Universidad Complutense de Madrid

Facultad de Ciencias Matemáticas



TESIS DOCTORAL

Ecuaciones elípticas y parabólicas en dominios irregulares.

Elliptic and parabolic equations in rough domains.

Memoria para optar al grado de doctor presentada por

Pablo Hidalgo Palencia

Director:

José María Martell Berrocal (ICMAT)

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UNIVERSIDAD COMPLUTENSE MADRID

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Contents

Agradecimientos / Acknowledgments vii

Contents xi

Abstract xv

Resumen xvii

1 Introduction 1
- 1.1 Smooth Dirichlet data . . . 4
- 1.2 Singular Dirichlet data . . . 6
- 1.3 Parabolic PDE . . . 8
- 1.4 Neumann conditions and overdetermined problems . . . 10
- 1.5 Equations in non-divergence form . . . 11

2 General notation and preliminaries 15
- 2.1 Notation . . . 16
- 2.2 Harmonic Analysis . . . 17
- 2.3 Geometric Measure Theory . . . 24
- 2.4 Elliptic PDE . . . 28
- 2.5 Extension to parabolic PDE . . . 41
- 2.6 A short introduction to Functional Calculus . . . 47

3 The Dirichlet problem with smooth boundary data 51
- 3.1 Further historical context . . . 53
- 3.2 Main results . . . 54
- 3.3 Main ideas and outline of the chapter . . . 58
- 3.4 Preliminaries . . . 59
- 3.5 Growth functions and auxiliary lemmas . . . 60
- 3.6 Well-posedness of the Dirichlet problem in Hölder spaces . . . 63

3.7 Ill-posedness of the Dirichlet problem in Hölder spaces . . . 69
3.8 Optimality of the capacity density condition . . . 74
3.9 Applications to other boundary value problems . . . 87

4 The elliptic Dirichlet problem with singular data 103
4.1 Further historical context . . . 105
4.2 Main results . . . 106
4.3 Main ideas and outline of the chapter . . . 110
4.4 Preliminaries . . . 111
4.5 Proof of the main equivalences . . . 124
4.6 Proof of the abstract perturbation theory . . . 140
4.7 Applications in connection with Uniform Rectifiability . . . 144
4.8 Proof showing that our results extend previous ones . . . 149

5 The parabolic Dirichlet problem with singular data 157
5.1 Main results . . . 159
5.2 Further historical context . . . 160
5.3 Main ideas and outline of the chapter . . . 162
5.4 Preliminaries . . . 164
5.5 Boundary estimates in parabolic Lipschitz domains . . . 172
5.6 The assumptions in our main results . . . 175
5.7 Step 1: Initial stopping time regimes and the preliminary estimates . . . 177
5.8 Step 2: A local square function estimate and refined corona decomposition . . . 184
5.9 Step 3: Construction of approximating graphs via a stopping-time distance . . . 194
5.10 Step 4: Regularity of the approximating graphs . . . 201
5.11 Cut-off function construction . . . 213
5.12 Comments on smoothness assumptions . . . 217
5.13 A linear algebra fact about elliptic matrices . . . 218

6 Overdetermined elliptic boundary value problems 221
6.1 Further historical context . . . 223
6.2 Main results . . . 224
6.3 Main ideas and outline of the chapter . . . 227
6.4 Assumptions and auxiliary results . . . 228
6.5 Some basic properties of minimizers . . . 232
6.6 Asymptotic minimality at small scales . . . 237
6.7 Lipschitz continuity and non-degeneracy . . . 241
6.8 Compact support of minimizers . . . 245
6.9 The free boundary . . . 246
6.10 Proof of the main result . . . 262
6.11 Overdetermined problems on manifolds . . . 265
6.12 Non-periodic or fastly growing counterexamples . . . 267

7 The Kato square root problem for elliptic operators in non-divergence form 271
7.1 Further historical context . . . 272
7.2 Preliminaries . . . 273
7.3 Main result . . . 276

7.4 Main ideas and outline of the chapter 278
7.5 Auxiliary results 279
7.6 The Kato problem for L 286
7.7 The adjoint Kato problem, for $\widetilde{L}$ 289

Bibliography **301**

Abstract

The Laplacian is probably the most famous second order differential operator because it helps model diffusion processes, which are ubiquitous in life, and hence in many Sciences. In Mathematics, the Laplacian is the germ of elliptic and parabolic Partial Differential Equations (PDEs), which are the central thread of this thesis.

One of the key properties of the Laplacian is that it possesses many symmetries. This is usually an advantage because these symmetries make it the easiest operator to work with. But there are also downsides: diffusion processes in real life are often *not* symmetric. For instance, differences in the composition of the substances or materials in which heat is transmitted, affect the way the heat itself is distributed. Similarly, the way a liquid flows through a pipe, or the way water moves within the ocean, are influenced by many agents, like different physical forces or temperature. If instead we are looking at a chemical substance spread, the movement will also depend on many different physical and chemical quantities, and how our substance reacts to their presence.

Therefore, it would be desirable that, as mathematicians, we also establish a theory where we substitute the Laplacian by similar operators, which model diffusion, but do not assume extra symmetries, and hence can model general versions of anisotropies or inhomogeneities (i.e. that may behave differently if we vary the position or the direction). This is the main goal of this thesis: to extend results that were known for the Laplacian (because it is the simplest operator to work with) to more general operators accounting for (possibly wild) anisotropies and/or inhomogeneities. These extensions, although always based on results that were known for the Laplacian, will demand careful and novel work because of the generality of the operators that we will consider, aside from the lack of symmetry of the problems. The advantage is that we will create new methods that are much more robust, because they will no longer depend on specific symmetries, as was the case for the Laplacian.

More concretely, in this thesis we will focus on extending several results that had already been established for the Laplacian, to (much) wider families of operators. These results will often be related with behavior of solutions to PDEs near the boundary of the domain in which the PDE is posed: the behavior near the boundary is known to be much more subtle and hard to analyze than in the interior of the domain. Actually, many of the properties that one expects for solutions of PDEs to hold in the interior of the domain, were extended from the Laplacian to general operators many years ago, and can

be nowadays found in any textbook.

The behavior of solutions near the boundary will be very much connected with the behavior of the boundary itself: there will be an intense interaction between the theory of PDE and geometry. Moreover, we will strive to work with operators whose behavior may change or oscillate in a wild way. To account for these oscillations, we will make intensive use of heavy machinery from Harmonic Analysis (precisely the branch of Analysis specializing on oscillations). Furthermore, we will work with domains whose boundaries may have very oscillating parts, or that may not be differentiable at all: the branch of Geometry (which is actually much closer to Analysis) that will help us deal with these kind of boundaries is called Geometric Measure Theory. Therefore, the results of this thesis will lie in the intersection between these 3 areas (PDE, Harmonic Analysis, and Geometric Measure Theory), although there will be at times glimpses to other areas like Calculus of Variations.

Resumen

El Laplaciano es, muy probablemente, el operador diferencial de segundo orden más conocido, ya que frecuentemente ayuda a modelizar procesos de difusión, que son ubicuos en la vida y, por tanto, en la Ciencia. En Matemáticas, el Laplaciano es la piedra angular de las Ecuaciones en Derivadas Parciales (EDPs), que son el hilo conductor de esta tesis.

Una propiedad fundamental del Laplaciano es que muestra muchas simetrías. Esto suele ser una ventaja, pues estas simetrías hacen que sea el operador más sencillo con el que trabajar. Pero también existen inconvenientes: los procesos de tipo difusivo en la realidad *no* son siempre simétricos. Por ejemplo, la composición de las sustancias o materiales pueden afectar en cómo se difunde y distribuye el calor a través de ellos. De forma similar, muchos agentes, sean fuerzas (en el sentido de la Física) o temperatura, por ejemplo, afectan en cómo los líquidos fluyen por tuberías o cómo se mueve el agua del océano. Algo parecido sucede si observamos cómo se propagan las sustancias químicas: su movimiento dependerá de diversos factores físico-químicos, y de cómo esa sustancia concreta reacciona ante ellos.

Por ello, sería deseable que nosotros, como matemáticos, creáramos una teoría donde el Laplaciano quedara sustituido por operadores similares, en tanto que modelizan procesos de difusión, pero sin asumir ninguna simetría, y así poder modalizar cualquier tipo de anisotropía o inhomogeneidad (o sea, que el operador pueda ser diferente dependiendo de la posición y la dirección). Este es precisamente el principal objetivo de esta tesis: extender resultados que se conocían para el Laplaciano (pues es el operador más fácil de analizar) a operadores más generales, que tengan en cuenta anisotropías y/o inhomogeneidades (y de hecho, que estas puedan ser muy notables). Para extender estos resultados, aunque nos basemos siempre en resultados que se conocían para el Laplaciano, necesitaremos hacer un análisis cuidadoso y novedoso de cada situación, pues trabajaremos con operadores muy generales que no poseen simetrías. La ventaja es que, de esta forma, crearemos nuevos métodos que son mucho más robustos que los que ya existían, pues ya no dependerán de propiedades o simetrías concretas, como pudiera ser el caso para el Laplaciano.

Más concretamente, en esta tesis extenderemos varios resultados que ya se conocían para el Laplaciano a familias de operadores (mucho) más grandes. Estos resultados tendrán habitualmente relación con cómo se comportan las soluciones de EDPs cerca del borde del dominio en el que hemos resuelto la EDP: es conocido que el estudio del comportamiento de soluciones cerca del borde es mucho más

sutil y difícil de entender que en el interior del dominio. De hecho, muchas de las propiedades de EDPs en el interior del dominio ya se extendieron a operadores generales hace muchos años, y hoy se pueden encontrar en cualquier libro.

Como es esperable, el comportamiento de soluciones cerca del borde tiene mucha relación con cómo es el borde en sí: habrá una relación intensa entre EDPs y geometría al estudiar esto. Además, nos enfocaremos en trabajar con operadores que pueden cambiar u oscilar mucho. Para lidiar con estas oscilaciones, usaremos frecuentemente maquinaria proveniente del Análisis Armónico (que es precisamente la rama del Análisis que está especializada en entender oscilaciones). Asimismo, trabajaremos en dominios cuyos bordes tendrán partes que oscilan mucho, y pueden no ser ni siquiera diferenciables: la rama de la Geometría (que realmente es mucho más cercana al Análisis) que nos ayudará para este cometido será la denominada Teoría Geométrica de la Medida. Así, los resultados que obtendremos en esta tesis estarán en la intersección de estas 3 áreas (EDPs, Análisis Armónico y Teoría Geométrica de la Medida), aunque habrá breves incursiones a otras ramas como el Cálculo de Variaciones.

CHAPTER 1

Introduction

Partial Differential Equations (PDEs) are ubiquitous in Science, and concretely in Mathematics, for they model a wide variety of processes and evolutions of different quantities. More concretely, one is often interested in Boundary Value Problems: finding the evolution of some physical quantity given the law (an equation) that it follows, the initial conditions that it was subject to, and how it interacts with the environment (boundary conditions).

The theory of Boundary Value Problems in Mathematics is now a classic and well-established subject. During the last centuries, several methods have been developed to find solutions to certain interesting PDEs prescribing their boundary behavior. Concretely, a big portion of the research efforts have focused on studying the heat equation

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = 0 & \text{for } (x,t) \in \Omega \subset \mathbb{R}^{n+1}, \\ u(x,t) = f(x,t) & \text{for } (x,t) \in \partial\Omega, \end{cases}$$

where $\Delta = \sum_{j=1}^{n} \partial_{x_j} \partial_{x_j}$, and its stationary state, the Laplace equation,

$$\begin{cases} -\Delta u(x) = 0 & \text{for } x \in \Omega \subset \mathbb{R}^{n}, \\ u(x) = f(x) & \text{for } x \in \partial\Omega, \end{cases}$$

because they are probably the easiest to understand second-order PDEs, and often serve as a role model for many other second-order PDEs which are very important for applications in other sciences. Among the properties that have been studied throughout the years, it is already known (since a number of decades now) that if the boundary of the domain Ω is smooth enough, then solutions of the associated Boundary Value Problem are very well-behaved, too (we will make this more precise at due time).

The goal of this work is to get to the limit in the understanding of regularity for second-order PDEs arising from the heat equation: we want to find the minimal conditions for domains and operators which are still sufficient to obtain analogue properties to those of solutions of the heat equation. Concretely, our study will focus on properties regarding regularity of solutions close to the boundary $\partial\Omega$, which is known to be a really delicate matter, for it is not only an analytical question, but is also affected by geometrical features of the boundary $\partial\Omega$ of the domain.

Since we want to study *minimal* conditions supporting several properties, we will work in domains which are potentially very irregular; at least more than what has been considered classically, like domains with C^2 boundaries which one can find in books like [GT77]. In fact, we will deal with boundaries of domains which are not necessarily differentiable: to study these and their relationship with PDEs, we will make intensive use of tools from Geometric Measure Theory (GMT) and Potential Theory, which will be introduced at due time.

Apart from fine properties of sets (studied by GMT and Potential Theory), we will also need to analyze properties of potentially very irregular functions, because they will be solutions of PDEs with very oscillating parameters, in domains which may oscillate widly, too. Therefore, at least a priori, our functions could present many oscillations. Thus, the help of Harmonic Analysis, which is precisely the branch of Analysis that carefully studies functions with regards to their oscillations, will be really necessary and useful throughout this work.

As a summary, the goal of this thesis, trying to determine the most general conditions in which some desirable PDE properties hold, lies in the intersection between PDE, Harmonic Analysis and Geometric Measure Theory.

Concretely, we will be interested in showing that very general Boundary Value Problems are solvable/ well-posed, that is, that solutions exist, they are unique, and they satisfy some continuity estimate with respect to boundary data (when studying this, boundary behavior of solutions will be of key relevance). Of course, this will depend on the properties of the parameters defining our equation, and the functional spaces where we want our solutions to live. Our intention will be to consider parameters and spaces which are as general as possible, yet maintaining suitable well-posedness.

Actually, in this thesis, we will show several results which were already known for the heat or the Laplace equation, and extend them to (much) more general equations with (much) more irregular coefficients. Sometimes, the cases of the heat equation or the Laplace equation will almost be trivial, while the general cases that we consider will require much work. In other cases, the heat/Laplace equation will need very novel ideas, and we will refine and extend the method carefully while maintaining the key ideas.

Ideally, we would like to obtain the exact minimal conditions ensuring well-posedness of the Boundary Value Problems under consideration. In some cases (see Chapters 3, 5) we will obtain results that are characterizations, so they identify correctly the minimal conditions for those PDEs. In other cases, we can provide some novel partial progress towards those characterizations (see Chapters 4, 6 and 7). For the moment, it is an open problem in the field to find (and prove) the minimal conditions in these cases.

The concrete problems that we will face in this work will always be related to second-order PDE, but will have slightly different flavors:

- In Chapter 3, we will consider second-order elliptic PDE in divergence form, with *smooth* Dirichlet boundary data. This chapter is based on the publication [CHMPZ25]:

  > M. Cao, P. Hidalgo-Palencia, J.M. Martell, C. Prisuelos-Arribas, and Z. Zhao,
  > *Elliptic operators in rough sets, and the Dirichlet problem with boundary data in Hölder spaces*,
  > J. Funct. Anal. **288** (2025), 68pp.

- In Chapter 4, we will consider second-order elliptic PDE in divergence form, with *singular* Dirich-

let boundary data. This chapter is based on the publication [CHM24]:

> M. Cao, P. Hidalgo-Palencia, and J.M. Martell,
> *Carleson measure estimates, corona decompositions, and perturbation of elliptic operators without connectivity*,
> Math. Ann. **390** (2024), 95–156.

- In Chapter 5, we will consider second-order *parabolic* PDE in divergence form, with singular Dirichlet boundary data. This chapter is based on the publication [BFHH25]:

  > S. Bortz, S. Ferris, P. Hidalgo-Palencia, and S. Hofmann,
  > *A variable coefficient free boundary problem for L^p solvability of parabolic Dirichlet problem in graph domains*.
  > Arxiv preprint 2503.00873.

- In Chapter 6, we will consider second-order elliptic PDE in divergence form, with both Dirichlet and *Neumann* boundary data. This chapter is based on the publication [EHR24]:

  > A. Enciso, P. Hidalgo-Palencia, and X. Ros-Oton,
  > *Positive solutions to general semilinear overdetermined boundary problems*,
  > Arxiv preprint 2412.11902.

- In Chapter 7, we will consider second-order elliptic PDE in *non-divergence* form. This chapter is based on the publication [EHH24]:

  > L. Escauriaza, P. Hidalgo-Palencia, and S. Hofmann,
  > *On the Kato problem for elliptic operators in non-divergence form*,
  > Vietnam J. Math **52** (2024), 1067–1096.

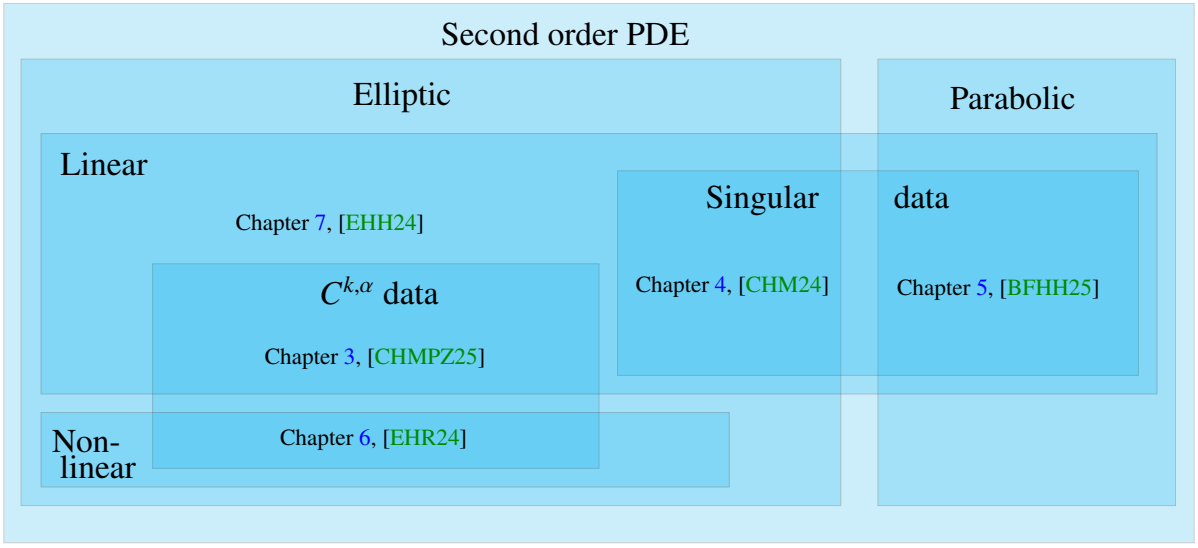


Figure 1.1: A scheme of the different areas of the theory of PDE where the different chapters could be framed.

As just noted, in every chapter we will use different formulations for the PDEs, and different boundary conditions, too. This will have a clear impact on the kind of behavior that one may expect for solutions

near the boundary of our domain, and this is one of the key aspects when showing well-posedness of the appropriate boundary value problems. Moreover, since our goal is to obtain minimal conditions (not only sufficient ones), we will work in geometrical frameworks that are adapted to the formulation of the boundary value problem of each chapter. The interest (and at the same time, difficulty) of this thesis relies on correctly identifying which geometrical conditions are important for each concrete PDE formulation, will require a careful study of the interplay between Analysis and Geometry.

At the beginning of every one of the subsequent chapters, we will spend some time explaining the historical and mathematical context of the result. We will end up being very concrete in our explanations, so that the reader knows exactly the evolution of the field before the result, and the gap that the result fills within the theory. But we prefer to also provide a more informal discussion of the historical framework and the main result, so that the reader starts to become acquainted with the topic before having to dive deeper in the details. We will do so in the remaining part of this chapter. We will keep the discussion in the most elementary terms possible, but the reader may consult Chapter 2 for further notation and definitions.

1.1 Smooth Dirichlet data

The first problem that we will treat in detail in this work, in Chapter 3, will have a formulation that makes us dive deep into the very fundamentals of PDE. Since the very early ages of PDE, researchers wondered about what conditions would guarantee that solutions of second-order elliptic PDE would be continuous up to the boundary. Questions about regularity in the interior of the domain were known to be easier and are nowadays quite classical; but questions about regularity at the boundary, which entail geometrical considerations about the boundary itself, have always been more subtle. Even if the boundary data is assumed to live in spaces of classical smoothness, for example continuous or $C^{k,\alpha}$, matters are highly subtle and technical.

The first groundbreaking result was given by Wiener in 1924 [Wi24]: he found a potential theoretic (so in some sense geometrical, as we shall see) characterization of the domains for which harmonic functions attain their Dirichlet boundary values continuously. Informally,

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \subset \mathbb{R}^{n+1} \\ u = f & \text{on } \partial\Omega, \end{cases} \quad + \quad f \in C(\partial\Omega) \ : \ \ \Omega \text{ satisfies Wiener's criterion} \iff u \in C(\overline{\Omega}).$$

This exemplifies how important are potential theoretic concepts, such as capacities, for harmonic functions, and more generally for solutions of second-order elliptic equations. We will revisit this in Section 2.4.5.

After Wiener's result, Littman, Stamppachia and Weinberger [LSW63] performed a pioneering analysis on very general elliptic equations in divergence form, where $-\Delta u = 0$ is generalized to $-\operatorname{div}(A\nabla u) = 0$ in Ω, where $A = A(x)$ is merely an uniformly elliptic and bounded matrix. Concretely, apart from a very important development of Green's functions that we will revisit later (see Sections 2.4.3 and 2.4.4), they showed that the property of solutions arriving to the boundary in a continuous way is local, and (most importantly) does not depend on the operator taken into consideration. Thus, the same characterization that Wiener gave for the Laplace equation is valid for these general equations in divergence form.

Nevertheless, most of the regularity theory for elliptic PDE has always considered solutions living in spaces which are in some sense quantitative, like $C^{k,\alpha}$ or $W^{k,p}$ (see e.g. Section 2.4.3 and the references

therein). In this direction, Wiener's criterion did not give the complete answer because it only dealt with continuity, which is a qualitative property. Much more recently, Aikawa [Ai02] filled the gap by finding the exact condition, the so-called CDC (for Capacity Density Condition, which is in fact very similar to Wiener's condition), that ensures that solutions get to the boundary in an appropriate Hölder continuous way. However, he only provided a proof for the Laplace equation in a bounded domain, so there was plenty of room to generalize and refine the result.

In a joint work with Cao, Martell, Prisuelos-Arribas and Zhao [CHMPZ25], which we address in all detail in Chapter 3, we show that the quantitative Wiener's condition that Aikawa proposed characterizes solutions getting to the boundary in a Hölder continuous way, not only for the Laplacian in bounded domains, but actually for all elliptic operators in divergence form with bounded coefficients (not even necessarily continuous), and for all kinds of open sets (not necessarily bounded). The main results that we will prove in Chapter 3 are:

> Let $\Omega \subset \mathbb{R}^{n+1}$, where $n \geqslant 2$, be a bounded open set satisfying the Capacity Density Condition (see Section 2.4.5). Let $L = -\operatorname{div} A\nabla$ be a second order elliptic operator in divergence form, with bounded coefficients (see Section 2.4.1). Then, for $\alpha > 0$ small enough, the $\dot{C}^\alpha$-Dirichlet problem is well-posed, namely: for every $f \in \dot{C}^\alpha(\partial\Omega)$ there exists a unique u satisfying (see Section 2.4.1)
>
> $$\begin{cases} u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap \dot{C}^\alpha(\Omega), \\ Lu = 0 \text{ in the weak sense in } \Omega, \\ u = f \text{ on } \partial\Omega, \end{cases}$$
>
> and actually, this solution u is given by an integration of the boundary datum f against the elliptic measure (see Theorem 2.29), and satisfies $\|u\|_{\dot{C}^\alpha(\Omega)} \leqslant C\,\|u\|_{\dot{C}^\alpha(\partial\Omega)}$ for $C > 0$ uniform on our hypotheses.
> Moreover, the same is true when Ω is unbounded with $\partial\Omega$ unbounded; and when Ω is unbounded with $\partial\Omega$ bounded, uniqueness may fail, but there is always a solution satisfying the above properties.
> Furthermore, the Capacity Density Condition is the minimal condition needed for these properties to hold.

Informally what this means is (compare to the Wiener criterion above):

$$\begin{cases} -\operatorname{div}(A\nabla u) = 0 & \text{in } \Omega \subset \mathbb{R}^{n+1} \\ u = f & \text{on } \partial\Omega, \end{cases} + f \in \dot{C}^\alpha(\partial\Omega) \ : \ \Omega \text{ satisfies CDC} \iff \begin{cases} u \in \dot{C}^\alpha(\overline{\Omega}) \text{ with} \\ \|u\|_{\dot{C}^\alpha(\overline{\Omega})} \lesssim \|f\|_{\dot{C}^\alpha(\partial\Omega)}. \end{cases}$$

Our treatment of the problem in Chapter 3 needs to be much more careful as compared to Aikawa's in [Ai02], because the properties of solutions to these very general equations are significantly more subtle.

As a last remark, the reader may have noted that the main result exposed in Chapter 3 is only stated for $n \geqslant 2$. Indeed, the main difference between the planar case ($n = 1$) and the higher dimensional case $n \geqslant 2$ is the shape of the fundamental solution of the Laplace operator: it follows a logarithmic profile when $n = 1$, and a polynomial profile when $n \geqslant 2$. This makes it more difficult, for instance, to work outside compact sets when $n = 1$, because the fundamental solution does not offer any decay (as opposed to the situation when $n \geqslant 2$), hence making the arguments sometimes harder, as can be

seen in [Ai02, Section 5]. We have preferred to keep the exposition simpler (in this and most of the remaining chapters) by restricting to the higher dimensional setting ($n \geqslant 2$), but we have actually no reason to believe that the main result in Chapter 3 should be false in the planar case $n = 1$; for example, in the related works [LSW63, Ai02], no restriction on n was made.

1.2 Singular Dirichlet data

The most widely studied Dirichlet boundary conditions are probably those in spaces of classical regularity like $C^{k,\alpha}$: in any classical book (e.g. [GT77, Ev98]) one usually finds results stated within these regularity classes. Beyond these spaces, most probably the next ones more widely considered in modern Analysis are the Lebesgue L^p spaces. Studying PDE with boundary data in Lebesgue spaces is a much more delicate matter because functions in L^p may even have singularities, as opposed to the smooth boundary data with which we dealt in the previous pages.

Since the early stages of the second half of the last century, the Dirichlet problem with boundary data in L^p has attracted lots of interest. And not only because of trying to generalize naively the PDE results that were already known (in fact, results known for smooth data are often not even true for singular data), but also because it was known that it is intrinsically tied to Harmonic Analysis. Indeed, it was one of the main problems in the so-called Alberto Calderón's program (see e.g. [Ca78]), who proposed that modern techniques from Harmonic Analysis should have a deep influence on the theory of PDE and pseudo-differential operators, among others, which later turned out to be the case.

The first very relevant result in this setting was proved by Dahlberg in 1977 [Da77]: he showed that if the boundary of a domain is (locally) Lipschitz, then the associated boundary value problem for the Laplace equation was well-posed in L^2, if defined appropriately. Informally,

$$\begin{cases} -\operatorname{div}(A\nabla u) = 0 & \text{in } \Omega \subset \mathbb{R}^{n+1} \\ u = f & \text{on } \partial\Omega, \end{cases} : \partial\Omega \text{ is Lipschitz} \implies \begin{array}{c}\text{Well-posedness} \\ \text{if } f \in L^2(\partial\Omega).\end{array}$$

Let us not be completely rigorous for the moment in our definition of well posedness: it will entail estimates like $\|u\| \lesssim \|f\|_2$, but we will not specify yet which norm one should use to measure u, because it is a more technical matter (indeed, one should take the L^2 norm of the so-called non-tangential maximal function, which will be introduced at due time).

Since Dahlberg's result, there has been a great interest in the Dirichlet problem with data in L^p, which integrates Harmonic Analysis, PDE and Geometric Measure Theory. For instance, it has been strongly tied to the development of the theory of Muckenhoupt weights, McIntosh's Functional Calculus, $T1$ and Tb theorems, the boundedness of (operator adapted) Riesz transforms and other singular integral operators, among many others.

More recently, the most complete answer describing when the L^p Dirichlet problem is well-posed has been given by Azzam, Hofmann, Martell, Mourgoglou and Tolsa in [AHMMT20]: they proved that, under very mild background assumptions, the Dirichlet for the Laplacian problem with boundary data in L^p is well-posed if and only if the boundary is Uniformly Rectifiable and non-tangentially accessible in a concrete weak sense (the so-called Weak Local John condition). Informally

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \subset \mathbb{R}^{n+1} \\ u = f & \text{on } \partial\Omega, \end{cases} : \begin{array}{c}\partial\Omega \text{ is Uniformly Rectifiable} \\ \text{and } \Omega \text{ Weak Local John}\end{array} \iff \begin{array}{c}\text{Well-posedness} \\ \text{if } f \in L^p(\partial\Omega).\end{array}$$

This result is sharp (it is a characterization), and is the culmination of many relevant partial progresses along the last decades, like [Da77, CFMS81, JK82, DS91, DS93, FKP91, KP01, HM14, HMU14, NTV14, HM15, HKMP15a, HMaM15, HMM16, AHMM+16, HLMN17, AHMNT17, HMT17, GMT18, AGMT23], among many others.

The condition of a boundary being Uniformly Rectifiable means that this boundary is somehow flat because it has many Lipschitz portions, although these portions may be very small and sparse; in fact Uniformly Rectifiable sets are much more general than Lipschitz. Uniform rectifiability and its qualitative analogue, rectifiability, are very important notions in Geometric Measure Theory. They are often the sets which are the threshold in which we know how to carry out some Analysis, and they generalize Lipschitz sets, which are sets in which doing Analysis is sometimes not trivial at all (see [Da77, CMM82]). Thus, they are central notions in Calderón's program.

After the result in [AHMMT20], it is natural to strive to understand to what extent that characterization is exclusive or not to the Laplace equation, as we did in [CHMPZ25] (but for other property). This case is more subtle because it is clear that the characterization cannot hold for just any elliptic operator because there exist counterexamples like those in [MM80] and [CFK81]. But the important work [FKP91] of R. Fefferman, Kenig and Pipher identified a class of operators for which a similar result to [AHMMT20] holds, at least under more beneficial background assumptions, which have been later generalized in [HMMTZ21]. Therefore, there is some evidence that the result of [AHMMT20] may be generalized to wider classes of operators beyond the Laplacian, probably to that of [FKP91].

In a joint work with Cao and Martell [CHM24], which we explain in detail in Chapter 4, we give a further step in this direction by proving and discussing some implications from [AHMMT20] for a class of operators intermediate between that of [AHMMT20] and [FKP91] or [KP01], similar to what had been done by Azzam, Garnett, Mourgoglou and Tolsa in [AGMT23] (heavily influenced by the work of a subset of the same authors in [GMT18]). The main novelty is that we address for the first time the crucial issue in [AHMMT20], which is to work without any background connectivity assumption, but for the first time for operators beyond the Laplacian.

A key idea in Chapter 4 is that if we work with operators which are not the Laplacian, we need to distinguish the regions and scales where the operator behaves well and not: it is very likely that there are some regions where we cannot expect PDE solutions to behave well because the operator itself does not behave well. But if we somehow ensure that our operator only behaves poorly in a small portion of the scales (which is essentially the meaning of the condition in [FKP91] or [KP01]), there is some hope that we can just disregard those regions (usually via convolved structures like corona decompositions, intimately connected with Carleson measures) and still obtain nice overall properties of solutions. Indeed, the main results that we prove in Chapter 4 is:

> Let $\Omega \subset \mathbb{R}^{n+1}$, where $n \geqslant 2$, be an open set satisfying the interior corkscrew condition and with Ahlfors-David regular boundary (see Sections 2.3.1 and 2.3.3). Let $L = -\operatorname{div} A\nabla$ be a second order elliptic operator in divergence form, with bounded coefficients (see Section 2.4.1). Then, the following are equivalent:
>
> - The elliptic measure is comparable to the surface measure $\mathcal{H}^n|_{\partial\Omega}$ in the sense of corona decompositions (see Definition 4.10),
>
> - The Green's function (associated to L inside Ω) is comparable to the distance to the boundary in the sense of corona decompositions (see Definition 4.10),

- Bounded weak solutions of $Lu = 0$ satisfy weak Carleson measure estimates (see Definition 4.15).

Moreover, all these properties are stable under certain Fefferman-Kenig-Pipher perturbations, namely: if $L_0 = -\operatorname{div} A_0\nabla$ and $L_1 = -\operatorname{div} A_1\nabla$, and we define

$$\sup_{\substack{x\in\partial\Omega\\ 0<r<\operatorname{diam}(\partial\Omega)}} \frac{1}{\mathcal{H}^n(B(x,r)\cap\partial\Omega)} \iint_{B(x,r)\cap\Omega} \frac{\left(\sup_{Y\in B(X,\operatorname{dist}(X,\partial\Omega)/2)} |A_0(Y)-A_1(Y)|\right)^2}{\operatorname{dist}(X,\partial\Omega)} dX < \infty,$$

then all the equivalent properties listed before hold for L_0 if and only if they hold for L_1. As a consequence, if it is the case that $A \in \operatorname{Lip}_{\text{loc}}(\Omega)$, $\left\||\nabla A|\operatorname{dist}(\cdot,\partial\Omega)\right\|_{L^\infty(\Omega)} < \infty$ and

$$\sup_{\substack{x\in\partial\Omega\\ 0<r<\operatorname{diam}(\partial\Omega)}} \frac{1}{\mathcal{H}^n(B(x,r)\cap\partial\Omega)} \iint_{B(x,r)\cap\Omega} |\nabla A(X)| dX < \infty,$$

then all the above properties are also equivalent to $\partial\Omega$ being Uniformly Rectifiable (see Section 2.3.1).

Anyway, what we explain in Chapter 4 is only a step towards the whole extension of [AHMMT20] to operators beyond the Laplacian: there is still substantial work to be done, but we believe that our efforts will help pave the way towards removing connectivity in settings beyond the Laplacian.

Let us lastly mention that apart from the Dirichlet problem, one could of course consider other boundary conditions like Neumann or the regularity problem. It turns out that there are some abstract methods to transfer the information about the solvability of those problems and the Dirichlet one, which have been explored in recent works like [DHP23, MT24a, MT24b]. We will not dive deeper here, but will just mention that the problem with Neumann boundary conditions has not yet been solved and is a major open problem in the area.

1.3 Parabolic PDE

So far, we have only considered elliptic PDEs. But these are the stationary states of the more physically relevant parabolic PDEs. These equations, whose model is often the heat equation $\partial_t u - \Delta u = 0$, are well-known to be more difficult than their elliptic counterpart because of the special role of time: for example, since time is not reversible, there are many properties which are only true if one looks in the correct direction in time (for instance the ubiquitous Harnack's inequalities).

Naturally, the Dirichlet problem with smooth data for the heat equation (compare to Section 1.1) has attracted much attention. Suitable analogues of the elliptic Wiener's criterion hold in the parabolic setting, characterizing the points of the boundary where solutions to the heat equation are continuous (i.e. *regular* points): these were shown in the works of Lanconelli [La73] and Evans and Gariepy [EG82]. However, the parabolic analogue of the result from [LSW63] is not true: if we consider operators with non-homogeneous diffusion like $\partial_t - \operatorname{div}(A\nabla)$, the regularity of boundary points may depend on the coefficients A, as shown by Petrowsky in [Pe35]. Since then, there have been several works that explain that both the values and the smoothness of A play a crucial role when determining regularity of boundary points (see [GL88, FGL89]), but of course it is clear that the situation is more subtle than the elliptic one. Concretely, quantitative versions of Wiener-like criteria, as those treated in Chapter 3, have not yet been

obtained in general domains, although there have recently been partial advances, as in [MP21, Section 3].

Furthermore, in the case of the Dirichlet problem with data in L^p, the distinguished role of time is not the only issue when dealing with parabolic PDE, there is another major difficulty as compared to elliptic PDE: Uniformly Rectifiable sets in the parabolic sense are not only geometrically flat (in the sense that they contain many Lipschitz portions, as their elliptic counterparts), but also need to satisfy certain analytical properties. Concretely, it is the case that if the boundary of our domain is a (parabolic) Lipschitz graph, which would be trivially elliptic Uniformly Rectifiable, may not be parabolic Uniformly Rectifiable: in order for the graph to be parabolic Uniformly Rectifiable, its half-time derivative in the direction of time must be somehow well-behaved, in the sense that it must belong to the BMO space. Indeed, otherwise there would exist counterexamples to the desired properties that an Uniformly Rectifiable set should fulfill, as shown by Kaufmann and Wu in [KW80]. Moreover, the definition that we just (informally) gave was shown to be the correct one by the early works of Lewis and Murray [LM95] and Hofmann and Lewis [HL96].

This new and more demanding definition of our central notion of Uniform Rectifiability makes the transition from elliptic to parabolic equations highly non-trivial: most of the interesting results that have been known for some time in the elliptic world are notably difficult to extend to their parabolic counterpart both because of the special role of time and the need to show BMO estimates for the half-time derivatives of the parametrizations of the boundary. Interestingly enough, the fact that this new condition is related to the BMO space, which is a central object in modern Harmonic Analysis, brings again this branch of Analysis into play.

To illustrate all these differences, let us comment on the recent result of Bortz, Hofmann, Martell and Nyström in [BHMN25]. They prove that the heat equation is well-posed with data in L^p in the domain above a Lipschitz graph if and only if this graph actually has half-time derivatives in BMO (i.e. the boundary of the domain is parabolic Uniformly Rectifiable). Informally,

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \Omega \subset \mathbb{R}^n \times \mathbb{R} \\ u = f & \text{on } \partial\Omega, \end{cases} \quad : \quad \begin{array}{c} \partial\Omega \text{ is parabolic} \\ \text{Uniformly Rectifiable} \end{array} \iff \begin{array}{c} \text{Well-posedness} \\ \text{if } f \in L^p(\partial\Omega). \end{array}$$

$+\ \partial\Omega$ is a parabolic Lipschitz graph

Again, this result would be much easier (essentially dating back to [Da77]) in the elliptic world because Lipschitz graphs are automatically (elliptic) Uniformly Rectifiable.

In a joint work with Bortz, Ferris and Hofmann [BFHH25], explained in detail in Chapter 5, we extend [BHMN25] beyond the heat equation. More precisely, our main result is:

> Let $\Omega \subset \mathbb{R}^{n+1}$, where $n \geqslant 2$, be the region above a $\mathrm{Lip}(1, 1/2)$ (parabolic Lipschitz) graph (see Section 2.5). Let A be a symmetric, uniformly elliptic and bounded matrix satisfying
>
> $$\sup_{\Omega} \left(\operatorname{dist}(\cdot, \partial\Omega)^k \left|\nabla_X^k A(\cdot)\right| + \operatorname{dist}(\cdot, \partial\Omega)^{2k} \left|\partial_t^k A(\cdot)\right| + \operatorname{dist}(\cdot, \partial\Omega)^3 \left|\partial_t \nabla_X A(\cdot)\right| \right) < \infty,$$
>
> where $k = 1, 2$, and for which $\left(\left|\nabla_X A(X,t)\right| + \left|\partial_t A(X,t)\right| \operatorname{dist}((X,t), \partial\Omega)\right) dXdt$ is a Carleson measure (see Section 2.2). If the Dirichlet problem in Ω with boundary data in L^p (for some $p < \infty$) is well posed for $\partial_t - \operatorname{div} A\nabla$ (see Section 5.6), then $\partial\Omega$ is Uniformly Rectifiable in the parabolic sense (namely, the graph defining Ω has half-order time derivatives belonging to parabolic BMO, see Section 5.4).

Informally, this means (compare to [BHMN25])

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = 0 & \text{in } \Omega \subset \mathbb{R}^n \times \mathbb{R} \\ u = f & \text{on } \partial\Omega, \end{cases} \quad : \quad \begin{array}{c} \partial\Omega \text{ is parabolic} \\ \text{Uniformly Rectifiable} \end{array} \quad \Longleftrightarrow \quad \begin{array}{c} \text{Well-posedness} \\ \text{if } f \in L^p(\partial\Omega). \end{array}$$

$+$ $\partial\Omega$ is a parabolic Lipschitz graph
$+$ A behaves poorly in few regions

In the spirit of what was explained in the last subsection, there is some evidence that the result should be true for the class of operators introduced in [FKP91], and we provide some intermediate result in this direction. Again, the key consideration is that one needs to handle and disregard the regions in which the operator may behave poorly, and hence be able to overall run the argument in [BHMN25]. This entails much technical work, again dealing with Carleson measures and corona decompositions to analyze good and bad scales.

1.4 Neumann conditions and overdetermined problems

As already commented before, there are other very meaningful boundary conditions beyond Dirichlet, and Neumann conditions (i.e. prescribing the normal derivative along the boundary of our domain) play a very predominant role among those. One may wonder whether it is possible for a solution to satisfy both Dirichlet and Neumann boundary conditions. This would be called an overdetermined boundary value problem.

Of course, even with 1-dimensional examples (ODE), we can convince ourselves that typically one cannot expect to prescribe both the value of a function and its derivative at the boundary of a domain where it satisfies a second-order equation. But, although generally false, it may happen in some cases. And it turns out that the domains in which we can pose meaningful overdetermined problems actually enjoy different desirable properties because of this unlikely property, so they usually deserve our attention.

In his groundbreaking paper [Se71], Serrin proved that for some simple equations, only trivial domains with many symmetries (in his case, balls) support overdetermined problems. Informally,

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = -\text{ct}. & \text{on } \partial\Omega, \end{cases} \quad + \quad \partial\Omega \text{ is } C^2 \quad \Longrightarrow \quad \Omega \text{ is a ball.}$$

Very recently, the hypothesis that $\partial\Omega$ is C^2 has been relaxed by Figalli and Zhang in [FZ24].

The result of [Se71], which is non-trivial at all, informs us that solutions of very symmetric equations must also enjoy some symmetry. In that regard, one may of course wonder what happens if one drops out the symmetry assumptions. In such case, it is not even clear that solutions exist, because explicit solutions as in the ball cannot be constructed.

In [EHR24], explained in detail in Chapter 6, we show that there are solutions to overdetermined boundary value problems for very general equations in $\mathbb{R}^{n+1}$, and also in manifolds. This effectively removes symmetry assumptions from our problems. Our main result reads:

Let $m > 0$ and f, A, q be (simultaneously) periodic, smooth functions satisfying the admissibility conditions in Definition 6.3, so essentially:

- A is symmetric and uniformly elliptic,
- $q(x) > 0$ for every $x \in \mathbb{R}^{n+1}$,
- $f(x,u) \geqslant 0$, and f is at most linear on u.

Then there exists an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfying $|\Omega| \approx m$, a constant $c > 0$, and a positive function u satisfying the overdetermined boundary value problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ A(x)\nabla u \cdot \nabla u = cq(x) & \text{on } \Omega. \end{cases}$$

Moreover, u is $C^\infty(\overline{\Omega})$ and can be extended as a Lipschitz function to the whole $\mathbb{R}^{n+1}$, and Ω is smooth up to a subset of dimension at most $n-4$.

One may informally think about this result as:

$$\exists \Omega \subset \mathbb{R}^{n+1}, \text{ of any size, such that } \exists u \geqslant 0 \text{ s.t. } \begin{cases} -\operatorname{div}(A\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = -\text{ct.} & \text{on } \partial\Omega. \end{cases}$$

An important feature of our proof is that we provide domains which have any size, which contrasts other techniques which find solutions in very small domains by perturbing solutions in balls, like in [PS09]. Moreover, we show that the boundaries of our domains are regular.

The method to find these solutions is very abstract and robust, and follows the very well-established variational program of Alt and Caffarelli [AC81], which has been later improved and refined by many authors, as can be seen in the recent monograph [Ve23] by Velichkov. It relies on the minimization of an energy over very abstract Sobolev functions, to take advantage of the compactness properties of Sobolev spaces. But after finding these solutions, one needs to show that they are actually somehow regular because of enjoying this special property of being minimizers of an energy related with an overdetermined problem. This involves a careful and technical program bringing together fine properties of PDE and Geometric Measure Theory through the Calculus of Variations.

1.5 Equations in non-divergence form

As a last way to generalize results that are known for the Laplace equation and not for wider classes of equations, we will consider linear elliptic equations with coefficients in non-divergence form. These and the divergence form ones already discussed, are the families of coefficients that one may find in any classical book. Although not as very well suited for weak formulations, there is a vast theory of equations in non-divergence form and is very relevant, for instance, as an intermediate step for nonlinear equations (a classical reference for those is the book by Caffarelli and Cabré [CC95]). When the coefficients are regular, it is essentially equivalent to work in divergence or non-divergence form (up to drift lower order terms, which are frequently harmless). But when the coefficients are not regular, the formulation of the problem makes a difference.

The theory of elliptic equations in non-divergence form is, at least for rough coefficients, still very far in its development from the divergence form counterpart. Before tackling successfully hard problems like the Dirichlet problem with data in L^p (which has received a lot of attention in the last 50 years in the divergence form case, as explained above), one needs to know more about non-divergence form operators.

Among the properties of divergence form equations that stand out for their applications in the last decades, is the so-called Kato square root property. After many (very relevant) partial progresses by different authors (see [CMM82, CDM83, FJK85, DJ84, Mc85, AT98, AHLT01, HLM02]), it was finally shown by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [AHLMT02]. Essentially, the Kato property tells us that the square root of any second-order elliptic operator (in divergence form) behaves, in a precise quantitative way, like the most preeminent first-order operator, the gradient. Informally,

$$L = -\operatorname{div}(A\nabla) \ + \ A \in L^\infty \quad \Longrightarrow \quad \operatorname{dom}(\sqrt{L}) = H^1 \ \text{ and } \ \|\sqrt{L}f\|_2 \approx \|\nabla f\|_2 \, .$$

This was actually one of the main questions in Calderón's program, posed by Kato in the early 60s, which remained unsolved for almost 40 years. Indeed, its solution integrates many modern techniques from Harmonic Analysis (like $T1$ and Tb theorems, extrapolation of Carleson measures...) that were developed along those 40 years, some of them even created ad-hoc to solve the problem (like McIntosh's Functional Calculus for sectorial operators) and have become a more general tool since then. All the techniques involved in the proof have proved to be useful in other problems afterwards. They have even proved to be part of a much more abstract theory, which for example includes the so-called *first order method* to view second order equations as first order systems, hence being able to use the powerful semigroup machinery (see e.g. [AKM06, AAM10a, AAM10b, AA11]).

Moreover, not only the techniques, but the main result in [AHLMT02] has been useful for many works ever since. For instance, let us mention its clear connection with the boundedness of the Riesz transforms and the subsequent applications to regularity of PDE via Calderón-Zygmund estimates.

It is clear that it would be very desirable to know whether the Kato square root property also holds for non-divergence form operators. This is still an open problem, but it would allow for a strong development of the theory, and probably the techniques used for the proof would be useful to examine other properties of solutions to PDEs in non-divergence form.

For the moment, a joint work with Escauriaza and Hofmann in [EHH24], that we will discuss in detail in Chapter 7, is the furthest development in this direction: we establish the Kato square root property for a very general family of coefficients of non-divergence form equations. Our main result is:

> Let L be a second order elliptic operator in non-divergence form (see Section 7.1), with smooth coefficients. Denote by W the global non-negative adjoint solution provided by Lemma 7.4. If $W \in A_2$ (see Section 2.2.1), then
>
> $$\|\sqrt{L}f\|_{L^2(W(x)dx)} \approx \|\nabla f\|_{L^2(W(x)dx)} \approx \|\sqrt{\widetilde{L}}f\|_{L^2(W(x)dx)}$$
>
> with uniform implicit constants, where $\widetilde{L}$ denotes the normalized adjoint of L (see (7.5)).

As explained in Chapter 7, this result covers at least the operators whose matrix of coefficients has small enough BMO norm. This result covers many relevant operators, but it is not yet the complete answer, the main difficulties being the fact that the adjoints of non-divergence form equations can behave very

poorly and the need to work in weighted Banach spaces. However, developing techniques to overcome these difficulties and lack of symmetry may be influential in itself, not only for the sake of solving the Kato problem for non-divergence form operators.

CHAPTER 2

General notation and preliminaries

Let us use the next few pages to set the stage properly for all the results which will come next. The reader will next find the notation that we will consistently follow along the document, and later all the necessary preliminary knowledge needed to face the next chapters (some of them will need a more specific preparation, which will be explained at due time).

2.1 Notation . . . 16
2.2 Harmonic Analysis . . . 17
2.2.1 Maximal functions and Muckenhoupt weights . . . 18
2.2.2 BMO and Carleson measures . . . 19
2.2.3 Littlewood-Paley theory . . . 22
2.3 Geometric Measure Theory . . . 24
2.3.1 Hausdorff measures, and boundaries of domains . . . 24
2.3.2 Dyadic analysis . . . 25
2.3.3 Quantitative openness and connectivity . . . 26
2.4 Elliptic PDE . . . 28
2.4.1 Basic framework and existence of solutions . . . 28
2.4.2 Interior estimates . . . 31
2.4.3 Inhomogeneous PDE . . . 32
2.4.4 Fine estimates near the boundary . . . 34
2.4.5 Potential theory . . . 36
2.4.6 Extensions to unbounded domains . . . 38
2.5 Extension to parabolic PDE . . . 41
2.5.1 Geometrical framework . . . 41
2.5.2 Existence of parabolic measure, and classification of boundary points . . . 43
2.5.3 Regularity of solutions . . . 44
2.6 A short introduction to Functional Calculus . . . 47

2.1 Notation

- We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on the hypotheses of the corresponding theorems (the "allowable parameters"). We shall also write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \leqslant Cb$ and $0 < c \leqslant a/b \leqslant C$, where the constants c and C are as above, unless explicitly noted to the contrary. Explicit dependencies will be denoted using brackets like $C(n)$ or subscripts like $\lesssim_n$.

- We shall use capital letters $X, Y, Z \ldots$ to denote generic points in $\mathbb{R}^{n+1}$, especially those in the domain of the PDE. In turn, we use lowercase letters $x, y, z \ldots$ when we want to emphasize that the points lie on the boundary of the domain where our PDE is posed.

- The open $(n+1)$-dimensional Euclidean ball of radius r will be denoted $B(x,r)$, or sometimes also $B_r(x)$. A "surface ball" (namely, a ball relative to the boundary of our domain) is sometimes denoted $\Delta(x,r) := B(x,r) \cap \partial\Omega$.

- Given a Euclidean ball $B = B(X,r)$, we will denote $r_B := r$, and its concentric dilate by a factor of $\kappa > 0$ will be denoted by $\kappa B = B(X, \kappa r)$.

- We denote by $|\cdot|$ the $(n+1)$-dimensional Lebesgue measure in $\mathbb{R}^{n+1}$, and by $\mathcal{H}^n$ the n-dimensional Hausdorff measure. We will frequently abbreviate $\sigma := \mathcal{H}^n|_{\partial\Omega}$, the surface measure on $\partial\Omega$, the boundary of our domain.

- We will use the dot $\cdot$ to denote the usual inner product in $\mathbb{R}^{n+1}$.

- We often denote $\delta(X) := \operatorname{dist}(X, \partial\Omega)$ for $X \in \mathbb{R}^{n+1}$, the distance to the boundary of the domain.

- We also emphasize by using different integral signs, like $\int$ or $\iint$, the dimension of the set over which we are integrating ($\int$ refers to n-dimensional sets, and $\iint$ to $(n+1)$-dimensional ones).

- Modified integral signs like $⨍$ denote averages. For instance, if $A \subset \partial\Omega$ with $0 < \sigma(A) < \infty$, we mean $⨍_A f\, d\sigma := \sigma(A)^{-1} \int_A f\, d\sigma$. Similarly, given a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and an Euclidean ball $B \subset \mathbb{R}^{n+1}$ with $|B| > 0$, we set $u_B := ⨌_B u\, dX := ⨌_B u\, dX := \frac{1}{|B|} \iint_B u\, dX$.

- Given $E \subset \mathbb{R}^{n+1}$ we write $\operatorname{diam}(E) = \sup_{x,y\in E} |x-y|$ to denote its diameter.

- If $E, F \subset \mathbb{R}^{n+1}$ are arbitrary subsets, we write $\operatorname{dist}(E,F) := \inf\{|x-y| : x \in E, y \in F\}$.

- We will use the symbol $\bigsqcup$ to denote a union comprised of pairwise disjoint sets.

- ∂_ν denotes the outer normal derivative, usually of the boundary of our domain.

- $\mathbf{1}_E$ denotes the indicator function of the set $E \subset \mathbb{R}^{n+1}$.

- We will denote by $\lambda_1(E)$ the first Dirichlet eigenvalue of the set E.

- We denote differentiation with respect to the different variables by ∂_{x_j} or $\frac{\partial}{\partial x_j}$.

- $\mathcal{S}$ will denote the usual Schwarz class of smooth, rapidly decaying functions on $\mathbb{R}^n$.
- We utilize the typical notation for Fourier transform, like $\widehat{f}$.
- Given a Borel set $E \subset \mathbb{R}^{n+1}$ and $0 < p < \infty$, we denote by $L^p(E)$ the usual Lebesgue space of functions whose p-th power is integrable, namely $\|f\|_{L^p(E)} := \left(\int_E \left|f(X)\right|^p \, dX\right)^{1/p}$. For $p = \infty$, $\|f\|_{L^\infty(E)} := \operatorname{ess\,sup}_{X\in E} \left|f(X)\right|$. We will frequently abuse notation and use sup instead of ess sup.
- Given a set $E \subset \mathbb{R}^{n+1}$ and $0 < \alpha \leqslant 1$, we denote by $C^\alpha(E)$ and $\dot{C}^\alpha(E)$ the inhomogeneous and homogeneous, respectively, Hölder spaces. Namely, $f \in \dot{C}^\alpha(E)$ is tested with the seminorm

$$[f]_{\dot{C}^\alpha(E)} := \sup_{\substack{X,Y\in E \\ X\neq Y}} \frac{\left|f(X) - f(Y)\right|}{|X-Y|^\alpha}.$$

  We will often abuse notation and call it $\|f\|_{\dot{C}^\alpha(E)}$. In turn, the inhomogeneous version is

$$\|f\|_{C^\alpha(E)} := \|f\|_{L^\infty(E)} + [f]_{\dot{C}^\alpha(E)}.$$

  The spaces $C^{k,\alpha}(E)$ for $k \geqslant 0$ are comprised of the functions which are k times differentiable and their derivatives of order k belong to $C^\alpha(E)$.
- We denote by $C_c^\infty(E)$ the space of smooth (infinitely differentiable) functions with compact support inside E.
- We say that the boundary $\partial\Omega$ of a set $\Omega \subset \mathbb{R}^{n+1}$ is $C^{k,\alpha}$ (for some $k \geqslant 0$ and $\alpha \in [0,1]$) if it is locally parametrized by $C^{k,\alpha}$ functions, i.e. we can cover $\partial\Omega$ with balls B so that (up to a change of coordinates depending on B) $\partial\Omega \cap B$ is the graph of a $C^{k,\alpha}$ function. We sometimes abuse notation saying that Ω is $C^{k,\alpha}$.

2.2 Harmonic Analysis

Along this work, we will deal with rough PDE in rough domains. This means that both the operators and the sets in which we pose the PDE may have lots of oscillations at arbitrary small scales, and this ends up implying that we need to do very careful analysis on how our solutions may oscillate at various scales.

The best tools to analyze how functions oscillate are given by Harmonic Analysis. Therefore, we will make frequent use of Harmonic Analysis throughout the whole work. So let us explain the main tools that will be used.

The reader will note that, opposite to a classical reference of Harmonic Analysis, we will avoid using the Fourier transform essentially at all times. This is one of the key points of this thesis: Fourier transform methods are extremely powerful when dealing with smooth functions or operators like the Laplacian or the heat equation. But it is well-known that they fail (or are basically useless) as soon as we use more general (possibly rough) operators or restrict ourselves to domains. Since our main focus is put in these situations (working both with rough operators and inside domains), we need tools

from Harmonic Analysis not relying directly on the Fourier transform. Luckily enough, a big part of modern Harmonic Analysis has been developed precisely to accomplish that, as we will see in the next subsections.

We will assume basic knowledge like L^p spaces, Fourier transform and the Plancherel theorem, the Schwartz space $\mathcal{S}$. All the facts below and needed throughout the text can be consulted in classical textbooks: we will follow [Du01, Gr08a, Gr08b], but one can also consult other books like [St70, St93].

2.2.1 Maximal functions and Muckenhoupt weights

Let us start by the definition of probably the most classical operator in Harmonic Analysis, the Hardy-Littlewood maximal function. We will define a centered and an uncentered version. See [Du01, Chapter 2] and [Gr08a, Section 2.1] for more details.

Definition 2.1 (Hardy-Littlewood maximal function)**.** Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define

$$\mathcal{M}f(x) := \sup_{r>0} \fint_{B(x,r)} \big|f(x-y)\big|\, dy, \qquad \widetilde{\mathcal{M}}f(x) := \sup_{\substack{B \text{ ball} \\ B \ni x}} \fint_B \big|f(x-y)\big|\, dy, \qquad x \in \mathbb{R}^n.$$

An easy argument using only the triangle inequality shows that $\mathcal{M}f(x) \leqslant \widetilde{\mathcal{M}}f(x) \leqslant C(n)\mathcal{M}f(x)$ for any $x \in \mathbb{R}^{n+1}$, so both versions of the maximal function are essentially interchangeable. Similarly, one could consider cubes instead of balls.

The main feature of this function is the following well-known boundedness property, which can is typically shown using classical covering lemmas.

Theorem 2.2. *$\mathcal{M}$ is bounded $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $1 < p \leqslant \infty$, and also satisfies the weak (1, 1) estimate*

$$\big|\{x \in \mathbb{R}^n : \big|\mathcal{M}f(x)\big| \geqslant \lambda\}\big| \leqslant C(n)\frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda}, \qquad \forall\, \lambda > 0.$$

The boundedness of the maximal function is a is very important fact in Harmonic Analysis because maximal functions control many classical operators (we will see some examples later on). This, at least at an intuitive level, is due to the presence of the supremum in the definition of the maximal functions, which makes them somehow "large" operators. Therefore, if we somehow manage to show that a given operator is controlled by some kind of averages, then it is likely that the operator will be bounded, which is of course a very desirable property.

One may wonder whether the maximal function is only bounded because of the nice properties of the Lebesgue measure, namely that it is constant/translation invariant. This is a reasonable question since we could be in the position of studying an operator which does not behave the same way everywhere or in different directions, so maybe we would like to take averages that are somehow weighted.

In fact, it is the case that maximal functions are actually flexible in the sense that they can incorporate weighted averages and still be bounded. To know exactly which averages are permitted, we introduce the Muckenhoupt weights. The reader may read [Du01, Chapter 7] or [Gr08b, Chapter 9] for all the information coming next.

Definition 2.3. A function w is called weight if $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e. in $\mathbb{R}^n$. Moreover, for a given $1 < p < \infty$, we say that w is in the Muckenhoupt A_p class, written $w \in A_p$, if it holds

$$[w]_{A_p} := \sup_{B \text{ ball}} \left(\frac{1}{|B|} \int_B w(x)\, dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'}\, dx \right)^{p-1} < \infty.$$

Moreover, we denote $A_\infty := \bigcup_{1<p<\infty} A_p$.

We will frequently use weights w as measures, because they are naturally associated with the measure $w(x)dx$.

It is also not difficult to check (just use Hölder's inequality) that $A_p \subset A_q$ whenever $p < q$, so the Muckenhoupt classes are ordered, and smaller indices correspond to better properties of the weight. In this direction, it is a much deeper fact that A_p weights always self-improve to a better Muckenhoupt class, which is a consequence of the following Reverse Hölder inequality, first discovered by Gehring.

Theorem 2.4 (Reverse Hölder)**.** *Given $w \in A_p$, for some $1 < p < \infty$, there exists $1 < q = q(n, p, [w]_{A_p}) < p$ such that $w \in A_q$. More precisely, there exists $r = r(n, p, [w]_{A_p}) > 1$ such that*

$$\left(\frac{1}{|B|} \int_B w(x)^r\, dx \right)^{\frac{1}{r}} \leqslant \frac{C(n, p, [w]_{A_p})}{|B|} \int_B w(x)\, dx, \qquad \forall B \subset \mathbb{R}^{n+1} \text{ ball.}$$

A proof of this result can be carried out by iterating a Calderón-Zygmund decomposition, see [Du01, Theorem 7.4], along with the following relationships between weights and Lebesgue measure, which are usually handier in practice than the definition of an A_p weight.

Lemma 2.5. *Let $w \in A_p$ for $1 < p < \infty$. Then w is doubling, i.e., $w(B(x, 2r)) \leqslant Cw(B(x, r))$ for every $x \in \mathbb{R}^{n+1}$, $r > 0$. Moreover, for every $\alpha \in (0, 1)$ there exists $\beta \in (0, 1)$ such that*

$$\frac{|S|}{|B|} \leqslant \alpha \quad \Longrightarrow \quad \frac{w(S)}{w(B)} \leqslant \beta, \qquad \forall B \text{ ball, and } S \subset B.$$

A very important consequence of the Reverse Hölder property is the fact that the Muckenhoupt weights are the precise class of weights for which the Hardy-Littlewood maximal function is bounded. Indeed, the classical argument (see [Du01, Theorem 7.3]) consists on first using the Reverse Hölder property (to get that $w \in A_q$ for some $q < p$), then showing weak (p, p) boundedness in $L^p(w(x)dx)$ (which follows very closely the proof of Theorem 2.2, merely using a covering argument), and finishing by the Marcinkiewicz interpolation theorem. The precise result reads us follows.

Theorem 2.6. *Given $1 < p < \infty$, $\mathcal{M} : L^p(\mathbb{R}^n, w(x)dx) \longrightarrow L^p(\mathbb{R}^n, w(x)dx)$ is bounded if and only if $w \in A_p$.*

2.2.2 BMO and Carleson measures

The ubiquitous Lebesgue spaces L^p are very well behaved when $1 < p < \infty$. However, there is usually trouble at the endpoints 1 and ∞. Very often, a good substitute for L^∞ is the John-Nirenberg's space of functions with bounded mean oscillation (BMO).

Definition 2.7 (BMO space)**.** Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we say that f belongs to BMO if

$$\sup_{Q \text{ cube}} \fint_Q \left| f(x) - \fint_Q f \right| dx < \infty. \tag{2.1}$$

There are many interesting properties about the BMO space, which can be consulted in [Du01, Chapter 6] or [Gr08b, Chapter 7]. It clearly holds $L^\infty \subset \text{BMO}$, and therefore it is usually simpler to prove that an operator is bounded $L^\infty \to \text{BMO}$ than to prove that it is bounded $L^\infty \to L^\infty$. This added with the fact that there are some interpolation schemes that work well with BMO (see [Du01, Theorem 6.8]), makes the BMO space very useful in Harmonic Analysis.

The property that a function f belongs to BMO is essentially a matter of *cancellations*: given a cube Q, there is a subset of Q where f is greater than $\fint_Q f$ (which contributes positively to the integral in (2.1)), and another subset where f is smaller (and contributes negatively). A suitable balance between these positive and negative contributions is what is needed for the function to belong to BMO. All this works because we subtract the average in (2.1) *inside* the absolute value: if we used the triangle inequality to bring the absolute values inside, we would end up with the definition of L^∞ (by Lebesgue Differentiation Theorem), hence losing all the new information.

However, detecting cancellations is usually difficult to prove or formalize. It is often easier to show that some quantity is *small* or not-very-big (i.e. talking about *sizes*) than to show that it carries some cancellation. For that purpose, if we want to study cancellation properties for functions $f(x)$ defined in $\mathbb{R}^n$ as above, it is useful to associate them with other functions $F(x,t)$ defined in $\mathbb{R}^{n+1}_+$ which are positive and, at (x,t), reflect some property of f in the ball $B(x,t)$ (like the deviation from its average as in (2.1)). In this way, studying cancellation properties for functions in $\mathbb{R}^n$ will somehow reduce to studying sizes of positive functions in $\mathbb{R}^{n+1}_+$.

To find the right conditions that help us understand which functions $F(x,t)$ are good in the sense that they yield good properties for their counterparts $f(x)$, we introduce Carleson measures. For this part, the reader may consult [Du01, Chapter 9] or [Gr08b, Chapter 7].

Definition 2.8 (Carleson measures)**.** Given a positive measure ν on $\mathbb{R}^{n+1}_+$, we say that it is a Carleson measure if

$$\|\nu\|_{\mathscr{C}} := \sup_{Q \text{ cube}} \frac{\nu\big(Q \times (0, \ell(Q))\big)}{|Q|} < \infty.$$

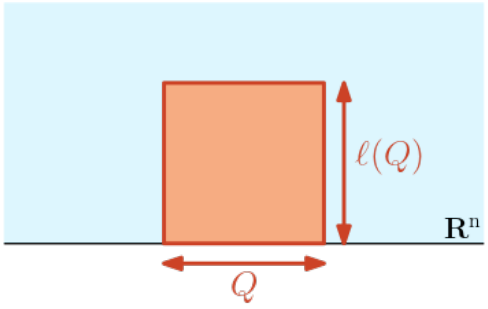


Figure 2.1: A Carleson measure lives in $\mathbb{R}^{n+1}_+$, but sees every Carleson box (namely $Q \times (0, \ell(Q))$) as if it had 1 dimension less, because its measure is controlled by the measure of the "basis" Q.

The intuition it that a measure ν is a Carleson measure if it is somehow n-dimensional, although it is originally defined in an $(n+1)$-dimensional space, which again connects the functions $f(x)$ defined in $\mathbb{R}^n$ with functions $F(x,t)$ defined in $\mathbb{R}^{n+1}_+$. To make this relationship explicit, let us define the following non-tangential maximal function:

Definition 2.9 (Non-tangential maximal function)**.** Given $F(x,t)$ defined in $\mathbb{R}^{n+1}_+$ we define its non-tangential maximal function $\mathcal{N}F : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\mathcal{N}F(x) := \sup_{(y,t)\in\Gamma(x)} \big|F(y,t)\big|, \qquad x \in \mathbb{R}^n,$$

where $\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < t\}$ is a cone with vertex at x.

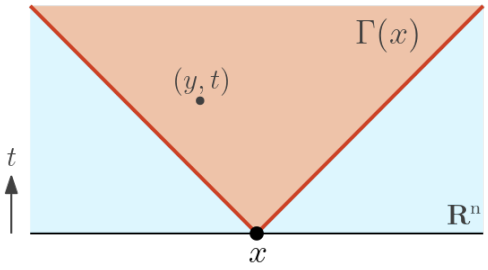


Figure 2.2: A depiction of a cone living in $\mathbb{R}^{n+1}_+$, with vertex at $x \in \mathbb{R}^n$: the non-tangential maximal function $\mathcal{N}F$ takes the supremum of F over the colored region. For our purposes, the aperture of the cones is not relevant (this is classical, although non-trivial).

One could have taken any other aperture for the cones using $|y-x| < (1+\alpha)t$ for $\alpha > 0$, but one often ends up with a (up to constants) equivalent definition for the sake of applications.

Carleson measures and non-tangential maximal functions are intimately related by duality. This duality is a deep property that actually extends to much more abstract settings like the theory of tent spaces, initiated by [CMS85]. The precise result is as follows (the reader can see a proof in [St93, Section II.2])

Theorem 2.10 (Carleson's lemma)**.** *Let ν be a Carleson measure in $\mathbb{R}^{n+1}_+$ and $F(x,t)$ be measurable in $\mathbb{R}^{n+1}_+$. Then, for any $0 < p < \infty$,*

$$\int_{\mathbb{R}^{n+1}_+} \big|F(x,t)\big|^p \, d\nu(x,t) \leqslant C(n)\, \|\nu\|_{\mathscr{C}} \int_{\mathbb{R}^n} (\mathcal{N}F(x))^p \, dx.$$

In the case that one chooses $F(x,t) := \phi_t * f(x)$ where ϕ_t is the approximate identity associated to ϕ, namely $\phi_t(x) := t^{-n}\phi\left(\frac{x}{t}\right)$, one obtains very meaningful results, which are known as Littlewood-Paley theory, which we will explore next. These are very connected with taking (smooth) averages of f at the scale of the ball $B(x,t)$, which brings us again to the connection between BMO and Carleson measures. A result which turns out to be very practical is the following (see [St93, Section II.2] or [Gr08a, Section 2.1.2]), that brings us again the intuition that maximal functions are *large* in some sense.

