 1)$-*category* is an $(\infty, 1)$-category such that the maps (7.6) are bijections for $n > m$, $1 \leq j \leq n-1$, and an m-*groupoid* is an is an $(\infty, 1)$-category such that the maps (7.6) are bijections for $n > m$, $0 \leq j \leq n$.

The motivation for Definition 7.5 is that, as it is clear from the informal description of the nerve of an $(\infty, 1)$-category, imposing surjectivity of (7.6) for $n = 2$ and $j = 1$ amounts to imposing existence of composites of 1-arrows, while imposing surjectivity for $n > 2$ and $1 \leq j \leq n-1$ amounts to imposing existence of composites and inverses of $(n-1)$-cells, and imposing that (7.6) is a bijection amounts to imposing uniqueness of such composites.

Example 7.6. For X a topological space, we define its *fundamental* ∞-*groupoid* $\Pi_\infty(X)$ as follows. First, recall that there is a functor $|\cdot| : \Delta \to \mathrm{Top}$, called the *standard cosimplicial topological space of topological simplices*, where Top is the category of topological spaces and continuous maps, such that

$$\begin{aligned} |\Delta^n| &:= \{(x_0, ..., x_n) \in [0,1]^{n+1} \mid \textstyle\sum_i x_i = 1\}, \\ |\delta_j|(x_0, ..., x_{n-1}) &= (x_0, ..., x_{j-1}, 0, x_j, ..., x_{n-1}), \\ |\sigma_j|(x_0, ..., x_{n+1}) &= (x_0, ..., x_{j-1}, x_j + x_{j+1}, x_{j+2}, ..., x_{n+1}). \end{aligned} \tag{7.7}$$

Then $\Pi_\infty(X)$ is defined by

$$\begin{aligned} \Pi_\infty(X)_n &:= \mathrm{Top}(|\Delta^n|, X), \\ d_j &:= |\delta_j|^* : \mathrm{Top}(|\Delta^n|, X) \to \mathrm{Top}(|\Delta^{n-1}|, X), \\ s_j &:= |\sigma_j|^* : \mathrm{Top}(|\Delta^n|, X) \to \mathrm{Top}(|\Delta^{n+1}|, X). \end{aligned} \tag{7.8}$$

This construction can be enhanced to give a functor $\Pi_\infty : \mathrm{Top} \to \mathrm{Set}_\Delta$. Moreover, Π_∞ has an adjoint $|\cdot| : \mathrm{Set}_\Delta \to \mathrm{Top}$, called *fat geometric realization*, and defined on objects as

$$||X_\bullet|| := \bigsqcup_{n\in\mathbb{N}} X_n \times |\Delta^n|/\sim, \tag{7.9}$$

where the equivalence relation is $(p, d_j^\Delta(x)) \sim (d_j(p), x)$. A theorem of Quillen states that for any compactly generated topological space X and for any Kan complex $X_\bullet$ the counit and unit of this adjunction induce a weak homotopy equivalence of topological spaces $||\Pi_\infty(X)|| \cong X$ and a weak homotopy equivalence of simplicial sets $X_\bullet \cong \Pi_\infty(||X_\bullet||)$.

Recall that, given categories A, B, then functors $A \to B$ form a category $Fun(A,B)$, where arrows are given by natural transformations. It is easy to check that $N(Fun(A,B))_n = \mathrm{Set}_\Delta(N(A)_\bullet \times \Delta^n, N(B)_\bullet)$, which justifies the following definition.

Definition 7.7 ([181])**.** Let A, B be $(\infty,1)$-categories. The $(\infty,1)$-*category of functors* from A to B is the simplicial set $Fun(A,B)$ with

$$\begin{aligned} Fun(A,B)_n &:= \mathrm{Set}_\Delta(A \times \Delta^n, B), \\ d_j &:= (s_j^\Delta)^*, \\ s_j &:= (d_j^\Delta)^*, \end{aligned} \tag{7.10}$$

where $d_j^\Delta : \Delta^{n+1} \to \Delta^n$, $s_j^\Delta : \Delta^{n-1} \to \Delta^n$ are the face and degeneracy maps between the combinatorial simplices (see Example 7.2). Element of $Fun(A,B)_0$ are called *functors*, elements of $Fun(A,B)_1$ are called *natural transformations* and elements of $Fun(A,B)_2$ are called *modifications*.

For Definition 7.7 to make sense, one must check that when A, B satisfy the Kan condition from Definition 7.5 then the simplicial set defined by (7.10) also satisfies it;

this is proven in [181, Prop 1.2.7.3]. In fact, it is also true that when B is an ∞-groupoid then so is $Fun(A,B)$, and that when B is an $(n,1)$-category then so is $Fun(A,B)$.

7.1.2 Simplicial categories and localization

In the following definition, and for the rest of this chapter, we shall abuse language by using the term *simplicial categories* for certain objects which are not equivalent to arbitrary functors $\Delta^{op} \to \mathrm{Cat}$ (cf. Definition 7.1), although they can be identified with those functors $\Delta^{op} \to \mathrm{Cat}$ which are 'constant on objects' [185, Remark 3.6].

Definition 7.8 ([181]). A *simplicial category* or *simplicially enriched category* A is the following data.

1. A class of *objects* A_0
2. For each pair of objects x, $y \in A_0$, a simplicial set of *arrows* $A(x,y) \in \mathrm{Set}_\Delta$,
3. For each triple of objects x, y, $z \in A_0$, a *composition* morphism of simplicial sets $\circ : A(x,y) \times A(y,z) \to A(x,z)$ such that for x, y, z, $t \in A_0$ we have a commutative diagram
$$\begin{tikzcd} A(x,y)\times A(y,z)\times A(z,t) \arrow[r, "id\times\circ"] \arrow[d, "\circ\times id"] & A(x,z)\times A(z,t) \arrow[d, "\circ"] \\ A(x,y)\times A(y,t) \arrow[r, "\circ"] & A(x,t). \end{tikzcd}, \qquad (7.11)$$
4. For each object $x \in A_0$, a map $id_x : \Delta^0 \to A(x,x)$ such that for x, $y \in A_0$ we have commutative diagramas
$$\begin{tikzcd} \Delta(x,y) \arrow[r, "id_x"] \arrow[dr, "id"'] & \Delta(x,x)\times\Delta(x,y) \arrow[d, "\circ"] \\ & \Delta(x,y) \end{tikzcd}, \quad \begin{tikzcd} \Delta(x,y) \arrow[r, "id_y"] \arrow[dr, "id"'] & \Delta(x,y)\times\Delta(y,y) \arrow[d, "\circ"] \\ & \Delta(x,y) \end{tikzcd}. \qquad (7.12)$$

A *fibrant simplicial category* is a simplicial category A such that all the simplicial sets $A(x,y)$ are ∞-groupoids. The *underlying fibrant simplicial category* $A_{(\infty,1)}$ of a simplicial category A is the fibrant simplicial category with same objects as A but with $A_{(\infty,1)}(x,y) = \Pi_\infty(||A(x,y)||)$, where $||\cdot||$ denotes fat geometric realization (cf. Example 7.6). The *homotopy category* of a simplicial category A is the category $Ho(A)$ with same objects as A but with $Ho(A)(x,y) = \pi_0(||A(x,y)||)$.

As discussed in [181, Section 1.1.5], a simplicial category A has a *simplicial nerve* $N(A)_\bullet$, which is a simplicial set constructed similarly as the nerve of a category. If A is a fibrant simplicial category, then $N(A)_\bullet$ is an $(\infty,1)$-category as in Definition 7.5. Up to set-theoretic considerations, and with the right notion of equivalence, the simplicial nerve construction induces an equivalence between fibrant simplicial categories and $(\infty,1)$-categories.

Most of the (large) ∞-categories that we consider in this thesis are presented as fibrant simplicial categories, and the only reason why we have defined $(\infty,1)$-categories as in Definition 7.5, instead of as fibrant simplicial categories, is that the $(\infty,1)$-category of functors between two $(\infty,1)$-categories is much easier to define in the model from Definition 7.5. For this reason, given fibrant simplicial categories A, B we write $Fun(A,B)$ for the $(\infty,1)$-category of functors between the $(\infty,1)$-categories associated to A, B.

Example 7.9. The fibrant simplicial category *Spc* of *spaces* is the simplicial category whose class of objects is the class of all ∞-groupoids as in Definition 7.5, with $Spc(A,B) = Fun(A,B)$ and where composition and identities are defined in an obvious way. Note that this is a combinatorial model for the (informal) $(\infty,1)$-category of topological spaces, continuous maps, homotopies and higher homotopies. Indeed, Quillen's theorem that we recalled in Example 7.6 implies that the homotopy category of *Spc* coincides with the localization of the category of compactly generated topological spaces at the weak homotopy equivalences. [181, Section 1.1.4]

We define *homotopy limits* and *homotopy colimits* in *Spc* by taking them in topological spaces; i.e., given a small category C and a functor $F : C \to \mathrm{Spc}$, then its homotopy limit is $\Pi_\infty(\lim ||\cdot|| \circ F) \in Spc_0$, where $\lim ||\cdot|| \circ F \in \mathrm{Top}_0$ denotes the homotopy limit in the sense of topology [146] of the functor $||\cdot|| \circ F : C \to \mathrm{Top}$.

A common technique in category theory is *localization*; i.e., constructing a category $A[W]$ out of a category A and a class of morphisms W (*weak equivalences*) in A that are formally inverted in $A[W]$. For example, a span $x \overset{w}{\leftarrow} x' \overset{f}{\to} y$ with $w \in W$ induces an arrow $(w,f) : x \to y$ in $A[W]$, and given a second span $x \overset{w_2}{\leftarrow} x_2' \overset{f_2}{\to} y$ with $w_2 \in W$ one has $(w,f) = (w_2,f_2)$ if there exists some $\alpha : x' \to x_2'$ commuting with the two spans. This identification, which is necessary for the composition on $A[W]$ to be well-defined, loses the information of the arrow α, which is important in some contexts.

If we are working in one such context, then we might try to construct instead an $(\infty,1)$-category $A[W]_\infty$ in which pairs (w,f) as before induce arrows, and in which α induces a 2-cell from (w,f) to (w_2,f_2). The systematic way to do this for an arbitrary class of morphisms W is called *hammock*, *Dwyer-Kan* or *simplicial* localization [104]. If (A,W) satisfy additional conditions, then the ∞-category obtained from Dwyer-Kan localization is equivalent to an easier to present fibrant simplicial category [204, Section 3.6.2], and for the purposes of this thesis this situation is general enough.

Definition 7.10 ([204])**.** Let A be a category, let W, $F \subset A$ be two subcategories, whose arrows we call *weak equivalences* and *fibrations*, respectively, and call the arrows that are both in F and W *acyclic fibrations*. The data (A, W, F) is a *category of fibrant objects* if the following axioms are satisfied.

1. Isomorphisms are acyclic fibrations.
2. For $x \xrightarrow{f} y \xrightarrow{g} z$, if any two of $\{f, g, g \circ f\}$ is a weak equivalence, then so is the other one.
3. For every $x \in A_0$, $x \to 1$ is a fibration.
4. Any diagram
$$\begin{tikzcd} & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z \end{tikzcd} \tag{7.13}$$
in A such that g is a fibration has a pull-back $x\,{}_f\!\times_g y \to z$ in A. This is again a fibration and, if g is an acyclic fibration, then so is the pull-back.
5. Any arrow $x \xrightarrow{f} y$ in A factorizes as $x \xrightarrow{\lambda} x' \xrightarrow{f'} y$, where f' is a fibration and λ is a weak equivalence that is a section of an acyclic fibration $x' \to x$.

The *localization* $A[W]_{(\infty,1)}$ of a category of fibrant objects (A, W, F) is the underlying fibrant simplicial category of the simplicial category $A[W]_\infty$ defined as follows.

1. The class of objects of $A[W]_\infty$ is A_0 .
2. For x, $y \in A_0$ the simplicial set $A[W]_\infty(x, y)$ is the nerve of the following category:
   (a) Its objects are spans $x \xleftarrow{w} x' \xrightarrow{f} y$ in A with w an acyclic fibration.
   (b) An arrow between the spans $x \xleftarrow{w} x' \xrightarrow{f} y$ and $x \xleftarrow{w_2} x'_2 \xrightarrow{f_2} y$ is a weak equivalence $\alpha : x' \to x'_2$ in A such that
$$\begin{tikzcd} & x' \arrow[dl, "w"'] \arrow[dr, "f"] \arrow[dd, "\alpha"] & \\ x & & y \\ & x'_2 \arrow[ul, "w_2"] \arrow[ur, "f_2"'] & \end{tikzcd} \tag{7.14}$$
   commutes.
   (c) Composition and identities are induced by composition and identities in A.

3. For x, y, $z \in A_0$, composition $A[W]_\infty(x,y) \times A[W]_\infty(y,z) \to A[W]_\infty(x,z)$ is defined by

$$\begin{tikzcd}[column sep=small] & x' \arrow[dl, "w_x"'] \arrow[dr, "f"] & & y' \arrow[dl, "w_y"'] \arrow[dr, "g"] & \\ x & & y & & z \end{tikzcd} = \begin{tikzcd}[column sep=small] & x' {}_f\times_{w_y} y' \arrow[dl, "w_x"'] \arrow[dr, "g"] & \\ x & & z \end{tikzcd} \tag{7.15}$$

4. The identity of $x \in A_0$ is $x \overset{id_x}{\leftarrow} x \overset{id_x}{\rightarrow} x$.

Remark 7.11. Since the simplicial sets $A[W]_\infty(x,y)$ in Definition 7.10 are nerves of categories, one can think of $A[W]_\infty$ as a category internal to categories, or equivalently as a strict $(2,2)$-category. It might seem counter-intuitive that it is possible to construct an $(\infty,1)$-category $A[W]_{(\infty,1)}$ with non-trivial k-cells for $k \geq 3$ out of a $(2,2)$-category $A[W]_\infty$. However, this is related to the surprising fact that the geometric realization of any ∞-groupoid (such as $A[W]_{(\infty,1)}(x,y)$) is weakly homotopy equivalent to the geometric realization of some category (such as $A[W]_\infty(x,y)$) [260]. For example, note that $A[W]_\infty(x,y)$ are groupoids if and only if every weak equivalence is already invertible in A, in which case $A[W]_\infty = A[W]_{(\infty,1)}$ is a $(2,1)$-category which is actually canonically equivalent to the category A.

The motivation for Definition 7.10 is that, although arbitrary fibered products need not exist on the nose on A, they do exist 'up to weak equivalences'. Hence, if we pass from A to $A[W]_{(\infty,1)}$, then we can take arbitrary fibered products. More precisely, if (A,W,F) is a category of fibrant objects and we are given a diagram in A

$$\begin{tikzcd} & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z \end{tikzcd}, \tag{7.16}$$

we may choose a factorization $x \overset{\lambda}{\to} x' \overset{f'}{\to} z$ of f, with f' a fibration and λ a weak equivalence. Then we define the corresponding *homotopy fibered product* to be $x\, {}_f\times^h_g\, y := x'\, {}_{f'}\times_g\, y$. Since this depends on the choice of factorization, the homotopy fibered product is not really a well-defined fibered product on A, but it is well-defined as an operation on $A[W]_{(\infty,1)}$. In fact, it satisfies an analogous universal property (it is an $(\infty,1)$-limit [181, Section 1.2.13]). Notice the similarity with how derived functors in abelian categories are computed in terms of resolutions.

Example 7.12. Let A be the category whose objects are ∞-groupoids and whose morphisms are functors between them. We define *weak equivalences* in A to be functors $f_\bullet : X_\bullet \to Y_\bullet$ such that $|f_\bullet| : |X_\bullet| \to |Y_\bullet|$ is a weak homotopy equivalence of topological spaces and *fibrations* in A to be Kan fibrations; i.e., functors $f_\bullet : X_\bullet \to Y_\bullet$ such that for

$n \geq 1$ and $0 \leq j \leq n$, the map

$$\mathrm{Set}_\Delta(\Delta^n, X) \to \mathrm{Set}_\Delta(\Lambda^n_j, X) \times_{\mathrm{Set}_\Delta(\Lambda^n_j, Y)} \mathrm{Set}_\Delta(\Delta^n, Y) \tag{7.17}$$

induced by the inclusion $\Lambda^n_j \to \Delta^n$ and the map f, is surjective. This is a structure of category of fibrant objects on A, and its localization coincides with *Spc* [214, Section II.3].

Example 7.13. For a field k of characteristic 0, write *dgAlg* for the category of differential $\mathbb{Z}^{\leq 0}$-graded commutative k-algebras, with degree and differential-preserving morphisms of algebras as arrows. This has structure of category of fibrant objects [50] if we define weak equivalences to be quasi-isomorphisms (i.e., morphisms that induce isomorphisms in cohomology) and fibrations to be morphisms that are surjections on each degree. The localization of *dgAlg* is by definition (the opposite of) the $(\infty, 1)$-category $AffdSch$ of affine derived schemes. $AffdSch$ is an enhancement of the category of smooth schemes in which fibered products always exist and behave better than the standard fibered products of schemes.

7.1.3 ∞-stacks

The easiest way to define rigourously the $(\infty, 1)$-category of derived differentiable ∞-stacks is to adopt the functor of points approach. For this we need some basic notions of sheaf theory on $(\infty, 1)$-categories.

Definition 7.14 ([261])**.** Let A be a fibrant simplicial category. Then its *opposite* is the fibrant simplicial category A^{op} with same objects as A but $A^{op}(x, y) = A(y, x)$. The $(\infty, 1)$-category of ∞*-presheaves* on A is $P(A) := Fun(A^{op}, \mathrm{Spc})$. For each $j \in \mathbb{N}$, the j*th homotopy presheaf* of $\mathfrak{X} \in P(A)$ is $\pi_j(\mathfrak{X}) \in Fun(A^{op}, \mathrm{Set})$ obtained by composing $\mathfrak{X}$ with the jth homotopy group functor $\pi_j : \mathrm{Spc} \to \mathrm{Set}$. The *homotopy limit* or *homotopy colimit* of a diagram in $P(A)$ is the presheaf that assigns to each $x \in A_0$ the homotopy limit or homotopy colimit of the corresponding diagram in Spc.

A *sieve* on an object $x \in A$ is a sieve on $Ho(A)$; that is, a collection of arrows in $Ho(A)$

$$\tau^\alpha_x := \{U^\alpha_i \to x\}_{i \in I}$$

such that for $l \in \tau^\alpha_x$ and $V \xrightarrow{f} U^\alpha_i$ an arrow in $Ho(A)$ we have $l \circ f \in \tau^\alpha_x$. A *Grothendieck topology* on A is a Grothendieck topology τ on $Ho(A)$. Equivalently, it is the data of, for each $x \in A$, a class of distinguished sieves on x, called *covering sieves* and denoted τ^α_x, subject to the following axioms.

1. The class of all arrows in $Ho(A)$ with target x is a covering sieve on x.

2. If $f : x \to y$ is an arrow in $Ho(A)$ and τ_y^α is a covering sieve on y, then $f^*\tau_y^\alpha := \{h : s(h) \to x \mid f \circ h \in \tau_y^\alpha\}$ is a covering sieve on x.

3. If τ_x^α is a covering sieve on x and τ_x^β is an arbitrary sieve such that for any $l : U_i^\alpha \to x$ in τ_x^α the sieve $l^*\tau_x^\beta$ is a covering sieve on U_i^α, then τ_x^β is a covering sieve on x.

Example 7.15. Let $A = Man$ be the category of manifolds. Define a *good open cover* of $X \in Man$ to be a collection $\{U_i \to X\}_{i \in I}$ of morphisms in Man such that $U_i \to X$ are inclusions of open sets with $X = \cup_{i \in I} U_i$ and all finite intersections of the open sets $\{U_i\}_{i \in I}$ are contractible. A Grothendieck topology on Man is given by letting a sieve be a covering sieve if and only if it contains a good open cover.

Example 7.16. Let A be the fibrant simplicial category of affine derived schemes from Example 7.13. There is a notion of *étale morphism* between affine derived schemes [261] that can be used to define a Grothendieck topology by letting $\{U_i \to X\}_{i \in I}$ be a covering sieve if each $U_i \to X$ is étale and the induced morphism $\bigsqcup_{i \in I} t_0(U_i) \to t_0(X)$ is a surjective morphism of schemes, where for

$$Z = Spec(\mathcal{O}(Z)_0 \overset{d}{\leftarrow} \mathcal{O}(Z)_{-1} \overset{d}{\leftarrow} \dots) \in AffdSch$$

we define $t_0(Z) := Spec(\mathcal{O}(Z)_0 / d\,\mathcal{O}(Z)_{-1}) \in AffSch$.