Lemma 2.11. *Let $\phi \in L^1(\mathbb{R}^n)$ be radial, non-negative, radially decreasing, and $\phi(0) < \infty$. If we define $\phi_t(x) := t^{-n}\phi\left(\frac{x}{t}\right)$, then*

$$\mathcal{N}(\phi_t * f)(x) \leqslant C(n,\phi)\, \mathcal{M}f(x), \qquad x \in \mathbb{R}^n.$$

2.2.3 Littlewood-Paley theory

The so-called Littlewood-Paley theory is a collection of tools and facts that illustrate that one can modify the frequencies of a function in some precise ways, and still preserve the norm, hence somehow extending Plancherel's theorem. A first statement illustrating this, that one should always have in mind is the following result. We follow [Du01, Chapter 8], [Gr08a, Chapter 5] (or also [St70, Section IV.5]), where they prove the following result using the theory of vector-valued Calderón-Zygmund operators, central objects in modern Harmonic Analysis.

Theorem 2.12 (Littlewood-Paley in $\mathbb{R}$, non-smooth)**.** *Let $f \in L^p(\mathbb{R})$ for $1 < p < \infty$. For $j \in \mathbb{N}$, define the following multiplier via the Fourier transform*

$$(S_j f)\widehat{\ }(\xi) = \mathbf{1}_{(-2^{j+1},-2^j]\cup[2^j,2^{j+1})}(\xi)\widehat{f}(\xi), \qquad \xi \in \mathbb{R}.$$

Then

$$c(p)\,\|f\|_{L^p(\mathbb{R})} \leqslant \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} \leqslant C(p)\,\|f\|_{L^p(\mathbb{R})}\,.$$

As commented above, the moral of this result is that as long as one modifies the frequencies in dyadic annuli separately, the norm is (essentially) preserved. Note that this would be true automatically in L^2 by Plancherel's theorem, and gives the impression of having some kind of orthogonality in L^p: it is as if the frequencies in different dyadic annuli lived in spaces which are orthogonal to each other, so that if one is willing to take norms overall, one could measure in each of these spaces separately. In that sense, this result is central to the theory of Harmonic Analysis, as so is the Plancherel theorem in Fourier Analysis, or in general any way of expanding a vector with respect to an orthogonal basis in a general vector space. The power of Littlewood-Paley theory is that it will allow us to study operators distinguishing how they behave on small and large frequencies, which in turn relate to very different properties of functions, like those which are local or global.

It turns out that to generalize the last result to higher dimensions, it is better to use smooth truncations in the space of frequencies, rather than just using indicator functions as above. This is very much related to the famous Fefferman's counterexample regarding the ball multiplier [Fe71], which illustrates that non-smooth functions are not good multipliers in higher dimensions.

Therefore, we prefer to use smooth truncations in the space of frequencies. For instance, given $\psi \in \mathcal{S}$ with $\psi(0) = 0$ (somewhat more flexible than a cut-off function supported on an annulus), we can use $(\widetilde{S}_j)\widehat{\ }(\xi) = \psi(2^j\xi)\widehat{f}(\xi)$, and this should do the role of S_j. Note that this operator can also be seen in the physical side as convolving against $(2^j)^{-n}\check{\psi}\left(\frac{\cdot}{2^j}\right)$, which should remind the reader of the previous section. With these ideas in mind, we have the following theorem, where we use a continuous family of truncations instead of a discrete one as above. We restrict our attention to $p = 2$ for this preliminary exposition.

Theorem 2.13 (Littlewood-Paley in $\mathbb{R}^n$, smooth version)**.** *Let $\phi \in C_c^\infty(B(0,1))$ be radial, nontrivial, and satisfy $\int_{\mathbb{R}^n} \phi(x)\,dx = 0$. Then it is clear that $\widehat{\phi}(0) = 0$ and $\widehat{\phi} \in \mathcal{S}\backslash\{0\}$, and let us (upon multiplication by a constant) normalize so that*

$$\int_0^\infty |\widehat{\phi}(r)|^2 \frac{dr}{r} = 1. \tag{2.2}$$

Denote $Q_s f(x) := \phi_s * f(x)$, *where* $\phi_s(x) := s^{-n}\phi(x/s)$ *(so* $(Q_s f)\widehat{\ }(\xi) = \widehat{\phi}(sx)\widehat{f}(\xi)$*). Then*

$$\int_{\mathbb{R}^n} Q_s^2 f \, \frac{ds}{s} = f, \qquad \forall f \in L^2(\mathbb{R}^n). \tag{2.3}$$

Moreover, the following identity holds:

$$\int_0^\infty \int_{\mathbb{R}^n} \left|Q_s f(x)\right|^2 \, dx \, \frac{ds}{s} = \int_{\mathbb{R}^n} \left|f(x)\right|^2 \, dx, \qquad \forall f \in L^2(\mathbb{R}^n). \tag{2.4}$$

The proof of both conclusions follows easily after using the Plancherel's theorem to work on the Fourier side, the reader can see a self-contained proof in [BHL20, Section 2]. In general, an upper bound in (2.2) is related with the upper bound in (2.4), and similarly for lower bounds (see [Du01, Theorem 8.6]).

Note that the vertical square function estimate (2.4) is a continuous version of the estimate in Theorem 2.12 when $p = 2$, and has the feature of working in any dimension. Again, both (2.3) (known as Calderón's reproducing formula) and (2.4) give us a sense that the operators Q_s are some kind of orthogonal projections because to measure the size of a function f, we may measure the sizes of each $Q_s f$, and then add everything up in an L^2 sense (and Q_s projects to a space where frequencies are small or large, depending on the size of s).

A very important result from the point of view of applications is the following generalization of the last theorem, which gives us some flexibility when trying to prove square function estimates for other families of operators, not necessarily acting by convolution as the Q_s.

Theorem 2.14 (Almost orthogonality)**.** *Let* R_t, *for* $t > 0$, *be a family of operators which are uniformly bounded in* $L^2(\mathbb{R}^n)$, *i.e.*

$$\|R_t f\|_{L^2(\mathbb{R}^n)} \leqslant C_1 \|f\|_{L^2(\mathbb{R}^n)}, \qquad \forall\, f \in L^2(\mathbb{R}^n),\ t > 0,$$

and also satisfy, where Q_s *are as defined in Theorem 2.13,*

$$\|R_t Q_s\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leqslant C_2 \min\left\{\frac{t}{s}, \frac{s}{t}\right\}^\alpha \tag{2.5}$$

for some $\alpha > 0$. *Then* R_t *satisfy the square function estimate*

$$\int_0^\infty \int_{\mathbb{R}^n} \left|R_t f(x)\right|^2 \, dx \frac{dt}{t} \leqslant C(C_1, C_2, \alpha) \int_{\mathbb{R}^n} \left|f(x)\right|^2 \, dx.$$

The reader can see a proof in [Gr08b, Theorem 8.6.4], or a shorter and self-contained one in [BHL20, Theorem 2.10]. It is based on a reduction of the boundedness of the square functions associated to Q_s to the ones of R_t via (2.5), and the latter are bounded by Theorem 2.13 (or also Lemma 2.11).

This theorem informs us that if the operators R_t are "almost" orthogonal projections to different frequencies because (2.5) holds (note that the meaning of (2.5) is that if t and s have very different magnitudes, then $R_t Q_s$ should be very small), then we are still on the favorable position of having small interactions between the norms of R_t for different t so that adding all up is still a bounded operator. Recall that, since Q_s is basically a projection onto frequencies of size roughly s, the assumption (2.5) is informing us about how R_t behave over different frequencies.

A clear advantage of Theorem 2.14 is that, although internally it somehow considers frequencies, it only requires information of our object of interest (the operators R_t) on the physical side. Indeed, as commented at the beginning of this section, Fourier transform methods are not available whenever one does PDE in domains and/or with non-smooth coefficients, and both are our main goals in this work. Thus, it is of crucial importance to be able to draw conclusions which would be available in more favorable settings via Fourier transform methods, which have shown in the last centuries to be extremely powerful. In this sense, the modern Littlewood-Paley theory that we just presented is a very powerful theory that generalizes Fourier transform methods to rougher settings, hence letting us somehow work with frequencies, but only using the physical side, which is the one which is always available.

2.3 Geometric Measure Theory

Along the whole work, our focus will be put on working on the most general sets possible, in which we can still obtain desirable properties for solutions to PDE. Certainly, the kind of sets which we will be able to explore will go beyond the classical framework of domains whose boundary is C^2 or $C^{2,\alpha}$, which is for instance the usual assumption in classical PDE references like [GT77].

2.3.1 Hausdorff measures, and boundaries of domains

Our main object of study will be boundaries of open sets in $\mathbb{R}^{n+1}$. At least for smooth domains, their boundaries will be lower dimensional (codimension 1 actually). Therefore, to carry out analysis or topology with those, we need to be able to measure them correctly. The Hausdorff measure does this role, analog to what Lebesgue measure does with open sets (of the dimension of the ambient space).

Definition 2.15 (Hausdorff measure)**.** Given $s, \delta > 0$ and $E \subset \mathbb{R}^{n+1}$, we define

$$\mathcal{H}^s_\delta(E) := \inf\left\{ \sum_j \operatorname{diam}(U_j)^s : E \subset \bigcup_j U_j,\ \operatorname{diam}(U_j) \leqslant \delta \right\},$$

and the s-dimensional Hausdorff measure by

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

We also define the Hausdorff dimension as

$$\dim_{\mathcal{H}}(E) = \inf\left\{s > 0 : \mathcal{H}^s(E) = 0\right\} = \sup\left\{s > 0 : \mathcal{H}^s(E) = \infty\right\}.$$

As commented above, if $\Omega \subset \mathbb{R}^{n+1}$ is a smooth domain, meaning that its boundary is locally parametrized by a smooth function, then $\partial\Omega$ will be n-dimensional, as measured by Hausdorff dimension. However, in this work we are interested in sets which are not necessarily smooth. In that sense, a typical way to generalize this condition is to allow for Ahlfors-David regular boundaries. These have the advantage of providing a very general framework in which one can still perform a lot of Analysis.

Definition 2.16 (Ahlfors-David regular)**.** We say that a closed set $E \subset \mathbb{R}^{n+1}$ is n-dimensional Ahlfors-David regular (or simply ADR) if there is some uniform constant $C \geqslant 1$ such that

$$C^{-1} r^n \leqslant \mathcal{H}^n(E \cap B(x,r)) \leqslant C r^n, \qquad \forall\, x \in E,\ r \in (0, 2\operatorname{diam}(E)).$$

Moreover, the class of Uniformly Rectifiable sets, which has appeared very often in the Introduction, will be one of our central objects of study. It is a generalization of Lipschitz sets in which one can, after careful work, still can prove meaningful analytic properties. This is the reason why these kind of sets, and their qualitative counterparts (rectifiable sets), are central objects in Geometric Measure Theory.

Definition 2.17 (Uniformly Rectifiable)**.** A set $E \subset \mathbb{R}^{n+1}$ is uniformly rectifiable (or simply *UR*) if it is ADR and has big pieces of Lipschitz images of $\mathbb{R}^n$ (BPLI for short). The latter means that there exist $\theta, M > 0$ such that for every $x \in E$ and $r \in (0, \operatorname{diam}(E))$ there is a Lipschitz mapping $\rho = \rho_{x,r} : B_{\mathbb{R}^n}(0,r) \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ with Lipschitz constant controlled by M, such that

$$\mathcal{H}^n(E \cap B(x,r) \cap \rho(B_{\mathbb{R}^n}(0,r))) \geqslant \theta r^n.$$

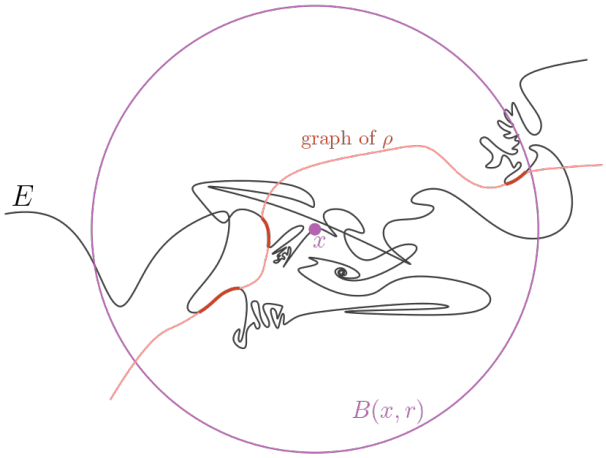


Figure 2.3: One should imagine a typical Uniformly Rectifiable set as very oscillating and with many cusps, but still having some tiny Lipschitz portions at every scale. Here, we depict a very oscillating set which still has some flat parts that we may cover with a Lipschitz graph.

For more details about these concepts and related ones, one may consult [DS93], or any reference on Geometric Measure Theory like [Ma95] (or even the foundational [Fe69]). However, these go well beyond the scope of the present work, so let us keep matters as simple as possible.

2.3.2 Dyadic analysis

Our emphasis in all this work will be made on dealing with non-smooth functions in non-smooth domains. These are both objects that exhibit non-trivial oscillations at various places and possibly very small scales. Therefore, we need tools to examine scales separately. For open sets, the main tool are Whitney decompositions (see e.g. [Gr08a, Appendix J])

Theorem 2.18 (Whitney decomposition)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Then there exists a family of closed cubes $\{Q_j\}_j$ such that*

1. *$\Omega = \bigcup_j Q_j$ and the Q_j have disjoint interiors,*
2. *$\sqrt{n+1}\,\ell(Q_j) \leqslant \operatorname{dist}(Q_j, \mathbb{R}^{n+1}\backslash\Omega) \leqslant 4\sqrt{n+1}\,\ell(Q_j)$,*
3. *if $\partial Q_j \cap \partial Q_k \neq \varnothing$, then $1/4 \leqslant \ell(Q_j)/\ell(Q_k) \leqslant 4$,*

4. *given Q_j, there are at most 12^{n+1} cubes Q_k such that $\partial Q_j \cap \partial Q_k \neq \varnothing$.*

This theorem effectively distinguishes the relevant scales of an open set: in our setting, the main focus must be put on what happens at the boundary $\partial\Omega$ of our domain Ω, so it is important to distinguish regions depending on their distance to $\partial\Omega$, as the previous Whitney decomposition does. To decompose the boundary $\partial\Omega$ in different relevant scales, we have the following result, not as elementary as before, dating back to [Ch90] (see also [DS91, DS93])

Theorem 2.19 (Dyadic grid)**.** *Let $E \subset \mathbb{R}^{n+1}$ be n-dimensional ADR. Then there exists $C_1 = C_1(n, ADR) \geqslant 1$ such that, for each $k \in \mathbb{Z}$, there is a collection of Borel sets (which we call "cubes")*

$$\mathbb{D}_k = \{Q_j^k \subset E : j \in \mathfrak{I}_k\}$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying:

(a) *$E = \bigcup_j Q_j^k$, for each $k \in \mathbb{Z}$.*

(b) *If $m \leqslant k$, then either $Q_i^k \subset Q_j^m$ or $Q_i^k \cap Q_j^m = \varnothing$.*

(c) *For each $k \in \mathbb{Z}$, $j \in \mathfrak{I}_k$, and each $m < k$, there is a unique $i \in \mathfrak{I}_k$ such that $Q_j^k \subset Q_i^m$.*

(d) *For each $k \in \mathbb{Z}$, $j \in \mathfrak{I}_k$, there is $x_j^k \in E$ such that*

$$B(x_j^k, C_1^{-1} 2^{-k}) \cap E \ \subset \ Q_j^k \ \subset \ B(x_j^k, C_1 2^{-k}) \cap E.$$

This construction does the role of a dyadic decomposition in cubes, which would be easy to perform in the whole $\mathbb{R}^{n+1}$, but is non-trivial to do in objects of smaller dimension (hence the need of some regularity, imposed by the ADR condition). Indeed, this decomposition resembles the classical dyadic one in that it is stratified in generations of cubes of size roughly 2^{-k} (properties (a) and (d)), cubes of different generations are either nested or disjoint (property (b)), and there is only one way to go up in the hierarchy of generations of nested cubes (equivalently, that there is a unique parent cube, property (c)).

We will explore other more convolved dyadic constructions, like corona decompositions or cut-off functions to integrate by parts in non-smooth domains, at due time. We defer the discussion for the next chapters because of its technicality.

2.3.3 Quantitative openness and connectivity

Apart from the study of boundaries of domains, we will often require some (fine) geometrical or topological properties to the domains themselves. The point is that we want to carry out Analysis of quantitative nature while studying our PDEs, so we need our open sets to behave well in a quantitative way. In this direction, let us first define a quantitative notion of being open.

Definition 2.20 (Corkscrew condition)**.** We say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the (interior) corkscrew condition if for some uniform constant $c \in (0, 1)$, and for every surface ball $\Delta := B(x, r) \cap \partial\Omega$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, cr) \subset B(x, r) \cap \Omega$. The point $X_\Delta \in \Omega$ is called a "corkscrew point" relative to Δ.

On top of this quantitative openness notion, let us also define a quantitative version of path-connectedness. This condition has been extensively used in the literature, and we will often try to remove it from our arguments to make the results stronger.

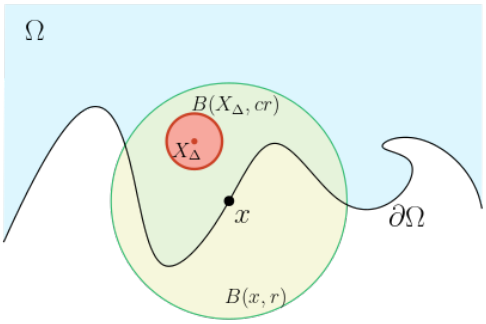


Figure 2.4: The corkscrew condition means that, inside every ball centered at the boundary of our domain, we can always find a smaller ball of comparable size which is fully contained in the domain.

Definition 2.21 (Harnack chains)**.** We say that an open set Ω satisfies the Harnack chain condition if there is a uniform constant C such that for every $\rho > 0$, $\Lambda \geqslant 1$, and every pair of points $X, X' \in \Omega$ with $\min\{\delta(X), \delta(X')\} \geqslant \rho$ and $|X - X'| < \Lambda\rho$, there is a chain of open balls $B_1, \ldots, B_N \subset \Omega$, $N \leqslant C(\Lambda)$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \varnothing$, $C^{-1} \operatorname{diam}(B_k) \leqslant \operatorname{dist}(B_k, \partial\Omega) \leqslant C \operatorname{diam}(B_k)$.

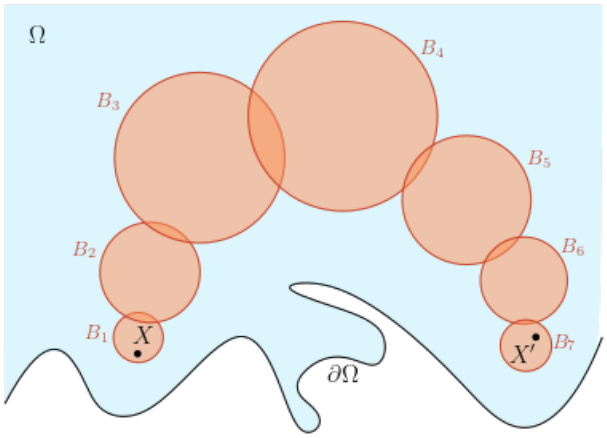


Figure 2.5: The Harnack chain condition asserts that, between any two points $X, X' \in \Omega$, we can always find a sequence of balls connecting them within Ω, and each of these balls are far from $\partial\Omega$ (so we can apply Harnack's inequality with uniform constants inside each of these balls). We also ask for the number of balls in the sequence to be uniformly bounded in terms of the distance between X and X', and their distances to the boundary: this ensures that, by repeatedly applying Harnack's inequality along the sequence of balls, the values of any solution at X and X' are comparable.

The conditions explained above allow us to define several classes of nicely behaved domains.

Definition 2.22 (1-sided NTA and CAD domains)**.** We say that Ω is a 1-sided NTA (non-tangentially accessible) domain if Ω satisfies both the corkscrew and Harnack chain conditions. If a 1-sided NTA domain has an ADR boundary, then it is called a 1-sided CAD (chord-arc domain).

As a remark, 1-sided NTA are often also called *uniform* in the literature.

2.4 Elliptic PDE

After all the preliminary explanations on Harmonic Analysis and Geometric Measure Theory, let us come to the main object of study of this work: Partial Differential Equations. Let us start by an introduction to elliptic PDEs, which are easier to understand than parabolic ones. We will first state a number of results (with which any graduate student will be familiar) that explain the modern way to approach PDE: showing existence via weak solutions, and then using different kinds of regularity theory to improve their properties. We will finish the section with finer estimates about solutions close to the boundary (which is the part where geometry plays the major role), which are probably not so well-known to people outside the area, but should nevertheless be understandable for anyone with some background in Functional Analysis and Measure Theory.

2.4.1 Basic framework and existence of solutions

As commented above, the modern way to approach PDE is to show existence of solutions in very big spaces with nice compactness properties (the Sobolev spaces, that we present next), and later improve their regularity by other means. Hence, let us start by defining the Sobolev spaces because they constitute the framework in which we work. One may consult the following properties in any classical book, like [GT77, Chapter 7] or [Ev98, Chapter 5].

Definition 2.23 (Sobolev spaces)**.** Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Given $u, v_j \in L^1_{\text{loc}}(\Omega)$, we say that v_j is the weak derivative (or distributional derivative) of u in the j-th direction if

$$\int_\Omega u \partial_{x_j} \phi \, dX = -\int_\Omega v_j \phi \, dX, \qquad \forall \, \phi \in C_c^\infty(\Omega).$$

We denote $\nabla u = (v_1, \dots, v_n)$ the weak gradient of u if weak derivatives exist for all directions.

We define $W^{1,2}(\Omega)$ as the Hilbert space of functions $u \in L^2(\Omega)$ for which their weak gradient satisfies $\nabla u \in L^2(\Omega)$, endowed with the inner product $\langle u, v \rangle := \int_\Omega uv \, dX + \int_\Omega \nabla u \cdot \nabla v \, dX$.

We also define $W^{1,2}_0(\Omega)$ as the closure of $C_c^\infty(\Omega)$ under the $W^{1,2}(\Omega)$ topology.

One can also define higher order Sobolev spaces by imposing conditions on higher derivatives (which are defined inductively in the naive way), and also with integrability conditions in L^p rather than L^2 as above. Indeed, $W^{k,p}$ will denote the space of functions whose derivatives up to order k are L^p integrable.

Sobolev spaces enjoy many desirable properties. The following is among the most important ones.

Theorem 2.24 (Rellich-Kondrachov)**.** *Let $1 \leqslant p < n+1$ and $\Omega \subset \mathbb{R}^{n+1}$ be a bounded and open set with C^1 boundary. Then, for every $1 \leqslant q < p^*$, $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$.*

Here and everywhere else, p^*, the Sobolev conjugate of p (if $p < n+1$), is defined by

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n+1}.$$

This compactness result and the Banach-Alaoglu theorem from abstract Functional Analysis, turn out to be usually the crucial tool that allows us to find solutions to PDE in Sobolev spaces, as opposed to showing the existence of solutions in spaces of classical differentiability. This is the reason why we usually prefer to first solve PDE in Sobolev spaces (very large spaces) instead of trying to show directly

existence of solutions in spaces of "nice" functions (e.g. spaces of classical differentiability), which can be done for specific PDE, but not in general.

The Rellich-Kondrachov embedding theorem is actually a consequence of a family of inequalities, known as Sobolev inequalities, that illustrate how having integrable derivatives usually improves the integrability of the function itself. These are summarized as follows:

Theorem 2.25 (Sobolev inequalities)**.** *If $1 \leqslant p < n+1$, then*

$$\|u\|_{L^{p^*}(\mathbb{R}^{n+1})} \leqslant C(n,p)\,\|\nabla u\|_{L^p(\mathbb{R}^{n+1})}\,.$$

If $n+1 < p \leqslant \infty$, then, for $\gamma := 1-(n+1)/p$, it holds

$$\|u\|_{C^\gamma(\mathbb{R}^{n+1})} \leqslant C(n,p)\,\|u\|_{W^{1,p}(\mathbb{R}^{n+1})}\,.$$

The previous Sobolev inequalities can also be extended to hold in (somewhat smooth, for instance C^1) domains $\Omega \subset \mathbb{R}^{n+1}$, simply by using $W^{1,p}(\Omega)$ norms in all the right hand sides. An important version of these inequalities in domains is the Poincaré inequality, ubiquitous in Analysis because it encapsulates minimization of energies in domains via the smallest eigenvalue λ_1.

Lemma 2.26 (Poincaré's inequality)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Then there exists $\lambda_1(\Omega) > 0$ such that*

$$\int_\Omega u^2\,dX \leqslant \lambda_1(\Omega)^{-1}\int_\Omega |\nabla u|^2\,dX, \qquad u \in W^{1,2}_0(\Omega).$$

To complement this lemma, it is simply a matter of homogeneity to show that $\lambda_1(B(X_0,r)) \approx_n r^{-2}$ for any $X_0 \in \mathbb{R}^n$ and $r > 0$. With this in mind, it is also useful to have the following extension of Poincaré's inequality, that works for functions which do not necessarily vanish on the boundary:

$$\int_{B(X_0,r)} \left|u(X) - \fint_{B(X_0,r)} u\right|^2 dX \lesssim r^2 \int_{B(X_0,r)} \left|\nabla u(X)\right|^2 dX, \qquad u \in W^{1,2}(B(X_0,r)).$$

Let us next define the class of operators in divergence form, that we will consider most of the time to generalize the Laplacian $-\Delta = -\sum_{i=1}^n \partial_{x_i}\partial_{x_i}$, which is the prototypical differential operator in PDE theory. The basic theory about these operators can be consulted in [GT77, Chapter 8] or [Ev98, Chapter 6].

Definition 2.27 (Elliptic operator)**.** Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. We define the elliptic operator L of the form $Lu = -\operatorname{div}(A\nabla u)$ if $A(\cdot) = (a_{i,j}(\cdot))_{i,j=1}^{n+1}$ is a real (non-necessarily symmetric) matrix such that $a_{i,j} \in L^\infty(\Omega)$ and there exists $\Lambda \geqslant 1$ such that the uniform ellipticity condition

$$\Lambda^{-1}|\xi|^2 \leqslant A(X)\xi\cdot\xi \quad \text{ and } \quad |A(X)\xi\cdot\eta| \leqslant \Lambda|\xi|\,|\eta| \tag{2.6}$$

holds for all $\xi,\eta \in \mathbb{R}^{n+1}$ and for almost every $X \in \Omega$.

After these preliminaries, we are ready to state what we understand by a solution to a PDE.

Definition 2.28 (Weak solution)**.** Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and $L = -\operatorname{div}(A\nabla)$ be an elliptic operator. We say that u is a weak solution to $Lu = 0$ in Ω, or that $Lu = 0$ in the weak sense in Ω, if $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega A(X)\nabla u(X)\cdot\nabla\phi(X)\,dX = 0, \qquad \forall \phi \in C^\infty_c(\Omega).$$

Given an open set $\Omega \subset \mathbb{R}^{n+1}$ and an elliptic operator $L = -\operatorname{div} A\nabla$, there are many weak solutions to $Lu = 0$ in Ω (any constant function, for example). It is more interesting to focus on those solutions which have a prescribed behavior at the boundary $\partial\Omega$. For instance, we can prescribe the values of u at $\partial\Omega$, the so-called Dirichlet boundary conditions. These boundary conditions are among the most relevants that one can consider, and the existence of solutions satisfying these boundary conditions is a very abstract fact (see e.g. [HKM06, Chapter 11], a manifold of methods that apply to different families of operators in [HL11, Chapter 6], and a careful exposition for the Laplacian in [PT23, Chapter 3]).

Theorem 2.29 (Existence of solutions and elliptic measure)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and $L = -\operatorname{div} A\nabla$ an elliptic operator. Assume further that Ω is bounded. Then there exists a family of positive Borel regular measures (the so-called elliptic measure) $\{\omega_L^X\}_{X\in\Omega}$ supported on $\partial\Omega$ such that $\omega_L^X(\partial\Omega) = 1$, and if we define*

$$u(X) := \int_{\partial\Omega} f(y)\, d\omega_L^X(y), \qquad X \in \Omega,$$

for $f \in C(\partial\Omega)$, then $Lu = 0$ in Ω in the weak sense and u satisfies the maximum principle

$$\inf_{\partial\Omega} f \leqslant \inf_{\Omega} u \leqslant \sup_{\Omega} u \leqslant \sup_{\partial\Omega} f.$$

Moreover, if $f \in \operatorname{Lip}(\partial\Omega)$, then u as constructed above is the unique weak solution of $Lu = 0$ in Ω satisfying $u = f$ in the $W^{1,2}(\Omega)$ sense on $\partial\Omega$ (i.e. $u - F \in W_0^{1,2}(\Omega)$ for any $F \in \operatorname{Lip}_c(\mathbb{R}^{n+1})$ extending f, that is, $F|_{\partial\Omega} = f$ on $\partial\Omega$).

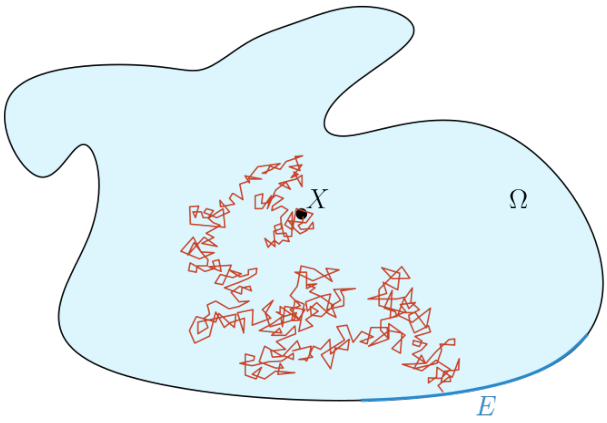


Figure 2.6: It is well-known (Kakutani's theorem) that harmonic measure has a probabilistic interpretation: given $E \subset \partial\Omega$ and $X \in \Omega$, $\omega_{-\Delta}^X(E)$ represents the probability that a Brownian motion started at X first exits Ω through E. This gives us a lot of intuition regarding why conditions on "largeness" of $\partial\Omega$ (or sometimes $\mathbb{R}^{n+1}\backslash\overline{\Omega}$) are relevant: the parts of $\partial\Omega$ occupying the most space, at least intuitively, are the most likely ones to be hit by random (Brownian) travelers, and this will manifest rigorously by means of the harmonic measure.

The proof of this result is well-known, but let us sketch it. Given a Lipschitz boundary data f, one can easily extend it to $F \in \operatorname{Lip}_c(\mathbb{R}^{n+1})$ (see e.g. [EG92, Chapter 3]). That allows us to search for $u - F$ satisfying an elliptic PDE with boundary data identically 0 (homogeneous), instead of our initial inhomogeneous boundary datum f. Then, the existence (and uniqueness) of solutions of the form $u - F$ is guaranteed by the Lax-Milgram theorem, which is a very abstract result from Functional Analysis (alternatively, one can also use the Perron method). After adding up F again, we obtain the unique solution

u. The extension to continuous boundary data is a matter of approximating continuous functions by Lipschitz ones. After that, to ensure the existence of the elliptic measure, we use the Riesz representation theorem, stating that the dual of the space of continuous functions in the compact set $\partial\Omega$ is a space of measures satisfying the properties in the statement of the above theorem.

2.4.2 Interior estimates

Now that we have discussed existence and uniqueness of solutions, let us summarize some of the most classical properties that solutions to elliptic PDE enjoy. Let us start by several so-called interior estimates, which are true as long as we stay away from $\partial\Omega$. Concretely, we first have the following standard energy estimate, whose easy proof (just an integration by parts using a cut-off) can be consulted in [HL11, Lemma 1.36].

Lemma 2.30 (Caccioppoli's inequality)**.** *Let L be an elliptic operator and u a weak solution to $Lu = 0$ in $B(X_0, 2r)$. Then*

$$\int_{B(X_0,r)} |\nabla u|^2 \, dX \leqslant C(n,\lambda)\, r^2 \int_{B(X_0,2r)} u^2 \, dX.$$

A much deeper fact is the following local self-improvement in the integrability of weak solutions. The following results can be found in [GT77, Sections 8.6 and 8.7] and, even more pertinently and clearly, in [HL11, Chapter 4].

Lemma 2.31 (Local boundedness)**.** *Let L be an elliptic operator and u a weak solution to $Lu = 0$ in $B(X_0, R)$. Then, for every $0 < r < R$ and $0 < p < \infty$,*

$$\|u\|_{L^\infty(B(X_0,r))} \leqslant \frac{C(n,\lambda,p)}{(R-r)^{n/p}} \, \|u\|_{L^p(B(X_0,R))}.$$

This result was key in the proof of one of the milestones in the theory of PDE in the last century, summarized in the next result.

Theorem 2.32 (De Giorgi/Nash: Interior Hölder continuity)**.** *Let L be an elliptic operator. Then there exist $C \geqslant 1$ and $\alpha \in (0,1)$ (depending only on n and λ) such that for every ball $B = B(X_0, r) \subset \mathbb{R}^{n+1}$ and for every $u \in W^{1,2}(2B)$ so that $Lu = 0$ in the weak sense in $2B$, one can redefine u in a set of null measure (abusing the notation, the new function is called again u) in such a way that $u \in C^\alpha(\overline{B}) \subset C(\overline{B}) \cap L^\infty(B)$ satisfying*

$$\sup_{X\in\overline{B}} |u(X)| + \sup_{\substack{X,Y\in\overline{B}\\ X\neq Y}} \frac{|u(X)-u(Y)|}{(|X-Y|/r)^\alpha} \leqslant C\left(\fint\!\!\fint_{2B} |u(Y)|^2 \, dY \right)^{\frac{1}{2}}.$$

This result, proved independently by De Giorgi and Nash at the end of the 1950s, and shortly after refined by Moser, was the final step to solve the 19th Hilbert problem. It is a key fact in the theory of non-smooth PDE, with deep applications in the realm of non-linear PDE: in principle, solutions to PDE only lived in Sobolev spaces, so they were very general functions. This theorem asserts that the fact that these functions solve a PDE makes them actually better, they live in a space of classical regularity, a Hölder space (although maybe for small α). One should compare this with the classical fact that harmonic functions (i.e. whenever A is the constant identity matrix) are locally real analytic: although

in the setting of a general elliptic operator we can only hope for the local Hölder continuity, it is a very non-trivial result because the matrices A can be very poorly behaved: we are only assuming that they are bounded, let alone any kind of continuity or classical regularity.

As a consequence of the above, we can get another classical property of solutions to PDE.

Theorem 2.33 (Harnack's inequality)**.** *Let L be an elliptic operator and u a weak solution to $Lu = 0$ in the ball $2B$. If $u \geqslant 0$ in $2B$, then*

$$\sup_{X \in B} u \leqslant C(n, \lambda) \inf_{X \in B} u.$$

2.4.3 Inhomogeneous PDE

So far, we have only considered homogeneous PDE, namely of the kind $Lu = 0$ in some open set Ω. But one can also consider more general equations with a right hand side like $Lu = f$, and this is actually crucial from the perspective of applications. A very important tool to study these kind of inhomogeneous equations are fundamental solutions. These have been known for the simplest cases (Poisson and heat kernels) for centuries, but the development of a general theory that works for any elliptic operator is much more recent. In this direction, we want to highlight the works by Littman, Stampacchia and Weinberger in [LSW63], Grüter and Widman in [GW82], and Hofmann and Kim in [HK07], that we summarize in the next result.

Theorem 2.34 (Fundamental solutions)**.** *Let L be an elliptic operator. Then there exists a unique fundamental solution $\Gamma_L(X, Y)$ satisfying*

1. *Γ_L is continuous in $\{(X, Y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : X \neq Y\}$,*
2. *$\Gamma_L(X, \cdot)$ is integrable for a.e. $X \in \mathbb{R}^{n+1}$,*
3. *for every $f \in C_c^\infty(\mathbb{R}^{n+1})$, the function defined by*

$$u(X) := \int_{\mathbb{R}^{n+1}} \Gamma_L(X, Y) f(Y)\, dY$$

solves $Lu = f$ in the weak sense, namely

$$\int_{\mathbb{R}^{n+1}} A\nabla u \cdot \nabla \phi \, dX = \int_{\mathbb{R}^{n+1}} f\phi \, dX, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^{n+1}).$$

One usually thinks (and can actually be formalized in the language of distributions) that $L\Gamma_L(X, \cdot) = \delta_X$, a Dirac's delta, so that integrating as above (similarly as in a convolution) against any function f gives $Lu = f$. There are a number of additional properties that the fundamental solutions satisfy, regarding for example their integrability and smoothness (see [HK07, Theorem 3.1]). We highlight the decay property (if $n + 1 \geqslant 3$) that

$$\Gamma_L(X, Y) \leqslant C(n, \lambda) \, |X - Y|^{1-n},$$

which is truly important. The homogeneity in this estimate is the same as for the simplest operator, the Laplacian $L = -\Delta$, for which the fundamental solution can be (easily) computed explicitly (see e.g. [GT77, (4.1)]) $\Gamma_{-\Delta}(X, Y) = C(n) \, |X - Y|^{1-n}$. Actually, it also holds (see [LSW63, (7.9)])

$$\Gamma_L(x, y) \approx \Gamma_{-\Delta}(X, Y) \approx |X - Y|^{1-n},$$

with implicit constants only depending on n and λ.

Although not as good as solutions to homogeneous PDE, solutions to PDE with a right hand side are also very nicely behaved if so is the right hand side. For instance, one can show (at least if the right hand side of the equation is bounded locally) some *approximate* mean value formulae that can also play the role of Harnack inequalities, hence becoming a very basic tool in the theory.

Lemma 2.35. *Let L be an elliptic operator with C^α coefficients, for some $\alpha > 0$. Let $r \in (0,1)$ and $X_0 \in \mathbb{R}^{n+1}$, and suppose that $u \in H^1(B(X_0,r))$ satisfies $u \geqslant 0$ on $\partial B(X_0,r)$. For any $M \geqslant 0$, the following hold:*

1. *If $Lu \leqslant M$ in $B(X_0,r)$, then*

$$u(X) \lesssim_n \fint_{\partial B(X_0,r)} u \, d\mathcal{H}^{n-1} + Mr^2, \qquad X \in B(X_0, r/2).$$

2. *If $|Lu| \leqslant M$ in $B(X_0,r)$, then*

$$\left|\nabla u(0)\right| \lesssim_n \frac{1}{r} \fint_{\partial B(X_0,r)} u \, d\mathcal{H}^{n-1} + Mr.$$

The proof of these estimates for the Laplacian is well-known because it follows at once from the explicit form of the Poisson kernel, but it is more subtle for general L with merely C^α coefficients. We will provide a complete proof in Chapter 7.

Apart from basic tools to deal with solutions, a major question in the world of PDE is to know how regular are solutions to a given equation. For example, it is very well-known that harmonic functions, although only assumed initially to be twice differentiable, are actually real analytic. This kind of self-improvement in the regularity of functions because of solving a PDE is not exclusive to harmonic functions/the Laplace operator, and manifests itself in many ways. Let us present two of the most classical, Schauder and Calderón-Zygmund estimates, that illustrate how solutions to PDE "gain two derivatives" with respect to the regularity of the right hand side of the equation, as would be our initial intuition because the equation is of second order. The reader may read about these in [GT77, Chapters 6 and 9], [HL11, Chapter 5], and the very understandable exposition in [FR22, Chapter 2].

Theorem 2.36 (Schauder regularity)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded open set, and L an elliptic operator. If, for some $\alpha > 0$, it holds that $\partial\Omega$ is C^2, the coefficients of L are $C^\alpha(\Omega)$, $f \in C^\alpha(\Omega)$, and $\varphi \in C(\partial\Omega)$, then there exists a solution $u \in C^{2,\alpha}(\Omega)$ of $Lu = f$ in Ω satisfying $u = \varphi$ on $\partial\Omega$.*

In the previous result, it is necessary to use Hölder spaces for $\alpha > 0$, for otherwise the result can fail (see a discussion in [FR22, Chapter 2]). Moreover, there are analogues of the same result that provide higher regularity of u, namely if $f \in C^{k,\alpha}$, then $u \in C^{k+2,\alpha}$ as long as the other parameters are also smooth enough (see [GT77, Theorem 6.17] or [FR22, Corollary 2.29]).

If instead of gaining classical derivatives, we think on the scale of weak derivatives and their integrability, the following result also showcases that there is a gain of two derivatives.

Theorem 2.37 (Calderón-Zygmund regularity)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and L an elliptic operator. If, for some $1 < p < \infty$, it holds that $\partial\Omega$ is $C^{1,1}$, the coefficients of L are uniformly continuous, $f \in L^p(\Omega)$ and $\varphi \in W^{2,p}(\Omega)$, then there exists a solution $u \in W^{2,p}(\Omega)$ of $Lu = f$ in Ω satisfying $u - \varphi \in W^{1,p}_0(\Omega)$ (so it is the Dirichlet boundary datum).*

Again, there are similar results implying higher regularity of u in this scales of Sobolev spaces, see [GT77, Theorem 9.19]. Note that Sobolev embeddings imply that the solutions u with right hand side $f \in L^p$ actually belong to L^q for $q > p$: this constitutes an improvement in the integrability, that one can often bootstrap (i.e. iterate) to obtain strong conclusions like $u \in L^\infty$.

2.4.4 Fine estimates near the boundary

Although we have stated both Schauder and Calderón-Zygmund regularity methods in a global way, that is, asserting regularity on the whole domain Ω, they are actually results of local nature. Namely, their full power is obtained when we restrict ourselves to compact subsets of Ω (we may just think on small balls fully contained inside Ω), and they lose their *sharpness* as soon as we touch the boundary. In fact, this is a key point: obtaining estimates for solutions of PDE that are true up to the boundary $\partial\Omega$ is often much harder than obtaining interior estimates. This is usually the case because one needs some regularity of $\partial\Omega$ itself to obtain regularity of solutions near $\partial\Omega$.

The classical method (see [GT77, Section 6.4] or [FR22, Section 2.6]) to deal with estimates at the boundary is the following: if the boundary is smooth enough (usually C^2), one can locally flatten it (locally) to a hyperplane, and reflect solutions in an appropriate way (usually as odd functions). In that way, one often obtains an extension of the original solution u to a slightly bigger set than Ω, and then estimates in $\overline{\Omega}$ become interior estimates in this slightly bigger set. This is what has been done in Theorems 2.36 and 2.37.

Nevertheless, we can be much finer and precise if we need better control on behavior of solutions near the boundary. In fact, since this is the main goal of this work, let us provide some tools that will appear repeatedly from here on. These require careful analysis on the geometry of the boundary and suitable analogues to fundamental solutions adapted to our domains Ω.

As a first example, let us present a non-degeneracy estimate for the elliptic measure, which dates back to Bourgain's work [Bo87], although in a slightly different form. For the whole proof in the form that we present (actually under a weaker assumption on the boundary), in the recent (and easier to understand) survey by Prats and Tolsa (concretely, see [PT23, Remark 7.24]).

Lemma 2.38 (Bourgain's estimate)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with ADR boundary, and L an elliptic operator. Then, there exists $c = c(n, \lambda, ADR) > 0$ such that, for $x \in \partial\Omega$ and $r > 0$ it holds*

$$\omega_L^X(\partial\Omega \cap B(x,r)) \geqslant c, \qquad \forall X \in B(x, cr).$$

This lemma informs us that if we are close enough to a surface ball (i.e. a ball at the boundary), the elliptic measure is always non-degenerate. Given the parallelism of elliptic PDE with Brownian travelers (namely, the harmonic measure $\omega^X(E)$ for $E \subset \partial\Omega$ measures the probability that a Brownian traveler starts moving at X and first exits Ω through E, see Figure 2.6), this is basically telling us that if we release a Brownian traveler very close to a portion of the boundary, it is very likely that it will hit the boundary nearby (see Figure 2.7). Although this would seem intuitively very natural, note that we have made an assumption on the geometry of the boundary, namely the ADR property from Definition 2.16, which somehow ensures that there is a sufficient amount of boundary around x. This will always be the case whenever we study properties around the boundary: some assumption needs to be made, for otherwise we cannot even control the size or dimension of the boundary, which can make very intuitive things fail.

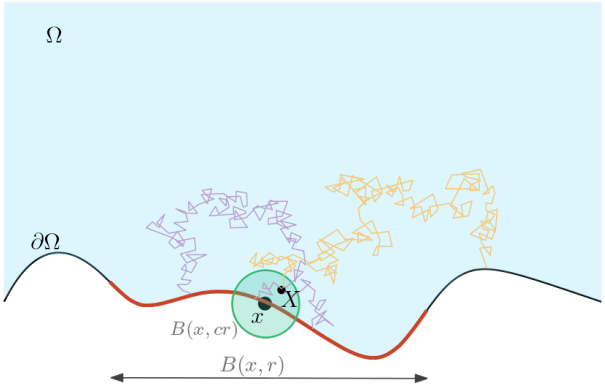


Figure 2.7: The result in Lemma 2.38 should be intuitive given the probabilistic interpretation of harmonic measure (recall Figure 2.6): if the boundary $\partial\Omega$ is somewhat large (this is the meaning of the ADR condition here) and we start a Brownian motion very close to the boundary (indeed, $\operatorname{dist}(X, \partial\Omega) \leqslant cr$, where c is a small constant at our disposal), it should be likely that a Brownian traveler hits the boundary nearby, so definitely within $B(x, r)$, which is a much larger ball. Actually, the probabilistic interpretation of harmonic measure was very important in Bourgain's paper [Bo87], where Lemma 2.38 first appeared (actually, in a sharper version than the one stated here).

As commented above, we also introduce the suitable analogues to fundamental solutions which behave well close to the boundary $\partial\Omega$, the so-called Green's functions, developed in the present generality by Grüter and Widman in [GW82] and Hofmann and Kim in [HK07]. The main difference with respect to fundamental solutions is that Green's functions vanish at the boundary $\partial\Omega$, so they are very connected with solutions within Ω.

Theorem 2.39 (Green's functions)**.** *Let L be an elliptic operator and $\Omega \subset \mathbb{R}^{n+1}$ an open, bounded and connected set. Then there exists a unique Green's function $G_L(X, Y)$ satisfying*

1. *G_L is continuous in $\{(X, Y) \in \Omega \times \mathbb{R}^{n+1} : X \neq Y\}$,*
2. *$G_L(X, \cdot)$ is locally integrable in Ω for all $X \in \Omega$,*
3. *for every $f \in C_c^\infty(\Omega)$, the function defined by*

$$u(X) := \int_\Omega G_L(X, Y) f(Y)\, dY$$

solves $Lu = f$ in Ω in the weak sense and belongs to $W_0^{1,2}(\Omega)$, namely

$$\int_\Omega A\nabla u \cdot \nabla\phi\, dX = \int_\Omega f\phi\, dX, \qquad \forall \phi \in C_c^\infty(\Omega),$$

4. *$G_L(\cdot, Y)$ vanishes on $\partial\Omega$ for every $Y \in \Omega$.*

The Green's functions are key tools when studying general solutions to PDE in $\partial\Omega$. The moral is always that Green's functions exhibit the model behavior of solutions which vanish at the boundary, and one always tries to compare solutions to Green's functions for that reason. For instance, the following result is frequently very useful, and is kind of a generalized version of results like the well known Hopf's lemma that asserts that harmonic functions arrive to the boundary in a linear way.

Lemma 2.40 (Upper bound in CFMS). *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with ADR boundary, and L an elliptic operator. If $X, Y \in \Omega$ with $|X - Y| \geqslant \delta(Y)/2$, then*

$$\frac{G_L(X,Y)}{\delta(Y)} \leqslant C(n,\lambda,ADR)\, \frac{\omega_L^X(B(\hat{y}, 2\delta(Y)) \cap \partial\Omega)}{\mathcal{H}^n(B(\hat{y}, 2\delta(Y)) \cap \partial\Omega)},$$

where $\hat{y} \in \partial\Omega$ satisfies $|Y - \hat{y}| = \delta(Y)$.

Indeed, this lemma informs us that, wherever ω_L behaves like $\mathcal{H}^n|_{\partial\Omega}$ (as would be the case in a smooth domain for nice L), then the Green's functions arrive linearly to the boundary. If this were the case, we would hope for the rest of solutions of PDE to satisfy the same kind of behavior because Green's functions exhibit the model behavior. In favorable settings, this is made rigorous by the so-called Boundary Harnack inequalities.

Apart from the previous result, in favorable settings one has a stronger version of Lemma 2.40.

Lemma 2.41 (Full CFMS). *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with ADR boundary, and L an elliptic operator. Assume further that Ω is an 1-sided NTA domain (so that Ω is actually a 1-sided CAD). If $X \in \Omega$, $y \in \partial\Omega$ and $r > 0$ satisfy $X \notin 2B(y,r)$, then*

$$G_L(X, X_{B(y,r)}) \approx r^{1-n}\omega_L^X(B(y,r) \cap \partial\Omega),$$

where we recall that $X_{B(y,r)}$ is the corkscrew point associated to the surface ball $B(y,r) \cap \partial\Omega$.

For a proof of Lemma 2.40 in its current form (actually, under a weaker background assumption on the boundary), see [PT23, Remark 7.34]. In fact, Lemma 2.40 is a modern refinement of Lemma 2.41, which has been known for decades now. After being shown in Lipschitz domains by Dahlberg in [Da77], in the more general setting of 1-sided CAD domains, it first appeared in the work of Caffarelli, Fabes, Mortola and Salsa [CFMS81, Lemma 2.2] (it takes the name from these initials). It was shortly after applied successfully for the complete study of NTA/CAD domains by Jerison and Kenig [JK82, Lemma 4.8], and it has been extensively used ever since (see also the monograph of Kenig [Ke94, Section 1.3], for instance). The same happens with Bourgain's estimate (Lemma 2.38): before the generalization of Bourgain [Bo87] to very general sets, it had been used in more favorable settings like 1-sided CAD in the foundational works [Da77, Lemma 2], [CFMS81, Lemma 2.1] and [JK82, Lemma 4.2] (see also [Ke94, Section 1.3]). These works paved the way to study boundary behavior of harmonic functions in non-smooth domains, which has been perfected since then in works like this one.

2.4.5 Potential theory

To obtain even finer estimates near the boundary, we need appropriate *geometric* conditions on $\partial\Omega$. It turns out that the correct way to measure properties of $\partial\Omega$ which are important for PDE is not always given by objects from Geometric Measure Theory, but from other branch of Analysis called Potential Theory. The basic object in this field are capacities. Let us define the classical ones for elliptic PDE, the Newtonian capacities.

Definition 2.42. Given an open set $D \subset \mathbb{R}^{n+1}$ (where we recall that we always assume that $n \geqslant 2$) and a compact set $K \subset D$ we define the capacity of K relative to D as

$$\mathrm{Cap}(K,D) := \inf\left\{ \iint_D |\nabla v(X)|^2 dX : \ v \in C_c^\infty(D),\ v \geqslant 1 \text{ in } K \right\}.$$

There are analogues for $n = 1$, like the logarithmic capacities, but we will not discuss them here.

The usefulness of capacities to model boundary behavior of solutions of PDE is exemplified by the following result, Wiener's criterion, already known for a century now, which characterizes when solutions arrive to the boundary in a continuous way. Many of the classical sources that we have cited until now contain proofs of this criterion (see e.g. [GT77, Theorem 8.31], [HKM06, Theorem 6.27]), but we recommend the exposition in [PT23, Chapter 6] if the reader is new to the realm of capacities. That works for the Laplacian, and the extension to general elliptic operators was first shown in [LSW63, Theorem 9.2].

Theorem 2.43 (Wiener's criterion)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and L an elliptic operator. The following conditions are equivalent:*

1. *Ω satisfies the Wiener condition, namely: for every $x \in \partial\Omega$ it holds*

$$\int_0^1 \frac{\mathrm{Cap}(\overline{B(x,r)}\setminus\Omega, B(x,2r))}{\mathrm{Cap}(\overline{B(x,r)}, B(x,2r))}\frac{dr}{r} = \infty,$$

2. *For every $\varphi \in C(\partial\Omega)$, the solution u to $Lu = 0$ in Ω with $u = \varphi$ on $\partial\Omega$, satisfies $u \in C(\overline{\Omega})$.*

This criterion is indeed the right condition to obtain continuity up to the boundary. But continuity is only a qualitative property, and most of the theory of PDE that we will work with is actually quantitative. Therefore, it will be useful to introduce the natural quantitative analogue of the Wiener's condition as follows.

Definition 2.44 (**Capacity density condition, CDC**)**.** An open set Ω is said to satisfy the *capacity density condition* (CDC for short) if there exists a uniform constant $c > 0$ such that

$$\frac{\mathrm{Cap}(\overline{B(x,r)}\setminus\Omega, B(x,2r))}{\mathrm{Cap}(\overline{B(x,r)}, B(x,2r))} \geqslant c$$

for all $x \in \partial\Omega$ and $0 < r < \mathrm{diam}(\partial\Omega)$.

This will indeed turn out to be the right condition that allows us to characterize which functions arrive to the boundary in a Hölder continuous way, which is nothing more than the natural quantification of continuity. We will discuss this in more detail in Chapter 3.

Moreover, the CDC condition is a *geometric* condition on our domain (indeed, it has some geometric meaning because it is intimately connected to Hausdorff contents, see e.g. [PT23, Corollary 7.30]) which can be used to generalize the fine estimates in the previous subsection. For instance, one can show the following result.

Lemma 2.45 (Boundary Hölder continuity)**.** *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set satisfying the CDC, and let $L = -\mathrm{div}(A\nabla)$ be an elliptic operator. There exist $C \geqslant 1$ and $\alpha \in (0,1)$ (depending only on n, λ and the CDC constant) such that for every ball $B = B(x,r)$ with $x \in \partial\Omega$, $0 < r < \mathrm{diam}(\partial\Omega)$; and for every $0 \leqslant u \in W^{1,2}_{\mathrm{loc}}(2B \cap \Omega) \cap \mathscr{C}(\overline{2B \cap \Omega})$ with $u \equiv 0$ in $2B \cap \partial\Omega$ satisfying $Lu = 0$ in the weak sense in $2B \cap \Omega$, there holds*

$$u(X) \leqslant C\left(\frac{|X-x|}{r}\right)^{\alpha} \sup_{B\cap\Omega} u, \quad \forall X \in B \cap \Omega.$$

Indeed, this is a very fine estimate that generalizes the classical Hopf's lemma and Lemma 2.41 to very rough settings. Indeed, we knew that, in "nice" domains,0 solutions that vanish at the boundary arrive linearly to it. This result informs us that in very general sets, this linear behavior may not hold, but we always have at least some Hölder-like decay when approaching the boundary. In fact, this non-linear behavior is well-known to be expected for instance around corners or cusps of a domain (think e.g. on the harmonic function $u(X) = X_1X_2$ in $\Omega = (0,\infty)^2 \subset \mathbb{R}^2$).

The proof of Lemma 2.45 in its current form is a consequence of [HKM06, Theorem 6.18], as shown in [Zh18, Proposition 3.4] (see also [PT23, Section 7.5.2] for the Laplacian). Again, the foundational works in the area already made use of the same Hölder continuity around the boundary, at least in more favorable domains like NTA or CAD, see e.g. [JK82, Lemma 4.1] and [Ke94, Section 1.3].

2.4.6 Extensions to unbounded domains

Some of the questions and results discussed in the previous subsections become more subtle when our domain of reference Ω is unbounded. Regularity results or properties of solutions close to the boundary are usually true without a modification because they are of local nature. However, more fundamental questions like existence and uniqueness of solutions or whether they satisfy maximum principles (see Theorem 2.29), become much more challenging.

For instance, even in the easiest possible example, when one is willing to solve the problem $-\Delta u = 0$ in $\mathbb{R}^{n+1}_+$ with $u = 0$ on $\partial\mathbb{R}^{n+1}_+$, there are at least two very easy solutions: the trivial solution $u \equiv 0$, and also $u(X) = X_{n+1}$. And both solutions are very meaningful: the trivial solution is the one we would expect if we worked in bounded sets, and $u(X) = X_{n+1}$ is very connected to Green's functions (actually, it is the so-called Green's function with pole at infinity) and exhibits a desirable lineal growth at the boundary.

It is well-known that there are ways to limit the growth of solutions or prescribe in an appropriate way their value at infinity, so that uniqueness is ensured. But for the purpose of this work, let us only explain which solutions (or equivalently, elliptic measures) will be our canonical choices when solving PDE in unbounded domains. This will be sufficient for the next chapters.

Indeed, our solution of interest will be the one that emerges naturally when taking limits with elliptic measures. Namely, the elliptic measure associated to an unbounded set Ω will be the one satisfying

$$\omega_L^X(F) = \lim_{R\to\infty} \omega^X_{L,\Omega\cap B(0,R)}(F), \qquad \forall\, F \subset \partial\Omega \text{ Borel.} \tag{2.7}$$

This is the usual definition of elliptic measure in an unbounded domain (see e.g. [PT23, Remark 5.44] for a slightly less general situation than ours) because it leads naturally to many desirable properties for solutions, like maximum principles. A full development of this theory in our generality (for very general elliptic operators and open sets) has been pursued in [Hi21, Chapter 4] and will appear in [HMT]. The reader can also consult [PT23, Section 5.5] for a less general framework.

On top of the above, this elliptic measure can be constructed following another method, slightly modifying what we sketched in Section 2.4.1, by simply considering other spaces rather than $W^{1,2}$: first, for Lipschitz boundary data, use the Lax-Milgram theorem to construct (unique) solutions $u \in L^{2^*}(\Omega)$ with $\nabla u \in L^2(\Omega)$, which agree with the data in the Sobolev sense; and then extend this construction to compactly supported continuous boundary data by density. This allows us to invoke the Riesz representation

theorem to obtain the associated elliptic measure, and it actually coincides with the one constructed via (2.7). Each construction has its own benefits, and together, they give more power to the theory.

A key point to consider for unbounded domains is the question whether the elliptic measure is a probability or not, i.e., whether $\omega_L^X(\partial\Omega) = 1$ for every $X \in \Omega$. This is the case when Ω is bounded because the only solution (it is crucial that there is uniqueness of solutions, see Theorem 2.29) with boundary datum constantly 1 is the function which is constantly 1, too. But in unbounded domains, we have chosen one elliptic measure among possibly other options, so the question is not clear.

Let us start with the best known case, when Ω is unbounded but $\partial\Omega$ is bounded (e.g. the complement of a ball). In this case, it is well-known that the elliptic measure is not a probability (it even has a probabilistic reason, because the Brownian motion is transient, meaning that Brownian travelers almost surely never return to a place already visited, and actually escape to infinity).

Lemma 2.46. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and L be an elliptic operator. If Ω is unbounded and $\partial\Omega$ is bounded, then the elliptic measure is not a probability if $n \geqslant 2$.*

We can give a short proof as follows. We know that $X \longmapsto \omega_L^X(\partial\Omega)$ is the solution associated to the boundary data which is identically one. The construction explained above gives that the solution belongs to $L^{2^*}(\Omega)$, hence $\omega_L^X(\partial\Omega)$ cannot be identically 1 in the unique unbounded connected component of Ω. Then, invoking the strong maximum principle, we conclude that $\omega_L^X(\partial\Omega) < 1$ for such X.

It is less known that the other case, when both Ω and $\partial\Omega$ are unbounded, is more subtle. One may think that the elliptic measure is always a probability in this setting because that is well-known to happen for smooth boundaries. But it turns out that one needs a mild assumption for that. We addressed this issue in [CHMPZ25].

Lemma 2.47. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and L be an elliptic operator. If Ω and $\partial\Omega$ are unbounded, and Ω satisfies the CDC, then the elliptic measure is a probability.*

Proof. Consider the solution $u(X) := 1 - \omega_L^X(\partial\Omega)$, where $X \in \Omega$. It satisfies $0 \leqslant u \leqslant 1$ in Ω with $u \in C(\overline{\Omega})$ and $u|_{\partial\Omega} \equiv 0$. As a result, fixing $x_0 \in \partial\Omega$, Lemma 2.45 easily yields for every $X \in \Omega$

$$0 \leqslant u(X) \lesssim \left(\frac{|X - x_0|}{r}\right)^{\alpha_2} \sup_{B(x_0,r)\cap\Omega} u \leqslant \left(\frac{|X - x_0|}{r}\right)^{\alpha_2} \longrightarrow 0, \quad \text{as } r \to \infty.$$

□

The sharpness of the assumption is exemplified by the following result. It shows that we can find counterexamples that still satisfy the Wiener criterion, which is a slightly milder condition than CDC, as discussed before.

Lemma 2.48. *There are open sets $\Omega \subset \mathbb{R}^{n+1}$, with Ω and $\partial\Omega$ unbounded, and Ω satisfying the Wiener criterion, such that for any elliptic operator L the associated elliptic measure is not a probability.*

Proof. Let us present an example when L is the Laplacian (which can be easily generalized to any general L using (3.32)). Define $X_j := (2^j, 0, \dots, 0)$ and $r_j := 2^{-j}$ for $j \geqslant 1$. Set $B_j := B(X_j, r_j)$ and $\Omega := \mathbb{R}^{n+1} \setminus \cup_{j\geqslant 0} \overline{B_j}$. Fix $N \gg 1$ and we claim that if $R \gg N$, then

$$\omega^X_{\Omega\cap B(0,R)}(\partial\Omega \cap B(0,N)) \leqslant \sum_{j=1}^{\infty} r_j^{n-1} \left|X - X_j\right|^{1-n}, \qquad X \in \Omega \cap B(0,R). \tag{2.8}$$

To show this, fix J with $2^J \gg R$. Set $v_J(X) := \sum_{j=1}^{J} r_j^{n-1} \left|X - X_j\right|^{1-n}$ for $X \in \Omega$. Let us observe that $v_J \geqslant 0$ is harmonic in Ω (it is a finite sum of fundamental solutions centered at points outside Ω), continuous in $\overline{\Omega}$, and satisfies $v_J(X) \geqslant r_k^{n-1} |X - X_k|^{1-n} = 1$ for every $X \in \partial B_k$ and $1 \leqslant k \leqslant J$. Therefore, by the classical maximum principle in the bounded and Wiener regular domain $\Omega \cap B(0,R)$ we get for every $X \in \Omega \cap B(0,R)$,

$$\omega^X_{\Omega\cap B(0,R)}(\partial\Omega \cap B(0,N)) \leqslant \int_{\partial(\Omega\cap B(0,R))} v_J \, d\omega^X = v_J(X) \leqslant \sum_{j=1}^{\infty} r_j^{n-1} \left|X - X_j\right|^{1-n},$$

which finishes the proof of (2.8).

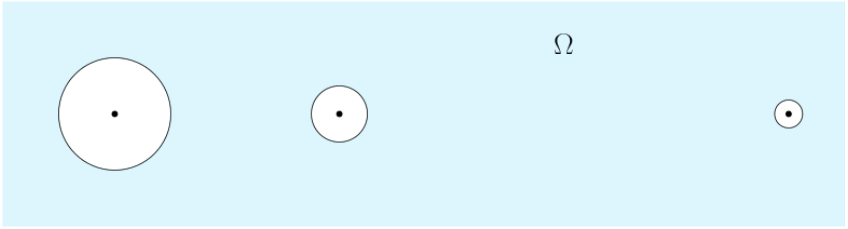


Figure 2.8: The domain under consideration is defined by removing infinitely many aligned balls, exponentially separated, and with exponentially decreasing radii.

Consider now $X_0 := (0, x_0, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ with $x_0 > 1$. Then $\left|X_0 - X_j\right| \geqslant x_0$ for every $j \geqslant 1$. Therefore, (2.8) gives at once

$$\omega^{X_0}_{\Omega\cap B(0,R)}(\partial\Omega \cap B(0,N)) \leqslant \sum_{j=1}^{\infty} r_j^{n-1} \left|X_0 - X_j\right|^{1-n} \leqslant x_0^{1-n} \sum_{j=1}^{\infty} (2^{-j})^{n-1} \leqslant x_0^{1-n}.$$

Using this and taking into account the construction of the harmonic/elliptic measure explained above we conclude that

$$\omega^{X_0}(\partial\Omega) = \lim_{N\to\infty} \omega^{X_0}_{\Omega}(\partial\Omega \cap B(0,N)) = \lim_{N\to\infty} \lim_{R\to\infty} \omega^{X_0}_{\Omega\cap B(0,R)}(\partial\Omega \cap B(0,N)) \leqslant x_0^{1-n} < 1$$

since $x_0 > 1$. This shows that the harmonic measure for Ω with pole X_0 is not a probability, hence the same occurs with any arbitrary pole by the strong maximum principle.

Let us further observe that Ω is Wiener regular: any point in $\partial\Omega$ is in the boundary of B_j for some $j \geqslant 1$ and there exist exterior corkscrew points for scales smaller than 2^{-j}, which readily imply Wiener regularity. One can also show that Ω does not satisfy the CDC as stated in Definition 2.44. To see this, for every $j \geqslant 2$, let $y_j = (2^j - 2^{-j}, 0, \ldots 0) \in \partial B_j \subset \partial\Omega$. Note that $B(y_j, 1)\backslash\Omega = B_j = B(X_j, 2^{-j})$ and that $B(X_j, 1) \subset B(y_j, 2)$. These and the properties of the capacity, see [HKM06, Theorem 2.2, Example 2.12], give

$$\frac{\operatorname{Cap}(\overline{B(y_j,1)}\backslash\Omega, B(y_j,2))}{\operatorname{Cap}(\overline{B(y_j,1)}, B(y_j,2))} \leqslant \frac{\operatorname{Cap}(\overline{B(X_j,2^{-j})}\backslash\Omega, B(X_j,1))}{\operatorname{Cap}(\overline{B(y_j,1)}, B(y_j,2))} \approx (2^{(n-1)\,j} - 1)^{-1}.$$

This quantity decays to 0 as $j \to \infty$, and as a result the CDC fails. □

2.5 Extension to parabolic PDE

Essentially all what we have seen so far in this chapter has an extension to the realm of parabolic PDE. However, there are several subtleties that end up making a relevant difference sometimes. Let us sketch now some of the most important differences, and leave some other more specific details to Chapter 5, where we will work with parabolic equations.

Our model equation when thinking on parabolic equations will be the heat equation $\partial_t u - \Delta u = 0$, where $u(X,t)$ is now a function $u : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ (i.e. $X \in \mathbb{R}^n, t \in \mathbb{R}$), and Δ only considers spatial derivatives. We shall also consider more general equations $\partial_t u - \operatorname{div}(A\nabla u) = 0$, where A is a matrix as in Definition 2.27, and again the divergence and the gradient are taken only in the spatial variables.

2.5.1 Geometrical framework

A key difference with respect to the elliptic world is that the appropriate norm in which it is natural to work in the setting of parabolic equations is not given by the ℓ^2 norm of $\mathbb{R}^{n+1}$, but rather

$$\operatorname{dist}((X,t),(Y,s)) := \big\|(X,t)-(Y,s)\big\| := |X-Y| + |t-s|^{1/2},$$

where $|X-Y|$ refers to the ℓ^2 norm in $\mathbb{R}^n$. This has to do with the scaling of the heat equation: if $u(X,t)$ is a solution of the heat equation, so are the rescalings $u(\lambda X, \lambda^2 t)$ for any $\lambda > 0$. Since the homogeneity in space and time is not the same, the most convenient metric to tackle these problems is the one exposed above.

Accordingly, the shape of cubes (which we used extensively in Section 2.3.2) change. Given $(X,t) \in \mathbb{R}^{n+1}$ and $r > 0$, we define

$$Q_r(X,t) := \big\{(Y,s) \in \mathbb{R}^{n+1} : \ |Y_i - X_i| < r \text{ for } 1 \leqslant i \leqslant n, \ |t-s| < r^2\big\}, \tag{2.9}$$

and these will also serve as substitutes for parabolic balls, because understanding their geometry is easier. See Figure 2.9.

Although the metric of parabolic $\mathbb{R}^{n+1}$ is different to the usual metric in $\mathbb{R}^{n+1}$, one can still define a meaningful notion of Hausdorff measures, which we call $\mathcal{H}^s_P$ for $s > 0$ (indeed, the theory of Hausdorff measures is flexible enough to allow any norm). A significant change is that the ambient dimension is not $n+1$ anymore, but rather $\dim_{\mathcal{H}_P}(\mathbb{R}^{n+1}) = n+2$. This again has to do with the special role of time: it is easy to see that a hyperplane not depending on t has its natural dimension (e.g. $\dim_{\mathcal{H}_P}([0,1]^n \times \{0\}) = n$), but a line moving in the time direction has dimension 2 despite being a line (e.g. $\dim_{\mathcal{H}_P}(\{(0,\dots,0)\} \times [0,1])) = 2$). See Figure 2.10.

With this in mind, the correct generalization of ADR sets as defined in Section 2.3.1 are sets $E \subset \mathbb{R}^{n+1}$ with

$$C^{-1} r^{n+1} \leqslant \sigma(E \cap Q_r(x,t)) \leqslant C r^{n+1}, \qquad \text{for every } (x,t) \in E,$$

where σ is the surface measure (recall that E is meant to be the boundary of the open set in which we pose the PDE), defined by $d\sigma := \mathcal{H}^{n-1}_P|_{E_t}\, dE_t\, dt$, where $E_t := \{(y,t) : y \in E\}$ denotes a time slice.