For A a fibrant simplicial category and an object $X \in A_0$, we write $j(X) \in P(A)$ for the presheaf associated to j by the Yoneda embedding (i.e., $j(X)(Y) = A(X, Y)$). It is proved in [181, Prop 6.2.2.5] that, as in ordinary category theory, sieves on X are in bijection with pairs (U, s) of a presheaf $U \in P(A)$ and a monomorphism $U \overset{s}{\to} j(X)$. For example, when $A = \text{Man}$, a sieve $\tau_X = \{U_i \overset{l_i}{\to} X\}_{i \in I}$ over a manifold X determines the presheaf

$$U(M) := \{\{f_i : M \to U_i\}_{i \in I} \mid \forall i,\, j \in I,\, l_i \circ f_i = l_j \circ f_j\}, \tag{7.18}$$

which has an obvious monomorphism $U \to j(X)$. When A is an arbitrary $(\infty, 1)$-category, the presheaf U can be constructed as follows. First, construct the *Čech resolution* of the sieve τ_X; this is a simplicial object $\check{C}(\tau_X) : \Delta^{op} \to P(A)$ in $P(A)$ defined as

$$\check{C}(\tau_X)_n := \bigsqcup_{i_0, \dots, i_n} j(U_{i_0}) \times^h_{j(X)} \dots \times^h_{j(X)} j(U_{i_n}), \tag{7.19}$$

with obvious face and degeneracy maps. Then $U \in P(A)$ is defined as the *homotopy colimit* of $\check{C}(\tau_X)$; that is,

$$U(M) := \lim_{[n]\in\Delta} \left(\coprod_{i_0,\dots,i_n} j(U_{i_0})(M) \times^h_{j(X)(M)} \cdots \times^h_{j(X)(M)} j(U_{i_n})(M) \right), \tag{7.20}$$

where $\times^h_{j(X)(M)}$ denotes homotopy fibered products of spaces and lim denotes homotopy colimit of spaces. We can then formulate the descent condition of a sheaf as follows.

Definition 7.17 ([181, Def. 6.2.2.6])**.** Let A be a fibrant simplicial category with a Grothendieck topology τ. A *sheaf* on A is a presheaf $\mathfrak{X} : A^{op} \to \mathrm{Spc}$ such that for every covering sieve $\tau_X = \{U_i \xrightarrow{l_i} X\}_{i\in I}$ over $X \in A_0$ determining the monomorphism $U \to j(X)$ in $P(A)$ we have that the induced map

$$P(A)(j(X), \mathfrak{X}) \to P(A)(U, \mathfrak{X}) \tag{7.21}$$

is a weak homotopy equivalence of topological spaces. The fibrant simplicial category of sheaves $Sh(A, \tau)$ (or simply $Sh(A)$) is the full simplicial subcategory of the simplicial category of presheaves spanned by the sheaves. A *subcanonical Grothendieck topology* on A is a Grothendieck topology τ such that for every $X \in A_0$ we have $j(X) \in Sh(A, \tau)$.

For any Grothendieck topology, the inclusion functor $Sh(A) \to P(A)$ has a left exact (in the sense of $(\infty, 1)$-categories) adjoint functor $L : P(A) \to Sh(A)$, called *sheafification* [181, Lem. 6.2.2.7]. This implies in particular that $Sh(A)$ is closed under the homotopy limits of $P(A)$, while homotopy colimits on $Sh(A)$ are defined by taking them on $P(A)$ and then sheafifying with L. Moreover, [181, Lem. 6.2.2.7] also implies that Sh(A) has all internal homs, meaning that for any $\mathfrak{X}$, $\mathfrak{Y} \in$ Sh(A) there exists some $\underline{Sh(A)}(\mathfrak{X}, \mathfrak{Y}) \in$ Sh(A) such that $\mathrm{Sh(A)}(\mathfrak{Z}, \underline{Sh(A)}(\mathfrak{X}, \mathfrak{Y})) = \mathrm{Sh(A)}(\mathfrak{Z} \times \mathfrak{X}, \mathfrak{Y})$.

In the applications we have in mind, A is an $(\infty, 1)$-category of (possibly very singular) geometric objects and τ is a subcanonical Grothendieck topology. In this case, an arbitrary sheaf $\mathfrak{X} \in Sh(A)$ represents the functor of points of a moduli problem on A, and we want to characterize which of these sheaves are *geometric*, in the sense that they can be represented in some way by objects of A.

Definition 7.18 ([261])**.** Let $\tilde{A}$ be a fibrant simplicial category and let $j : \tilde{A} \to P(\tilde{A})$ be the Yoneda embedding. Then a *groupoid object* in $\tilde{A}$ is a functor $\mathfrak{X} : \Delta^{op} \to \tilde{A}$ satisfying the *Segal condition*: for $n \geq 2$ and for every subdivision $[n] = S \cup S'$ with $S \cap S' = \{s\}$, the map

$$\mathfrak{X}_n \to \mathfrak{X}(S) \times \mathfrak{X}(S') \tag{7.22}$$

induced by the inclusions S, $S' \to [n]$ exhibits $j(\mathfrak{X}_n)$ as the homotopy fibered product in $P(\tilde{A})$

$$j(\mathfrak{X}_n) = j(\mathfrak{X}(S)) \times^h_{j(\mathfrak{X}(\{s\}))} j(\mathfrak{X}(S')). \tag{7.23}$$

A *morphism of groupoid objects* is a natural transformation of functors.

Let (A, τ) be a fibrant simplicial category with a subcanonical Grothendieck topology. Let $(\mathcal{A}, \tau')$ be a full simplicial sub-category of $Sh(A)$ containing the image of the Yoneda embedding $j : A \to Sh(A)$, with a Grothendieck topology τ' restricting to τ over A. A morphism $U \to \mathfrak{X}$ in $\mathcal{A}$ is called *smooth* if $\{U \to \mathfrak{X}\}$ is a covering sieve for τ'. We define inductively, for each $n \geq 0$, the fibrant simplicial category of n-geometric stacks and their smooth morphisms as follows.

1. The fibrant simplicial category of 0-*geometric stacks* $0gSt(\mathcal{A})$ is $\mathcal{A}$.

2. For $n \geq 1$, a *smooth groupoid in* $(n-1)$-*geometric stacks* $\mathfrak{X} : \Delta^{op} \to (n-1)gSt(\mathcal{A})$ is a groupoid object in the simplicial category of $(n-1)$-geometric stacks such that $\mathfrak{X}_1 \overset{d_j}{\to} \mathfrak{X}_0$ is a smooth arrow for $j = 0, 1$. A *smooth morphism* between two such groupoids $\mathfrak{X} \to \mathfrak{Y}$ is a morphism of groupoid objects such that the corresponding map $\mathfrak{X}_0 \to \mathfrak{Y}_0$ is a smooth morphism of $(n-1)$-geometric stacks.

3. For $n \geq 1$, the fibrant simplicial category $ngSt(\mathcal{A})$ of n-*geometric stacks* is the full simplicial sub-category of $Sh(A)$ spanned by objects that can be obtained as colimits of $i \circ \mathfrak{X} : \Delta^{op} \to Sh(A)$, for $\mathfrak{X}$ a smooth groupoid in $(n-1)$-geometric stacks and $i : (n-1)gSt(\mathcal{A}) \to Sh(A)$ the inclusion functor. A *smooth morphisms of* n-*geometric stacks* is an arrow in $ngSt(\mathcal{A})$ arising as the colimit of a smooth morphism of groupoid objects in $(n-1)gSt(\mathcal{A})$.

4. A *geometric stack* is $\mathfrak{X} \in Sh(A)_0$ such that $\mathfrak{X} \in ngSt(\mathcal{A})$ for some $n \in \mathbb{N}$.

Remark 7.19. When $\tilde{A}$ is actually an ordinary category, then groupoid objects on $\tilde{A}$ as in Definition 7.18 coincide with groupoid objects as in Definition 7.5 (replacing Set by $\tilde{A}$ and bijection by isomorphism). In this case, n-geometric stacks in the setting of Definition 7.22 form an $(n+1, 1)$-category [213]. Another interesting remark is that the homotopy presheaves of a sheaf are actually sheaves, but even if the original sheaf was geometric then its homotopy presheaves might not be so, if $\mathcal{A}$ does not have all colimits.

Example 7.20. If we take A to be the category $AffSch := Ring^{op}$ of affine schemes, with étale covers as covering sieves, and $\mathcal{A}$ to be the image under the Yoneda embedding of the category of all schemes, also with étale covers as covering sieves, then the corresponding n-geometric stacks are precisely the Artin n-stacks that model higher algebraic geometry. If we take A to be the fibrant simplicial category of affine derived schemes with the étale Grothendieck topology from Example 7.16, and we let $\mathcal{A}$ be the image under the Yoneda embedding of the fibrant simplicial category of all derived schemes in the sense of [261] with étale topology, then the resulting geometric stacks are called derived Artin stacks and provide the framework for higher derived algebraic geometry.

The following proposition provides a way to present examples of geometric stacks over a fibrant simplicial category A in terms of simplicial objects on A.

Proposition 7.21 ([213]). *Let (A, τ) be a fibrant simplicial category with a subcanonical Grothendieck topology and a fully faithful embedding Set $\to A$. Let $(\mathcal{A}, \tau')$ be a full simplicial sub-category of $Sh(A)$ containing the image of the Yoneda embedding $j : A \to Sh(A)$, scuh that τ' restricts to τ over A, and such that homotopy fibered products along smooth maps remain in $\mathcal{A}$. Let $m \geq 1$ and let $\mathfrak{X} : \Delta^{op} \to \mathcal{A}$ be a functor such that*

1. *Face maps $\mathfrak{X}_n \to \mathfrak{X}_{n-1}$ are smooth. In particular, for every horn Λ^n_j, the internal hom $\underline{Fun(\Delta^{op}, \tilde{A})}(\Lambda^n_j, \mathfrak{X})$ exists in $\mathcal{A}$.*

2. *The arrows in $\mathcal{A}$*

$$\mathfrak{X}_n = \underline{Fun(\Delta^{op}, A)}(\Delta^n, \mathfrak{X}) \to \underline{Fun(\Delta^{op}, A)}(\Lambda^n_j, \mathfrak{X}) \tag{7.24}$$

induced by the inclusion $\Lambda^n_j \to \Lambda^n$ are isomorphisms for $n > m$, $0 \leq j \leq n$ and smooth for $2 \leq n \leq m$, $0 \leq j \leq n$.

Then the colimit of $j \circ \mathfrak{X} : \Delta^{op} \to Sh(A)$ is an m-geometric stack. Moreover, if A, $\mathcal{A}$ are ordinary categories and the class of smooth morphisms satisfies the conditions in [213, Properties 1.8], then all m-geometric stacks arise in this way.

7.2 Higher derived differential geometry

7.2.1 Lie ∞-groupoids

Definition 7.22 ([213, 282]). For $m \in \mathbb{N}$, the $(m+1,1)$*-category of differentiable m-stacks* is the $(m+1,1)$-category of geometric m-stacks $\mathfrak{X} : \mathrm{Man}^{op} \to \mathrm{Spc}$ in the sense of Definition 7.18, for $(A, \tau) = (\mathcal{A}, \tau')$ the category of manifolds with the Grothendieck topology from Example 7.15. The $(\infty, 1)$*-category $DiffSt$ of differentiable stacks* is the $(\infty, 1)$-category of all stacks that are m-geometric for some $m \in \mathbb{N}$.

For $m \in \mathbb{N} \cup \{\infty\}$, a *Lie m-groupoid* is a simplicial manifold $\mathfrak{X} : \Delta^{op} \to \mathrm{Man}$ (cf. Definition 2.1) such that the restriction maps

$$\mathfrak{X}_n = \mathrm{Man}_{\Delta}(\Delta^n, \mathfrak{X}) \to \mathrm{Man}_{\Delta}(\Lambda^n_j, \mathfrak{X}) \tag{7.25}$$

are diffeomorphisms for $n > m$, $0 \leq j \leq n$ and surjective submersions for $1 \leq n \leq m$, $0 \leq j \leq n$.

Remark 7.23. [282] proves that, for any $N \in \mathbb{N}$, the right-hand side of (7.25) is a manifold for $n = N$ and any $0 \leq j \leq N$ if for $n < N$ and all $0 \leq j \leq n$ the right-hand side of (7.25) is a manifold and the restriction maps (7.25) are surjective submersions. Thus, it follows by induction that Definition 7.22 makes sense. It is also useful to know that, as shown in [177, Def. 2.39, Lem. 2.44], all face maps of a Lie m-groupoid are submersions. Moreover, the tangent complex of a Lie m-groupoid as defined in Section 2.1.1 is indeed a complex of vector bundles, as the quotients (2.17) have constant rank in this case.

Example 7.24. For $\mathfrak{G}$ a Lie 2-group acting on a Lie 1-groupoid $\mathfrak{P}$, the quotient 2-groupoid constructed in Section 3.1.2 is a Lie 2-groupoid in the sense of Definition 7.22.

By definition, the category of manifolds is equivalent to the category of differentiable 0-stacks. The bicategory of Lie groupoids, anafunctors and smooth transformations from Section 3.1.1 is equivalent to the bicategory of differentiable 1-stacks, as shown in [33]. An explicit way of assigning a differentiable 1-stack $\underline{\mathfrak{X}}$ to a Lie groupoid $\mathfrak{X}$ is to construct its nerve $N(\mathfrak{X})_\bullet$ as in Section 3.1.1, which is a Lie 1-groupoid in the sense of Definition 7.22, and then take the homotopy colimit $\underline{\mathfrak{X}} := \lim_{\Delta}(N(\mathfrak{X})_\bullet)$ as in Section 7.1.3. Note that, for Lie groupoids $\mathfrak{X}$, $\mathfrak{Y}$, the simplicial set $Fun(N(\mathfrak{X}_\bullet), N(\mathfrak{Y}_\bullet))$ is (the nerve of) the groupoid of smooth functors $\mathfrak{X} \to \mathfrak{Y}$ with smooth transformations between them, and not the groupoid of all smooth anafunctors.

For arbitrary $m \in \mathbb{N}$, Proposition 7.21 implies that all differentiable m-stacks admit a description as colimits of Lie m-groupoids in the sense of Definition 7.22. In particular, Lie m-groupoids defining equivalent differentiable m-stacks have weak homotopy equivalent geometric realizations (but equivalence as geometric m-stacks is stronger). As we have just seen for $m = 1$, given two Lie m-groupoids $\mathfrak{X}$, $\mathfrak{Y} : \Delta^{op} \to \mathrm{Man}$ defining stacks $\underline{\mathfrak{X}}$, $\underline{\mathfrak{Y}}$, the space $\mathrm{DiffSt}(\underline{\mathfrak{X}}, \underline{\mathfrak{Y}})$ contains but can be strictly bigger than the simplicial set $Fun(\mathfrak{X}, \mathfrak{Y})$.

The idea is that the Lie m-groupoid $\mathfrak{X}$ is only a presentation of its corresponding m-stack $\underline{\mathfrak{X}}$, and so an arbitrary morphism of stacks might only be defined as a functor of m-groupoids in some other presentation $\mathfrak{X}'$. Thus, a rigorous presentation of the whole $(m+1,1)$-category of differentiable m-stacks purely in terms of Lie m-groupoids requires allowing for resolutions of $\mathfrak{X}$ and then localizing in an appropriate way to remove the dependence on the resolution, similarly as in the construction in Section 7.1.2. A rigorous construction for arbitrary $m \in \mathbb{N}$ in the algebraic setting is in [213], while [222] solves the problem in the smooth setting through the notion of *incomplete categories of fibrant objects*. The result for $m = 1$ is the bicategory of Lie groupoids and anafunctors from Section 3.1.1. A useful intermediate result is the following.

Proposition 7.25. *Let M be a manifold and let $\mathfrak{Y}$ be a Lie m-groupoid defining differentiable stacks $\underline{M}$ and $\underline{\mathfrak{Y}}$. For $\{U_i\}_{i\in I}$ a good cover of M, there is a canonical equivalence of m-groupoids*

$$Fun(\check{C}(M,\mathcal{U})_\bullet,\mathfrak{Y}) \to DiffSt(\underline{M},\underline{\mathfrak{Y}}), \tag{7.26}$$

where $\check{C}(M,\mathcal{U})_\bullet$ is the Čech groupoid from Example 2.7.

Proof. It follows directly from the sheaf condition from Definition 7.17 and the definition of the Grothendieck topology on Man. □

Remark 7.26. *Complex differentiable m-stacks* are defined analogously as in 7.22, replacing Man by the category of complex manifolds with similar Grothendieck topology, and similarly for *complex Lie m-groupoids*. Then Proposition 7.21 implies that every complex differentiable m-stack can be represented as the colimit of a complex Lie m-groupoid.

7.2.2 Lie m-groups and principal m-bundles

Given a Lie group G we can associate to it two simplicial manifolds.

1. The one that is just G at all levels. This is a Lie 0-groupoid in the sense of Definition 7.22.
2. Its delooping $BG_\bullet$ (cf. Example 2.6). This is a Lie 1-groupoid in the sense of Definition 7.22.

Note Proposition 7.25 implies that for any manifold M the groupoid DiffSt(M, BG) is equivalent to the groupoid of principal G-bundles on M. Moreover, for $\{*\} \to BG_\bullet$ the unique map of simplicial manifolds extending the map $\{*\} \to \{*\}$ at level 0, one can easily see that a map of differentiable stacks $M \to BG$ factorices through $\{*\} \to BG_\bullet$ if and only if the corresponding G-bundle $P \to M$ is trivial. This means that

$$G = \{*\} \times^h_{BG_\bullet} \{*\}, \tag{7.27}$$

because for any manifold M we have $\underline{G}(M) = (\{*\} \times^h_{BG_\bullet} \{*\})(M)$, meaning that the set of functions from M to G equals the set of automorphisms of the trivial G-bundle over M. Similarly, given a Lie 2-group $\mathfrak{G}$ in the sense of Definition 3.9 we can associate to it the Lie 1-groupoid $N(\mathfrak{G})_\bullet$ and the Lie 2-groupoid $B\mathfrak{G}_\bullet$, and their corresponding differentiable stacks satisfy $N(\mathfrak{G})_\bullet = \{*\} \times^h_{B\mathfrak{G}_\bullet} \{*\}$.

Definition 7.27 ([239])**.** For $m \geq 1$, a *Lie m-group* is a Lie $(m-1)$-groupoid $\mathfrak{G}$ in the sense of Definition 7.22 such that there exists a Lie m-groupoid, denoted $B\mathfrak{G}$ and called its *delooping*, with

$$B\mathfrak{G}_0 = \{*\}, \tag{7.28}$$

$$\mathfrak{G} = \{*\} \times^h_{B\mathfrak{G}} \{*\} \in \mathrm{Sh}(\mathrm{Man}), \tag{7.29}$$

where the map $\{*\} \to B\mathfrak{G}$ is the inclusion of the point $\{*\} \in B\mathfrak{G}_0$. For $\mathfrak{X}$ a Lie l-groupoid $\mathfrak{X}$ and a Lie m-group $\mathfrak{G}$, the m-groupoid of $\mathfrak{G}$*-principal bundles* over $\mathfrak{X}$ is $\mathrm{DiffSt}(\mathfrak{X}, B\mathfrak{G})$.

The equivalence between Lie 1-groups in the sense of Definition 7.27 and ordinary Lie groups follows from the familiar fact that a Lie groupoid with one object is determined by a Lie group. The equivalence between Lie 2-groups in the sense of Definition 7.27 and Lie 2-groups in the sense of Definition 3.9 is proven in [281]. By spelling out in detail the definition of the homotopy fibered product of sheaves, one sees that imposing (7.29) is equivalent to imposing that for any manifold M there is a canonical equivalence of $(m-1)$-groupoids between $\mathrm{DiffSt}(M, \mathfrak{G})$ and the $(m-1)$-groupoid of automorphisms inside $\mathrm{DiffSt}(M, B\mathfrak{G})$ of the unique functor $M \to B\mathfrak{G}$ that extends the smooth map $M \to \{*\}$ at level 0.

We proceed to present examples of Lie m-groups for higher $m \in \mathbb{N}$. We define a Lie m-group $\mathfrak{G}$ by presenting its delooping $B\mathfrak{G}$, from which $\mathfrak{G}$ can be recovered by definition. One of the advantages of this approach is that cocycle data for $\mathfrak{G}$-bundles is obtained immediately. We also comment on how to define connections on $\mathfrak{G}$-bundles in each case, for which we use some extra data on $\mathfrak{G}$ generalizing the Maurer-Cartan forms from Definition 3.23.

Example 7.28 ([178])**.** For T an abelian Lie group and $l \geq 1$, recall the simplicial manifold $B^lT_\bullet$ from Example 2.8. It is clear that $B^lT_\bullet$ is a Lie l-groupoid with $B^lT_0 = \{*\}$. We claim that $B^{l-1}T = \{*\} \times^h_{B^lT} \{*\}$, implying that $B^{l-1}T$ is a Lie l-group for each l. One way to see this is by noting that for a manifold M we can compute $\mathrm{DiffSt}(M, B^lT)$ as in Proposition 7.25, by taking a good cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M. Then an object in $\mathrm{DiffSt}(M, B^lT)$ is precisely a T-valued Čech l-cocycle, an arrow between two l-cocycles is a coboundary for the difference of the l-cocycles, a 2-cell is a coboundary between coboundaries, etc. In particular, the $(l-1)$-groupoid of automorphisms of any given object in $\mathrm{DiffSt}(M, B^l T)$ is equivalent to $\mathrm{DiffSt}(M, B^{l-1}T)$, which means precisely that $\{*\} \times^h_{B^l T} \{*\} = B^{l-1}T$.

In conclusion, $\mathrm{DiffSt}(M, B^l T)$ is the l-groupoid of T-$(l-1)$-gerbes over M in the sense of [57, 82, 133]. The preceeding discussion implies that we can think of T-$(l-1)$-gerbes as $B^{l-1}T$-principal bundles over M. If T is connected, so that the exponential sequence $1 \to Z \to \mathfrak{t} \overset{exp}{\to} T \to 1$ is exact (for $Z := \ker(exp)$), then for a $B^{l-1}T$-bundle described by a T-valued cocycle $t_{i_0 \dots i_l}$ we may choose a lift to $\mathfrak{t}$-valued functions $f_{i_0 \dots i_l}$ with $exp(f_{i_0 \dots i_l}) = t_{i_0 \dots i_l}$. One can easily prove that the Z-valued cocycle $\lambda_{i_0 \dots i_{l+1}} := \delta f$ determines a class in $H^{l+1}(M, Z)$, which we call the *Chern* or *Dixmier-Douady class*, and which classifies the $B^{l-1}T$-bundle completely.

Note that the Maurer-Cartan form on T defines a canonical 1-form $\theta \in \Omega^1((B^l T)_l, \mathfrak{t})$ such that $\delta\theta = 0$ and $d\theta = 0$. This means that we can define connections on a $B^{l-1}T$-principal bundle described by coycle data $t_{i_0 \dots i_l} : U_{i_0 \dots i_l} \to T$ as follows. Since $\delta\theta = 0$, $t^*\theta \in \Omega^1(\sqcup_{i_0 \dots i_l} U_{i_0 \dots i_l}, \mathfrak{t})$ is a Čech cocycle and so there exist $A^1_{i_0 \dots i_{l-1}} \in \Omega^1(U_{i_0 \dots i_{l-1}}, \mathfrak{t})$ with $\delta A^1 = t^*\theta$. Then $d\theta = 0$ implies $\delta dA^1 = 0$ and so there exist $A^2_{i_0 \dots i_{l-2}} \in \Omega^2(U_{i_0 \dots i_{l-2}}, \mathfrak{t})$ with $\delta A^2 = dA^1$. This procedure continues until we have $A^l_i \in \Omega^l(U_i, \mathfrak{t})$ with $A^l_j - A^l_i = dA^{l-1}_{ij}$; these define a global closed $(l+1)$-form $F := dA^l_i$, which is a de Rham representative for the Chern class. In the literature $(A^1, ..., A^l)$ is called a *connection* on the l-gerbe and F is its *curvature*.

Given a Lie m-group $\mathfrak{G}$, one can compose the sheaf $\underline{\mathfrak{G}} : \mathrm{Man} \to \mathrm{Spc}$ with the Postnikov tower construction [189, Theorem 4.35] on Spc, as explained in [181, Section 6.5.1]. It follows that the composition of $\underline{\mathfrak{G}}$ with each homotopy group functor $\mathrm{Spc} \to \mathrm{Grp}$ defines (possibly not representable) sheaves $G_0 : \mathrm{Man} \to \mathrm{Grp}$ and $T_1, T_2, ..., T_{m-1} : \mathrm{Man} \to \mathrm{AbGrp}$ such that G_0 acts on each T_j and $\mathfrak{G}$ can be decomposed as

$$\mathfrak{G} = \mathfrak{G}_{m-1} \to \mathfrak{G}_{m-2} \to \dots \to \mathfrak{G}_1 \to \mathfrak{G}_0 = G_0, \tag{7.30}$$

where the fiber of each $\mathfrak{G}_j \to \mathfrak{G}_{j-1}$ is $B^j T_j$. It follows that one way to construct Lie m-groups is by characterizing extensions of Lie $(m-1)$-groups by Lie m-groups of the form $B^{m-1}T_{m-1}$ and then proceeding inductively. A cocycle characterization of such extensions is in [178, Ex. 4.7], which can be used to construct many examples of relevance in physics.

Example 7.29 ([119])**.** Given vector spaces V_0, V_1 with lattices $\Lambda_0 \subset V_0$, $\Lambda_1 \subset V_1$ and a bilinear form $\langle \cdot, \cdot \rangle : \Lambda_0 \otimes \Lambda_0 \to \Lambda_1$, the Lie 2-group $\mathcal{T}$ from [119] presented in Example

3.14 has delooping $B\mathcal{T}$ defined by

$$
\begin{aligned}
(B\mathcal{T})_n := \{\{v^0_{ij}\}_{i<j\in[n]}, \{\lambda^0_{ijk}\}_{i<j<k\in[n]}, \{[v^1_{ijk}]\}) \in V_0^{\binom{n+1}{2}} \times \Lambda_0^{\binom{n+1}{3}} \times (V_1/\Lambda_1)^{\binom{n+1}{3}} \mid \\
\forall i<j<k,\quad v^0_{ij} - v^0_{ik} + v^0_{jk} = \lambda^0_{ijk}, \\
\forall i<j<k<l,\quad [v^1_{ijk} - v^1_{ijl} + v^1_{ikl} - v^1_{jkl}] = [\langle v^0_{ij}, \lambda^0_{jkl}, \rangle]\},
\end{aligned}
\tag{7.31}
$$

with simplicial maps defined as in Example 7.28. Using that $\langle\cdot,\cdot\rangle$ is Λ_1-valued over $\Lambda_0\otimes\Lambda_0$, one can see that $(B\mathcal{T})_n = V_0^n \times \Lambda_0^{\binom{n}{2}} \times (V_1/\Lambda_1)^{\binom{n}{2}}$ and that the Kan conditions are satisfied. As proven in [119], the 2-group constructed from the bilinear form $\langle\cdot,\cdot\rangle^t$ defined by $\langle u, v\rangle^t := \langle v, u\rangle$ is isomorphic to the one constructed from $\langle\cdot,\cdot\rangle$. In this approach this can be seen from the fact that, if $\{[v^1_{ijk}]\}_{i,j,k}$ satisfies $(\delta v^1)_{ijkl} = [\langle v^0_{ij}, \lambda^0_{jkl}\rangle]$, then

$$
w^1_{ijk} := [v^1_{ijk} - \langle v^0_{ij}, v^0_{jk}\rangle + \langle \lambda^0_{ijk}, v^0_{ik}\rangle] \tag{7.32}
$$

satisfies $(\delta w^1)_{ijkl} = [\langle \lambda^0_{jkl}, v^0_{ij}\rangle]$.

Let M be a manifold and let v^0_{ij}, λ^0_{ijk}, $[v^1_{ijk}]$ be cocycle data for a $\mathcal{T}$-bundle on a cover $\{U_i\}_{i\in I}$ of M (i.e., these are local functions satisfying the same relations as in (7.31)). This has an underlying V_0/Λ_0-bundle P whose Chern class $c(P) \in H^2(M, \Lambda_0)$ is represented in Čech cohomology by the cocycle λ^0_{ijk}. We may choose lifts $v^1_{ijk} : U_{ijk} \to V_1$ of $[v^1_{ijk}]$ and define $\lambda^1_{ijkl} : U_{ijkl} \to \Lambda_1$ by

$$
v^1_{ijk} - v^1_{ijl} + v^1_{ikl} - v^1_{jkl} = \langle v^0_{ij}, \lambda^0_{jkl}, \rangle - \lambda^1_{ijkl}; \tag{7.33}
$$

then we see that $(\delta\lambda^1)_{ijklm} = \langle \lambda^0_{ijk}, \lambda^0_{klm}\rangle$, which means precisely that $0 = \langle c(P)\wedge c(P)\rangle \in H^4(M, \Lambda_1)$. This recovers Proposition 4.5 for the Lie 2-group $\mathcal{T}$.

The Maurer-Cartan form on $\mathcal{T}$ determined by Example 3.46 and Proposition 3.50 is given by the 1-forms $\theta^0 \in \Omega^1(B\mathcal{T}_1, V_0)$, $\theta^1 \in \Omega^1(B\mathcal{T}_2, V_1)$ defined by

$$
\theta^0 := dv_{01}, \tag{7.34}
$$

$$
\theta^1 := dv^1_{012} - \langle dv^0_{01}, v_{12}\rangle. \tag{7.35}
$$

They satisfy

$$
\delta\theta^0 = 0, \quad \delta\theta^1 = 0, \quad d\theta^0 = 0, \quad d\theta^1 = \langle d_2^*\theta^0 \wedge d_0^*\theta^0\rangle. \tag{7.36}
$$

Given a $\mathcal{T}$-bundle over a manifold M described by cocycle data v^0_{ij}, λ^0_{ijk}, $[v^1_{ijk}]$ on a cover $\{U_i\}_{i\in I}$ of M, Proposition 4.12 implies that a connection on it is described by a family $(\{A_i\}_i, \{\Lambda_{ij}\}_{i,j}, \{B_i\}_i)$ with

$$A_i \in \Omega^1(U_i, V_0),\ \Lambda_{ij} \in \Omega^1(U_{ij}, V_1),\ B_i \in \Omega^2(U_i, V_1)$$

such that

$$A_j - A_i = (v^0_{ij})^*\theta^0, \tag{7.37}$$
$$\Lambda_{ij} - \Lambda_{ik} + \Lambda_{jk} = (v^1_{ijk})^*\theta^1, \tag{7.38}$$
$$B_j - B_i = d\Lambda_{ij} + \frac{1}{2}\langle dv^0_{ij} \wedge dv^0_{ij}\rangle - \langle A_i \wedge (v^0_{ij})^*\theta^0\rangle_{sy}. \tag{7.39}$$

We can do a change of variables $B'_i := B_i - \frac{1}{2}\langle A_i \wedge A_i\rangle$ to obtain a simpler relation

$$B'_j - B'_i = d\Lambda_{ij} - \langle A_i \wedge (v^0_{ij})^*\theta^0\rangle. \tag{7.40}$$

The curvature of the connection is the pair $(F, H) \in \Omega^2(M, V_0) \oplus \Omega^3(M, V_1)$ defined by

$$F = dA_i, \qquad H = dB'_i + \langle dA_i \wedge A_i\rangle \tag{7.41}$$

and one can check directly that it satisfies the Bianchi identity

$$dF = 0, \tag{7.42}$$
$$dH - \langle F \wedge F\rangle = 0, \tag{7.43}$$

giving an explicit proof of Proposition 4.11 for $\mathcal{T}$. When $V_0/\Lambda_0 = \mathbb{R}^n \oplus (\mathbb{R}^n)^*/\mathbb{Z}^n \oplus (\mathbb{Z}^n)^*$, $V_1 = \mathbb{R}/\mathbb{Z}$ and

$$\langle \cdot, \cdot \rangle : (\mathbb{Z}^n \oplus (\mathbb{Z}^n)^*) \otimes (\mathbb{Z}^n \oplus (\mathbb{Z}^n)^*) \to \mathbb{Z} \tag{7.44}$$
$$(v_1 + A_1, v_2 + A_2) \mapsto \iota_{v_1} A_2, \tag{7.45}$$

the corresponding Lie 2-group is denoted $T\mathbb{D}_n$ and plays a role in the description of T-Duality from [166, 207, 274].

Example 7.30. The same data from Example 7.29 can be used to construct for each $p \in \mathbb{N}$ a $(2p+2)$-group $\mathcal{T}$ which extends $B^p(V_0/\Lambda_0)$ by $B^{2p+1}(V_1/\Lambda_1)$. It is defined by

$$\begin{aligned}(B\mathcal{T})_n := \{&\{v^0_{i_0,\dots,i_{p+1}}\}, \{\lambda^0_{i_0,\dots,i_{p+2}}\}, \{[v^1_{i_0,\dots,i_{2p+2}}]\}) \subset V_0 \times \Lambda_0 \times V_1/\Lambda_1 \\ &\mid \delta v^0 = \lambda^0,\ [\delta v^1] = [\langle v^0, \lambda^0\rangle]\},\end{aligned} \tag{7.46}$$

where the indices belong to $[n]$ and the equations are defined similarly as in (7.31), suppresing the indices from the notation. Similar arguments as in Example 7.29 show that a $B^p(V_0/\Lambda_0)$-bundle with Chern class $c \in H^{p+2}(M, \Lambda_0)$ lifts to a $\mathcal{T}$-bundle if and only if $0 = \langle c \wedge c \rangle \in H^{2p+4}(M, \Lambda_1)$. This implies that when p is even, then two $(2p+2)$-groups constructed in this way are isomorphic if and only if the symmetric parts of the corresponding bilinear forms $\langle \cdot, \cdot \rangle$ coincide, and when p is odd then the same is true but for the skew-symmetric part of $\langle \cdot, \cdot \rangle$. Indeed, in these cases we have that for any $c \in H^{p+2}(M, \Lambda_1)$ the classes $\langle c \wedge c \rangle \in H^{2p+4}(M, \Lambda_1)$ coincide for the two bilinear forms, and so a coboundary relating them can be used to give an isomorphism between the corresponding $(2p+2)$-groups as in Example 7.29.

The 1-forms $\theta^0 \in \Omega^1((B\mathcal{T})_{p+1}, V_0)$, $\theta^1 \in \Omega^1((B\mathcal{T})_{2p+2}, V_1)$ defined by

$$\begin{aligned} \theta^0 &:= dv^0_{i_0 \dots i_{p+1}}, \\ \theta^1 &:= dv^1_{i_0 \dots i_{2p+2}} - \langle dv^0_{i_0, \dots, i_{p+1}}, v^0_{i_{p+1} \dots i_{2p+2}} \rangle, \end{aligned} \tag{7.47}$$

can be used to define a connection on a $\mathcal{T}$-bundle described by cocycle data $v^0_{i_0 \dots i_{p+1}}$, $\lambda^0_{i_0, \dots, i_{p+2}}$, $[v^1_{i_0 \dots i_{2p+2}}]$ as a sequence of differential forms

$$\begin{aligned} A^r_{i_0 \dots i_{p+1-r}} &\in \Omega^r(U_{i_0 \dots i_{p+1-r}}, V_0), \quad r = 1, \dots, p+1, \\ \Lambda^s_{i_0 \dots i_{2p+2-s}} &\in \Omega^s(U_{i_0 \dots i_{2p+2-s}}, V_1), \quad s = 1, \dots, 2p+2, \end{aligned} \tag{7.48}$$

such that

$$\delta A^1 = (v^0)^* \theta^0, \tag{7.49}$$

$$\delta A^r = dA^{r-1}, \qquad r = 2, \dots, p+1, \tag{7.50}$$

$$\delta \Lambda^1 = (v^1)^* \theta^1, \tag{7.51}$$

$$\delta \Lambda^s = d\Lambda^{s-1} + (-1)^{s+1} \langle A^s \wedge (v^0)^* \theta^0 \rangle, \qquad s = 2, \dots, p+2, \tag{7.52}$$

$$\delta \Lambda^s = d\Lambda^{s-1} + (-1)^{s+1} \langle dA^{p+1} \wedge A^{s-p-2} \rangle, \qquad s = p+3, \dots, 2p+2, \tag{7.53}$$

where δ denotes the Čech differential and the products $\langle \cdot \wedge \cdot \rangle$ are defined by taking the indices similarly as in (7.47). Its curvature is $(F, H) \in \Omega^{p+2}(M, V_0) \oplus \Omega^{2p+3}(M, V_1)$ defined locally by

$$F := dA^{p+1}_i, \qquad H := d\Lambda^{2p+2}_i + \langle dA^{p+1}_i \wedge A^{p+1}_i \rangle \tag{7.54}$$

and it satisfies the *Bianchi identity*

$$dF = 0, \qquad dH - \langle F \wedge F \rangle = 0. \tag{7.55}$$

The 6-group corresponding to $p = 2$, $V_0/\Lambda_0 = V_1/\Lambda_1 = \mathbb{R}/\mathbb{Z}$ and $\langle \cdot, \cdot \rangle : \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ the canonical product provides a model for prequantizing $M5$-branes [234]. The $(4k+2)$-groups for $p = 2k$, $k \geq 1$ and the same choice of V_0/Λ_0, V_1/Λ_1, $\langle \cdot, \cdot \rangle$ that gives $T\mathbb{D}_n$ in

Example 7.29 are called *higher T-duality groups* in [108], where their role in string and M-theory is discussed.

Example 7.31. Let $\mathcal{T}$ be the Lie 2-group constructed from $\Lambda_0 \subset V_0$, $\Lambda_1 \subset V_1$, $\langle\cdot,\cdot\rangle_0 : \Lambda_0 \otimes \Lambda_0 \to \Lambda_1$ as in Example 7.29. Given another vector space with a lattice $\Lambda_2 \subset V_2$, we construct a Lie 3-group $\mathcal{T}^2$ extending $\mathcal{T}$ by $B^2(V_2/\Lambda_2)$ from the data of a bilinear form $\langle\cdot,\cdot\rangle_1 : \Lambda_0 \otimes \Lambda_1 \to \Lambda_2$ such that

$$\langle u_0, \langle v_0, w_0\rangle_0\rangle_1 + \langle v_0, \langle u_0, w_0\rangle_0\rangle_1 = 0. \tag{7.56}$$

We also choose a section of the map of $\mathbb{Z}$-modules $\Lambda_0^* \otimes \Lambda_0^* \to \Lambda^2\Lambda_0^*$ defined by $B \mapsto B^{sk}$, where

$$B^{sk}(u,v) := B(u,v) - B(v,u). \tag{7.57}$$

Then for $\lambda_0 \in \Lambda_0$ we write

$$\langle\cdot, \langle\cdot, \lambda_0\rangle_0\rangle_1^{low} \in \Lambda_0^* \otimes \Lambda_0^* \otimes \Lambda_2 \tag{7.58}$$

for the corresponding bilinear form such that

$$\langle u_0, \langle v_0, \lambda_0\rangle_0\rangle_1 = \langle u_0, \langle v_0, \lambda_0\rangle_0\rangle_1^{low} - \langle v_0, \langle u_0, \lambda_0\rangle_0\rangle_1^{low}. \tag{7.59}$$

Then we define $\mathcal{T}^2$ by

$$\begin{aligned}(B\mathcal{T}^2)_n := \{(\{v^0_{ij}\}, \{\lambda^0_{ijk}\}, \{v^1_{ijk}\}, \{\lambda^1_{ijkl}\}, \{[v^2_{ijkl}]\}) \subset V_0 \times \Lambda_0 \times V_1 \times \Lambda_1 \times V_2/\Lambda_2 \\ \mid v^0_{ij} - v^0_{ik} + v^0_{jk} = \lambda^0_{ijk}, \\ v^1_{ijk} - v^1_{ijl} + v^1_{ikl} + v^1_{jkl} = \langle v^0_{ij}, \lambda^0_{jkl}\rangle_0 - \lambda^1_{ijkl}, \\ [v^2_{ijkl} - v^2_{ijkm} + v^2_{ijlm} - v^2_{iklm} + v^2_{jklm}] \\ = [\langle v^0_{ij}, \lambda^1_{jklm}\rangle_1 + \langle \lambda^0_{ijk}, \langle v^0_{ik}, \lambda^0_{klm}\rangle_0\rangle_1^{low} - \langle v^0_{ij}, \langle v^0_{jk}, \lambda^0_{klm}\rangle_0\rangle_1^{low}]\}.\end{aligned} \tag{7.60}$$

Given a $\mathcal{T}$-bundle $\mathcal{P} \to M$, the obstruction to lift it to a $\mathcal{T}^2$-bundle can be described as follows. First, recall from Example 7.29 that the topological class of $\mathcal{P}$ is characterized on a good open cover $\{U_i\}_{i\in I}$ of M by a Λ_0-cocycle $\lambda^0_{ijk} : U_{ijk} \to \Lambda_0$ defining a class $[\lambda^0] \in H^2(M, \Lambda_0)$ and a coboundary $\lambda^1_{ijkl} : U_{ijkl} \to \Lambda_1$ for $\langle \lambda^0 \wedge \lambda^0\rangle_0$; i.e., $(\delta\lambda^1)_{ijklm} = \langle \lambda^0_{ijk}, \lambda^0_{klm}\rangle_0$. Given this data, one can check that

$$-\langle \lambda^0_{ijk}, \lambda^1_{klmn}\rangle_1 + \langle \lambda^0_{ijk}, \langle \lambda^0_{ikl}, \lambda^0_{lmn}\rangle_0\rangle_1^{low} - \langle \lambda^0_{jkl}, \langle \lambda^0_{ijl}, \lambda^0_{lmn}\rangle_0\rangle_1^{low} \tag{7.61}$$

is a Čech cocycle defining a class in $H^5(M, \Lambda_2)$ which vanishes precisely when $\mathcal{P}$ can be lifted to a $\mathcal{T}^2$-bundle. The image of this class in $H^5(M, V_2)$ is represented in de Rham cohomology by $[-\langle F\wedge H\rangle_1]\in H^5(M,V_2)$, where $(F,H)\in\Omega^2(M,V_0)\oplus\Omega^3(M,V_1)$ is the curvature of any connection on the $\mathcal{T}$-bundle. This follows from the following discussion.