However, the correct generalization to the parabolic world of the Uniformly Rectifiable sets that we defined in Section 2.3.1 is not so easy. First note that the analogue of Lipschitz functions in the parabolic setting, are functions $f : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$\big|f(X,t) - f(Y,s)\big| \leqslant C\,\big\|(X,t)-(Y,s)\big\| = C\,|X-Y| + C\,|t-s|^{1/2}.$$

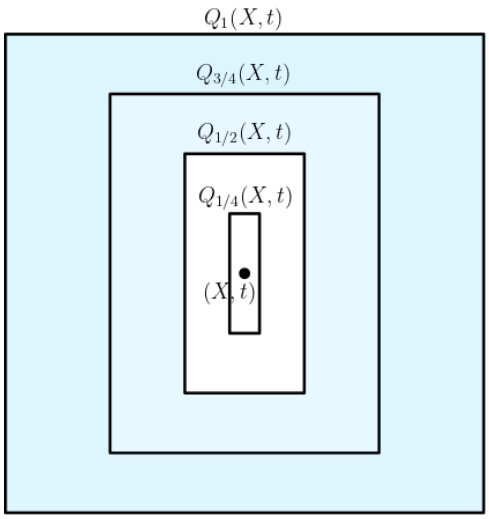


Figure 2.9: Parabolic cubes do not always have the same shape, because they scale different in the space and time variables. Recall that we always depict time as flowing left-to-right (past-to-future): under this convention, cubes get elongated vertically as the radius decreases. Similarly, cubes with very large radii are elongated horizontally.

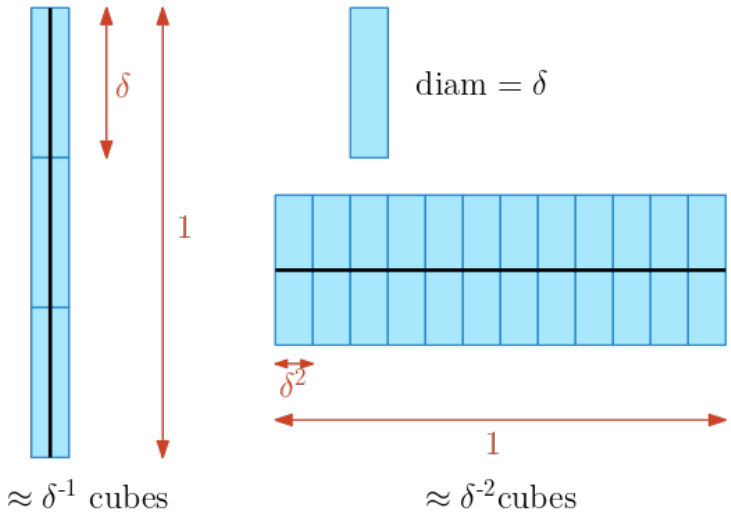


Figure 2.10: Consider the case $n = 1$ (and recall that time flows from left to the right). Then, if we try to cover a time slice like $[0,1] \times \{0\}$ (depicted on the left) with parabolic cubes of radius δ, we will need (approximately) δ^{-1} because the side-length in the space variable of a parabolic cube is just its radius. Therefore, this segment is 1-dimensional. However, if we cover a line moving only in time like $\{0\} \times [0,1]$ (depicted on the right) with parabolic cubes, we will need (around) δ^{-2} because the side-length in the time variable of parabolic cubes of radius δ is δ^2. This is basically the reason behind this segment being 2-dimensional (although being a segment).

We call these functions $\mathrm{Lip}(1,1/2)$ because, indeed, these functions are Lipschitz in space, but only $1/2$-Hölder continuous in time (which is clearly weaker). This is a key difference when it comes to doing analysis or geometry over graphs of $\mathrm{Lip}(1,1/2)$ functions. It turns out that one needs some extra

regularity in the time variable to obtain meaningful results: we will explain this later in more detail, but the appropriate analogue of uniform rectifiability in the parabolic setting is defined with the aid of *regular* $\mathrm{Lip}(1,1/2)$ graphs, which are graphs of functions f which are $\mathrm{Lip}(1,1/2)$ and, in addition, they satisfy

$$\partial_t^{1/2} f \in \mathrm{BMO}_P\,. \tag{2.10}$$

Here, BMO_P simply refers to the BMO space that we introduced in Definition 2.7, but using parabolic cubes. But more importantly, $\partial_t^{1/2}$ refers to a half-order derivative in the time direction. This is a non-local object that can be defined in several ways, but it turns out that the most classical ones (via Fourier multipliers or singular integrals) are good enough for our purposes. For instance, one may consider

$$\partial_t^{1/2} f(x,t) := \int_{\mathbb{R}} \frac{f(x,s)-f(x,t)}{|s-t|^{3/2}}\,ds.$$

This new definition of parabolic uniform rectifiability (note that the change has been produced by the change of metric and, ultimately, by the homogeneity of the heat equation) really sets a difference between the elliptic and parabolic worlds. Since Uniformly Rectifiable sets are the boundaries of the appropriate sets in which one wants to meaningfully pose boundary value problems with singular data, working with parabolic equations will always be much more technical and subtle because of having to deal with the extra condition (2.10). This will make the parabolic theory challenging, even if we are trying to establish the analogues to well-known results in the elliptic setting; we will see this clearly in Chapter 5.

Let us continue to comment on the generalization of all the previous preliminaries to the parabolic setting. Regarding Section 2.3.2, there is not much difference: one need only work with parabolic cubes as in (2.9), and the results essentially adapt without a change. Something similar will happen with the corkscrew condition in Section 2.3.3, but Harnack chains are significantly different because Harnack's inequality is notably different for parabolic equations.

2.5.2 Existence of parabolic measure, and classification of boundary points

Let us comment on the different Harnack inequalities in the parabolic setting, and other specifics of the equation, following Section 2.4.

First, let us just say that there is an appropriate version of the theory of Sobolev spaces and inequalities, and weak solutions (see [Li96, Chapter VI]). Moreover, at least for continuous boundary data, solutions can be found using the Perron-Wiener-Brelot method as in [Wa12, Chapter 8] (also available in the elliptic setting, see [HKM06, Chapter 9]), which gives rise (using the Riesz representation theorem) to the so-called parabolic measure (called *caloric* measure in the precise case of the heat equation). See [Li96, Section 3.4], [Wa12, Section 8.4], [GH20, Section 1] or [MP21, Section 3] for more details.

However, there are some caveats when solving the boundary value problem with continuous boundary data for parabolic equations in a domain $\Omega \subset \mathbb{R}^{n+1}$. The first one is that it is not reasonable to impose data in some parts of $\partial\Omega$. Concretely, the data that we impose at some point, will only affect points in the future (indeed, that is the nature of the heat equation and all its analogues). Therefore, if we desire to work in an arbitrary domain $\Omega \subset \mathbb{R}^{n+1}$, we must distinguish the nature of the points in $\partial\Omega$: it will only be reasonable to prescribe the values in some of them. Such a classification is performed

in detail in [Wa12, Section 8.1] (for cylindrical domains it is much easier, see [Li96, Section 2.1]), and also explained in [GH20, Section 1] and [MP21, Section 2]. Basically, it informs us that the parabolic measure is always supported on the so-called essential boundary $\partial_e\Omega$ (so this is precisely the subset of the boundary in which it is reasonable to prescribe values), which is defined as follows:

$$\partial_e\Omega := \partial\Omega \setminus \left\{(x,t) \in \partial\Omega : \exists \varepsilon > 0 \text{ with } Q_\varepsilon^-(x,t) \subset \Omega, Q_\varepsilon^+(x,t) \subset \Omega^c\right\},$$

where $Q_\varepsilon^-(x,t) := \{(y,s) \in Q_\varepsilon(x,t) : s < t\}$, and similarly, $Q_\varepsilon^+(x,t) := \{(y,s) \in Q_\varepsilon(x,t) : s > t\}$.

2.5.3 Regularity of solutions

Many of the results about regularity of solutions that we presented in Sections 2.4.2, 2.4.3 and 2.4.4 (both at the interior of the domain and near the boundary) have analogues in the parabolic setting.

First, estimates like Cacciopoli's inequality (Lemma 2.30) are readily adapted to parabolic equations (see [FGS86, Theorem 0.1]). In turn, Lemma 2.31 and Theorem 2.32 are also generalizable to general parabolic equations with bounded and measurable coefficients (see [Li96, Section 6.6]). In fact, Nash [Na58] originally obtained the elliptic version of Theorem 2.32 as a byproduct of his parabolic proof. Similarly, Schauder regularity (i.e. the correspondent version of Theorem 2.36) also holds in the parabolic setting. A proof can be found in [Li96, Section 5.2], but let us not state here the result because of the technicality of the functional spaces that are used (see [Li96, Section 4.1] for that). The same happens for Calderón-Zygmund regularity (analogue to Theorem 2.37), see [Li96, Chapter 7]. We will only need these estimates in very specific parts of Chapter 5: there, we will make everything clear.

Apart from all these estimates, which are generalized in a rather intuitive way from the elliptic to the parabolic world (only the proofs change, but the statements are nearly identical), there are other properties which change in a fundamental way. The most well-known one is Harnack's inequality: in contrast to the elliptic version (Theorem 2.33), the parabolic estimate is necessarily time-directed (otherwise, one may find counterexamples, see [Li96, Exercise 6.7]). A first version of Harnack's inequality, which exemplifies this time-directedness, is the following one (originating from the work of Moser [Mo64, Theorem 1], and formulated more recently in [Li96, Theorem 6.27]; see a necessary definition before [Li96, Theorem 6.18]). See a picture in Figure 2.11.

Theorem 2.49 (Parabolic Harnack's inequality, version 1)**.** *Let $u \geqslant 0$ be a solution of $\partial_t u - \operatorname{div}(A\nabla u) = 0$ in $Q_{4r}(Z,\tau)$, where A satisfies (2.6). Then*

$$\sup_{(Y,s)\in Q_r(Z,\tau-4r^2)} u(Y,s) \leqslant C(\lambda,n) \inf_{(X,t)\in Q_r(Z,\tau)} u(X,t).$$

Another version of Harnack's inequality, which already appears in the original work of Moser [Mo64, Theorem 2] (find it with more modern notation in [FGS86, Theorem 0.2]) is the following:

Theorem 2.50 (Parabolic Harnack's inequality, version 2)**.** *Let $u \geqslant 0$ be a solution of $\partial_t u - \operatorname{div}(A\nabla u) = 0$ in $\Omega \subset \mathbb{R}^{n+1}$, where A satisfies (2.6). Let $D \subset \Omega$ be convex. Then, if $s < t$:*

$$u(Y,s) \leqslant u(X,t) \exp\left[C(\lambda,n)\left(\frac{|X-Y|^2}{t-s} + \frac{t-s}{c} + 1\right)\right], \qquad (Y,s),(X,t) \in D,$$

where $c := \min\{1, \operatorname{dist}(D,\partial\Omega)^2, s - T_{\min}(\Omega)\}$, and $T_{\min}(\Omega)$ is the minimum time of Ω (possibly $-\infty$).

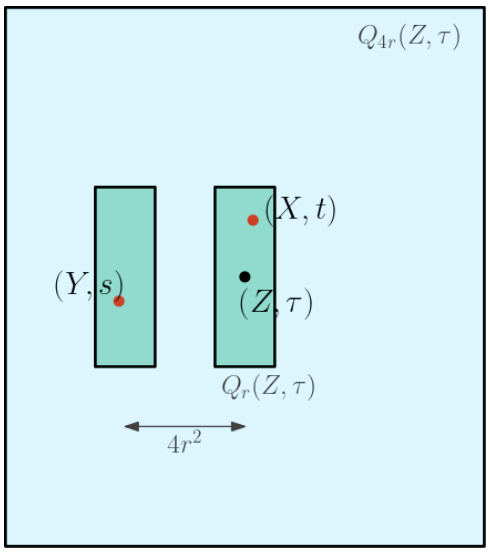


Figure 2.11: The so-called Moser's Harnack inequality always requires that the region with which we compare the value of a solution at a given point is strictly forward in time. If one wants to keep constants uniform, there must also be an uniform separation in time between these regions.

The way to interpret this inequality is that, as long as we stay away from the boundary of Ω, values of a solution u can always be controlled at a point (Y,s) by the values of u at any point in a parabola with vertex at (Y,s), pointing forward in time. Importantly, the constants will (basically) only depend on the aperture of the parabola. See an illustration in Figure 2.12.

As already commented, Harnack inequalities are naturally directed towards the future. Nevertheless, there are times when one can establish *reverse/backwards* Harnack inequalities, i.e. control the value at a point by values at the past. This is much more subtle and requires further hypotheses to be true. For example, the main result of an important work by Fabes, Garofalo and Salsa [FGS86, Theorem 1.3], asserts that a backwards Harnack inequality is true under the hypotheses that the underlying domain is a Lipschitz cylinder (which is very connected with the fact that the caloric measure is doubling) and the solution vanishes on a portion of the boundary (indeed, the possible creation of extra heat via the boundary conditions is one of the reasons why Harnack inequalities are necessarily time-directed). We will not enter more details here.

A second crucial difference of parabolic equations with respect to elliptic ones is the shape of fundamental solutions. We already saw in Section 2.4.3 that, up to constants (at least if $n \geqslant 2$), elliptic fundamental solutions behave like powers, namely $|X-Y|^{1-n}$. The situation for parabolic equations is radically different. For instance, the fundamental solution of the heat equation is the well-known heat kernel, which is nothing more than a Gaussian:

$$\frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{|X-Y|^2}{4(t-s)}}.$$

As in the elliptic case, for general parabolic operators with bounded coefficients like $\partial_t - \operatorname{div} A\nabla$, the associated fundamental solution $\Gamma((X,t),(Y,s))$ may be different, but still preserves the shape of the

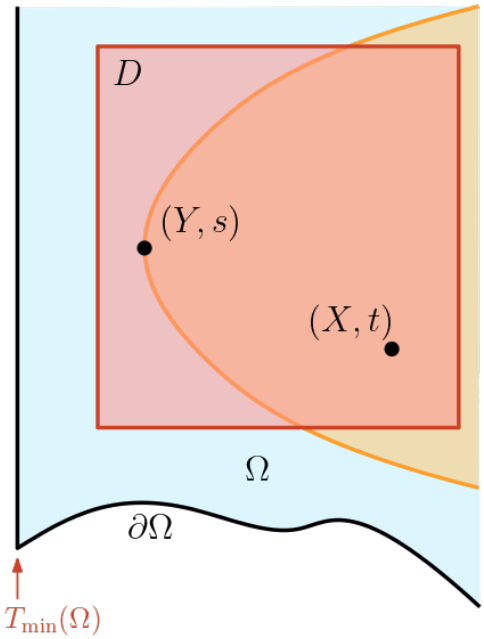


Figure 2.12: A more elaborated version of the Harnack inequality asserts that one can even compare to points which lie in a parabola pointing forward in time, and the constants will depend on the aperture on the parabola (and also how much we travel forward in time, as in Figure 2.11).

Gaussian:

$$\frac{1}{N}\frac{1}{(t-s)^{n/2}}e^{-N\frac{|X-Y|^2}{t-s}}\mathbf{1}_{\{t>s\}} \leqslant \Gamma((X,t),(Y,s)) \leqslant N\frac{1}{(t-s)^{n/2}}e^{-\frac{1}{N}\frac{|X-Y|^2}{t-s}}\mathbf{1}_{\{t>s\}},$$

where $N = N(\lambda, n)$. A proof of this fact for general coefficients A can be found in [QX19], but originates in the works of Aronson [Ar67] (in fact, these Gaussian estimates are known as Aronson's estimates) and Nash [Na58]. The reader can consult a fairly complete historical overview in [QX19, Section 2].

A very important fact about fundamental solutions is that they are only non-zero if one of the points is to the right of the other point, and is zero otherwise. This has to do with the fact that fundamental solutions allow us to represent solutions of an equation with a source/right-hand-side, as in Theorem 2.34. In the parabolic setting, sources/boundary values can only affect the values of our solutions in the future; and since fundamental solutions roughly inform us about how important the value is at (Y, s) to determine the value at (X, t), it must necessarily hold $t > s$.

Lastly, let us comment that the fine estimates near the boundary that we obtained in Sections 2.4.4 and 2.4.5 have natural analogues in the parabolic setting. Good references for this are [GH20, Section 2], [MP21, Section 3], and the detailed exposition in [Ge20, Section 2]. A difference with respect to the results in Sections 2.4.4 and 2.4.5 is that the assumptions have to be, again, time-directed. As an example, to obtain the boundary Hölder continuity in Lemma 2.45, one only needs to assume the so-called Time Backwards CDC (instead of the whole CDC): this amounts to checking capacities only to the past of the region in which we are working, because this is actually the only part of the boundary that will affect the problem. For more details, just check [GH20, Ge20, MP21] and the references therein.

2.6 A short introduction to Functional Calculus

Although we will only use it at very specific places in Chapter 7, let us give a brief introduction to Functional Calculus for sectorial operators, a topic initiated primarily by McIntosh, because it is a very powerful tool due its abstract nature. For this, we will follow the book of Haase [Ha06] and the very understandable lecture notes of Egert, Haller, Monniaux and Tolksdorf [EHMT24].

The motivation of this section is the following. A very powerful method when dealing with problems related with the Laplace equation is to use Fourier Analysis: indeed, since the multiplier (in the Fourier side) associated to the Laplacian is simply (up to constants) $|\xi|^2$, it sometimes suffices to work on the Fourier side algebraically with this multiplier, to draw conclusions about harmonic functions. Since it is often much easier to analyze these algebraic properties of the multiplier than the properties of abstract harmonic functions, there is frequently a substantial gain by working on the Fourier side. However, if instead of the Laplacian, we consider a more general operator as in Definition 2.27, Fourier transform methods are not directly applicable because we lack regularity. Functional Calculus is precisely the tool that allows us to mimic some Fourier transform methods, but only using the physical side (similarly to what we did in Section 2.2.3 with Littlewood-Paley theory, creating a sense of orthogonality without going to the frequency side explicitly).

Along the section, H will denote a Hilbert space, and $L : H \longrightarrow H$ will be an operator acting on H. Our objective is to meaningfully define what it means $f(L)$ for a function $f : \mathbb{C} \longrightarrow \mathbb{C}$, to generalize Fourier multipliers to L. At some points, we will need to restrict to elliptic operators in divergence form with bounded coefficients $L = -\operatorname{div} A\nabla$ as in Definition 2.27 and put $H = L^2$, because for general operators L in a Hilbert space, the theory that we are about to present is not as rich as for elliptic operators, and these are ultimately our main objects of interest.

Let us begin with a central definition: that of sectorial operators. Apart from this definition, we would need many more definitions (closed, injective or densely defined operators, the domain of an operator...) to make all the forthcoming theory complete. To avoid possible confusion and just focus on the main features of the theory, let us forget about those technicalities (which are actually crucial, but would confuse readers new to the topic) to be able to provide a short and simple introduction to the topic. We first follow [EHMT24, Chapter 2] (see also [Ha06, Section 2.1]).

Definition 2.51 (Sectorial operator)**.** Let L be a linear operator in H. We say that L is a sectorial operator of angle $\varphi_L \in [0, \pi)$ if:

1. $\sigma(L) \subset \overline{S_{\varphi_L}}$, where $\sigma(L)$ denotes the spectrum of L and $S_{\varphi_L} := \{z \in \mathbb{C}\backslash\{0\} : \left|\arg(z)\right| < \varphi_L\}$ denotes a sector of aperture φ (see Figure 2.13),

2. We have the following resolvent bounds: for every $\psi \in (\varphi_L, \pi)$ there exists $C_\psi \geqslant 0$ such that

$$\left\|(\lambda - L)^{-1}\right\|_{H\to H} \leqslant \frac{C_\psi}{|\lambda|}, \qquad \lambda \in \mathbb{C}\backslash\overline{S_{\varphi_L}}.$$

With this definition in mind, it is not difficult to check that any second-order elliptic operator in divergence form L as in Definition 2.27 is sectorial, with angle $\varphi_L < \pi/2$ (see [EHMT24, Theorem 3.29]). This allows us to define the notion of what it means to apply a function to L. For this, we follow [EHMT24, Chapter 5] (see also [Ha06, Section 2.3]).

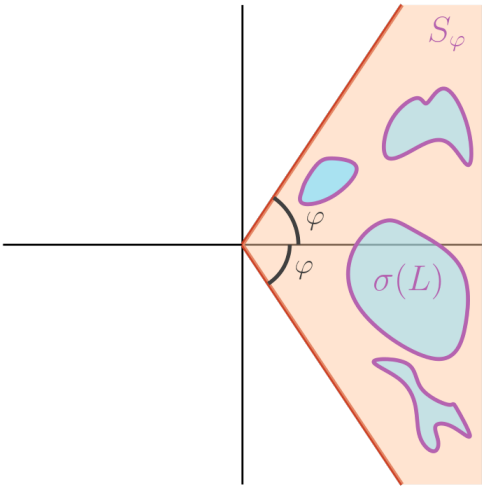


Figure 2.13: Depiction of the sector S_φ of angle φ, inside the complex plane $\mathbb{C}$. For an operator to be sectorial, its spectrum must be contained in one of these sectors.

Definition 2.52 (Elementary functional calculus)**.** Let L be an operator as in Definition 2.27, and fix $\varphi_L < \psi < \varphi < \pi$. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a bounded holomorphic function with polynomial growth at 0 and infinity, namely

$$\exists\, C, s > 0 : \quad \left|f(z)\right| \leqslant C \min\{|z|^s, |z|^{-s}\} \qquad \forall\, z \in S_\varphi.$$

Then, we define $f(L) : H \longrightarrow H$ as the operator defined by

$$f(L) := \frac{1}{2\pi i} \int_{\gamma_\psi} f(z)(z - L)^{-1}\, dz,$$

where γ_ψ is the curve (see Figure 2.14)

$$\gamma_\psi(t) := \begin{cases} -te^{i\psi} \text{ for } t \in (-\infty, 0), \\ te^{-i\psi} \text{ for } t \in [0, \infty) \end{cases}$$

Note that this is just a generalization of Cauchy's integral formula for holomorphic functions: since the curve γ_ψ surrounds the spectrum of L (so all the possible singularities of $(z - L)^{-1}$), evaluating the integral against the function f should recover $f(L)$. The assumptions on f can be relaxed to allow for much wider families of holomorphic functions [EHMT24, Chapter 5], but with the assumptions above it is easier to show that the integrals converge. This allows us to, for instance, define fractional powers of elliptic operators L^α, or semigroups associated to them like e^{-tL} (see [EHMT24, Chapter 6], [Ha06, Chapter 3]).

In the case that L is the Laplacian, the above method is just the same as applying functions on the frequency side: if m is the multiplier of the Laplacian, then (see [EHMT24, Example 5.8])

$$\widehat{(f(-\Delta)u)}(\xi) = f(m(\xi))\hat{u}(\xi).$$

A relevant question within this framework is to know whether, for a given holomorphic f, the associated operator $f(L)$ is bounded. In analogy with last paragraph, it is very easy to show (just an application

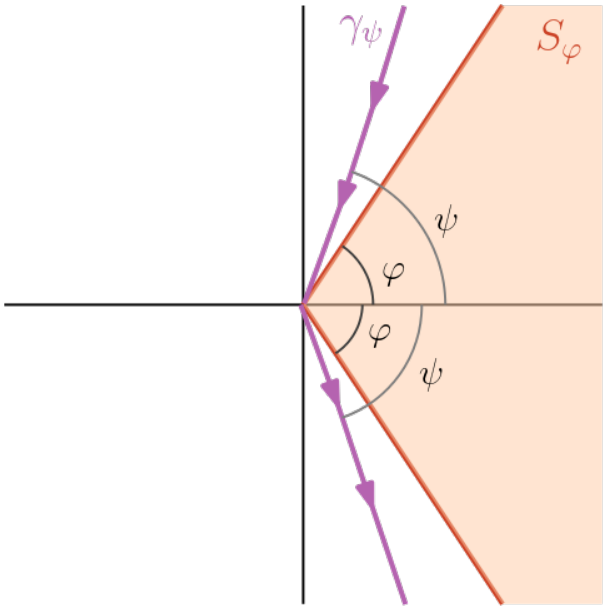


Figure 2.14: The path γ_ψ should make us remember Cauchy integrals because it somehow surrounds the sector S_φ because $\psi > \varphi$. The growth conditions on f and the resolvent bounds for L from Definition 2.51 (we just discussed that L is sectorial) allow us to show easily that the integral defining $f(L)$ is meaningfully defined by an absolutely convergent integral.

of Plancherel) that the Fourier multipliers that give rise to bounded operators are precisely those whose symbol is bounded as a function. We would like something similar to hold for any L, so we introduce a new notion. We follow [EHMT24, Section 7.1] (see also [Ha06, Chapter 5]).

Definition 2.53 (Bounded H^∞-calculus)**.** We say that an operator L has a bounded H^∞-calculus, with angle $\varphi \in (\varphi_L, \pi)$, if there exists $C > 0$ such that

$$\left\| f(L) \right\|_{H \to H} \leqslant C \, \|f\|_{L^\infty(S_\varphi)}$$

for every f in the class of functions from Definition 2.52.

The situation for general operators L (not necessarily elliptic operators) is not as easy for the Laplacian: in the early stages of the development of this Functional Calculus, McIntosh and Yagi found invertible sectorial operators L for which the associated H^∞-calculus is never bounded (see [EHMT24, Section 7.1]). However, for elliptic operators in divergence form, the situation is better (see [EHMT24, Section 7.2] or [Ha06, Section 7.1]):

Theorem 2.54 (Von Neumann's inequality)**.** *Let L be an elliptic operator as in Definition 2.27. Then the H^∞-calculus associated to L, of any angle between $\pi/2$ and π, is bounded. Concretely,*

$$\left\| f(L) \right\|_{H \to H} \leqslant \|f\|_{L^\infty(S_{\pi/2})} ,$$

for any f holomorphic and bounded in the sector S_φ, where $\varphi \in (\pi/2, \pi)$.

This informs us that, somehow, any elliptic operator as in Definition 2.27 inherits what happens for the Laplacian: the operators that arise as multipliers/via the Functional Calculus as in Definition 2.52

are bounded (as operators) if the function that we use as multiplier/in the Functional Calculus is itself bounded (as a function). However, many more technicalities are needed in order to define everything properly for a general L, as we have just seen, because we need to avoid the usage of the Fourier transform.

As a last thing in this introduction to Functional Calculus, let us provide a very general result that makes Functional Calculus useful when needing to show concrete estimates that may arise, for instance, when dealing with solutions to PDE associated to our operator. See [EHMT24, Chapter 8] (or also [Ha06, Section 7.3]) for more details.

Theorem 2.55 (McIntosh's theorem)**.** *Let L be a sectorial operator. Then, the following are equivalent:*

1. *L has a bounded H^∞ calculus of some angle $\varphi \in (\varphi_L, \pi)$,*
2. *L has a bounded H^∞ calculus of any angle $\varphi \in (\varphi_L, \pi)$,*
3. *The square function estimate*

$$\int_0^\infty \left\| f(tLu) \right\|_H^2 \frac{dt}{t} \approx \|u\|_H^2, \qquad \forall u \in H, \tag{2.11}$$

   holds for some function $f \not\equiv 0$ as in Definition 2.52,
4. *The square function estimate (2.11) holds for any function $f \not\equiv 0$ as in Definition 2.52.*

This is very powerful that, along with the orthogonality arguments already presented in Section 2.2.3 about Littlewood-Paley theory, can somehow serve as a substitute for Plancherel or the rest of knowledge about Fourier Analysis that we know for the Laplacian. The advantage is that it works for just any elliptic operator in divergence form with bounded coefficients (as in Definition 2.27), so it is a very robust and powerful theory. In turn, a clear disadvantage is that it is much more technical and subtle to be used.

Functional Calculus is a tool that helped notably when showing the original Kato square root for operators as in Definition 2.27 (shown in the paper [AHLMT02]). It will help us, too, when trying to extend the proof of this result to non-divergence form operators in Chapter 7, for instance allowing us to represent $\sqrt{L}$ by an integration of semigroups, for which many good properties are known. We leave further discussion to Chapter 7.

CHAPTER 3

The Dirichlet problem with smooth boundary data

As discussed in Chapter 2 (concretely, Section 2.4.4), boundary behavior of solutions to elliptic PDE is usually a fairly delicate matter, even when the boundary data belong to spaces of classical smoothness, as will be in this chapter. If one desires to show certain behavior of solutions around a portion of the boundary, one should also assume that the portion itself is regular in some geometrical sense, for otherwise it is common that counterexamples exist.

There are very few instances where anyone knows the exact minimal condition that a portion of the boundary must satisfy in order for solutions of PDE to exhibit a concrete desirable behavior. One of these examples is the famous Wiener's criterion, which characterizes the points of the boundary where we can expect harmonic functions to be continuous. We stated this in full detail in Theorem 2.43. A very important result complementing Wiener's criterion is what was obtained by Littman, Stampacchia and Weinberger in [LSW63]: they showed that this property (harmonic functions arriving continuously to a certain point of the boundary) does not depend on the elliptic operator (see Definition 2.27) taken into consideration, whence the same characterization of boundary points works for any elliptic operator, not only the Laplacian.

Wiener's criterion is a crucial tool in the theory of PDE because continuity is often needed to run very elementary and ubiquitous arguments, like maximum principles. However, a very big portion of the regularity theory of elliptic PDE (we unraveled a small part in Sections 2.4.2 and 2.4.3) is of quantitative nature, meaning that it requires that the boundary data or solutions have an specific modulus of continuity. Wiener's criterion does not give the final answer in that regard because it is only of qualitative nature.

For that reason, the problem of obtaining quantitative versions of the Wiener's criterion becomes relevant. The goal would be to find the correct geometric conditions ensuring that solutions have some specific modulus of continuity when approaching the boundary. Since probably the easiest possible moduli of continuity are those making the solutions belong to the classical Hölder continuity spaces, a first question would be to understand the minimal geometrical condition ensuring Hölder-like behavior around the boundary. Aikawa [Ai02] gave the answer in the simplest case: for harmonic functions in bounded domains (under the background assumption of Wiener).

Theorem 3.1 (Aikawa, [Ai02]). *Let $\Omega \subset \mathbb{R}^{n+1}$, where $n \geqslant 1$, be a bounded open set satisfying the Wiener criterion. Then, Ω satisfies the Capacity Density Condition (CDC, see Definition 3.9) if and only if for any $0 < \alpha \leqslant 1$ it holds*

$$\sup_{0 \not\equiv f \in C^\alpha(\partial\Omega)} \frac{\|u_f\|_{C^\alpha(\Omega)}}{\|f\|_{C^\alpha(\partial\Omega)}} < \infty,$$

where u_f denotes, for $f \in C^\alpha(\partial\Omega)$, the function satisfying $-\Delta u_f = 0$ in Ω with $u_f = f$ on $\partial\Omega$.

This chapter is devoted to a deep generalization of Aikawa's result. We show that the same geometrical condition (the CDC, a very natural quantitative version of the condition found by Wiener) ensures Hölder continuity up to the boundary of solutions to *any* elliptic PDE, not only the Laplace equation; and we show that it is indeed the minimal condition leading to that behavior.

A key point when generalizing Aikawa's result to general operators is that for harmonic functions one may even consider Lipschitz continuity (i.e. $\alpha = 1$), whereas for solutions to general PDE only Hölder continuity with possibly small $\alpha > 0$ is expected, and this is really inherent to the theory by De Giorgi/Nash's celebrated result (see Theorem 2.32).

Furthermore, to further extend Aikawa's work, we show that everything extends appropriately to unbounded domains, which introduces several notable subtleties (see Section 2.4.6). On top of that, we are also able to work with much richer families of moduli of continuity generalizing those which arise in Hölder spaces, hence analyzing much finer boundary behaviors.

This chapter is based on the paper [CHMPZ25], done in collaboration with Mingming Cao, José María Martell, Cruz Prisuelos-Arribas and Zihui Zhao.

We suggest the reader to have Section 3.3 handy while reading the chapter, for it has the key ideas and provides a simplified and understandable sketch of the proofs that will be carried out in all detail along the rest of the chapter.

3.1 Further historical context . . . 53
3.2 Main results . . . 54
3.3 Main ideas and outline of the chapter . . . 58
3.4 Preliminaries . . . 59
3.5 Growth functions and auxiliary lemmas . . . 60
3.6 Well-posedness of the Dirichlet problem in Hölder spaces . . . 63
3.6.1 Proof of Theorem 3.21: Existence . . . 64
3.6.2 Proof of Theorem 3.21: Uniqueness . . . 68
3.6.3 Proof of Corollary 3.8: Boundary traces . . . 69
3.7 Ill-posedness of the Dirichlet problem in Hölder spaces . . . 69
3.8 Optimality of the capacity density condition . . . 74
3.8.1 Proof of Theorem 3.7: optimality of the CDC . . . 79
3.8.2 Adaptation of the proof of optimality of the CDC for growth functions . . . 86
3.9 Applications to other boundary value problems . . . 87
3.9.1 The Dirichlet problem with solutions satisfying a fractional Carleson measure condition . . . 88
3.9.2 The Morrey-Campanato Dirichlet problem . . . 99

3.1 Further historical context

Before diving into the detailed results and proofs, let us give a more complete overview of the state of the art regarding the solvability of the Dirichlet problem or the boundary behavior of harmonic functions (or solutions to elliptic PDE) with different kinds of boundary data. This complements the explanations in the last pages and Section 1.1.

In the (classical) Dirichlet problem for the Laplacian in the upper half-space with a continuous boundary datum f, one seeks a harmonic function $u \in C^\infty(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+})$ satisfying $u|_{\mathbb{R}^n} = f$. As noted in [He69, p. 42 and p. 158], the solution to this classical Dirichlet problem is not unique even if the boundary datum f is a bounded continuous function in $\mathbb{R}^n$. Typically, the uniqueness can be ensured by specifying the behavior of $u(x,t)$ as $t \to \infty$. Indeed, the uniqueness was established in [ST96] for harmonic functions $u \in C^\infty(\mathbb{R}^{n+1}_+) \cap C(\overline{\mathbb{R}^{n+1}_+})$ satisfying $u(x) = o(|x| \sec^\gamma \theta)$ as $|x| \to \infty$ (where $\theta := \arccos(x_n/|x|)$ and $\gamma \in \mathbb{R}$ is arbitrary), by proving a Phragmén-Lindelöf principle under the latter growth condition. For more related work in this regard see also [Si87, Yo91, YM98].

Much attention has also been paid to the case of the L^p-Dirichlet problem in the upper half-space $\mathbb{R}^{n+1}_+$ for variable coefficient elliptic operators in divergence form $L = -\operatorname{div}(A\nabla)$ with t-independent coefficients $A(x,t) = A(x)$ for $(x,t) \in \mathbb{R}^{n+1}_+$. For a complex bounded coefficient matrix A, the method of layer potentials was used to establish the solvability of the Dirichlet problem with $L^2(\mathbb{R}^n)$ data, for a small complex L^∞ type perturbation of a real symmetric matrix in [AAAHK11], which was extended to the case with $L^p(\mathbb{R}^n)$ data in [HMiM15], and then to the case with $L^p(\mathbb{R}^n)$, $\mathrm{BMO}(\mathbb{R}^n)$, and $\dot{C}^\alpha(\mathbb{R}^n)$ data, for a small complex Carleson type perturbation of a real symmetric matrix in [HMaM15]. Here, we mention that this kind of work crucially relies on the De Giorgi-Nash-Moser theory. On the other hand, the harmonic/elliptic measure is a powerful tool to deal with elliptic operators with real bounded coefficient matrices. Indeed, Hofmann, Kenig, Mayboroda, and Pipher [HKMP15a] proved that for each t-independent coefficients matrix A in $\mathbb{R}^{n+1}_+$, there exists some $p \in (1,\infty)$ such that the Dirichlet problem with $L^p(\mathbb{R}^n)$-boundary data is solvable, which complemented the previous result by Kenig, Koch, Pipher, and Toro [KKPT00] in dimension $n = 1$. Conversely, counterexamples in [KKPT00] showed that for each $p \in (1,\infty)$, there exists a uniformly elliptic coefficient matrix A for which the Dirichlet problem is not solvable for $L^p(\mathbb{R}^n)$-boundary data. A significant portion of that theory in [HKMP15a] was recently extended to the degenerate elliptic case by Hofmann, Le, and Morris [HLM19], see also the results due to Auscher, Rosén, and Rule [ARR15] for $L^2(\mathbb{R}^n)$ data.

As regards the L^p-Dirichlet problem in rough domains, its solvability is closely linked with the absolute continuity properties of elliptic measures. Indeed, the importance of the quantitative absolute continuity of the elliptic measure with respect to the surface measure comes from the fact that $\omega_L \in RH_p(\sigma)$ (the reverse Hölder class with respect to the surface measure σ) is equivalent to the $L^{p'}(\sigma)$-solvability of the Dirichlet problem (see, e.g. [FKP91, Theorem 1.5] and [HL18]). In this direction, Dahlberg [Da77] first established its solvability for the Laplacian with L^p data and $p \in [2,\infty)$ in Lipschitz domains. This result was extended to BMO_1 domains by Jerison and Kenig [JK82], to chord-arc domains by David and Jerison [DJ90], and further studied in 1-sided chord-arc domains by Hofmann et al. [AHMNT17, HM14, HMU14]. Under some optimal background hypothesis (an open set satisfying the corkscrew condition with an n-dimensional Ahlfors-David regular boundary), it was culminated in the recent results of [AHMMT20], which gives a geometric characterization of the solvability of the L^p-Dirichlet problem for the Laplacian and for some finite p via the rectifiability of the boundary plus a local accessibility by non-tangential paths to some pieces of the boundary. It is worth pointing out that

all the aforementioned results are restricted to the n-dimensional boundaries of domains in $\mathbb{R}^{n+1}$. Some analogues have been obtained in [DFM19, DM23, MZ19] on lower-dimensional sets.

Other types of boundary data have also been studied in the literature. For example, Fabes et al. [FJN76] dealt with some boundary value problems in the upper-half space with data in BMO and Morrey-Campanato spaces, which were extended to more general domains and operators in the works [DK87, FN80, Zh18]. Shen [Sh03] also established well-posedness for elliptic systems in Lipschitz domains with boundary data in Morrey spaces when $p = 2$. Recently, Martell et al. [MMMM19] have taken advantage of the properties of the Poisson kernel associated with constant complex elliptic systems [ADN59, ADN64] to obtain well-posedness of the Dirichlet boundary value problem in the upper-half space with data in BMO, VMO and Morrey-Campanato spaces. Some extensions are even possible to more general Banach spaces [MMMM16] that include variable exponent Lebesgue spaces, Lorentz spaces, Zygmund spaces, and weighted versions of them in [CMM22, MMMMM22].

3.2 Main results

The main goal of this chapter is to study the well-posedness of the $\dot{C}^\alpha$-Dirichlet boundary value problem

$$\begin{cases} u \in W^{1,2}_{\text{loc}}(\Omega), \\ Lu = 0 \text{ in the weak sense in } \Omega, \\ u \in \dot{C}^\alpha(\Omega), \\ u|_{\partial\Omega} = f \in \dot{C}^\alpha(\partial\Omega). \end{cases} \tag{3.1}$$

Let us recall the definition of the homogeneous Hölder spaces that will be our focus in this chapter (they were introduced in Section 2.1). Indeed, given an arbitrary set $E \subset \mathbb{R}^{n+1}$ and $\alpha \in (0,1)$, the homogeneous Hölder space of order α on E is defined as

$$\dot{C}^\alpha(E) := \big\{u : E \longrightarrow \mathbb{R} : \|u\|_{\dot{C}^\alpha(E)} < \infty\big\},$$

where $\|\cdot\|_{\dot{C}^\alpha(E)}$ stands for the semi-norm

$$\|u\|_{\dot{C}^\alpha(E)} := \sup_{\substack{X,Y\in E\\ X\neq Y}} \frac{|u(X)-u(Y)|}{|X-Y|^\alpha}.$$

As in Definition 2.27, the operator L is a second order uniformly elliptic operator in divergence form defined as $Lu = -\operatorname{div}(A\nabla u)$, with A being a bounded real (non-necessarily symmetric) uniformly elliptic matrix:

$$\Lambda^{-1}|\xi|^2 \leqslant A(X)\xi\cdot\xi \quad \text{ and } \quad |A(X)\xi\cdot\eta| \leqslant \Lambda|\xi|\,|\eta|,$$

for all $\xi,\eta \in \mathbb{R}^{n+1}$, for almost every $X \in \Omega$, and for some $\Lambda \geqslant 1$.

A particular case of (3.1) is $\Omega = \mathbb{R}^{n+1}_+$, which was treated in [MMM20, Theorem 1.4] and [MMMM19, Theorem 1.21] for homogeneous second-order strongly elliptic systems of differential operators with constant complex coefficients. In the current chapter, we would like to investigate what geometrical features of the domain will guarantee the well-posedness of (3.1). The method used in [MMM20, MMMM19] is invalid in rough domains since one cannot ensure the existence and good properties of

the associated Poisson kernel. Recently, the area has witnessed the availability and powerfulness of the method of harmonic/elliptic measure in studying the Dirichlet boundary value problem in rough domains (cf. [AHMMT20, AHMNT17, CHMT20, HLMN17, HM14, HMMTZ21, HMU14]). Motivated by these, we are interested in finding solutions, given by the associated harmonic/elliptic measure, which also satisfy the natural bounds. We also consider the problem of uniqueness and investigate under what circumstances such elliptic measure solutions are indeed unique.

Our main results establish that provided α is sufficiently small (depending only on the allowable parameters) we can always find a solution to problem (3.1) expressed in terms of the boundary data and the associated elliptic measure. On the other hand, the uniqueness of such solution depends on the topological features of the domain and its boundary, more specifically, on their boundedness. We have two possibilities. In the first one, where Ω is either bounded or it is unbounded with unbounded boundary (equivalently, $\mathbb{R}^{n+1}\backslash\Omega$ unbounded), the $\dot{C}^\alpha$-Dirichlet problem is well-posed, that is, the associated elliptic measure solution is the unique solution. In the second scenario, where Ω is unbounded with bounded boundary (equivalently, $\mathbb{R}^{n+1}\backslash\Omega$ bounded), solutions cease to be unique in general and one can find different solutions to the same boundary value problem, written in terms of the elliptic measure. Our proof is very robust so that we can work in open sets without connectivity assumptions. More precisely, our first result establishes the following:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set satisfying the capacity density condition (cf. Definition 3.9) so that either* **Ω is bounded or Ω is unbounded with $\partial\Omega$ being unbounded**. *Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator, and let ω_L be the associated elliptic measure. Then there exists $\alpha_0 \in (0,1)$[1] (depending only on n, the CDC constant, and the ellipticity constant of L) such that for every $\alpha \in (0,\alpha_0)$, the $\dot{C}^\alpha$-Dirichlet problem (3.1) is well-posed. More specifically, there is a unique solution which is given by*

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y), \qquad X \in \Omega. \tag{3.2}$$

Moreover, $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap \dot{C}^\alpha(\overline{\Omega})$ and satisfies $u|_{\partial\Omega} = f$, and there is a constant C_α (depending only on n, the CDC constant, the ellipticity constant of L, and α) such that

$$\|f\|_{\dot{C}^\alpha(\partial\Omega)} \leqslant \|u\|_{\dot{C}^\alpha(\Omega)} \leqslant C_\alpha \|f\|_{\dot{C}^\alpha(\partial\Omega)}. \tag{3.3}$$

Remark 3.3. Compared to previous work on bounded domains (see, for example, [HKM06, Theorem 6.44]), we remark that the constant C_α above (as well as in the next result) depends on the CDC constant of Ω, but not on $\operatorname{diam}(\partial\Omega)$.

Our second result concerns the case Ω being unbounded with $\partial\Omega$ bounded (e.g. $\Omega = \mathbb{R}^{n+1}\backslash B$, where B is a ball in $\mathbb{R}^{n+1}$). In this scenario, solutions of (3.1) fail to be unique in general.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set satisfying the capacity density condition so that* **Ω is unbounded with $\partial\Omega$ being bounded**. *Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator, and let ω_L be the associated elliptic measure. Then there exists $\alpha_0 \in (0,1)$[2] (depending*

[1] One can take $\alpha_0 = \min\{\alpha_1, \alpha_2\}$, where $\alpha_1 \in (0,1)$ is the exponent in the De Giorgi/Nash estimates (cf. Lemma 3.10) and $\alpha_2 \in (0,1)$ is the exponent in the Hölder continuity at the boundary (cf. Lemma 3.11).

[2] One can take $\alpha_0 = \min\{\alpha_1, \alpha_2\}$, where $\alpha_1 \in (0,1)$ is the exponent in the De Giorgi/Nash estimates (cf. Lemma 3.10) and $\alpha_2 \in (0,1)$ is the exponent in the Hölder continuity at the boundary (cf. Lemma 3.11).

only on n, the CDC constant, and the ellipticity constant of L) such that for every $\alpha \in (0, \alpha_0)$*, for every* $f \in \dot{C}^\alpha(\partial\Omega)$*, and for every* $y_0 \in \partial\Omega$*, if we define*

$$u(X) = f(y_0) + \int_{\partial\Omega} (f(y) - f(y_0))\, d\omega_L^X(y), \qquad X \in \Omega, \tag{3.4}$$

then $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap \dot{C}^\alpha(\overline{\Omega})$*, satisfies* $u|_{\partial\Omega} = f$ *and solves the* $\dot{C}^\alpha$*-Dirichlet problem for L in* Ω *as formulated in* (3.1)*. Moreover, there exists* C_α *(depending on n, the CDC constant, the ellipticity constant of L, and* α*) such that*

$$\|f\|_{\dot{C}^\alpha(\partial\Omega)} \leqslant \|u\|_{\dot{C}^\alpha(\Omega)} \leqslant C_\alpha \|f\|_{\dot{C}^\alpha(\partial\Omega)}. \tag{3.5}$$

Nonetheless, the $\dot{C}^\alpha$*-Dirichlet problem for L in* Ω *is ill-posed since* (3.1) *always has more than one solution if f is non-constant.*

In Remark 3.26 below we discuss how the previous problem can be uniquely solved provided one prescribed the value of the solution at infinity.

Remark 3.5. We would like to call the reader's attention to the apparently different expressions (3.2) and (3.4) for solutions to the $\dot{C}^\alpha$-Dirichlet problem. First, in the scenario of Theorem 3.2 (after showing that u is well-defined via an absolutely convergent integral), one can use the fact that $\omega_L^X(\partial\Omega) = 1$ for every $X \in \Omega$ (see Section 2.4.6) to see that (3.2) and (3.4) give rise to the same function u. On the other hand, in the scenario of Theorem 3.4, we have $0 < \omega_L^X(\partial\Omega) < 1$ for every X in the unique unbounded connected component of Ω (again, see Section 2.4.6), hence (3.2) and (3.4) are different in general. Moreover, if u were chosen as in (3.2), then the second estimate in (3.5) could fail. Indeed, if (3.5) were true and we took $f \equiv 1$, then the solution would need to be constantly 1. Since the solution given by (3.2) is exactly $\omega_L^X(\partial\Omega)$, we would have $\omega_L^X(\partial\Omega) = 1$ in Ω, which is false in this case, as already noted.

As an immediate consequence of Theorems 3.2 and 3.4, we have the following.

Corollary 3.6. *Let* $\Omega \subset \mathbb{R}^{n+1}$*,* $n \geqslant 2$*, be an open set satisfying the capacity density condition and* $L = -\operatorname{div}(A\nabla)$ *be a real (non-necessarily symmetric) uniformly elliptic operator. Then, the* $\dot{C}^\alpha$*-Dirichlet problem for L in* Ω *formulated in* (3.1) *is well-posed if and only if either* Ω ***is bounded*** *or* Ω ***is unbounded with*** $\partial\Omega$ ***being unbounded****.*

It is important to emphasize that the results above are optimal in the sense that the capacity density condition is the minimal assumption needed to obtain well-posedness of the problem (3.1). In [Ai02], Aikawa proved (see Theorem 3.1) that for the Laplacian in a bounded regular domain Ω, solvability of the Dirichlet problem in Hölder spaces (along with the bound (3.6)) already implies that Ω satisfies the capacity density condition. Our next result, which extends [Ai02] in several ways, is stated as follows:

Theorem 3.7. *Let* $\Omega \subset \mathbb{R}^{n+1}$*,* $n \geqslant 2$*, be an open set, and let* $L = -\operatorname{div}(A\nabla)$ *be a real (non-necessarily symmetric) elliptic operator. Let* $\alpha \in (0, 1)$*. Assume that for every* $f \in \dot{C}^\alpha(\partial\Omega)$*, the function u defined by* (3.4) *is a solution*[3] *to* (3.1) *satisfying*

$$\|u\|_{\dot{C}^\alpha(\Omega)} \leqslant C_\alpha \|f\|_{\dot{C}^\alpha(\partial\Omega)}. \tag{3.6}$$

Then Ω *satisfies the capacity density condition, with constants depending only on n, the ellipticity of L,* α*, and* C_α*.*

[3]Here and elsewhere, whenever (3.4) (which agrees with (3.2) when elliptic measure is a probability) is assumed to be a solution to (3.1), we implicitly understand that the integral is absolutely convergent.

This result (along with the previous statements) is a quantitative analogue of the main results in [LSW63]. Indeed, we show that solving the Dirichlet problem with data in Hölder spaces relies on the CDC property which is of a geometrical nature (this should be compared with Wiener regularity for the continuous Dirichlet problem). Besides, we obtain that solving the Dirichlet problem with data in Hölder spaces for some given elliptic operator implies such solvability for any operator in this class (with a possibly different exponent depending on ellipticity). Moreover, we can prove that for any regular open set Ω (cf. Section 3.8) and for any elliptic operator L, the following holds: solutions defined by (3.2) or (3.4) satisfy the estimate (3.6) if and only if non-negative solutions with vanishing boundary data enjoy boundary Hölder continuity (see the precise statement in Lemma 3.11), with *almost* the same exponent α (see Remark 3.37).

On the other hand, Theorem 3.2 allows us to provide a characterization of Hölder spaces in terms of the boundary traces of solutions. To discuss this, we need to introduce some notation. We first observe that $\|\cdot\|_{\dot{C}^\alpha(\partial\Omega)}$ is only a semi-norm, and as a matter of fact, $\|f\|_{\dot{C}^\alpha(\partial\Omega)} = 0$ if and only if f is constant everywhere in $\partial\Omega$. In view of this, given two functions f, g defined on $\partial\Omega$, we say that $f \sim g$ provided $f - g$ is constant in $\partial\Omega$. This induces an equivalence relation and for every function $f : \partial\Omega \to \mathbb{R}$ we introduce the equivalence class of f as $[f] := \{g : \partial\Omega \to \mathbb{R} : g \sim f\}$, which ultimately defines the quotient space $\dot{C}^\alpha(\partial\Omega)/\mathbb{R} := \{[f] : f \in \dot{C}^\alpha(\partial\Omega)\}$. And defining $\big\|[f]\big\|_{\dot{C}^\alpha(\partial\Omega)/\mathbb{R}} := \|f\|_{\dot{C}^\alpha(\partial\Omega)}$ for $f \in \dot{C}^\alpha(\partial\Omega)$, we get a genuine norm on this quotient space. It is not hard to show that $\dot{C}^\alpha(\partial\Omega)/\mathbb{R}$ equipped with the norm $\big\|[\cdot]\big\|_{\dot{C}^\alpha(\partial\Omega)}$ is complete, hence a Banach space.

Let us also introduce the space

$$\dot{C}^\alpha_L(\Omega) := \big\{u \in W^{1,2}_{\rm loc}(\Omega) \cap \dot{C}^\alpha(\Omega) : Lu = 0 \text{ in the weak sense in } \Omega\big\}.$$

This is clearly a linear space on which $\|\cdot\|_{\dot{C}^\alpha(\Omega)}$ is also a semi-norm with null-space $\mathbb{R}$. Much as before we can introduce $\big\|[u]\big\|_{\dot{C}^\alpha_L(\Omega)/\mathbb{R}} := \|u\|_{\dot{C}^\alpha(\Omega)}$, which defines a genuine norm on the quotient space $\dot{C}^\alpha_L(\Omega)/\mathbb{R}$.

Corollary 3.8. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set satisfying the capacity density condition so that either Ω **is bounded** or Ω **is unbounded with** $\partial\Omega$ **being unbounded**. Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator. There exists $\alpha_0 \in (0,1)$[4] (depending only on n, the CDC constant, and the ellipticity constants of L) such that for every $\alpha \in (0, \alpha_0)$, there exists $C_\alpha \geqslant 1$ (depending on the same parameters and additionally on α) with the property that*

$$u \in \dot{C}^\alpha_L(\Omega) \Longrightarrow \begin{cases} u \in \dot{C}^\alpha(\overline{\Omega}), \text{ hence } u|_{\partial\Omega} \in \dot{C}^\alpha(\partial\Omega), \\ u(X) = \displaystyle\int_{\partial\Omega} \big(u|_{\partial\Omega}\big)(y)\, d\omega^X_L(y), \quad X \in \Omega, \\ \big\|u|_{\partial\Omega}\big\|_{\dot{C}^\alpha(\partial\Omega)} \leqslant \|u\|_{\dot{C}^\alpha(\Omega)} \leqslant C_\alpha \big\|u|_{\partial\Omega}\big\|_{\dot{C}^\alpha(\partial\Omega)}. \end{cases} \tag{3.7}$$

In addition,

$$\dot{C}^\alpha(\partial\Omega) = \big\{u|_{\partial\Omega} : u \in \dot{C}^\alpha_L(\Omega)\big\}. \tag{3.8}$$

Moreover, $\dot{C}^\alpha_L(\Omega)/\mathbb{R}$ equipped with the norm $\big\|[\cdot]\big\|_{\dot{C}^\alpha(\Omega)/\mathbb{R}}$ is a Banach space and the operator

$$\dot{C}^\alpha_L(\Omega)/\mathbb{R} \ni [u] \longmapsto \big[u|_{\partial\Omega}\big] \in \dot{C}^\alpha(\partial\Omega)/\mathbb{R} \tag{3.9}$$

is a well-defined isomorphism between Banach spaces.

[4] One can take $\alpha_0 = \min\{\alpha_1, \alpha_2\}$, where $\alpha_1 \in (0,1)$ is the exponent in the De Giorgi/Nash estimates (cf. Lemma 3.10) and $\alpha_2 \in (0,1)$ is the exponent in the Hölder continuity at the boundary (cf. Lemma 3.11).

As applications of the above results, we will also establish well-posedness results for Dirichlet boundary value problems associated with different functional spaces, both for solutions and boundary data. These include Morrey-Campanato spaces, and spaces of solutions satisfying a fractional Carleson measure condition, which we will show to be closely related to the $\dot{C}^\alpha$ spaces.

3.3 Main ideas and outline of the chapter

In Section 3.4, we present and recall the definitions and auxiliary results that will be central for all the subsequent proofs: the CDC property, De Giorgi/Nash interior estimates, and Hölder estimates for solutions that vanish on a portion of the boundary. The fact that solutions to very general equations are Hölder continuous in the interior of our domain (De Giorgi/Nash, Lemma 3.10) is one of the milestones of modern PDE theory, and has been exploited very successfully for the last half-century. In contrast, the fact that solutions with zero boundary data attain this boundary value in a Hölder way (Lemma 3.11) is more subtle because it is not true in just any domain. Here, we state it under the CDC assumption (which will ultimately turn out to be optimal), which asserts that the exterior of our domain is, in some sense, *large*. It will be important to have in mind that the CDC (our main hypothesis) will manifest in the proofs through Lemma 3.11.

The proofs of the main results, that is, those of well-posedness and ill-posedness of the $\dot{C}^\alpha$-Dirichlet problem (Theorems 3.2, 3.4, and also Corollary 3.8), are presented in Sections 3.6 and 3.7, respectively. The basic idea to prove that solutions satisfy the Hölder estimate (3.3) (or related ones) is that the estimate should follow from Lemma 3.10 when far from $\partial\Omega$, and from Lemma 3.11 when close to $\partial\Omega$. Apart from technical difficulties (like the fact that all the integrals are well-defined as in Step 1 in Subsection 3.6.1), the key is to understand how to extract the power from Lemma 3.11 (which is really where the CDC property plays a role), which only works for solutions vanishing on certain parts of the boundary: the main point will be to introduce suitable truncations (as in Lemma 3.19) or cancellations (as in Step 2 in Subsection 3.6.1) to be able to take advantage of Lemma 3.11.

Next, Section 3.8 is devoted to proving optimality of the capacity density condition (Theorem 3.7). The proof extends that in [Ai02]: the main idea is that Hölder solvability implies some decay for the elliptic measure (as in Definition 3.34) which is very similar to Lemma 3.11. This decay allows us to find many boundary points (hence also exterior) close to any other given boundary point (with the probabilistic interpretation in mind, if the elliptic measure decays at a certain rate as we approach the boundary, it must be the case that there are "lots of boundary" around, or "lots of exterior"), which ultimately give us information about capacities. We will explore this relationship using maximum principles between the capacitary potentials (the solutions which "represent" capacity of sets) and suitably adapted/truncated elliptic measures.

Lastly, in Section 3.9 we explore some applications of the previous results to Morrey-Campanato spaces and spaces of functions satisfying a fractional Carleson measure condition. We will find that these spaces, under suitable assumptions, are related (or even contained) with Hölder spaces, which will allow us to use the methods already developed to obtain well-posedness of the Dirichlet problem in these new spaces. Concretely, the spaces defined by Carleson measure conditions in Subsection 3.9.1 will only prove to be related to Hölder spaces in presence of some extra connectivity (1-sided CAD), and for the Morrey-Campanato spaces from Subsection 3.9.2, it will suffice to have the ADR condition for the boundary instead of the CDC.

A relevant difference as compared to the previous sections is that the formulation of these new prob-

lems (see (3.76)) involves non-tangential convergence in certain large subset of the boundary. However, this subset may not include the closest point $\hat{x}$ that we used in Subsection 3.6.2 to show uniqueness of solutions. This forces us to consider more convolved forms to approach the boundary and mimic that proof: we will use carrot paths to accomplish this in Proposition 3.53.

We mention that all the results above can be extended to generalized Hölder and Morrey-Campanato spaces under certain assumptions on the growth function/modulus of continuity (see Section 3.5 for more details). Our proofs will be carried out in this general scenario, and we will repeatedly use some elementary properties of our classes of growth functions, that will be presented in Subsection 3.5.

Anyway, for a first read, we suggest the reader to focus on the case of pure Hölder spaces, where the modulus of continuity is $\varphi(t) = t^\alpha$.

3.4 Preliminaries

We suggest the reader to review the notation that will be used in this chapter in Section 2.1.

We already introduced the general framework in which we will work in Chapter 2, but let us recall the key concepts and tools that will appear very frequently in this chapter, for the convenience of the reader. First, we recall the CDC (see Section 2.4.5), which is the key geometric assumption.

Definition 3.9 (Capacity density condition, CDC). An open set Ω is said to satisfy the *capacity density condition* (CDC for short) if there exists a uniform constant $c_1 > 0$ such that

$$\frac{\mathrm{Cap}(\overline{B(x,r)}\setminus\Omega, B(x,2r))}{\mathrm{Cap}(\overline{B(x,r)}, B(x,2r))} \geqslant c_1$$

for all $x \in \partial\Omega$ and $0 < r < \mathrm{diam}(\partial\Omega)$.

Regarding the PDE framework, we advice the reader to revisit Section 2.4.1 for the definition of elliptic operator, weak solution and elliptic measure; and Section 2.4.6 for the extension of the elliptic measure to unbounded domains. Apart from these sections, the reader should remind the following regularity results, whose careful combination will give the results in this chapter.

Lemma 3.10 (De Giorgi/Nash). *Let $L = -\mathrm{div}(A\nabla)$ be a real (non-necessarily symmetric) uniformly elliptic operator. There exist $C_1 \geqslant 1$ and $\alpha_1 \in (0,1)$ (depending only on dimension and on the ellipticity constant of L) such that for every ball $B = B(X_0, R) \subset \mathbb{R}^{n+1}$ and for every $u \in W^{1,2}(2B)$ so that $Lu = 0$ in the weak sense in $2B$, one can redefine u in a set of null measure (abusing the notation, the new function is called again u) in such a way that $u \in \dot{C}^{\alpha_1}(\overline{B})$ (hence $u \in C(\overline{B}) \cap L^\infty(B)$) satisfying*

$$\sup_{X\in\overline{B}} |u(X)| + \sup_{\substack{X,Y\in\overline{B}\\ X\neq Y}} \frac{|u(X)-u(Y)|}{(|X-Y|/R)^{\alpha_1}} \leqslant C_1 \left(\fint\!\!\!\fint_{2B} |u(Y)|^2\, dY \right)^{\frac{1}{2}}.$$

Lemma 3.11 (Boundary Hölder continuity). *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set satisfying the CDC, and let $L = -\mathrm{div}(A\nabla)$ be a real (non-necessarily symmetric) uniformly elliptic operator. There exist $C_2 \geqslant 1$ and $\alpha_2 \in (0,1)$ (depending only on dimension, the CDC constant, and on the ellipticity constant of L) such that for every ball $B = B(x,r)$ with $x \in \partial\Omega$, $0 < r < \mathrm{diam}(\partial\Omega)$, and $\Delta = B \cap \partial\Omega$; and for*

every $0 \leqslant u \in W^{1,2}_{\mathrm{loc}}(2B \cap \Omega) \cap C(\overline{2B \cap \Omega})$ *with* $u \equiv 0$ *in* 2Δ *satisfying* $Lu = 0$ *in the weak sense in* $2B \cap \Omega$*, there holds*

$$u(X) \leqslant C_2 \left(\frac{|X - x|}{r} \right)^{\alpha_2} \sup_{\overline{B \cap \Omega}} u, \quad \forall X \in B \cap \Omega.$$

3.5 Growth functions and auxiliary lemmas

In the rest of the chapter, we will be working with moduli of continuity/growth functions that generalize $\varphi(t) = t^\alpha$ with $\alpha \in (0,1)$, hence generalizing the $\dot{C}^\alpha$ spaces.

Definition 3.12 (Growth function, $\mathcal{G}_\alpha$ class). Call a given function $\varphi : (0, +\infty) \longrightarrow (0, +\infty)$ a *growth function* if φ is non-decreasing and $\varphi(t) \to 0$ as $t \to 0^+$. Moreover, we say that the growth function φ belongs to the class $\mathcal{G}_\alpha^\infty$ for some $\alpha \in (0,1)$ if there exists some C_φ such that

$$t^\alpha \int_t^\infty \frac{\varphi(s)}{s^\alpha} \frac{ds}{s} \leqslant C_\varphi \, \varphi(t), \qquad \forall t > 0. \tag{3.10}$$

Analogously, we say that the growth function φ belongs to the class $\mathcal{G}_\alpha$ for some $\alpha \in (0,1)$ if there exists some C_φ such that

$$\int_0^t \varphi(s) \frac{ds}{s} + t^\alpha \int_t^\infty \frac{\varphi(s)}{s^\alpha} \frac{ds}{s} \leqslant C_\varphi \, \varphi(t), \qquad \forall t > 0. \tag{3.11}$$

Definition 3.13 (Generalized Hölder spaces). Let φ be a growth function. Define the homogeneous φ-Hölder space on a set $E \subset \mathbb{R}^{n+1}$ as

$$\dot{C}^\varphi(E) := \left\{ u : E \to \mathbb{R} : \|u\|_{\dot{C}^\varphi(E)} := \sup_{\substack{X,Y \in E \\ X \neq Y}} \frac{|u(X) - u(Y)|}{\varphi(|X - Y|)} < \infty \right\}.$$

Observe that $\| \cdot \|_{\dot{C}^\varphi(E)}$ is only a semi-norm, and as a matter of fact, $\|f\|_{\dot{C}^\varphi(E)} = 0$ if and only if f is a constant everywhere in E. In view of this, given two functions f, g defined in E, we say that $f \sim g$ provided $f - g$ is a constant in E. This induces an equivalence relation and for every function $f : E \to \mathbb{R}$ we introduce the equivalence class of f as $[f] := \{g : E \to \mathbb{R} : g \sim f\}$. For any $f \in \dot{C}(E)$ we can now define $\|[f]\|_{\dot{C}^\varphi(E)/\mathbb{R}} := \|f\|_{\dot{C}^\varphi(E)}$, which defines a genuine norm on the quotient space $\dot{C}^\varphi(E)/\mathbb{R} := \{[f] : f \in \dot{C}^\varphi(E)\}$. It is not hard to show that $\dot{C}^\varphi(E)/\mathbb{R}$ equipped with the norm $\| \cdot \|_{\dot{C}^\varphi(E)/\mathbb{R}}$ is complete, hence a Banach space.

Let us denote, for $\alpha \in (0,1)$ and φ a growth function,

$$Q_\alpha \varphi(t) := t^\alpha \int_t^\infty \frac{\varphi(s)}{s^\alpha} \frac{ds}{s}. \tag{3.12}$$

Observe that $Q_\alpha \varphi$ is a well-defined function whenever $Q_\alpha \varphi(1) < \infty$, because then $Q_\alpha \varphi(t) < \infty$ for any $t > 0$. We would like to observe that Q_α is a generalization of the so-called Hardy operators, see [BS88, Chapter 1].

Lemma 3.14. *Let* φ *be a growth function and* $\alpha \in (0,1)$ *be such that* $Q_\alpha \varphi(1) < \infty$*. Then the following hold:*

(a) *$Q_\alpha\varphi$ is a growth function.*

(b) *$t \longmapsto t^{-\alpha}\, Q_\alpha\varphi(t)$ is a decreasing function that decays to 0 as $t \to \infty$.*

(c) *$Q_\alpha\varphi$ is doubling. Concretely, $Q_\alpha\varphi(2\,t) \leqslant 2^\alpha Q_\alpha\varphi(t)$ for every $t > 0$.*

(d) *$\varphi(t) \leqslant \alpha\, Q_\alpha\varphi(t) \leqslant Q_\alpha\varphi(t)$ for any $t > 0$.*

(e) *$t^\alpha \displaystyle\sum_{k=0}^{\infty} \frac{\varphi(2^k\,t)}{(2^k\,t)^\alpha} \lesssim_\alpha Q_\alpha\varphi(t)$ for any $t > 0$.*

(f) *$Q_\alpha\varphi$ is decreasing in α, that is, $Q_\alpha\varphi \leqslant Q_{\alpha'}\varphi$ whenever $0 < \alpha' < \alpha < 1$.*

Proof. After changing variables, we have

$$Q_\alpha\varphi(t) = \int_1^\infty \frac{\varphi(t\tau)}{\tau^\alpha}\frac{d\tau}{\tau}. \tag{3.13}$$

Therefore, since φ is non-decreasing, $Q_\alpha(t)$ is also non-decreasing on t. Moreover, letting $t \to 0^+$ with the Dominated Convergence Theorem (because $Q_\alpha\varphi(1) < \infty$), and using that $\varphi(t) \to 0$ as $t \to 0^+$, it follows that $Q_\alpha\varphi(t) \to 0$ as $t \to 0^+$. This shows (a). To show (b), invoke the Dominated Convergence Theorem in (3.12), again using $Q_\alpha\varphi(1) < \infty$. In turn, property (c) follows from the fact that φ is positive:

$$Q_\alpha\varphi(2t) = 2^\alpha t^\alpha \int_{2t}^\infty \frac{\varphi(s)}{s^\alpha}\frac{ds}{s} \leqslant 2^\alpha t^\alpha \int_t^\infty \frac{\varphi(s)}{s^\alpha}\frac{ds}{s} = 2^\alpha Q_\alpha\varphi(t).$$

Property (d) is an easy consequence of the monotonicity of φ:

$$Q_\alpha\varphi(t) \geqslant t^\alpha\varphi(t) \int_t^\infty \frac{1}{s^\alpha}\frac{ds}{s} = \frac{\varphi(t)}{\alpha},$$

and we may also obtain (e) as follows:

$$\sum_{k=0}^{\infty} \frac{\varphi(2^k\,t)}{(2^k\,t)^\alpha} \approx \sum_{k=0}^{\infty} \frac{\varphi(2^k\,t)}{(2^k\,t)^\alpha} \int_{2^k t}^{2^{k+1}t} \frac{ds}{s} \lesssim \sum_{k=0}^{\infty} \int_{2^k t}^{2^{k+1}t} \frac{\varphi(s)}{s^\alpha}\frac{ds}{s} = t^{-\alpha} Q_\alpha(t).$$

Lastly, (f) follows at once from (3.13). □

Lemma 3.15. *Let φ be a growth function. Assume that φ is doubling, i.e., there exists C such that $\varphi(2t) \leqslant C\varphi(t)$ for every $t > 0$. Then, for each set $E \subset \mathbb{R}^{n+1}$ and function $u \in \dot{C}^\varphi(E)$, there exists an extension —which we call again u— to the closure of E, so that $u \in \dot{C}^\varphi(\overline{E})$ and the following estimate holds:*

$$\|u\|_{\dot{C}^\varphi(E)} \leqslant \|u\|_{\dot{C}^\varphi(\overline{E})} \leqslant C\|u\|_{\dot{C}^\varphi(E)}.$$

Proof. Note first that if $u \in \dot{C}^\varphi(E)$, then u is uniformly continuous in E. Hence, we can extend u continuously to a function v on $\overline{E}$. Let $y \neq z \in \overline{E}$ and pick $(y_k)_k, (z_k)_k \subset E$ with $y_k \to y$ and $z_k \to z$. Compute now

$$|v(y)-v(z)| = \lim_{k\to\infty} |u(y_k)-u(z_k)| \leqslant \|u\|_{\dot{C}^\varphi(E)} \limsup_{k\to\infty} \varphi(|y_k-z_k|)$$
$$\leqslant \|u\|_{\dot{C}^\varphi(E)} \varphi(2|y-z|) \leqslant C\|u\|_{\dot{C}^\varphi(E)} \varphi(|y-z|)$$

using the doubling property of φ. This finishes the proof. □

The following result generalizes [MMM20, Lemma 2.1], and follows easily from Lemma 3.14:

Lemma 3.16. *Given a growth function $\varphi \in \mathcal{G}_\alpha^\infty$ for some $\alpha \in (0,1)$, the following hold:*

(a) *For every $0 < t_1 \leqslant t_2 < \infty$ we have $t_2^{-\alpha}\varphi(t_2) \leqslant \beta C_\varphi\, t_1^{-\alpha}\varphi(t_1)$.*

(b) *For every $t > 0$ we have $\varphi(2t) \leqslant 2^\alpha \beta C_\varphi\, \varphi(t) \leqslant 2C_\varphi\, \varphi(t)$.*

(c) *For every $t_1, t_2 > 0$ we have $\varphi(t_1+t_2) \leqslant 2C_\varphi\, (\varphi(t_1)+\varphi(t_2))$.*

(d) *$t^{-\alpha}\varphi(t) \to 0$ as $t \to \infty$.*

(e) *$\varphi \in \mathcal{G}_\beta^\infty$ for every $\alpha \leqslant \beta < 1$.*

Proof. Using Lemma 3.14 (concretely, (d), (b), and (3.10)), (a) follows. From this, we get (b) just plugging $t_2 = 2t$ and $t_1 = t$. Moreover, (b) implies that for all $t_1, t_2 > 0$,

$$\varphi(t_1+t_2) \leqslant \varphi(2\max\{t_1,t_2\}) \leqslant 2C_\varphi\, \varphi(\max\{t_1,t_2\}) \leqslant 2C_\varphi\, (\varphi(t_1)+\varphi(t_2)),$$

which shows (c). Furthermore, (d) follows at once from Lemma 3.14 (concretely, (d) and (b)). Finally, (e) follows at once from Lemma 3.14 part (f). □

Remark 3.17. Let us note that whenever $\varphi \in \mathcal{G}_\alpha^\infty$, it follows that $Q_\alpha\varphi(1) < \infty$ and φ is doubling, hence the conclusions of Lemmas 3.14 and 3.15 are also satisfied.