We define the 1-forms

$$\theta^0\in\Omega^1((B\mathcal{T}^2)_1,V_0),\quad \theta^1\in\Omega^1((B\mathcal{T}^2)_2,V_1),\quad \theta^2\in\Omega^1((B\mathcal{T}^2)_3,V_2)$$

by

$$\begin{aligned}\theta^0&:=dv^0_{01},\\ \theta^1&:=dv^1_{012}-\langle dv^0_{01},v^0_{12}\rangle_0,\\ \theta^2&:=dv^2_{0123}+\langle dv^0_{01},v^1_{123}\rangle_1-\langle v^0_{01},\langle dv^0_{01},\lambda^0_{123}\rangle_0\rangle_1^{low}.\end{aligned}\tag{7.62}$$

These satisfy

$$\begin{aligned}\delta\theta^0&=0, & d\theta^0&=0,\\ \delta\theta^1&=0, & d\theta^1&=\langle d_2^*\theta^0\wedge d_0^*\theta^0\rangle_0,\\ \delta\theta^2&=0, & d\theta^2&=-\langle dv^0_{01}\wedge dv^1_{123}\rangle_1-\langle dv^0_{01}\wedge\langle dv^0_{01},\lambda^0_{123}\rangle_0\rangle_1^{low}.\end{aligned}\tag{7.63}$$

Then we define a connection on a $\mathcal{T}^2$-bundle to be the data of

$$\begin{aligned}A_i\in\Omega^1(U_i,V_0),\quad \Lambda_{ij}\in\Omega^1(U_{ij},V_1),\quad B_i\in\Omega^2(U_i,V_1),\\ \Xi_{ijk}\in\Omega^1(U_{ijk},V_2),\quad \Sigma_{ij}\in\Omega^2(U_{ij},V_2),\quad C_i\in\Omega^3(U_i,V_2)\end{aligned}\tag{7.64}$$

satisfying the relations

$$A_j-A_i=(v^0_{ij})^*\theta^0,\tag{7.65}$$
$$\Lambda_{ij}-\Lambda_{ik}+\Lambda_{jk}=(v^1_{ijk})^*\theta^1,\tag{7.66}$$
$$B_j-B_i=d\Lambda_{ij}-\langle A_i\wedge(v^0_{ij})^*\theta^0\rangle_0,\tag{7.67}$$
$$\Xi_{ijk}-\Xi_{ijl}+\Xi_{ikl}-\Xi_{jkl}=(v^2_{ijk})^*\theta^2,\tag{7.68}$$
$$\Sigma_{ij}-\Sigma_{ik}+\Sigma_{jk}=d\Xi_{ijk}+\langle dv^0_{ij}\wedge\Lambda_{jk}\rangle_1+\langle dv^0_{ij}\wedge\langle dv^0_{ij},v^0_{jk}\rangle_0\rangle_1^{low},\tag{7.69}$$
$$C_j-C_i=d\Sigma_{ij}-\langle dv^0_{ij}\wedge B_j\rangle_1+\langle A_i\wedge\langle A_i\wedge dv^0_{ij}\rangle_0\rangle_1^{low}.\tag{7.70}$$

Its curvature is the triple $(F,H,G)\in\Omega^2(M,V_0)\oplus\Omega^3(M,V_1)\oplus\Omega^4(M,V_2)$ defined locally by

$$F:=dA_i,\tag{7.71}$$
$$H:=dB_i+\langle dA_i\wedge A_i\rangle_0,\tag{7.72}$$
$$G:=dC_i-\langle A_i\wedge dB_i\rangle_1,\tag{7.73}$$

and it satisfies the Bianchi identity

$$dF = 0, \tag{7.74}$$

$$dH - \langle F \wedge F \rangle_0 = 0, \tag{7.75}$$

$$dG + \langle F \wedge H \rangle_1 = 0. \tag{7.76}$$

In [117] we study a certain duality for 2-branes in M-theory which is closely related to U-Duality, and we model it in terms of Lie 3-groups $T_2\mathbb{D}_n^{F_3}$, $T_2\mathbb{D}_n^{F_3}$ which can be constructed as above from the following data.

1. $T_2\mathbb{D}_n^{F_3}$ is the Lie 3-group constructed from

$$V_0 = \mathbb{R}^n, \qquad V_1 = (\mathbb{R}^n)^*, \qquad V_2 = \mathbb{R},$$
$$\Lambda_0 = \mathbb{Z}^n, \qquad \Lambda_1 = (\mathbb{Z}^n)^*, \qquad \Lambda_2 = \mathbb{Z},$$

$$\langle v_1, v_2 \rangle_0 = 0, \qquad \langle v, A \rangle_1 = \iota_v A. \tag{7.77}$$

2. $T_2\mathbb{D}_n^{F_2}$ is the Lie 3-group constructed from

$$V_0 = \mathbb{R}^n \oplus \Lambda^2(\mathbb{R}^n)^*, \qquad V_1 = (\mathbb{R}^n)^*, \qquad V_2 = \mathbb{R},$$
$$\Lambda_0 = \mathbb{Z}^n \oplus \Lambda^2(\mathbb{Z}^n)^*, \qquad \Lambda_1 = (\mathbb{Z}^n)^*, \qquad \Lambda_2 = \mathbb{Z},$$

$$\langle v_1 + B_1, v_2 + B_2 \rangle_0 = \iota_{v_1} B_2, \qquad \langle v + B, A \rangle_1 = \iota_v A. \tag{7.78}$$

7.2.3 The ∞-categories of derived manifolds and derived differentiable stacks

There is an $(\infty, 1)$-category dMan_∞ which can be characterized axiomatically as the minimal $(\infty, 1)$-category in which manifolds can be embedded and in which fibered products always exist [76]; this is the framework of derived differential geometry. The results of [32, 75] imply that dMan_∞ can be presented in terms of the derived manifolds from Section 2.2.2.

Definition 7.32 ([32])**.** Let $\mathcal{M}^1 = (M^1, E^1, Q^1)$, $\mathcal{M}^2 = (M^2, E^2, Q^2)$ be derived manifolds. A morphism $(\varphi, \psi) : \mathcal{M}^1 \to \mathcal{M}^2$ is

1. a *fibration* if $\psi^1 : E^1 \to E^2$ is a submersion,
2. *étale* if $\psi_* : T\mathcal{M}^1 \to \varphi^* T\mathcal{M}^2$ is a quasi-isomorphism of chain complexes of vector bundles over $Z(\mathcal{M}^1)$,

3. a *weak equivalence* if it is étale and $\varphi : M^1 \to M^2$ restricts to a bijection $Z(M^1) \to Z(M^2)$.

The following theorem recaps the main results of [32] and [75].

Theorem 7.33 ([32, 75]). *The category dMan from Definition 2.17 is a category of fibrant objects in the sense of Definition 7.10 with the fibrations and weak equivalences from Definition 7.32. Its localization is the $(\infty, 1)$-category dMan$_\infty$.*

As it follows from Section 7.1.2, Theorem 7.33 states the following.

1. Given morphisms of derived manifolds

$$\begin{tikzcd} & \mathcal{M}^1 \arrow[d, "(\varphi_1,\psi_1)"] \\ \mathcal{M}^2 \arrow[r, "(\varphi_2,\psi_2)"] & \mathcal{N} \end{tikzcd} \tag{7.79}$$

   such that (φ_1, ψ_1) is a fibration, then the fibered product $\mathcal{M}^1 \times_{\mathcal{N}} \mathcal{M}_2$ exists. Indeed, we define this fibered product by the vector bundle $L := E^1 \times_{E^{\mathcal{N}}} E^2 \to M^1 \times_N M^2$, which satisfies $S^\bullet L[1]^* = S^\bullet E^1[1]^* \otimes_{S^\bullet E^{\mathcal{N}}[1]^*} S^\bullet E^2[1]^*$ and so it has a natural homological vector field Q, since tensor product is a coproduct in the category of dg-algebras.

2. Any morphism of graded manifolds $\mathcal{M}^1 \xrightarrow{f} \mathcal{N}$ factorizes as $\mathcal{M}^1 \xrightarrow{\lambda} \widetilde{\mathcal{M}}^1 \xrightarrow{f'} \mathcal{N}$, where λ is a weak equivalence and f' is a fibration. Thus, given a diagram (7.79) where (φ_i, ψ_i) are arbitrary morphisms, we may choose a factorization $(\varphi_1, \psi_1) = (\varphi_1', \psi_1') \circ \lambda_1$ and construct the fibered product of (φ_1', ψ_1') and (φ_2, ψ_2) as in 1. This is called the *homotopy fibered product* of (φ_1, ψ_1) and (φ_2, ψ_2), as it is only well-defined up to weak equivalences.

We show how to perform homotopy fibered products in dMan$_\infty$ with some examples.

Example 7.34. Let $\Phi : M \to E$ be a section of a vector bundle $\pi : E \to M$ and write $0 : M \to E$ for the zero section. We claim that the derived manifold constructed in Example 2.20 coincides with the homotopy fibered product $M\,_\Phi{\times}_0^h M$ of Φ and 0. This can be seen by factorizing $0 : M \to E$ as $M \xrightarrow{\lambda} M' \xrightarrow{0'} E$, where M' is the derived manifold associated to the vector bundle $\pi^* E[-2] \to E$ with homological vector field given by the identity section $E \to \pi^* E[-2]$, λ is defined by the zero section and $0'$ is the identity on E. In particular, one can easily show that when Φ is transversal to 0 and $rk(E) = \dim(M)$ then $M\,_\Phi{\times}_0^h M$ is weakly equivalent to the manifold given simply by the standard zero set of Φ.

Moreover, if there is a map of vector bundles $d : E \to F$ such that $d\Phi = 0$, then we claim that the corresponding derived manifold from Example 2.20 is obtained by defining first the homotopy fibered product $E_d := E\,{}_d\!\times_0^h M$, letting Φ_d be the map $M \to E_d$ induced by Φ and then taking the homotopy fibered product $Z := M\,{}_{\Phi_d}\!\times_0^h M$. This can be checked similarly as before, resolving the zero sections $0 : M \to F$ by the derived manifold $M' = (F, \pi^*F[-2], id)$ and the zero section $0 : M \to E_d$ by the derived manifolds $M'_d = (E, \pi^*E[-2] \oplus F[-2] \oplus F[-3]), Q)$, with Q defined by the identity section $E \to \pi^*E[-2]$ and the identity map $F[-2] \to F[-3]$.

Example 7.35. Let $i : X \to Y$ be an embedded submanifold. We construct a derived manifold modelling the fibered product $X \times_Y^h X$, called the *self-intersection* of X within Y. For this we use the tubular neighborhood theorem to obtain a neighborhood V of the zero section of the normal bundle $\pi : N(X) \to X$, defined by $N(X) := i^*TY/TX$, and a diffeomorphism $V \xrightarrow{\phi} U$ onto an open neighborhood $U \subset Y$ of $i(X)$ such that $i = \phi \circ 0$, for $0 : X \to V$ the zero section. Then we factorize i as $X \xrightarrow{\lambda} X' \xrightarrow{i'} Y$, where X' is the derived manifold associated to the vector bundle $\pi^*N(X)[-2] \to V$ with homological vector field given by the identity section $V \to \pi^*N(X)[-2]$, λ is defined by the zero section of $N(X)$ and i' is defined by ϕ. Then we can compute $X \times_Y^h X = X' \times_Y X$, which is the derived manifold with underlying vector bundle $N(X)[-2] \to X$ and 0 as homological vector field.

Finally, once we have a model for the $(\infty, 1)$-category dMan_∞ of derived manifolds, the construction of the $(\infty, 1)$-category dDiffSt of derived differentiable stacks is a straightforward analog of its algebraic geometric counterpart.

Definition 7.36. Let $\mathcal{M} := (M, E, Q)$ be a derived manifold. A *covering sieve* on $\mathcal{M}$ is a family $\{\mathcal{U}_i \to \mathcal{M}\}_{i\in I}$ of étale morphisms of derived manifolds such that the induced map $\sqcup_{i\in I} Z(\mathcal{U}_i) \to Z(\mathcal{M})$ is surjective.

For $m \in \mathbb{N}$, the $(\infty, 1)$-category of *derived differentiable m-stacks* is the $(\infty, 1)$-category of m-geometric stacks as in Definition 7.18, for $(A, \tau) = (\mathcal{A}, \tau')$ the $(\infty, 1)$-category dMan_∞ with the Grothendieck topology induced by the above covering sieves. The $(\infty, 1)$-category dDiffSt of *derived differentiable stacks* is the $(\infty, 1)$-category of stacks that are m-geometric for some $m \in \mathbb{N}$.

For $m \in \mathbb{N} \cup \{\infty\}$, a *derived Lie m-groupoid* is a simplicial derived manifold $\mathfrak{X} : \Delta^{op} \to$ dMan (cf. Definition 2.22) such that the restriction maps

$$\mathfrak{X}_n = \mathrm{Man}_\Delta(\Delta^n, \mathfrak{X}) \to \mathrm{Man}_\Delta(\Lambda_j^n, \mathfrak{X}) \tag{7.80}$$

are weak equivalences for $n > m$, $0 \le j \le n$ and fibrations for $1 \le n < \infty$, $0 \le j \le n$.

Remark 7.37. The right-hand side of (7.80) is a derived manifold, as it follows from the proof in [282] applied to the category dMan with the Grothendieck topology given by fibrations.

Since all homotopy fibered products exist in dMan_∞, it follows by general sheaf theory that dDiffSt is an enrichment of the category of manifolds in which all fibered products and quotients exist. Derived Lie m-groupoids provide examples of derived differentiable ∞-stacks by Proposition 7.21 but, as opposed to the case of (not derived) differentiable stacks, it is not necessarily true that all derived differentiable ∞-stacks arise in this way. It is easy to check that the simplicial derived manifolds from Examples 6.3, 6.4 and 6.5 are derived Lie 2-groupoids in the sense of Definition 7.36, and so they should actually be regarded as derived differentiable 2-stacks.

Chapter 8

Discussions, open problems and conclusions

8.1 Discussions

8.1.1 On the need for Maurer-Cartan forms

Maurer-Cartan forms on Lie 2-groups play a prominent role in our work. We proceed to discuss some thoughts on whether these are actually necessary and/or fundamental. For a Lie 2-group $\mathfrak{G}$ with Lie 2-algebra $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$, our approach requires choosing a Maurer-Cartan form on $\mathfrak{G}$ for the following purposes.

1. Defining a 1-shifted symplectic structure on $(\mathfrak{g}^* \xrightarrow{t_*^*} \mathfrak{h}^*)//\mathfrak{G}$ which can be used for defining Hamiltonian actions of $\mathfrak{G}$ (see Propositions 3.27 and 6.11).
2. Defining an L_∞-structure on $\mathfrak{h} \xrightarrow{t_*} \mathfrak{g}$ that maps into the Lie 2-algebra of vector fields on $\mathfrak{G}$, at least when $\mathfrak{G}$ arises from a multiplicative gerbe (see Proposition 3.51).
3. Defining connections on $\mathfrak{G}$-bundles, at least when $\mathfrak{G}$ arises from a multiplicative gerbe or from a Lie crossed module (see Sections 4.1.2 and 4.1.4).

Item 1 seems to be related through Theorem 3.43 to the observation in [66] that the Lie algebra of a Lie group acting on a Courant algebroid in a way that Hamiltonian reduction can be performed is naturally equipped with an Ad-invariant, symmetric bilinear form. Item 2 reflects the problem that an L_∞-algebroid structure on the tangent complex of a simplicial manifold can only be obtained after choosing connections [178]. Item 3 reflects the problem that a consistent, fully non-abelian notion of connection is for now only available for Lie 2-groups with an additional structure [220, 237, 273] (Chern-Simons forms or adjustments).

What we seem to be experiencing in all situations is that, while the tangent bundle of a Lie group is trivial, and the Maurer-Cartan form provides a canonical trivialization that behaves well with respect to the group product, a trivialization of the tangent complex of a Lie 2-group is only available after *choosing* a Maurer-Cartan form (which we have not proved to always be possible).

For example, as we mentioned in Example 2.33, the 1-shifted symplectic structure on $\mathfrak{g}^*//G$ for G a Lie group exists essentially because the Maurer-Cartan form on G induces an isomorphism $\mathfrak{g}^*//G = T^*[1](BG)$; thus, it seems reasonable to expect that the 1-shifted symplectic structure from Proposition 3.27 exists because a choice of Maurer-Cartan form on the Lie 2-group $\mathfrak{G}$ induces an isomorphism $(\mathfrak{g}^* \xrightarrow{t^*} \mathfrak{h}^*)//\mathfrak{G} = T^*[1]B\mathfrak{G}$ (although here we have not defined $T^*[1]B\mathfrak{G}$ rigorously, this is perhaps possible following [90, 223]). Similarly, the fact that a connection on the trivial G-bundle $P := M \times G \to M$ (thought of as a splitting of $TP/G \to TM$) is given by a $\mathfrak{g}$-valued 1-form on M follows from the isomorphism $TG = G \times \mathfrak{g}$, and so it is natural that a connection on a $\mathfrak{G}$-bundle can only be defined in terms of $\mathfrak{g}$- and $\mathfrak{h}$-valued forms once we have chosen an isomorphism $T\mathfrak{G} = \mathfrak{G} \times (\mathfrak{h} \to \mathfrak{g})$.

Following this line of thought, a natural objection to our approach is that we should just acknowledge the fact that $T\mathfrak{G}$ is not canonically trivial, instead of trying to fix this with a seemingly arbitrary choice of trivialization. Indeed, it is perhaps possible to define Hamiltonian actions of $\mathfrak{G}$ and connections on $\mathfrak{G}$-bundles in this way, probably leading to moment maps that take values in some non-trivial vector bundle with fiber $\mathfrak{g}^* \to \mathfrak{h}^*$ and to connections described locally by differential forms with values in some non-trivial vector bundle with fiber $\mathfrak{h} \to \mathfrak{g}$, and this is perhaps the only way to develop a natural theory that is valid for any Lie 2-group (or even higher Lie group).