Remark 3.18. Given $\alpha \in (0,1)$, the function $\varphi(t) := t^\alpha$ satisfies $Q_\beta\varphi(t) \approx t^\alpha$ for every $\beta > \alpha$. Hence, $\varphi \in \mathcal{G}_\beta^\infty$ and $\varphi \in \mathcal{G}_\alpha$ for every $\beta > \alpha$. This is the particular case we are most interested in.

The following lemma will be key in the forthcoming arguments.

Lemma 3.19. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set satisfying the CDC, and let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) uniformly elliptic operator. Then there exists C (depending only on dimension, the CDC constant, and on the ellipticity constant of L) such that, for any $x \in \partial\Omega$ and $r > 0$,*

$$\omega_L^X(\partial\Omega\setminus\Delta(x,4r)) \leqslant C\left(\frac{|X-x|}{r}\right)^{\alpha_2}, \qquad X \in B(x,r)\cap\Omega, \tag{3.14}$$

where $\alpha_2 \in (0,1)$ is the exponent in Lemma 3.11. Furthermore, for each growth function φ, there exists some C (depending only on dimension, the CDC constant and the ellipticity constant of L) such that

$$\int_{\partial\Omega} \varphi(|y-x|)\, d\omega_L^X(y) \leqslant C Q_{\alpha_2}\varphi(\delta(X)), \quad \forall X \in \Omega, x \in \partial\Omega \text{ such that } |X-x| \leqslant 3\delta(X). \tag{3.15}$$

Therefore, if $\varphi \in \mathcal{G}_\beta^\infty$ for some $0 < \beta \leqslant \alpha_2$ there exists some C (depending only on dimension, the CDC constant, the ellipticity constant of L, and C_φ) such that

$$\int_{\partial\Omega} \varphi(|y-x|)\, d\omega_L^X(y) \leqslant C\varphi(\delta(X)), \quad \forall X \in \Omega, x \in \partial\Omega \text{ such that } |X-x| \leqslant 3\delta(X). \tag{3.16}$$

Remark 3.20. We remark that even though we assume that Ω satisfies the CDC, we only need the fact that Ω satisfies the boundary Hölder continuity property stated in Lemma 3.11 and regularity of the boundary (cf. Section 3.8).

Proof. Given $j \gg 1$, choose $\phi_j \in C_c^\infty(\mathbb{R}^{n+1})$ such that $\phi_j \equiv 1$ in $B(x,2^j r)\backslash B(x,4r)$, $\phi_j \equiv 0$ in $B(x,2r) \cup (\partial\Omega\backslash B(x,2^{j+1}r))$, and $0 \leqslant \phi_j \leqslant 1$ everywhere. Hence

$$\omega_L^X(\Delta(x,2^j r)\backslash\Delta(x,4r)) \leqslant \int_{\partial\Omega} \phi_j \, d\omega_L^X =: v_j(X), \qquad X \in \Omega.$$

By boundary Hölder continuity (cf. Lemma 3.11, which we can use because Ω satisfies the CDC, whence $\partial\Omega$ is Wiener regular), we have, for every $X \in B(x,r) \cap \Omega$,

$$v_j(X) \lesssim \left(\frac{|X-x|}{r}\right)^{\alpha_2} \sup_\Omega v_j \leqslant \left(\frac{|X-x|}{r}\right)^{\alpha_2},$$

because $0 \leqslant v_j \leqslant 1$ since $0 \leqslant \phi_j \leqslant 1$. Taking limits as $j \to \infty$ and using the Monotone Convergence Theorem, (3.14) follows.

To proceed, let φ be a growth function. Let $X \in \Omega$ and $x \in \partial\Omega$ be such that $|X-x| \leqslant 3\delta(X)$. Write $\Delta_k := \Delta(x, 2^k\delta(X))$ for any $k \in \mathbb{Z}$. Then we may compute

$$\begin{aligned}\int_{\partial\Omega} \varphi(|y-x|) d\omega_L^X(y) &= \int_{\Delta_4} \varphi(|y-x|) d\omega_L^X(y) + \sum_{k=4}^\infty \int_{\Delta_{k+1}\backslash\Delta_k} \varphi(|y-x|) d\omega_L^X(y) \\ &\lesssim \omega_L^X(\Delta_4)\varphi(16\delta(X)) + \sum_{k=4}^\infty \varphi(2^{k+1}\delta(X)) \left(\frac{3\delta(X)}{2^{k-2}\delta(X)}\right)^{\alpha_2} \lesssim Q_{\alpha_2}\varphi(\delta(X)),\end{aligned}$$

where we have used that φ is non-decreasing, $\omega_L^X(\Delta_4) \leqslant \omega_L^X(\partial\Omega) \leqslant 1$, and Lemma 3.14. This shows (3.15). Lastly, (3.16) follows at once from (3.15), Lemma 3.14, $\beta \leqslant \alpha_2$, and that $\varphi \in \mathcal{G}_\beta^\infty$. □

3.6 Well-posedness of the Dirichlet problem in Hölder spaces

In this section, we will prove Theorem 3.2. We will in fact obtain a more general version (see Remark 3.18), using growth functions $\varphi \in \mathcal{G}_\beta$.

Theorem 3.21. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set satisfying the CDC so that either Ω* **is bounded or Ω is unbounded with $\partial\Omega$ being unbounded**. *Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator, and let ω_L be the associated elliptic measure. Then there exists $\beta \in (0,1)$*[5] *(depending only on n, the CDC constant, and the ellipticity constant of L) such that for every growth function $\varphi \in \mathcal{G}_\beta^\infty$, the $\dot{C}^\varphi$-Dirichlet problem*

$$\begin{cases} u \in W^{1,2}_{\mathrm{loc}}(\Omega), \\ Lu = 0 \text{ in the weak sense in } \Omega, \\ u \in \dot{C}^\varphi(\Omega), \\ u|_{\partial\Omega} = f \in \dot{C}^\varphi(\partial\Omega) \end{cases} \tag{3.17}$$

[5] One can take $\alpha_0 = \min\{\alpha_1, \alpha_2\}$, where $\alpha_1 \in (0,1)$ is the exponent in the De Giorgi/Nash estimates (cf. Lemma 3.10) and $\alpha_2 \in (0,1)$ is the exponent in the Hölder continuity at the boundary (cf. Lemma 3.11).

is well-posed. More specifically, there is a unique solution which is given by

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y), \qquad X \in \Omega. \tag{3.18}$$

Moreover, $u \in W^{1,2}_{\rm loc}(\Omega) \cap \dot{C}^\varphi(\overline{\Omega})$ and satisfies $u|_{\partial\Omega} = f$, and there is a constant C (depending only on n, the CDC constant, the ellipticity constant of L, and C_φ) such that

$$\|f\|_{\dot{C}^\varphi(\partial\Omega)} \leqslant \|u\|_{\dot{C}^\varphi(\Omega)} \leqslant C\|f\|_{\dot{C}^\varphi(\partial\Omega)}. \tag{3.19}$$

3.6.1 Proof of Theorem 3.21: Existence

We first show that u defined in (3.18) solves (3.17). If f is a constant on $\partial\Omega$, then it follows from (3.18) and the fact that in the current case ω_L is a probability that $u \equiv f(y_0)$ for any $y_0 \in \partial\Omega$. Then u satisfies (3.17) and (3.19). Thus, without loss of generality, by normalizing, we may assume that $\|f\|_{\dot{C}^\varphi(\partial\Omega)} = 1$.

Step 1: u is a well-defined solution. When $\partial\Omega$ is bounded, $f \in \dot{C}^\varphi(\partial\Omega)$ implies that f is continuous and bounded in the compact set $\partial\Omega$, hence the integral in (3.18) is certainly well-defined. When $\partial\Omega$ is unbounded, we need to consider smooth cut-offs to show that the integral in (3.18) is absolutely convergent. To that end, cover Ω with a sequence of balls of the form $\Omega = \bigcup_{i=0}^\infty B(X_i, \delta(X_i)/4)$. We first work locally inside any of these Whitney balls, say $B(X_0, \delta(X_0)/4)$. Consider $\widehat{x}_0 \in \partial\Omega$ such that $\delta(X_0) = |X_0 - \widehat{x}_0|$.

Choose now $\phi \in C_c^\infty(\mathbb{R})$ so that $\mathbf{1}_{[0,1]}(t) \leqslant \phi(t) \leqslant \mathbf{1}_{[0,2]}(t)$ for every $t \geqslant 0$. For any $j \in \mathbb{Z}$, if we write $\phi_j^{X_0}(X) := \phi(2^{-j}|X - \widehat{x}_0|)$, then $\phi_j^{X_0} \in C_c^\infty(\mathbb{R}^{n+1})$ and $\mathbf{1}_{B(\widehat{x}_0, 2^j)}(X) \leqslant \phi_j^{X_0}(X) \leqslant \mathbf{1}_{B(\widehat{x}_0, 2^{j+1})}(X)$ for every $X \in \mathbb{R}^{n+1}$. Observe that for each $j \in \mathbb{Z}$,

$$|\phi_j^{X_0}(y) - \phi_{j+1}^{X_0}(y)| \leqslant \mathbf{1}_{\Delta(\widehat{x}_0, 2^{j+2}) \setminus \Delta(\widehat{x}_0, 2^j)}(y), \qquad y \in \partial\Omega.$$

If we define

$$u_j^{X_0}(X) := f(\widehat{x}_0) + \int_{\partial\Omega} (f(y) - f(\widehat{x}_0))\phi_j^{X_0}(y)\, d\omega_L^X(y), \qquad X \in \Omega,$$

then for any $X \in B(X_0, \delta(X_0)/2)$ and $j \in \mathbb{Z}$ such that $2^j > 4\delta(X_0)$,

$$\begin{aligned}
|u_j^{X_0}(X) - u_{j+1}^{X_0}(X)| &\leqslant \int_{\partial\Omega} |f(y) - f(\widehat{x}_0)|\big|\phi_j^{X_0}(y) - \phi_{j+1}^{X_0}(y)\big| d\omega_L^X(y) \\
&\leqslant \int_{\Delta(\widehat{x}_0, 2^{j+2}) \setminus \Delta(\widehat{x}_0, 2^j)} |f(y) - f(\widehat{x}_0)| d\omega_L^X(y) \\
&\leqslant \int_{\Delta(\widehat{x}_0, 2^{j+2}) \setminus \Delta(\widehat{x}_0, 2^j)} \varphi(|y - \widehat{x}_0|)\, d\omega_L^X(y) \\
&\leqslant \varphi(2^{j+2})\, \omega_L^X(\Delta(\widehat{x}_0, 2^{j+2}) \setminus \Delta(\widehat{x}_0, 2^j)) \\
&\approx \varphi(2^{j+2})\, \omega_L^{X_0}(\Delta(\widehat{x}_0, 2^{j+2}) \setminus \Delta(\widehat{x}_0, 2^j)) \\
&\lesssim \varphi(2^{j+2}) \left(\frac{|X_0 - \widehat{x}_0|}{2^{j-1}}\right)^{\alpha_2} \\
&\lesssim \varphi(2^{j+2}) \left(\frac{\delta(X_0)}{2^{j+2}}\right)^{\alpha_2},
\end{aligned} \tag{3.20}$$

where we used that φ is non-decreasing, Harnack's inequality (for $X \in B(X_0, \delta(X_0)/2)$), and Lemma 3.19. Telescoping for $k' > k$ with $2^k > 4\delta(X_0)$, we obtain from (3.20) and Lemma 3.14 (e), (f) and (b), assuming that $\beta \leqslant \alpha_2$:

$$\begin{aligned}\left|u_k^{X_0}(X) - u_{k'}^{X_0}(X)\right| &\leqslant \sum_{j=k}^{k'-1} \left|u_j^{X_0}(X) - u_{j+1}^{X_0}(X)\right| \lesssim \sum_{j=k}^{\infty} \varphi(2^{j+2}) \left(\frac{\delta(X_0)}{2^{j+2}}\right)^{\alpha_2} \\ &= \delta(X_0)^{\alpha_2} \sum_{j=0}^{\infty} \frac{\varphi(2^j 2^{k+2})}{(2^j 2^{k+2})^{\alpha_2}} \lesssim \delta(X_0)^{\alpha_2} (2^{k+2})^{-\alpha_2} Q_{\alpha_2} \varphi(2^{k+2}) \underset{k\to\infty}{\longrightarrow} 0, \quad (3.21)\end{aligned}$$

as a result $\{u_j^{X_0}\}_j$ is a Cauchy sequence uniformly in $B(X_0, \delta(X_0)/2)$. Since we also know that $u_j^{X_0} \in C(B(X_0, \delta(X_0)/2))$, we deduce that there exists $u_\infty^{X_0} \in C(B(X_0, \delta(X_0)/2))$ such that $u_j^{X_0} \to u_\infty^{X_0}$ uniformly in $B(X_0, \delta(X_0)/2)$ as $j \to \infty$. On the other hand, Caccioppoli's inequality implies

$$\begin{aligned}\iint_{B(X_0,\delta(X_0)/4)} \left|\nabla u_j^{X_0} - \nabla u_{j'}^{X_0}\right|^2 dX &\lesssim (\delta(X_0))^{-2} \iint_{B(X_0,\delta(X_0)/2)} \left|u_j^{X_0} - v_{j'}^{X_0}\right|^2 dX \\ &\lesssim (\delta(X_0))^{n-1} \sup_{B(X_0,\delta(X_0)/2)} \left|u_j^{X_0} - u_{j'}^{X_0}\right|^2,\end{aligned}$$

thus $\{\nabla u_j^{X_0}\}_j$ is a Cauchy sequence in $L^2(B(X_0, \delta(X_0)/4))$. Thus $u_\infty^{X_0} \in W^{1,2}(B(X_0, \delta(X_0)/4))$ and $u_j^{X_0} \to u_\infty^{X_0}$ in $W^{1,2}(B(X_0, \delta(X_0)/4))$ as $j \to \infty$.

We now turn to see that $u_\infty^{X_0}$ coincides with u (as defined in (3.18)) for $X \in B(X_0, \delta(X_0)/4)$. Recall that

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y) = f(\widehat{x}_0) + \int_{\partial\Omega} (f(y) - f(\widehat{x}_0))\, d\omega_L^X(y)$$

because the elliptic measure is a probability in this case, and the integral in the previous equation is well defined since it is absolutely convergent:

$$\int_{\partial\Omega} |f(y) - f(\widehat{x}_0)|\, d\omega_L^X(y) \leqslant \int_{\partial\Omega} \varphi(|y - \widehat{x}_0|)\, d\omega_L^X(y) \lesssim \varphi(\delta(X)) < \infty$$

by Lemma 3.19, where we also use the fact that $|X - \widehat{x}_0| < \frac{5}{4}\delta(X_0) < \frac{5}{3}\delta(X)$. In particular, for any $X \in B(X_0, \delta(X_0)/4)$ and $j \in \mathbb{Z}$ with $2^j > 5\delta(X_0)$ we invoke Lemma 3.19, and Harnack's inequality later, to conclude that

$$\begin{aligned}\left|u(X) - u_j^{X_0}(X)\right| &\leqslant \int_{\partial\Omega} |f(y) - f(\widehat{x}_0)| \left|1 - \phi_j^{X_0}(y)\right| d\omega_L^X(y) \\ &\leqslant \int_{\partial\Omega\setminus\Delta(\widehat{x}_0,2^j)} |f(y) - f(\widehat{x}_0)|\, d\omega_L^X(y) \\ &\leqslant \sum_{k=j}^{\infty} \varphi(2^{k+1}) \omega_L^X(\Delta(\widehat{x}_0, 2^{k+1}) \setminus \Delta(\widehat{x}_0, 2^k)) \\ &\approx \sum_{k=j}^{\infty} \varphi(2^{k+1}) \omega_L^{X_0}(\Delta(\widehat{x}_0, 2^{k+1}) \setminus \Delta(\widehat{x}_0, 2^k))\end{aligned}$$

$$\lesssim \sum_{k=j}^{\infty} \varphi(2^{k+1}) \left(\frac{\delta(X_0)}{2^{k-2}}\right)^{\alpha_2},$$

which converges to 0 as $j \to \infty$ by Lemma 3.14 (the computation is analogous to that in (3.21)), provided $\beta \leqslant \alpha_2$. Therefore, we conclude that $u(X) = u_\infty^{X_0}(X)$ for all $X \in B(X_0, \delta(X_0)/4)$.

We can proceed in a similar fashion in $B(X_i, \delta(X_i)/4)$ for each $i \in \mathbb{N}$ to see that there exists some $u_\infty^{X_i} \in C(B(X_i, \delta(X_i)/4)) \cap W^{1,2}(B(X_i, \delta(X_i)/4))$ that coincides with u in $B(X_i, \delta(X_i)/4)$. This yields easily that $u \in C(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega)$ since $\Omega = \bigcup_{i=0}^{\infty} B(X_i, \delta(X_i)/4)$.

Now we are going to show that u is a weak solution. For that purpose, pick $\psi \in C_c^\infty(\Omega)$ and decompose $\psi = \sum_{i=0}^{\infty} \psi_i$ for some $\psi_i \in C_c^\infty(\Omega)$ with $\operatorname{supp}\psi_i \subset B(X_i, \delta(X_i)/4)$ (this can be easily done with a partition of unity subordinated to the open cover of Ω we have been using). In fact, since ψ is compactly supported, actually we only need finitely many terms and will say $\psi = \sum_{i=0}^{N} \psi_i$ for some large enough N, so that $\operatorname{supp}\psi \subset \bigcup_{i=0}^{N} B(X_i, \delta(X_i)/4)$, to highlight that there are no convergence issues. We can then compute

$$\begin{aligned}
\iint_\Omega A\nabla u \cdot \nabla\psi &= \sum_{i=0}^{N} \iint_\Omega A\nabla u \cdot \nabla\psi_i = \sum_{i=0}^{N} \iint_{B(X_i,\delta(X_i)/4)} A\nabla u \cdot \nabla\psi_i \\
&= \sum_{i=0}^{N} \iint_{B(X_i,\delta(X_i)/4)} A\nabla u_\infty^{X_i} \cdot \nabla\psi_i = \sum_{i=0}^{N} \lim_{j\to\infty} \iint_{B(X_i,\delta(X_i)/4)} A\nabla u_j^{X_i} \cdot \nabla\psi_i \\
&= \sum_{i=0}^{N} \lim_{j\to\infty} \iint_\Omega A\nabla u_j^{X_i} \cdot \nabla\psi_i = 0
\end{aligned}$$

because $Lu_j^{X_i} = 0$ in the weak sense for any $i, j \in \mathbb{N}$. With all these, it is now clear that u is indeed a well defined weak solution in $W^{1,2}_{\text{loc}}(\Omega) \cap C(\Omega)$.

Step 2: $u \in \dot{C}^\varphi(\Omega)$**.** Let $X, Y \in \Omega$. Without loss of generality we may assume that $\delta(X) \leqslant \delta(Y)$. Denote by $\widehat{x} \in \partial\Omega$ a point for which $|X - \widehat{x}| = \delta(X)$, and $\widehat{y} \in \partial\Omega$ satisfying $|Y - \widehat{y}| = \delta(Y)$. Let us distinguish some cases.

Case 1: $|X - Y| < \delta(Y)/4$. In this case $X \in B(Y, \delta(Y)/4)$, and De Giorgi/Nash (Lemma 3.10) estimate for $u - f(\widehat{y})$ yields

$$\begin{aligned}
|u(X) - u(Y)| &= |(u(X) - f(\widehat{y})) - (u(Y) - f(\widehat{y}))| \qquad (3.22)\\
&\lesssim \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \left(\fint\!\!\fint_{B(Y,\delta(Y)/2)} |u(Z) - f(\widehat{y})|^2 dZ\right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \left(\fint\!\!\fint_{B(Y,\delta(Y)/2)} \left(\int_{\partial\Omega} |f(z) - f(\widehat{y})| d\omega_L^Z(z)\right)^2 dZ\right)^{\frac{1}{2}} \\
&\lesssim \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \left(\fint\!\!\fint_{B(Y,\delta(Y)/2)} \left(\int_{\partial\Omega} \varphi(|z - \widehat{y}|)\, d\omega_L^Z(z)\right)^2 dZ\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, for $Z \in B(Y, \delta(Y)/2)$, we clearly have $|Z - \widehat{y}| < 3\delta(Y)/2$ and $\delta(Z) < 3\delta(Y)/2$. Hence, we can use Lemma 3.19 and Lemma 3.14 to obtain

$$\int_{\partial\Omega} \varphi(|z-\widehat{y}|)\, d\omega_L^Z(z) \lesssim Q_{\alpha_2}\varphi(\delta(Z)) \leqslant Q_{\alpha_2}\varphi(3\delta(Y)/2) \lesssim Q_{\alpha_2}\varphi(\delta(Y)).$$

This, back in (3.22), and setting $\alpha_0 := \min\{\alpha_1, \alpha_2\}$,

$$|u(X)-u(Y)| \lesssim \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} Q_{\alpha_2}\varphi(\delta(Y)) \leqslant \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_0} Q_{\alpha_0}\varphi(\delta(Y)) \leqslant Q_{\alpha_0}\varphi(|X-Y|),$$

where we have used Lemma 3.14 repeatedly. Therefore, assuming that $\beta \leqslant \alpha_0$ and that $\varphi \in \mathcal{G}_\beta^\infty$, we obtain by Lemma 3.16

$$|u(X)-u(Y)| \lesssim \varphi(|X-Y|).$$

Case 2: $\delta(Y) \leqslant 4|X-Y|$. In this scenario, we have

$$|\widehat{x}-\widehat{y}| \leqslant |\widehat{x}-X| + |X-Y| + |Y-\widehat{y}| = \delta(X) + |X-Y| + \delta(Y) \leqslant 2\delta(Y) + |X-Y| \leqslant 9|X-Y|.$$

Using Lemmas 3.19 and 3.14, it follows that

$$\begin{aligned}|u(X)-f(\widehat{x})| &\leqslant \int_{\partial\Omega} |f(z)-f(\widehat{x})| d\omega_L^X(z) \leqslant \int_{\partial\Omega} \varphi(|z-\widehat{x}|) d\omega_L^X(z) \\ &\lesssim Q_{\alpha_2}\varphi(\delta(X)) \leqslant Q_{\alpha_2}\varphi(4|X-Y|) \lesssim Q_{\alpha_2}\varphi(|X-Y|).\end{aligned}$$

A similar computation shows

$$|u(Y)-f(\widehat{y})| \lesssim Q_{\alpha_2}\varphi(|X-Y|).$$

Then the triangle inequality and Lemma 3.14 yield

$$\begin{aligned}|u(X)-u(Y)| &\leqslant |u(X)-f(\widehat{x})| + |f(\widehat{x})-f(\widehat{y})| + |f(\widehat{y})-u(Y)| \\ &\lesssim Q_{\alpha_2}\varphi(|X-Y|) + \varphi(9|X-Y|) \lesssim Q_{\alpha_2}\varphi(|X-Y|).\end{aligned}$$

Therefore, assuming that $\beta \leqslant \alpha_2$ and since $\varphi \in \mathcal{G}_\beta^\infty$, we obtain by Lemma 3.16

$$|u(X)-f(\widehat{x})| \lesssim \varphi(|X-Y|).$$

In either of the situations we have shown that $|u(X)-u(Y)| \lesssim \varphi(|X-Y|)$ with constants independent of the chosen points X and Y, hence $u \in \dot{C}^\varphi(\Omega)$ provided $\beta \leqslant \alpha_0 = \min\{\alpha_1, \alpha_2\}$.

Step 3: $u \in \dot{C}^\varphi(\overline{\Omega})$, $u|_{\partial\Omega} = f$, **and** (3.19) **holds.** By Lemma 3.15 (along with Remark 3.17), u can be extended to $\overline{\Omega}$ (abusing the notation we denote such an extension by u), satisfying $u \in \dot{C}^\varphi(\overline{\Omega})$ and (3.19). We only need to check that $u|_{\partial\Omega} = f$.

Let $X \in \Omega$ and $y \in \partial\Omega$. Pick $\widehat{x} \in \partial\Omega$ such that $|X-\widehat{x}| = \delta(X)$. Then it follows from Lemma 3.19 (concretely from (3.15)) that

$$\begin{aligned}|u(X)-f(y)| &\leqslant |u(X)-f(\widehat{x})| + |f(\widehat{x})-f(y)| \leqslant \int_{\partial\Omega} |f(z)-f(\widehat{x})| d\omega_L^X(z) + |f(\widehat{x})-f(y)| \\ &\leqslant \int_{\partial\Omega} \varphi(|z-\widehat{x}|) d\omega_L^X(z) + \varphi(|\widehat{x}-y|) \lesssim Q_{\alpha_2}\varphi(\delta(X)) + \varphi(|\widehat{x}-y|),\end{aligned}$$

which converges to 0 if we let X approach y, recalling that $\varphi(t) \to 0$ as $t \to 0^+$ (for the first term, see Lemma 3.14, which states that $Q_{\alpha_2}\varphi$ is a growth function). With this, the proof is complete.

Remark 3.22. The reader may review the previous proof (along with Lemma 3.14) to observe that we have actually shown that, given $f \in \dot{C}^{\varphi}(\partial\Omega)$, the associated solution u constructed by (3.18) satisfies $u \in \dot{C}^{Q_{\alpha_0}\varphi}(\overline{\Omega})$ with $\alpha_0 = \min\{\alpha_1, \alpha_2\}$, provided that φ is merely a growth function for which $Q_{\alpha_0}\varphi(1) < \infty$ (this always happens whenever $\varphi \in \mathcal{G}^{\infty}_{\alpha_0}$, see Remark 3.17).

3.6.2 Proof of Theorem 3.21: Uniqueness

Given $f \in \dot{C}^{\varphi}(\partial\Omega)$, suppose that u_1 and u_2 are two solutions of (3.17). Write $u := u_1 - u_2$, which solves (3.17) with $f \equiv 0$. We are going to show that $u \equiv 0$. When Ω is bounded, the fact that $u \in \dot{C}^{\varphi}(\overline{\Omega}) \subset C(\overline{\Omega})$ and the classical maximum principle readily gives $u \equiv 0$.

We next consider the case Ω unbounded with $\partial\Omega$ unbounded. If $\|u\|_{\dot{C}^{\varphi}(\Omega)} = 0$, then u is a constant and this constant must be zero since $u \in C(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$. Therefore, by homogeneity we may assume that $\|u\|_{\dot{C}^{\varphi}(\Omega)} = 1$ and $u|_{\partial\Omega} = 0$.

Fix $X \in \Omega$ and let $\widehat{x} \in \partial\Omega$ be such that $|X - \widehat{x}| = \delta(X)$ and let $X_j = \widehat{x} + 2^{-j}(X - \widehat{x})$ for $j \geqslant 0$. Note that $|X - X_j| = (1 - 2^{-j})|X - \widehat{x}|$, hence $X_j \in \Omega$. Since $u \in \dot{C}^{\varphi}(\overline{\Omega}) \subset C(\overline{\Omega})$, $\|u\|_{\dot{C}^{\varphi}(\Omega)} = 1$, and $u|_{\partial\Omega} = 0$ we have

$$|u(X)| = |u(X) - u(\widehat{x})| = \lim_{j\to\infty} |u(X) - u(X_j)| \leqslant \sup_{j\geqslant 0} \varphi(|X - X_j|) \leqslant \varphi(|X - \widehat{x}|) = \varphi(\delta(X)).$$

Hence,

$$\sup_{B(x,r)\cap\Omega} |u| \leqslant \varphi(r), \quad \forall x \in \partial\Omega, r > 0. \tag{3.23}$$

Now fix $X_0 \in \Omega$ and pick $\widehat{x}_0 \in \partial\Omega$ such that $|X_0 - \widehat{x}_0| = \delta(X_0)$. Invoking (3.23) and boundary Hölder continuity from Lemma 3.11 applied to u (with $u|_{\partial\Omega} \equiv 0$), we get, for R big enough so that $X_0 \in B(\widehat{x}_0, R/2)$:

$$|u(X_0)| \lesssim \left(\frac{|X_0 - \widehat{x}_0|}{R}\right)^{\alpha_2} \sup_{B(\widehat{x}_0,R)\cap\overline{\Omega}} |u| \lesssim \delta(X_0)^{\alpha_2} \frac{\varphi(R)}{R^{\alpha_2}} \lesssim \delta(X_0)^{\alpha_2} \frac{Q_{\alpha_2}\varphi(R)}{R^{\alpha_2}} \underset{R\to\infty}{\longrightarrow} 0$$

by Lemmas 3.14 and 3.16 since $\beta \leqslant \alpha_2$. Since X_0 was chosen arbitrarily, this shows that $u \equiv 0$, and finishes the proof of the uniqueness part in Theorem 3.21.

Remark 3.23. We may also generalize this uniqueness result in the spirit of Remark 3.22. In fact, the reader may check from the above proof that we have shown the following: if ψ is a growth function for which

$$R^{-\alpha_2}\psi(R) \underset{R\to\infty}{\longrightarrow} 0, \tag{3.24}$$

then there is uniqueness of solutions u to the version of the problem (3.17) with $u \in \dot{C}^{\psi}(\Omega)$.

Following Remark 3.22, let us consider the case $\psi := Q_\beta\varphi$ for some $\beta \in (0,1)$ and φ a growth function. Then, it turns out that (3.24) holds (so that one has uniqueness of solutions to (3.17) with $u \in \dot{C}^{Q_\beta\varphi}(\Omega)$) if $\beta \leqslant \alpha_2$ and $Q_\beta\varphi(1) < \infty$ by Lemma 3.14.

3.6.3 Proof of Corollary 3.8: Boundary traces

Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap \dot{C}^{\alpha}(\Omega)$ satisfy $Lu = 0$ in the weak sense in Ω. By Lemma 3.15 (along with Remark 3.17), u extends uniquely to a function defined in $\overline{\Omega}$, which is called again u, so that $u \in \dot{C}^{\alpha}(\overline{\Omega})$ and $\|u|_{\partial\Omega}\|_{\dot{C}^{\alpha}(\partial\Omega)} \leqslant \|u\|_{\dot{C}^{\alpha}(\overline{\Omega})} \lesssim \|u\|_{\dot{C}^{\alpha}(\Omega)}$. In particular, u solves (3.1) with $f = u|_{\partial\Omega}$. Theorem 3.2 readily implies (3.7).

To see (3.8) let $f \in \dot{C}^{\alpha}(\Omega)$ and invoke Theorem 3.2 to solve (3.1). Then, $u \in \dot{C}^{\alpha}_{L}(\Omega)$ and $f = u|_{\partial\Omega}$. Thus, $f = u|_{\partial\Omega}$ belongs to the set on the right hand side of (3.8). The converse inclusion follows trivially from Lemma 3.15.

To continue, we note that (3.7) and (3.8) easily imply that the linear mapping (3.9) is a well-defined bounded operator. Note also that if $[u|_{\partial\Omega}] = 0$ then $u|_{\partial\Omega}$ is constant, say c_0. This, (3.7), and the fact that elliptic measure is a probability in the current scenario imply that u must be identically to c_0. Eventually, $[u] = 0$ and we have shown that the operator in (3.9) is injective. It is additionally surjective by Theorem 3.2 and invoking again (3.7) the converse of the mapping (3.9) is also bounded. Finally, given that $\|[\cdot]\|_{\dot{C}^{\alpha}(\partial\Omega)/\mathbb{R}}$ is complete we readily obtain that $\dot{C}^{\alpha}_{L}(\Omega)/\mathbb{R}$ equipped with the norm $\|[\cdot]\|_{\dot{C}^{\alpha}(\Omega)/\mathbb{R}}$ is a Banach space. This completes the proof of Corollary 3.8.

3.7 Ill-posedness of the Dirichlet problem in Hölder spaces

The goal of this section is to show Theorem 3.4. This will be done using Remark 3.18 and the following more general result.

Theorem 3.24. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set satisfying the CDC so that Ω* **is unbounded with $\partial\Omega$ being bounded**. *Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator, and let ω_L be the associated elliptic measure. Then there exists $\beta \in (0,1)$[6] (depending only on n, the CDC constant, and the ellipticity constant of L) such that for every growth function $\varphi \in \mathcal{G}^{\infty}_{\beta}$, for every $f \in \dot{C}^{\varphi}(\partial\Omega)$, and for every $y_0 \in \partial\Omega$, if we define*

$$u(X) = f(y_0) + \int_{\partial\Omega} (f(y) - f(y_0))\, d\omega^{X}_{L}(y), \qquad X \in \Omega, \tag{3.25}$$

then $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap \dot{C}^{\varphi}(\overline{\Omega})$, satisfies $u|_{\partial\Omega} = f$, and solves the $\dot{C}^{\varphi}$-Dirichlet problem for L in Ω as formulated in (3.17). Moreover, there exists C (depending on n, the CDC constant, the ellipticity constant of L, and C_{φ}) such that

$$\|f\|_{\dot{C}^{\varphi}(\partial\Omega)} \leqslant \|u\|_{\dot{C}^{\varphi}(\Omega)} \leqslant C\|f\|_{\dot{C}^{\varphi}(\partial\Omega)}.$$

Nonetheless, the $\dot{C}^{\varphi}$-Dirichlet problem for L in Ω is ill-posed because (3.17) always has more than one solution if f is non-constant.

Proof. As in the proof of Theorem 3.21, we may assume $\|f\|_{\dot{C}^{\varphi}(\partial\Omega)} = 1$ because the case f being constant is trivial. As before, let $\beta \leqslant \alpha_0 := \min\{\alpha_1, \alpha_2\}$, where $\alpha_1 \in (0,1)$ is the exponent in Lemma 3.10 and $\alpha_2 \in (0,1)$ is the exponent in Lemma 3.11.

[6] One can take $\alpha_0 = \min\{\alpha_1, \alpha_2\}$, where $\alpha_1 \in (0,1)$ is the exponent in the De Giorgi/Nash estimates (cf. Lemma 3.10) and $\alpha_2 \in (0,1)$ is the exponent in the Hölder continuity at the boundary (cf. Lemma 3.11).

Fix some arbitrary $y_0 \in \partial\Omega$. We first show that the function u defined in (3.25) is a solution to the problem (3.17). Much as in the proof of Theorem 3.21, we proceed in a series of steps. Let us denote

$$u(X;z) := f(z) + \int_{\partial\Omega} (f(y) - f(z))\, d\omega_L^X(y), \quad X \in \Omega,\ z \in \partial\Omega.$$

Step 1: u is a well defined solution. The reader may observe that in this case we are defining u via an integration of a bounded and continuous function in the compact set $\partial\Omega$, hence, the integral is a well defined weak solution with $u \in W^{1,2}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ and $u|_{\partial\Omega} = f$.

Step 2: $u \in \dot{C}^{\varphi}(\Omega)$. Since $\varphi \in \mathcal{G}^{\infty}_{\beta}$ for $\beta \leqslant \alpha_0$, by Lemmas 3.14 and 3.16 it suffices to show that

$$|u(X;y_0) - u(Y;y_0)| \lesssim Q_{\alpha_0}\varphi(|X-Y|), \qquad X, Y \in \Omega.$$

For that purpose, fix $X, Y \in \Omega$. Without loss of generality we may assume that $\delta(X) \leqslant \delta(Y)$. Let $\widehat{x}, \widehat{y} \in \partial\Omega$ be points such that $|X - \widehat{x}| = \delta(X)$ and $|Y - \widehat{y}| = \delta(Y)$, respectively. Let us distinguish some cases.

Case 1: $|X-Y| < \delta(Y)/4$ and $\operatorname{diam}(\partial\Omega) < \delta(Y)$. We first note that for any $Z \in \Omega$, since $\omega_L^Z(\partial\Omega) < 1$,

$$|u(Z;y_0) - f(y_0)| \leqslant \int_{\partial\Omega} |f(y) - f(y_0)|\, d\omega_L^Z(y) \leqslant \int_{\partial\Omega} \varphi(|y-y_0|)\, d\omega_L^Z(y) \leqslant \varphi(\operatorname{diam}(\partial\Omega)).$$

With this estimate and De Giorgi/Nash (see Lemma 3.10) we deduce that

$$\begin{aligned} |u(X;y_0) - u(Y;y_0)| &= |(u(X;y_0) - f(y_0)) - (u(Y;y_0) - f(y_0))| \\ &\lesssim \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \left(\fint\!\!\fint_{B(Y,\delta(Y)/2))} |u(Z;y_0) - f(y_0)|^2 dZ\right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \varphi(\operatorname{diam}(\partial\Omega)). \end{aligned}$$

Then, using the monotonicity of φ, and Lemma 3.14 repeatedly, we obtain

$$|u(X;y_0) - u(Y;y_0)| \lesssim \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \varphi(\delta(Y)) \leqslant \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_0} Q_{\alpha_0}\varphi(\delta(Y)) \lesssim Q_{\alpha_0}\varphi(|X-Y|),$$

Case 2: $|X-Y| < \delta(Y)/4$ and $\delta(Y) \leqslant \operatorname{diam}(\partial\Omega)$. In this case $X \in B(Y, \delta(Y)/4)$, so we will use interior estimates. We can first compute

$$\begin{aligned} |u(X;y_0) - u(Y;y_0)| &= \left| \int_{\partial\Omega} (f(y) - f(y_0))\, d\omega_L^X(y) - \int_{\partial\Omega} (f(y) - f(y_0))\, d\omega_L^Y(y) \right| \\ &\leqslant \left| \int_{\partial\Omega} (f(y) - f(\widehat{y}))\, d\omega_L^X(y) - \int_{\partial\Omega} (f(y) - f(\widehat{y}))\, d\omega_L^Y(y) \right| \\ &\quad + \left| \int_{\partial\Omega} (f(\widehat{y}) - f(y_0))\, d\omega_L^X(y) - \int_{\partial\Omega} (f(\widehat{y}) - f(y_0))\, d\omega_L^Y(y) \right| \\ &\leqslant |u(X;\widehat{y}) - u(Y;\widehat{y})| + \varphi(\operatorname{diam}(\partial\Omega))|\omega_L^X(\partial\Omega) - \omega_L^Y(\partial\Omega)|. \end{aligned} \tag{3.26}$$

Let us estimate each term separately. For the first term, we proceed as in Case 1 in the proof of Theorem 3.21 (using De Giorgi/Nash, Harnack, Lemma 3.19, $\alpha_0 = \min\{\alpha_1, \alpha_2\}$ and Lemma 3.14):

$$\begin{aligned}
|u(X;\widehat{y}) - u(Y;\widehat{y})| &= \left| \int_{\partial\Omega} (f(y) - f(\widehat{y}))\, d\omega_L^X(y) - \int_{\partial\Omega} (f(y) - f(\widehat{y}))\, d\omega_L^Y(y) \right| \\
&\lesssim \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \left(\iint_{B(Y,\delta(Y)/2))} \left(\int_{\partial\Omega} |f(z) - f(\widehat{y})| d\omega_L^Z(z) \right)^2 dZ \right)^{\frac{1}{2}} \\
&\approx \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \left(\iint_{B(Y,\delta(Y)/2))} \left(\int_{\partial\Omega} |f(z) - f(\widehat{y})| d\omega_L^Y(z) \right)^2 dZ \right)^{\frac{1}{2}} \\
&= \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \int_{\partial\Omega} |f(z) - f(\widehat{y})| d\omega_L^Y(z) \\
&\leqslant \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \int_{\partial\Omega} \varphi(|z-\widehat{y}|)\, d\omega_L^Y(z) \\
&\lesssim \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} Q_{\alpha_2}\varphi(\delta(Y)) \\
&\leqslant \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_0} Q_{\alpha_0}\varphi(\delta(Y)) \\
&\leqslant Q_{\alpha_0}\varphi(|X-Y|).
\end{aligned}$$

Therefore it is only left to estimate the second term in (3.26). We use first De Giorgi/Nash (Lemma 3.10), then Harnack's inequality applied to the positive solution $1 - \omega_L^Z(\partial\Omega)$ and boundary Hölder continuity (cf. Lemma 3.11):

$$\begin{aligned}
|\omega_L^X(\partial\Omega) - \omega_L^Y(\partial\Omega)| &= |(1 - \omega_L^X(\partial\Omega)) - (1 - \omega_L^Y(\partial\Omega))| \\
&\lesssim \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \left(\iint_{B(Y,\delta(Y)/2)} |1 - \omega_L^Z(\partial\Omega)|^2 dZ \right)^{1/2} \\
&\approx \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} (1 - \omega_L^Y(\partial\Omega)) \\
&\lesssim \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \left(\frac{\delta(Y)}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_2} \sup_{Z\in B(\widehat{y},\operatorname{diam}(\partial\Omega))} (1 - \omega_L^Z(\partial\Omega)) \\
&\leqslant \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_1} \left(\frac{\delta(Y)}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_2} \\
&\leqslant \left(\frac{|X-Y|}{\delta(Y)}\right)^{\alpha_0} \left(\frac{\delta(Y)}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_0} \\
&= \left(\frac{|X-Y|}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_0},
\end{aligned}$$

where we have also used that $\alpha_0 = \min\{\alpha_1, \alpha_2\}$. As a result, by Lemma 3.14,

$$\varphi(\operatorname{diam}(\partial\Omega))|\omega_L^X(\partial\Omega) - \omega_L^Y(\partial\Omega)| \lesssim Q_{\alpha_0}\varphi(\operatorname{diam}(\partial\Omega)) \left(\frac{|X-Y|}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_0} \lesssim Q_{\alpha_0}\varphi(|X-Y|).$$

Case 3: $\delta(Y)/4 \leqslant |X-Y|$ **and** $|X-Y| \leqslant \operatorname{diam}(\partial\Omega)$**.** As in the proof of Theorem 3.21, we may

suppose $|\hat{x}-\hat{y}| \leqslant 10|X-Y|$ and write

$$|u(X;y_0)-u(Y;y_0)| \leqslant |u(X;y_0)-u(X;\hat{x})-u(Y;y_0)+u(Y;\hat{y})| + |u(X;\hat{x})-u(Y;\hat{y})|. \tag{3.27}$$

Let us first estimate the second term. Using the triangle inequality we get

$$|u(X;\hat{x})-u(Y;\hat{y})| \leqslant |u(X;\hat{x})-f(\hat{x})| + |u(Y;\hat{y})-f(\hat{y})| + |f(\hat{x})-f(\hat{y})|. \tag{3.28}$$

The first term in this sum can be estimated much as before using Lemmas 3.19 and 3.14:

$$\begin{aligned}|u(X;\hat{x})-f(\hat{x})| &\leqslant \int_{\partial\Omega} |f(y)-f(\hat{x})|\,d\omega_L^X(y) \leqslant \int_{\partial\Omega} \varphi(|y-\hat{x}|)\,d\omega_L^X(y)\\ &\lesssim Q_{\alpha_2}\varphi(\delta(X)) \leqslant Q_{\alpha_2}\varphi(4|X-Y|) \lesssim Q_{\alpha_2}\varphi(|X-Y|) \leqslant Q_{\alpha_0}\varphi(|X-Y|),\end{aligned}$$

because $\alpha_0 \leqslant \alpha_2$. The same ideas apply to the second term in (3.28). In turn, the last term is easy to estimate with Lemma 3.14:

$$|f(\hat{x})-f(\hat{y})| \leqslant \varphi(|\hat{x}-\hat{y}|) \leqslant \varphi(10|X-Y|) \lesssim Q_{\alpha_0}\varphi(10|X-Y|) \lesssim Q_{\alpha_0}\varphi(|X-Y|).$$

We turn now our attention to the first term in (3.27) and use again the triangle inequality to introduce some additional boundary terms (along with the monotonicity of φ):

$$\begin{aligned}&|u(X;y_0)-u(X;\hat{x})-u(Y;y_0)+u(Y;\hat{y})| \\ &\quad= |(f(\hat{y})-f(\hat{x}))(1-\omega_L^X(\partial\Omega)) + (f(\hat{y})-f(y_0))(\omega_L^X(\partial\Omega)-\omega_L^Y(\partial\Omega))| \\ &\quad\leqslant |f(\hat{x})-f(\hat{y}))| + |f(\hat{y})-f(y_0)||\omega_L^X(\partial\Omega)-\omega_L^Y(\partial\Omega)| \\ &\quad\leqslant \varphi(|\hat{x}-\hat{y}|) + \varphi(|\hat{y}-y_0|)|\omega_L^X(\partial\Omega)-\omega_L^Y(\partial\Omega)| \\ &\quad\leqslant \varphi(10|X-Y|) + \varphi(\operatorname{diam}(\partial\Omega))|\omega_L^X(\partial\Omega)-\omega_L^Y(\partial\Omega)|.\end{aligned} \tag{3.29}$$

Next, the boundary Hölder continuity from Lemma 3.11 yields

$$\begin{aligned}|\omega_L^X(\partial\Omega)-\omega_L^Y(\partial\Omega)| &\leqslant |1-\omega_L^X(\partial\Omega)| + |1-\omega_L^Y(\partial\Omega)| \\ &\lesssim \left(\frac{\delta(X)}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_2} \sup_{Z\in B(\hat{x},\operatorname{diam}(\partial\Omega))} (1-\omega_L^Z(\partial\Omega)) \\ &\qquad + \left(\frac{\delta(Y)}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_2} \sup_{Z\in B(\hat{y},\operatorname{diam}(\partial\Omega))} (1-\omega_L^Z(\partial\Omega)) \\ &\leqslant \left(\frac{\delta(X)}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_2} + \left(\frac{\delta(Y)}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_2} \\ &\lesssim \left(\frac{|X-Y|}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_2},\end{aligned}$$

since $\omega_L^Z(\partial\Omega) \in [0,1]$ for any Z. Thus, returning to (3.29) we conclude that

$$\begin{aligned}&|u(X;y_0)-u(Y;y_0)-u(X;\hat{x})+u(Y;\hat{y})| \\ &\qquad\lesssim \varphi(10|X-Y|) + \varphi(\operatorname{diam}(\partial\Omega))\left(\frac{|X-Y|}{\operatorname{diam}(\partial\Omega)}\right)^{\alpha_0} \lesssim Q_{\alpha_0}\varphi(|X-Y|)\end{aligned}$$

using $\alpha_0 \leqslant \alpha_2$, Lemma 3.14, and $|X-Y| \leqslant \operatorname{diam}(\partial\Omega)$. Collecting all the estimates, the proof in Case 3 is complete.

Case 4: $\delta(Y)/4 \leqslant |X-Y|$ **and** $|X-Y| > \operatorname{diam}(\partial\Omega)$. This case is easy:

$$\begin{aligned}|u(X;y_0)-u(Y;y_0)| &= \left|\int_{\partial\Omega}(f(y)-f(y_0))\,d\omega_L^X(y)-\int_{\partial\Omega}(f(y)-f(y_0))\,d\omega_L^Y(y)\right| \\ &\leqslant \int_{\partial\Omega}\varphi(|y-y_0|)\,d\omega_L^X(y)+\int_{\partial\Omega}\varphi(|y-y_0|)\,d\omega_L^Y(y) \\ &\lesssim \varphi(\operatorname{diam}(\partial\Omega)) \leqslant \varphi(|X-Y|) \leqslant Q_{\alpha_0}\varphi(|X-Y|),\end{aligned}$$

where we have used the monotonicity of φ, the fact that $\omega_L^Z(\partial\Omega) \leqslant 1$ for any $Z \in \Omega$, and Lemma 3.14.

Collecting all the cases above, we conclude that $u \in \dot{C}^{Q_{\alpha_0}\varphi}(\Omega)$. By Lemmas 3.14 and 3.16, and the fact that $\varphi \in \mathcal{G}_\beta^\infty$ with $\beta \leqslant \alpha_0$ one can easily see that $Q_{\alpha_0}\varphi \approx \varphi$, hence $u \in \dot{C}^\varphi(\Omega)$.

Step 3: $u \in \dot{C}^\varphi(\overline{\Omega})$ **and** (3.19) **holds**. This follows at once from the previous steps and Lemma 3.15 (along with Remark 3.17).

Step 4: Lack of uniqueness. As a final remark, let us show that the constructed solutions are different whenever f is not constant in the boundary of the unique unbounded connected component. Indeed, if f is not constant in such a set, we can pick $x_0, y_0 \in \partial\Omega$ for which $f(x_0) \neq f(y_0)$. If we consider the solutions $u(\cdot;x_0)$ and $u(\cdot;y_0)$, it turns out that for any $X \in \Omega$ we have

$$\begin{aligned}u(X;x_0)-u(X;y_0) &= f(x_0)+\int_{\partial\Omega}(f(y)-f(x_0))\,d\omega_L^X(y)-f(y_0)-\int_{\partial\Omega}(f(y)-f(y_0))\,d\omega_L^X(y) \\ &= (f(x_0)-f(y_0))(1-\omega_L^X(\partial\Omega)) \neq 0,\end{aligned}$$

since the elliptic measure is not a probability in this situation. This shows the ill-posedness of the problem in the case when f is not constant, and the proof is complete. □

Remark 3.25. As the reader may check from the above proof, generalizations of this result to more general classes of growth functions φ, in the spirit of Remarks 3.22 and 3.23, are also possible.

Remark 3.26. We have just shown that the $\dot{C}^\varphi$-Dirichlet problem for L in Ω is not uniquely solvable. However, it is easy to show that the solution is unique provided we prescribe its value at ∞. To elaborate on this we first observe that, as mentioned above, the CDC implies that every boundary point is Wiener regular. Hence, we can invoke Lemma 3.31 below to obtain in the present situation (Ω unbounded with $\partial\Omega$ bounded) that, for every $x_0 \in \partial\Omega$,

$$|u(X;x_0)-f(x_0)| \leqslant \|f\|_{\dot{C}^\varphi(\partial\Omega)}\varphi(\operatorname{diam}(\partial\Omega))\,\omega_L^X(\partial\Omega) \longrightarrow 0, \quad \text{as} \quad \Omega \ni X \to \infty.$$

This means that under the hypothesis of Theorem 3.24, and using the notation above, for every $x_0 \in \partial\Omega$, it follows that $u(X;x_0) \longrightarrow f(x_0)$ as $\Omega \ni X \to \infty$. Taking this into account, given $x_0 \in \partial\Omega$, we claim the $\dot{C}^\varphi$-Dirichlet problem as formulated in (3.17) is uniquely solvable provided we further impose that $u(X) \longrightarrow f(x_0)$ as $\Omega \ni X \to \infty$, and in that case the unique solution must be $u(\cdot;x_0)$. To see this, let us suppose that we have two solutions u_1 and u_2 of (3.17) both converging to $f(x_0)$ at infinity. Write $u := u_1 - u_2$, which solves (3.17) with $f \equiv 0$ and $u(X) \longrightarrow 0$ as $\Omega \ni X \to \infty$. Given then $\varepsilon > 0$, there

exists $R_\varepsilon \gg \operatorname{diam}(\partial\Omega)$ so that if $|X - x_0| \geqslant R_\varepsilon$ then $|u(X)| < \varepsilon$. Let $\Omega_R = \Omega \cap B(x_0, R)$ for an arbitrary $R > R_\varepsilon$. Since $u \in \dot{C}^\varphi(\overline{\Omega}) \subset C(\overline{\Omega_R})$, $u|_{\partial\Omega} = 0$, and $|u(X)| \leqslant \varepsilon$ if $|X - X_0| = R$, the classical maximum principle implies that $|u(X)| \leqslant \varepsilon$ for every $X \in \Omega_R$. Letting $R \to \infty$ we conclude that $|u(X)| < \varepsilon$ for every $X \in \Omega$ and since $\varepsilon > 0$ is arbitrary we conclude that $u \equiv 0$ as desired. Once we have shown that the corresponding problem has a unique solution one can derive that such a solution is $u(\cdot; x_0)$ since we have already shown that it satisfies all the required properties including the convergence to $f(x_0)$ at infinity.

3.8 Optimality of the capacity density condition

In this section we prove Theorem 3.7. Our proof extends [Ai02] and contains several novelties and new technical difficulties because we are working with a general elliptic operator L rather than the Laplacian, and Ω is not assumed to be connected nor bounded. In fact, the case when Ω is unbounded with $\partial\Omega$ being bounded has some additional subtle points to be taken into account.

Unlike in the other sections, for the sake of clarity and of a cleaner exposition, we are going to show Theorem 3.7 only for the $\dot{C}^\alpha$ spaces, not for the more general $\dot{C}^\varphi$. It is not hard to check that our proof can be adapted to the $\dot{C}^\varphi$ setting with small modifications, and we give a sketch of such proof in Subsection 3.8.2.

Let us start with some auxiliary results. We first introduce a notion of regularity of points in the boundary, which will be used frequently in the subsequent proofs. See Theorem 2.43 for a connection.

Definition 3.27 ((Wiener) regular set). Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set, and let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator. We say that Ω is *(Wiener) regular for L* if for any function $f : \partial\Omega \longrightarrow \mathbb{R}$ which is uniformly continuous and bounded, the associated solution u defined by (3.2) satisfies $u \in C(\overline{\Omega})$.

Remark 3.28. Note that in the case Ω is unbounded with $\partial\Omega$ being bounded, if Ω is regular for L, then the solutions defined by (3.4) are also $C(\overline{\Omega})$. We will use this fact repeatedly from now on. The short proof is as follows: indeed, any solution defined by (3.4) can be rewritten as

$$u(X; y_0) = f(y_0)(1 - \omega_L^X(\partial\Omega)) + \int_{\partial\Omega} f(y)\, d\omega_L^X(y),$$

where $X \mapsto \int_{\partial\Omega} f(y)\, d\omega_L^X(y)$ belongs to $C(\overline{\Omega})$ by regularity of Ω, and $X \mapsto \omega_L^X(\partial\Omega)$ clearly belongs to $C(\overline{\Omega})$ because it is the solution given by (3.2) associated with the boundary datum constantly 1, using again that Ω is regular. Thus, $u(\cdot\,; y_0) \in C(\overline{\Omega})$, as desired.

It is well known that in the context of Definition 3.27, Ω is regular for L if and only if every boundary point is Wiener regular (see (2.43)), hence the regularity of Ω does not depend on the operator L (see [LSW63]). Hence, we will often simply say that Ω *is regular*.

Before proving Theorem 3.7, let us first present a qualitative version of it.

Lemma 3.29. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set, and let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator. Let $\alpha \in (0, 1)$. Assume that for every $f \in \dot{C}^\alpha(\partial\Omega)$, the function u constructed in (3.4) is well-defined via an absolutely convergent integral and moreover is a solution to (3.1) satisfying (3.6). Then Ω is regular for L.*

Remark 3.30. We want to point out that our assumption above rules out the possibility that $\partial\Omega$ contains a polar set (trivial boundary points). For example, consider Ω to be the unit ball with the origin removed, i.e., $\Omega = B\backslash\{0\}$. Then it is not true that for any $f \in \dot{C}^\alpha(\partial\Omega) = \dot{C}^\alpha(\partial B \cup \{0\})$, we can find a solution u to (3.1), which in particular assumes the boundary datum pointwise, because any such u must also be a classical solution to L in the unit ball, and thus by the uniqueness of classical solutions we would have that $f(0) = u(0) = \int_{\partial B} f\, d\omega^0_{L,B}$. See also [Ai02, Theorem 1] for comparison.

Proof. Let $f : \partial\Omega \longrightarrow \mathbb{R}$ be uniformly continuous and bounded. By a simple adaptation of [He01, Lemma 6.8], there exists a sequence $\{f_j\} \subset \dot{C}^\alpha(\partial\Omega)$ such that $\|f - f_j\|_{L^\infty(\partial\Omega)} \longrightarrow 0$ as $j \to \infty$. Denote by u_j and u the functions defined by (3.4) with some fixed $y_0 \in \partial\Omega$, associated with f_j and f, respectively. We know that u_j is well-defined by hypothesis and also that u is well defined since f is uniformly continuous and bounded. Since $f_j \in \dot{C}^\alpha(\partial\Omega)$, by (3.6) we know that $u_j \in \dot{C}^\alpha(\Omega)$. Therefore it can be extended (denoting again the extension by u_j) to $u_j \in \dot{C}^\alpha(\overline{\Omega}) \subset C(\overline{\Omega})$, and moreover, since u_j is a solution to (3.1) with boundary datum f_j, we also have that $u_j = f_j$ on $\partial\Omega$.

Write

$$\widehat{u}(X) := \begin{cases} u(X) & \text{if } X \in \Omega, \\ f(X) & \text{if } X \in \partial\Omega. \end{cases}$$

Our goal is to show $\widehat{u} \in C(\overline{\Omega})$. To this end, fix $\varepsilon > 0$ and choose $N \gg 1$ such that $\|f - f_N\|_{L^\infty(\partial\Omega)} < \varepsilon/9$. Since $u_N \in \dot{C}^\alpha(\overline{\Omega})$, take $\delta > 0$ (which depends on N and ε) such that $|X - Y| < \delta$ implies $\big|u_N(X) - u_N(Y)\big| < \varepsilon/3$ for any $X, Y \in \overline{\Omega}$. Then, for any $X, Y \in \overline{\Omega}$ such that $|X - Y| < \delta$, we may compute

$$\big|\widehat{u}(X) - \widehat{u}(Y)\big| \leqslant \big|\widehat{u}(X) - u_N(X)\big| + \big|u_N(X) - u_N(Y)\big| + \big|u_N(Y) - \widehat{u}(Y)\big| =: I + II + III. \tag{3.30}$$

The second term satisfies $II < \varepsilon/3$. In turn, for the first term, if $X \in \Omega$, then (3.4) shows

$$\begin{aligned} \big|\widehat{u}(X) - u_N(X)\big| = \big|u(X) - u_N(X)\big| &\leqslant \big|f(y_0) - f_N(y_0)\big| \\ &+ \int_{\partial\Omega} \big(\big|f(y) - f_N(y)\big| + \big|f(y_0) - f_N(y_0)\big|\big)\, d\omega^X_L(y) \leqslant 3\,\|f - f_N\|_{L^\infty(\partial\Omega)} < \frac{\varepsilon}{3}. \end{aligned}$$

Whereas if $X \in \partial\Omega$ we have

$$\big|\widehat{u}(X) - u_N(X)\big| = \big|f(X) - f_N(X)\big| \leqslant \|f - f_N\|_{L^\infty(\partial\Omega)} < \frac{\varepsilon}{9}.$$

In any case, there holds $I < \varepsilon/3$. In the same fashion, $III < \varepsilon/3$. In view of (3.30), these yield $\big|\widehat{u}(X) - \widehat{u}(Y)\big| < \varepsilon$, which finally shows $\widehat{u} \in C(\overline{\Omega})$ (indeed, we have shown that $\widehat{u}$ is uniformly continuous), as desired. □

The following lemma will help us deal with the case of unbounded open sets (without any connectivity assumption), which is one of the main novelties of this section in comparison to the work of Aikawa [Ai02] in bounded domains.

Lemma 3.31. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be a regular open set, and let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator. Then there exists some $C > 0$ (depending only on n and the ellipticity of L) such that for any $a \in \partial\Omega$ and $r > 0$ we have*

$$\omega^X_L(\partial\Omega \cap B(a,r)) \leqslant C\left(\frac{|X - a|}{r}\right)^{1-n}, \qquad X \in \Omega\backslash B(a, 2r). \tag{3.31}$$

As a consequence, in the case that Ω is unbounded, for any $a \in \partial\Omega$ and $r > 0$, we have

$$\omega_L^X(\partial\Omega \cap B(a,r)) \longrightarrow 0 \quad \text{as} \quad X \to \infty.$$

Proof. The second assertion clearly follows from the first one, so let us focus on that first one. Note that we may assume $\Omega\backslash B(a,2r) \neq \varnothing$, for otherwise (3.31) is an empty statement. Write $\Gamma_L(\cdot,\cdot)$ for the fundamental solution to L. It is well known (see Section 2.4.3) that, for some $c_0 \in (0,1)$ depending only on n and the ellipticity of L,

$$c_0\,|X-Y|^{1-n} \leqslant \Gamma_L(X,Y) \leqslant c_0^{-1}\,|X-Y|^{1-n}, \qquad X \neq Y. \tag{3.32}$$

Using the definition of the elliptic measure in unbounded domains (and in bounded domains this is trivially true), we have, for any $X \in \Omega$,

$$\omega_L^X(B(a,r)\cap\partial\Omega) = \lim_{N\to\infty} \omega^X_{L,\Omega\cap B(a,N)}(B(a,r)\cap\partial\Omega), \tag{3.33}$$

where (here and elsewhere) $\omega_{L,D}$ refers to the elliptic measure relative to the set D; as opposed to ω_L, which is always constructed relative to Ω. See Section 2.4.6.

Now, for $N \gg 2r$, choose $\phi \in C_c^\infty(\mathbb{R}^{n+1})$ so that $\mathbf{1}_{B(a,r)} \leqslant \phi \leqslant \mathbf{1}_{B(a,2r)}$. Denote by

$$u_\phi(X) := \int_{\partial(\Omega\cap B(a,N))} \phi\, d\omega^X_{L,\Omega\cap B(a,N)}$$

the solution associated with ϕ, which is continuous up to the boundary by the regularity of Ω.

For $X \in \Omega \cap \partial B(a,2r)$, it follows from (3.32) that

$$u_\phi(X) \leqslant 1 \leqslant \frac{\Gamma_L(X,a)}{c_0(2r)^{1-n}}.$$

Moreover, for $x \in \partial\Omega \cap \overline{B(a,N)}\backslash B(a,2r)$ or $x \in \Omega\cap\partial B(a,N)$, the continuity of u_ϕ up to the boundary gives

$$u_\phi(x) = \phi(x) = 0 < \frac{\Gamma_L(x,a)}{c_0(2r)^{1-n}}.$$

Therefore, by the maximum principle in $(\Omega\cap B(a,N))\backslash B(a,2r)$, we obtain

$$\omega^X_{L,\Omega\cap B(a,N)}(B(a,r)\cap\partial\Omega) \leqslant u_\phi(X) \leqslant \frac{\Gamma_L(X,a)}{c_0(2r)^{1-n}} \leqslant \frac{|X-a|^{1-n}}{c_0^2(2r)^{1-n}}, \tag{3.34}$$

for all $X \in (\Omega\cap B(a,N))\backslash B(a,2r)$, where (3.32) was used in the last step. As a consequence, (3.31) follows from (3.33) and (3.34). □

As a byproduct of the preceding lemma, we get the absolute continuity of the elliptic measure with respect to codimension-two Hausdorff measure, namely sets of lower dimensions are invisible to the elliptic measure. Although we will not use it in the sequel, we consider it to be interesting on its own. For the Laplacian, the result is known to experts (and the proof is not simpler than ours), but there is no easy reference for it (see [Ba11, Appendix A]).

Corollary 3.32. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be a regular open set, and let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator. Then $\omega_L^X \ll \mathcal{H}^{n-1}$ for any $X \in \Omega$.*

Proof. Fix $X \in \Omega$. Let $E \subset \partial\Omega$ such that $\mathcal{H}^{n-1}(E) = 0$. It is well known that

$$\liminf_{\delta \to 0} \left\{ \sum_{i=1}^{\infty} r_i^{n-1} : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), 2r_i < \delta \right\} = 0.$$

This means that given $\varepsilon > 0$, there exists some $\delta > 0$ small enough (we may assume that $\delta < \delta(X)/2$) such that for some $x_i \in \mathbb{R}^{n+1}$ and $r_i \in (0, \delta/2)$, we have $E \subset \bigcup_i B(x_i, r_i)$ and $\sum_i r_i^{n-1} < \varepsilon$. Without loss of generality assume that $B(x_i, r_i) \cap \partial\Omega \neq \varnothing$. Note that then $E \subset \bigcup_i B(y_i, 2r_i)$ for some $y_i \in \partial\Omega$. Therefore, using the fact that $\delta(X) > 2\delta$ and Lemma 3.31, we obtain

$$\omega_L^X(E) \leqslant \sum_{i=1}^{\infty} \omega_L^X(\partial\Omega \cap B(y_i, 2r_i)) \leqslant C \sum_{i=1}^{\infty} \left(\frac{|X - y_i|}{2r_i} \right)^{1-n} \leqslant \frac{2^{n-1}C}{\delta(X)^{n-1}} \sum_{i=1}^{\infty} r_i^{n-1} < \frac{2^{n-1}C}{\delta(X)^{n-1}} \varepsilon.$$

By the arbitrariness of ε, this indeed implies $\omega_L^X(E) = 0$. □

The following estimate establishes the relationship between the capacitary potential and the elliptic measure, which plays an important ingredient to show CDC.

Lemma 3.33. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be a regular open set, and let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator. Fix $a \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. Denote $F := \overline{B(a,r)} \backslash \Omega$, and $v := \mathcal{R}_L(F; B(a,2r))$ the capacitary potential (cf. [HKM06, Chapter 6]) in $B(a,2r)$ of the compact set F with respect to the operator L. Then*

$$v(X) \geqslant 1 - \omega^X_{L,\Omega\cap B(a,r)}(\partial B(a,r) \cap \overline{\Omega}) \quad \text{for any} \quad X \in \Omega \cap B(a,r).$$

Proof. Write

$$u(X) := 1 - \omega^X_{L,\Omega\cap B(a,r)}(\partial B(a,r) \cap \overline{\Omega}) = \omega^X_{L,\Omega\cap B(a,r)}(\partial\Omega \cap B(a,r)).$$

Given $0 < \eta < r$, choose $\phi_\eta \in C_c^\infty(\mathbb{R}^{n+1})$ such that $0 \leqslant \phi_\eta \leqslant 1$, $\phi_\eta \equiv 1$ in $B(a, r-\eta)$, and $\phi_\eta \equiv 0$ outside $B(a,r)$. Set

$$u_\eta(X) := \int_{\partial(\Omega\cap B(a,r))} \phi_\eta \, d\omega^X_{L,\Omega\cap B(a,r)}$$

and the Dominated Convergence Theorem implies $u_\eta(X) \longrightarrow u(X)$ as $\eta \to 0$ for any $X \in \Omega \cap B(a,r)$. Hence, to conclude the proof it suffices to prove

$$v \geqslant u_\eta \text{ in } \Omega \cap B(a,r) \text{ for any } \eta > 0. \tag{3.35}$$

Since $\Omega \cap B(a,r) \subset \Omega$ and Ω is regular, by the Wiener criterion $\Omega \cap B(a,r)$ is also regular. Then, the smoothness of ϕ_η implies that $u_\eta \in C(\overline{\Omega \cap B(a,r)})$. However v may not be continuous or even defined pointwise near $\partial B(a,r) \cap \partial\Omega$, so we can not directly apply the classical maximum principle to prove (3.35).