A possible answer to this objection is that results such as Corollary 3.45 show that, for important families of Lie 2-groups, there is a canonical choice of Maurer-Cartan form, up to a notion of isomorphism that does not change the notion of Hamiltonian actions or connections. Moreover, our examples of higher Lie groups from Section 7.2.2 are also equipped with canonical Maurer-Cartan forms (see Section 8.2.1 for a discussion of how likely it is that this is in fact true for all higher Lie groups). Hence, in these situations, there is no reason not to use Maurer-Cartan forms, since they are essentially canonical, they simplify the presentation of definitions and examples with respect to an hypothetical abstract theory that does not trivialize $T\mathfrak{G}$, and their corresponding notion of connection behaves exactly as expected from the physics literature.

A more elaborate answer is that some of our results and the intuition from physics suggest that Maurer-Cartan forms are not just useful gadgets, but actually fundamental objects. The first observation in this direction is Proposition 3.52, which follows from the fact that the Maurer-Cartan form associated to a connective structure on a multiplicative $U(1)$-gerbe over G (defining a 2-shifted presymplectic structure on BG through the pairing $\langle\cdot,\cdot\rangle$) provides a prequantization of this 2-shifted presymplectic structure on $B\mathfrak{G} \to BG$. This is an important remark, as Theorem 2.29 suggests that the structure from Proposition 3.52 is essentially the one that leads to the shifted symplectic structures in our moduli spaces (Theorem 6.7 and 6.8), hence that the Maurer-Cartan form on $\mathfrak{G}$ is a fundamental object.

On the other hand, to illustrate the meaning of Maurer-Cartan forms in physics, recall from Theorem 3.48 that a multiplicative T-gerbe $\mathcal{G}$ over G defines a central extension of Lie 2-groups

$$1 \to BT \to \mathcal{G} \to G \to 1. \tag{8.1}$$

This can also be read as a sequence of differentiable 1-stacks, where $\mathcal{G} : \mathrm{Man} \to \mathrm{Gpd}$ is the stack with $\mathcal{G}(M)$ the groupoid of pairs (g, σ), where $g : M \to G$ is a smooth map and σ is a trivialization of $g^*\mathcal{G}$. Now a connective structure on $\mathcal{G}$ determines a lift of this sequence to a sequence of stacks

$$1 \to BT_\nabla \to \mathcal{G}_\nabla \to G \to 1, \tag{8.2}$$

where $BT_\nabla : \mathrm{Man} \to \mathrm{Gpd}$ is the stack with $BT_\nabla(M)$ the groupoid of T-bundles with connection over M. Namely, $\mathcal{G}_\nabla(M)$ is the groupoid of pairs (g, σ_∇) of a function $g : M \to G$ and σ_∇ a trivialization with connection of $g^*\mathcal{G}_\nabla$. Then $\mathcal{G}_\nabla$ is a stack of 2-groups, in the sense that pairs (g^1, σ^1_∇), (g^2, σ^2_∇) can be multiplied in a weakly associative way. Proposition 4.12 can be interpreted as saying that a principal bundle for $\mathcal{G}_\nabla$ is a $\mathcal{G}$-bundle with connective structure (hence the terminology used throughout the thesis).

At different points throughout the thesis we have decided to fix a connective structure on a $\mathcal{G}$-bundle for convenience (for example, for constructing the Atiyah algebroid in Theorem 4.23 or for presenting the moduli spaces in Theorems 6.7 and 6.8). From the discussion above, this can be interpreted as the idea of fixing a $\mathcal{G}_\nabla$-bundle as background geometry, and then seeing the data (A, B) as a connection on it, instead of fixing a $\mathcal{G}$-bundle and then seeing the data of a connective structure and (A, B) as a connection on it.

This is arguably the natural way to proceed from the point of view of 2-dimensional sigma-models. Indeed, a 2-form B on M is supposed to model the interaction of a physical field with a closed oriented surface $\Sigma \to M$, defined as $exp(2\pi i \int_\Sigma \Sigma^* B) \in U(1)$.

Since this expression does not change under $B \mapsto B + F$, for F the curvature of a line bundle, it leads naturally to determine that the physical field is actually given by locally defined 2-forms B_i that satisfy the gluing condition $B_i - B_j = F_{ij}$, for F_{ij} the curvature of local $U(1)$-bundles with connection $(P_{ij}, A_{ij}) \to U_{ij}$. In short, the 2-forms B_i describe the physical field, while the $U(1)$-bundles with connection (P_{ij}, A_{ij}) describe how it transforms. This suggests that the 2-group describing the local symmetries of these fields is BT_∇, instead of BT; thus why in the context in which B is coupled to an ordinary G-connection we need to consider extensions of the form (8.2) instead of extensions of the form (8.1). Using a related physics terminology, for our theory to have local 1-form symmetries, we must have a 'gauge group' containing the information of how to glue those 1-form symmetries, which is what (8.2) provides.

A philosophical discussion of whether a connective structure on a principal 2-bundle should be considered as part of the principal 2-bundle or as part of a connection on it might seem superficial, considering that for quantizing the theory all that we care about is the moduli space of all connections modulo gauge, and this coincides in both points of view by Proposition 4.17. However, it becomes more relevant once we impose equations of motion: our Theorem 5.26 (generalizing the abelian version from [142]) states that supersymmetric configurations in heterotic string theory are related to holomorphic $\mathcal{G}$-bundles with holomorphic connective structure (i.e. holomorphic $\mathcal{G}_\nabla$-bundles and not plain holomorphic $\mathcal{G}$-bundles). This can be compared with how holomorphic G-bundles are related to the Yang-Mills equations, but holomorphic connections on G-bundles do not often play a significant role in physics, and is important for constructing geometric structures on moduli spaces that are natural from the point of view of complex geometry.

We summarize our conclusion in the following slogan. *The local symmetries of higher gauge theory are described not by plain higher Lie groups, but rather by higher Lie groups equipped with Maurer-Cartan forms* (or some generalization of these, such as the cleavages in [265]). The observations above show in which sense this is the natural point of view in physics, and how it leads to a well-behaved mathematical theory.

8.1.2 On the interplay between higher geometry, generalized geometry and string theory

Our main results are obtained from a fruitful interaction between higher geometry and generalized geometry, inspired by string theory. We comment on some generalities about the relation between these theories. Firstly, we note that the bijection between connections on a principal 2-bundle and splittings of its associated Courant-Dorfman algebroid from Theorem 4.23 might suggest that higher geometry is not necessary to describe heterotic string theory coupled to supergravity, as long as one uses Courant algebroids. However, this is not entirely true. The reason is that, as already noted in [245], automorphisms of principal 2-bundles do not coincide with automorphisms of the corresponding Courant algebroid, leading to different moduli spaces when quotienting fields by symmetries. We proceed to justify that higher geometry describes the symmetries of 2-dimensional field theories in a way that is closer to the original literature on string theory than the approach from generalized geometry.

Consider, for example, the description of parallel transport along surfaces associated to a $U(1)$-gerbe with connection $(\mathcal{L}, \Lambda, B) \to M$ in [131] (see also [116, 186]). This assigns to each smooth map $\gamma : S^1 \to M$ a $U(1)$-torsor L_γ, and to each smooth map $\Sigma \to M$, where Σ is a connected, oriented, compact, 2-dimensional manifold, a morphism of $U(1)$ torsors $PT(\Sigma, B) : \otimes_{\gamma\in\#\partial\Sigma^+} L_\gamma \to \otimes_{\gamma\in\#\partial\Sigma^-} L_\gamma$, where $\#\partial\Sigma^+$ (resp. $\#\partial\Sigma^-$) denotes the set of positively (resp. negatively) oriented connected components of the boundary of Σ. For example, for $(\mathcal{L}, \Lambda)$ the trivial gerbe with trivial connective structure and $B \in \Omega^2(M)$ a connection on it, L_γ is the trivial torsor for every γ and $PT(\Sigma, B)$ is the automorphism of the trivial torsor given by $exp(2\pi i \int_\Sigma B)$.

An automorphism of the gerbe with connective structure $(\mathcal{L}, \Lambda)$ given by a $U(1)$-bundle with connection $(P, A) \to M$ induces for each $\gamma : S^1 \to M$ an automorphism $L_\gamma \to L_\gamma$ (given by acting with the holonomy of A around γ), and these automorphisms induce a commutative diagram relating $PT(\Sigma, B)$ and $PT(\Sigma, B + F_A)$ for any $\Sigma \to M$. In particular, if (P, A) and (P', A') are related by an isomorphism $(P, A) \to (P', A')$, then they induce the same automorphisms $L_\gamma \to L_\gamma$.

We learn two things from this construction. The first one is that the isomorphisms $(P, A) \to (P', A')$ (i.e., 2-isomorphisms of gerbes) are natural symmetries of the theory and so a mathematical framework that accounts for them is desirable. The second one, perhaps more interesting, is that, while a Kalb-Ramond field B can be equivalently modelled by a connection on $\mathcal{L}$ or by a splitting of its associated Courant algebroid E, from the point of view of parallel transport the natural symmetries of B are the automorphisms of $(\mathcal{L}, \Lambda)$; i.e., the $U(1)$-bundles with connection (P, A). Automorphisms of E are general closed 2-forms $b_0 \in \Omega^2_{cl}(M)$, acting on B as $B + b_0$, but this is not a symmetry of the parallel transport construction above unless b_0 has integral periods (i.e. $exp(2\pi i \int_\Sigma b_0) = 1$ for every Σ), which happens precisely when it is the curvature of a $U(1)$-bundle with connection (P, A). Restricting to *exact* or *Hamiltonian* symmetries of E as in, for example, [127], would amount to restricting b_0 to be exact, but this does not capture all the symmetries of the theory.

To sum up, we can conclude that higher geometry is faithful to physics on encoding the symmetries of 2-dimensional sigma-models, while generalized geometry is not. Thus, moduli spaces associated to these field theories are more rigorously defined in terms of the symmetries dictated by higher geometry. We can also see that this pays off with moduli spaces that behave better than those in [84, 125, 127], in the sense that the shifted symplectic structures from Theorems 6.7 and 6.8 only exist because we are taking this approach: the tangent complex of a similar moduli space not keeping track of 2-isomorphisms of $\mathcal{G}$-bundles will never have the right dimensions to admit a shifted symplectic structure, if the last term in the deformation complex of volume forms is included.

However, this is not to say that generalized geometry is of no use for dealing with these moduli spaces. In fact, our construction of simplicial derived manifolds in Section 6.1.2 relies completely on the construction of the Atiyah algebroid E of a principal 2-bundle $\mathcal{P}$ from Theorem 4.23. Indeed, Theorem 4.26 uses E to give a smooth structure on the gauge 2-group of $\mathcal{P}$, which is responsible for the 'higher' smooth structure on the moduli spaces, while the 'derived' structure follows from the deformation theory of sub-bundles of Courant-Dorfman algebroids developed in Section 6.1.1.

The preceeding discussion shows what higher geometry offers to string theory, and what generalized geometry offers to higher geometry. As for what string theory and higher geometry offer to generalized geometry, we can only make some general comments based on the work of other authors. A key construction is the result from [226, 243] that Courant algebroids are equivalent to degree 2 symplectic *dg*-manifolds, which is used in [242], based on the AKSZ construction from [1], to associate a 2-dimensional sigma-model (the *Courant sigma-model*) to any Courant algebroid $E \to M$. This implies that essentially every construction in generalized geometry has a counterpart within string theory, with the interpretation of generalized Kähler geometry as the imposition of $N = (2, 2)$ supersymmetry on the corresponding field theory [141] as its prime example.

This suggests that the construction of invariants in generalized geometry can be approached by a deep understanding of the Courant sigma-model and its quantization, and so that such invariants will naturally be of higher categorical nature. Examples of this line of thought can be traced back to Weinstein's observation that Poisson manifolds are integrated by symplectic groupoids [275], and the complex analog of this result [22], while a more recent construction is the interpretation of the generalized Kähler potential within a double symplectic groupoid in [3].

8.2 Open problems

8.2.1 On the theory of higher Lie groups and connections on higher principal bundles

In this thesis we have developed some original tools for dealing with Lie 2-groups. As we had concrete applications within the theory of moduli spaces in mind, some of these tools might seem ad hoc from a more abstract point of view, or only applicable to certain families of Lie 2-groups. Thus, a general problem to address in the future is to translate our tools to a language that allows for generalizations to all Lie 2-groups, or even to all higher Lie groups. A concrete problem in this direction is the following.

Problem 8.1. *Find a notion of Maurer-Cartan form on a Lie ∞-group $\mathfrak{G}$ that subsumes Definition 3.23 and which has the following properties.*

1. *The choice of a Maurer-Cartan form on $\mathfrak{G}$ leads to a good notion of connection on a $\mathfrak{G}$-bundle $\mathfrak{P} \to M$, based on local differential forms with values on the tangent complex of $B\mathfrak{G}$, and with an associated parallel transport map from the fundamental ∞-groupoid of M to $B\mathfrak{G}$.*

2. *The choice of a Maurer-Cartan form on $\mathfrak{G}$ leads to an L_∞-structure on the tangent complex of $B\mathfrak{G}$ that maps to an L_∞-algebra of vector fields on $\mathfrak{G}$.*

3. *The choice of a Maurer-Cartan form on $\mathfrak{G}$ leads to a good notion of exponential map for $\mathfrak{G}$.*

4. *Every Lie ∞-group admits a Maurer-Cartan form, and any two choices of Maurer-Cartan forms lead to equivalent notions of connections on $\mathfrak{G}$-bundles, L_∞-structures and exponential maps.*

It could seem that property 4 of Problem 8.1 contradicts Proposition 3.42, which describes an obstruction to the existence of connective structures on a multiplicative T-gerbe over G (living in $H^2_{gr,cont}(G, \mathfrak{g}^* \otimes \mathfrak{t})$), and identifies the space of inequivalent choices of connective structures as $H^1_{gr,cont}(G, \mathfrak{g}^* \otimes \mathfrak{t})$, which could be larger than a point. However, there could be a notion of Maurer-Cartan form solving Problem 8.1 which for the case of a multiplicative T-gerbe over G is more flexible than the notion of connective structures, allowing for a twist by an element in $H^2_{gr,cont}(G, \mathfrak{g}^* \otimes \mathfrak{t})$, and whose corresponding notion of connection is unaffected by $H^1_{gr,cont}(G, \mathfrak{g}^* \otimes \mathfrak{t})$.

We wish to emphasize here that there are consistent notions of connections on bundles for general Lie ∞-groups in the literature [110], with associated notions of parallel transport.

However, as already noticed in [235], the connections defined in this way are in general too strict, as they satisfy additional Bianchi identites (such as *fake flatness*) which are not expected from physics. This is only solved in [235] for Lie ∞-groups with an additional structure. This additional structure is axiomatized in the notion of adjustments for Lie ∞-algebras [230] and for strict Lie 2-groups [220] (we interpreted the latter in terms of Maurer-Cartan forms in Proposition 3.59). We expect a solution to Problem 8.1 to be a global, perhaps also weaker, version of the adjustments from [230]. A very recent alternative approach to define general connections uses Atiyah L_∞-algebroids defined abstractly as formal moduli problems [61]; perhaps Theorem 4.23 and Proposition 4.21, or the work of [111], help on relating both approaches.

A good candidate to solve Problem 8.1 is the notion of *cleavage* from [95, 265]. This is a connection-like object on the simplicial manifold $B\mathfrak{G}$, which provides a 'horizontal transport' of vectors along the products of $\mathfrak{G}$. A cleavage on a Lie ∞-group always exists and, as shown in [265], it induces an adjoint (homotopy) action on the tangent complex of $B\mathfrak{G}$ which is independent (up to homotopy) of the choice of cleavage. It is thus natural to expect that this can be used to define connections, as local differential forms with values on the tangent complex of $B\mathfrak{G}$ that are glued in terms of this adjoint action and the horizontal projections of the cleavage. We believe, however, that for a complex Lie 2-group it is not reasonable to expect existence of *holomorphic* cleavages in general.

Asides from the satisfaction of developing the theory in its most general possible form, solving Problem 8.1 could lead to applications in high-dimensional field theories. Examples of physical phenomena that have been modelled with higher Lie groups include the 3-forms and 6-forms that couple to fivebranes in M-theory [11, 236], topological defects in topological phases of matter [25] and higher form symmetries in axion electrodynamics [150]. However, the study of geometric structures in moduli spaces associated to these theories is yet to be done. Some aspects of M-theory have also been treated with the language of *exceptional generalized geometry* [23, 60, 154, 211], in which Courant algebroids are replaced by more complicated objects called *exceptional algebroids*. Thus, an interesting problem to approach is the following.

Problem 8.2. *Find a family of higher Lie groups $\mathfrak{G}$ with the following properties.*

1. *There is a good notion of connection on $\mathfrak{G}$-bundles.*
2. *A $\mathfrak{G}$-bundle with connection $\mathfrak{P}$ determines an exceptional algebroid E with a splitting of $\pi : E \to TM$.*
3. *Sections of E can be exponentiated to give automorphisms of $\mathfrak{P}$.*

In relation to Problem 8.2, it is worth mentioning that some of the examples of higher Lie groups that we presented in this thesis (namely, Examples 7.30 and 7.31), with a notion of connection on their associated principal bundles, play a role in M-theory. It is thus plausible that a solution to Problem 8.2 is related to these, as well as to the fivebrane 6-group from [236, 237].

Finally, recall that we also used Maurer-Cartan forms to define Hamiltonian actions of a Lie 2-group $\mathfrak{G}$ on a symplectic manifold M (see Proposition 6.11). Although the constructions of moduli spaces in Section 6.2.2 provide interesting examples of these, the arrows of $\mathfrak{G}$ play a rather trivial role, as M is a manifold. For a more interesting action, M must be replaced by a Lie groupoid $\mathfrak{M}$, which is the natural object where a Lie 2-group acts.

Problem 8.3. *Find a notion of Hamiltonian action of a Lie* 2*-group* $\mathfrak{G}$ *equipped with a Maurer-Cartan form on a quasi-symplectic groupoid* $\mathfrak{M}$ *such that a symplectic quotient* $\mathfrak{M}//_{\mu}\mathfrak{G}$ *can be defined. Provide natural examples of this construction.*

The formalism of [210] suggests how to proceed for solving Problem 8.3. Namely, one can construct in a similar way to Proposition 3.27 a 2-shifted symplectic structure on the simplicial manifold

$$\mathfrak{Y} := (\mathfrak{h}^*//\mathfrak{g}^*)//\mathfrak{G}, \tag{8.3}$$

where $\mathfrak{g}^*$ acts on $\mathfrak{h}^*$ by t_*^* and $\mathfrak{G}$ acts on $\mathfrak{h}^*//\mathfrak{g}^*$ by the adjoint action. It is such that the canonical morphisms $\mathfrak{h}^*//\mathfrak{g}^* \to \mathfrak{Y}$ and $B\mathfrak{G} \to \mathfrak{Y}$ have canonical Lagrangian structures. Then a Hamiltonian map for the action of a Lie 2-group $\mathfrak{G}$ on a quasi-symplectic groupoid $\mathfrak{M}$ is the following data.

1. A morphism with a Lagrangian structure $\mu : \mathfrak{M}//\mathfrak{G} \to (\mathfrak{h}^*//\mathfrak{g}^*)//\mathfrak{G}$.
2. An equivalence of 1-shifted symplectic derived stacks $\mathfrak{M} \cong \mathfrak{M}//\mathfrak{G}\,{}_{\mu}\!\times_{id}\mathfrak{h}^*//\mathfrak{g}^*$.

The corresponding symplectic reduction is the 1-shifted symplectic derived differentiable stack $\mathfrak{M}//\mathfrak{G}\,{}_{\mu}\!\times_{id}B\mathfrak{G}$. So the problem is actually to characterize this construction in terms of more classical data, perhaps for the case in which $\mathfrak{G}$ is a multiplicative gerbe with connective structure or an adjusted Lie crossed module. Whether there are already natural constructions in the literature that can be interpreted within this formalism is unknown to us, but it is possible that the work on reduction of Courant algebroids [66], homotopy moment maps [71] or reduction of symplectic graded manifolds [21, 65, 194] is related to this.

8.2.2 On the theory of moduli spaces in higher gauge theory

In Section 6.1.2 we construct simplicial derived manifolds presenting moduli spaces of flat connections, holomorphic structures and holomorphic structures with holomorphic connective structure on principal 2-bundles, and in Section 6.2.1 we show that we can equip the first two moduli spaces with shifted symplectic structures, if we also include a new parameter in the moduli representing, respectively, a smooth or holomorphic volume form on the base manifold. The most natural problem that arises from this thesis is thus to perform a similar construction with the third moduli space.