By [HKM06, Lemma 7.9], we may extend u_η by 0 to $B(a,2r)\backslash F$, and if we denote by $\hat{u}_\eta$ the extension, $\hat{u}_\eta \in C(\overline{B(a,2r)\backslash F})$ is a weak subsolution. By the weak maximum principle for subsolutions in

$B(a,2r)\backslash F$ (see [Ke94, Lemma 1.1.17]), to prove (3.35) it suffices to show that $\hat{u}_\eta - v \leqslant 0$ in the $W^{1,2}$-sense on $\partial(B(a,2r)\backslash F)$. In fact, an easy modification of [Ke94, Definition 1.1.13] lets us reduce to finding a sequence $\{f_j\} \subset C^\infty(B(a,2r)\backslash F) \cap W^{1,2}(B(a,2r)\backslash F)$ such that $f_j \to \hat{u}_\eta - v$ in $W^{1,2}(B(a,2r)\backslash F)$ and $f_j \leqslant 0$ in an open neighborhood U_j of $\partial(B(a,2r)\backslash F)$. Let us next prove this.

Pick $\Phi \in C_c^\infty(\mathbb{R}^{n+1})$ such that $\mathbf{1}_{B(a,\frac{3}{2}r)} \leqslant \Phi \leqslant \mathbf{1}_{B(a,2r)}$. Since v is the solution to $Lv = 0$ in $B(a,2r)\backslash F$ with data $\mathbf{1}_{\partial F}$, and Φ is a smooth extension of $\mathbf{1}_{\partial F}$, we have $v - \Phi \in W_0^{1,2}(B(a,2r)\backslash F)$. This means that there exists a sequence $\{\varphi_j\} \subset C_c^\infty(B(a,2r)\backslash F)$ such that $\varphi_j \to v - \Phi$ in $W^{1,2}(B(a,2r)\backslash F)$ as $j \to \infty$.

On the other hand, since $u_\eta - \phi_\eta \in W_0^{1,2}(\Omega \cap B(a,r))$ because u_η is a solution with smooth boundary data ϕ_η, we can find $\psi_j \in C_c^\infty(\Omega \cap B(a,r))$ so that $\psi_j \to u_\eta - \phi_\eta$ in $W^{1,2}(\Omega \cap B(a,r))$. Denoting by $\hat{\psi}_j$ the extension of ψ_j by 0, we get $\hat{\psi}_j \in C_c^\infty(\mathbb{R}^{n+1})$ and $\hat{\psi}_j \to \hat{u}_\eta - \phi_\eta$ in $W^{1,2}(B(a,2r)\backslash F)$.

As a consequence, $\hat{\psi}_j + \phi_\eta - \varphi_j - \Phi \to \hat{u}_\eta - v$ in $W^{1,2}(B(a,2r)\backslash F)$ and $\hat{\psi}_j + \phi_\eta - \varphi_j - \Phi \in W^{1,2}(B(a,2r)\backslash F) \cap C^\infty(B(a,2r)\backslash F)$. Hence, to complete the proof, it remains to check that

$$\hat{\psi}_j + \phi_\eta - \varphi_j - \Phi \leqslant 0 \text{ in a neighborhood } U_j \text{ of } \partial(B(a,2r)\backslash F). \tag{3.36}$$

Indeed, choose $U_j := \mathbb{R}^{n+1}\backslash(\operatorname{supp}\varphi_j \cup \operatorname{supp}\hat{\psi}_j)$ so that $\varphi_j \equiv 0 \equiv \hat{\psi}_j$ in U_j. In $B(a,r) \cap \Omega$, we see that $\Phi \equiv 1$ and $\phi_\eta \leqslant 1$, so overall $\hat{\psi}_j + \phi_\eta - \varphi_j - \Phi \leqslant 0$ in $U_j \cap (B(a,r) \cap \Omega)$. Additionally, in $\mathbb{R}^{n+1}\backslash B(a,r)$, there holds $\phi_\eta \equiv 0$ and $\Phi \geqslant 0$, so $\hat{\psi}_j + \phi_\eta - \varphi_j - \Phi \leqslant 0$ in $U_j\backslash B(a,r)$. This justifies (3.36) and the proof is complete. □

Before showing Theorem 3.7 with the tools developed above, let us introduce some extra definitions to make the proof more understandable. The definitions below are indeed a generalization of those in [Ai02].

Definition 3.34. Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set. Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator, and let ω_L be the associated elliptic measure.

1. Given $\alpha \in (0,1)$, we say that Ω satisfies the *global elliptic measure decay property* with exponent α (abbreviated to GEMD(α)) if there exists some $M > 0$ such that

$$\omega_L^X(\partial\Omega\backslash B(x,r)) \leqslant M\left(\frac{|X-x|}{r}\right)^\alpha, \tag{3.37}$$

   for all $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and $X \in \Omega$.

2. Given $\alpha \in (0,1)$, we say that Ω satisfies the *local elliptic measure decay property* with exponent α (abbreviated to LEMD(α)) at $x \in \partial\Omega$ with scale $0 < r < \operatorname{diam}(\partial\Omega)$ if there exists some $M > 0$ such that for every $X \in \Omega \cap B(x,r)$

$$\omega^X_{L,\Omega\cap B(x,r)}(\overline{\Omega} \cap \partial B(x,r)) \leqslant M\left(\frac{|X-x|}{r}\right)^\alpha. \tag{3.38}$$

Remark 3.35. We are going to show that the solvability of (3.1) along with (3.6) implies GEMD(α), as defined above. But in the proof of GEMD(α) $\implies$ CDC, the only relevant case is the estimate (3.37) for X close enough to x, say $X \in \Omega \cap B(x, \frac{1}{4}r)$ (in fact, upon changing the value of M, the GEMD(α) condition is vacuous for $X \notin \Omega \cap B(x, \frac{1}{4}r)$). Then (3.37) is the same as (3.14) in Lemma 3.19, which is the key property allowing us to obtain the α-Hölder-solvability in Sections 3.6 and 3.7. We also remark that, whenever Ω is regular, by the maximum principle, (3.38) implies (3.37) for $X \in \Omega \cap B(x,r)$.

Remark 3.36. LEMD(α) implies boundary Hölder continuity as in Lemma 3.11 with the same Hölder exponent α and constant. Indeed, fix any ball $B = B(x,r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and suppose that $0 \leqslant u \in W^{1,2}_{\rm loc}(2B \cap \Omega) \cap C(\overline{2B \cap \Omega})$ satisfies $Lu = 0$ in the weak sense in $2B \cap \Omega$, with $u \equiv 0$ in $2\Delta = 2B \cap \partial\Omega$. Then, for every $X \in B \cap \Omega$, we have

$$u(X) = \int_{\partial(\Omega\cap B)} u|_{\partial(\Omega\cap B)}\, d\omega^X_{L,\Omega\cap B} \leqslant \omega^X_{L,\Omega\cap B}(\overline{\Omega} \cap \partial B) \sup_{\overline{\Omega\cap B}} u \leqslant M \left(\frac{|X-x|}{r}\right)^{\alpha} \sup_{\overline{\Omega\cap B}} u.$$

We are going to prove in this section that $\dot{C}^\alpha(\partial\Omega)$-solvability of (3.1) along with (3.6) imply LEMD(α).[7] Therefore, these two conditions together yield the boundary Hölder continuity property with the same Hölder exponent α.

We can now proceed to prove the main result of this section. It will be convenient to split the proof in two cases, depending whether or not the elliptic measure is a probability.

3.8.1 Proof of Theorem 3.7: optimality of the CDC

Our goal is to prove the CDC estimate for $a \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$:

$$\frac{\operatorname{Cap}(\overline{B(a,r)}\setminus\Omega, B(a,2r))}{\operatorname{Cap}(\overline{B(a,r)}, B(a,2r))} \geqslant c, \tag{3.39}$$

with c independent of a and r.

We proceed in a series of steps for clarity of exposition. Recall that by Lemma 3.29 we may assume that Ω is regular.

Step 1: Non-degeneracy of elliptic measure. We show that, there exists a constant $0 < c^* \leqslant 1$ (depending only on ellipticity, α and C_α in (3.6)) such that

$$\omega^X_L(\partial\Omega) \geqslant c^*, \qquad \text{for any } X \in \Omega \cap B(a,100r), a \in \partial\Omega, 0 < r < \operatorname{diam}(\partial\Omega). \tag{3.40}$$

This is clearly true with $c^* = 1$ whenever the elliptic measure is a probability, and that is the case when Ω is bounded. Consider next the case Ω unbounded, and note that since we only have that Ω is regular, we do not know (yet) that the elliptic measure is a probability when $\partial\Omega$ unbounded (see Section 2.4.6).

Take two different points $x_0, y_0 \in \partial\Omega$. Define $\psi(x) := |x-y_0|^\alpha / |x_0-y_0|^\alpha$ so that

$$\psi(x_0) = 1, \quad \psi(y_0) = 0, \quad \text{and} \quad \psi \in \dot{C}^\alpha(\partial\Omega) \quad \text{with} \quad \|\psi\|_{\dot{C}^\alpha(\partial\Omega)} = |x_0-y_0|^{-\alpha}.$$

As in (3.4) we construct the solution as associated with this boundary datum ψ:

$$u_1(X) := u_1(X; y_0, \psi) := \psi(y_0) + \int_{\partial\Omega} (\psi(y) - \psi(y_0))\, d\omega^X_L(y) = \int_{\partial\Omega} \psi(y)\, d\omega^X_L(y),$$

which, by (3.6), satisfies

$$\|u_1\|_{\dot{C}^\alpha(\Omega)} \leqslant C_\alpha \|\psi\|_{\dot{C}^\alpha(\partial\Omega)} = C_\alpha\, |x_0-y_0|^{-\alpha}. \tag{3.41}$$

In a similar fashion, we can also define another solution for the datum $1-\psi$:

[7] To be more precise, our argument below shows LEMD(α) for the pair a and r satisfying the first condition in (3.48). However, when the second condition in (3.48) holds, we have $\overline{\Omega} \cap \partial B(a,r) = \varnothing$ and thus (3.38) holds trivially.

$$u_2(X) := u_2(X; x_0, 1-\psi) := (1-\psi)(x_0) + \int_{\partial\Omega} [(1-\psi)(y) - (1-\psi)(x_0)]\, d\omega_L^X(y)$$
$$= \omega_L^X(\partial\Omega) - u_1(X),$$

which satisfies

$$\|u_2\|_{\dot{C}^\alpha(\Omega)} \leqslant C_\alpha \|1-\psi\|_{\dot{C}^\alpha(\partial\Omega)} = C_\alpha \|\psi\|_{\dot{C}^\alpha(\partial\Omega)} = C_\alpha\, |x_0 - y_0|^{-\alpha}\,. \tag{3.42}$$

Gathering (3.41) and (3.42), and setting $u(X) := u_1(X) + u_2(X) = \omega_L^X(\partial\Omega)$, we conclude that

$$u \in \dot{C}^\alpha(\Omega) \quad \text{with} \quad \|u\|_{\dot{C}^\alpha(\Omega)} \leqslant M|x_0 - y_0|^{-\alpha}, \tag{3.43}$$

where $M = 2C_\alpha$.

To proceed, given $X \in \Omega$, we pick $\widehat{x} \in \partial\Omega$ so that $|X - \widehat{x}| = \delta(X)$. Then, by (3.43) and the regularity of Ω,

$$|\omega_L^X(\partial\Omega) - 1| = |u(X) - u(\widehat{x})| \leqslant M\left(\frac{|X-\widehat{x}|}{|x_0 - y_0|}\right)^\alpha. \tag{3.44}$$

In the case $\partial\Omega$ unbounded, we may choose $x_0, y_0 \in \partial\Omega$ satisfying $|x_0 - y_0| \longrightarrow \infty$, which by the previous estimate implies that $\omega_L^X(\partial\Omega) = 1$ for any $X \in \Omega$. Therefore, ω_L^X is a probability in this case, and (3.40) holds trivially with $c^* = 1$.

Therefore, we are left with the case Ω unbounded with $\partial\Omega$ bounded. In this case, the elliptic measure is not a probability (see Section 2.4.6) and we need to work more. Take arbitrary $a \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. Fix $x_0, y_0 \in \partial\Omega$ such that $|x_0 - y_0| = \operatorname{diam}(\partial\Omega)$. Then, if we pick $X \in \Omega$ with $\delta(X) \leqslant (2M)^{-1/\alpha} \operatorname{diam}(\partial\Omega) =: r_0$, (3.44) implies

$$|\omega_L^X(\partial\Omega) - 1| \leqslant M\left(\frac{|X-\widehat{x}|}{\operatorname{diam}(\partial\Omega)}\right)^\alpha \leqslant \frac{1}{2},$$

and, as a consequence,

$$\omega_L^X(\partial\Omega) \geqslant \frac{1}{2}, \qquad \text{for any } X \in \Omega \text{ with } \delta(X) \leqslant r_0. \tag{3.45}$$

Introduce $\Omega' := \{X \in \Omega : \delta(X) > r_0\}$, and observe that (3.32) and Lemma 3.31 clearly imply (recalling that $a \in \partial\Omega$)

$$\frac{\Gamma_L(X,a)}{2\, c_0^{-1} r_0^{1-n}} \leqslant \frac{1}{2} \ \text{ for } X \in \partial\Omega', \quad \text{and} \quad \omega_L^X(\partial\Omega) = \omega_L^X(\partial\Omega \cap B(a, 3\operatorname{diam}(\partial\Omega))) \underset{X\to\infty}{\longrightarrow} 0.$$

Hence, (3.45) and the classical maximum principle in Ω' yield

$$\omega_L^X(\partial\Omega) \geqslant \frac{\Gamma_L(X,a)}{2c_0^{-1} r_0^{1-n}}, \qquad X \in \Omega'.$$

This, together with (3.32), gives that for $X \in \Omega' \cap B(a, 100r) \subset \Omega' \cap B(a, 100\operatorname{diam}(\partial\Omega))$,

$$\omega_L^X(\partial\Omega) \geqslant c_0^2 \frac{(100\operatorname{diam}(\partial\Omega))^{1-n}}{2r_0^{1-n}} = c_0^2 \frac{[100(2M)^{1/\alpha}]^{1-n}}{2} =: c^* > 0. \tag{3.46}$$

As a result, (3.40) is a consequence of (3.45) and (3.46). Step 1 is then complete.

Step 2: Solvability of (3.1) with bounds (3.6) implies GEMD(α).

Take arbitrary $x \in \partial\Omega$ and $0 < \rho < \operatorname{diam}(\partial\Omega)$. Set, for $\xi \in \partial\Omega$ and $X \in \Omega$,

$$\phi(\xi) := \min\left\{\left(\frac{|\xi - x|}{\rho}\right)^{\alpha}, 1\right\} \quad \text{and} \quad u_\phi(X) := \int_{\partial\Omega} \phi \, d\omega_L^X.$$

The bound in (3.6) (note that $u_\phi(X)$ is given by (3.4) using $y_0 = x$ and the fact that $\phi(x) = 0$) gives

$$\frac{|u_\phi(X) - u_\phi(Y)|}{|X - Y|^{\alpha}} \leqslant C_\alpha \|\phi\|_{\dot{C}^{\alpha}(\partial\Omega)} \leqslant C_\alpha \rho^{-\alpha}, \qquad X, Y \in \Omega.$$

Letting $Y \to x$ and using the regularity of Ω (because ϕ is clearly bounded and uniformly continuous for it belongs to $\dot{C}^{\alpha}(\partial\Omega)$), we obtain

$$0 \leqslant u_\phi(X) \leqslant C_\alpha \left(\frac{|X - x|}{\rho}\right)^{\alpha}. \tag{3.47}$$

Note that $\mathbf{1}_{\partial\Omega\setminus B(x,\rho)}(\xi) \leqslant \phi(\xi)$ for every $\xi \in \partial\Omega$. Hence, (3.47) implies

$$\omega_L^X(\partial\Omega\setminus B(x,\rho)) = \int_{\partial\Omega} \mathbf{1}_{\partial\Omega\setminus B(x,\rho)} \, d\omega_L^X \leqslant \int_{\partial\Omega} \phi \, d\omega_L^X = u_\phi(X) \leqslant C_\alpha \left(\frac{|X - x|}{\rho}\right)^{\alpha}.$$

Since x, ρ, and X are arbitrary we conclude GEMD(α) as desired.

Step 3: GEMD(α) implies the geometrical condition: if $a \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, then

$$\text{either } \partial\Omega \cap S(a; cr, r) \neq \varnothing \quad \text{or} \quad S(a; cr, r) \subset \mathbb{R}^{n+1}\setminus\overline{\Omega}, \tag{3.48}$$

for some constant $c \in (0, 1/2)$ independent of a and r, where $S(a; cr, r)$ denotes the closed annulus

$$S(a; cr, r) := \{X \in \mathbb{R}^{n+1} : cr \leqslant |X - a| \leqslant r\}.$$

We extend the proof of [Ai02, Lemma 1] for the Laplacian to the case of general elliptic operators (we remark that the second case of (3.48) does not occur in [Ai02] due to the connectedness assumption on Ω). Take arbitrary $x \in \partial\Omega$ and $0 < \rho < \operatorname{diam}(\partial\Omega)$. Fix $\kappa = \min\{(c^*/(3M))^{1/\alpha}, 4^{-1}\}$ with M being the constant in GEMD(α) and c^* is the constant in (3.40). Then, GEMD(α) yields

$$\omega_L^X(\partial\Omega\setminus B(a,r)) \leqslant c^*/3, \qquad X \in \Omega \cap \overline{B(a, \kappa r)}. \tag{3.49}$$

Let us prove (3.48) with $c := \kappa(c_0^2 c^*/3)^{1/(n-1)} \in (0, \kappa)$, by contradiction.

Indeed, assume $\partial\Omega \cap S(a; cr, r) = \varnothing$ and $S(a; cr, r) \not\subset \mathbb{R}^{n+1}\setminus\overline{\Omega}$. The former implies (simply because $S(a; cr, r)$ is connected) that either $S(a; cr, r) \subset \Omega$ or $S(a; cr, r) \subset \mathbb{R}^{n+1}\setminus\overline{\Omega}$. Since the latter case has already been ruled out, we are left with $S(a; cr, r) \subset \Omega$. Let us obtain a contradiction in this case.

Now, let $v \in C(\Omega) \cap W^{1,2}(\Omega)$ be the solution to $Lv = 0$ in Ω with boundary data $\mathbf{1}_{B(a,r)\cap\partial\Omega}$, that is, $v(X) = \omega_L^X(B(a,r) \cap \partial\Omega)$ for $X \in \Omega$. Note that since $S(a; cr, r) \subset \Omega$, the connected components of $\partial\Omega$

either lie inside $B(a,cr)$ or outside of $\overline{B(a,r)}$. Thus we have that $\mathbf{1}_{B(a,r)\cap\partial\Omega}$ is uniformly continuous on $\partial\Omega$ and hence $v\in C(\overline{\Omega})$ by the regularity of Ω.

For $X\in\Omega$ such that $|X-a|=cr$, (3.32) yields

$$\frac{\Gamma_L(X,a)}{c_0(cr)^{1-n}}\geqslant 1\geqslant \omega_L^X(B(a,r)\cap\partial\Omega)=v(X).$$

Note that if $x\in\partial\Omega\backslash B(a,cr)$ then $x\notin\overline{B(a,r)}$: otherwise, $x\in S(a;cr,r)\subset\Omega$ which contradicts the fact that $x\in\partial\Omega$. Thus,

$$\frac{\Gamma_L(x,a)}{c_0(cr)^{1-n}}>0=v(x).$$

Moreover, in the case that Ω is unbounded, (3.32) and Lemma 3.31 give

$$\frac{\Gamma_L(X,a)}{c_0(cr)^{1-n}}\longrightarrow 0\quad\text{ and }\quad \omega_L^X(B(a,r)\cap\partial\Omega)\longrightarrow 0\quad\text{as } X\to\infty.$$

With all the above estimates, the maximum principle in $\Omega\backslash\overline{B(a,cr)}$ yields

$$\frac{\Gamma_L(X,a)}{c_0(cr)^{1-n}}\geqslant\omega_L^X(B(a,r)\cap\partial\Omega),\qquad X\in\Omega\backslash\overline{B(a,cr)}. \tag{3.50}$$

Now take $Y\in\mathbb{R}^{n+1}$ such that $|Y-a|=\kappa r$. Then, since $0<c<\kappa<\frac{1}{2}$, we have $Y\in S(a;cr,r)$, hence $Y\in\Omega\cap\overline{B(a,\kappa r)}$.

Therefore, invoking (3.49), (3.40), (3.50), and (3.32), we obtain

$$\frac{c^*}{3}\geqslant\omega_L^Y(\partial\Omega\backslash B(a,r))=\omega_L^Y(\partial\Omega)-\omega_L^Y(\partial\Omega\cap B(a,r))\geqslant c^*-\frac{\Gamma_L(Y,a)}{c_0(cr)^{1-n}}\geqslant c^*-c_0^{-2}\left(\frac{\kappa}{c}\right)^{1-n}=\frac{2c^*}{3},$$

which is a contradiction. This finally shows that the dichotomy (3.48) holds.

Step 4: The second condition in (3.48) implies the CDC for a and r.

Let us briefly show how the condition $S(a;cr,r)\subset\mathbb{R}^{n+1}\backslash\overline{\Omega}$ in (3.48) implies (3.39) for a and r. Indeed, if $S(a;cr,r)\subset\mathbb{R}^{n+1}\backslash\overline{\Omega}$, then we may find a ball

$$B(X_0,cr/4)\subset S(a;cr,r)\subset\mathbb{R}^{n+1}\backslash\overline{\Omega}\subset\mathbb{R}^{n+1}\backslash\Omega.$$

This implies, by the monotonicity of capacity, that

$$\frac{\operatorname{Cap}(\overline{B(a,r)}\backslash\Omega,B(a,2r))}{\operatorname{Cap}(\overline{B(a,r)},B(a,2r))}\geqslant\frac{\operatorname{Cap}(B(X_0,cr/4),B(X_0,3r))}{\operatorname{Cap}(\overline{B(a,r)},B(a,2r))}\approx_c 1,$$

where in the last estimate we have used the well known fact that $\operatorname{Cap}(B(Y,R);B(Y,\lambda R))\approx_\lambda R^{1-n}$ for any $Y\in\mathbb{R}^{n+1}$, $R>0$ and $\lambda>1$ (see for instance [HKM06, Chapter 2]). Since by the previous steps, c does not depend on a nor r, we have shown (3.39) in the case that $S(a;cr,r)\subset\mathbb{R}^{n+1}\backslash\overline{\Omega}$ in (3.48). We are therefore left with the case $\partial\Omega\cap S(a;cr,r)\neq\varnothing$, which we handle in the next steps.

Step 5: GEMD(α) and the first condition in (3.48) imply LEMD(α) at a with scale r. Assuming that the GEMD(α) condition is satisfied and that $\partial\Omega \cap S(a; cr, r) \neq \varnothing$, we claim that there exists $\varepsilon > 0$, independent of a and r, such that

$$\omega_L^X(\partial\Omega \backslash B(a, cr/2)) \geqslant \varepsilon \qquad \text{for every } X \in \Omega \cap \partial B(a, r). \tag{3.51}$$

Assuming this momentarily, let us see how (3.51) implies LEMD(α) at a and scale r, with uniform constants on a and r. Choose $\phi \in C^\infty(\mathbb{R}^{n+1})$ such that $\mathbf{1}_{\mathbb{R}^{n+1}\backslash B(a,cr/2)} \leqslant \phi \leqslant \mathbf{1}_{\mathbb{R}^{n+1}\backslash B(a,cr/4)}$. Let

$$u_\phi(X) := \int_{\partial\Omega} \phi \, d\omega_L^X$$

be the associated solution in Ω. By regularity of Ω, $u_\phi \in C(\overline{\Omega})$. Moreover, the choice of ϕ along with (3.51) yield

$$u_\phi(Y) \geqslant \omega_L^Y(\partial\Omega \backslash B(a, cr/2)) \geqslant \varepsilon, \qquad \text{for every } Y \in \Omega \cap \partial B(a, r).$$

Moreover, for $y \in \partial\Omega \cap \partial B(a, r)$, we have, by continuity of u_ϕ up to the boundary, $u_\phi(y) = \phi(y) = 1 \geqslant \varepsilon$. Thus we have shown

$$u_\phi(Y) \geqslant \varepsilon, \qquad \text{for every } Y \in \overline{\Omega} \cap \partial B(a, r).$$

Hence, for any $X \in \Omega \cap B(a, r)$,

$$\begin{aligned} \omega^X_{L,\Omega\cap B(a,r)}(\overline{\Omega} \cap \partial B(a, r)) &= \int_{\partial(\Omega\cap B(a,r))} \mathbf{1}_{\overline{\Omega}\cap\partial B(a,r)}(y) \, d\omega^X_{L,\Omega\cap B(a,r)}(y) \\ &\leqslant \int_{\partial(\Omega\cap B(a,r))} \varepsilon^{-1} u_\phi(y) \, d\omega^X_{L,\Omega\cap B(a,r)}(y) = \varepsilon^{-1} u_\phi(X), \end{aligned} \tag{3.52}$$

where in the last step we have used the uniqueness of classical solutions in $\Omega \cap B(a, r)$ and the fact that u_ϕ is a solution in $\Omega \cap B(a, r)$, continuous in the whole $\overline{\Omega}$.

On the other hand, the GEMD(α) condition gives that for any $X \in \Omega \cap B(a, r)$,

$$\begin{aligned} u_\phi(X) = \int_{\partial\Omega} \phi \, d\omega_L^X &\leqslant \int_{\partial\Omega} \mathbf{1}_{\partial\Omega\backslash B(a,cr/4)} \, d\omega_L^X \\ &= \omega_L^X(\partial\Omega \backslash B(a, cr/4)) \leqslant M\left(\frac{|X-a|}{cr/4}\right)^\alpha = \frac{4^\alpha M}{c^\alpha}\left(\frac{|X-a|}{r}\right)^\alpha. \end{aligned} \tag{3.53}$$

Therefore, (3.52) and (3.53) imply the desired LEMD(α) at a and scale r with constant $4^\alpha M \varepsilon^{-1} c^{-\alpha}$, which is indeed independent of a and r by the previous steps.

It remains to prove our claim (3.51). The proof is similar to that of [Ai02, Lemma 2]. Since $\partial\Omega \cap S(a; cr, r) \neq \varnothing$, there exist $\rho \in [cr, r)$ and $x^* \in \partial\Omega \cap \partial B(a, \rho)$. Morally speaking, the existence of such boundary point x^* away from $B(a, cr/2)$ attracts the Brownian traveller and this is what gives us a non-zero lower bound in (3.51). The GEMD(α) condition gives

$$\omega_L^X(\partial\Omega \backslash B(x^*, \rho/4)) \leqslant M\left(\frac{|X-x^*|}{\rho/4}\right)^\alpha, \qquad X \in \Omega \cap B(x^*, \rho/4). \tag{3.54}$$

To proceed, choose $\psi \in C^\infty(\mathbb{R}^{n+1})$ such that $\mathbf{1}_{\mathbb{R}^{n+1}\backslash B(a,3\rho/4)} \leqslant \psi \leqslant \mathbf{1}_{\mathbb{R}^{n+1}\backslash B(a,\rho/2)}$, and let

$$u_\psi(X) := \int_{\partial\Omega} \psi \, d\omega_L^X, \qquad X \in \Omega,$$

be the associated solution in Ω, so that $u_\psi \in C(\overline{\Omega})$ by the regularity of Ω (note that u_ψ agrees with (3.4) with $y_0 = a$ since $\psi(a) = 0$). Then, noting that $|x^* - a| = \rho$ implies $B(a, 3\rho/4) \subset \mathbb{R}^{n+1}\backslash B(x^*, \rho/4)$, we use (3.54) to obtain that for all $X \in \Omega \cap B(x^*, \rho/4)$,

$$\omega_L^X(\partial\Omega \cap B(a, 3\rho/4)) \leqslant \omega_L^X(\partial\Omega\backslash B(x^*, \rho/4)) \leqslant M\left(\frac{|X - x^*|}{\rho/4}\right)^\alpha, \tag{3.55}$$

whence for some $0 < d < 1/12$ small enough, independent of r, ρ and x^*, it holds (recalling that $\psi \equiv 1$ on $\partial\Omega\backslash B(a, 3\rho/4)$, and also (3.40)) for any $X \in \Omega \cap B(x^*, d\rho/4)$

$$u_\psi(X) \geqslant \omega_L^X(\partial\Omega\backslash B(a, 3\rho/4)) = \omega_L^X(\partial\Omega) - \omega_L^X(\partial\Omega \cap B(a, 3\rho/4)) \geqslant c^*/2. \tag{3.56}$$

Define

$$u(X) = \begin{cases} u_\psi(X) & \text{if } X \in \Omega, \\ 1 & \text{if } X \notin \Omega. \end{cases}$$

It is well known that u is a supersolution to $Lu = 0$ in $\mathbb{R}^{n+1}\backslash B(a, 3\rho/4)$ (see [HKM06, Lemma 7.9]), and it is clear that $0 \leqslant u \leqslant 1$ in $\mathbb{R}^{n+1}$. Also, (3.56) implies that $u \geqslant 1/2$ in $B(x^*, d\rho/4)$. By regularity of Ω, it is not hard to see that $u \in C(\mathbb{R}^{n+1}\backslash B(a, 3\rho/4))$.

Take now any $x_1^* \in \partial B(a, \rho)$ with $\left|x_1^* - x^*\right| < \rho/12$, and denote $B := B(x_1^*, \frac{\rho}{12} + \frac{d\rho}{4})$. These clearly imply that $B(x^*, d\rho/4) \subset B$, and so $|\{X \in B : u(X) \geqslant 1/2\}| \geqslant |B(x^*, d\rho/4)| \approx |B|$, with constants depending only on n and d. Moreover, using the facts that $|x_1^* - a| = \rho$ and $d < 1/12$, we easily get that $2B \subset \mathbb{R}^{n+1}\backslash B(a, 3\rho/4)$. These allow us to apply (a trivial modification of) [HL11, Theorem 4.9] to deduce

$$u(X) \geqslant \delta, \qquad X \in B(x_1^*, d\rho/4),$$

for some $\delta > 0$ independent of x^*, x_1^* and ρ, because $B(x_1^*, d\rho/4) \subset \frac{1}{2}B$ since $d < 1/12$. Then repeat the argument using x_1^* in place of x^*, and repeat until we have covered $\partial B(a, \rho)$ by a finite amount (depending only on d and n) of balls $B(x^*, d\rho/4), B(x_1^*, d\rho/4), \ldots, B(x_k^*, d\rho/4)$, ultimately obtaining

$$u(X) \geqslant \varepsilon, \qquad \text{for all } X \in \partial B(a, \rho),$$

for some $\varepsilon > 0$ independent of x^*, a and ρ. This implies, by definition of u,

$$u_\psi(X) \geqslant \varepsilon, \qquad X \in \Omega \cap \partial B(a, \rho).$$

Therefore, recalling that $cr \leqslant \rho$ and that $\psi \equiv 0$ on $\partial\Omega \cap B(a, \rho/2)$, we obtain that for all $X \in \Omega \cap \partial B(a, \rho)$,

$$\omega_L^X(\partial\Omega\backslash B(a, cr/2))) \geqslant \omega_L^X(\partial\Omega\backslash B(a, \rho/2)) \geqslant u_\psi(X) \geqslant \varepsilon. \tag{3.57}$$

Let $\Omega' := \Omega\backslash\overline{B(a, \rho)}$. Note that $v(Y) := \omega_L^Y(\partial\Omega\backslash B(a, cr/2))$, $Y \in \Omega$,

clearly satisfies $v \in C(\overline{\Omega'})$ since $B(a, cr/2) \subset B(a, \rho/2)$. Therefore, if we let

$$\widehat{v}(Y) := \int_{\partial\Omega'} v(z)\, d\omega_{L,\Omega'}^Y(z), \qquad Y \in \Omega', \tag{3.58}$$

it follows that $\widehat{v} \in C(\overline{\Omega'})$ (since Ω' is clearly regular) with $\widehat{v} \equiv v$ in $\partial\Omega'$. Since both v and $\widehat{v}$ are L-solutions in Ω' continuous up to the boundary, the classical maximum principle yields that $\widehat{v} \equiv v$ in

Ω' —here we note that when Ω is unbounded, $v - \widehat{v}$ decays at infinity by Lemma 3.31, hence we can legitimately invoke the maximum principle. As a result, since $\rho < r$, for any $Y \in \Omega \cap \partial B(a,r) \subset \Omega'$ we obtain

$$\omega_L^Y(\partial\Omega\backslash B(a,cr/2)) = \int_{\partial\Omega'} \omega_L^X(\partial\Omega\backslash B(a,cr/2))\, d\omega_{L,\Omega'}^Y(X) \geqslant \omega_{L,\Omega'}^Y(\partial\Omega') \inf_{X\in\partial\Omega'} \omega_L^X(\partial\Omega\backslash B(a,cr/2)).$$

Note that $\omega_L^X(\partial\Omega\backslash B(a,cr/2)) = 1$ for $X \in \partial\Omega\backslash B(a,\rho)$, and, by (3.57), $\omega_L^X(\partial\Omega\backslash B(a,cr/2)) \geqslant \varepsilon$ for $X \in \partial B(a,\rho) \cap \Omega$. These clearly imply that

$$\inf_{X\in\partial\Omega'} \omega_L^X(\partial\Omega\backslash B(a,cr/2)) \geqslant \varepsilon.$$

On the other hand, since $\Omega' = \Omega\backslash\overline{B(a,\rho)} \subset \mathbb{R}^{n+1}\backslash\overline{B(a,\rho)} =: \Omega''$, by the maximum principle in Ω' (its use is justified because if Ω' is unbounded with $\partial\Omega'$ bounded, we can use Lemma 3.31 to show that solutions decay at infinity; and otherwise $\omega_{L,\Omega'}$ is a probability and the result is trivial) we obtain, for $Y \in \partial B(a,r) \cap \Omega \subset \Omega'$,

$$\omega_{L,\Omega'}^Y(\partial\Omega') \geqslant \omega_{L,\Omega''}^Y(\partial\Omega'').$$

To continue, we next observe that $\Omega'' = \mathbb{R}^{n+1}\backslash\overline{B(a,\rho)}$ satisfies the CDC with a constant independent of ρ, whence the Hölder continuity at the boundary (cf. Lemma 3.11) for Ω'' and a straightfoward use of Harnack's inequality (moving radially towards a) gives that $\omega_{L,\Omega''}^Z(\partial\Omega'') \approx 1$ for every $Z \in B(a,\rho/c) \cap \Omega''$ where the implicit constants depend on the harmless constant c. Recalling then that $cr \leqslant \rho < r$, we obtain $\omega_{L,\Omega''}^Y(\partial\Omega'') \gtrsim 1$ for every $Y \in \partial B(a,r)$. Collecting all these observations we have shown that

$$\omega_L^Y(\partial\Omega\backslash B(a,cr/2)) \gtrsim \varepsilon, \qquad \text{for } Y \in \partial B(a,r) \cap \Omega,$$

with a harmless implicit constant.

Upon renaming ε, this finishes the proof of the claim (3.51).

Step 6: LEMD(α) at a with scale r implies CDC for a and r. This is a generalization of [An86, Lemma 3]. For the sake of convenience, set $F := \overline{B(a,r)}\backslash\Omega$ and $v := \mathcal{R}_L(F; B(a,2r))$ the capacitary potential in $B(a,2r)$ of the compact set F with respect to the operator L. Then, Lemma 3.33 along with the LEMD(α) condition at a and scale r (obtained in Step 4) gives that for any $X \in \Omega \cap B(a,r/N)$, for $N \gg 1$,

$$v(X) \geqslant 1 - \omega_{L,\Omega\cap B(a,r)}^X(\partial B(a,r) \cap \overline{\Omega}) \geqslant 1 - M\left(\frac{|X-a|}{r}\right)^\alpha \geqslant 1 - \frac{M}{N^\alpha} =: \eta > 0.$$

where η and N are independent of a and r by the previous steps. Moreover, since $B(a,r/N)\backslash\Omega \subset F$, we actually obtain

$$\eta^{-1} v(X) \geqslant 1, \qquad X \in B(a,r/N).$$

By the definition of capacity and the ellipticity of A, we deduce

$$\operatorname{Cap}(B(a,r/N); B(a,2r)) \leqslant \int_{B(a,2r)} \left|\nabla(\eta^{-1}v)\right|^2 dX \lesssim \eta^{-2} \int_{B(a,2r)} A\nabla v \cdot \nabla v\, dX. \tag{3.59}$$

Observe that the results in [LSW63, Section 4] (or [HKM06, Chapter 8]) assert that $v = \mathcal{R}_L(F; B(a,2r))$ is precisely the function minimizing the energy $\int_{B(a,2r)} A\nabla v \cdot \nabla v \, dX$ among those which are at least 1 on F. This means

$$\int_{B(a,2r)} A\nabla v \cdot \nabla v \, dX = \inf_{\substack{w \in W_0^{1,2}(B(a,2r)) \\ w \geqslant 1 \text{ on } F}} \int_{B(a,2r)} A\nabla w \cdot \nabla w \, dX$$

$$\lesssim \inf_{\substack{w \in W_0^{1,2}(B(a,2r)) \\ w \geqslant 1 \text{ on } F}} \int_{B(a,2r)} |\nabla w|^2 \, dX = \mathrm{Cap}(F; B(a,2r)). \quad (3.60)$$

From (3.59), (3.60), and the fact that

$$\mathrm{Cap}(B(a, r/N), B(a,2r)) \approx_N r^{1-n} \approx \mathrm{Cap}(\overline{B(a,r)}, B(a,2r)),$$

we immediately get

$$\mathrm{Cap}(F, B(a,2r)) \gtrsim_N \varepsilon^2 \, \mathrm{Cap}(\overline{B(a,r)}, B(a,2r)),$$

which shows (3.39) with constants depending on N and ε, so independent of a and r, as desired. This completes the proof of Theorem 3.7.

Remark 3.37. From our proof we can see that for any regular open set Ω and for any elliptic operator L, solutions of the $\dot{C}^\alpha$-Dirichlet problem defined by (3.2) or (3.4) satisfy the estimate (3.6), if and only if, non-negative solutions with vanishing boundary data enjoy boundary Hölder continuity (as in Lemma 3.11) with *almost* the same exponent α.

To see this, take $\alpha \leqslant \alpha_1$, where α_1 is the exponent in the De Giorgi/Nash estimates (cf. Lemma 3.10). First note that $\dot{C}^\alpha$-solvability along with (3.6) implies LEMD(α) by Steps 2-5 above (noting again that some cases are trivial as explained in the footnote in Remark 3.36), so by Remark 3.36 we obtain boundary Hölder continuity (as in Lemma 3.11) with exponent α.

Conversely, if boundary Hölder continuity (from Lemma 3.11) with exponent α holds, then Lemma 3.19 is available (see also Remark 3.20), and then the machinery in Sections 3 and 4 runs to obtain $\dot{C}^{\alpha'}$-solvability with bounds (3.6), for any $\alpha' < \alpha$. To conclude, we remark that it is in general not possible to take $\alpha' = \alpha$ by the example in [Ma14, p. 3].

3.8.2 Adaptation of the proof of optimality of the CDC for growth functions

Let us provide a brief sketch of how the preceding proof would work with an arbitrary growth function $\varphi \in \mathcal{G}_\beta$. First, we should replace the quotients $\left(\frac{|X-x|}{r}\right)^\alpha$ in Definition 3.34 by $\frac{\varphi(|X-x|)}{\varphi(r)}$.

Then, we observe that $Q_\beta\varphi$ is sub-additive. Indeed, for every $0 < t_1, t_2 < \infty$ there holds

$$Q_\beta\varphi(t_1+t_2) = (t_1+t_2)^\beta \int_{t_1+t_2}^\infty \frac{\varphi(s)}{s^\alpha}\frac{ds}{s} \leqslant (t_1^\beta + t_2^\beta) \int_{t_1+t_2}^\infty \frac{\varphi(s)}{s^\alpha}\frac{ds}{s} \leqslant Q_\beta\varphi(t_1) + Q_\beta\varphi(t_2).$$

Taking this into account, and given $x_0 \in \partial\Omega$, if we define $\psi_{x_0}(x) := Q_\beta\varphi(|x-x_0|)$ for $x \in \partial\Omega$, we clearly have, by the assumption $\varphi \in \mathcal{G}_\beta$ and Lemma 3.14,

$$\|\psi_{x_0}\|_{\dot{C}^\varphi(\partial\Omega)} \approx \|\psi_{x_0}\|_{\dot{C}^{Q_\beta\varphi}(\partial\Omega)} \leqslant 1.$$

Our second observation is that $\varphi(t) \longrightarrow \infty$ as $t \to \infty$, fact that follows easily from $\varphi \in \mathcal{G}_\beta$. Finally, (3.11) implies

$$\varphi(\lambda^{-1}t) \leqslant \frac{1}{\log\lambda}\int_{\lambda^{-1}t}^{t}\varphi(s)\,\frac{ds}{s} \leqslant \frac{1}{\log\lambda}\int_{0}^{t}\varphi(s)\,\frac{ds}{s} \leqslant \frac{C_\varphi}{\log\lambda}\varphi(t), \qquad \text{for all } t>0 \text{ and } \lambda>1.$$

With the three observations the proof can be carried out with minor changes. In Step 1 we take $\psi(x) := Q_\beta\varphi(|x-y_0|)/Q_\beta\varphi(|x_0-y_0|)$ to obtain the analogue of (3.44):

$$|\omega_L^X(\partial\Omega)-1| \lesssim \frac{Q_\beta\varphi(|X-\widehat{x}|)}{Q_\beta\varphi(|x_0-y_0|)} \approx \frac{\varphi(|X-\widehat{x}|)}{\varphi(|x_0-y_0|)},$$

and the rest of the argument can be easily adapted. In Step 2, we set $\phi(\xi) := \min\left\{\frac{Q_\beta\varphi(|\xi-x|)}{Q_\beta\varphi(\rho)}, 1\right\}$ and show that

$$\|\phi\|_{\dot{C}^{\varphi}(\partial\Omega)} \approx \|\phi\|_{\dot{C}^{Q_\beta\varphi}(\partial\Omega)} \lesssim Q_\beta\varphi(\rho)^{-1} \leqslant \varphi(\rho)^{-1},$$

which in turn gives the appropriate analogue of the GEMD(α) condition:

$$\omega_L^X(\partial\Omega\backslash B(x,\rho)) \lesssim \frac{\varphi(|X-x|)}{\varphi(\rho)}, \qquad x\in\partial\Omega, 0<\rho<\operatorname{diam}(\partial\Omega), X\in\Omega.$$

From here the proof can be easily adapted, further details are left to the reader.

3.9 Applications to other boundary value problems

In this section, we will explore some other Dirichlet boundary value problems whose well-posedness, under suitable underlying geometrical conditions, can be shown as a consequence of that of the $\dot{C}^\alpha$-Dirichlet problem that we have extensively studied in the previous sections.

We first introduce the following definition:

Definition 3.38 (Non-tangential limit). Given $\theta>1$ and $y\in\partial\Omega$, we say that $u|_{\partial\Omega}^{\text{nt. lim},\theta}(y) = f(y)$ if

$$\lim_{\substack{X\in\Gamma^\theta(y)\\ X\to y}} u(X) = f(y).$$

Here and elsewhere, $\Gamma^\theta(y) := \{X\in\Omega : |X-y| < \theta\delta(X)\}$ denotes the cone with vertex $y\in\partial\Omega$ and aperture $\theta>1$.

In this section, we shall impose the boundary value condition

$$u|_{\partial\Omega}^{\text{nt. lim},\theta} = f, \quad \sigma\text{-a.e. on } \partial\Omega,$$

for function f in suitable spaces. Compared to (3.1), we impose a weaker boundary value condition here because the functions u and/or f are only defined in Sobolev or L^p spaces a priori, and it does not make sense to impose a pointwise boundary condition. A posteriori we shall indeed prove that $u=f$ on $\partial\Omega$.

3.9.1 The Dirichlet problem with solutions satisfying a fractional Carleson measure condition

For the first application of the results in the previous sections, we introduce some spaces:

Definition 3.39. Let φ be a growth function. Define a (generalized) Carleson type φ-Hölder space on an open set $\Omega \subset \mathbb{R}^{n+1}$ as

$$C^{*,\varphi}(\Omega) := \big\{u \in W^{1,2}_{\mathrm{loc}}(\Omega) : \|u\|_{C^{*,\varphi}(\Omega)} < \infty\big\},$$

where $\|\cdot\|_{C^{*,\varphi}(\Omega)}$ stands for

$$\|u\|_{C^{*,\varphi}(\Omega)} := \sup_{\substack{x\in\partial\Omega\\ 0<r<\infty}} \frac{1}{\varphi(r)}\left(\frac{1}{r^n}\iint_{B(x,r)\cap\Omega} |\nabla u(X)|^2\delta(X)\,dX\right)^{\frac{1}{2}},$$

where here and elsewhere $\delta(X) := \operatorname{dist}(X,\partial\Omega)$.

Remark 3.40. When $\varphi(r) \equiv 1$, the quantity above is usually referred to as a "Carleson measure condition", and it is used to study the absolute continuity of harmonic/elliptic measure (cf. [CHMT20, HKMP15a, HLM19, HMMTZ21]).

In very general frameworks, this space is related to $\dot{C}^{\varphi}(\Omega)$, as the following result shows.

Lemma 3.41. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with ADR boundary. Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) uniformly elliptic operator. Let $\varphi \in \mathcal{G}_\beta$ for some $\beta \in (0,1)$. Then there exists C (depending only on dimension, the ADR constant, the ellipticity constant of L, and C_φ) such that for any weak solution u of $Lu = 0$ in Ω,*

$$\|u\|_{C^{*,\varphi}(\Omega)} \leqslant C\|u\|_{\dot{C}^{\varphi}(\Omega)}.$$

Proof. Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote the collection of (closed) dyadic Whitney cubes of Ω, so that the cubes in $\mathcal{W}$ form a pairwise non-overlapping covering of Ω, which satisfy

$$4\operatorname{diam}(I) \leqslant \operatorname{dist}(4I,\partial\Omega) \leqslant \operatorname{dist}(I,\partial\Omega) \leqslant 40\operatorname{diam}(I), \quad \forall I \in \mathcal{W}, \tag{3.61}$$

and

$$\operatorname{diam}(I_1) \approx \operatorname{diam}(I_2), \qquad \text{whenever } I_1 \text{ and } I_2 \text{ touch.}$$

Take an arbitrary ball $B = B(x,r)$ with $x \in \partial\Omega$ and $0 < r < \infty$. Set $\mathcal{W}_B := \{I \in \mathcal{W} : I \cap B \neq \varnothing\}$. For any $I \in \mathcal{W}_B$, we choose $Y_I \in I \cap B$. Then Caccioppoli's inequality applied to $u - u(Y_I)$ gives

$$\begin{aligned}\iint_{B\cap\Omega} |\nabla u(Y)|^2\delta(Y)\,dY &\lesssim \sum_{I\in\mathcal{W}_B} \ell(I)\iint_I |\nabla(u-u(Y_I))(Y)|^2\,dY\\ &\lesssim \sum_{I\in\mathcal{W}_B} \ell(I)^{-1}\iint_{\frac{3}{2}I} |u(Y)-u(Y_I)|^2\,dY \lesssim \|u\|^2_{\dot{C}^{\varphi}(\Omega)}\sum_{I\in\mathcal{W}_B}\ell(I)^n\varphi(\ell(I))^2,\end{aligned} \tag{3.62}$$

where we have used the doubling property of φ (cf. Lemma 3.16 part (b)).

Pick $y_I \in \partial\Omega$ so that $\delta(Y_I) = |Y_I - y_I|$. Set $B_I := B(y_I,\ell(I))$ and $\Delta_I := B_I \cap \partial\Omega$. For each $I \in \mathcal{W}_B$, it follows from (3.61) that

$$\operatorname{diam}(I) \leqslant \operatorname{dist}(I,\partial\Omega) \leqslant \delta(Y_I) \leqslant \operatorname{diam}(I) + \operatorname{dist}(I,\partial\Omega) \leqslant 41\operatorname{diam}(I), \tag{3.63}$$

and

$$\ell(I) \leqslant \operatorname{diam}(I) \leqslant \operatorname{dist}(I, \partial\Omega) \leqslant \delta(Y_I) \leqslant |Y_I - x| < r. \tag{3.64}$$

Note that

$$|y_I - x| \leqslant |y_I - Y_I| + |Y_I - x| = \delta(Y_I) + |Y_I - x| \leqslant 2|Y_I - x| < 2r,$$

which implies $B_I \subset 3B$. Moreover, we claim that we even have some bounded overlap property, in the sense that for any $k \in \mathbb{Z}$ with $2^{-k} < r$, there holds

$$\sum_{I \in \mathcal{W}_B : \ell(I) = 2^{-k}} \mathbf{1}_{\Delta_I} \leqslant C_n \, \mathbf{1}_{3\Delta}. \tag{3.65}$$

Indeed, if $\Delta_I \cap \Delta_J \neq \varnothing$ with $I, J \in \mathcal{W}_B$ and $\ell(I) = \ell(J)$, then we pick $z_{I,J} \in \Delta_I \cap \Delta_J$ and obtain

$$\begin{aligned}\operatorname{dist}(I, J) \leqslant |Y_I - Y_J| &\leqslant |Y_I - y_I| + |y_I - z_{I,J}| + |z_{I,J} - y_J| + |y_J - Y_J| \\ &\leqslant \delta(Y_I) + \ell(I) + \ell(J) + \delta(Y_J) \leqslant 84\sqrt{n+1}\,\ell(I).\end{aligned}$$

Thus, given $I \in \mathcal{W}_B$ we clearly have that

$$\#\big\{J \in \mathcal{W}_B : \Delta_J \cap \Delta_I \neq \varnothing,\, \ell(J) = \ell(I)\big\} \leqslant C_n.$$

This shows (3.65).

On the other hand, if $\operatorname{diam}(\partial\Omega) \leqslant 2^{-k} < r$, we have for every $I \in \mathcal{W}_B$ with $\ell(I) = 2^{-k}$,

$$|Y_I - x| \leqslant |Y_I - y_I| + |y_I - x| \leqslant \delta(Y_I) + \operatorname{diam}(\partial\Omega) \leqslant 41 \operatorname{diam}(I) + \operatorname{diam}(\partial\Omega) < 42\sqrt{n+1}\,2^{-k},$$

using (3.63) and therefore $I \subset B(x, 43\sqrt{n+1}\,2^{-k})$. Thus,

$$\begin{aligned}\#\{I \in \mathcal{W}_B : \ell(I) = 2^{-k}\} &= 2^{k(n+1)} \sum_{I \in \mathcal{W}_B : \ell(I) = 2^{-k}} |I| \\ &= 2^{k(n+1)} \Big| \bigcup_{I \in \mathcal{W}_B : \ell(I) = 2^{-k}} I \Big| \leqslant 2^{k(n+1)} |B(x, 43\sqrt{n+1}\,2^{-k})| = C_n,\end{aligned} \tag{3.66}$$

whenever $\operatorname{diam}(\partial\Omega) \leqslant 2^{-k} < r$. Then, recalling (3.64), we use (3.65) and (3.66) to obtain that

$$\begin{aligned}\sum_{I \in \mathcal{W}_B} \ell(I)^n \varphi(\ell(I))^2 &= \sum_{\substack{I \in \mathcal{W}_B \\ \ell(I) < \operatorname{diam}(\partial\Omega)}} \ell(I)^n \varphi(\ell(I))^2 + \sum_{\substack{I \in \mathcal{W}_B \\ \ell(I) \geqslant \operatorname{diam}(\partial\Omega)}} \ell(I)^n \varphi(\ell(I))^2 \\ &= \sum_{k:\, 2^{-k} < \min\{r, \operatorname{diam}(\partial\Omega)\}} \sum_{I \in \mathcal{W}_B : \ell(I) = 2^{-k}} \ell(I)^n \varphi(\ell(I))^2 \\ &\qquad + \sum_{k:\, \operatorname{diam}(\partial\Omega) \leqslant 2^{-k} < r} \sum_{I \in \mathcal{W}_B : \ell(I) = 2^{-k}} \ell(I)^n \varphi(\ell(I))^2 \\ &\lesssim \sum_{k:\, 2^{-k} < r} \varphi(2^{-k})^2 \sum_{I \in \mathcal{W}_B : \ell(I) = 2^{-k}} \sigma(\Delta_I) \\ &\qquad + \sum_{k:\, \operatorname{diam}(\partial\Omega) \leqslant 2^{-k} < r} r^n \varphi(2^{-k})^2 \#\{I \in \mathcal{W}_B : \ell(I) = 2^{-k}\}\end{aligned} \tag{3.67}$$

$$\lesssim (\sigma(3\Delta) + r^n)\Big(\sum_{k:\,2^{-k}<r} \varphi(2^{-k})\Big)^2$$
$$\lesssim r^n\Big(\int_0^r \varphi(s)\frac{ds}{s}\Big)^2$$
$$\lesssim r^n\varphi(r)^2,$$

where in the last steps we have used the ADR property of $\partial\Omega$, and

$$\sum_{k:\,2^{-k}<r} \varphi(2^{-k}) \approx \sum_{k:\,2^{-k}<r} \varphi(2^{-k}) \int_{2^{-k}}^{2^{-k+1}} \frac{ds}{s}$$
$$\leqslant \sum_{k:\,2^{-k}<r} \int_{2^{-k}}^{2^{-k+1}} \varphi(s)\frac{ds}{s} = \int_0^{2r} \varphi(s)\frac{ds}{s} \lesssim \varphi(2r) \lesssim \varphi(r), \quad (3.68)$$

since φ is non-decreasing and doubling, and we have also used (3.11). As a consequence, the desired estimate immediately follows from (3.62) and (3.67). □

Equivalence of spaces in 1-sided CAD

Before exploring the boundary value problem associated with the space $C^{*,\varphi}(\Omega)$, let us first show that this space can in fact be identified with $\dot{C}^{\varphi}(\Omega)$ under some geometrical assumptions on Ω. To this end, we need some definitions and lemmas. We already introduced some in Section 2.3.3, but let us restate the definitions to give more details that will be needed later.

Definition 3.42 (Corkscrew condition). We say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *corkscrew condition* if for some uniform constant $0 < c_0 < 1$ and for every $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there exists a ball $B(X_{\Delta(x,r)}, c_0 r) \subset B(x,r) \cap \Omega$. The point $X_{\Delta(x,r)} \subset \Omega$ is called a *corkscrew point relative to* $\Delta(x,r)$ (or relative to $B(x,r)$). We note that we may allow $r < C\operatorname{diam}(\partial\Omega)$ for any fixed C, simply by adjusting the constant c_0.

Let us observe that, in general, different boundary points and radii may produce the same surface ball. Hence, in the previous definition there is a slight abuse of notation, as the corkscrew point depends on the particular boundary point and radius. In what follows when we consider with a Corkscrew point relative to Δ it is always understood that $\Delta := \Delta(x,r)$ with some fixed $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$ and that $X_\Delta = X_{\Delta(x,r)}$ is adapted to $B(x,r)$.

Definition 3.43 (Harnack Chain condition). We say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the *Harnack chain condition* if there are uniform constants $C_1, C_2 > 1$ so that for every pair of points $X, X' \in \Omega$ there is a chain of balls $B_1, B_2, \ldots, B_N \subset \Omega$ with $N \leqslant C_1(2 + \log_2^+ \Pi(X,X'))$, where

$$\Pi(X,X') := \frac{|X - X'|}{\min\{\delta(X), \delta(X')\}}, \quad (3.69)$$

such that $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \varnothing$ for each $1 \leqslant k \leqslant N-1$, and

$$C_2^{-1}\operatorname{diam}(B_k) \leqslant \operatorname{dist}(B_k, \partial\Omega) \leqslant C_2\operatorname{diam}(B_k), \qquad 1 \leqslant k \leqslant N. \quad (3.70)$$

The chain of balls is called a *Harnack chain*.

We note that in the context of the previous definition if $\Pi(X,X') \leqslant 1$ we can trivially form the Harnack chain $B_1 = B(X, 3\delta(X)/5)$ and $B_2 = B(X', 3\delta(X')/5)$ so that (3.70) holds with $C_2 = 3$. Hence the Harnack chain condition is non-trivial only when $\Pi(X,X') > 1$.

Remark 3.44. If $B_1, \dots B_N$ is a Harnack chain connecting X and Y in Ω, we clearly have that for every $1 \leqslant k \leqslant N-1$, using that $B_k \cap B_{k+1} \neq \varnothing$,

$$C_2^{-1} \operatorname{diam}(B_k) \leqslant \operatorname{dist}(B_k, \partial\Omega) \leqslant \operatorname{diam}(B_{k+1}) + \operatorname{dist}(B_{k+1}, \partial\Omega) \leqslant (1+C_2)\operatorname{diam}(B_{k+1})$$

and

$$C_2^{-1} \operatorname{diam}(B_{k+1}) \leqslant \operatorname{dist}(B_{k+1}, \partial\Omega) \leqslant \operatorname{diam}(B_k) + \operatorname{dist}(B_k, \partial\Omega) \leqslant (1+C_2)\operatorname{diam}(B_k).$$

Besides, there hold

$$C_2^{-1} \operatorname{diam}(B_1) \leqslant \operatorname{dist}(B_1, \partial\Omega) \leqslant \delta(X) \leqslant \operatorname{diam}(B_1) + \operatorname{dist}(B_1, \partial\Omega) \leqslant (1+C_2)\operatorname{diam}(B_1)$$

and

$$C_2^{-1} \operatorname{diam}(B_N) \leqslant \operatorname{dist}(B_N, \partial\Omega) \leqslant \delta(Y) \leqslant \operatorname{diam}(B_N) + \operatorname{dist}(B_N, \partial\Omega) \leqslant (1+C_2)\operatorname{diam}(B_N).$$

All these easily imply that for every $1 \leqslant j \leqslant N$,

$$(1+C_2)^{-N} C_2^{N-1} \max\{\delta(X), \delta(Y)\} \leqslant \operatorname{diam}(B_j) \leqslant (1+C_2)^{N-1} C_2^N \min\{\delta(X), \delta(Y)\}. \tag{3.71}$$

Definition 3.45 (1-sided CAD). We say that an open set $\Omega \subset \mathbb{R}^{n+1}$ is a *1-sided CAD* (chord-arc domain) if it satisfies the corkscrew and Harnack chain conditions, and has an ADR boundary.

Let us state an auxiliary lemma that will be used later.

Lemma 3.46. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 1$, be an unbounded open set with bounded boundary and assume that it satisfies the corkscrew condition with constant $c_0 \in (0,1)$. Then, for every $x \in \partial\Omega$ and $r \geqslant \operatorname{diam}(\partial\Omega)$ there exists a ball $B(X_{x,r}, c_0\, r/4) \subset B(x,r) \cap \Omega$.*

Proof. Fix $x \in \partial\Omega$ and let us first consider the case $\operatorname{diam}(\partial\Omega) \leqslant r \leqslant 2\operatorname{diam}(\partial\Omega)$. If we write $\Delta' = \Delta(x, \operatorname{diam}(\partial\Omega)/2)$, by the corkscrew condition we can find $X_{\Delta'}$ such that $B(X_{\Delta'}, c_0 \operatorname{diam}(\partial\Omega)/2) \subset B(x, \operatorname{diam}(\partial\Omega)/2) \cap \Omega$. If we set $X_{x,r} := X_{\Delta'}$ then we get as desired

$$B(X_{x,r}, c_0\, r/4) \subset B(X_{\Delta'}, c_0 \operatorname{diam}(\partial\Omega)/2) \subset B(x, \operatorname{diam}(\partial\Omega)/2) \cap \Omega \subset B(x,r) \cap \Omega.$$

Consider next the case $r > 2\operatorname{diam}(\partial\Omega)$ and pick an arbitrary $X_{x,r} \in \partial B(x, 3r/4)$ so that $B(X_{x,r}, r/4) \subset B(x,r)$. We claim that $B(X_{x,r}, r/4) \subset \Omega$. To see this, note that $x \in \partial\Omega$ implies that $\partial\Omega \subset \overline{B(x, \operatorname{diam}(\partial\Omega))}$ and hence $\partial\Omega \backslash \overline{B(x, \operatorname{diam}(\partial\Omega))} = \varnothing$. This means that either $\mathbb{R}^{n+1} \backslash \overline{B(x, \operatorname{diam}(\partial\Omega))} \subset \Omega$ or instead $\mathbb{R}^{n+1} \backslash \overline{B(x, \operatorname{diam}(\partial\Omega))} \subset \mathbb{R}^{n+1} \backslash \overline{\Omega}$. In the latter case we have $\overline{\Omega} \subset \overline{B(x, \operatorname{diam}(\partial\Omega))}$ which implies that Ω is bounded and this contradicts our hypothesis. Consequently, $\mathbb{R}^{n+1} \backslash \overline{B(x, \operatorname{diam}(\partial\Omega))} \subset \Omega$ and as a result $B(X_{x,r}, r/4) \subset \mathbb{R}^{n+1} \backslash \overline{B(x, \operatorname{diam}(\partial\Omega))} \subset \Omega$. All these, together with the fact that $c_0 \in (0,1)$, imply that $B(X_{x,r}, c_0 r/4) \subset B(x,r) \cap \Omega$. □

The main result in this subsection is formulated as follows.

Theorem 3.47. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be a 1-sided CAD, and let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator. Then for every $\beta \in (0, \alpha_1]$ (where $\alpha_1 \in (0,1)$ is the exponent in the De Giorgi/Nash estimates, see Lemma 3.10), and for every growth function $\varphi \in \mathcal{G}_\beta$, we have*

$$\|u\|_{\dot{C}^\varphi(\Omega)} \approx \|u\|_{C^{*,\varphi}(\Omega)}, \qquad \forall\, u \in W^{1,2}_{\rm loc}(\Omega),\ Lu = 0 \text{ in the weak sense in } \Omega, \tag{3.72}$$

where the implicit constants depend only on n, the 1-sided CAD constants, the ellipticity constant of L, and C_φ. In particular, if we define

$$C^{*,\varphi}_L(\Omega) := \big\{u \in W^{1,2}_{\rm loc}(\Omega) \cap C^{*,\varphi}(\Omega) : Lu = 0 \text{ in the weak sense in } \Omega\big\},$$

then

$$\dot{C}^\varphi_L(\Omega) = C^{*,\varphi}_L(\Omega) \quad \text{and} \quad \dot{C}^\varphi_L(\Omega)/\mathbb{R} = C^{*,\varphi}_L(\Omega)/\mathbb{R}, \quad \text{with equivalent norms.}$$

To show Theorem 3.47, we establish two auxiliary results, the second one depending heavily on the Harnack chain condition.

Lemma 3.48. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set, and let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) uniformly elliptic operator. Let $\Upsilon \geqslant 1$ and assume that the ball $B = B(X,R) \subset \Omega$ satisfies $\Upsilon^{-1}R \leqslant \operatorname{dist}(B, \partial\Omega) \leqslant \Upsilon R$. Then for every $\beta \in (0, \alpha_1]$ (where α_1 is the exponent in the De Giorgi/Nash estimates, see Lemma 3.10), for every growth function $\varphi \in \mathcal{G}_\beta$, and for every $u \in W^{1,2}_{\rm loc}(\Omega)$ satisfying $Lu = 0$ in the weak sense in Ω there holds*

$$\sup_{\substack{X,Y \in \overline{B} \\ X \neq Y}} \frac{|u(X) - u(Y)|}{\varphi(|X-Y|)} \lesssim \|u\|_{C^{*,\varphi}(\Omega)},$$

where the implicit constant depends only on n, the ellipticity of L, Υ, and C_φ.

Proof. Assume first that $X, Y \in \overline{B}$ with $|X - Y| < R/(8\Upsilon)$. Write $B_X = B(X, R/(8\Upsilon))$ and note that

$$\frac{R}{\Upsilon} \leqslant \operatorname{dist}(B, \partial\Omega) \leqslant \delta(X) \leqslant \operatorname{dist}(4B_X, \partial\Omega) + \frac{R}{2\Upsilon},$$

hence, $\operatorname{dist}(4B_X, \partial\Omega) \geqslant R/(2\Upsilon)$ and $4B_X \subset \Omega$. Also, if $\widehat{x} \in \partial\Omega$ is chosen so that $\operatorname{dist}(B, \partial\Omega) = \operatorname{dist}(B, \widehat{x})$, for every $Z \in 2B_X$, we have

$$|Z - \widehat{x}| \leqslant |Z - X| + \operatorname{diam}(B) + \operatorname{dist}(B, \widehat{x}) < (3 + \Upsilon)R,$$

thus $2B_X \subset B(\widehat{x}, (3+\Upsilon)R)$. Using all these, Lemma 3.10 applied in B_X to the solution $u(\cdot) - u_{2B_X}$ (where u_{2B_X} is the average of u in $2B_X$), and Poincaré's inequality we conclude that (we allow constants depend on Υ)

$$\begin{aligned} |u(X) - u(Y)| &\lesssim \left(\frac{|X-Y|}{R}\right)^{\alpha_1} \left(\frac{1}{|2B_X|} \iint\limits_{2B_X} |u(Y) - u_{2B_X}|^2\, dY\right)^{\frac{1}{2}} \\ &\lesssim \left(\frac{|X-Y|}{R}\right)^{\alpha_1} \left(\frac{1}{R^{n-1}} \iint\limits_{2B_X} |\nabla u(Y)|^2\, dY\right)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\frac{|X-Y|}{R}\right)^{\alpha_1}\left(\frac{1}{R^n}\iint\limits_{B(\hat{x},(3+\Upsilon)R)}|\nabla u(Y)|^2\,\delta(Y)dY\right)^{\frac{1}{2}}\\
&\lesssim \left(\frac{|X-Y|}{R}\right)^{\alpha_1}\varphi((3+\Upsilon)R)\|u\|_{C^{*,\varphi}(\Omega)}\\
&\lesssim \left(\frac{|X-Y|}{R}\right)^{\beta}\varphi(R)\|u\|_{C^{*,\varphi}(\Omega)}\\
&\lesssim \varphi(|X-Y|)\|u\|_{C^{*,\varphi}(\Omega)},
\end{aligned}
$$

where we have used that $|X-Y|<R$, $\beta\leqslant\alpha_1$, and Lemma 3.16.

To consider the case $X,Y\in\overline{B}$ with $|X-Y|\geqslant R/(8\Upsilon)$, we can easily find a sequence $X_0=X,X_1,X_2,\ldots,X_N=Y$ with $|X_j-X_{j+1}|<R/(8\Upsilon)$ for $0\leqslant j\leqslant N-1$ and with N depending just on Υ. We can then apply what we have just obtained to X_j and X_{j+1} and eventually conclude the desired estimate. □

Lemma 3.49. *Let $\Omega\subset\mathbb{R}^{n+1}$, $n\geqslant 2$, be a 1-sided CAD and let $L=-\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) uniformly elliptic operator. Given $c_1\in(0,1)$, $\beta\in(0,\alpha_1]$ (where α_1 is the exponent in the De Giorgi/Nash estimates, see Lemma 3.10), and a growth function $\varphi\in\mathcal{G}_\beta$, there exists a constant C (depending only on n, the 1-sided CAD constants, the ellipticity constants of L, c_1, and C_φ) such that for every $x\in\partial\Omega$, $0<r\leqslant s<\infty$, and $X,Y\in\Omega$ so that $B(X,c_1r)\subset B(x,r)\cap\Omega$ and $B(Y,c_1s)\subset B(x,s)\cap\Omega$, and for every $u\in W^{1,2}_{\rm loc}(\Omega)$ satisfying $Lu=0$ in the weak sense in Ω there holds*

$$
|u(X)-u(Y)|\leqslant C\,\varphi(s)\,\|u\|_{C^{*,\varphi}(\Omega)}.
$$

Proof. Note that if $\|u\|_{C^{*,\varphi}(\Omega)}=\infty$ there is nothing to prove. If $\|u\|_{C^{*,\varphi}(\Omega)}=0$ then u is constant and the desired inequality is also trivial. Thus, we may assume that $0<\|u\|_{C^{*,\varphi}(\Omega)}<\infty$, and in that case we can replace u by $u/\|u\|_{C^{*,\varphi}(\Omega)}$ and suppose that $\|u\|_{C^{*,\varphi}(\Omega)}=1$. Let $k_0\geqslant 0$ be such that $2^{-k_0-1}s<r\leqslant 2^{-k_0}s$.

We first work under the restriction $s<2\operatorname{diam}(\partial\Omega)/c_1$ (if $\partial\Omega$ is unbounded, this imposes no restriction). Using that Ω satisfies the corkscrew condition, for every $0\leqslant k\leqslant k_0$ we pick $X_k:=X_{\Delta(x,2^{-k}s)}$ (cf. Definition 3.42) so that

$$
B(X_k,c_0 2^{-k}s)\subset B(x,2^{-k}s)\cap\Omega. \tag{3.73}
$$

Note that for every $0\leqslant k\leqslant k_0$ we have $c_0 2^{-k}s\leqslant\delta(X_k)<2^{-k}s$. Also,

$$
2^{-1}c_1\delta(X_{k_0})<2^{-k_0-1}c_1 s<c_1 r\leqslant\delta(X)<r\leqslant 2^{-k_0}s\leqslant c_0^{-1}\delta(X_{k_0})
$$

and

$$
c_1\delta(X_0)<c_1 s\leqslant\delta(Y)<s\leqslant c_0^{-1}\delta(X_0).
$$

Moreover,

$$
\begin{aligned}
|Y-X_0|&\leqslant|Y-x|+|x-X_0|<2s,\\
|X-X_{k_0}|&\leqslant|X-x|+|x-X_{k_0}|<r+2^{-k_0}s<3r,
\end{aligned}
$$

and

$$
|X_{k-1}-X_k|\leqslant|X_{k-1}-x|+|x-X_k|<2^{-k+1}s+2^{-k}s<2^{-k+2}s,\qquad 1\leqslant k\leqslant k_0.
$$

All this show that with the notation introduced in (3.69)

$$\Pi(X, X_{k_0}),\ \Pi(Y, X_0),\ \Pi(X_{k-1}, X_k) \lesssim 1, \qquad 1 \leqslant k \leqslant k_0.$$

Set $X_{k_0+1} := X$ and $X_{-1} := Y$ and fix an arbitrary $0 \leqslant k \leqslant k_0 + 1$. Use that Ω satisfies the Harnack chain condition to construct a chain of balls $B_1^k, \dots, B_{N_k}^k$ joining X_{k-1} and X_k, satisfying (3.70), and with $N_k \lesssim (2+\log_2^+ \Pi(X_k, X_{k-1})) \lesssim 1$. Pick $Y_j^k \in B_j^k \cap B_{j+1}^k$ for $1 \leqslant j \leqslant N_k - 1$ and note that by Lemma 3.48 and (3.71) in Remark 3.44 we obtain

$$\begin{aligned}
|u(X_{k-1}) - u(X_k)| &\leqslant |u(X_{k-1}) - u(Y_1^k)| + \sum_{j=1}^{N_k-1} |u(Y_j^k) - u(Y_{j+1}^k)| + |u(Y_{N_k}^k) - u(X_k)| \\
&\lesssim \varphi(|X_{k-1} - Y_1^k|) + \sum_{j=1}^{N_k-1} \varphi(|Y_j^k - Y_{j+1}^k|) + \varphi(|Y_{N_k}^k - X_k|) \\
&\leqslant (N_k + 1) \sup_{0\leqslant j\leqslant N_k} \varphi(\operatorname{diam}(B_j^k)) \\
&\lesssim (N_k + 1)\varphi\big((1 + C_2)^{N_k-1} C_2^{N_k} \min\{\delta(X_{k-1}), \delta(X_k)\}\big) \\
&\lesssim \varphi(2^{-k}s),
\end{aligned}$$

where we have used that $N_k \lesssim 1$ and Lemma 3.16. Then by telescoping, (3.11), and invoking again Lemma 3.16, we conclude that

$$\begin{aligned}
|u(Y) - u(X)| = |u(X_{-1}) - u(X_{k_0+1})| &\leqslant \sum_{k=0}^{k_0+1} |u(X_{k-1}) - u(X_k)| \\
&\lesssim \sum_{k=0}^{k_0+1} \varphi(2^{-k}s) \lesssim \sum_{k=0}^{\infty} \int_{2^{-k}s}^{2^{-k+1}s} \varphi(t)\frac{dt}{t} \lesssim \int_0^{2s} \varphi(t)\frac{dt}{t} \lesssim \varphi(2s) \lesssim \varphi(s).
\end{aligned}$$

This is the desired estimate under the assumption that $s < 2\operatorname{diam}(\partial\Omega)/c_1$.

Let us next consider the case $s \geqslant 2\operatorname{diam}(\partial\Omega)/c_1$, which is meaningful only when $\partial\Omega$ is bounded. Note that $B(Y, c_1 s) \subset B(x, s) \cap \Omega$ implies that $2\operatorname{diam}(\partial\Omega) \leqslant c_1 s \leqslant \operatorname{diam}(\Omega)$, hence Ω must be unbounded (otherwise, $\operatorname{diam}(\partial\Omega) = \operatorname{diam}(\Omega)$). This means that we are in the situation Ω unbounded with $\partial\Omega$ bounded. For any $0 \leqslant k \leqslant k_0$ so that $2^{-k}s < \operatorname{diam}(\partial\Omega)$, we use that Ω satisfies the corkscrew condition to find $X_k := X_{\Delta(x,2^{-k}s)}$. If $0 \leqslant k \leqslant k_0$ is so that $2^{-k}s \geqslant \operatorname{diam}(\partial\Omega)$, we invoke Lemma 3.46 and let $X_k := X_{x,2^{-k}s}$. These choices guarantee that $B(X_k, c_0 2^{-k}s/4) \subset B(x, 2^{-k}s) \cap \Omega$, which is analogous to (3.73). We can then repeat the previous argument replacing c_0 by $c_0/4$ and conclude the desired estimate. Further details are left to the reader. □

Proof of Theorem 3.47. In view of Lemma 3.41, it suffices to show the upper bound in (3.72). First note that if $\|u\|_{C^{*,\varphi}(\Omega)} = \infty$ there is nothing to prove. If $\|u\|_{C^{*,\varphi}(\Omega)} = 0$ then u is constant and the desired inequality is also trivial. Thus, as before, we may assume that $\|u\|_{C^{*,\varphi}(\Omega)} = 1$.

Fix $X, Y \in \Omega$ and $0 < \beta \leqslant \alpha_1$. We may assume that $\delta(X) \leqslant \delta(Y)$. We first consider the case $|X - Y| < \max\{\delta(X), \delta(Y)\}/2 = \delta(Y)/2$. Let $B_Y = B(Y, \delta(Y)/2)$ so that $X \in B_Y$ and $\operatorname{dist}(B_Y, \partial\Omega) = \delta(Y)/2$. We can then apply Lemma 3.48 with $B = B_Y$ and $\Upsilon = 1$ to see that

$$|u(X) - u(Y)| \lesssim \varphi(|X - Y|).$$

To continue, we next assume that $|X-Y| \geqslant \max\{\delta(X),\delta(Y)\}/2 = \delta(Y)/2$. Pick $\widehat{x}$, $\widehat{y} \in \partial\Omega$ so that $\delta(X) = |X-\widehat{x}|$ and $\delta(Y) = |Y-\widehat{y}|$. If $|X-Y| < 2\operatorname{diam}(\partial\Omega)$ we choose $\widetilde{X} := X_{\Delta(\widehat{x},4|X-Y|)}$ and $\widetilde{Y} := X_{\Delta(\widehat{y},4|X-Y|)}$ (here we use the corkscrew condition, see Definition 3.42). On the other hand, if $|X-Y| \geqslant 2\operatorname{diam}(\partial\Omega)$ we then have that $\partial\Omega$ is necessarily bounded and $2\operatorname{diam}(\partial\Omega) \leqslant |X-Y| \leqslant \operatorname{diam}(\Omega)$, hence Ω must be unbounded (otherwise, $\operatorname{diam}(\Omega) = \operatorname{diam}(\partial\Omega)$). We can then use Lemma 3.46 to find $\widetilde{X} := X_{\widehat{x},4|X-Y|}$ and $\widetilde{Y} := X_{\widehat{y},4|X-Y|}$. In either scenario and by construction the following hold:

$$B(X,\delta(X)) \subset B(\widehat{x},2\delta(X)) \cap \Omega, \quad B(\widetilde{X},c_0|X-Y|) \subset B(\widehat{x},4|X-Y|) \cap \Omega,$$

and

$$B(Y,\delta(Y)) \subset B(\widehat{y},2\delta(Y)) \cap \Omega, \quad \text{and} \quad B(\widetilde{Y},c_0|X-Y|) \subset B(\widehat{y},4|X-Y|) \cap \Omega.$$

Observing that $2\delta(X) \leqslant 2\delta(Y) \leqslant 4|X-Y|$, we can then invoke Lemma 3.49 with $c_1 = c_0/4$ to obtain

$$|u(X)-u(\widetilde{X})| \lesssim \varphi(|X-Y|) \qquad \text{and} \qquad |u(Y)-u(\widetilde{Y})| \lesssim \varphi(|X-Y|). \tag{3.74}$$

On the other hand we note that $\delta(\widetilde{X}) \geqslant c_0|X-Y|$, $\delta(\widetilde{Y}) \geqslant c_0|X-Y|$, and

$$\begin{aligned}|\widetilde{X}-\widetilde{Y}| &\leqslant |X-\widehat{x}| + |\widehat{x}-\widetilde{X}| + |X-Y| + |Y-\widehat{y}| + |\widehat{y}-\widetilde{Y}| \\ &\leqslant \delta(X) + 9|X-Y| + \delta(Y) \leqslant 13|X-Y|.\end{aligned}$$

Thus, $\Pi(\widetilde{X},\widetilde{Y}) \leqslant 13c_0^{-1}$. We can then form a Harnack chain $B_1,\dots,B_N$ joining $\widetilde{X}$ and $\widetilde{Y}$ with $N \lesssim (2+\log_2^+\Pi(\widetilde{X},\widetilde{Y})) \lesssim 1$. Pick $Z_j \in B_j \cap B_{j+1}$ for $1 \leqslant j \leqslant N$ and note that by Lemma 3.48 and (3.71) in Remark 3.44 we obtain

$$\begin{aligned}|u(\widetilde{X})-u(\widetilde{Y})| &\leqslant |u(\widetilde{X})-u(Z_1)| + \sum_{j=1}^{N-1}|u(Z_j)-u(Z_{j+1})| + |u(Z_N)-u(\widetilde{Y})| \\ &\lesssim \varphi(|\widetilde{X}-Z_1|) + \sum_{j=1}^{N-1}\varphi(|Z_j-Z_{j+1}|) + \varphi(|Z_N-\widetilde{Y}|) \\ &\leqslant (N+1)\sup_{1\leqslant j\leqslant N}\varphi(\operatorname{diam}(B_j)) \\ &\lesssim (N+1)\varphi\big((1+C_2)^{N-1}C_2^N\min\{\delta(\widetilde{X}),\delta(\widetilde{Y})\}\big) \\ &\lesssim \varphi\big(\min\big\{|\widetilde{X}-\widehat{x}|,|\widetilde{Y}-\widehat{x}|\big\}\big) \\ &\leqslant \varphi(4|X-Y|) \\ &\lesssim \varphi(|X-Y|),\end{aligned}$$

where we have used that $N \lesssim 1$ and Lemma 3.16. This and (3.74) readily imply the desired estimate:

$$|u(X)-u(Y)| \leqslant |u(X)-u(\widetilde{X})| + |u(\widetilde{X})-u(\widetilde{Y})| + |u(\widetilde{Y})-u(Y)| \lesssim \varphi(|X-Y|).$$

This shows the upper bound in (3.72) and completes the proof of Theorem 3.47. □

Existence of solutions without connectivity

After working in 1-sided CAD to obtain Theorem 3.47, let us go back to a more general setting, where we assume no connectivity on Ω, and consider the Dirichlet boundary value problem associated with the $C^{*,\varphi}$ space.

We may now state an existence result for the Dirichlet boundary value problem associated with the $C^{*,\varphi}$ space. Again, we remark the fact that no connectivity is assumed on Ω.

Theorem 3.50. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with ADR boundary so that either* **Ω is bounded or Ω is unbounded with $\partial\Omega$ being unbounded**. *Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator, and let ω_L be the associated elliptic measure. Then there exists $\beta \in (0,1)$ (depending only on n, the ADR constant, and the ellipticity constant of L) such that for any $\theta > 1$ and for any growth function $\varphi \in \mathcal{G}_\beta$, the function*

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y), \qquad X \in \Omega \tag{3.75}$$

is a solution to the $C^{*,\varphi}$-Dirichlet problem

$$\begin{cases} u \in W^{1,2}_{\text{loc}}(\Omega), \\ Lu = 0 \text{ in the weak sense in } \Omega, \\ u \in C^{*,\varphi}(\Omega), \\ u|_{\partial\Omega}^{\text{nt. lim},\theta} = f \in \dot{C}^{\varphi}(\partial\Omega),\ \sigma\text{-a.e. on } \partial\Omega. \end{cases} \tag{3.76}$$

Moreover, $u \in C(\overline{\Omega})$ and satisfies $u|_{\partial\Omega} = f$, and there is a constant C (depending on n, the ADR constant, the ellipticity constant of L, and C_φ) such that

$$\|u\|_{C^{*,\varphi}(\Omega)} \leqslant C\|f\|_{\dot{C}^{\varphi}(\partial\Omega)}.$$

Proof. Since $\partial\Omega$ is ADR, it follows that Ω satisfies the CDC. Then, from Theorem 3.21 we see that, given $f \in \dot{C}^{\varphi}(\partial\Omega)$, the function u defined in (3.75) solves the problem (3.17). Moreover, $u \in \dot{C}^{\varphi}(\Omega) \cap C(\overline{\Omega})$ satisfies $u|_{\partial\Omega} = f$, and

$$\|u\|_{\dot{C}^{\varphi}(\Omega)} \lesssim \|f\|_{\dot{C}^{\varphi}(\partial\Omega)}.$$

On the other hand, invoking Lemma 3.41 we infer that $u \in C^{*,\varphi}(\Omega)$ and

$$\|u\|_{C^{*,\varphi}(\Omega)} \lesssim \|u\|_{\dot{C}^{\varphi}(\Omega)} \lesssim \|f\|_{\dot{C}^{\varphi}(\partial\Omega)}.$$

This completes the proof. □

Remark 3.51. We observe that from the previous result one obtains $u \in C(\overline{\Omega})$. This implies that $u|_{\partial\Omega}^{\text{nt. lim},\theta} = f$ everywhere on $\partial\Omega$ in the sense of Definition of 3.38, for any $\theta > 1$. This and the fact that none of the constants involved in the result depend on θ show that the precise value of θ does not play any relevant role in the theorem.

A sufficient condition for uniqueness

In this subsection we will show that, under a very mild connectivity assumption, the solutions to the $C^{*,\varphi}$ problem (3.76) are in fact unique. Before stating the result, let us recall a definition.

Definition 3.52 (Carrot path). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. Given a point $X \in \Omega$, and a point $y \in \partial\Omega$, we say that a connected rectifiable path $\gamma = \gamma(y, X)$, with endpoints y and X, is a *carrot path* (more precisely, a λ-carrot path) connecting y to X if $\gamma\backslash\{y\} \subset \Omega$ and, for some $\lambda \in (0,1)$ and for all $Z \in \gamma$, there holds $\lambda\ell(\gamma(y,Z)) \leqslant \delta(Z)$.

Proposition 3.53. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with ADR boundary so that either* **Ω is bounded or Ω is unbounded with $\partial\Omega$ being unbounded**. *Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator, and let ω_L be the associated elliptic measure. Given $\theta > 1$, assume that*

$$\begin{gathered} \text{there is an open neighborhood } U \text{ of } \partial\Omega \text{ in } \Omega \text{ so that} \\ \text{for every } X \in U \text{ there exists some } \lambda = \lambda(X) \in (0,1) \text{ satisfying } \lambda^{-1} < \theta \text{ and} \\ \sigma\Big(\big\{y \in \partial\Omega \cap B(X, 2\delta(X)) : \text{there exists a } \lambda\text{-carrot path joining } y \text{ and } X\big\}\Big) > 0. \end{gathered} \tag{3.77}$$

Then there exists $\beta \in (0,1)$ (depending only on n, the ADR constant, and on the ellipticity constant of L) such that for any growth function $\varphi \in \mathcal{G}_\beta$, the solution given by (3.75) is the unique solution to the problem (3.76), and hence the problem is well-posed.

Remark 3.54. The assumption $\lambda^{-1} < \theta$ in (3.77) implies that the λ-carrot path joining y and X is contained in the non-tangential cone $\Gamma^\theta(y)$. We also remark that the condition (3.77) is much weaker than the usual weak local John condition where one assumes non-tangentially accessibility to portions of the boundary (see [AHMMT20] where the weak local John condition plays a crucial role).

Proof. Suppose that u_1 and u_2 are two solutions of (3.76). Write $u := u_1 - u_2$ which solves (3.76) with $f \equiv 0$. We are going to show that $u \equiv 0$. If $\|u\|_{C^{*,\varphi}(\Omega)} = 0$, then u is a constant on $\partial\Omega$. This together with the fact that $u|_{\partial\Omega}^{\text{nt. lim},\theta} = 0$ implies that $u \equiv 0$ in Ω. On the other hand, if $\|u\|_{C^{*,\varphi}(\Omega)} \neq 0$, by homogeneity we may assume that $\|u\|_{C^{*,\varphi}(\Omega)} = 1$. We want to show that $u = 0$, and we proceed in a series of steps, which we state as claims.

Claim 3.55. *In the current scenario, there exists a constant C (depending on n, the ADR constant, the ellipticity constant of L, C_φ in (3.11), and θ) such that*

$$u(X) \leqslant C\,\varphi(\delta(X)) \text{ for all } X \in U.$$

Proof of the claim. Fix $X \in U$. Using (3.77) and the fact that $u|_{\partial\Omega}^{\text{nt. lim},\theta} = 0$, σ-a.e. on $\partial\Omega$, there exist $y \in B(X, 2\delta(X)) \cap \partial\Omega$ such that $u|_{\partial\Omega}^{\text{nt. lim},\theta}(y) = 0$ and a λ-carrot path $\gamma(y,X)$ connecting y and X. The assumption $\lambda^{-1} < \theta$ guarantees that $\gamma(y,X) \subset \Gamma^\theta(y)$. Indeed, for any $Z \in \gamma(y,X)$, by Definition 3.52 we have

$$\delta(Z) \geqslant \lambda\ell(\gamma(y,Z)) \geqslant \lambda|Z - y|.$$

Thus $|Z - y| \leqslant \lambda^{-1}\delta(Z) < \theta\delta(Z)$, and $Z \in \Gamma^\theta(y)$.

Let $\rho > 1$ be fixed such that $1 - 1/\rho < \lambda/3$. We define the sequence $\{X_k\}_{k\geqslant 0} \subset \gamma(y,X)$ as the points which satisfy $\ell_k := \ell(\gamma(y,X_k)) = \rho^{-k}\ell(\gamma(y,X)) =: \rho^{-k}\ell$, so concretely $X_0 = X$. Note that $\delta(X_k) \geqslant \lambda\ell_k$. We also define $B_k := B(X_k, \frac{\lambda}{3}\ell_k)$ and $\widetilde{B}_k := 2B_k$, which are clearly Whitney regions inside Ω. Observe that

$$|X_{k+1} - X_k| \leqslant \ell(\gamma(X_{k+1}, X_k)) = \ell_k - \ell_{k+1} = (1 - 1/\rho)\ell_k < \frac{\lambda}{3}\ell_k,$$

by our assumption on ρ. Thus $X_{k+1} \in B_k$.

Using Lemma 3.10 (applied to the function $u - u_{\widetilde{B}_k}$), Poincaré's inequality, the fact that $\delta(X_k) \geqslant \lambda\ell_k$ and $\widetilde{B}_k \subset B(y, 2\ell_k)$, $\|u\|_{\dot{C}^{*,\varphi}(\partial\Omega)} = 1$, and $\lambda^{-1} < \theta$, we have that

$$\sup_{B_k}|u-u_{\widetilde{B}_k}| \lesssim \left(\fint\!\!\!\fint_{\widetilde{B}_k}|u-u_{\widetilde{B}_k}|^2\,dZ\right)^{\frac12} \lesssim \lambda\ell_k\left(\fint\!\!\!\fint_{\widetilde{B}_k}|\nabla u|^2\,dZ\right)^{\frac12}$$
$$\lesssim \left(\frac{1}{(\lambda\ell_k)^n}\iint_{\widetilde{B}_k}|\nabla u|^2\delta(Z)\,dZ\right)^{\frac12} \leqslant \left(\frac{\theta^n}{(\ell_k)^n}\iint_{B(y,2\ell_k)}|\nabla u|^2\delta(Z)\,dZ\right)^{\frac12} \lesssim_\theta \varphi(2\ell_k)\lesssim\varphi(\ell_k).$$

In the last inequality we have used the doubling property of φ (see Lemma 3.16). Since both $X_k, X_{k+1}\in B_k$, the previous estimate implies

$$|u(X_k)-u(X_{k+1})|\leqslant |u(X_k)-u_{\widetilde{B}_k}|+|u(X_{k+1})-u_{\widetilde{B}_k}|\leqslant 2\sup_{B_k}|u-u_{\widetilde{B}_k}|\lesssim\varphi(\ell_k),$$

for every $k\geqslant 0$. Thus,

$$|u(X)|\leqslant\sum_{k=0}^{N}|u(X_k)-u(X_{k+1})|+|u(X_{N+1})|\lesssim\sum_{k=0}^{N}\varphi(\ell_k)+|u(X_{N+1})|. \tag{3.78}$$

For the first term, using (3.11), we obtain:

$$\sum_{k=0}^{\infty}\varphi(\ell_k)\lesssim\varphi(\ell)+\sum_{k=1}^{\infty}\int_{\rho^{-k}\ell}^{\rho^{-k+1}\ell}\varphi(s)\frac{ds}{s}=\varphi(\ell)+\int_0^{\ell}\varphi(s)\frac{ds}{s}\lesssim\varphi(\ell). \tag{3.79}$$

For the second term, since $\{X_k\}\subset\Gamma^\theta(y)$ and $X_k\to y$, the assumption $u|_{\partial\Omega}^{\text{nt. lim},\theta}(y)=0$ gives $|u(X_{N+1})|\to 0$ as $N\to\infty$. On the other hand, since $\delta(X)=\delta(X_0)\geqslant\lambda\ell$, by the monotonicity of φ and its doubling property (see Lemma 3.16), we also have

$$\varphi(\ell)=\varphi(\lambda^{-1}\cdot\lambda\ell)\leqslant\varphi(\theta\lambda\ell)\lesssim_\theta\varphi(\lambda\ell)\leqslant\varphi(\delta(X)).$$

Combined with (3.78) and (3.79), we conclude that

$$|u(X)|\lesssim_\theta\varphi(\delta(X)),$$

where the implicit constant only depends on θ, so in particular it is independent of X. □

Clearly, Claim 3.55 implies that $u\in C(\overline{\Omega})$ and $u=0$ on $\partial\Omega$. When Ω is bounded, the maximum principle guarantees that $u=0$ in Ω, thus the proof of the uniqueness of solutions is complete when Ω is bounded. When Ω is unbounded with $\partial\Omega$ unbounded, we need the following global estimate.

Claim 3.56. *In the current scenario, there exists a constant C (depending on n, the ADR constant, the ellipticity constant of L, C_φ in (3.11), and θ) such that*

$$u(X)\leqslant C\varphi(\delta(X))\ \text{for all } X\in\Omega. \tag{3.80}$$

Proof of the claim. The proof is very similar to that of Claim 3.55, but since we have already proven that $u\in C(\overline{\Omega})$ and $u=0$ on $\partial\Omega$, we can choose a more straightforward *carrot path*, that is the line segment to the closest point at the boundary.

Fix $X\in\Omega$. Take $\widehat{x}\in\partial\Omega$ such that $\delta(X)=|X-\widehat{x}|$. Clearly the line segment connecting X to $\widehat{x}$, denoted as γ, is contained in Ω, and every point $Z\in\gamma$ satisfies that $|Z-\widehat{x}|=\delta(Z)$. Define the sequence

of points $X_0 = X$ and $X_{k+1} = (X_k + \widehat{x})/2$ for each $k \geqslant 0$. Since U is an open neighborhood of $\partial\Omega$ in Ω, there exists some $\varepsilon = \varepsilon(\widehat{x})$ such that $B(\widehat{x}, \varepsilon) \cap \Omega \subset U$. Pick $k_0 \in \mathbb{N}$ so that $\delta(X_{k_0}) = \delta(X)/2^{k_0} \leqslant \varepsilon_0/2$, which yields $X_{k_0} \subset B(\widehat{x}, \varepsilon) \cap \Omega \subset U$. Hence by Claim 3.55, we have that

$$|u(X_{k_0})| \leqslant C\varphi(\delta(X_{k_0})) \leqslant C\varphi(\delta(X)).$$

Similarly as (3.78) and (3.79), we have that

$$|u(X)| \leqslant \sum_{k=0}^{k_0-1} |u(X_k) - u(X_{k+1})| + |u(X_{k_0})| \lesssim \varphi(\delta(X)) + |u(X_{k_0})| \lesssim \varphi(\delta(X)),$$

with a constant bigger than C in Claim 3.55. □

To conclude the proof of Proposition 3.53 for the case when Ω is unbounded with $\partial\Omega$ unbounded, we can proceed much as in the proof of Theorem 3.21. Fix $X_0 \in \Omega$ and pick $\widehat{x}_0 \in \partial\Omega$ such that $|X_0 - \widehat{x}_0| = \delta(X_0)$. Since (3.80) works for any point in Ω, the monotonicity of φ yields

$$\sup_{B(\widehat{x}_0,R)\cap\overline{\Omega}} |u| \lesssim \varphi(R), \qquad \forall\, R > 0.$$

With this and boundary Hölder continuity from Lemma 3.11 (since u is a weak solution and $u|_{\partial\Omega} \equiv 0$ by Claim 3.56) we get, for R big enough so that $X_0 \in B(\widehat{x}_0, R/2)$:

$$|u(X_0)| \lesssim \left(\frac{|X_0 - \widehat{x}_0|}{R}\right)^{\alpha_2} \sup_{B(\widehat{x}_0,R)\cap\overline{\Omega}} |u| \lesssim \delta(X_0)^{\alpha_2} \frac{\varphi(R)}{R^{\alpha_2}} \lesssim \delta(X_0)^{\beta} \frac{\varphi(R)}{R^{\beta}} \underset{R\to\infty}{\longrightarrow} 0$$

by Lemma 3.16 and recalling that $\beta \leqslant \alpha_2$. Since X_0 was chosen arbitrarily, this shows that $u \equiv 0$ in Ω, and the proof is complete. □

3.9.2 The Morrey-Campanato Dirichlet problem

For our last application, we introduce the Morrey-Campanato spaces. The point of introducing these spaces is their connection with the $\dot{C}^{\alpha}$ spaces, from which we will also be able to reduce their associated Dirichlet problems to the study of a $C^{*,\alpha}$ problem.

Throughout the section we assume the surface measure $\sigma := \mathcal{H}^n|_{\partial\Omega}$ is doubling.

Definition 3.57 (Generalized Morrey-Campanato spaces). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and assume that σ is doubling. Given a growth function φ along with some integrability exponent $p \in [1, \infty)$, the associated *generalized Morrey-Campanato space* $\mathcal{E}^{\varphi,p}(\partial\Omega)$ on $\partial\Omega$ is defined as the collection of all functions $f \in L^1_{\mathrm{loc}}(\partial\Omega, d\sigma)$ satisfying

$$\|f\|_{\mathcal{E}^{\varphi,p}(\partial\Omega)} := \sup_{\substack{x\in\partial\Omega \\ r>0}} \frac{1}{\varphi(r)} \left(\fint_{\Delta(x,r)} |f - f_{\Delta(x,r)}|^p \, d\sigma \right)^{\frac{1}{p}} < \infty,$$

where we recall that $\Delta(x, r) := B(x, r) \cap \partial\Omega$ and $f_{\Delta(x,r)} := \fint_{\Delta(x,r)} f \, d\sigma$.

For each $p \in [1,\infty)$, $r \in (0,\infty)$, and $f \in L^1_{\rm loc}(\partial\Omega)$, we define the L^p-based mean oscillations of f at a given scale r as

$$\operatorname{osc}_p(f;r) := \sup_{\substack{x\in\partial\Omega\\ 0<s\leqslant r}} \left(\fint_{\Delta(x,s)} |f - f_{\Delta(x,s)}|^p \, d\sigma \right)^{\frac{1}{p}}.$$

Lemma 3.58. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and assume that σ is doubling measure. Let $\varphi \in \mathcal{G}_\beta$ for some $\beta \in (0,1)$. For each $f \in L^1_{\rm loc}(\partial\Omega, d\sigma)$, the following properties hold:*

(a) *Fix $p \in [1,\infty)$. Then, as a function of r, the quantity $\operatorname{osc}_p(f;r)$ is non-decreasing in r.*

(b) *Given $p \in [1,\infty)$, the function f belongs to $\mathscr{E}^{\varphi,p}(\partial\Omega)$ if and only if $\operatorname{osc}_p(f;r)/\varphi(r)$, as a function in r, is bounded on $(0,\infty)$. More specifically,*

$$\|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)} = \sup_{r>0} \frac{\operatorname{osc}_p(f;r)}{\varphi(r)}.$$

(c) *If $\Delta(x,r) \subset \Delta(y,s)$ for some $x,y \in \partial\Omega$ and $r,s>0$, we have*

$$|f_{\Delta(x,r)} - f_{\Delta(y,s)}| \leqslant \frac{\sigma(\Delta(y,s))}{\sigma(\Delta(x,r))} \operatorname{osc}_p(f;s), \quad 1 \leqslant p < \infty.$$

If in addition $f \in \mathscr{E}^{\varphi,p}(\partial\Omega)$, we deduce

$$|f_{\Delta(x,r)} - f_{\Delta(y,s)}| \lesssim \varphi(s)\|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)}, \quad 0 < s \leqslant 2r.$$

(d) *For any $0<r<s$ and $x \in \partial\Omega$,*

$$|f_{\Delta(x,r)} - f_{\Delta(x,s)}| \lesssim \int_0^{4s} \operatorname{osc}_p(f;t)\frac{dt}{t}, \quad 1 \leqslant p < \infty.$$

If in addition $f \in \mathscr{E}^{\varphi,p}(\partial\Omega)$, we have

$$|f_{\Delta(x,r)} - f_{\Delta(x,s)}| \lesssim \varphi(s)\|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)}, \quad 1 \leqslant p < \infty.$$

Proof. Parts (a) and (b) are trivial. To show (c), we use Jensen's inequality:

$$\begin{aligned}|f_{\Delta(x,r)} - f_{\Delta(y,s)}| &\leqslant \fint_{\Delta(x,r)} |f - f_{\Delta(y,s)}| d\sigma = \frac{\sigma(\Delta(y,s))}{\sigma(\Delta(x,r))} \fint_{\Delta(y,s)} |f - f_{\Delta(y,s)}| d\sigma \\ &\leqslant \frac{\sigma(\Delta(y,s))}{\sigma(\Delta(x,r))} \left(\fint_{\Delta(y,s)} |f - f_{\Delta(y,s)}|^p d\sigma \right)^{\frac{1}{p}} \leqslant \frac{\sigma(\Delta(y,s))}{\sigma(\Delta(x,r))} \operatorname{osc}_p(f;s).\end{aligned}$$

And if $f \in \mathscr{E}^{\varphi,p}(\partial\Omega)$ and $0 < s \leqslant 2r$, noting that $\Delta(x,r) \subset \Delta(y,s) \subset \Delta(x,2s) \subset \Delta(x,4r)$, we use the doubling property of σ and (b) to get

$$|f_{\Delta(x,r)} - f_{\Delta(y,s)}| \leqslant \frac{\sigma(\Delta(x,4r))}{\sigma(\Delta(x,r))} \operatorname{osc}_p(f;s) \lesssim \varphi(s)\|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)}.$$

In turn, the first part of (d) follows by iterating of (c). Indeed, let $k \geqslant 0$ be such that $2^k r < s \leqslant 2^{k+1} r$. Denoting $\Delta_\tau := \Delta(x,\tau)$ for any $\tau > 0$, since σ is doubling and $\operatorname{osc}_p(f;\cdot)$ is non-decreasing from (a), a similar computation to the one in (3.68) leads to

$$\begin{aligned}
|f_{\Delta_r} - f_{\Delta_s}| &\leqslant \sum_{j=0}^{k} \left|f_{\Delta_{2^j r}} - f_{\Delta_{2^{j+1} r}}\right| + \left|f_{\Delta_{2^{k+1} r}} - f_{\Delta_s}\right| \\
&\leqslant \sum_{j=0}^{k} \frac{\sigma(\Delta_{2^{j+1} r})}{\sigma(\Delta_{2^j r})} \operatorname{osc}_p(f; 2^{j+1} r) + \frac{\sigma(\Delta_{2^{k+1} r})}{\sigma(\Delta_s)} \operatorname{osc}_p(f; 2^{k+1} r) \\
&\lesssim \sum_{j=0}^{k} \operatorname{osc}_p(f; 2^{j+1} r) \lesssim \int_{2r}^{2^{k+2} r} \operatorname{osc}_p(f;t) \frac{dt}{t} \leqslant \int_0^{4s} \operatorname{osc}_p(f;t) \frac{dt}{t}.
\end{aligned}$$

If in addition $f \in \mathscr{E}^{\varphi,p}(\partial\Omega)$, we further obtain, using (3.11) and the doubling property of φ:

$$|f_{\Delta_r} - f_{\Delta_s}| \leqslant \int_0^{4s} \varphi(t) \frac{dt}{t} \|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)} \lesssim \varphi(4s) \|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)} \lesssim \varphi(s) \|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)},$$

which completes the proof. □

The following proposition establishes the equivalences of the spaces we have been dealing with. This has been shown in [MS79, Theorem 4], but our proof is self-contained and simpler.

Proposition 3.59. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and assume that σ is a doubling measure. Let $\varphi \in \mathcal{G}_\beta$ for some $\beta \in (0,1)$. Then the spaces $\dot{C}^\varphi(\partial\Omega)$ and $\mathscr{E}^{\varphi,p}(\partial\Omega)$ are the same, with equivalent norms. (In the inclusion $\mathscr{E}^{\varphi,p}(\partial\Omega) \hookrightarrow \dot{C}^\varphi(\partial\Omega)$ we understand that for any $f \in \mathscr{E}^{\varphi,p}(\partial\Omega)$, we can modify f in a set of zero σ-measure so that $f \in \dot{C}^\varphi(\partial\Omega)$).*

Proof. The fact that $\|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)} \lesssim \|f\|_{\dot{C}^\varphi(\partial\Omega)}$ follows easily from the definitions, hence we only prove the converse inequality. First, Lemma 3.58 part (d) implies that for each fixed $x \in \partial\Omega$, the sequence $\{f_{\Delta(x,r)}\}_r$ is Cauchy for each fixed $x \in \partial\Omega$. Hence, the limit exists and we can define $g(x) := \lim_{r \to 0^+} f_{\Delta(x,r)}$. By Lebesgue Differentiation Theorem (which we may apply because σ is a doubling measure and $f \in L^1_{\mathrm{loc}}(\partial\Omega, d\sigma)$), we obtain that $f = g$ σ-a.e. in $\partial\Omega$.

Fix $x, y \in \partial\Omega$ and call $s := |x-y|$. By definition of g, we have

$$|g(x) - g(y)| = \lim_{r \to 0} |f_{\Delta(x,r)} - f_{\Delta(y,r)}|. \tag{3.81}$$

Now compute, for r small enough, say $2r < s$,

$$\begin{aligned}
|f_{\Delta(x,r)} - f_{\Delta(y,r)}| &\leqslant |f_{\Delta(x,r)} - f_{\Delta(x,s)}| + |f_{\Delta(x,s)} - f_{\Delta(y,2s)}| + |f_{\Delta(y,2s)} - f_{\Delta(y,r)}| \\
&\lesssim \varphi(2s) \|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)} \lesssim \varphi(s) \|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)},
\end{aligned} \tag{3.82}$$

where we have applied Lemma 3.58 part (d) to the first and last terms, and part (c) to the second one since $\Delta(x,s) \subset \Delta(y,2s)$ by definition of s. Plugging (3.82) into (3.81) yields $\|g\|_{\dot{C}^\varphi(\partial\Omega)} \lesssim \|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)}$, as desired. □

With all these tools in hand, we finally consider the Dirichlet problem in generalized Morrey-Campanato spaces.

Theorem 3.60. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with ADR boundary so that either* **Ω is bounded or Ω is unbounded with $\partial\Omega$ being unbounded**. *Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator, and let ω_L be the associated elliptic measure. Then there exists $\beta \in (0,1)$ (depending only on n, the ADR constant, and the ellipticity constant of L) such that for any $\theta > 1$ and for any growth function $\varphi \in \mathcal{G}_\beta$, the Morrey-Campanato Dirichlet problem*

$$\begin{cases} u \in W^{1,2}_{\mathrm{loc}}(\Omega), \\ Lu = 0 \text{ in the weak sense in } \Omega, \\ u \in C^{*,\varphi}(\Omega), \\ u|^{\text{nt. lim},\theta}_{\partial\Omega} = f \in \mathscr{E}^{\varphi,p}(\partial\Omega), \ \sigma\text{-a.e. on } \partial\Omega, \end{cases} \tag{3.83}$$

has a solution. Moreover, $u \in C(\overline{\Omega})$ satisfies $u|_{\partial\Omega} = f$ σ-a.e. in $\partial\Omega$, and there is a constant C (depending on n, the ADR constant, the ellipticity constant of L, and C_φ) such that

$$\|u\|_{C^{*,\varphi}(\Omega)} \leqslant C\|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)}.$$

Furthermore, if $\omega_L \ll \sigma$ (in the sense that $\omega_L^X \ll \sigma$ for all $X \in \Omega$), such a solution can be represented as

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_L^X(y), \qquad X \in \Omega. \tag{3.84}$$

Proof. Since $\partial\Omega$ is ADR, the surface measure σ is doubling. Therefore, Proposition 3.59 implies that there exists $g \in \dot{C}^\varphi(\partial\Omega)$ such that $f = g$ σ-a.e. in $\partial\Omega$, with $\|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)} \approx \|g\|_{\dot{C}^\varphi(\partial\Omega)}$. By Theorem 3.50, the function

$$\widetilde{u}(X) = \int_{\partial\Omega} g(y)\, d\omega_L^X(y), \qquad X \in \Omega$$

is also a solution to the problem (3.76). Since $f = g$ σ-a.e. in $\partial\Omega$, this means that $\widetilde{u}$ is a solution to the problem (3.83). Moreover, Theorem 3.50 also yields $u \in C(\overline{\Omega})$ with $u|_{\partial\Omega} = g$ in $\partial\Omega$, and

$$\|u\|_{C^{*,\varphi}(\Omega)} \lesssim \|g\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)} \approx \|f\|_{\mathscr{E}^{\varphi,p}(\partial\Omega)}$$

For the second part, assume that $\omega_L \ll \sigma$ (in the sense that $\omega_L^X \ll \sigma$ for all $X \in \Omega$). Since $f = g$ σ-a.e. on $\partial\Omega$, we have $f = g$ ω_L^X-a.e. on $\partial\Omega$ for all $X \in \Omega$. This easily implies that

$$\int_{\partial\Omega} f\, d\omega_L^X = \int_{\partial\Omega} g\, d\omega_L^X,$$

so $u(X)$, as defined by (3.84), coincides with $\widetilde{u}(X)$ for every $X \in \Omega$. This finishes the proof. □

Proposition 3.61. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with ADR boundary so that either* **Ω is bounded or Ω is unbounded with $\partial\Omega$ being unbounded**. *Let $L = -\operatorname{div}(A\nabla)$ be a real (non-necessarily symmetric) elliptic operator, and let ω_L be the associated elliptic measure. Given $\theta > 1$, assume that (3.77) holds. Then there exists $\beta \in (0,1)$ (depending only on n, the ADR constant, and the ellipticity constant of L) such that for any growth function $\varphi \in \mathcal{G}_\beta$, the problem (3.83) has a unique solution, and hence it is well-posed.*

Proof. As in the proof of Theorem 3.60, matters can be reduced to the proof of Proposition 3.53 by means of Proposition 3.59. In particular, note that the proof of Lemma 3.55 only uses the fact that the boundary datum vanishes σ-a.e. on $\partial\Omega$. Details are left to the reader. □

CHAPTER 4

The elliptic Dirichlet problem with singular boundary data

In this chapter, we will be interested in the elliptic Dirichlet boundary problem with singular boundary data, as opposed to data in spaces of classical smoothness as in Chapter 3. Concretely, we will consider boundary data in the scale of Lebesgue spaces L^p, which are the simplest in which functions may develop singularities and, at the same time, have a vast literature supporting their analytical properties, often related with PDE.

The first milestone in the theory of PDE with singular data is due to Dahlberg [Da77], who showed that the problem for harmonic functions with data in L^2 is well-posed if the boundary of the domain under consideration is Lipschitz.

As explained in the introduction, Dahlberg's result, although not optimal in the assumption that the boundary is Lipschitz, correctly identifies the necessity of the boundary to be somehow flat if we desire to obtain well-posedness in Lebesgue spaces. After many years of intense research, the minimal assumptions were recently found by Azzam, Hofmann, Martell, Mourgoglou and Tolsa in their very influential paper [AHMMT20]. Essentially, they show that one may relax the Lipschitz assumption used by Dahlberg to a weaker flatness condition, namely uniform rectifiability, as long as one assumes also some very mild connectivity (the so-called Weak Local John condition, which is automatically satisfied in Lipschitz domains). The precise result is as follows.

Theorem 4.1 (Azzam, Hofmann, Martell, Mourgoglou, Tolsa [AHMMT20])**.** *Let $\Omega \subset \mathbb{R}^{n+1}$, where $n \geqslant 1$, be an open set satisfying the corkscrew condition (see Section 2.3.3), and with ADR boundary (see Section 2.3.1). Then the Dirichlet problem for the Laplacian is well-posed in L^p for some $p < \infty$ (see [AHMMT20, (1.4)]) if and only if $\partial\Omega$ is Uniformly Rectifiable (see Section 2.3.1) and Ω satisfies the Weak Local John condition (see [AHMMT20, Definition 2.13]).*

Additionally, in [AHMMT20] (and the references therein) it is shown that any of these conditions, so concretely the L^p solvability, is equivalent to some other properties, including the so-called weak-A_∞ property of the harmonic measure (with respect to the harmonic measure). This is a generalization of the A_∞ condition explained in Section 2.2.1 that is better suited to non-doubling settings (the harmonic measure is not doubling in general open sets). This strong links between L^p solvability of the Dirichlet problem and quantitative absolute continuity/A_∞ conditions for the harmonic measure (see more e.g.

in [HL18]) will be ubiquitous within this chapter.

Although optimal in terms of geometrical considerations (indeed, it is a characterization), the work [AHMMT20] does not give us the complete picture because it only works for harmonic functions, i.e. when the operator is the Laplacian. It would be very useful to extend their result to more general classes of operators in order for it to be more applicable in other contexts.

However, unlike in Chapter 3, there is no hope to extend the result in [AHMMT20] to just any elliptic operator as in Definition 2.27: there are well-known counterexamples like those in [MM80, CFK81]. Nevertheless, there is a class of operators, introduced by Fefferman, Kenig and Pipher in [FKP91], for which one expects the result to be true. Moreover, in [FKP91, Section 4] the authors show that, under stronger background connectivity assumptions (definitely stronger than those in [AHMMT20] and later in this chapter) their class is optimal for the sake of characterizing the well-posedness of the Dirichlet problem in L^p, which makes it clear that it is the correct class in which we should look for generalizations of [AHMMT20].

In this chapter, we show some partial progress towards extending the result of [AHMMT20] to the operators in the class from [FKP91]. More concretely, we prove that if the L^p problem is well-posed for a subclass of those operators, then the boundary must be Uniformly Rectifiable. This will in fact be a consequence of a more abstract perturbation theory that we will develop, hanging on some new results in which we establish that different features of boundary behavior of solutions are in fact equivalent.

The novelty of the work resides on the generality in which we work: we will consider very general operators and, at the same time, we will not assume any connectivity in our domains. In this sense, we are establishing the most general perturbation theory known so far. For sure, this is at least the first perturbation theory carried out without any background connectivity assumption.

This chapter is based on the paper [CHM24], done in collaboration with Mingming Cao and José María Martell.

We suggest the reader to have Section 4.3 handy while reading the chapter, for it has the key ideas and provides a simplified and understandable sketch of the proofs that will be carried out in all detail along the rest of the chapter.

4.1 Further historical context . . . 105
4.2 Main results . . . 106
4.3 Main ideas and outline of the chapter . . . 110
4.4 Preliminaries . . . 111
4.4.1 Geometric framework and Corona decompositions . . . 111
4.4.2 Dyadic grids and sawtooths . . . 115
4.4.3 PDE framework . . . 120
4.4.4 Extension to unbounded boundaries . . . 122
4.5 Proof of the main equivalences . . . 124
4.5.1 Proof of (b) $\Longrightarrow$ (c): corona for ω_L implies corona for G_L . . . 124
4.5.2 Proof of (c) $\Longrightarrow$ (d): corona for G_L implies partial CME . . . 128
4.5.3 Proof of (d) $\Longrightarrow$ (a): partial CME implies corona for ω_L . . . 132
4.6 Proof of the abstract perturbation theory . . . 140
4.7 Applications in connection with Uniform Rectifiability . . . 144

4.7.1 Proof of Theorem 4.5 for the Laplacian 144
4.7.2 Proof of Theorem 4.5 for the operators in (a) 148
4.7.3 Proof of Theorem 4.5 for the operators in (b) 149
4.7.4 Proof of Theorem 4.5 for the operators in (c) 149
4.8 Proof showing that our results extend previous ones 149
4.8.1 Proof of Theorem 4.6, part (i) 150
4.8.2 Proof of Theorem 4.6, part (ii) 152
4.8.3 Proof of Theorem 4.6, part (iii) 156

4.1 Further historical context

Before getting into the specifics of the main results and their proofs, let us give some more concrete historical background, in addition to the last pages and Section 1.2.

The study of the relationship between the geometry of an open set and the absolute continuity properties of its harmonic measure (which are intimately tied with the solvability of the associated Dirichlet problem in L^p) has attracted considerable interest over the last century: the philosophy is that rectifiability along with suitable connectivity hypothesis is sufficient for absolute continuity of harmonic measure with respect to the surface measure, and that the rectifiability of the boundary is necessary. The first evidence comes from F. and M. Riesz [RR16], who showed that for a simply connected domain in the complex plane, rectifiability of the boundary implies that harmonic measure is absolutely continuous with respect to arclength measure on the boundary. The converse was established in [AHMM+16], where it was proved that, in any dimension and in the absence of any connectivity condition, every piece of the boundary with finite surface measure is rectifiable, provided surface measure is absolutely continuous with respect to harmonic measure on that piece.

On the other hand, some connectivity property seems to be necessary for absolute continuity to occur, since Bishop and Jones in [BJ90] constructed a uniformly rectifiable set E on the plane and some subset of F with zero arc-length which carries positive harmonic measure relative to the domain $\mathbb{R}^2\backslash E$. Considering this point, a big effort has been recently made to understand in what open sets $\Omega \subset \mathbb{R}^{n+1}$ and for what elliptic operators L the elliptic measure ω_L is quantitatively absolutely continuous with respect to surface measure $\sigma := \mathcal{H}^n|_{\partial\Omega}$ on the boundary, which is formulated as $\omega_L \in A_\infty(\sigma)$ (cf. Section 4.8). One context where this theory has been satisfactorily developed is that of 1-sided chord-arc domains, that is, open sets which satisfy the interior corkscrew and Harnack chain conditions (quantitative openness and connectedness respectively, as explained in Section 2.3.3) with an Ahlfors-David regular boundary. Under such background hypothesis, for the Laplacian, [DJ90, HM14, HMU14, AHMNT17] give a characterization of $\omega \in A_\infty(\sigma)$ in terms of uniform rectifiability of the boundary (a quantitative version of rectifiability, as seen in Section 2.3.1), which is also equivalent to the fact that Ω satisfies an exterior corkscrew condition and hence Ω is a chord-arc domain. Recently, in [HMMTZ21] (see also [HMT17] and [KP01]), these characterizations have been extended to the so-called Kenig-Pipher operators, that is, elliptic operators with variable coefficients whose gradient satisfies an L^2-Carleson measure condition. On the other hand, in the setting of 1-sided chord-arc domains it has been established that for general elliptic operators one can characterize $\omega_L \in A_\infty(\sigma)$ in terms of the fact that all bounded null solutions satisfy Carleson measure estimates (cf. [CHMT20]). Additionally, the papers [CHM19, CHMT20] extend the fundamental work of [FKP91] to the setting of 1-sided chord-arc domains and establish that the A_∞ property is stable under Carleson perturbations. Finally, the 1-codimensional case of [DM22]

establishes that uniform rectifiability of the boundary is equivalent to the fact that the associated Green functions for some class of elliptic operators are well approximated by affine functions.

However, in the absence of the Harnack chain condition, the quantitative absolute continuity of the harmonic or elliptic measure becomes delicate as can be seen from the Bishop-Jones counterexample. In [HLMN17] (see also [HM15, MT20]) it was shown that in an open set with Ahlfors-David regular boundary, if the associated harmonic measure satisfies the weak-A_∞ property (a quantitative version of the absolute continuity) then the boundary is uniformly rectifiable. This latter property is in turn equivalent to the fact that all bounded harmonic functions satisfy Carleson measure estimates or are ε-approximable (see [HMM16, GMT18] and also [AGMT23] for a class of elliptic operators with variable coefficients). Thus, even without strong connectivity assumptions (like in the Bishop-Jones counterexample), some estimates on harmonic functions —which in more topologically benign environments are equivalent to the A_∞ property of harmonic measure— remain valid.

The geometrical characterization of the weak-A_∞ property of harmonic measure, and hence the solvability of the L^p-Dirichlet problem for finite p, see for instance [HL18], was obtained in [AHMMT20], as explained before in this chapter: assuming that an open set Ω satisfies natural (and optimal in a certain sense) background conditions (more precisely, interior corkscrew condition and Ahlfors-David regularity of the boundary), it is shown that harmonic measure belongs to weak-$A_\infty(\sigma)$ if and only if $\partial\Omega$ is uniformly rectifiable and Ω satisfies the weak local John condition, a condition that guarantees local non-tangential access to an ample portion of the boundary. Additionally, any of these properties is equivalent to the fact that Ω has "interior big pieces of chord-arc domains". In this regard we also mention Azzam [Az21], who obtained a similar characterization for the stronger A_∞ condition.

4.2 Main results

The main goal in this chapter is to consider open sets with no connectivity assumption, and at the same time general elliptic operators. As explained above, the weak-A_∞ property (which is strongly tied with the L^p solvability of the Dirichlet problem, as explained at the beginning of this chapter) is not expected to hold in this very general framework. Our main objetive is to relax the weak-A_∞ condition so that we find a new condition with the following features:

1. First, it is equivalent to the A_∞ property in better settings.
2. Second, it gives Carleson measure type estimates for bounded solutions.
3. Third, in the case of the Laplacian or Kenig-Pipher operators, it allows one to characterize the uniform rectifiability of the boundary.
4. Last but not least, it is stable under Fefferman-Kenig-Pipher Carleson perturbations, that is, if the disagreement of two matrices satisfies a Fefferman-Kenig-Pipher Carleson measure condition, then the desired condition should be transferable from one operator to the other.

The condition that we consider here is a corona decomposition adapted to the elliptic measure that appeared implicitly in [HLMN17], playing a fundamental role in the proof of its main result, and was formalized in [GMT18] (see also [AGMT23]) in a somehow weaker form. This corona decomposition (see Definition 4.7) is a partition of the collection of the dyadic cubes associated with the Ahlfors-David regular boundary (see Section 2.3.2) in good cubes and bad cubes, where the good cubes are organized in trees, and the collection of bad cubes and the tops of the trees satisfy a Carleson packing condition.

Moreover, in each fixed tree, the averages of the elliptic measure, with some fixed pole at a scale above the top and normalized appropriately, is comparable to a constant (see Definition 4.10 below). We show that any of the two versions of this corona decomposition is equivalent to the fact that the Green function is comparable to the distance to the boundary (with an appropriate normalization) in the corona sense (see Definition 4.10) or equivalent to partial/weak Carleson measure estimates for bounded solutions, that is, Carleson measure estimates that only take into account one of the components of the associated Whitney region (see Definition 4.15). The precise result is as follows:

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with Ahlfors-David regular boundary satisfying the corkscrew condition, and let $Lu = -\operatorname{div}(A\nabla u)$ be a real (not necessarily symmetric) uniformly elliptic operator. Write ω_L and G_L for the associated elliptic measure and Green function, respectively. Then the following statements are equivalent:*

(a) *ω_L admits a strong corona decomposition (cf. Definition 4.10).*

(b) *ω_L admits a corona decomposition (cf. Definition 4.10).*

(c) *G_L is comparable to the distance to the boundary in the corona sense (cf. Definition 4.10).*

(d) *L satisfies partial/weak Carleson measure estimates (cf. Definition 4.15).*

Theorem 4.2 can be applied to consider Fefferman-Kenig-Pipher perturbations of elliptic operators extending [FKP91, MPT13, CHMT20] to sets without any connectivity assumption. The result that we state next is, as far as we know, the first Fefferman-Kenig-Pipher perturbation in open sets which does not assume the Harnack chain condition:

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with Ahlfors-David regular boundary satisfying the corkscrew condition. Let $L_0u = -\operatorname{div}(A_0\nabla u)$ and $L_1u = -\operatorname{div}(A_1\nabla u)$ be real (not necessarily symmetric) uniformly elliptic operators so that L_1 is a Fefferman-Kenig-Pipher perturbation of L_0, that is, the following Carleson measure estimate holds:*

$$\sup_{\substack{x\in\partial\Omega \\ 0<r<\operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r)\cap\partial\Omega)} \iint_{B(x,r)\cap\Omega} \frac{\varrho(A_0,A_1)(X)^2}{\delta(X)}dX < \infty, \tag{4.1}$$

where the disagreement between A_0 and A_1 in Ω is given by

$$\varrho(A_0,A_1)(X) := \sup_{Y\in B(X,\delta(X)/2)} |A_0(Y) - A_1(Y)|, \quad X \in \Omega.$$

Then the following statements are equivalent:

(a) *ω_{L_0} admits a (strong) corona decomposition (equivalently, G_{L_0} is comparable to the distance to the boundary in the corona sense or L_0 satisfies partial/weak Carleson measure estimates).*

(b) *ω_{L_1} admits a (strong) corona decomposition (equivalently, G_{L_1} is comparable to the distance to the boundary in the corona sense or L_1 satisfies partial/weak Carleson measure estimates).*

Much as in [CHMT20] we can see that any of the equivalent conditions in Theorem 4.2 can be transferred from L to its transpose $L^\top$ or to its symmetric part L^{sym} when the matrix of coefficients are regular and some Carleson measure condition is assumed on its antisymmetric part:

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with Ahlfors-David regular boundary satisfying the corkscrew condition. Let $Lu = -\operatorname{div}(A\nabla u)$ be a real (not necessarily symmetric) uniformly elliptic operator, let $L^\top$ denote the transpose of L, and let $L^{\mathrm{sym}} = \frac{L+L^\top}{2}$ be the symmetric part of L. Assume that $(A - A^\top) \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ and that the following Carleson measure estimate holds*

$$\sup_{\substack{x\in\partial\Omega \\ 0<r<\operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r)\cap\partial\Omega)} \iint_{B(x,r)\cap\Omega} |\operatorname{div}_C(A-A^\top)(X)|^2 \delta(X)\,dX < \infty, \tag{4.2}$$

where

$$\operatorname{div}_C(A-A^\top)(X) = \left(\sum_{i=1}^{n+1} \partial_i(a_{i,j}-a_{j,i})(X)\right)_{1\leqslant j\leqslant n+1}, \quad X\in\Omega.$$

Then the following statements are equivalent:

(a) *ω_L admits a (strong) corona decomposition (equivalently, G_L is comparable to the distance to the boundary in the corona sense or L satisfies partial/weak Carleson measure estimates).*

(b) *$\omega_{L^\top}$ admits a (strong) corona decomposition (equivalently, $G_{L^\top}$ is comparable to the distance to the boundary in the corona sense or $L^\top$ satisfies partial/weak Carleson measure estimates).*

(c) *$\omega_{L^{\mathrm{sym}}}$ admits a (strong) corona decomposition (equivalently, $G_{L^{\mathrm{sym}}}$ is comparable to the distance to the boundary in the corona sense or L^{sym} satisfies partial/weak Carleson measure estimates).*

Back to the Laplacian, by means of Theorem 4.2, we can characterize the uniform rectifiability of the boundary of an open set with an Ahlfors-David regular boundary in terms of any of the preceding equivalent properties. In the case of (full) Carleson measure estimates, [HMM16, Theorem 1.1] shows that any bounded harmonic function in $\mathbb{R}^{n+1}\backslash E$, with E uniformly rectifiable, satisfies (full) Carleson measure estimates. As a consequence, if $\Omega \subset \mathbb{R}^{n+1}$ with $\partial\Omega$ is uniformly rectifiable then all bounded harmonic functions in Ω satisfy (full) Carleson measure estimates. The reciprocal is obtained in [GMT18, Theorem 1.3] and the proof uses that uniform rectifiability is equivalent to the boundedness of the Riesz transform in L^2 (cf. [NTV14]). Here we show that with the help of Theorem 4.2 the argument in [HLMN17] can be easily adapted to obtain such a characterization. We would like to highlight that as a consequence of our result, we do not need full Carleson measure estimates, that is, partial/weak Carleson measure estimates suffice. Moreover, using some integration by parts argument from [HMT17] and exploiting the fact that for "good" coefficients we can invoke Theorem 4.4, one can reduce matters to symmetric matrices and characterize the uniform rectifiability of the boundary of an open set with an Ahlfors-David regular boundary in terms of any of the preceding equivalent properties for a natural class of elliptic operators with variable coefficients. We would like to emphasize that our result extends [HMT17], where the Harnack chain condition was assumed, and, more notably, improves [AGMT23], where the authors need a control of the oscillation of the matrix in points that might be in different connected components (in our case the oscillations are always in a connected component of the set).

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with Ahlfors-David regular boundary satisfying the corkscrew condition, and let ω be the associated harmonic measure. Then $\partial\Omega$ is uniformly rectifiable if and only if ω admits a (strong) corona decomposition. Moreover any of the previous conditions is equivalent to the fact that G, the associated Green function, is comparable to the distance to the boundary in*

the corona sense and/or all bounded harmonic functions satisfy partial/weak (or full) Carleson measure estimates.

Furthermore, the same equivalences hold for real (not necessarily symmetric) uniformly elliptic operators $Lu = -\operatorname{div}(A\nabla u)$ *with* A *satisfying one of the following conditions:*

(a) $A \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, $\big\| |\nabla A|\,\delta(\cdot)\big\|_{L^\infty(\Omega)} < \infty$ *where* $\delta(\cdot) = \operatorname{dist}(\cdot, \partial\Omega)$, *and*

$$\sup_{\substack{x\in\partial\Omega\\ 0<r<\operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r)\cap\partial\Omega)} \iint_{B(x,r)\cap\Omega} |\nabla A(X)|\,dX < \infty. \tag{4.3}$$

(b) *A satisfies*

$$\sup_{\substack{x\in\partial\Omega\\ 0<r<\operatorname{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r)\cap\partial\Omega)} \iint_{B(x,r)\cap\Omega} \frac{\operatorname{osc}(A,X)}{\delta(X)}\,dX < \infty, \tag{4.4}$$

where $\operatorname{osc}(A,X) := \sup_{Y,Z\in B(X,\delta(X)/2)} |A(Y)-A(Z)|$, *for* $X \in \Omega$.

(c) *A is a Fefferman-Kenig-Pipher perturbation (c.f. (4.1)) of the Laplacian or more generally any of the operators in (a) or (b).*

Let us mention that in the previous result the corkscrew condition cannot be removed, see Remark 4.32 below.

Our last result studies the relationship between the corona decomposition associated with ω_L and the A_∞ or weak-A_∞ properties. Under strong connectivity, that is, in 1-sided chord-arc domains (see Section 4.8), these notions turn out to be equivalent and, in view of [CHMT20, Theorem 1.1], we conclude that full and partial/weak Carleson measure estimates are equivalent properties.

Theorem 4.6. *Let* $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, *be an open set with Ahlfors-David regular boundary satisfying the corkscrew condition and let* $Lu = -\operatorname{div}(A\nabla u)$ *be a real (not necessarily symmetric) uniformly elliptic operator.*

(i) *If* $\omega_L \in A_\infty^{\mathrm{weak}}(\sigma)$ *(cf. Section 4.8) then* ω_L *admits a (strong) corona decomposition. As a result (c) and (d) in Theorem 4.2 hold.*

(ii) *If* Ω *is a* 1*-sided CAD (cf. Section 4.8), then following are equivalent:*

  (a) $\omega_L \in A_\infty(\sigma)$ *(cf. Section 4.8).*

  (b) ω_L *admits a (strong) corona decomposition*

  (c) G_L *is comparable to the distance to the boundary in the corona sense.*

  (d) L *satisfies partial/weak Carleson measure estimates.*

  (e) L *satisfies full Carleson measure estimates (cf. Definition 4.15).*

(iii) *Let* Ω *be a* 1*-sided CAD and let* L *be the Laplacian; or a Kenig-Pipher operator, that is, an operator as in Theorem 4.5 part (a) but where (4.3) is relaxed by replacing* $|\nabla A|$ *by* $|\nabla A|^2\delta(\cdot)$;

or an operator as in Theorem 4.5 part (b) but where (4.4) is relaxed by replacing $\operatorname{osc}(A,\cdot)$ *by* $\operatorname{osc}(A,\cdot)^2$*; or a Fefferman-Kenig-Pipher perturbation of one of the previous operators. Then any of the conditions (a)–(e) is equivalent to the fact that* Ω *is a CAD or* $\partial\Omega$ *is uniformly rectifiable.*

We would like to observe that in (iii) above we show that uniform rectifiability of the boundary can be expressed in terms of the fact that the Green function is comparable to the distance to the boundary. This result should be compared with [DM22] where the authors compare the Green function with the distance to an n-dimensional hyperplane.

4.3 Main ideas and outline of the chapter

In Section 4.4, we recall and present some preliminaries, definitions, and some background results that will be used throughout the chapter. Concretely, we will introduce corona decompositions, which are a central tool in the chapter: they involve a very technical construction, but morally speaking they only explain that certain property is *prevalent* in Ω, meaning that it happens at most (dyadic) scales. However, it is of great relevance for the results exposed here the fact that in different regimes $\mathbf{S} \in \mathbb{F}$ of a corona decomposition, the pole (of the elliptic measure or the Green's function) $X_{\mathbf{S}}$ is adapted to the regime: this reflects the absence of connectivity, in the sense that parts of Ω that are well connected with $X_{\mathbf{S}}$ may be inaccessible from $X_{\mathbf{S}'}$. This is also the motivation behind or definition of partial/weak Carleson measure estimates.

Later, Section 4.5 is devoted to proving our main result, Theorem 4.2.

- In Subsection 4.5.1, the most subtle part is the lower bound (Step 1) to find a point where the Green's function is non-degenerate (the upper bound is an easier consequence of the CFMS estimate presented in Section 4.4.3). Actually, finding such point is not difficult by (4.18), but one needs an argument later to show that such point is not very close to the boundary: for that, the key is to hide some quantities using the fact that the elliptic measure behaves like the surface measure (hypothesis (b)), and the latter satisfies the ADR condition.
- In Subsection 4.5.2, we carry out a well-established integration by parts scheme, in which a "sawtooth region" is created by joining appropriate Whitney cubes, in such a way that the "boundary" of the region is somehow small/of smaller dimension (as seen in Lemma 4.20). The most relevant novelty is that the careful definition of partial/weak Carleson measure estimates allows us to exchange the distance to the boundary by the Green's function (Step 1). Having the Green's function in our quantities to estimate, we can just carry out the integration by parts using the fact that the Green's function solves a PDE, and always taking care that the poles are far enough.
- In Subsection 4.5.3, we run a stopping-time argument to find regions where the elliptic measure behaves like the surface measure (precisely the condition (b)). The difficulty resides in proving a packing condition for the "bad cubes", which essentially informs us that the "bad" scales at which the elliptic measure does not behave as expected, are few. One also needs to take into account that the lack of connectivity may result on the elliptic measure behaving poorly simply because the regions become inaccessible from previous poles. The cubes in F_Q^+ (see Proposition 4.27) are easy to pack via a geometric series. In turn, the difficult cubes to be packed are those in F_Q^-, which are seen as very small by the elliptic measure: this allows (as in [GMT18]) to construct oscillating solutions (i.e. with large gradients) somehow supported on those "small" sets, via Lemma 4.23, but at the same time the Carleson measure estimates (our hypothesis (d)) prevent large gradients

to exist at many scales, which ultimately implies that the bad cubes cannot be very prevalent. The argument gets fairly technical because one stratifies bad cubes in two steps: one first finds a corona decomposition only with F^+ cubes, and afterwards, inside each of the regimes of that corona decomposition, one finds another corona decomposition with only F^- cubes. Doing this is important, because the packing argument for the F^- cubes ceases to work as soon as we find some F^+ cube in between (this breaks the disjointness of $\{Q\backslash F_Q^-\}$ in Step 3.2, and ultimately creates undesired interactions between the different oscillating functions that we constructed before).

Next, in Section 4.6 we prove Theorems 4.3 and 4.4. These results turn out to be particular cases of a more general result, Theorem 4.30. The proof is similar in spirit to that in Subsection 4.5.2 because it follows a similar integration by parts using different dyadic constructions and Green's functions. The difference is that there are two operators now, and therefore one has to deal with extra terms taking into account the discrepancy between the operators that we are comparing (Step 4 in the proof).

The proof of Theorem 4.5 is given in Section 4.7, where the argument is just sketched because our results allow us to reduce matters to an argument essentially contained in [HLMN17]. Basically, one shows the so-called WHSA condition, which is equivalent to UR, but more convenient to use because it basically reduces to showing some packing conditions for the scales at which tangent planes change too much. Since one can define (approximate) tangent planes as those perpendicular to the gradient of the Green's function, because Green's functions are (often) linear near the boundary, the point is to reduce the packing condition for the bad WHSA cubes to a packing condition for scales where the Green's function does not behave in the desired linear way, which are essentially those where it exhibits some non-trivial oscillation. If we note that these oscillations will contribute to suitable square functions like that in our definition of Carleson measure estimates, we end up reducing the WHSA condition to the objects studied in Theorem 4.2.

Finally, in Section 4.8 we obtain Theorem 4.6. For part (ii) the open set satisfies the Harnack chain condition, so the connectivity is much better: this implies the existence of very powerful tools like Harnack's inequalities, and concretely "the change of pole formula" is at our disposal, which largely simplifies the arguments. Moreover, in this framework with connectivity, the powerful machinery of extrapolation of Carleson measures is also available to simplify all our dyadic arguments, because the elliptic measure is doubling (which is not true in settings without connectivity, as the ones considered in the previous sections): concretely, this allows us to have very ample contact sets of our regimes at the boundary.

4.4 Preliminaries

We suggest the reader to revise the Section 2.1 for the notation that will be used in this chapter. Concretely, we recall the reader of the specific notations

$$\delta(X) := \operatorname{dist}(X, \partial\Omega) \quad \text{for } X \in \mathbb{R}^{n+1}, \qquad \text{and} \qquad \sigma := \mathcal{H}^n|_{\partial\Omega},$$

and that surface balls are denoted $\Delta(x,r) = B(x,r) \cap \partial\Omega$ for $x \in \partial\Omega$ and $r > 0$.

4.4.1 Geometric framework and Corona decompositions

At this point, we advice the reader to revisit Section 2.3.1 to review the definitions of ADR and Uniformly Rectifiable boundaries, and also Section 2.3.3 for the definition of corkscrews. All of them will be ubiquitous within this chapter.

Apart from these basic definitions, a central concept along this chapter are corona decompositions. These are dyadic constructions (recall Section 2.3.2) which help us deal with properties which happen very frequently or rarely at different scales around the boundary. Let us give the precise definitions to have a rigorous idea of what we are handling. First of all we proceed to define what we understand by the different corona decompositions present in Theorems 4.2 and 4.6.

Definition 4.7 (General corona decomposition)**.** Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with Ahlfors-David regular boundary and let $\mathbb{D} := \mathbb{D}(\partial\Omega)$ denote a dyadic grid on $\partial\Omega$ (cf. Section 4.4.2).

- A subcollection $\mathbf{S} \subset \mathbb{D}$ is semi-coherent if the following properties hold:

  (a) $\mathbf{S}$ contains a unique maximal element, denoted by $\mathrm{Top}(\mathbf{S})$, which contains all other elements of $\mathbf{S}$ as subsets, that is, if $Q \in \mathbf{S}$ then $Q \subset \mathrm{Top}(\mathbf{S})$.

  (b) If $Q \in \mathbf{S}$ and $Q \subset Q' \subset \mathrm{Top}(\mathbf{S})$, then $Q' \in \mathbf{S}$.