Problem 8.4. *Let $\mathcal{G}_\nabla$ be a holomorphic multiplicative $\mathbb{C}^*$-gerbe with holomorphic connective structure over a complex Lie group G such that the induced bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}$ from Theorem 3.43 is non-degenerate. Let X be a smooth, compact manifold with $\dim_{\mathbb{R}} X = 2n$ admitting $SU(n)$-structures, and let $\mathcal{P}_\nabla \to X$ be a smooth $\mathcal{G}_\nabla$-bundle. Construct a simplicial derived manifold with a $(2-n)$-shifted holomorphic symplectic structure parameterizing $SU(n)$-structures on X and holomorphic structures with holomorphic connective structure on $\mathcal{P}_\nabla$.*

The idea that $SU(n)$-structures are the parameters to include in order to obtain a shifted symplectic structure is motivated by two observations. The first one is that, as shown in [136], the natural deformation complex associated to a $SU(n)$-structure on X (i.e. a complex structure J and a holomorphic volume form $\Omega\in\Omega^{(n,0)}(X,\mathbb{C})$) is

$$\begin{aligned}\Omega^{(n-1,0)}(X,\mathbb{C}) &\xrightarrow{d} \Omega^{(n,0)+(n-1,1)}(X,\mathbb{C}) \xrightarrow{d} \Omega^{(n,1)+(n-1,2)}(X,\mathbb{C}) \xrightarrow{d} \dots \\ &\dots \xrightarrow{d} \Omega^{(n,n-2)+(n-1,n-1)}(X,\mathbb{C}) \xrightarrow{d} \Omega^{(n,n-1)+(n-1,n)}(X,\mathbb{C}) \xrightarrow{d} \Omega^{(n,n)}(X,\mathbb{C}),\end{aligned} \tag{8.4}$$

which is canonically dual to the deformation complex of holomorphic structures with holomorphic connective structure on a gerbe

$$\begin{aligned}\Omega^{0}(X,\mathbb{C}) &\xrightarrow{d} \Omega^{1}(X,\mathbb{C}) \xrightarrow{d} \Omega^{(0,2)+(1,1)}(X,\mathbb{C}) \xrightarrow{d} \dots \\ &\dots \xrightarrow{d} \Omega^{(0,n-1)+(1,n-2)}(X,\mathbb{C}) \xrightarrow{d} \Omega^{(0,n)+(1,n-1)}(X,\mathbb{C}) \xrightarrow{d} \Omega^{(1,n)}(X,\mathbb{C}).\end{aligned} \tag{8.5}$$

The second one is that, for $n=3$, this is exactly what the literature on heterotic string theory expects [13, 14, 94, 174]. These references also suggest how to construct the moduli when $n=3$ using the derived critical locus construction from Proposition 6.10. Namely, let M be the space of equivalence classes $[(\Omega,(A,B))]$, where Ω is a totally decomposable form on X in the sense of [153] (in particular, it defines an almost complex structure J_Ω on X), (A,B) is a connection on $\mathcal{P}_\nabla$, and we identify $(\Omega,(A,B))\sim(\Omega,(A,B)\cdot(a,b))$ for $(a,b)\in\Omega^{1,0}(ad\,P)\times_{\langle\cdot,\cdot\rangle}\Omega^{2,0}(ad\,P)$, where the types are with respect to J_Ω. Then the *superpotential* is the function $S : M \to \mathbb{C}$ defined by

$$S(\Omega, (A, B)) = \int_X H \wedge \Omega. \tag{8.6}$$

Its critical points are the $[(\Omega, (A, B))]$ such that J_Ω is integrable, Ω is a holomorphic volume form and (A, B) is an integrable 2-semiconnection. Moreover, it is invariant by the action of the 2-group $Aut(\mathcal{P}_\nabla)$ of automorphisms of $\mathcal{P}_\nabla$ covering possibly non-identity diffeomorphisms of X. If we had a smooth structure with a Maurer-Cartan form on $Aut(\mathcal{P}_\nabla)$, which should probably be modelled on the space of all sections of the Atiyah algebroid of $\mathcal{P}_\nabla$, then we could apply Proposition 6.10 to obtain the desired moduli space when $n = 3$.

The idea of working with derived moduli stacks is not to replace classical moduli spaces of stable objects, as discussed in Section 1.3, but rather to produce new geometric structures on them. For example, [34, 160, 164] construct a perverse sheaf on the moduli space of stable bundles over a Kähler Calabi-Yau threefold using the -1-shifted holomorphic symplectic structure from Section 2.3.3, which categorifies the Donaldson-Thomas invariants from [102, 259]. The existence of a well-behaved moduli space of stable objects, which can be obtained either by algebraic-geometric methods or from the Donaldson-Uhlenbeck-Yau theorem [100, 267], is crucial for their construction.

Similarly, the shifted holomorphic symplectic structure from Theorem 6.8, or the conjectural shifted holomorphic symplectic from Problem 8.4, could be used in a similar way to produce categorified invariants, provided we have a good notion of stable holomorphic principal 2-bundles (with holomorphic connective structure). An 'infinitesimal Donaldson-Uhlenbeck-Yau' theorem from [127] suggests that the gauge-theoretic analog of such stability condition is given by the Hull-Strominger system from Definition 6.14.

Problem 8.5. *Let K be a compact, connected Lie group and let $\mathcal{K} \to K$ be a multiplicative $U(1)$-gerbe whose associated pairing $\langle\cdot,\cdot\rangle : \mathfrak{k} \otimes \mathfrak{k} \to \mathbb{R}$ is non-degenerate. Let (X, Ω) be a complex manifold with a holomorphic volume form and let $\mathcal{P}_{h,\nabla_h} \to X$ be a $\mathcal{K}$-bundle with connective structure. Write $\mathcal{P}_\nabla$ for its complexification and let*

$$\begin{aligned} \mathcal{M} &:= \{((A_h, B_h), g) \in \mathcal{A}_+^{en}(\mathcal{P}_{h,\nabla_h}) \mid ((A_h, B_h), g) \textit{ solves } (6.90), (6.91)\}/Gauge(\mathcal{P}_{h,\nabla_h}), \\ \mathcal{H} &:= \mathcal{D}'_{int}(\mathcal{P}_\nabla)/Gauge(\mathcal{P}_\nabla). \end{aligned}$$

Prove that the map $\mathcal{M} \to \mathcal{H}$ given by Theorem 5.26 induces a homeomorphism between large open subsets $U \subset \mathcal{M}$ and $V \subset \mathcal{H}$ with manifold structure. Use this and the holomorphic shifted symplectic structure from Theorem 6.8 and Problem 8.4 to produce invariants.

The problem of relating the Hull-Strominger system to complex-geometric stability conditions has been approached with the language of Courant algebroids in [122, 123]; it is expected that the natural geometric flows of generalized geometry can help solve this problem [124]. A solution to Problem 8.5 could also relate our shifted holomorphic symplectic structures with the presymplectic structure from [127] by means of some hyperKähler structure, as sketched in Section 6.2.3. Note hyperKähler structures associated to the Hull-Strominger system have been considered in [122].

Other geometric structures on higher gauge moduli spaces could also arise from relating connections with representations of fundamental 2-groupoids by the parallel transport construction from Problem 8.1. This, together with a good description of fundamental 2-groupoids of manifolds in terms of generators and relations, could also give rise to global finite-dimensional descriptions for the moduli spaces, as in the case of ordinary Lie groups [31, 42]. This point of view has been adopted for studying moduli spaces of flat connections on principal 2-bundles with finite structure 2-group in [41].

It would also be interesting to quantize the shifted symplectic structures from Theorems 6.7 and 6.8 in some sense that recovers constructions in mathematical physics. For example, BV quantization in its different incarnations has been interpreted as a form of either deformation quantization or geometric quantization for shifted symplectic derived stacks [70, 145, 231] which we could try to apply. Another approach for quantizing these systems, based on the theory of vertex algebras [191], is also related to Courant algebroids [4, 5, 51, 148] and should therefore have an interpretation within the language of higher gauge theory. Perhaps a necessary step to perform these constructions is to understand first the representation theory of the main Lie 2-groups of interest [172, 173].

Finally, developing the representation theory of Lie 2-groups could also lead to an explanation and generalization of the construction of *instanton towers* in [93]. As shown there, a solution to the gravitino equations over a spin manifold M with a G-bundle $P \to M$ induces for each $k \in \mathbb{N}^{\geq 1}$ an instanton ∇_k on a vector bundle V_k of the form $V_k = T^*M \oplus \Lambda^2 V_{k-1}^* \oplus \Lambda^2 V_{k-1}^*$, with $V_1 = ad\, P$. A potential explanation for this is that there is a Lie 2-group $\mathcal{G}_\nabla$ extending G such that a solution to the gravitino equations can be understood as a special type of connection (A, B) on a principal 2-bundle $\mathcal{P}_\nabla \to M$ extending P, and that each instanton (V_k, ∇_k) is obtained from (A, B) by some *associated vector bundle* construction.

Problem 8.6. *Let $\mathfrak{G}$ be a Lie 2-group (perhaps with a Maurer-Cartan form) with a homotopy representation in the 2-vector space $V_{-1} \xrightarrow{d} V_0$ in the sense of [265]. Define, for each $\mathfrak{G}$-bundle $\mathfrak{P} \to M$ (with connection), an* associated vector bundle *(with connection) with the following properties.*

1. *For $\mathfrak{G}$ the Lie 2-group associated to a multiplicative T-gerbe $\mathcal{G}_\nabla$ with connective structure over G and its adjoint representation from Proposition 3.50, the associated vector bundle of a $\mathcal{G}_\nabla$-bundle $\mathcal{P}_\nabla$ is $Ker(\pi) \subset E$, for E the Atiyah algebroid of $\mathcal{P}_\nabla$ from Theorem 4.23.*

2. *For each Lie 2-group $\mathfrak{G}$, there is a sequence of Lie 2-group homomorphisms $\mathfrak{G} \stackrel{Ad}{\to} \mathfrak{G}_1 \stackrel{Ad}{\to} \mathfrak{G}_2 \stackrel{Ad}{\to} \ldots$ such that each $\mathfrak{G}_j$ has a natural homotopy representation in the Lie 2-algebra of $\mathfrak{G}_{j-1}$.*

3. *The tower of instantons from [93] is constructed from the associated vector bundle construction, applied to some initial connection on a $\mathcal{G}_\nabla$-bundle and the corresponding sequence of Lie 2-group homomorphisms starting with $\mathcal{G}_\nabla$.*

8.3 Conclusions

The main goal of this thesis has been the development and application of general tools for studying the geometry of moduli spaces that can be expressed in terms of equations for connections on categorified principal bundles. More precisely, we have studied principal bundles for Lie 2-groups $\mathfrak{G}$ that can be decomposed as central extensions [238] of the form $1 \to BT \to \mathfrak{G} \to G \to 1$ for Lie groups G, T with T abelian. This family of Lie 2-groups is relevant in string theory and supergravity [235, 253].

A mathematical study of this sort of moduli spaces has only been carried out until now for the case in which $\mathfrak{G} = BT$ for an abelian Lie group T [63, 109, 110, 192, 256]. One of the main ingredients of our work is the fact that, for $\mathfrak{G}$ a Lie 2-group as above, a $\mathfrak{G}$-bundle $\mathfrak{P} \to M$ with connection determines a transitive Courant algebroid $E \to M$ with a splitting. This was proven in [245] for a specific choice of Lie 2-group. Apart from extending their result to a larger family of Lie 2-groups (Theorem 4.23), we have used it for the following two purposes.

1. The construction of a smooth structure on the automorphism 2-group of $\mathfrak{P}$, modelled on the space of sections of a sub-bundle of E (Theorem 4.26).

2. The identification of the deformation theory for connections on $\mathfrak{P}$ with the deformation theory for sub-bundles of E (Section 6.1.2).

These two results constitute fundamental blocks in higher gauge theory, as they can be applied to the construction of any (derived) moduli space that can be expressed in terms of equations for connections on $\mathfrak{G}$-bundles. We have applied them in Section 6.1.2 to construct derived moduli spaces of flat $\mathfrak{G}$-connections, of holomorphic $\mathfrak{G}$-bundles and of holomorphic $\mathfrak{G}$-bundles with holomorphic connective structures. This last moduli space parameterizes supersymmetric configurations in heterotic string theory, as it follows from our Theorem 5.26, inspired by analogous results for Courant algebroids [127] and gerbes [142]. This means that our constructions are crucial for a rigorous mathematical understanding of string theory.

Besides constructing these derived moduli spaces, we have shown that they can be equipped with canonical shifted symplectic structures (Theorems 6.7 and 6.8). These are geometric structures that were introduced in [210] and which have attracted a lot of attention in recent years for their relation with the obtainment of invariants for manifolds [69, 160, 208], suggesting that our constructions also have potential applications in this respect. Moduli spaces of flat connections and holomorphic structures on ordinary principal bundles also have shifted symplectic structures which generalize work of Atiyah-Bott [15] and Mukai [196]. While these have inspired our results, an interesting difference is that, in order to be equipped with a shifted symplectic structure, our moduli spaces need to include an additional parameter representing a volume form on the base manifold. This agrees with the expectations from string theory, where this volume form can be interpreted as a dilaton.

We wish to emphasize that, for $\mathfrak{G}$ a Lie 2-group, defining connections on $\mathfrak{G}$-bundles is already a subtle point in the literature. Over the last fifteen years, different authors [220, 235, 273] have managed to formalize previous work on supergravity [38, 81] to obtain a satisfactory notion of connection on $\mathfrak{G}$-bundles, whenever $\mathfrak{G}$ arises from either a multiplicative gerbe with connection or from a Lie crossed module with an adjustment. However, there is still no conceptual framework for defining connections on fully general Lie 2-groups in such a way that the work of [220, 235, 273] is recovered as a special case. This is a fundamental problem, as it shows a conceptual insufficiency in our understanding of connections.

We have also contributed to this general problem by proving that the different approaches from [273] and [220] are equivalent. Moreover, we have shown that connections on multiplicative gerbes and adjustments on Lie crossed modules are both examples of a more general object that can be defined for any Lie 2-group, and which we have called *Maurer-Cartan forms*. These play a very important role in all our main results, and we expect that they can be used in the future for defining connections on principal bundles for arbitrary Lie 2-groups.

8.4 Conclusiones

El objetivo principal de esta tesis ha sido el desarrollo y la aplicación de herramientas generales para el estudio de la geometría de espacios de móduli que pueden ser descritos en términos de ecuaciones para conexiones en fibrados principales categorificados. Concretamente, hemos estudiado fibrados principales para 2-grupos de Lie $\mathfrak{G}$ que pueden ser descompuestos como extensiones centrales [238] de la forma $1 \to BT \to \mathfrak{G} \to G \to 1$ para grupos de Lie G, T con T abeliano. Esta familia de 2-grupos de Lie es relevante en teoría de cuerdas y supergravedad [235, 253].

El estudio matemático de este tipo de espacios de móduli solo se ha llevado a cabo hasta ahora en el caso en que $\mathfrak{G} = BT$ para un grupo abeliano T [63, 109, 110, 192, 256]. Uno de los ingredientes principales de nuesto trabajo es el hecho de que, para $\mathfrak{G}$ un 2-grupo de Lie del tipo mencionado anteriormente, un $\mathfrak{G}$-fibrado $\mathfrak{P} \to M$ con una conexión determina un algebroide de Courant transitivo $E \to M$ con una escisión. Esto fue demostrado en [245] para un 2-grupo concreto. Además de extender este resultado para toda una familia de 2-grupos de Lie (Teorema 4.23), lo hemos utilizado para los siguientes propósitos.

1. La construcción de una estructura suave en el 2-grupo de automorfismos de $\mathfrak{P}$, modelada en el espacio de secciones de un sub-fibrado de E (Teorema 4.26).
2. La identificación de la teoría de deformaciones de conexiones en $\mathfrak{P}$ con la teoría de deformaciones de sub-fibrados de E (Sección 6.1.2).

Estos dos resultados constituyen pilares fundamentales para la teoría gauge de tipo superior, ya que pueden ser aplicados a la construcción de cualquier espacio de móduli derivado que pueda expresarse en términos de conexiones en $\mathfrak{G}$-fibrados. Nosotros los hemos aplicado en la Sección 6.1.2 para construir espacios de móduli derivados de $\mathfrak{G}$-conexiones planas, $\mathfrak{G}$-fibrados holomorfos y $\mathfrak{G}$-fibrados holomorfos con estructuras conectivas holomorfas. Este último espacio de móduli parametriza configuraciones supersimétricas en teoría de cuerdas heterótica, como se sigue de nuestro Teorema 5.26, inspirado por resultados análogos para algebroides de Courant [127] y gerbes [142]. Por tanto, nuestras construcciones son cruciales para entender la teoría de cuerdas de manera matemáticamente rigurosa.

Además de construir estos espacios de móduli derivados, hemos demostrado que están equipados con estructuras simplécticas desplazadas canónicas (Teoremas 6.7 y 6.8). Estas son estructuras geométricas que fueron introducidas en [210] y que han atraído mucha atención en los últimos años por su relación con la obtención de invariantes para variedades [69, 160, 208], sugeriendo que nuestras construcciones también podrían ser aplicadas para este propósito. Los espacios de móduli de conexiones planas y estructuras holomorfas en fibrados principales ordinarios también tienen estructuras simplécticas desplazadas que generalizan trabajos de Atiyah-Bott [15] y Mukai [196]. Si bien esto ha inspirado nuestros resultados, una diferencia interesante es que, para aceptar estructuras simplécticas desplazadas, nuestros espacios de móduli deben incluir un parámetro adicional representando una forma de volumen en la variedad base. Esto se corresponde con las previsiones de teoría de cuerdas, donde esta forma de volumen puede ser interpretada como un dilatón.

Queremos destacar que, para $\mathfrak{G}$ un 2-grupo de Lie, incluso definir conexiones en $\mathfrak{G}$-fibrados constituye ya una tarea sutil en la literatura. En los últimos quince años, distintos autores [220, 235, 273] han logrado formalizar trabajos previos sobre supergravedad [38, 81] para obtener una noción satisfactoria de conexión en un $\mathfrak{G}$-fibrado, si $\mathfrak{G}$ puede construirse a partir de un gerbe multiplicativo con conexión o de un módulo cruzado con un ajuste. Sin embargo, no existe aún un paradigma conceptual capaz de definir conexiones para 2-grupos de Lie totalmente generales que recupere el trabajo de [220, 235, 273] como un caso particular. Este es un problema fundamental, ya que muestra una deficiencia conceptual en nuestra comprensión de las conexiones.

También hemos contribuido a este problema general demostrando que los distintos enfoques de [273] y [220] son equivalentes. Además, hemos demostrado que tanto conexiones en gerbes multiplicativos como ajustes en módulos cruzados son ejemplos de objetos más generales que pueden definirse para cualquier 2-grupo de Lie, a los que hemos llamado *formas de Maurer-Cartan*. Estas juegan un papel fundamental en todos nuestros resultados, y confiamos en que puedan usarse en el futuro para definir conexiones en fibrados principales para 2-grupos de Lie arbitrarios.

Bibliography

[1] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronski, *The Geometry of the Master Equation and Topological Quantum Field Theory*, Int. J. Mod. Phys. **12** 7 (1997).

[2] E. Aldrovandi, *Hermitian-holomorphic (2)-Gerbes and tame symbols*, Journal of Pure and Applied Algebra **200** (2003).

[3] D. Álvarez, M. Gualtieri, and Y. Jiang, *Symplectic double groupoids and the generalized Kähler potential*, arXiv:2407.00831 [math.DG] (2024).

[4] L. Álvarez-Cónsul, A. De Arriba de la Hera, and M. Garcia-Fernandez, *(0,2) Mirror symmetry on homogeneous Hopf surfaces*, International Mathematics Research Notices (2024) 2 1211-1298.

[5] L. Álvarez-Cónsul, A. De Arriba de la Hera, and M. Garcia-Fernandez, *Vertex algebras from the Hull-Strominger system*, arXiv:2305.06836 [math.DG].

[6] B. Anchouche, and I. Biswas, *Einstein-Hermitian Connections on Polystable Principal Bundles over a Compact Kähler Manifold* American Journal of Mathematics **123** (2001) 2 207-228.