- A subcollection $\mathbf{S} \subset \mathbb{D}$ is coherent if it is semi-coherent and additionally satisfies that for every $Q \in \mathbf{S}$ either all of its children belong to $\mathbf{S}$, or else none of them do.

- A semi-coherent (resp. coherent) corona decomposition with constant $M_0 \geqslant 1$ is a triple $(\mathbb{B}, \mathbb{G}, \mathbb{F})$, where $\mathbb{B}$ and $\mathbb{G}$ are two subsets of $\mathbb{D}$ (the "bad cubes" and the "good cubes") and $\mathbb{F}$ is a family of subsets of $\mathbb{G}$ which satisfy the following conditions:

  (a) $\mathbb{D} = \mathbb{G} \bigsqcup \mathbb{B}$.

  (b) $\mathbb{G} = \bigsqcup_{\mathbf{S}\in\mathbb{F}} \mathbf{S}$, where each $\mathbf{S}$ is semi-coherent (resp. coherent).

  (c) The collection $\mathbb{B}$ and the family of top cubes $\mathrm{Top}(\mathbb{F}) := \{\mathrm{Top}(\mathbf{S}) : \mathbf{S} \in \mathbb{F}\}$ satisfy a Carleson packing condition:

$$\sup_{Q\in\mathbb{D}} \Big(\frac{1}{\sigma(Q)} \sum_{Q'\in\mathbb{B}:Q'\subset Q} \sigma(Q') + \frac{1}{\sigma(Q)} \sum_{Q'\in\mathrm{Top}(\mathbb{F}):Q'\subset Q} \sigma(Q') \Big) \leqslant M_0.$$

Let $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$ be a semi-coherent corona decomposition with constant $M_0 \geqslant 1$. As observed in [DS93, pp. 56–57], there exists a different partition $\mathbb{F}_1$ of $\mathbb{G}_0$ such that $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_1)$ is a coherent corona decomposition with constant $M_1 \geqslant 1$ depending on n, Ahlfors-David regularity, and M_0; and, additionally, for every $\mathbf{S} \in \mathbb{F}_1$, there exists $\mathbf{S}' \in \mathbb{F}_0$ such that $\mathbf{S} \subset \mathbf{S}'$.

Definition 4.8 (2^N-close cubes)**.** Given $N \geqslant 0$ we say that two cubes $Q_1, Q_2 \in \mathbb{D}$ are 2^N-close if

$$2^{-N}\,\ell(Q_1) \leqslant \ell(Q_2) \leqslant 2^N\,\ell(Q_1), \qquad \mathrm{dist}(Q_1, Q_2) \leqslant 2^N\,(\ell(Q_1) + \ell(Q_2)).$$

With this definition in mind, for every $N \geqslant 0$ we can invoke [DS93, Lemma 3.26] to find a new coherent corona decomposition $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ (depending on N) so that for every $\mathbf{S} \in \mathbb{F}$ there exists $\mathbf{S}' \in \mathbb{F}_1$ such that

$$\mathbf{S} \subset \bigcup_{Q\in\mathbf{S}} \big\{Q' \in \mathbb{D} : Q \text{ and } Q' \text{ are } 2^N\text{-close}\big\} \subset \mathbf{S}'.$$

Moreover, the associated constant for the new corona decomposition depends on n, Ahlfors-David regularity, N, and M_1. Putting everything together one has the following:

Lemma 4.9. *Let $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$ be a semi-coherent corona decomposition with constant $M_0 \geqslant 1$. For every $N \geqslant 0$, there exists a coherent corona decomposition $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ (depending on N), called the 2^N-refinement of $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$, such that the associated constant depends on n, Ahlfors-David regularity, N, and M_0; and, moreover, for every $\mathbf{S} \in \mathbb{F}$ there exists $\mathbf{S}' \in \mathbb{F}_0$ such that*

$$\mathbf{S} \subset \mathbf{S}(N) := \bigcup_{Q \in \mathbf{S}} \{Q' \in \mathbb{D} : Q \text{ and } Q' \text{ are } 2^N\text{-close}\} \subset \mathbf{S}'.$$

Definition 4.10 (Specific corona decompositions)**.** Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with Ahlfors-David regular boundary satisfying the corkscrew condition, and let $Lu = -\operatorname{div}(A\nabla u)$ be a real (non-necessarily symmetric) uniformly elliptic operator. Denote by ω_L and G_L the associated elliptic measure and the Green function respectively. Let $\mathbb{D} := \mathbb{D}(\partial\Omega)$ denote a dyadic grid on $\partial\Omega$.

(i) We say that ω_L admits a (coherent/semi-coherent) strong corona decomposition if there exists a (coherent/semi-coherent) corona decomposition $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ such that for each $\mathbf{S} \in \mathbb{F}$ there exist $Q_{\mathbf{S}} \in \mathbb{D}$ and $X_{\mathbf{S}} \in \Omega$ with

$$\operatorname{Top}(\mathbf{S}) \subset Q_{\mathbf{S}}, \qquad \delta(X_{\mathbf{S}}) \approx \ell(Q_{\mathbf{S}}) \approx \operatorname{dist}(X_{\mathbf{S}}, Q_{\mathbf{S}}), \tag{4.5}$$

and

$$\frac{\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})}{\sigma(Q_{\mathbf{S}})} \lesssim \frac{\omega_L^{X_{\mathbf{S}}}(Q)}{\sigma(Q)} \lesssim \left(\fint_Q (\mathcal{M}\omega_L^{X_{\mathbf{S}}})^{\frac{1}{2}} d\sigma \right)^2 \lesssim \frac{\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})}{\sigma(Q_{\mathbf{S}})}, \quad \forall Q \in \mathbf{S}. \tag{4.6}$$

(ii) We say that ω_L admits a (coherent/semi-coherent) corona decomposition if there exists a (coherent/semi-coherent) corona decomposition $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ such that for each $\mathbf{S} \in \mathbb{F}$ there exist $Q_{\mathbf{S}} \in \mathbb{D}$ and $X_{\mathbf{S}} \in \Omega$ with

$$\operatorname{Top}(\mathbf{S}) \subset Q_{\mathbf{S}}, \qquad \delta(X_{\mathbf{S}}) \approx \ell(Q_{\mathbf{S}}) \approx \operatorname{dist}(X_{\mathbf{S}}, Q_{\mathbf{S}}), \tag{4.7}$$

and

$$\frac{\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})}{\sigma(Q_{\mathbf{S}})} \lesssim \frac{\omega_L^{X_{\mathbf{S}}}(Q)}{\sigma(Q)} \lesssim \frac{\omega_L^{X_{\mathbf{S}}}(2\widetilde{\Delta}_Q)}{\sigma(2\widetilde{\Delta}_Q)} \lesssim \frac{\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})}{\sigma(Q_{\mathbf{S}})}, \quad \forall Q \in \mathbf{S} \tag{4.8}$$

(see (4.10) for the definition of $\widetilde{\Delta}_Q$).

(iii) We say that G_L is comparable to the distance to the boundary in the corona sense if there exists a (coherent/semi-coherent) corona decomposition $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ such that for each $\mathbf{S} \in \mathbb{F}$ there exist $\Lambda_{\mathbf{S}} > 0$, $Q_{\mathbf{S}} \in \mathbb{D}$, and $X_{\mathbf{S}} \in \Omega$ with

$$\operatorname{Top}(\mathbf{S}) \subset Q_{\mathbf{S}}, \qquad \delta(X_{\mathbf{S}}) \geqslant 4\Xi\ell(Q_{\mathbf{S}}), \qquad \operatorname{dist}(X_{\mathbf{S}}, Q_{\mathbf{S}}) \lesssim \ell(Q_{\mathbf{S}}),$$

and

$$\sup_{\substack{X \in 2\widetilde{B}_Q \cap \Omega \\ \delta(X) \geqslant c\,\ell(Q)}} \frac{G_L(X_{\mathbf{S}}, X)}{\delta(X)} \approx \Lambda_{\mathbf{S}}, \quad \forall Q \in \mathbf{S},$$

for some $c \in (0, \frac{1}{2})$ (see (4.10) for the meaning of $\widetilde{B}_Q$ and the parameter Ξ).

In the previous conditions it is understood that the implicit constants are all uniform, that is, the same constant in each estimate is valid for all $Q \in \mathbf{S}$ and for all $\mathbf{S} \in \mathbb{F}$.

Remark 4.11. In the previous definitions, we may always assume that the corona decompositions are formed by coherent subregimes. As a matter of fact, if $(\mathbb{B},\mathbb{G},\mathbb{F})$ is a semi-coherent corona decomposition with constant M_0 as in (i) (the cases (ii) and (iii) are treated identically) we can invoke [DS93, pp. 56–57], to see that there is a different partition $\mathbb{F}'$ of $\mathbb{G}$ such that $(\mathbb{B},\mathbb{G},\mathbb{F}')$ is a coherent corona decomposition with constant $M_1 \geqslant 1$ depending on n, Ahlfors-David regularity, and M_0; and, additionally, for every $\mathbf{S}\in\mathbb{F}'$, there exists $\widehat{\mathbf{S}}\in\mathbb{F}$ such that $\mathbf{S}\subset\widehat{\mathbf{S}}$. Then, given $\mathbf{S}\in\mathbb{F}'$, we take $\widehat{\mathbf{S}}\in\mathbb{F}$ such that $\mathbf{S}\subset\widehat{\mathbf{S}}$. Set $Q_{\mathbf{S}} := Q_{\widehat{\mathbf{S}}}$ and $X_{\mathbf{S}} := X_{\widehat{\mathbf{S}}}$. Since $\mathbf{S}\subset\widehat{\mathbf{S}}$ we have that $\mathrm{Top}(\mathbf{S})\subset\mathrm{Top}(\widehat{\mathbf{S}})\subset Q_{\widehat{\mathbf{S}}} = Q_{\mathbf{S}}$ and the other two conditions in (4.5) follow automatically by construction. The same occurs with (4.6), which holds for the cubes in $\widehat{\mathbf{S}}$, thus for those in $\mathbf{S}$. This shows that ω_L admits a coherent strong corona decomposition.

The same can be done with (ii) and (iii) (details are left to the interested reader), hence from now on in the previous definitions we will drop the adjective "coherent" or "semi-coherent", with the understanding that, if needed, the corona decomposition we start with can be formed by coherent subregimes.

Remark 4.12. One can also refine the corona decompositions in the previous definitions so that the required conditions not only hold for the cubes in the good sub-regimes but also in all the nearby cubes. More precisely, let $(\mathbb{B}_0,\mathbb{G}_0,\mathbb{F}_0)$ be a corona decomposition with constant M_0 as in (ii) (the cases (i) and (iii) are treated identically). Taking into account Lemma 4.9 for every $N\geqslant 0$ we can then find $(\mathbb{B},\mathbb{G},\mathbb{F})$, the 2^N-refinement of $(\mathbb{B}_0,\mathbb{G}_0,\mathbb{F}_0)$, which is a coherent corona decomposition with associated constant depending on n, Ahlfors-David regularity, M_0, and N. Moreover, for every $\mathbf{S}\in\mathbb{F}$ there exists $\mathbf{S}'\in\mathbb{F}_0$ such that $\mathbf{S}\subset\mathbf{S}(N)\subset\mathbf{S}'$. Set $Q_{\mathbf{S}} := Q_{\mathbf{S}'}$ and $X_{\mathbf{S}} := X_{\mathbf{S}'}$. Write $Q^{(N)}$ for the N-th dyadic ancestor of Q (that is the unique dyadic cube containing Q and with sidelenth $2^N\ell(Q)$). It is clear that $\mathrm{Top}(\mathbf{S})^{(N)}$ and $\mathrm{Top}(\mathbf{S})$ are 2^N-close. Hence, $\mathrm{Top}(\mathbf{S})^{(N)}\in\mathbf{S}(N)\subset\mathbf{S}'$ and $\mathrm{Top}(\mathbf{S})\subset\mathrm{Top}(\mathbf{S})^{(N)}\subset\mathrm{Top}(\mathbf{S}')\subset Q_{\mathbf{S}'} = Q_{\mathbf{S}}$. The other two conditions in (4.7) follow by construction and we have

$$\mathrm{Top}(\mathbf{S})\subset\mathrm{Top}(\mathbf{S})^{(N)}\subset Q_{\mathbf{S}},\qquad \delta(X_{\mathbf{S}})\approx\ell(Q_{\mathbf{S}})\approx\mathrm{dist}(X_{\mathbf{S}},Q_{\mathbf{S}}),$$

Finally, since $\mathbf{S}(N)\subset\mathbf{S}'$ and by construction $Q_{\mathbf{S}} = Q_{\mathbf{S}'}$ and $X_{\mathbf{S}} = X_{\mathbf{S}'}$, we have

$$\frac{\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})}{\sigma(Q_{\mathbf{S}})}\lesssim\frac{\omega_L^{X_{\mathbf{S}}}(Q)}{\sigma(Q)}\lesssim\frac{\omega_L^{X_{\mathbf{S}}}(2\widetilde{\Delta}_Q)}{\sigma(2\widetilde{\Delta}_Q)}\lesssim\frac{\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})}{\sigma(Q_{\mathbf{S}})},\quad\forall\,Q\in\mathbf{S}(N).$$

This means that we can refine the initial corona decomposition so that for every new good sub-regime $\mathbf{S}$ the desired property holds for every $Q\in\mathbf{S}(N)$, that is, for all the cubes that are N-close to the ones in $\mathbf{S}$. Also, $\ell(\mathrm{Top}(\mathbf{S}))\leqslant 2^{-N}\,\ell(Q_{\mathbf{S}})$. Of course the same can be done with the other two definitions, the precise statements are left to the interested reader.

Remark 4.13. The reader may wonder whether in (i), (ii), or (iii) it would have been reasonable or convenient to impose Bourgain's estimate for the family of poles $\{X_{\mathbf{S}}\}_{\mathbf{S}\in\mathbb{F}}$, that is, whether for each $\mathbf{S}\in\mathbb{F}$ one would have needed to assume $\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})\gtrsim 1$. As we will show below, see Remark 4.29, at the cost of possibly changing the given corona decomposition, we may always assume (in each of the items in the definition) that $\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})\gtrsim 1$ for every $\mathbf{S}\in\mathbb{F}$.

Remark 4.14. We would like to emphasize that from the proof of Theorem 4.2 it follows that assuming any of the conditions (a)-(d) (hence, all of them), then one gets an explicit expression for the parameter $\Lambda_{\mathbf{S}}$ in (iii), indeed the argument reveals that $\Lambda_{\mathbf{S}} = \omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})/\sigma(Q_{\mathbf{S}})$.

Definition 4.15 (Carleson measure estimates)**.** Let $\Omega\subset\mathbb{R}^{n+1}$, $n\geqslant 2$, be an open set with Ahlfors-David regular boundary satisfying the corkscrew condition, and let $Lu = -\operatorname{div}(A\nabla u)$ be a real (non-necessarily symmetric) uniformly elliptic operator. Let $\mathbb{D} := \mathbb{D}(\partial\Omega)$ denote a dyadic grid on $\partial\Omega$.

(i) We say that L satisfies full Carleson measure estimates if

$$\sup_{\substack{x\in\partial\Omega\\ 0<r<\infty}} \frac{1}{r^n} \iint_{B(x,r)\cap\Omega} |\nabla u(X)|^2 \delta(X)\, dX \lesssim \|u\|^2_{L^\infty(\Omega)},$$

for every bounded weak solution $u \in W^{1,2}_{\rm loc}(\Omega) \cap L^\infty(\Omega)$ of $Lu = 0$ in Ω.

(ii) We say that L satisfies partial/weak Carleson measure estimates (with parameter $\tau \in (0, \frac{1}{2})$)[2] if for every $Q \in \mathbb{D}$ there exists $P_Q \in \Omega$ with $\delta(P_Q) \approx \ell(Q) \approx \operatorname{dist}(P_Q, Q)$ such that

$$\sup_{Q_0\in\mathbb{D}} \frac{1}{\sigma(Q_0)} \sum_{Q\in\mathbb{D}_{Q_0}} \iint_{B(P_Q,(1-\tau)\delta(P_Q))} |\nabla u(X)|^2 \delta(X)\, dX \lesssim_\tau \|u\|^2_{L^\infty(\Omega)},$$

for every bounded weak solution $u \in W^{1,2}_{\rm loc}(\Omega) \cap L^\infty(\Omega)$ of $Lu = 0$ in Ω.

Remark 4.16. Much as in Remark 4.13, at the cost of possibly changing the collection of points $\{P_Q\}_{Q\in\mathbb{D}}$, we may always assume that $\omega_L^{P_Q}(Q) \gtrsim 1$ for every $Q \in \mathbb{D}$, see Remark 4.29.

4.4.2 Dyadic grids and sawtooths

In Section 2.3.2 we explained several dyadic tools that help us handling different scales, both in the case of open sets and boundaries. We will use them at all times along this chapter. Actually, we will need further (more convolved) constructions based on the Whitney decomposition and the dyadic grid. Therefore, we state again the existence and properties of the dyadic grid in order to have all the notation handy in this chapter.

Lemma 4.17. *Suppose that $E \subset \mathbb{R}^{n+1}$ is an n-dimensional ADR set. Then there exist constants $C_1 \geqslant 1$ and $\gamma > 0$ depending only on n and the ADR constant such that, for each $k \in \mathbb{Z}$, there is a collection of Borel sets (cubes)*

$$\mathbb{D}_k = \{Q_j^k \subset E : j \in \mathfrak{I}_k\}$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying:

(a) *$E = \bigcup_j Q_j^k$, for each $k \in \mathbb{Z}$.*

(b) *If $m \leqslant k$, then either $Q_i^k \subset Q_j^m$ or $Q_i^k \cap Q_j^m = \varnothing$.*

(c) *For each $k \in \mathbb{Z}$, $j \in \mathfrak{I}_k$, and each $m < k$, there is a unique $i \in \mathfrak{I}_k$ such that $Q_j^k \subset Q_i^m$.*

(d) *For each $k \in \mathbb{Z}$, $j \in \mathfrak{I}_k$, there is $x_j^k \in E$ such that*

$$B(x_j^k, C_1^{-1} 2^{-k}) \cap E \subset Q_j^k \subset B(x_j^k, C_1 2^{-k}) \cap E.$$

[2] Regarding Theorem 4.2, in (c) $\Longrightarrow$ (d) we actually show that partial/weak Carleson measure estimates hold for every $\tau \in (0, \frac{1}{2})$, and for (d) $\Longrightarrow$ (a) one just needs some fixed τ, provided it is small enough depending on n, Ahlfors-David regularity, and ellipticity, namely, $0 < \tau < \tau_0$ with τ_0 from Lemma 4.23. In turn, Theorem 4.2 shows a posteriori that partial/weak Carleson measure estimates for some fixed parameter τ_1 implies the same property for all values of τ provided τ_1 is sufficiently small depending on the allowable parameters.

(*e*) $\mathcal{H}^n(\{x \in Q_j^k : \operatorname{dist}(x, E\backslash Q_j^k) \leqslant 2^{-k}\tau\}) \leqslant C_1\tau^\gamma \mathcal{H}^n(Q_j^k)$ *for all* $k \in \mathbb{Z}$, $j \in \mathfrak{I}_k$, *and* $\tau \in (0,1)$.

A few remarks are in order concerning this lemma.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [Ch90], with the dyadic parameter $1/2$ replaced by some constant $\delta \in (0,1)$. In fact, one may always take $\delta = 1/2$ (cf. [HMMM17, Proof of Proposition 2.12]). In the presence of the Ahlfors-David property, the result already appears in [DS91, DS93].

- Since in the present scenario we have that $\operatorname{diam}(\Delta(x,r)) \approx r$ we clearly have that if E is bounded and $k \in \mathbb{Z}$ is such that $\operatorname{diam}(E) < C_1\, 2^{-k}$, then there cannot be two distinct cubes in $\mathbb{D}_k$. Thus, $\mathbb{D}_k = \{Q^k\}$ with $Q^k = E$. Therefore, for our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$, in the case that the latter is finite. Hence, we shall denote by $\mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e.,

$$\mathbb{D}(E) := \bigcup_{k\in\mathbb{Z}} \mathbb{D}_k,$$

  where, if $\operatorname{diam}(E)$ is finite, the union runs over $k \geqslant -k_0$ with $2^{k_0} \approx \operatorname{diam}(E)$ and there exits $Q \in \mathbb{D}_{-k_0}$ so that $E = Q$.

- For a dyadic cube $Q \in \mathbb{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the "length" of Q. Evidently, $\ell(Q) \approx \operatorname{diam}(Q)$. We set $k(Q) = k$ to be the dyadic generation to which Q belongs if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$. One can easily see that if $Q \in \mathbb{D}_k$ and $Q' \in \mathbb{D}_{k'}$ with $Q \subsetneq Q'$ then necessarily $k' < k$. However, it is possible to have two cubes $Q \in \mathbb{D}_k$ and $Q' \in \mathbb{D}_{k'}$, such that $k \neq k'$ and $Q = Q'$. In that case the ADR condition implies that $k \approx k'$. To avoid some technicalities whenever we write $Q \subset Q'$ we will understand (unless otherwise is specified) that matters are organized so that $Q \in \mathbb{D}_k$, $Q' \in \mathbb{D}_{k'}$, and $k' < k$.

- Write $\Xi = 2C_1^2$. Property (d) implies that for each cube $Q \in \mathbb{D}$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r_Q)$ and a surface ball $\Delta(x_Q, r_Q) := B(x_Q, r_Q) \cap E$, with $\Xi^{-1}\ell(Q) \leqslant r_Q \leqslant \ell(Q)$ (indeed $r_Q = (2C_1)^{-1}\ell(Q)$), such that

$$\Delta(x_Q, 2r_Q) \subset Q \subset \Delta(x_Q, \Xi r_Q), \tag{4.9}$$

  We shall write

$$\begin{aligned} B_Q &:= B(x_Q, r_Q), & \Delta_Q &:= \Delta(x_Q, r_Q), \\ \widetilde{B}_Q &:= B(x_Q, \Xi r_Q), & \widetilde{\Delta}_Q &:= \Delta(x_Q, \Xi r_Q), \end{aligned} \tag{4.10}$$

  and we shall refer to the point x_Q as the "center" of Q.

- Let $Q \in \mathbb{D}_k$ and consider the family of its dyadic children $\{Q' \in \mathbb{D}_{k+1} : Q' \subset Q\}$. Note that for any two distinct children Q', Q'', one has $|x_{Q'} - x_{Q''}| \geqslant r_{Q'} = r_{Q''} = r_Q/2$, otherwise $x_{Q''} \in Q'' \cap \Delta_{Q'} \subset Q'' \cap Q'$, contradicting the fact that Q' and Q'' are disjoint. Also $x_{Q'}, x_{Q''} \in Q \subset \Delta(x_Q, r_Q)$, hence by the geometric doubling property we have a purely dimensional bound for the number of such $x_{Q'}$ and hence the number of dyadic children of a given dyadic cube is uniformly bounded.

We next introduce the notation of "Carleson region" and "discretized sawtooth" from [HM14, Section 3]. Given a dyadic cube $Q \in \mathbb{D}(E)$, the "discretized Carleson region" $\mathbb{D}_Q$ relative to Q is defined by

$$\mathbb{D}_Q := \{Q' \in \mathbb{D}(E) : Q' \subset Q\}.$$

Let $\mathcal{F} = \{Q_j\} \subset \mathbb{D}(E)$ be a family of pairwise disjoint cubes. The "global discretized sawtooth" relative to $\mathcal{F}$ is the collection of cubes $Q \in \mathbb{D}(E)$ that are not contained in any $Q_j \in \mathcal{F}$, that is,

$$\mathbb{D}_{\mathcal{F}} := \mathbb{D}(E) \setminus \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j}.$$

For a given cube $Q \in \mathbb{D}(E)$, we define the "local discretized sawtooth" relative to $\mathcal{F}$ is the collection of cubes in $\mathbb{D}_Q$ that are not contained in any $Q_j \in \mathcal{F}$ of, equivalently,

$$\mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \setminus \bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} = \mathbb{D}_{\mathcal{F}} \cap \mathbb{D}_Q.$$

We also introduce the "geometric" Carleson regions and sawtooths. In the sequel, $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, is an open set with ADR boundary and satisfying the corkscrew condition. Given $Q \in \mathbb{D} := \mathbb{D}(\partial\Omega)$, we define the "corkscrew point relative to Q" as $X_Q := X_{\Delta_Q}$. We then note that

$$\delta(X_Q) \approx \operatorname{dist}(X_Q, Q) \approx \operatorname{diam}(Q).$$

Our next goal is to define some associated regions which inherit the good properties of Ω. Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of Ω, so that the cubes in $\mathcal{W}$ form a covering of Ω with non-overlapping interiors, which satisfy

$$4\operatorname{diam}(I) \leqslant \operatorname{dist}(4I, \partial\Omega) \leqslant \operatorname{dist}(I, \partial\Omega) \leqslant 40\operatorname{diam}(I), \quad \forall\, I \in \mathcal{W},$$

and also

$$(1/4)\operatorname{diam}(I_1) \leqslant \operatorname{diam}(I_2) \leqslant 4\operatorname{diam}(I_1), \quad \text{whenever } I_1 \text{ and } I_2 \text{ touch.}$$

Let $X(I)$ be the center of I and $\ell(I)$ denote the sidelength of I.

Given $0 < \lambda < 1$ and $I \in \mathcal{W}$, we write $I^* = (1+\lambda)I$ for the "fattening" of I. By taking λ small enough, we can arrange matters, so that for any $I, J \in \mathcal{W}$,

$$\operatorname{dist}(I^*, J^*) \approx \operatorname{dist}(I, J),$$
$$\operatorname{int}(I^*) \cap \operatorname{int}(J^*) \neq \varnothing \Longleftrightarrow \partial I \cap \partial J \neq \varnothing.$$

(The fattening thus ensures overlap of I^* and J^* for any pair $I, J \in \mathcal{W}$ whose boundaries touch, so that the Harnack chain property then holds locally, with constants depending upon λ, in $I^* \cap J^*$.) By choosing λ sufficiently small, say $0 < \lambda < \lambda_0$, we may also suppose that there is a $\tau \in (1/2, 1)$ such that for distinct $I, J \in \mathcal{W}$, we have that $\tau J \cap I^* = \varnothing$. In what follows we will need to work with the dilations $I^{**} = (1+2\lambda)I$ or $I^{***} = (1+4\lambda)I$, and in order to ensure that the same properties hold we further assume that $0 < \lambda < \lambda_0/4$.

Given $\vartheta \in \mathbb{N}$, for every cube $Q \in \mathbb{D}$ we set

$$\mathcal{W}_Q^\vartheta := \left\{I \in \mathcal{W} : 2^{-\vartheta}\ell(Q) \leqslant \ell(I) \leqslant 2^\vartheta \ell(Q), \text{ and } \operatorname{dist}(I, Q) \leqslant 2^\vartheta \ell(Q)\right\}. \tag{4.11}$$

We will choose $\vartheta \geqslant \vartheta_0$, with $\vartheta_0 \geqslant 6 + \log_2 n$ large enough depending on the constants of the corkscrew condition (cf. Definition 2.20) and in the dyadic cube construction (cf. Lemma 4.17), so that $X_Q \in I$ for some $I \in \mathcal{W}^\vartheta_Q$, and for each dyadic child Q^j of Q, the respective corkscrew points $X_{Q^j} \in I^j$ for some $I^j \in \mathcal{W}^\vartheta_Q$. Given $I \in \mathcal{W}$ with $\ell(I) \lesssim \operatorname{diam}(\partial\Omega)$ and let Q^*_I be one of the nearest dyadic cubes to I so that $\ell(I) = \ell(Q^*_I)$. Clearly, $\operatorname{dist}(I, Q^*_I) \leqslant 40\sqrt{n+1}\ell(I) = 40\sqrt{n+1}\ell(Q^*_I)$. Hence $I \in \mathcal{W}^\vartheta_{Q^*_I}$ since $\vartheta \geqslant \vartheta_0$.

Given $Q \in \mathbb{D}$ we define its associated Whitney regions U^ϑ_Q and $\widehat{U}^\vartheta_Q$ (not necessarily connected) as

$$U^\vartheta_Q := \bigcup_{I \in \mathcal{W}^\vartheta_Q} I^*, \qquad U^{\vartheta,*}_Q := \bigcup_{I \in \mathcal{W}^\vartheta_Q} I^{**},$$

For a given $Q \in \mathbb{D}$, the "Carleson boxes" relative to Q are defined by

$$T^\vartheta_Q := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_Q} U^\vartheta_{Q'}\right), \qquad T^{\vartheta,*}_Q := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_Q} U^{\vartheta,*}_{Q'}\right).$$

For a given family $\mathcal{F} = \{Q_j\}$ of pairwise disjoint cubes and a given $Q \in \mathbb{D}$, we define the "local sawtooth regions" relative to $\mathcal{F}$ by

$$\Omega^\vartheta_{\mathcal{F},Q} := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U^\vartheta_{Q'}\right), \qquad \Omega^{\vartheta,*}_{\mathcal{F},Q} := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U^{\vartheta,*}_{Q'}\right).$$

Following [HM14], one can easily see that there exist constants $0 < \kappa_1 < 1$ and $\kappa_0 \geqslant 16\Xi$ (with Ξ the constant in (4.9)), depending only on the allowable parameters and on ϑ, so that

$$\kappa_1 B_Q \cap \Omega \subset T^\vartheta_Q \subset T^{\vartheta,*}_Q \subset T^{\vartheta,**}_Q \subset \overline{T^{\vartheta,**}_Q} \subset \kappa_0 B_Q \cap \overline{\Omega} =: \frac{1}{2} B^*_Q \cap \overline{\Omega}, \tag{4.12}$$

where B_Q is defined as in (4.10).

We recall that Ω is an open set with ADR boundary and this allows us to define the open set $\Omega' := \mathbb{R}^{n+1}\backslash\partial\Omega$ whose boundary is $\partial\Omega' = \partial\Omega$. One can then proceed as above and define the associated local sawtooth regions $(\Omega')^\vartheta_{\mathcal{F},Q}$ with respect to Ω' (that is, we now have some underlying Whitney decomposition for Ω' which agrees with $\mathcal{W}(\Omega)$ when restricted to Ω). In [HMM16, Proposition A.2] it was shown that all $(\Omega')^\vartheta_{\mathcal{F},Q}$ have ADR boundaries and the constants are uniform and depend on n, the ADR constant of $\partial\Omega' = \partial\Omega$, and ϑ. One can easily see that $\partial\Omega^\vartheta_{\mathcal{F},Q} \subset \partial(\Omega')^\vartheta_{\mathcal{F},Q}$ since $\Omega^\vartheta_{\mathcal{F},Q} = (\Omega')^\vartheta_{\mathcal{F},Q} \cap \Omega$. Thus, it is trivial to obtain that all local sawtooth regions $\Omega^\vartheta_{\mathcal{F},Q}$ have boundary satisfying the upper ADR condition (that is only the upper estimate) albeit with bounds that are uniform and depend on n, the ADR constant of $\partial\Omega$, and ϑ. In short we have the following:

Lemma 4.18. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with Ahlfors-David regular boundary satisfying the corkscrew condition. For every $\vartheta \geqslant \vartheta_0$, all sawtooth domains $\Omega^\vartheta_{\mathcal{F},Q}$ and $\Omega^{\vartheta,*}_{\mathcal{F},Q}$ have upper ADR boundary (that is, they satisfy the upper bound in Definition 2.16). Moreover, the implicit constants are uniform and depend only on dimension, the ADR constant of $\partial\Omega$ and the parameter ϑ.*

Lemma 4.19. *Given $c, \tau \in (0, \frac{1}{2})$ there exist $\vartheta, C \gg 1$ (depending on n, ADR, τ, and c) with the following significance: for every $Q_0 \in \mathbb{D}$, every pairwise disjoint collection $\mathcal{F} \subset \mathbb{D}_{Q_0}$, and every*

sequence $\{P_Q\}_{Q\in\mathbb{D}_{\mathcal{F},Q_0}}$ *so that* $P_Q \in 2\,\widetilde{B}_Q \cap \Omega$ *with* $\delta(P_Q) \geqslant c\,\ell(Q)$ *there hold*

$$\bigcup_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} B(P_Q,(1-\tau)\delta(P_Q)) \subset \bigcup_{Q\in\mathbb{D}_{\mathcal{F},Q_0}}\bigcup_{I\in\mathcal{W}_Q^\vartheta} I \subset \operatorname{int}\Big(\bigcup_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} U_Q^\vartheta\Big) = \Omega^\vartheta_{\mathcal{F},Q_0}.$$

and

$$\sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} \mathbf{1}_{B(P_Q,(1-\tau)\delta(P_Q))} \leqslant C\,\mathbf{1}_{\Omega^\vartheta_{\mathcal{F},Q_0}}.$$

Proof. For each $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ write $V_Q := B(P_Q,(1-\tau)\delta(P_Q))$. Let $I \in \mathcal{W}$ be such that $I \cap V_Q \neq \varnothing$. Taking $Z_Q \in I \cap V_Q$ we note that

$$\ell(I) \approx \delta(Z_Q) \approx_\tau \delta(P_Q) \approx_c \ell(Q),$$

where the implicit constants depend on τ and the parameter c, and the other allowable parameters. Besides,

$$\operatorname{dist}(I,Q) \leqslant |Z_Q - P_Q| + |P_Q - x_Q| \leqslant (1-\tau)\delta(P_Q) + 2\Xi r_Q \lesssim \ell(Q)$$

where the implicit constant depends on the allowable parameters. This means that we can find $\vartheta \gg 1$ (depending on n, ADR, τ, and c) such that $I \in \mathcal{W}_Q^\vartheta$ (cf. (4.11)). As a consequence,

$$\bigcup_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} V_Q \subset \bigcup_{Q\in\mathbb{D}_{\mathcal{F},Q_0}}\bigcup_{I\in\mathcal{W}:I\cap V_Q\neq\varnothing} I \subset \bigcup_{Q\in\mathbb{D}_{\mathcal{F},Q_0}}\bigcup_{I\in\mathcal{W}_Q^\vartheta} I \subset \operatorname{int}\Big(\bigcup_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} U_Q^\vartheta\Big) = \Omega^\vartheta_{\mathcal{F},Q_0}.$$

To complete the proof let $X \in \bigcup_{Q\in\mathbb{D}_{\mathcal{F},Q_0}} V_Q$ and pick $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ so that $X \in V_Q$ with $Q \in \mathbb{D}_{\mathcal{F},Q_0}$. Note that if $X \in V'_Q$ for $Q' \in \mathbb{D}_{\mathcal{F},Q_0}$, then $\ell(Q) \approx_\tau \delta(X) \approx_\tau \ell(Q')$ and

$$\operatorname{dist}(Q,Q') \leqslant |x_Q - P_Q| + |P_Q - X| + |X - P_{Q'}| + |P_{Q'} - x_{Q'}| \lesssim_\tau \ell(Q) + \ell(Q'),$$

thus Q and Q' are 2^N-close with a uniform constant N depending just on n, ADR, τ, and c. As a result,

$$\sum_{Q'\in\mathbb{D}_{\mathcal{F},Q_0}} \mathbf{1}_{V_{Q'}}(X) = \#\{Q' \in \mathbb{D}_{\mathcal{F},Q_0} : V_{Q'} \ni X\} \leqslant \#\{Q' \in \mathbb{D} : Q' \text{ and } Q_0 \text{ are } 2^N\text{-close}\} \lesssim_N 1.$$

This completes the proof. □

We need the following auxiliary which adapts [HMT17, Lemma 4.44] and [CDMT22, Lemma 6.4] to our current setting. The proof is the same as that of [CDMT22, Lemma 6.4] and details are left to the interested reader.

Lemma 4.20. *Let* $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, *be an open set with Ahlfors-David regular boundary satisfying the corkscrew condition. Given* $Q_0 \in \mathbb{D}$, *a pairwise disjoint collection* $\mathcal{F} \subset \mathbb{D}_{Q_0}$, *and* $M \geqslant 4$ *let* $\mathcal{F}_M$ *be the family of maximal cubes of the collection* $\mathcal{F}$ *augmented by adding all the cubes* $Q \in \mathbb{D}_{Q_0}$ *such that* $\ell(Q) \leqslant 2^{-M}\ell(Q_0)$. *There exist* $\Psi_M \in \mathscr{C}^\infty_c(\mathbb{R}^{n+1})$ *and a constant* $C \geqslant 1$ *depending only on dimension n, the corkscrew and ADR constants, but independent of M,* $\mathcal{F}$, *and* Q_0 *such that the following hold:*

(*i*) $C^{-1}\,\mathbf{1}_{\Omega^\vartheta_{\mathcal{F}_M,Q_0}} \leqslant \Psi_M \leqslant \mathbf{1}_{\Omega^{\vartheta,*}_{\mathcal{F}_M,Q_0}}$.

(*ii*) $\sup_{X\in\Omega} |\nabla\Psi_M(X)|\,\delta(X) \leqslant C$.

(iii) Setting $\mathcal{W}_M^{\vartheta,\Sigma} := \big\{I \in \mathcal{W}_M^{\vartheta} : \exists\, J \in \mathcal{W}\backslash\mathcal{W}_M^{\vartheta} \text{ with } \partial I \cap \partial J \neq \varnothing\big\}$ *and*

$$\mathcal{W}_M^{\vartheta} := \bigcup_{Q\in\mathbb{D}_{\mathcal{F}_M,Q_0}} \mathcal{W}_Q^{\vartheta},$$

one has

$$\nabla\Psi_M \equiv 0 \quad \text{in} \quad \bigcup_{I\in\mathcal{W}_M^{\vartheta}\backslash\mathcal{W}_M^{\vartheta,\Sigma}} I^{**},$$

and there exists a family $\{\widehat{Q}_I\}_{I\in\mathcal{W}_M^{\vartheta,\Sigma}}$ *so that*

$$C^{-1}\,\ell(I) \leqslant \ell(\widehat{Q}_I) \leqslant C\,\ell(I), \qquad \operatorname{dist}(I,\widehat{Q}_I) \leqslant C\,\ell(I), \qquad \sum_{I\in\mathcal{W}_M^{\vartheta,\Sigma}} \mathbf{1}_{\widehat{Q}_I} \leqslant C.$$

4.4.3 PDE framework

Along this chapter, we will consider elliptic operators $Lu = \operatorname{div}(A\nabla u)$ as in Definition 2.27. We suggest the reader to revisit Sections 2.4.1 to review the concept of weak solutions and elliptic measure.

Moreover, we write $L^\top$ to denote the transpose of L, or, in other words, $L^\top u = -\operatorname{div}(A^\top\nabla u)$ with $A^\top$ being the transpose matrix of A.

Let us recall some very important properties of Green's functions, as presented in Section 2.4.4.

Lemma 4.21. *Suppose that* $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, *is an open set such that* $\partial\Omega$ *is ADR. Given an elliptic operator* L, *there exist* $C > 1$ *(depending only on dimension and on the ellipticity of* L*) and* $c_\theta > 0$ *(depending on the above parameters and on* $\theta \in (0,1)$*) such that* G_L, *the Green function associated with* L, *satisfies*

$$G_L(X,Y) \leqslant C|X-Y|^{1-n}; \tag{4.13}$$

$$c_\theta|X-Y|^{1-n} \leqslant G_L(X,Y), \quad \text{if } |X-Y| \leqslant \theta\delta(X), \theta \in (0,1); \tag{4.14}$$

$$G_L(\cdot,Y) \in C(\overline{\Omega}\backslash\{Y\}) \text{ and } G_L(\cdot,Y)|_{\partial\Omega} \equiv 0, \forall\, Y \in \Omega; \tag{4.15}$$

$$G_L(X,Y) \geqslant 0, \quad \forall\, X,Y \in \Omega, X \neq Y; \tag{4.16}$$

$$G_L(X,Y) = G_{L^\top}(Y,X), \quad \forall\, X,Y \in \Omega, X \neq Y. \tag{4.17}$$

Furthermore, $G_L(\cdot,Y) \in W^{1,2}_{\text{loc}}(\Omega\backslash\{Y\})$ *for every* $Y \in \Omega$, *and satisfies* $LG_L(\cdot,Y) = \delta_Y$ *in the weak sense in* Ω, *that is,*

$$\iint_\Omega A(X)\nabla_X G_L(X,Y)\cdot\nabla\Phi(X)dX = \Phi(Y), \quad \forall\,\Phi \in \mathscr{C}_c^\infty(\Omega).$$

Finally, the following Riesz formula holds

$$\iint_\Omega A^\top(Y)\nabla_Y G_{L^\top}(Y,X)\cdot\nabla\Phi(Y)dY = \Phi(X) - \int_{\partial\Omega}\Phi d\omega_L^X, \tag{4.18}$$

for a.e. $X \in \Omega$ *and for every* $\Phi \in \mathscr{C}_c^\infty(\mathbb{R}^{n+1})$.

Let us further recall other fine estimates for solutions around boundary points, which will be crucial along the chapter. Find further discussion in Sections 2.4.4 and 2.4.5.

Lemma 4.22. *Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, is an open set with ADR boundary. Let L be an elliptic operator. There exists a constant $C > 1$ (depending only on the dimension, the ADR constant and the ellipticity of L) such that we have the following properties:*

(a) *For every $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$ there holds*

$$\omega_L^Y(\Delta(x,r)) \geqslant 1/C, \quad \forall\, Y \in \Omega \cap B(x, C^{-1}r). \tag{4.19}$$

(b) *Given $X, Y \in \Omega$ such that $|X - Y| \geqslant \delta(Y)/2$, then*

$$\frac{G_L(X,Y)}{\delta(Y)} \leqslant C\, \frac{\omega_L^X(\Delta(\widehat{y}, 2\delta(Y)))}{\sigma(\Delta(\widehat{y}, 2\,\delta(Y)))} \tag{4.20}$$

with $\widehat{y} \in \Omega$ such that $|Y - \widehat{y}| = \delta(Y)$.

(c) *If $0 \leqslant u \in W^{1,2}_{\mathrm{loc}}(B_0 \cap \Omega) \cap \mathscr{C}(\overline{B_0 \cap \Omega})$ satisfies $Lu = 0$ in the weak-sense in $B_0 \cap \Omega$ and $u \equiv 0$ in Δ_0 then*

$$u(X) \leqslant C \left(\frac{|X - x_0|}{r_0}\right)^{\gamma} \sup_{Y \in \overline{B_0 \cap \Omega}} u(Y), \qquad \forall X \in \frac{1}{2}B_0 \cap \Omega.$$

Lastly, the following lemma from [GMT18] will be crucial in one of the implications of our main Theorem 4.2. It asserts that we can always find certain solutions that oscillate at a given concrete scale.

Lemma 4.23 ([GMT18, Lemma 3.3])**.** *There exists a small enough parameter $\tau_0 \in (0, \frac{1}{2})$ (depending on n, ADR and ellipticity) such that given $0 < \tau < \tau_0$ there exists $\varepsilon_\tau > 0$ small enough (depending on the same parameters and additionally on τ), so that for each $Q \in \mathbb{D}$, $\varepsilon \in (0, \varepsilon_\tau]$, $Y_Q \in \frac{1}{2}B_Q \cap \Omega$ (cf. (4.10)) with $\delta(Y_Q) \approx \varepsilon\ell(Q)$, and for each Borel set $E_Q \subset Q$ satisfying*

$$\omega_L^{Y_Q}(E_Q) \geqslant (1 - \varepsilon)\, \omega_L^{Y_Q}(Q),$$

there exists a Borel function $0 \leqslant f_Q \lesssim \mathbf{1}_{E_Q}$ so that

$$u_Q(X) := \int_{E_Q} f_Q \, d\omega_L^X, \qquad X \in \Omega,$$

satisfies $u_Q \in W^{1,2}_{\mathrm{loc}}(\Omega)$, $Lu_Q = 0$ in the weak sense in Ω, and

$$\iint\limits_{B(Y_Q, (1-\tau)\delta(Y_Q))} |\nabla u_Q(X)|^2\, \delta(X) \gtrsim_{\tau,\varepsilon} \sigma(Q),$$

where the implicit constants depend on n, ADR, ellipticity, τ, and ε.

4.4.4 Extension to unbounded boundaries

Much as in Definition 4.7, and fixed $Q_0 \in \mathbb{D}$ a semi-coherent corona decomposition relative to Q_0 with constant $M_0 \geqslant 1$ is a triple $(\mathbb{B}_{Q_0}, \mathbb{G}_{Q_0}, \mathbb{F}_{Q_0})$ where $\mathbb{D}_{Q_0} = \mathbb{G}_{Q_0} \sqcup \mathbb{B}_{Q_0}$, the "good" collection $\mathbb{G}_{Q_0}$ is further subdivided so that $\mathbb{G}_{Q_0} = \bigsqcup_{\mathbf{S}\in\mathbb{F}_{Q_0}} \mathbf{S}$, with each $\mathbf{S}$ being semi-coherent, and the "bad" collection $\mathbb{B}_{Q_0}$ and the family of top cubes $\mathrm{Top}(\mathbb{F}_{Q_0}) := \{\mathrm{Top}(\mathbf{S}) : \mathbf{S} \in \mathbb{F}_{Q_0}\}$ satisfy the Carleson packing condition:

$$\sup_{Q\in\mathbb{D}_{Q_0}} \Big(\frac{1}{\sigma(Q)} \sum_{Q'\in\mathbb{B}_{Q_0}\cap\mathbb{D}_Q} \sigma(Q') + \frac{1}{\sigma(Q)} \sum_{Q'\in\mathrm{Top}(\mathbb{F}_{Q_0})\cap\mathbb{D}_Q} \sigma(Q')\Big) \leqslant M_0.$$

Proposition 4.24. *Suppose that there exists $M_0 \geqslant 1$ such that for every $Q \in \mathbb{D}$ there is $(\mathbb{B}_Q, \mathbb{G}_Q, \mathbb{F}_Q)$, a (coherent/semi-coherent) corona decomposition relative to Q with constant $M_0 \geqslant 1$. Then, there exists a (coherent/semi-coherent) corona decomposition $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ with constant $M_0 + C$ (where C depends on dimension and the ADR constant) so that it is compatible with the given ones, that is, any coherent/semi-coherent sub-regime $\mathbf{S} \in \mathbb{F}$ belongs to $\mathbb{F}_Q$ for some $Q \in \mathbb{D}$.*

Proof. If $\partial\Omega$ is bounded, as observed above we have that $\mathbb{D} = \mathbb{D}_{Q_0}$ for some $Q_0 \in \mathbb{D}$ with $\ell(Q_0) \approx \mathrm{diam}(\partial\Omega)$. Thus, by hypothesis there exists $(\mathbb{B}_{Q_0}, \mathbb{G}_{Q_0}, \mathbb{F}_{Q_0})$, a (coherent/semi-coherent) corona decomposition, relative to Q_0 and with constant $M_0 \geqslant 1$. The desired conclusion follows at once by setting $\mathbb{G} := \mathbb{G}_{Q_0}$, $\mathbb{B} := \mathbb{B}_{Q_0}$, and $\mathbb{F} := \mathbb{F}_{Q_0}$.

We are now going to deal with the case $\partial\Omega$ unbounded, which requires a bit more work. Fix $x_0 \in \partial\Omega$ and some $Q_0 \in \mathbb{D}$ such that $x_0 \in Q_0$ and $\ell(Q_0) = 1$. For $k \geqslant 0$, call $Q_0^{(k)} \in \mathbb{D}$ the k-th dyadic ancestor of Q_0, that is $Q_0^{(k)}$ is the unique cube in $\mathbb{D}_{-k}$ containing Q_0. Recall that there exists $N \geqslant 1$ so that the number of dyadic children of a given dyadic cube is at most $N + 1$. For any $k \geqslant 0$, adding the null set if needed, we can label the dyadic children of $Q_0^{(k+1)}$ as $(Q_0^{(k)})^0, (Q_0^{(k)})^1, \ldots, (Q_0^{(k)})^N$ with $(Q_0^{(k)})^0 = Q_0^{(k)}$. In particular, $(Q_0^{(k)})^1, \ldots, (Q_0^{(k)})^N$ are the dyadic "siblings" of $Q_0^{(k)}$, that is, the dyadic children of $Q_0^{(k+1)}$ which are not $Q_0^{(k)}$ itself.

To simplify the notation, let $\{R_j\}_{j=1}^{\infty}$ be a enumeration of $\{(Q_0^{(k)})^i : k \geqslant 0, 1 \leqslant i \leqslant N\}$ (in this enumeration we drop the null sets that we may have added). Set also $R_0 := Q_0$. By construction it follows easily that we have a disjoint decomposition

$$\bigcup_{k=0}^{\infty} \mathbb{D}_{Q_0^{(k)}} = \Big(\bigsqcup_{j=0}^{\infty} \mathbb{D}_{R_j}\Big) \bigsqcup \Big(\bigsqcup_{k=1}^{\infty} \{Q_0^{(k)}\}\Big).$$

Using our hypothesis, for each R_j we let $(\mathbb{B}_{R_j}, \mathbb{G}_{R_j}, \mathbb{F}_{R_j})$ be the associated (coherent/semi-coherent) corona decomposition. This allows us to define a decomposition into good and bad cubes for the full family of cubes we are considering. Indeed, if we define

$$\mathbb{G}^{Q_0} := \bigsqcup_{j=0}^{\infty} \mathbb{G}_{R_j} \qquad \text{and} \qquad \mathbb{B}^{Q_0} := \Big(\bigsqcup_{j=0}^{\infty} \mathbb{B}_{R_j}\Big) \bigsqcup \Big(\bigsqcup_{k=1}^{\infty} \{Q_0^{(k)}\}\Big), \tag{4.21}$$

we have

$$\mathbb{D}^{Q_0} := \bigcup_{k=0}^{\infty} \mathbb{D}_{Q_0^{(k)}} = \mathbb{G}^{Q_0} \sqcup \mathbb{B}^{Q_0}.$$

If we also set

$$\mathbb{F}^{Q_0} := \bigsqcup_{j=0}^{\infty} \mathbb{F}_{R_j}$$

we have by construction,

$$\mathbb{G}^{Q_0} = \Big(\bigsqcup_{\mathbf{S}\in\mathbb{F}^{Q_0}} \mathbf{S} \Big) = \bigsqcup_{j=0}^{\infty} \Big(\bigsqcup_{\mathbf{S}\in\mathbb{F}_{R_j}} \mathbf{S} \Big)$$

where each $\mathbf{S}$ in the previous unions is coherent/semi-coherent.

Fix next $Q \in \mathbb{D}^{Q_0}$ and note that if $Q_0^{(k)} \subset Q$ for some $k \geqslant 1$ then $Q = Q_0^{(k_0)}$ with $k \leqslant k_0$. Hence,

$$\sum_{Q_0^{(k)}\subset Q} \sigma(Q_0^{(k)}) = \sum_{k=1}^{k_0} \sigma(Q_0^{(k)}) \leqslant C\,\sigma(Q_0^{(k_0)}) \sum_{k=0}^{k_0} 2^{(k-k_0)n} \leqslant C\,\sigma(Q_0^{(k_0)}) = C\,\sigma(Q),$$

where C depends on n and the ADR constants. As a result, by (4.21), the Carleson packing condition of $\mathbb{B}_{R_j}$ and the top cubes in $\mathbb{F}_{R_j}$, and the fact that the family $\{R_j\}_j$ is pairwise disjoint we obtain

$$\begin{aligned}
&\sum_{R\in \mathrm{Top}(\mathbb{F}^{Q_0})\cap\mathbb{D}_Q} \sigma(R) + \sum_{R\in\mathbb{B}^{Q_0}\cap\mathbb{D}_Q} \sigma(R) \\
&\qquad = \sum_{j=0}^{\infty} \Big(\sum_{R\in \mathrm{Top}(\mathbb{F}_{R_j})\cap\mathbb{D}_Q} \sigma(R) + \sum_{R\in\mathbb{B}_{R_j}\cap\mathbb{D}_Q} \sigma(R) \Big) + \sum_{Q_0^{(k)}\subset Q} \sigma(Q_0^{(k)}) \\
&\qquad\qquad \leqslant M_0 \sum_{j=0}^{\infty} \sigma(R_j\cap Q) + C\,\sigma(Q) \leqslant (M_0 + C)\,\sigma(Q).
\end{aligned}$$

We conclude as desired that the bad cubes and the top cubes of the good subregimes satisfy a packing condition with constant $M_0 + C$.

To complete the proof we are going to proceed inductively. Suppose that we have already constructed $Q_0, Q_1, \ldots, Q_N \in \mathbb{D}$ so that $\ell(Q_j) = 1$ for very $1 \leqslant j \leqslant N$; $\{\mathbb{D}^{Q_j}\}_{j=1}^N$ is a pairwise disjoint family (of families of dyadic cubes); and if we set $\partial\Omega_j = \bigcup_{k=0}^{\infty} Q_j^{(k)}$, one has that $\{\partial\Omega_j\}_{j=1}^N$ is a pairwise disjoint family of subsets of $\partial\Omega$. Let us explain how to take the next step. If $\mathbb{D} = \bigcup_{k=0}^N \mathbb{D}^{Q_k}$ then we stop the construction. Otherwise, we can pick $Q \in \mathbb{D}\setminus\bigcup_{k=0}^N \mathbb{D}^{Q_k}$.

If $\ell(Q) \leqslant 1$ let $Q_{N+1} \supset Q$ be so that $\ell(Q_{N+1}) = 1$ and note that clearly $Q_{N+1} \notin \bigcup_{k=0}^N \mathbb{D}^{Q_k}$. On the other hand, if $\ell(Q) > 1$ we select some $Q_{N+1} \subset Q$ with $\ell(Q_{N+1}) = 1$. We claim that in this case we also have $Q_{N+1} \notin \bigcup_{k=0}^N \mathbb{D}^{Q_k}$. Otherwise, $Q_{N+1} \subset Q_k^{(i)}$ for some $i \geqslant 0$ and $0 \leqslant k \leqslant N$. In particular, $Q \cap Q_k^{(i)} \neq \varnothing$. Then, either $Q \subset Q_k^{(i)}$, in which case $Q \in \mathbb{D}^{Q_k}$ and we have reached a contradiction; or $Q_k^{(i)} \subsetneq Q$, in which case $Q = Q_k^{(i')}$ for some $i' \geqslant 1$, and again $Q \in \mathbb{D}^{Q_k}$ leading to a contradiction. In either scenario we have found $Q_{N+1} \notin \bigcup_{k=0}^N \mathbb{D}^{Q_k}$ with $\ell(Q_{N+1}) = 1$.

We next show that $\mathbb{D}^{Q_j} \cap \mathbb{D}^{Q_{N+1}} = \varnothing$ for every $0 \leqslant j \leqslant N$. Assume otherwise that there exits $Q \in \mathbb{D}_{Q_j^{(k_0)}} \cap \mathbb{D}_{Q_{N+1}^{(k_1)}}$. In particular, $Q_j^{(k_0)}$ and $Q_{N+1}^{(k_1)}$ meet. This implies that either $Q_j^{(k_0)} \subset Q_{N+1}^{(k_1)}$ or $Q_{N+1}^{(k_1)} \subsetneq Q_j^{(k_0)}$. In both cases we conclude that $Q_{N+1} \in \mathbb{D}^{Q_j}$, a contradiction.

Set $\partial\Omega_{N+1} = \bigcup_{k=0}^{\infty} Q_{N+1}^{(k)}$ and we claim that $\partial\Omega_j \cap \partial\Omega_{N+1} = \varnothing$ for every $0 \leqslant j \leqslant N$. Indeed, if this were not the case, we could pick $y \in Q_j^{(k_0)} \cap Q_{N+1}^{(k_1)}$ for some $k_0, k_1 \geqslant 0$. Take then $Q \ni y$ with $\ell(Q) = 1$. Then, $Q \subset Q_j^{(k_0)} \cap Q_{N+1}^{(k_1)}$ and hence $Q \in \mathbb{D}^{Q_j} \cap \mathbb{D}^{Q_{N+1}}$, a contradiction.

The previous argument leads then to a (possible infinite) collection dyadic cubes $\{Q_j\}_{j\in\mathfrak{N}}$ such that $\ell(Q_j) = 1$ for every $j \in \mathfrak{N}$; $\mathbb{D}^{Q_j} \cap \mathbb{D}^{Q_{j'}} = \varnothing$ for every $j, j' \in \mathfrak{N}$; $\partial\Omega_j \cap \partial\Omega_{j'} = \varnothing$ for every $j, j' \in \mathfrak{N}$; and $\mathbb{D} = \sqcup_{j\in\mathfrak{N}} \mathbb{D}^{Q_j}$. For every $j \in \mathfrak{N}$ we can then repeat the argument above with Q_j in place of Q_0 to write $\mathbb{D}^{Q_j} = \mathbb{G}^{Q_j} \sqcup \mathbb{B}^{Q_j}$, so that $\mathbb{G}^{Q_j}$ splits further into coherent/semi-coherent subregimes, and the bad cubes and the top cubes of the good subregimes satisfy a Carleson packing condition with constant $M_0 + C$. To complete the proof we set $\mathbb{G} := \bigsqcup_{j\in\mathfrak{N}} \mathbb{G}^{Q_j}$, $\mathbb{B} := \bigsqcup_{j\in\mathfrak{N}} \mathbb{B}^{Q_j}$, and $\mathbb{F} := \bigsqcup_{j\in\mathfrak{N}} \mathbb{F}^{Q_j}$, and the reader can easily check that $\mathbb{D} = \mathbb{G} \sqcup \mathbb{B}$, so that $\mathbb{G}$ splits further into the semi-coherent subregimes in $\mathbb{F}$. Observe that for any $Q \in \mathbb{D}$ there exists a unique cube Q_j with $j \in \mathfrak{N}$ so that $\mathbb{D}_Q \subset \mathbb{D}^{Q_j}$, hence when proving the Carleson packing we have

$$\begin{aligned}\sup_{Q\in\mathbb{D}} \frac{1}{\sigma(Q)} \sum_{R\in \mathrm{Top}(\mathbb{F})\cap\mathbb{D}_Q} \sigma(R) + \frac{1}{\sigma(Q)} \sum_{R\in\mathbb{B}\cap\mathbb{D}_Q} \sigma(R)& \\ = \sup_{j\in\mathfrak{N}} \sup_{Q\in\mathbb{D}^{Q_j}} \frac{1}{\sigma(Q)} \sum_{R\in \mathrm{Top}(\mathbb{F}^{Q_j})\cap\mathbb{D}_Q} \sigma(R) + \frac{1}{\sigma(Q)} \sum_{R\in\mathbb{B}^{Q_j}\cap\mathbb{D}_Q} \sigma(R) \leqslant M_0 + C.&\end{aligned}$$

The fact that any (coherent/semi-coherent) subregime $\mathbf{S} \in \mathbb{F}$ belongs to $\mathbb{G}_Q$ for some $Q \in \mathbb{D}$ is clear from the construction and this completes the proof. □

4.5 Proof of Theorem 4.2: equivalence between corona decompositions and Carleson measure estimates

This section is devoted to showing Theorem 4.2. We will follow the scheme

$$\text{(a)} \Longrightarrow \text{(b)} \Longrightarrow \text{(c)} \Longrightarrow \text{(d)} \Longrightarrow \text{(a)}.$$

Since (a) $\Longrightarrow$ (b) is trivial, it suffices to show the other implications.

4.5.1 Proof of (b) $\Longrightarrow$ (c): if ω_L admits a corona decomposition, then G_L is comparable to the distance to the boundary in the corona sense

Let $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$ be the assumed corona decomposition associated with ω_L. Given $N \geqslant 0$, large enough to be chosen momentarily, we may proceed as in Remark 4.12 and pick $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ the 2^N-refinement of $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$. Recall that this implies that for every $\mathbf{S} \in \mathbb{F}$ there exist $Q_\mathbf{S} \in \mathbb{D}$ and $X_\mathbf{S} \in \Omega$ so that

$$\mathrm{Top}(\mathbf{S}) \subset \mathrm{Top}(\mathbf{S})^{(N)} \subset Q_\mathbf{S}, \qquad \delta(X_\mathbf{S}) \approx \ell(Q_\mathbf{S}) \approx \mathrm{dist}(X_\mathbf{S}, Q_\mathbf{S}),$$

and

$$\frac{\omega_L^{X_\mathbf{S}}(Q_\mathbf{S})}{\sigma(Q_\mathbf{S})} \lesssim \frac{\omega_L^{X_\mathbf{S}}(Q)}{\sigma(Q)} \lesssim \frac{\omega_L^{X_\mathbf{S}}(2\widetilde{\Delta}_Q)}{\sigma(2\widetilde{\Delta}_Q)} \lesssim \frac{\omega_L^{X_\mathbf{S}}(Q_\mathbf{S})}{\sigma(Q_\mathbf{S})}, \quad \forall\, Q \in \mathbf{S}(N). \tag{4.22}$$

Fix $\mathbf{S} \in \mathbb{F}$ and write

$$\mu := \sigma(Q_\mathbf{S})\omega_L^{X_\mathbf{S}} \quad \text{and} \quad \mathcal{G} := \sigma(Q_\mathbf{S})G_L(X_\mathbf{S}, \cdot). \tag{4.23}$$

Step 1: the lower bound. This is the most subtle part of the proof, for we need to find a region where the lower bound holds under no background connectivity assumptions. Let us isolate this relevant fact in the following lemma.

Lemma 4.25. *There exists $N \gg 1$, depending on the allowable parameters and on the implicit constants in (4.7), such that if $Q \in \mathbf{S} \in \mathbb{F}$, then there is a set $E_Q \subset \{Y \in 2\widetilde{B}_Q \cap \Omega : \delta(Y) \geqslant 2^{-N}\ell(Q)\}$ with $|E_Q| > 0$ so that*

$$\frac{\mathcal{G}(Y)}{\delta(Y)} \gtrsim \frac{\mu(Q)}{\sigma(Q)}, \quad \text{for every } Y \in E_Q, \qquad \text{and} \qquad \|\nabla \mathcal{G}\|_{L^\infty(E_Q)} \gtrsim \frac{\mu(Q)}{\sigma(Q)}. \tag{4.24}$$

Proof. We use some ideas from [HLMN17, Lemma 4.24]. Let $Q \in \mathbf{S}$ with and note that since $Q \subset \mathrm{Top}(\mathbf{S}) \subset \mathrm{Top}(\mathbf{S})^{(N)} \subset Q_{\mathbf{S}}$ it follows that $\ell(Q) \lesssim 2^{-N}\ell(Q_{\mathbf{S}})$. Let $\widetilde{B}_Q$ and $\widetilde{\Delta}_Q$ be as in (4.10) so that $Q \subset \widetilde{\Delta}_Q$. Choose a cut-off function $\Phi_Q \in \mathscr{C}^\infty_c(\mathbb{R}^{n+1})$ satisfying $\mathbf{1}_{\widetilde{B}_Q} \leqslant \Phi_Q \leqslant \mathbf{1}_{\frac{5}{4}\widetilde{B}_Q}$ and $\|\nabla \Phi_Q\|_{L^\infty(\mathbb{R}^{n+1})} \lesssim \ell(Q)^{-1}$. Let $N > 4$ be large enough to be chosen and note that by assumption

$$2^N \ell(Q) \lesssim \ell(Q_{\mathbf{S}}) \approx \delta(X_{\mathbf{S}}) \leqslant |X_{\mathbf{S}} - x_Q|.$$

Thus, taking N large enough one can guarantee that $X_{\mathbf{S}} \notin 4\widetilde{B}_Q$. Then, $\Phi_Q(X_{\mathbf{S}}) = 0$, and by (4.18),

$$\begin{aligned}
\ell(Q)\mu(Q) \leqslant \ell(Q) \int_{\partial\Omega} \Phi_Q \, d\mu &= -\ell(Q) \iint_\Omega A^\top \nabla \mathcal{G} \cdot \nabla \Phi_Q \, dX \\
&\lesssim \iint_{\frac{5}{4}\widetilde{B}_Q \cap \Omega} |\nabla \mathcal{G}| \mathbf{1}_{\{X \in \Omega : \delta(X) \geqslant 2^{-N+1}\ell(Q)\}} \, dX \\
&\qquad + \iint_{\frac{5}{4}\widetilde{B}_Q \cap \Omega} |\nabla \mathcal{G}| \, \mathbf{1}_{\{X : \delta(X) \leqslant 2^{-N+1}\ell(Q)\}} \, dX =: \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{4.25}$$

Let us first estimate II, the term close to the boundary. Picking N big enough, we have

$$\mathrm{II} \leqslant \sum_{\substack{I \in \mathcal{W} : I \subset \frac{3}{2}\widetilde{B}_Q \\ \ell(I) \lesssim 2^{-N}\ell(Q)}} \iint_I |\nabla \mathcal{G}| \, dX.$$

Given a cube I in the summation above, we take $\widehat{x}_I \in \partial\Omega$ such that $\big|\widehat{x}_I - X(I)\big| = \delta(X(I))$. Then, assuming that N is large enough,

$$\begin{aligned}
\delta(X_{\mathbf{S}}) \leqslant |X_{\mathbf{S}} - X(I)| + C\,\ell(I) &\leqslant |X_{\mathbf{S}} - X(I)| + C\,2^{-N+1}\,\ell(Q) \\
&\leqslant |X_{\mathbf{S}} - X(I)| + C\,2^{-2N+1}\,\ell(Q_{\mathbf{S}}) \leqslant |X_{\mathbf{S}} - X(I)| + 2^{-1}\,\delta(X_{\mathbf{S}})
\end{aligned}$$

and, as a result,

$$|X_{\mathbf{S}} - X(I)| \geqslant 2^{-1}\delta(X_{\mathbf{S}}) \approx \ell(Q_{\mathbf{S}}) \gtrsim 2^N \ell(Q) \gtrsim 4^N\,\ell(I) \approx 4^N\,\delta(X(I)).$$

By Caccioppoli's and Harnack's inequalities, and Lemma 4.22 we obtain

$$\mathrm{II} \leqslant \sum_{\substack{I\in\mathcal{W}:I\subset\frac{3}{2}\widetilde{B}_Q\\ \ell(I)\lesssim 2^{-N}\ell(Q)}} |I| \left(\fint\!\!\!\fint_I |\nabla\mathcal{G}|^2\,dX\right)^{\frac12} \lesssim \sum_{\substack{I\in\mathcal{W}:I\subset\frac{3}{2}\widetilde{B}_Q\\ \ell(I)\lesssim 2^{-N}\ell(Q)}} \ell(I)^n \left(\fint\!\!\!\fint_{I^*} |\mathcal{G}|^2\,dX\right)^{\frac12}$$
$$\approx \sum_{\substack{I\in\mathcal{W}:I\subset\frac{3}{2}\widetilde{B}_Q\\ \ell(I)\lesssim 2^{-N}\ell(Q)}} \ell(I)^n \mathcal{G}(X(I)) \lesssim \sum_{\substack{I\in\mathcal{W}:I\subset\frac{3}{2}\widetilde{B}_Q\\ \ell(I)\lesssim 2^{-N}\ell(Q)}} \ell(I)\,\mu(\Delta(\widehat{x}_I, 2\delta(X(I)))).$$

To proceed, we observe that the family $\{\Delta(\widehat{x}_I, 2\delta(X(I)))) : I \in \mathcal{W}, \ell(I) = 2^{-k}\ell(Q)\}$ has bounded overlap, and that each such surface ball is contained in $2\widetilde{\Delta}_Q$ provided N is large enough, $I \subset \frac{3}{2}\widetilde{B}_Q$, and $k \gtrsim N$. Thus, it follows from the corona for ω_L (more precisely from (4.22) and the fact that $\mathbf{S} \subset \mathbf{S}(N)$) and the ADR property that

$$\mathrm{II} \lesssim \sum_{k\gtrsim N} 2^{-k}\ell(Q) \sum_{\substack{I\in\mathcal{W}:I\subset\frac{3}{2}\widetilde{B}_Q\\ \ell(I)=2^{-k}\ell(Q)}} \mu(\Delta(\widehat{x}_I, 2\delta(X(I))))$$
$$\lesssim \sum_{k\gtrsim N} 2^{-k}\ell(Q)\,\mu\Bigg(\bigcup_{\substack{I\in\mathcal{W}:I\subset\frac{3}{2}\widetilde{B}_Q\\ \ell(I)=2^{-k}\ell(Q)}} \Delta(\widehat{x}_I, 2\delta(X(I)))\Bigg)$$
$$\leqslant \sum_{k\gtrsim N} 2^{-k}\ell(Q)\,\mu(2\widetilde{\Delta}_Q) \lesssim 2^{-N}\ell(Q)\,\mu(2\widetilde{\Delta}_Q) \leqslant \frac12\ell(Q)\mu(Q),$$

provided N is large enough. As a result, in (4.25) we can hide II to arrive at

$$\frac12\ell(Q)\mu(Q) \leqslant \mathrm{I} = \iint_{\frac{5}{4}\widetilde{B}_Q\cap\{X\in\Omega:\delta(X)\geqslant 2^{-N+1}\ell(Q)\}} |\nabla\mathcal{G}|\,dX.$$

Cover $\frac{5}{4}\widetilde{B}_Q$ with a family of balls $\{B_k\}_{k=1}^K$ with $B_k = B(X_k, 2^{-N-1}\,\ell(Q))$, $X_k \in \widetilde{B}_Q$, and where K is uniformly bounded and depends on n, ADR, and N. Then,

$$\frac12\ell(Q)\mu(Q) \leqslant \mathrm{I} \leqslant \sum_{k=1}^K \iint_{B_k\cap\{X\in\Omega:\delta(X)\geqslant 2^{-N+1}\ell(Q)\}} |\nabla\mathcal{G}|\,dX$$
$$\lesssim_N \max_{1\leqslant k\leqslant K} \iint_{B_k\cap\{X\in\Omega:\delta(X)\geqslant 2^{-N+1}\ell(Q)\}} |\nabla\mathcal{G}|\,dX.$$

Take $1 \leqslant k_Q \leqslant K$ such that the maximum is attained. Set $E_Q := B_{k_Q} \cap \{X \in \Omega : \delta(X) \geqslant 2^{-N+1}\ell(Q)\}$ and note that

$$\frac12\ell(Q)\mu(Q) \lesssim_N \iint_{E_Q} |\nabla\mathcal{G}|\,dX \lesssim (2^{-N}\,\ell(Q))^{n+1}\|\nabla\mathcal{G}\|_{L^\infty(E_Q)} \lesssim \ell(Q)\sigma(Q)\|\nabla\mathcal{G}\|_{L^\infty(E_Q)}.$$

This gives that $|E_Q| > 0$ and the second estimate in (4.24). Moreover, for any $Y \in E_Q$

$$\delta(Y) \leqslant |Y - x_Q| \leqslant |Y - X_{k_Q}| + |X_{k_Q} - x_Q| < 2^{-N-1}\ell(Q) + \frac54\,r(\widetilde{B}_Q) < \frac34\Xi r_Q + \frac54\,r(\widetilde{B}_Q) = 2r(\widetilde{B}_Q).$$

Consequently, $Y \in 2\widetilde{B}_Q \cap \Omega$ and $2^{-N+1}\ell(Q) \leqslant \delta(Y) \lesssim \ell(Q)$. Note also that if $X \in E_Q$, then

$$|X - Y| < 2^{-N}\ell(Q) \leqslant \delta(Y)/2.$$

As a result,

$$\ell(Q)\mu(Q) \lesssim_N \iint_{E_Q} |\nabla \mathcal{G}|\, dX \lesssim_N \ell(Q)^{\frac{n+1}{2}} \Big(\iint_{B(Y,\delta(Y)/2)} |\nabla \mathcal{G}|^2\, dX \Big)^{\frac{1}{2}}$$
$$\lesssim \ell(Q)^{\frac{n-1}{2}} \Big(\iint_{B(Y,3\delta(Y)/4)} |\mathcal{G}|^2\, dX \Big)^{\frac{1}{2}} \approx \sigma(Q)\mathcal{G}(Y),$$

where we have used Caccioppoli's and Harnack's inequalities. This readily leads to the desired estimate completing the proof. □

Remark 4.26. We need to make the following observation which will be used below to obtain Theorem 4.5. In the previous proof the fact that $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ is the 2^N-refinement of $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$ has been only used at the very beginning to make sure that $\ell(Q) \lesssim 2^{-N}\ell(Q_{\mathbf{S}})$. This means that one can state a version of Lemma 4.25 for the original corona decomposition $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$ as follows: Fixed $\mathbf{S} \in \mathbb{F}_0$ and defining μ and $\mathcal{G}$ as in (4.23), there exists $N \gg 1$, depending on the allowable parameters and on the implicit constants in (4.7), such that if $Q \in \mathbf{S} \in \mathbb{F}$ with $\ell(Q) \leqslant 2^{-N}\,\ell(\mathrm{Top}(\mathbf{S}))$, then (4.24) holds. Details are left to the interested reader.