[7] M. André, *Homologie des algebres commutatives*, Actes du Congres International des Mathématiciens (Nice, 1970) **1** Gauthier-Villars, Paris (1971) 301-308.

[8] M. Anel and D. Calaque, *Shifted symplectic reduction of derived critical loci*, Advances in Theoretical and Mathematical Physics, **26** (2022) 1543-1583.

[9] M. Artin, *Grothendieck Topologies*, Notes on a Seminar by M. Artin. Harvard University, Dep. of Mathematics. (1962).

[10] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974) 165-189.

[11] P. Aschieri and B. Jurčo, *Gerbes, M5-brane anomalies and E8 gauge theory*, JHEP **10** (2004) 068.

[12] P. Aschieri, L. Cantini, and B. Jurčo, *Nonabelian Bundle Gerbes, Their Differential Geometry and Gauge Theory*, Communications in Mathematical Physics **254** (2004) 367–400.

[13] A. Ashmore, X. de la Ossa, R. Minasian, C. Strickland-Constable, and E. E. Svanes, *Finite deformations from a heterotic superpotential: holomorphic Chern-Simons and an L_∞ algebra*, Journal of High Energy Physics, **10** (2018).

[14] A. Ashmore, C. Strickland-Constable, D. Tennyson, and D. Waldram, *Heterotic backgrounds via generalised geometry: moment maps and moduli*, Journal of High Energy Physics **11** (2020).

[15] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philosophical Transactions of the Royal Society London A **308** (1983) 523–615.

[16] J. C. Baez, *Higher Yang-Mills Theory*, arXiv:hep-th/0206130 (2002).

[17] J. C. Baez and J. Dolan, *Higher-dimensional algebra III. n-categories and the algebra of opetopes*, Adv. Math. **135** (1998), 145-206.

[18] J. C. Baez, A. S. Crans, U. Schreiber, and D. Stevenson, *From loop groups to 2-groups*, Homology, Homotopy and Applications, **9** (2007).

[19] J. C. Baez and A. D. Lauda, *Higher–dimensional algebra. V: 2–groups*, Theory and Applications of Categories **12** 423–491.

[20] J. C. Baez and U. Schreiber, *Higher gauge theory*, Contemp. Math. **431** (2007) 7.

[21] P. Baguis and T. Stavracou, *Marsden-Weinstein reduction on graded symplectic manifolds*, J. Math. Phys. **38** (1997) 1670-1684.

[22] M. Bailey and M. Gualtieri, *Integration of generalized complex structures*, J. Math. Phys. **64** (2023).

[23] D. Baraglia, *Leibniz algebroids, twistings and exceptional generalized geometry*, Journal of Geometry and Physics **62** (2012) 903-934.

[24] D. Baraglia and P. Hekmati, *Transitive Courant Algebroids, String Structures and T-duality*, Advances in Theoretical and Mathematical Physics, **19** (2015) 613–672.

[25] M. Barkeshli, Y.-A. Chen, S.-J. Huang, R. Kobayashi, N. Tantivasadakarn, and G. Zhu, *Codimension-2 defects and higher symmetries in (3+1)D topological phases*, SciPost Phys. **14** (2023) 065.

[26] C. Barwick, $(\infty, n)-$*Cat as a closed model category*, PhD Thesis at University of Pennsylvania (2005).

[27] C. Barwick and C. Schommer-Pries, *On the unicity of the theory of higher categories*, J. Amer. Math. Soc. **34** (2021) 1011-1058.

[28] I. A. Batalin and G. A. Vilkovisky, *Gauge algebra and quantization*, Phys. Lett. B **102** (1981) 1 27-31.

[29] M. Batanin, *Monoidal globular categories as a natural environment for the theory of weak n-categories*, Advances in Mathematics **136** 1 (1998), 39-103.

[30] C. Becchi, A. Rouet, and R. Stora, *Renormalization of gauge theories*, Ann. Phys. **98** (1976) 287.

[31] K. Behrend, I. Ciocan-Fontanine, J. Hwang, and M. Rose, *The derived moduli space of stable sheaves*, Algebra Number Theory **8** (2014) 4 781-812.

[32] K. Behrend, H.-Y. Liao, and P. Xu, *Derived Differentiable Manifolds*, arXiv:2006.01376 [math.DG] (2020).

[33] K. Behrend and P. Xu, *Differentiable Stacks and Gerbes*, J. Symplectic Geom. **9** (2011) 285-341.

[34] O. Ben-Bassat, C. Brav, V. Bussi, and D. Joyce, *A Darboux theorem for shifted symplectic structures on derived Artin stacks, with applications*, Geometry and Topology **19** (2015) 1287-1359.

[35] J. Bénabou, *Introduction to Bicategories*, Lecture Notes in Mathematics **47**, Springer-Verlag (1967).

[36] M. Benini, P. Safranov, and A. Schenkel, *Classical BV formalism for group actions*, Communications in Contemporary Mathematics **25** 1 (2021).

[37] F. A. Berezin and D. A. Leites, *Supermanifolds*, Dokl. Akad. Nauk SSSR **224** (1975) 3 505-508.

[38] E. Bergshoeff, M. de Roo, B. de Wit, and P. van Nieuwenhuizen, *Ten-dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields*, Nucl. Phys. B **195** (1982) 97.

[39] K. I. Berktav, *On Shifted Contact Derived Artin Stacks*, arXiv:2401.03334 [math.AG] (2024).

[40] D. Berwick-Evans and E. Lerman, *Lie 2-Algebras Of Vector Fields*, Pacific Journal of Mathematics, **309** (2020) 1 1-34.

[41] D. Berwick-Evans, E. Cliff, L. Murray, A. Nakade, and E. Phillips *String structures, 2-group bundles, and a categorification of the Freed-Quinn line bundle*, arXiv:2110.07571 [math.AT] (2021).

[42] F. Bischoff, *The derived moduli stack of logarithmic flat connections*, arXiv:2301.00962 [math.AG] (2023).

[43] F. Bischoff, M. Gualtieri, and M. Zabzine, *Morita equivalence and the generalized Kähler potential*, J. Diff. Geom. **121** (2018) 2.

[44] J. Boardman and R. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Math. **347**, Springer-Verlag (1973).

[45] D. Borisov, *Derived manifolds and Kuranishi models*, `arXiv:1212.1153 [math.DG]` (2012).

[46] D. Borisov and J. Noel, *Simplicial approach to derived differential manifolds*, arXiv:1112.0033 [math.DG] (2011).

[47] D. Borisov and D. Joyce, *Virtual fundamental classes for moduli spaces of sheaves on Calabi-Yau four-folds*, Geometry and Topology **21** (2017) 3231-3311.

[48] R. Bott, *On the Chern-Weil homomorphism and the continuous cohomology of Lie-groups*, Advances in Mathematics, **11** (1973) 289-303.

[49] R. Bott, H. Shulman, and J. Stasheff, *On the de Rham theory of certain classifying spaces*, Advances in Mathematics, **20** (1976), 43-56.

[50] A. Bousfield and V. Gugenheim, *On PL deRham theory and rational homotopy type*, Memoirs of the AMS **179** (1976).

[51] P. Bressler, *Vertex algebroids I*, `arXiv:math/0202185 [math.AG]`.

[52] P. Bressler, *The first Pontryagin class*, Compositio Mathematica, **143** (2007) 1127-1163.

[53] P. Bressler and A. Chervov, *Courant Algebroids*, Journal of Mathematical Science **128**, (2005) 3030-3053.

[54] K. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics **87**, Springer (1982).

[55] R. Brown and C. B. Spencer, *G-groupoids, crossed modules, and the classifying space of a topological group*, Proc. Kon. Akad. v. Wet. **79** (1976), 296-302.

[56] J. L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Modern Birkhäuser Classics (1993).

[57] J. L. Brylinski, *Holomorphic gerbes and the Beilinson regulator*, Astérisque, **226** (1994), 145-174.

[58] J. L. Brylinski, *Differentiable cohomology of gauge groups*, arXiv:math/0011069 [math.DG] (2000).

[59] J. L. Brylinski, *Gerbes on complex reductive Lie groups*, arXiv:math/0002158 [math.DG].

[60] M. Bugden, O. Hulik, F. Valach, and D. Waldram, *Exceptional Algebroids and Type IIB Superstrings*, Fortschritte der Physik **70** (2022) 1.

[61] S. Bunk, *Infinitesimal Higher Symmetries and Higher Connections (j.w. with L. Müller, J. Nuiten, and R. J. Szabo)*, Slides for a talk at New York University at Abu Dhabi (2023), available here.

[62] S. Bunk, L. Müller, and R. J. Szabo, *Smooth 2-Group Extensions and Symmetries of Bundle Gerbes*, Commun. Math. Phys. **384** (2021) 1829-1911.

[63] S. Bunk and C. S. Shahbazi, *Higher Geometric Structures on Manifolds and the Gauge Theory of Deligne Cohomology*, arXiv:2304.06633 [math.DG] (2023).

[64] U. Bunke, *String Structures and Trivialisations of a Pfaffian Line Bundle*, Communications in Mathematical Physics **307**, (2011) 675-712.

[65] H. Bursztyn, A. S. Cattaneo, R. A. Mehta, and M. Zambon, *Graded geometry and generalized reduction*, arXiv:2306.01508 [math.SG].

[66] H. Bursztyn, G. R. Cavalcanti, and M. Gualtieri, *Reduction of Courant algebroids and generalized complex structures*, Advances in Mathematics **211** (2007) 726-765.

[67] D. Calaque, *Shifted cotangent stacks are shifted symplectic*, Annales de la Faculté des sciences de Toulouse. Mathématiques. **28** 1 (2019) 67-90.

[68] D. Calaque, *Derived Stacks in Symplectic Geometry. in: New Spaces in Physics*, Cambridge University Press (2021) 155-201.

[69] D. Calaque, R. Haugseng, and C. Scheimbauer, *The AKSZ Construction in Derived Algebraic Geometry as an Extended Topological Field Theory*, To appear in Memoirs of the AMS, arXiv:2108.02473 [math.CT].

[70] D. Calaque, T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted Poisson Structures and deformation quantization*, Journal of Topology, London Mathematical Society **10** (2017) 2 483-584.

[71] M. Callies, Y. Frégier, C. L. Rogers, and M. Zambon, *Homotopy moment maps*, Advances in Mathematics **303** (2016) 954-1043.

[72] P. Candelas, X. de la Ossa, and X. J. McOrist, *A Metric for Heterotic Moduli*, Commun. Math. Phys. **356** (2017) 567-612.

[73] P. Candelas, X. de la Ossa, X. J. McOrist, and R. Sisca, *The Universal Geometry of heterotic vacua*, J. High Energ. Phys. **2019** (2019) 38.

[74] P. Candelas, G. Horowitz, A. Strominger, and E. Witten, *Vacuum configurations for superstrings*, Nuclear Physics B **258** (1985) 46-74.

[75] D. Carchedi, *Derived Manifolds as Differential Graded Manifolds*, arXiv:2303.11140 [math.DG] (2023).

[76] D. Carchedi and P. Steffens, *On the universal property of derived manifolds*, arXiv:1905.06195 [math.AT].

[77] A. Carey, S. Johnson, and M. Murray, *Holonomy on D-Branes*, Journal of Geometry and Physics **52** (2004) 2 186-216.

[78] A. Carey, S. Johnson, M. Murray, D. Stevenson, and B. L. Wang, *Bundle gerbes for Chern Simons and Wess Zumino Witten theories*, Communications in Mathematical Physics **259** (2005) 577–613.

[79] H. Cartan, *La transgression dans un groupe de Lie et dans un espace fibré principal*, Colloque de topologie (espaces fibrés). Bruxelles. (1950) 15–27.

[80] A. S. Cattaneo and N. Moshayedi, *Introduction to the BV-BFV formalism*, Reviews in Mathematical Physics **32** (2020).

[81] G. F. Chapline and N. S. Manton, *Unification of Yang Mills theory and supergravity in ten dimensions*, Phys. Lett. B **120** (1983) 105.

[82] D. S. Chatterjee, *On Gerbs*, PhD thesis, Trinity College, Cambridge. (1998).

[83] E. Cheng and A. Lauda, *Higher-Dimensional Categories. An Illustrated Guidebook.*, University of Sheffield (2004).

[84] A. Clarke, M. Garcia-Fernandez, and C. Tipler, *T-Dual solutions and infinitesimal moduli of the G_2-Strominger system*, arXiv:2005.09977 [math.DG] (2020).

[85] B. L. Collier, *Infinitesimal Symmetries of Dixmier-Douady Gerbes*, arXiv:1108.1525 [math.DG] (2011).

[86] K. Costello, *Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4*, Pure Appl. Math. Q. **9** (2013) 1 73-165.

[87] T. Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990) 631-661.

[88] M. Cueca, *The geometry of graded cotangent bundles*, Journal of Geometry and Physics **161** (2021).

[89] M. Cueca, F. Dorsch, R. Sjamaar, and C. Zhu, *Shifted symplectic structures on derived higher Lie groupoids*, In preparation.

[90] M. Cueca and S. Ronchi, *The 2-cotangent of a Lie 2-groupoid*, In preparation.

[91] M. Cueca and J. Schnitzer, *Deformations of Lagrangian Q-submanifolds*, arXiv:2309.05580 [math.SG].

[92] M. Cueca and C. Zhu, *Shifted symplectic higher Lie groupoids and classifying spaces*, Adv. Math. **413** (2023).

[93] A. A. da Silva Jr., M. Garcia-Fernandez, J. D. Lotay, and H. N. Sá Earp, *Coupled G2-instantons* arXiv:2404.12937 [math.DG]

[94] X. de la Ossa, M. Larfors, and E. E. Svanes, *The Infinitesimal Moduli Space of Heterotic G2 Systems*, Commun. Math. Phys. **360** (2018) 727-775.

[95] M. del Hoyo and G. Trentinaglia, *Simplicial vector bundles and representations up to homotopy*, arXiv:2109.01062v1 [math.CT] (2023).

[96] P. Deligne, *Théorie de Hodge : III*, Publications Mathématiques de l'IHÉS, **44** (1974), 5-77.

[97] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Publications Mathématiques de l'IHÉS **36** (1969) 75-109.

[98] G. A. Demessie and C. Sämann, *Higher gauge theory with string 2-groups*, Advances in Theoretical and Mathematical Physics **21** (2017) 1895–1952.

[99] D. Djounvouna and D. Krepski, *Infinitesimal symmetries of bundle gerbes and Courant algebroids*, arXiv:2304.05347 [math.DG].

[100] S. K. Donaldson, *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proceedings of the London Mathematical Society, **50** (1985), 1-26.

[101] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-manifolds*, Oxford mathematical monographs (1990).

[102] S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions*, In: The Geometric Universe: Science, Geometry. Oxford University Press. (1998).

[103] I. Ya. Dorfman, *Dirac structures of integrable evolution equations*, Phys. Lett. A **125** (1987) 240-246.

[104] W. Dwyer and D. Kan, *Simplicial localizations of categories*, J. Pure Appl. Algebra **17** 3 (1980) 267-284.

[105] C. Ehresmann, *Catégories topologiques et categories différentiables*, Colloque de Géométrie différentielle globale, Bruxelles, C.B.R.M., (1959) 137-150.

[106] S. Eilenberg and S. Maclane, *On the groups* $H(\Pi, n)$, Ann. Math. (2) **58** (1953) 55-106.

[107] M. Fairon, *Introduction to graded geometry*, European Journal of Mathematics **3** (2017) 208-222.

[108] D. Fiorenza, H. Sati, and U. Schreiber, *Higher T-duality of super M-branes*, Advances in Theoretical and Mathematical Physics **24** 3 (2020) 621-708.

[109] D. Fiorenza, C. L. Rogers, and U. Schreiber, *Higher* $U(1)$*-gerbe connections in geometric prequantization*, Rev. Math. Phys. **26** (2016) 72.

[110] D. Fiorenza, U. Schreiber, and J. Stasheff, *Cech cocycles for differential characteristic classes An infinity Lie theoretic construction*, Adv. Theor. Math. Phys. **16** (2012).

[111] S. R. Fischer, M. J. Farahani, H. Kim, and C. Sämann, *Adjusted Connections I: Differential Cocycles for Principal Groupoid Bundles with Connection*, arXiv:2406.16755 [math.DG].

[112] J. Fogarty, F. Kirwan, and D. Mumford, *Geometric Invariant Theory*, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Springer Berlin Heidelberg (1965).

[113] D. Freed, *Higher Algebraic Structures and Quantization*, Commun. Math. Phys. **159** (1994) 343-398.

[114] D. Freed and E. Witten, *Anomalies in String Theory with D-Branes*, Asian J. Math. **3** (1999) 4 819-852.

[115] Y. Frégier and M. Zambon, *Simultaneous deformations and Poisson geometry*, Compositio Mathematica, **151** (2015) 1763-1790.

[116] J. Fuchs, T. Nikolaus, C. Schweigert, and K. Waldorf, *Bundle Gerbes and Surface Holonomy*, European Congress of Mathematics (Amsterdam, 2008), European Mathematical Society (2010) 167-195.

[117] G. Gagliardo, C. Sämann, and R. Tellez, *Lie* 3*-groups and* U*-Duality.* In preparation.

[118] P. Gajer, *The geometry of Deligne cohomology*, Invent. Math. **127** (1997) 155-207.

[119] N. Ganter, *Categorical tori*, SIGMA arXiv:1406.7046 [math.RT] (2018).

[120] M. García Fernandez, *Torsion-Free Generalized Connections and Heterotic Supergravity*, Communications in Mathematical Physics, **332** (2014) 89-115.

[121] M. García Fernandez, *Ricci flow, Killing spinors, and T-duality in generalized geometry*, Advances in Mathematics **350** (2016) 3.

[122] M. Garcia-Fernandez and R. Gonzalez Molina, *Harmonic metrics for the Hull-Strominger system and stability*, International Journal of Mathematics **35** (2024) 9.

[123] M. Garcia-Fernandez and R. Gonzalez Molina, *Futaki Invariants and Yau's Conjecture on the Hull-Strominger system*, arXiv:2303.05274 [math.DG] (2023).

[124] M. Garcia-Fernandez, R. Gonzalez Molina, and J. Streets, *Pluriclosed flow and the Hull-Strominger system*, arXiv:2408.11674 [math.DG] (2024).

[125] M. Garcia-Fernandez, R. Rubio, and C. Tipler, *Infinitesimal moduli for the Strominger system and Killing spinors in generalized geometry*, Mathematische Annalen, **369** (2016) 539-595.

[126] M. Garcia-Fernandez, R. Rubio, and C. Tipler, *Holomorphic string algebroids*, Transactions of the American Mathematical Society, **373** (2020) 7347-7382.

[127] M. Garcia-Fernandez, R. Rubio, and C. Tipler, *Gauge Theory for String Algebroids*, Journal of Differential Geometry **128** (2024) 1 77-152.

[128] M. García-Fernández, R. Rubio, C. Shahbazi, and C. Tipler, *Canonical metrics on holomorphic Courant algebroids*, Proceedings of the London Mathematical Society **125** (2022) 700–758.

[129] A. Gastel, *Canonical Gauges in Higher Gauge Theory*, Communications in Mathematical Physics **376** (2019) 1053-1071.

[130] S. J. Gates, C. M. Hull, and M. Rocek, *Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models*, Nucl. Phys. B **248** (1984) 157.

[131] K. Gawedzki, *Topological Actions in two-dimensional Quantum Field Theories*, Nonperturbative quantum field theory, Nato Science Series B **185**, Springer (1986).

[132] K. Gawedzki and N. Reis, *WZW branes and gerbes*, Rev. Math. Phys. **14** (2002) 1281-1334.

[133] J. Giraud, *Cohomologie non abélienne*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Springer (1971).

[134] K. Gomi, *Equivariant smooth Deligne cohomology*, Osaka J. Math., **42** (2005) 309-337.

[135] R. Gordon, A. J. Power, and R. Street, *Coherence for tricategories*, Memoirs A.M.S. **117** (1995) 558.

[136] R. Goto, *Moduli spaces of topological calibrations. Calabi-Yau, Hyperkähler, G2 and Spin(7) structures.*, International Journal of Mathematics **15** (2004) 3 211-257.

[137] M. Green and J. Schwarz, *Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory*, Phys. Lett. B **149** (1984) 117-122.

[138] A. Grothendieck, *Pursuing Stacks*, Documents mathématiques **20**, Société mathématique de France (2022).

[139] A. Grothendieck, *Catégories cofibrées additives et complexe cotangent relatif*, Lecture Notes in Mathematics, **79**, Springer-Verlag, Berlin-New York (1968).

[140] M. Grützmann and T. Strobl, *General Yang–Mills type gauge theories for p-form gauge fields: From physics-based ideas to a mathematical framework or From Bianchi identities to twisted Courant algebroids*, International Journal of Geometric Methods in Modern Physics, **12** (2014).

[141] M. Gualtieri, *Generalized Complex Geometry*, PhD Thesis at University of Oxford (2004).

[142] M. Gualtieri, *Generalized Kähler geometry*, Communications in Mathematical Physics **331** (2014), 297-331.

[143] M. Gualtieri, M. Matviichuk, and G. Scott, *Deformation of Dirac structures via L_∞ algebras*, International Mathematics Research Notices, **2020** (2020), 4295-4323.

[144] N. Gurski, *An algebraic theory of tricategories*, PhD thesis at University of Chicago (2006).

[145] O. Gwilliam and R. Haugseng, *Linear Batalin-Vilkovisky quantization as a functor of ∞-categories*, Sel. Math. New Ser. **24** (2018) 1247-1313.

[146] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002).

[147] R. Haugseng, *Iterated spans and classical topological field theories*, Mathematische Zeitschrift, **289** (2017) 4 1427-1488.

[148] R. Heluani, *Supersymmetry of the chiral de Rham complex II: Commuting sectors*, Int. Math. Res. Not. **6** (2009) 953-987.

[149] A. Henriques, *Integrating L_∞-algebras*, Compositio Mathematica, **144** (2008) 4 1017-1045.

[150] Y. Hidaka, M. Nitta, and R. Yokokura, *Global 3-group symmetry and 't Hooft anomalies in axion electrodynamics*, J. High Energ. Phys. **173** (2021).

[151] N. J. Hitchin, *Non-linear problems in geometry*, Proceedings of the 6th international Taniguchi symposium at Katata (1979).

[152] N. J. Hitchin, *Lectures on special Lagrangian submanifolds*, Winter School on Mirror Symmetry and Vector Bundles, Cambridge, Massachusetts, January 4-15, 1999, 151-182.

[153] N. Hitchin, *Generalized Calabi-Yau Manifolds*, The Quarterly Journal of Mathematics, **54** (2003), 281–308.

[154] C. M. Hull, *Generalised Geometry for M-Theory*, JHEP **07** (2007) 079.

[155] L. Illusie, *Complexe cotangent et déformations*, Lecture Notes in Mathematics, **239**, Springer-Verlag, Berlin-New York (1971).

[156] X. Ji, *Deformation problems in Lie algebroids and extended Poisson geometry* PhD Thesis at Pennsylvania State University (2013).

[157] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Alg. **175** (2002).

[158] D. Joyce, *An introduction to d-manifolds and derived differential geometry*, In Moduli spaces, vol. 411 of London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge 230-281.

[159] D. Joyce and M. Upmeier, *Orientation data for moduli spaces of coherent sheaves over Calabi-Yau 3-folds*, arXiv:2001.00113 [math.AG].

[160] D. Joyce and Y. Song, *A theory of generalized Donaldson-Thomas invariants*, Memoirs of the American Mathematical Society **217** (2012).

[161] M. Kalb and P. Ramond, *Classical direct interstring action*, Phys. Rev. D. **9** (1974) 2273-2284.

[162] D. Kan, *Functors involving c.s.s complexes*, Transactions of the American Mathematical Society, **87** (1958) 2 330-346.

[163] F. Keller and S. Waldmann, *Formal deformations of Dirac structures*, Journal of Geometry and Physics, **57** (2007), 1015-1036.

[164] Y. H. Kiem and J. Li, *Categorification of Donaldson-Thomas invariants via Perverse Sheaves*, arXiv:1212.6444 [math.AG] (2016).

[165] T. P. Killingback, *World-sheet anomalies and loop geometry*, Nuclear Physics B., **288** (1987), 578-588.

[166] H. Kim and C. Sämann, *Non-Geometric T-Duality as Higher Groupoid Bundles with Connections*, arXiv:2204.01783 [hep-th] (2022).

[167] H. Kim and C. Sämann, *Adjusted parallel transport for higher gauge theories*, J. Phys. **52** (2020).

[168] S. Kobayashi, *Curvature and stability of vector bundles*, Proceedings of the Japan Academy. Mathematical Sciences. **58** (1982) 4 158-162.

[169] S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Princeton University Press (1987).

[170] M. Kontsevich, *Enumeration of rational curves via torus actions*, In The moduli space of curves (Texel Island, 1994) **129** Progr. Math. Birkhäuser Boston, Boston, MA (1995) 335-368.

[171] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, Differential geometrical methods in mathematical physics (Proc. Sympos. Univ. Bonn, Bonn, 1975), Lecture Notes in Math., **570**, Springer, Berlin, (1977) 177-306.

[172] P. Kristel, M. Ludewig, and K. Waldorf, *A representation of the string 2-group*, arXiv:2308.05139 [math.OA] (2023).

[173] P. Kristel, M. Ludewig, and K. Waldorf, *The stringor bundle*, arXiv:2206.09797 [math.AT] (2024).

[174] J. Kupka, C. Strickland-Constable, E. E. Svanes, D. Tennyson, and F. Valach, *BPS complexes and Chern-Simons theories from G-structures in gauge theory and gravity*, arXiv:2406.03550 [hep-th] (2024).

[175] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Communications in Algebra **23** 6 (1995).

[176] T. Leinster, *A survey of definitions of n-category*, Th. Appl. Cat. **10** (2002).

[177] D. Li, *Higher Groupoid Actions, Bibundles, and Differentiation*, PhD Thesis at University of Göttingen.

[178] D. Li, L. Ryvkin, A. Wessel, and C. Zhu, *Differentiating L_∞ groupoids – Part I*, arXiv:2309.00901 [math.DG] (2023).

[179] Z.-J. Liu, A. Weinstein, and P. Xu, *Manin Triples for Lie Bialgebroids*, Journal of Differential Geometry, **45** (1997), 547-574.

[180] M. Ludewig and K. Waldorf, *Lie 2-groups from loop group extensions*, arXiv:2303.13176 [math.DG] (2023).

[181] J. Lurie, *Higher Topos Theory*, Princeton University Press, Annals of mathematics studies (2009).

[182] J. Lurie, *Derived algebraic geometry V: Structured spaces*, available here.

[183] J. Lurie, *Derived algebraic geometry X: Formal moduli problems*, available here.

[184] J. Lurie, *On the Classification of Topological Field Theories*, Current Developments in Mathematics (2009) 129-280.

[185] M. Lydakis, *Simplicial functors and stable homotopy theory*, Preprint (1998).

[186] M. Mackaay and R. Picken, *Holonomy and parallel transport for Abelian gerbes*, Adv. Math. **170** (2002) 287-339.

[187] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Notes 124. Cambridge University Press, Cambridge, (1987).

[188] K. Mackenzie and P. Xu, *Classical Lifting Processes and Multiplicative Vector Fields*, The Quarterly Journal of Mathematics **49** (1998) 1 59-85.

[189] J. MacCleary, *A User's Guide to Spectral Sequences*, Cambridge University Press (2000).

[190] S. MacLane and J. H. C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. **36** 15 (1950), 41-48.

[191] F. Malikov, V. Schechtman, and A. Vaintrob, *Chiral de Rham complex*, Comm. Math. Phys. **204** (1999) 439-473.

[192] V. Mathai and D. Roberts, *Yang-Mills theory for bundle gerbes*, J. Phys. A **39** (2006) 6039-6044.

[193] V. Mathai and D. Stevenson, *Chern character in twisted K-theory. Equivariant and holomorphic cases*, Comm. Math. Phys. **236** 1 (2003), 161-186.

[194] R. A. Mehta, *On homotopy Poisson actions and reduction of symplectic Q-manifolds*, Differential Geometry and its Applications **29** (2011) 3 319-328.

[195] E. Meinrenken, *The basic gerbe over a compact simple Lie group*, Enseign. Math. **49** (2003), 3-4, 307–333.

[196] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984) 101-116.

[197] D. Mumford, *Picard groups of moduli problems*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ.) Harper and Row, New York (1965) 33-81.

[198] M. K. Murray, *Bundle gerbes*, Journal of the London Mathematical Society, **54** (1996), 403-416.

[199] M. K. Murray, D. M. Roberts, D. Stevenson, and R. F. Vozzo, *Equivariant bundle gerbes*, Advances in Theoretical and Mathematical Physics, **21** 4 (2017) 921-975.

[200] M. K. Murray and D. Stevenson, *The basic bundle gerbe on unitary groups*, J. Geom. Phys. **58** 11 (2008) 1571-1590.

[201] N. Nakahara, *Geometry, Topology and Physics*, Graduate student series in physics (2003).

[202] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Annals of Mathematics, **65** (1965), 540-67.

[203] T. Nikolaus, *Algebraic models for higher categories*, Indagationes Mathematicae, **21** (2011) 1 52-75.

[204] T. Nikolaus, U. Schreiber, and D. Stevenson, *Principal ∞-bundles: presentations*, Journal of Homotopy and Related Structures **10** (2014) 565-622.

[205] T. Nikolaus and K. Waldorf, *Four Equivalent Versions of Non-Abelian Gerbes*, Pacific Journal of Mathematics, **246** (2011).

[206] T. Nikolaus and K. Waldorf, *Lifting problems and transgression for non-abelian gerbes*, Advances in Mathematics, **242** (2013), 50-79.

[207] T. Nikolaus and K. Waldorf, *Higher Geometry for Non-Geometric T-Duals*, Communications in Mathematical Physics, **374** (2019) 317–366.

[208] J. Oh and R. P. Thomas, *Counting sheaves on Calabi-Yau 4-folds, I*, Duke Math. J. **172** (2023) 7 1333-1409.

[209] J. Oh and R. P. Thomas, *Complex Kuranishi structures and counting sheaves on Calabi-Yau 4-folds, II*, `arXiv:2305.16441 [math.AG]`.

[210] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*, Publications Mathématiques de L'IHÉS **117** (2013), 54.

[211] P. P. Pacheco and D. Waldram, *M-theory, exceptional generalised geometry and superpotentials*, Journal of High Energy Physics **123** (2008) 9.

[212] A. J. Power, *A general coherence result*, J. Pure Appl. Algebra **57** (1989) 165-173.

[213] J. P. Pridham, *Presenting higher stacks as simplicial schemes*, Advances in Mathematics **238** (2013) 184-245.

[214] D. Quillen, *Homotopical Algebra*, Springer Lecture Notes in Mathematics **43** (1967).

[215] D. Quillen, *The geometric realization of a Kan fibration is a Serre fibration*, Proc. Amer. Math. Soc. **19** (1968) 1499-1500.

[216] D. Quillen, *On the (co-)homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York) Amer. Math. Soc., Providence, R.I. (1968) 65-87.

[217] M. Reid, *The moduli space of 3-folds with* $K = 0$ *may nevertheless be irreducible*, Mathematische Annalen **278** (1987) 329-334.

[218] C. Rezk, *A cartesian presentation of weak n-categories*, Geom. Topol. **14** (2010) 1 521-571.

[219] E. Riehl and D. Verity, *Infinity category theory from scratch*, Higher Structures **4** 1 (2020).

[220] D. Rist, C. Saemann, and M. Wolf, *Explicit Non Abelian Gerbes with Connections*, arXiv:2203.00092 [hep-th].

[221] C. L. Rogers, *2-plectic geometry, Courant algebroids, and categorified prequantization*, arXiv:1009.2975 [math-ph] (2010).

[222] C. L. Rogers and C. Zhu, *On the homotopy theory for Lie* ∞*-groupoids, with an application to integrating* L_∞*-algebras*, Algebr. Geom. Topol. **20** 3 (2020) 1127-1219.

[223] S. Ronchi and C. Zhu, *Duals of Higher Vector Spaces*, arXiv:2407.03306 [math.DG] (2024).

[224] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, PhD thesis at UC Berkeley (1999).

[225] D. Roytenberg and A. Weinstein, *Courant Algebroids and Strongly Homotopy Lie Algebras*, Letters in Mathematical Physics **46** (1998) 81-93.

[226] D. Roytenberg, *On the structure of graded symplectic supermanifolds and Courant algebroids*, In: “Quantization, Poisson brackets and beyond (Manchester, 2001)” Contemp. Math., Amer. Math. Soc. **315** (2002) 169–185.

[227] D. Roytenberg, *Courant–Dorfman Algebras and their Cohomology*, Letters in Mathematical Physics, **90** (2009) 311–351.

[228] D. Roytenberg, *Quasi-Lie bialgebroids and twisted Poisson manifolds*, Letters in Mathematical Physics **61** (2002).

[229] R. Rubio and C. Tipler, *The Lie group of automorphisms of a Courant algebroid and the moduli space of generalized metrics*, Revista Matemática Iberoamericana **36** (2019).

[230] C. Sämann and L. Schmidt, *Towards an M5 Brane Model II Metric String Structures*, Fortschritte der Physik **68** (2020) 8.

[231] P. Safranov, *Shifted geometric quantization*, J. Geom. Phys. **194** (2023).

[232] P. Safronov, *Lectures on shifted Poisson geometry*, Panoramas et Syntheses **55** (2021) 187-228.

[233] S. Sandon, *On the Chern correspondence for principal fibre bundles with complex reductive structure group*, Master's Thesis, Universiteit Leiden.

[234] H. Sati and U. Schreiber, *Flux Quantization*, arXiv:2402.18473 [hep-th].

[235] H. Sati, U. Schreiber, and J. Stasheff, *L_∞ algebra connections and applications to String and Chern Simons n transport*, Quantum Field Theory, Birkhäuser (2009).

[236] H. Sati, U. Schreiber, and J. Stasheff, *Fivebrane Structures*, Reviews in Mathematical Physics **21** (2009) 10 1197-1240.

[237] H. Sati, U. Schreiber, and J. Stasheff, *Differential twisted string and fivebrane structures*, Commun. Math. Phys. **315** (2012).

[238] C. Schommer-Pries, *Central extensions of smooth 2–groups and a finite-dimensional string 2–group*, Geometry & Topology **15** (2011) 609–676.

[239] U. Schreiber, *Differential cohomology in a cohesive infinity-topos*, arXiv:1310.7930 [math-ph] (2013).

[240] A. Schwarz, *Geometry of Batalin-Vilkovisky quantization*, Comm. Math. Phys. **155** (1993) 2 249-260.

[241] A. Schwarz, *Semiclassical approximation in Batalin-Vilkovisky formalism*, Comm. Math. Phys. **158** (1993) 2 373-396.

[242] P. Ševera, *Some title containing the words homotopy and symplectic*, based on a talk at Poisson 2000, CIRM Marseille, (2000).

[243] P. Ševera, *Letters to Alan Weinstein about Courant algebroids*, arXiv:1707.00265 [math.DG] (2017).

[244] P. Ševera and A. Weinstein, *Poisson geometry with a 3-form background*, Progr. Theoret. Phys. **144** (2001) 145-154.

[245] Y. Sheng, X. Xu, and C. Zhu, *String Principal Bundles and Courant Algebroids*, International Mathematics Research Notices, **7** (2021) 5290-5348.

[246] H. B. Shulman, *Characteristic classes and foliations*, PhD Thesis at University of California, Berkeley (1972).

[247] C. Simpson, *Algebraic (geometric) n-stacks*, arXiv:alg-geom/9609014 (1996).

[248] C. Simpson, *Homotopy Theory of Higher Categories. From Segal Categories to n-Categories and Beyond.*, Cambridge University Press (2011).

[249] H. Sinh, *Gr-categories*, PhD Thesis at Université Paris VII (1975).

[250] D. I. Spivak, *Derived smooth manifolds*, Duke Math. J. **153** 1 (2010) 55-128.

[251] J. Stasheff, *Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras. in: Quantum groups*, Lecture Notes in Math. Springer, Berlin (1992).

[252] P. Steffens, *Derived Differential Geometry*, PhD thesis at University of Montpellier (2022).

[253] S. Stolz and P. Teichner, *What is an elliptic object?*, in: Topology, geometry and quantum field theory. London Math. Soc., Cambridge Univ. Press (2004) 247-343.

[254] R. Street, *The petit topos of globular sets*, JPAA **154** (2000) 299-315.

[255] R. Street, *Weak omega-categories, Diagrammatic Morphisms and Applications*, Contemporary Mathematics AMS **318**(2003) 207-213.

[256] R. J. Szabo, *Quantization of higher abelian gauge theory in generalized differential cohomology*, in Proceedings of ICMP **7** (2012).

[257] G. Tamme, *Introduction to étale cohomology*, Universitext. Springer-Verlag, Berlin (1994).

[258] R. Tellez-Dominguez, *Chern correspondence for higher principal bundles*, arXiv:2310.12738 [math.DG] (2023).

[259] R. P. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on K3 fibrations*, J. Differential Geom. **54** (2000) 2 367-438.

[260] R. W. Thomason, *Cat as a closed model category*, Cahiers Topologie Géom. Différentielle **21** 3 (1980) 305-324.

[261] B. Toën, *Derived algebraic geometry*, EMS Surveys in Mathematical Sciences, **1** (2014) 153-240.

[262] B. Toën and G. Vezzosi, *Homotopical algebraic geometry I. Topos theory.*, Adv. Math. **193** 2 (2005) 257-372.

[263] B. Toën and G. Vezzosi, *Homotopical algebraic geometry II. Geometric stacks and applications*, Mem. Amer. Math. Soc. **193** 902 (2008).

[264] T. tom Dieck, *On the homotopy type of classifying spaces*, Manuscripta Math **11** (1974) 41-49.

[265] G. Trentinaglia, *The Adjoint Representation of a Higher Lie Groupoid*, arXiv:2312.14908 [math.CT] (2024).

[266] T. Trimble, *Notes on tetracategories*, available here.

[267] K. K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable bundles*, Communications on Pure and Applied Mathematics **39-S** (1986) 257-293; **42** (1989) 703-707.

[268] M. Upmeier, *The canonical 2-gerbe of a holomorphic vector bundle*, Theory and Applications of Categories **30**, (2017), 1028-1049.

[269] T. Voronov, *Q-manifolds and Higher Analogs of Lie Algebroids*, Proceedings 29th Workshop on Geometric Methods in Physics. **1** (2010) 191-202.

[270] T. Voronov, *Graded Geometry, Q-Manifolds, and Microformal Geometry*, Symposium on Higher Structures in M-Theory. Fortschritte der Physik. (2019).

[271] K. Waldorf, *Multiplicative bundle gerbes with connection*, Differential Geometry and its Applications, **28** (2010) 313-340.

[272] K. Waldorf, *A Construction of String 2-Group Models using a Transgression-Regression Technique*, Contemp. Math. **584** (2012) 99-115.

[273] K. Waldorf, *String connections and Chern-Simons theory*, Transactions of the American Mathematical Society **365** (2013), 4393-4432.

[274] K. Waldorf, *Geometric T-duality. Buscher rules in general topology*, arXiv:2207.11799 [math.DG] (2023).

[275] A. Weinstein, *Symplectic groupoids and Poisson manifolds*, Bull. Amer. Math. Soc. **16** (1987) 101-104.

[276] E. Witten, *Two dimensional gravity and intersection theory on moduli space*, Surveys in Diff. Geom. Proceedings of the Conf. on Geom. and Top., Harvard Univ. AMS. (1991).

[277] E. Witten, *Global anomalies in String Theory*, Symposium on Anomalies, Geometry, Topology (1985).

[278] P. Xu, *Momentum Maps and Morita Equivalence*, Journal of Differential Geometry **67** 2 (2004).

[279] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampére equation.*, Communications on Pure and Applied Mathematics **31** 3 (1978) 339-411.

[280] S.-T. Yau, *Complex geometry: Its brief history and its future*, Science in China Series A Mathematics **48** (2005) 47-60.

[281] C. Zhu, *n–groupoids and stacky groupoids*, International Mathematics Research Notices, (2009) 4087–4141.

[282] C. Zhu, *Lie n-groupoids and stacky Lie groupoids*, International Mathematics Research Notices, (2006).

[283] B. Zwiebach, *Closed string field theory. Quantum action and the BV master equation*, Nucl. Phys. B390 (1993) 33.