We are now ready to continue with the proof. Apply Lemma 4.25 with N large enough. For $Q \in \mathbf{S}$, we have $|E_Q| > 0$, thus there exists $Y_Q \in E_Q$. On account of the normalization introduced in (4.23), we then obtain

$$\sup_{\substack{X \in 2\widetilde{B}_Q \cap \Omega \\ \delta(X) \geqslant 2^{-N}\ell(Q)}} \frac{G_L(X_{\mathbf{S}}, X)}{\delta(X)} \geqslant \frac{G_L(X_{\mathbf{S}}, Y_Q)}{\delta(Y_Q)} = \frac{1}{\sigma(Q_{\mathbf{S}})}\frac{\mathcal{G}(Y_Q)}{\delta(Y_Q)} \gtrsim \frac{1}{\sigma(Q_{\mathbf{S}})}\frac{\mu(Q)}{\sigma(Q)} = \frac{\omega_L^{X_{\mathbf{S}}}(Q)}{\sigma(Q)} \approx \frac{\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})}{\sigma(Q_{\mathbf{S}})},$$

Step 2: the upper bound. We next establish the converse inequality. Fix $X \in 2\widetilde{B}_Q \cap \Omega$ with $\delta(X) \gtrsim \ell(Q)$. Note that by (4.10)

$$\delta(X) \leqslant 2\,\Xi\, r_Q \leqslant 2\,\Xi\,\ell(Q) \lesssim 2^{-N}\,\Xi\,\ell(Q_{\mathbf{S}}) \approx 2^{-N}\,\Xi\,\delta(X_{\mathbf{S}}) < 2^{-1}\,\delta(X_{\mathbf{S}}),$$

provided N is large enough, and

$$2\delta(X) < \delta(X_{\mathbf{S}}) \leqslant |X_{\mathbf{S}} - X| + \delta(X).$$

Hence, $|X_{\mathbf{S}} - X| > \delta(X)$. Invoking then (4.20),

$$\frac{G_L(X_{\mathbf{S}}, X)}{\delta(X)} \lesssim \frac{\omega_L^{X_{\mathbf{S}}}(\Delta(\widehat{x}, 2\,\delta(X)))}{\delta(X)^n}, \tag{4.26}$$

where $\widehat{x} \in \partial\Omega$ is such that $|X - \widehat{x}| = \delta(X)$. For any $y \in \Delta(\widehat{x}, 2\,\delta(X))$ we observe that

$$|y - x_Q| \leqslant |y - \widehat{x}| + |\widehat{x} - X| + |X - x_Q| \leqslant 3\delta(X) + |X - x_Q| \leqslant 4|X - x_Q| \leqslant 8\Xi r_Q,$$

hence $\Delta(\widehat{x}, 2\,\delta(X)) \subset 8\widetilde{\Delta}_Q$. Also, if $z \in 8\widetilde{\Delta}_Q$, recalling that we write $Q^{(N)}$ for the N-th dyadic ancestor of Q (that is the unique dyadic cube containing Q and with sidelength $2^N \ell(Q)$), we have

$$|z - x_{Q^{(N)}}| \leqslant |z - x_Q| + |x_Q - x_{Q^{(N)}}| \leqslant 8\Xi r_Q + \Xi\, r_{Q^{(N)}} < 2\,\Xi\, r_{Q^{(N)}},$$

provided $N \geqslant 3$, since $r_Q = 2^{-N} r_{Q^{(N)}}$. Altogether,

$$\Delta(\widehat{x}, 2\delta(X)) \subset 8\widetilde{\Delta}_Q \subset 2\widetilde{\Delta}_{Q^{(N)}}. \tag{4.27}$$

Note that $Q \in \mathbf{S}$ and $Q^{(N)}$ are 2^N-close, hence $Q^{(N)} \in \mathbf{S}(N)$. Thus, we can invoke (4.26), (4.27), and (4.22) to conclude that

$$\frac{G_L(X_{\mathbf{S}}, X)}{\delta(X)} \lesssim \frac{\omega_L^{X_{\mathbf{S}}}(2\widetilde{\Delta}_{Q^{(N)}})}{\delta(X)^n} \approx \frac{\omega_L^{X_{\mathbf{S}}}(2\widetilde{\Delta}_{Q^{(N)}})}{\sigma(2\widetilde{\Delta}_{Q^{(N)}})} \lesssim \frac{\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})}{\sigma(Q_{\mathbf{S}})}.$$

This completes the proof of the current implication, with $\Lambda_{\mathbf{S}} := \omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})/\sigma(Q_{\mathbf{S}})$. □

4.5.2 Proof of (c) $\Longrightarrow$ (d): if G_L is comparable to the distance to the boundary in the corona sense, then L satisfies partial/weak Carleson measure estimates

Let $u \in L^\infty(\Omega)$ be a non-trivial weak solution of $Lu = -\operatorname{div}(A\nabla u) = 0$ in Ω. By homogeneity, we may assume that $\|u\|_{L^\infty(\Omega)} = 1$. Fix also $\tau \in (0, \frac{1}{2})$, the parameter which appears in the partial/weak Carleson measure estimate (see Definition 4.15). Assume that $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$ is the assumed corona decomposition associated with G_L.

Step 1: $\delta \lesssim G_L$ in carefully chosen regions. Let $N \geqslant 0$ be large enough to be chosen momentarily (depending on τ). We may proceed as in Remark 4.12 and pick $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ the 2^N-refinement of $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$. This implies that for every $\mathbf{S} \in \mathbb{F}$ there exist $\Lambda_{\mathbf{S}} > 0$, $Q_{\mathbf{S}} \in \mathbb{D}$, and $X_{\mathbf{S}} \in \Omega$ so that

$$\operatorname{Top}(\mathbf{S}) \subset \operatorname{Top}(\mathbf{S})^{(N)} \subset Q_{\mathbf{S}}, \quad \delta(X_{\mathbf{S}}) \geqslant 4\Xi\ell(Q_{\mathbf{S}}), \quad \operatorname{dist}(X_{\mathbf{S}}, Q_{\mathbf{S}}) \lesssim \ell(Q_{\mathbf{S}}), \tag{4.28}$$

and

$$\sup_{\substack{X \in 2\widetilde{B}_Q \cap \Omega \\ \delta(X) \geqslant c_0\,\ell(Q)}} \frac{G_L(X_{\mathbf{S}}, X)}{\delta(X)} \approx \Lambda_{\mathbf{S}}, \quad \forall Q \in \mathbf{S}(N), \tag{4.29}$$

for some $c_0 \in (0, \frac{1}{2})$. Write $G_{\mathbf{S}} := G_L(X_{\mathbf{S}}, \cdot)$. For each $Q \in \mathbf{S} \subset \mathbf{S}(N)$, we can find $P_Q \in 2\,\widetilde{B}_Q \cap \Omega$ with $\delta(P_Q) \geqslant c_0 \ell(Q)$ such that by (4.29) and Harnack's inequality there holds

$$\Lambda_{\mathbf{S}} \lesssim \frac{G_{\mathbf{S}}(P_Q)}{\delta(P_Q)} \approx_\tau \frac{G_{\mathbf{S}}(X)}{\delta(X)}, \quad \forall X \in V_Q := B(P_Q, (1-\tau)\delta(P_Q)). \tag{4.30}$$

This defines P_Q for every $Q \in \mathbb{G} = \bigsqcup_{\mathbf{S} \in \mathbb{F}} \mathbf{S}$. On the other hand, given $Q \in \mathbb{B} = \mathbb{D} \backslash \mathbb{G}$ we let $P_Q := X_Q$ be the corkscrew point relative to Q, in particular, $X_Q \in B_Q \cap \Omega \subset 2\,\widetilde{B}_Q \cap \Omega$ with $\delta(P_Q) \gtrsim \ell(Q)$.

Step 2: reduction to a single integral in a sawtooth region $\Omega^\vartheta_{\mathcal{F}, Q_0}$. Set next

$$\alpha_Q := \iint_{V_Q} |\nabla u(X)|^2 \delta(X)\, dX, \quad Q \in \mathbb{D},$$

and note that by Caccioppoli's inequality

$$\alpha_Q \lesssim \delta(P_Q) \iint_{V_Q} |\nabla u(X)|^2 dX \lesssim_\tau \delta(P_Q)^{-1} \iint_{B(P_Q,(1-\tau/2)\delta(P_Q))} |u(X)|^2 dX$$
$$\lesssim \delta(P_Q)^n \approx \ell(Q)^n \approx \sigma(Q). \quad (4.31)$$

We first claim that

$$\sup_{\mathbf{S}\in\mathbb{F}, Q\in\mathbf{S}} \frac{1}{\sigma(Q)} \sum_{Q'\in\mathbf{S}\cap\mathbb{D}_Q} \alpha_{Q'} \lesssim 1. \quad (4.32)$$

Assuming this momentarily let us continue as follows. Introduce some notation: if $Q_0 \in \mathbb{G}$ we define $\mathbf{S}_{Q_0}$ to be the unique $\mathbf{S} \in \mathbb{F}$ so that $Q_0 \in \mathbf{S}$, otherwise, if $Q_0 \in \mathbb{B}$ then we set $\mathbf{S}_{Q_0} = \varnothing$. We first see that

$$\mathbb{D}_{Q_0} \cap \mathbb{G} = \Big(\bigcup_{\mathbf{S}\in\mathbb{F}:\mathrm{Top}(\mathbf{S})\subsetneq Q_0} \mathbf{S} \Big) \bigcup \big(\mathbf{S}_{Q_0} \cap \mathbb{D}_{Q_0}\big). \quad (4.33)$$

Indeed, let $Q \in \mathbb{D}_{Q_0} \cap \mathbb{G}$, then $Q \in \mathbf{S}$ for some $\mathbf{S} \in \mathbb{F}$. In particular $Q \subset \mathrm{Top}(\mathbf{S}) \cap Q_0$. If $\mathrm{Top}(\mathbf{S}) \subsetneq Q_0$, then Q in the first set of the right hand side of the last expression. Otherwise, if $Q_0 \subset \mathrm{Top}(\mathbf{S})$, then $\mathbf{S} \ni Q \subset Q_0 \subset \mathrm{Top}(\mathbf{S})$ and the semi-coherency of $\mathbf{S}$ implies that $Q_0 \in \mathbf{S} \subset \mathbb{G}$, hence, necessarily $\mathbf{S} = \mathbf{S}_{Q_0}$ and $Q \in \mathbf{S}_{Q_0} \cap \mathbb{D}_{Q_0}$. Once (4.33) has been shown, we see that (4.31) and (4.32) yield

$$\begin{aligned}
\sum_{Q\in\mathbb{D}_{Q_0}} \alpha_Q &= \sum_{Q\in\mathbb{B}\cap\mathbb{D}_{Q_0}} \alpha_Q + \sum_{Q\in\mathbb{G}\cap\mathbb{D}_{Q_0}} \alpha_Q \\
&\leqslant \sum_{Q\in\mathbb{B}\cap\mathbb{D}_{Q_0}} \alpha_Q + \sum_{\mathbf{S}\in\mathbb{F}:\mathrm{Top}(\mathbf{S})\subsetneq Q_0} \sum_{Q\in\mathbf{S}} \alpha_Q + \sum_{Q\in\mathbf{S}_{Q_0}\cap\mathbb{D}_{Q_0}} \alpha_Q \\
&\lesssim \sum_{Q\in\mathbb{B}\cap\mathbb{D}_{Q_0}} \sigma(Q) + \sum_{\mathbf{S}\in\mathbb{F}:\mathrm{Top}(\mathbf{S})\subsetneq Q_0} \sigma(\mathrm{Top}(\mathbf{S})) + \sigma(Q_0) \\
&\leqslant \sum_{Q\in\mathbb{B}\cap\mathbb{D}_{Q_0}} \sigma(Q) + \sum_{Q\in\mathrm{Top}(\mathbb{F})\cap\mathbb{D}_{Q_0}} \sigma(Q) + \sigma(Q_0) \\
&\lesssim \sigma(Q_0), \quad (4.34)
\end{aligned}$$

where in the last estimate we have used that the families $\mathbb{B}$ and $\mathrm{Top}(\mathbb{F})$ satisfy a Carleson packing condition. Since $Q_0 \in \mathbb{D}$ is arbitrary this allows us to obtain our desired estimate modulo our claim (4.32) which we prove next.

To see (4.32) we fix $\mathbf{S} \in \mathbb{F}$ and $Q_0 \in \mathbf{S}$, in particular $\mathbf{S} \cap \mathbb{D}_{Q_0} \neq \varnothing$. Set $\mathbf{S}' := \mathbf{S} \cap \mathbb{D}_{Q_0}$ which is clearly semi-coherent with $\mathrm{Top}(\mathbf{S}') = Q_0$. Introduce $\mathcal{F}$, the family of maximal dyadic cubes (hence, pairwise disjoint) in $\mathbb{D}_{Q_0}\backslash\mathbf{S}'$. We claim that $\mathbb{D}_{\mathcal{F},Q_0} = \mathbf{S}'$. Indeed, if $Q \in \mathbb{D}_{\mathcal{F},Q_0}\backslash\mathbf{S}'$ then $Q \subset Q'$ for some $Q' \in \mathcal{F}$, a contradiction. On the other hand, if $Q \in \mathbf{S}'\backslash\mathbb{D}_{\mathcal{F},Q_0}$, then $Q \subset Q' \subset Q_0$ for some $Q' \in \mathcal{F}$. Noting that $Q \in \mathbf{S}'$ with $\mathbf{S}'$ being semi-coherent, we conclude that $Q' \in \mathbf{S}'$ which leads again to a contradiction.

To continue we observe that for every $Q \in \mathbf{S}' \subset \mathbf{S}$ we have by (4.30)

$$\delta(X) \lesssim_\tau \Lambda_{\mathbf{S}}^{-1} G_{\mathbf{S}}(X), \qquad \forall\, X \in V_Q.$$

This and Lemma 4.19 allow us to deduce that

$$\sum_{Q\in\mathbf{S}'}\alpha_Q=\sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}}\alpha_Q=\sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}}\iint_{V_Q}|\nabla u|^2\,\delta\,dX$$
$$\lesssim\Lambda_{\mathbf{S}}^{-1}\sum_{Q\in\mathbb{D}_{\mathcal{F},Q_0}}\iint_{V_Q}|\nabla u|^2\,G_{\mathbf{S}}\,dX\lesssim\Lambda_{\mathbf{S}}^{-1}\iint_{\Omega^{\vartheta}_{\mathcal{F},Q_0}}|\nabla u|^2\,G_{\mathbf{S}}\,dX,\quad(4.35)$$

where ϑ depends on the allowable constants but is independent of N.

Step 3: upper bounds for G_L within the sawtooth region. Observe that by (4.12) and (4.10) one has

$$\Omega^{\vartheta}_{\mathcal{F},Q_0}\subset T^{\vartheta,**}_{Q_0}\subset\frac{1}{2}B^*_{Q_0}\cap\Omega\subset\{X\in\Omega:\delta(X)\leqslant\kappa_0\Xi\ell(Q_0)\}.$$

On the other hand, from (4.28), and the fact that $Q_0\in\mathbf{S}$ we can readily see that

$$\delta(X_{\mathbf{S}})\approx\ell(Q_{\mathbf{S}})\gtrsim 2^N\ell(Q_0)\gg\kappa_0\Xi\ell(Q_0),$$

by choosing N large enough (depending eventually on τ). This means that $X_{\mathbf{S}}$ is far away from $B^*_{Q_0}$ and, in what comes after we never get to worry about the position of the pole of the Green function, $X_{\mathbf{S}}$, since it is always far from where the integrations take place. We warn the reader that, henceforth, we will make use of this observation repeatedly without explicitly mentioning it.

To proceed, let $I\in\mathcal{W}^{\vartheta}_Q$ with $Q\in\mathbb{D}_{\mathcal{F},Q_0}=\mathbf{S}\cap\mathbb{D}_{Q_0}$. Caccioppoli's and Harnack's inequalities yield

$$\iint_{I^{**}}\big(|\nabla G_{\mathbf{S}}|+|\nabla u|\,G_{\mathbf{S}}\big)\,dX$$
$$\lesssim\ell(I)^{\frac{n-1}{2}}\Big(\iint_{I^{***}}|G_{\mathbf{S}}|^2\,dX\Big)^{\frac12}+\ell(I)^{-1}\Big(\iint_{I^{***}}|u|^2\,dX\Big)^{\frac12}\Big(\iint_{I^{**}}|G_{\mathbf{S}}|^2\,dX\Big)^{\frac12}$$
$$\lesssim\ell(I)^n\,G_{\mathbf{S}}(X(I)).\quad(4.36)$$

Let $x_I\in\partial\Omega$ be such that $|X(I)-x_I|=\delta(X(I))$ and pick $Q_I\in\mathbb{D}$ satisfying $Q_I\ni x_I$ and $\delta(X(I))\leqslant\ell(Q_I)<2\delta(X(I))$. Note that (4.9)–(4.10) give

$$|X(I)-x_{Q_I}|\leqslant|X(I)-x_I|+|x_I-x_{Q_I}|<\delta(X(I))+\Xi\,r_{Q_I}\leqslant 2\Xi\,r_{Q_I},$$

hence $X(I)\in 2\widetilde{B}_{Q_I}$. Besides,

$$\ell(Q_I)\approx\delta(X(I))\approx\ell(I)\approx_{\vartheta}\ell(Q)$$

and

$$\operatorname{dist}(Q,Q_I)\leqslant\operatorname{dist}(Q,I)+\operatorname{diam}(I)+|X(I)-x_I|\lesssim_{\vartheta}\ell(Q)+\ell(Q_I).$$

By choosing (and fixing) N large enough (depending on ϑ) we therefore obtain that Q_I and $Q\in\mathbf{S}$ are 2^N-close, hence $Q_I\in\mathbf{S}(N)$. Using that $c_0<\frac12$, we can next invoke (4.29) to obtain that

$$\frac{G_{\mathbf{S}}(X(I))}{\delta(X(I))}\leqslant\sup_{\substack{X\in 2\widetilde{B}_{Q_I}\cap\Omega\\ \delta(X)\geqslant\frac12\ell(Q_I)}}\frac{G_L(X_{\mathbf{S}},X)}{\delta(X)}\leqslant\sup_{\substack{X\in 2\widetilde{B}_{Q_I}\cap\Omega\\ \delta(X)\geqslant c_0\ell(Q_I)}}\frac{G_L(X_{\mathbf{S}},X)}{\delta(X)}\approx\Lambda_{\mathbf{S}}.\quad(4.37)$$

This together with (4.36) leads to

$$\iint\limits_{I^{**}} \big(|\nabla G_{\mathbf{S}}| + |\nabla u|\, G_{\mathbf{S}}\big)\, dX \lesssim \Lambda_{\mathbf{S}}\, \ell(I)^{n+1}, \tag{4.38}$$

for every $I \in \mathcal{W}_Q^\vartheta$ with $Q \in \mathbb{D}_{\mathcal{F},Q_0}$.

Step 4: integration by parts in the sawtooth region. To continue, for every $M \geqslant 1$, we consider the pairwise disjoint collection $\mathcal{F}_M$ given by the family of maximal cubes of the collection $\mathcal{F}$ augmented by adding all the cubes $Q \in \mathbb{D}_{Q_0}$ such that $\ell(Q) \leqslant 2^{-M}\ell(Q_0)$. In particular, $Q \in \mathbb{D}_{\mathcal{F}_M,Q_0}$ if and only if $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ and $\ell(Q) > 2^{-M}\ell(Q_0)$. Moreover, $\mathbb{D}_{\mathcal{F}_M,Q_0} \subset \mathbb{D}_{\mathcal{F}_{M'},Q_0}$ for all $M \leqslant M'$, and hence $\Omega^\vartheta_{\mathcal{F}_M,Q_0} \subset \Omega^\vartheta_{\mathcal{F}_{M'},Q_0} \subset \Omega^\vartheta_{\mathcal{F},Q_0}$. Then the monotone convergence theorem implies

$$\iint\limits_{\Omega^\vartheta_{\mathcal{F},Q_0}} |\nabla u|^2\, G_{\mathbf{S}}\, dX = \lim_{M\to\infty} \iint\limits_{\Omega^\vartheta_{\mathcal{F}_M,Q_0}} |\nabla u|^2\, G_{\mathbf{S}}\, dX =: \lim_{M\to\infty} \mathcal{K}_M. \tag{4.39}$$

We claim that

$$\mathcal{K}_M \lesssim \Lambda_{\mathbf{S}}\, \sigma(Q_0), \tag{4.40}$$

where the implicit constant is independent of M, $\mathbf{S}$, and Q_0. Assuming that (4.40) holds momentarily, we obtain at once that (4.35), (4.39) and (4.40) give

$$\sum_{Q\in\mathbf{S}'} \alpha_Q \lesssim \sigma(Q_0).$$

This finally justifies (4.32) and, as explained above, the desired partial CME estimates follow.

The rest of this section is devoted to proving (4.40) and we borrow some ideas from [CHMT20]. Pick Ψ_M from Lemma 4.20 and use Leibniz's rule to arrive at

$$\begin{aligned} A\nabla u\cdot\nabla u\, G_{\mathbf{S}}\Psi_M^2 = A\nabla u\cdot\nabla(uG_{\mathbf{S}}\Psi_M^2) - \frac{1}{2}A\nabla(u^2\Psi_M^2)\cdot\nabla G_{\mathbf{S}} \\ + \frac{1}{2}A\nabla(\Psi_M^2)\cdot\nabla G_{\mathbf{S}}\, u^2 - \frac{1}{2}A\nabla(u^2)\cdot\nabla(\Psi_M^2)G_{\mathbf{S}}. \end{aligned} \tag{4.41}$$

Note that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)\cap L^\infty(\Omega)$, $G_{\mathbf{S}} \in W^{1,2}_{\mathrm{loc}}(\Omega\backslash\{X_{\mathbf{S}}\})$, and that $\overline{\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0}}$ is a compact subset of Ω away from $X_{\mathbf{S}}$. Hence, $u \in W^{1,2}(\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0})$ and $uG_{\mathbf{S}}\Psi_M^2 \in W^{1,2}_0(\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0})$. These, together with the fact that $Lu = 0$ in the weak sense in Ω, lead to

$$\iint\limits_{\Omega} A\nabla u\cdot\nabla(uG_{\mathbf{S}}\Psi_M^2)dX = \iint\limits_{\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0}} A\nabla u\cdot\nabla(uG_{\mathbf{S}}\Psi_M^2)dX = 0.$$

On the other hand, $G_{\mathbf{S}} \in W^{1,2}(\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0})$ and $L^\top G_{\mathbf{S}} = 0$ in the weak sense in $\Omega\backslash\{X_{\mathbf{S}}\}$. Thanks to the fact that $u^2\Psi_M^2 \in W^{1,2}_0(\Omega^{**}_{\mathcal{F}_M,Q_0})$, we then obtain

$$\iint\limits_{\Omega} A\nabla(u^2\Psi_M^2)\cdot\nabla G_{\mathbf{S}}\, dX = \iint\limits_{\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0}} A^\top\nabla G_{\mathbf{S}}\cdot\nabla(u^2\Psi_M^2)\, dX = 0. \tag{4.42}$$

By Lemma 4.20, the ellipticity of A, (4.41)–(4.42), the fact that $\|u\|_{L^\infty(\Omega)} = 1$, and (4.38) we arrive at

$$\mathcal{K}_M \lesssim \iint_\Omega A\nabla u\cdot\nabla u\, G_{\mathbf{S}}\Psi_M^2\, dX \lesssim \iint_\Omega \big(|\nabla G_{\mathbf{S}}| + |\nabla u|\, G_{\mathbf{S}}\big)\,|\nabla\Psi_M|\, dX$$
$$\lesssim \sum_{I\in\mathcal{W}_M^{\vartheta,\Sigma}} \ell(I)^{-1} \iint_{I^{**}} \big(|\nabla G_{\mathbf{S}}| + |\nabla u|\, G_{\mathbf{S}}\big)\, dX \lesssim \Lambda_{\mathbf{S}} \sum_{I\in\mathcal{W}_M^{\vartheta,\Sigma}} \ell(I)^n. \quad (4.43)$$

where we have used that if $I\in\mathcal{W}_M^{\vartheta,\Sigma}$ then $I\in\mathcal{W}_Q^{\vartheta}$ with $Q\in\mathbb{D}_{\mathcal{F}_M,Q_0}\subset\mathbb{D}_{\mathcal{F},Q_0}$. We use again Lemma 4.20 to observe that

$$\sum_{I\in\mathcal{W}_M^{\vartheta,\Sigma}} \ell(I)^n \approx \sum_{I\in\mathcal{W}_M^{\vartheta,\Sigma}} \sigma(\widehat{Q}_I) \lesssim \sigma\Big(\bigcup_{I\in\mathcal{W}_M^{\vartheta,\Sigma}} \widehat{Q}_I\Big) \leqslant \sigma(C\Delta_{Q_0}) \approx \sigma(Q_0), \quad (4.44)$$

where in the next-to-last inequality we have used that $\widehat{Q}_I\subset C\Delta_{Q_0}$ for every $I\in\mathcal{W}_M^{\vartheta,\Sigma}$. Indeed by Lemma 4.20 we have that if $x\in\widehat{Q}_I$ where $I\in\mathcal{W}_M^{\vartheta,\Sigma}$ and we let $Q\in\mathbb{D}_{\mathcal{F}_M,Q_0}$ be so that $I\in\mathcal{W}_Q^{\vartheta}$, then

$$|x-x_{Q_0}| \leqslant \operatorname{diam}(\widehat{Q}_I) + \operatorname{dist}(\widehat{Q}_I, I) + \operatorname{diam}(I) + \operatorname{dist}(I,Q) + \operatorname{diam}(Q_0)$$
$$\lesssim \ell(I) + \ell(Q) + \ell(Q_0) \approx \ell(Q) + \ell(Q_0) \lesssim \ell(Q_0).$$

Collecting (4.43) and (4.44) we conclude as desired (4.40), and the proof is then complete. □

4.5.3 Proof of (d) $\Longrightarrow$ (a): if L satisfies partial/weak Carleson measure estimates, then ω_L admits a strong corona decomposition

We introduce some notation, given $N\geqslant 1$, for any $Q\in\mathbb{D}$, we let $Q(N)\in\mathbb{D}_Q$ be the unique dyadic cube such that $Q(N)\ni x_Q$ and $\ell(Q(N)) = 2^{-N}\ell(Q)$.

Step 1: the basic construction, a single stopping-time regime. We begin with some auxiliary result which will be iterated to construct the desired corona decomposition:

Proposition 4.27. *Let $0<\tau<\tau_0$ (cf. Lemma 4.23) be the parameter implicitly assumed in (d). There exists N_τ (depending on n, ADR, ellipticity, and τ) such that for every $Q\in\mathbb{D}$ and for each $N\gg N_\tau$, if we set $Y_Q = P_{Q(N_\tau)}$ (where $P_{Q(N_\tau)}$ is the point associated with $Q(N_\tau)$ in (d)), one can then find possibly empty families of pairwise disjoint cubes $\mathcal{F}_Q = \mathcal{F}_Q^+\sqcup\mathcal{F}_Q^-\subset\mathbb{D}_Q\backslash\{Q\}$, a Borel function f_Q, and $u_Q\in W^{1,2}_{\rm loc}(\Omega)$ with $Lu_Q = 0$ in the weak sense in Ω, such that the following hold:*

$$\delta(Y_Q)\approx\ell(Q)\approx\operatorname{dist}(Y_Q,Q); \qquad \omega_L^{Y_Q}(Q)\approx 1; \quad (4.45)$$

$$2^{-N}\frac{\omega_L^{Y_Q}(Q)}{\sigma(Q)} \lesssim \frac{\omega_L^{Y_Q}(Q')}{\sigma(Q')} \leqslant \Big(\fint_{Q'}(\mathcal{M}\omega_L^{Y_Q})^{\frac12}\,d\sigma\Big)^2 \lesssim 2^{2N}\frac{\omega_L^{Y_Q}(Q)}{\sigma(Q)}, \quad (4.46)$$

for all $Q'\in\mathbb{D}_{\mathcal{F}_Q,Q}$. Moreover, if we set $F_Q^\pm := \bigcup_{Q'\in\mathcal{F}_Q^\pm} Q'$, then

$$\sigma(F_Q^+) \leqslant 2^{-N}\,\sigma(Q); \quad (4.47)$$

$$0 \leqslant f_Q \lesssim \mathbf{1}_{Q\setminus F_Q^-}, \qquad u_Q(X) := \int_{Q\setminus F_Q^-} f_Q \, d\omega_L^X, \quad X \in \Omega, \tag{4.48}$$

and

$$\sigma(Q) \lesssim \iint\limits_{B(Y_Q,(1-\tau)\delta(Y_Q))} |\nabla u_Q(X)|^2 \, \delta(X). \tag{4.49}$$

In the previous estimates the implicit constants depend on n, ADR, ellipticity, and τ, but they do not depend on N.

Proof. Write $C_0 > 1$ for the constant in Bourgain's estimate (see (4.19)). Let N_τ be large enough to be chosen momentarily so that $2^{N_\tau} \gg C_0$. Let $Y_Q := P_{Q(N_\tau)}$ be the point associated with $Q(N_\tau)$ in (d). Observe that since $x_Q \in Q(N_\tau)$ we have

$$\begin{aligned} |Y_Q - x_Q| &\leqslant \operatorname{dist}(Y_Q, Q(N_\tau)) + \operatorname{diam}(Q(N_\tau)) \\ &\approx \ell(Q(N_\tau)) = 2^{-N_\tau}\ell(Q) \ll \ell(Q). \end{aligned} \tag{4.50}$$

Hence, by taking N_τ large enough, Bourgain's estimate (see Lemma 4.22) and (4.9) imply

$$C_0^{-1} \leqslant \omega_L^{Y_Q}(Q) \leqslant 1. \tag{4.51}$$

Note that by (4.50) we have $Y_Q \in \frac{1}{2}B_Q \cap \Omega$ (cf. (4.10)) and also $\delta(Y_Q) = \delta(P_{Q(N_\tau)}) \approx \ell(Q(N_\tau)) = 2^{-N_\tau}\,\ell(Q)$. Hence, in the context of Lemma 4.23, applied to $\varepsilon = 2^{-N_\tau} \ll \varepsilon_\tau$ so that we have that $\delta(Y_Q) \approx \varepsilon\,\ell(Q)$.

Set $\omega := \sigma(Q)\omega_L^{Y_Q}$ and note that

$$C_0^{-1} \leqslant \frac{\omega(Q)}{\sigma(Q)} \leqslant \frac{\omega(\partial\Omega)}{\sigma(Q)} \leqslant 1. \tag{4.52}$$

Write $\mathcal{M}$ for the Hardy-Littlewood maximal operator on $\partial\Omega$ (with respect to σ) and observe that by the Hardy-Littlewood theorem and (4.52) one has

$$\|\mathcal{M}\omega\|_{L^{1,\infty}(\partial\Omega,\sigma)} \lesssim \omega(\partial\Omega) \lesssim \sigma(Q).$$

This and Kolmogorov's inequality imply

$$\left(\fint_Q (\mathcal{M}\omega)^{\frac{1}{2}}\, d\sigma\right)^2 \lesssim \sigma(Q)^{-1}\,\|\mathcal{M}\omega\|_{L^{1,\infty}(\partial\Omega,\sigma)} \lesssim 1. \tag{4.53}$$

Let $N \gg N_\tau$ be large enough to be chosen. Subdivide dyadically Q and stop the first time that one of the following two conditions occurs

$$\frac{\omega(Q')}{\sigma(Q')} < 2^{-N}, \qquad \left(\fint_{Q'} (\mathcal{M}\omega)^{\frac{1}{2}}\, d\sigma\right)^2 > 2^{2N}. \tag{4.54}$$

This stopping time family is denoted by $\mathcal{F}_Q$ and it is pairwise disjoint. Assuming that N is sufficiently large, (4.52) and (4.53) clearly give that $\mathcal{F}_Q \subset \mathbb{D}_Q\setminus\{Q\}$. Let $\mathcal{F}_Q^-$ be the subcollection of cubes in $\mathcal{F}_Q$

satisfying the first condition in (4.54) and set $\mathcal{F}_Q^+ = \mathcal{F}_Q \backslash \mathcal{F}_Q^-$, that is, the cubes that satisfy the second condition in (4.54), but not the first one. By construction $\mathcal{F}_Q := \mathcal{F}_Q^+ \sqcup \mathcal{F}_Q^-$ and

$$2^{-N} \leqslant \frac{\omega(Q')}{\sigma(Q')} \leqslant \left(\fint_{Q'} (\mathcal{M}\omega)^{\frac{1}{2}} \, d\sigma \right)^2 \leqslant 2^{2N}, \qquad \forall\, Q' \in \mathbb{D}_{\mathcal{F}_Q, Q}.$$

This readily implies (4.46) using (4.51) and $\omega := \sigma(Q)\omega_L^{Y_Q}$. If $\mathcal{F}_Q^+ \neq \varnothing$ we use that the cubes in $\mathcal{F}_Q^+$ are pairwise disjoint and satisfy the second condition in (4.54), (4.53), and (4.52), to arrive at (4.47):

$$\begin{aligned} \sigma(F_Q^+) = \sigma\Big(\bigcup_{Q' \in \mathcal{F}_Q^+} Q' \Big) = \sum_{Q' \in \mathcal{F}_Q^+} \sigma(Q') &\leqslant 2^{-N} \sum_{Q' \in \mathcal{F}_Q^+} \int_{Q'} (\mathcal{M}\omega)^{\frac{1}{2}} \, d\sigma \\ &\leqslant 2^{-N} \int_Q (\mathcal{M}\omega)^{\frac{1}{2}} \, d\sigma \lesssim 2^{-N} \sigma(Q), \end{aligned}$$

provided N is sufficiently large. This estimate clearly holds if $\mathcal{F}_Q^+ = \varnothing$. Hence we have shown (4.47).

To continue if $\mathcal{F}_Q^- \neq \varnothing$, use that the cubes in $\mathcal{F}_Q^-$ are pairwise disjoint and satisfy the first condition in (4.54), and (4.52), to see that

$$\omega(F_Q^-) = \omega\Big(\bigcup_{Q' \in \mathcal{F}_Q^-} Q' \Big) = \sum_{Q' \in \mathcal{F}_Q^-} \omega(Q') \leqslant 2^{-N} \sum_{Q' \in \mathcal{F}_Q^-} \sigma(Q') \leqslant 2^{-N} \sigma(Q) \leqslant C_0 2^{-N} \omega(Q).$$

Again this estimate clearly holds in the case $\mathcal{F}_Q^- = \varnothing$. Thus, in either scenario,

$$\omega_L^{Y_Q}(Q \backslash F_Q^-) \geqslant (1 - C_0\, 2^{-N}) \omega_L^{Y_Q}(Q).$$

Note that, as observed above, $Y_Q \in \frac{1}{2} B_Q \cap \Omega$ and $\delta(Y_Q) \approx \varepsilon\, \ell(Q)$. In view of Lemma 4.23, if one further assumes that N is large enough so that $2^{-N} C_0 < \varepsilon$, we can find a Borel function f_Q and $u_Q \in W^{1,2}_{\rm loc}(\Omega)$, with $Lu_Q = 0$ in the weak sense in Ω, such that (4.48) and (4.49) holds where the implicit constants depend on n, ADR, ellipticity, and τ, but they are independent of N. This completes the proof. □

Step 2: iteration to get the whole corona decomposition. Having Proposition 4.27 at our disposal, we are going to iterate that construction to obtain the desired corona decomposition. With this goal in mind we fix an arbitrary $Q^0 \in \mathbb{D}$. Set $\mathcal{F}_0 := \{Q^0\}$, $\mathcal{F}_0^+ := \{Q^0\}$, and $\mathcal{F}_0^- := \varnothing$. Let $\mathcal{F}_1 := \bigsqcup_{Q \in \mathcal{F}_0} \mathcal{F}_Q$ and $\mathcal{F}_1^\pm := \bigsqcup_{Q \in \mathcal{F}_0} \mathcal{F}_Q^\pm$ denote the first generation cubes. In general, we may define recursively

$$\mathcal{F}_{k+1} := \bigsqcup_{Q \in \mathcal{F}_k} \mathcal{F}_Q \quad \text{and} \quad \mathcal{F}_{k+1}^\pm := \bigsqcup_{Q \in \mathcal{F}_k} \mathcal{F}_Q^\pm, \qquad k \geqslant 0.$$

We also set

$$\mathcal{F} := \bigsqcup_{k=0}^{\infty} \mathcal{F}_k \quad \text{and} \quad \mathcal{F}^\pm := \bigsqcup_{k=0}^{\infty} \mathcal{F}_k^\pm.$$

With all these we construct a semi-coherent corona decomposition relative to Q_0 as follows. Observe first that

$$\mathbb{D}_{Q^0} = \mathbb{D}_{\mathcal{F}_{Q^0}, Q^0} \bigsqcup \Big(\bigsqcup_{Q \in \mathcal{F}_{Q^0}} \mathbb{D}_Q \Big) = \Big(\bigsqcup_{Q \in \mathcal{F}_0} \mathbb{D}_{\mathcal{F}_Q, Q} \Big) \bigsqcup \Big(\bigsqcup_{Q \in \mathcal{F}_1} \mathbb{D}_Q \Big).$$

In turn,

$$\bigsqcup_{Q\in\mathcal{F}_1}\mathbb{D}_Q=\bigsqcup_{Q\in\mathcal{F}_1}\Big(\mathbb{D}_{\mathcal{F}_Q,Q}\bigsqcup\Big(\bigsqcup_{Q'\in\mathcal{F}_Q}\mathbb{D}_{Q'}\Big)\Big)=\Big(\bigsqcup_{Q\in\mathcal{F}_1}\mathbb{D}_{\mathcal{F}_Q,Q}\Big)\bigsqcup\Big(\bigsqcup_{Q'\in\mathcal{F}_2}\mathbb{D}_{Q'}\Big).$$

Iterating this procedure we eventually obtain

$$\mathbb{D}_{Q^0}=\bigsqcup_{k=0}^{\infty}\Big(\bigsqcup_{Q\in\mathcal{F}_k}\mathbb{D}_{\mathcal{F}_Q,Q}\Big)=\bigsqcup_{Q\in\mathcal{F}}\mathbb{D}_{\mathcal{F}_Q,Q}.$$

We then set $\mathbb{B}_{Q^0}:=\varnothing$, $\mathbb{F}_{Q^0}=\{\mathbb{D}_{\mathcal{F}_Q,Q}\}_{Q\in\mathcal{F}}$, and $\mathbb{G}_{Q^0}=\bigsqcup_{\mathbf{S}\in\mathbb{F}_{Q^0}}\mathbf{S}$. Note that trees are of the form $\mathbb{D}_{\mathcal{F}_Q,Q}$ with $Q\in\mathcal{F}$, hence they are clearly semi-coherent with $\mathrm{Top}(\mathbf{S})=Q$, hence $\mathrm{Top}(\mathbb{F}_{Q^0})=\mathcal{F}$.

Step 3: packing the maximal cubes. Thus, in order to see that $(\mathbb{B}_{Q^0},\mathbb{G}_{Q^0},\mathbb{F}_{Q^0})$ is a semi-coherent corona decomposition relative to Q^0 we are left with showing that the family $\mathrm{Top}(\mathbb{F}_{Q^0})=\mathcal{F}$ satisfy a packing condition. This is the content of the following result:

Proposition 4.28. *Under the previous considerations, assuming* (d) *and taking N large enough (depending on n, ADR, ellipticity, and τ) the family $\mathrm{Top}(\mathbb{F}_{Q^0})=\mathcal{F}$ satisfies the Carleson condition*

$$\sup_{Q\in\mathbb{D}_{Q^0}}\frac{1}{\sigma(Q)}\sum_{Q'\in\mathcal{F}\cap\mathbb{D}_Q}\sigma(Q')\lesssim 1, \tag{4.55}$$

with an implicit constant which depends on n, ADR, ellipticity, and τ. As a consequence, $(\mathbb{B}_{Q^0},\mathbb{G}_{Q^0},\mathbb{F}_{Q^0})$ is a semi-coherent corona decomposition relative to Q^0.

Assume this result momentarily and take the corresponding large enough N. Pick an $\mathbf{S}\in\mathbb{F}_{Q^0}$, that is, $\mathbf{S}=\mathbb{D}_{\mathcal{F}_{Q_0},Q_0}$ for some $Q_0\in\mathcal{F}$. Set $X_{\mathbf{S}}:=Y_{Q_0}$ so that by (4.45)

$$\delta(X_{\mathbf{S}})\approx\ell(\mathrm{Top}(\mathbf{S}))\approx\mathrm{dist}(X_{\mathbf{S}},\mathrm{Top}(\mathbf{S})).$$

Invoking (4.46) and recalling that $\mathrm{Top}(\mathbf{S})=Q_0$, we get

$$2^{-N}\frac{\omega_L^{X_{\mathbf{S}}}(\mathrm{Top}(\mathbf{S}))}{\sigma(\mathrm{Top}(\mathbf{S}))}\lesssim\frac{\omega_L^{X_{\mathbf{S}}}(Q)}{\sigma(Q)}\leqslant\left(\fint_Q(\mathcal{M}\omega_L^{X_{\mathbf{S}}})^{\frac12}\,d\sigma\right)^2\lesssim 2^{2N}\frac{\omega_L^{X_{\mathbf{S}}}(\mathrm{Top}(\mathbf{S}))}{\sigma(\mathrm{Top}(\mathbf{S}))},$$

for all $Q\in\mathbf{S}$. All these, Proposition 4.28, the fact that Q^0 is arbitrary, and Proposition 4.24 give that ω_L admits a semi-coherent corona decomposition with $Q_{\mathbf{S}}=\mathrm{Top}(\mathbf{S})$. Note that with the help of Remark 4.11 we can refine transform this corona decomposition so that the new corona is coherent. This completes the proof of the current implication modulo proving Proposition 4.28. □

Remark 4.29. We would like to observe that in the previous proof we have obtained a semi-coherent corona decomposition such that $Q_{\mathbf{S}}=\mathrm{Top}(\mathbf{S})$ and with the additional property that $\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})\approx 1$, see (4.51). Invoking Remark 4.11 we can transform this into a coherent corona decomposition as in (i) in Definition 4.10 with the additional property that $\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})\approx 1$. A careful examination of the proofs of (a) $\Longrightarrow$ (b), (b) $\Longrightarrow$ (c), and (c) $\Longrightarrow$ (d) reveals that this extra property can be inherited in any of the implications, that is, both in (ii) and (iii) in Definition 4.10 we obtain the additional the property $\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})\approx 1$, and in in the partial/weak Carleson measure estimates the extra property $\omega_L^{P_Q}(Q)\approx 1$. All these together mean that in each of the conditions (a), (b), (c) we may assume that the corona decomposition has the extra property $\omega_L^{X_{\mathbf{S}}}(Q_{\mathbf{S}})\approx 1$ for every $\mathbf{S}\in\mathbb{F}$. Analogously, in (d) we may additionally assume that $\omega_L^{P_Q}(Q)\approx 1$. We observe however that adding this extra condition requires to possibly use a different decomposition in (a), (b), (c), or a different collection of points $\{P_Q\}_{Q\in\mathbb{D}}$ in (d).

Proof of Proposition 4.28. We borrow some ideas from [GMT18, AGMT23].

Step 3.1: stratification of the maximal cubes with respect to $\mathcal{F}^+$ cubes, so that we create regimes with only $\mathcal{F}^-$ cubes, whence $\{Q\backslash F_Q^-\}$ are disjoint. We start by arranging the cubes in $\mathcal{F}$ into some trees. To set the stage let $Q \in \mathcal{F}$ be an arbitrary cube. Write $\mathcal{S}_Q$ for the (possible empty) family of maximal cubes in $\mathbb{D}_Q^* \cap \mathcal{F}^+$, where, here and elsewhere, $\mathbb{D}_Q^* := \mathbb{D}_Q\backslash\{Q\}$. Note that by construction $\mathcal{S}_Q \subset \mathbb{D}_Q^* \cap \mathcal{F}^+$ is a pairwise disjoint family so that $(\mathbb{D}_{\mathcal{S}_Q,Q}\backslash\{Q\}) \cap \mathcal{F} \subset \mathcal{F}^-$.

We next iterate the previous selection procedure. Write $\mathcal{S}_0 = \{Q^0\}$ and define recursively $\mathcal{S}_{k+1} = \bigsqcup_{Q\in\mathcal{S}_k} \mathcal{S}_Q$ for $k \geqslant 0$. We then set $\mathcal{S} = \bigsqcup_{k=0}^\infty \mathcal{S}_k$. Recalling that $Q^0 \in \mathcal{F}^+$ it is easy to see that $\mathcal{S} = \mathcal{F}^+ \cap \mathbb{D}_{Q^0}$. On the other hand,

$$\mathbb{D}_{Q^0} = \big(\mathbb{D}_{\mathcal{S}_{Q^0},Q^0}\big) \bigsqcup \Big(\bigsqcup_{Q\in\mathcal{S}_{Q^0}} \mathbb{D}_Q \Big) = \Big(\bigsqcup_{Q\in\mathcal{S}_0} \mathbb{D}_{\mathcal{S}_Q,Q}\Big) \bigsqcup \Big(\bigsqcup_{Q\in\mathcal{S}_1} \mathbb{D}_Q\Big).$$

In turn,

$$\bigsqcup_{Q\in\mathcal{S}_1} \mathbb{D}_Q = \bigsqcup_{Q\in\mathcal{S}_1} \Big(\mathbb{D}_{\mathcal{S}_Q,Q} \bigsqcup \Big(\bigsqcup_{Q'\in\mathcal{S}_Q} \mathbb{D}_{Q'}\Big)\Big) = \Big(\bigsqcup_{Q\in\mathcal{S}_1} \mathbb{D}_{\mathcal{S}_Q,Q}\Big) \bigsqcup \Big(\bigsqcup_{Q'\in\mathcal{S}_2} \mathbb{D}_{Q'}\Big).$$

Iterating this procedure we eventually obtain

$$\mathbb{D}_{Q^0} = \bigsqcup_{k=0}^\infty \Big(\bigsqcup_{Q\in\mathcal{S}_k} \mathbb{D}_{\mathcal{S}_Q,Q}\Big) = \bigsqcup_{Q\in\mathcal{S}} \mathbb{D}_{\mathcal{S}_Q,Q}. \tag{4.56}$$

Step 3.2: pack the $\mathcal{F}^-$ cubes within a regime delimited by $\mathcal{F}^+$ cubes, using (4.48), (4.49) and a randomization argument. We first show that

$$\sup_{Q_0\in\mathcal{S}} \sup_{Q_0'\in\mathbb{D}_{\mathcal{S}_{Q_0},Q_0}\cap\mathcal{F}} \frac{1}{\sigma(Q_0')} \sum_{Q\in\mathbb{D}_{\mathcal{S}_{Q_0},Q_0'}\cap\mathcal{F}} \sigma(Q) \lesssim 1, \tag{4.57}$$

with a constant that is independent of N. Fix then $Q_0 \in \mathcal{S}$ and $Q_0' \in \mathbb{D}_{\mathcal{S}_{Q_0},Q_0} \cap \mathcal{F}$. For each $Q \in \mathcal{F}$ we set $V_Q = B(Y_Q, (1-\tau)\delta(Y_Q))$. Recalling the notation introduced in Proposition 4.27, we claim that the family $\{Q\backslash F_Q^-\}_{Q\in\mathbb{D}_{\mathcal{S}_{Q_0},Q_0}\cap\mathcal{F}}$ is pairwise disjoint. Suppose otherwise that there are two distinct cubes $Q, Q' \in \mathbb{D}_{\mathcal{S}_{Q_0},Q_0} \cap \mathcal{F}$ with $Q\backslash F_Q^-$ meeting $Q'\backslash F_{Q'}^-$. By relabeling if needed, we may assume that $Q' \subsetneq Q$. Let $Q'' \in \mathcal{F}$ be the maximal cube so that $Q' \subset Q'' \subsetneq Q$, and note that since $Q, Q' \in \mathbb{D}_{\mathcal{S}_{Q_0},Q_0}$ we necessarily have that $Q'' \in (\mathbb{D}_{\mathcal{S}_{Q_0},Q_0}\backslash\{Q_0\}) \cap \mathcal{F} \subset \mathcal{F}^-$ and $Q'' \in \mathcal{F}_Q^-$. Hence, $Q' \subset Q'' \subset F_Q^-$ and Q' cannot meet $Q\backslash F_Q^-$, which is a contradiction.

For any given $K \gg 1$ we take an arbitrary family $\{Q_k\}_{k=1}^K \subset \mathbb{D}_{\mathcal{S}_{Q_0},Q_0'} \cap \mathcal{F}$. Invoking Proposition 4.27 we can find f_{Q_k} and u_{Q_k} for each $1 \leqslant k \leqslant K$ so that (4.48) holds. Write $\{r_j(\cdot)\}_{j\in\mathbb{N}}$ for the Rademacher system in $[0,1)$ and for every $t \in [0,1)$ let us set

$$f_t := \sum_{k=1}^K r_k(t) f_{Q_k} \mathbf{1}_{Q_k\backslash F_{Q_k}^-}$$

and for every $X \in \Omega$

$$u_t(X) := \int_{\partial\Omega} f_t \, d\omega_L^X = \sum_{k=1}^K r_k(t) \int_{Q_k\backslash F_{Q_k}^-} f_{Q_k} \, d\omega_L^X = \sum_{k=1}^K r_k(t) u_{Q_k}(X).$$

Note that f_t is Borel measurable and by (4.48) and the fact that the family $\{Q\backslash F_Q^-\}_{Q\in\mathbb{D}_{S_{Q_0},Q_0}\cap\mathcal{F}}$ is pairwise disjoint we conclude that for every $y\in\partial\Omega$

$$|f_t(y)|\leqslant\sum_{k=1}^{K}|r_j(t)|\,|f_{Q_k}(y)|\,\mathbf{1}_{Q_k\backslash F_{Q_k}^-}(y)\lesssim\sum_{k=1}^{K}\mathbf{1}_{Q_k\backslash F_{Q_k}^-}(y)\leqslant 1.$$

Hence, $\|u_t\|_{L^\infty(\Omega)}\lesssim 1$, uniformly on t. As such, recalling Proposition 4.27 and that $Y_Q=P_{Q(N_\tau)}$, we can invoke (d) to obtain

$$\begin{aligned}\sum_{k=1}^{K}\iint_{V_{Q_k}}|\nabla u_t(X)|^2\,\delta(X)\,dX&=\sum_{k=1}^{K}\iint_{B(P_{Q_k(N_\tau)},(1-\tau)\delta(P_{Q_k(N_\tau)}))}|\nabla u_t(X)|^2\,\delta(X)\,dX\\&\leqslant\sum_{Q\in\mathbb{D}_{Q_0'}}\iint_{B(P_Q,(1-\tau)\delta(P_Q))}|\nabla u_t(X)|^2\,\delta(X)\,dX\lesssim\sigma(Q_0'),\end{aligned}$$

with implicit constants that are uniform on $t\in[0,1)$. Thus, by (4.49) and the orthogonality of the Rademacher system we arrive at

$$\begin{aligned}\sum_{k=1}^{K}\sigma(Q_k)&\lesssim\sum_{k=1}^{K}\iint_{V_{Q_k}}|\nabla u_{Q_k}(X)|^2\,\delta(X)\,dX\\&\leqslant\iint_{\bigcup_{k=1}^{K}V_{Q_k}}\sum_{k=1}^{K}|\nabla u_{Q_k}(X)|^2\delta(X)\,dX\\&=\sum_{j=1}^{n+1}\iint_{\bigcup_{k=1}^{K}V_{Q_k}}\sum_{k=1}^{K}|\partial_j u_{Q_k}(X)|^2\delta(X)\,dX\\&=\sum_{j=1}^{n+1}\iint_{\bigcup_{k=1}^{K}V_{Q_k}}\left(\int_0^1\Big|\sum_{k=1}^{K}r_k(t)\partial_j u_{Q_k}(X)\Big|^2dt\right)\delta(X)\,dX\\&=\int_0^1\left(\iint_{\bigcup_{k=1}^{K}V_{Q_k}}|\nabla u_t(X)|^2\,\delta(X)\,dX\right)dt\\&\leqslant\int_0^1\left(\sum_{k=1}^{K}\iint_{V_{Q_k}}|\nabla u_t(X)|^2\,\delta(X)\,dX\right)dt\\&\lesssim\sigma(Q_0').\end{aligned}$$

Recalling that $K\gg 1$ and the family $\{Q_k\}_{k=1}^K\subset\mathbb{D}_{S_{Q_0},Q_0'}\cap\mathcal{F}$ is arbitrary we can easily see that

$$\sum_{Q\in\mathbb{D}_{S_{Q_0},Q_0'}\cap\mathcal{F}}\sigma(Q)\lesssim\sigma(Q_0'),$$

hence (4.57) holds.

Step 3.3: pack the $\mathcal{F}^+$ cubes, using (4.47) and a geometric series. To proceed we next show that

$$\sup_{Q_0\in\mathcal{S}} \frac{1}{\sigma(Q_0)} \sum_{Q\in\mathcal{S}\cap\mathbb{D}_{Q_0}} \sigma(Q) \leqslant 2, \tag{4.58}$$

provided N is taken large enough.

For starters, fix $Q_0 \in \mathcal{S}$ and let $k_0 \geqslant 0$ be such that $Q_0 \in \mathcal{S}_{k_0}$. Then,

$$\sum_{Q\in\mathcal{S}\cap\mathbb{D}_{Q_0}} \sigma(Q) = \sum_{k=k_0}^{\infty} \sum_{Q\in\mathcal{S}_k\cap\mathbb{D}_{Q_0}} \sigma(Q) =: \sum_{k=k_0}^{\infty} \Sigma_k,$$

and we estimate each term in turn. Note first that

$$\Sigma_{k_0} = \sum_{Q\in\mathcal{S}_{k_0}\cap\mathbb{D}_{Q_0}} \sigma(Q) = \sigma(Q_0).$$

Additionally, for each $k \geqslant k_0 + 1$ we have

$$\Sigma_k = \sum_{Q\in\mathcal{S}_k\cap\mathbb{D}_{Q_0}} \sigma(Q) = \sum_{Q\in\mathcal{S}_{k-1}} \sum_{Q'\in\mathcal{S}_Q\cap\mathbb{D}_{Q_0}} \sigma(Q') = \sum_{Q\in\mathcal{S}_{k-1}\cap\mathbb{D}_{Q_0}} \sum_{Q'\in\mathcal{S}_Q} \sigma(Q').$$

We claim that for any fixed $Q \in \mathcal{S}_{k-1} \cap \mathbb{D}_{Q_0}$ there holds

$$\mathcal{S}_Q = \bigsqcup_{Q'\in\mathbb{D}_{\mathcal{S}_Q,Q}\cap\mathcal{F}} \mathcal{F}^+_{Q'}. \tag{4.59}$$

That the union is made of pairwise disjoint collections is clear from the iterative construction of the family $\mathcal{F}$. On the other hand, if $Q' \in \mathcal{S}_Q \subset \mathbb{D}^*_Q \cap \mathcal{F}^+$, let $Q'' \in \mathcal{F}$ be so that $Q' \in \mathcal{F}^+_{Q''} \subset \mathbb{D}^*_{Q''}$. Note that since $Q' \subsetneq Q \in \mathcal{S}_{k-1} \subset \mathcal{F}^+$ we must have $Q'' \subset Q$. Hence, $Q'' \in \mathbb{D}_{\mathcal{S}_Q,Q}$ and this shows the left-to-right inclusion.

Conversely, let $Q' \in \mathbb{D}_{\mathcal{S}_Q,Q} \cap \mathcal{F}$ and $Q'' \in \mathcal{F}^+_{Q'}$. Since $Q'' \subsetneq Q' \subset Q$ and, as observed before $(\mathbb{D}_{\mathcal{S}_Q,Q}\backslash\{Q\}) \cap \mathcal{F} \subset \mathcal{F}^-$, we obtain that $Q'' \notin \mathbb{D}_{\mathcal{S}_Q,Q}$. That is, $Q'' \subset Q''' \in \mathcal{S}_Q$. Since $Q'' \subset Q' \cap Q'''$ we must have $Q''' \subsetneq Q'$ —otherwise, $Q' \subset Q'''$ and, as a consequence, $Q' \notin \mathbb{D}_{\mathcal{S}_Q,Q}$, which is a contradiction. Using that $\mathcal{F}^+ \ni Q''' \subsetneq Q' \subsetneq Q$ and that $Q'' \in \mathcal{F}^+_{Q'}$, we arrive at $Q''' = Q''$, hence $Q'' \in \mathcal{S}_Q$ as desired.

Using then (4.59), (4.47), and (4.57) (whose implicit constant does not depend on N) we obtain

$$\begin{aligned}
\Sigma_k &= \sum_{Q\in\mathcal{S}_{k-1}\cap\mathbb{D}_{Q_0}} \sum_{Q'\in\mathbb{D}_{\mathcal{S}_Q,Q}\cap\mathcal{F}} \sum_{Q''\in\mathcal{F}^+_{Q'}} \sigma(Q'') \\
&= \sum_{Q\in\mathcal{S}_{k-1}\cap\mathbb{D}_{Q_0}} \sum_{Q'\in\mathbb{D}_{\mathcal{S}_Q,Q}\cap\mathcal{F}} \sigma(F^+_{Q'}) \leqslant 2^{-N} \sum_{Q\in\mathcal{S}_{k-1}\cap\mathbb{D}_{Q_0}} \sum_{Q'\in\mathbb{D}_{\mathcal{S}_Q,Q}\cap\mathcal{F}} \sigma(Q') \\
&\lesssim 2^{-N} \sum_{Q\in\mathcal{S}_{k-1}\cap\mathbb{D}_{Q_0}} \sigma(Q) = 2^{-N}\Sigma_{k-1} < \frac{1}{2}\Sigma_{k-1},
\end{aligned}$$

provided N is taken large enough. Iterating this we conclude as desired

$$\sum_{Q\in\mathcal{S}\cap\mathbb{D}_{Q_0}}\sigma(Q)=\sum_{k=k_0}^{\infty}\Sigma_k\leqslant\Sigma_{k_0}\sum_{k=k_0}^{\infty}2^{-(k-k_0)}=2\Sigma_{k_0}=2\sigma(Q_0).$$

Step 3.4: everything packs because within each regime, cubes pack by Step 3.2; and the regimes pack by 3.3. With (4.57) and (4.58) we are now ready to obtain (4.55). Fix then $Q_0\in\mathbb{D}_{Q^0}$ and assume first that $Q_0\in\mathcal{F}$. By (4.56) we can find $Q_0'\in\mathcal{S}$ such that $Q_0\in\mathbb{D}_{\mathcal{S}_{Q_0',Q_0'}}$. We note that if $Q\in\mathcal{S}$ with $Q_0'\subsetneq Q$ then $\mathbb{D}_{Q_0'}\cap\mathbb{D}_{\mathcal{S}_{Q,Q}}=\varnothing$, otherwise there is $Q'\in\mathbb{D}_{Q_0'}\cap\mathbb{D}_{\mathcal{S}_{Q,Q}}$ and since $Q'\subset Q_0'\subsetneq Q$ we conclude that $Q_0'\in\mathbb{D}_{\mathcal{S}_{Q,Q}}$, and this contradicts (4.56) and the fact that $Q_0'\in\mathbb{D}_{\mathcal{S}_{Q_0',Q_0'}}$. Thus, using again (4.56) we have

$$\begin{aligned}\sum_{Q\in\mathcal{F}\cap\mathbb{D}_{Q_0}}\sigma(Q)&=\sum_{Q\in\mathcal{S}}\sum_{Q'\in\mathbb{D}_{\mathcal{S}_{Q,Q}\cap Q_0}\cap\mathcal{F}}\sigma(Q')\\&=\sum_{Q\in\mathcal{S}\cap\mathbb{D}_{Q_0'}}\sum_{Q'\in\mathbb{D}_{\mathcal{S}_{Q,Q}\cap Q_0}\cap\mathcal{F}}\sigma(Q')\\&\leqslant\sum_{Q\in\mathcal{S}\cap\mathbb{D}_{Q_0'}:Q_0\subset Q}\sum_{Q'\in\mathbb{D}_{\mathcal{S}_{Q,Q_0}}\cap\mathcal{F}}\sigma(Q')+\sum_{Q\in\mathcal{S}\cap\mathbb{D}_{Q_0}}\sum_{Q'\in\mathbb{D}_{\mathcal{S}_{Q,Q}}\cap\mathcal{F}}\sigma(Q')\\&=\sum_{Q'\in\mathbb{D}_{\mathcal{S}_{Q_0',Q_0}}\cap\mathcal{F}}\sigma(Q')+\sum_{Q\in\mathcal{S}\cap\mathbb{D}_{Q_0}}\sum_{Q'\in\mathbb{D}_{\mathcal{S}_{Q,Q}}\cap\mathcal{F}}\sigma(Q'),\end{aligned}$$

where in the last equality we have used that if $Q\in\mathcal{S}\cap\mathbb{D}_{Q_0'}$ with $Q_0\subset Q$ and there is $Q'\in\mathbb{D}_{\mathcal{S}_{Q,Q_0}}\cap\mathcal{F}$ we have $Q'\subset Q_0\subset Q$ with $Q'\in\mathbb{D}_{\mathcal{S}_{Q,Q_0}}\subset\mathbb{D}_{\mathcal{S}_{Q,Q}}$, hence $Q_0\in\mathbb{D}_{\mathcal{S}_{Q,Q}}\cap\mathcal{F}$ and by (4.56) we conclude that $Q=Q_0'$. Let $\mathcal{S}^{Q_0}$ be the collection of maximal cubes (hence, pairwise disjoint) in $\mathcal{S}\cap\mathbb{D}_{Q_0}$. Then, invoking (4.57) and (4.58) we easily see that

$$\begin{aligned}\sum_{Q\in\mathcal{F}\cap\mathbb{D}_{Q_0}}\sigma(Q)\lesssim\sigma(Q_0)+\sum_{Q\in\mathcal{S}\cap\mathbb{D}_{Q_0}}\sigma(Q)&=\sigma(Q_0)+\sum_{Q\in\mathcal{S}^{Q_0}}\sum_{Q'\in\mathcal{S}\cap\mathbb{D}_Q}\sigma(Q')\\&\leqslant\sigma(Q_0)+2\sum_{Q\in\mathcal{S}^{Q_0}}\sigma(Q)=\sigma(Q_0)+2\,\sigma\Big(\bigsqcup_{Q\in\mathcal{S}^{Q_0}}Q\Big)\leqslant 3\sigma(Q_0).\end{aligned}$$

Recalling that $Q_0\in\mathbb{D}_{Q^0}\cap\mathcal{F}$ is arbitrary we have shown that

$$\sup_{Q\in\mathbb{D}_{Q^0}\cap\mathcal{F}}\frac{1}{\sigma(Q)}\sum_{Q'\in\mathcal{F}\cap\mathbb{D}_Q}\sigma(Q')\lesssim 1,\tag{4.60}$$

and we are left with the consider in the sup all cubes $Q\in\mathbb{D}_{Q^0}$. To see this we fix $Q_0\in\mathbb{D}_{Q^0}$ and let $\mathcal{F}^{Q_0}$ be the collection of maximal cubes (hence, pairwise disjoint) in $\mathcal{F}\cap\mathbb{D}_{Q_0}$. It is straightforward to see that (4.60) yields

$$\sum_{Q\in\mathcal{F}\cap\mathbb{D}_{Q_0}}\sigma(Q)=\sum_{Q\in\mathcal{F}^{Q_0}}\sum_{Q'\in\mathcal{F}\cap\mathbb{D}_Q}\sigma(Q')\lesssim\sum_{Q\in\mathcal{F}^{Q_0}}\sigma(Q)=\sigma\Big(\bigsqcup_{Q\in\mathcal{F}^{Q_0}}Q\Big)\leqslant\sigma(Q_0).$$

This completes the proof of Proposition 4.28, and hence the implication of this section. □

4.6 Proof of Theorems 4.3 and 4.4: the abstract perturbation theory

Theorems 4.3 and 4.4 will follow from the following general result. First we introduce some notation. Give a matrix $D = (d_{i,j})_{1\leqslant i,j\leqslant n+1} \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ we write $\mathrm{div}_C D$ to denote its divergence acting on columns, that is,

$$\mathrm{div}_C D := \big(\mathrm{div}(d_{\cdot,j})\big)_{1\leqslant j\leqslant n+1} = \Big(\sum_{i=1}^{n+1} \partial_i d_{i,j}\Big)_{1\leqslant j\leqslant n+1}.$$

Theorem 4.30. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with ADR boundary satisfying the corkscrew condition. Let $L_0 u = -\mathrm{div}(A_0\nabla u)$ and $L_1 u = -\mathrm{div}(A_1\nabla u)$ be real (not necessarily symmetric) elliptic operators. Assume that $A_0 - A_1 = A + D$ where $A, D \in L^\infty(\Omega)$ are real matrices verifying the following conditions:*

(i) *The function $a(X) := \sup_{Y\in B(X,\delta(X)/2)} |A(Y)|$, $X \in \Omega$ satisfies the Carleson measure estimate*

$$\sup_{\substack{x\in\partial\Omega\\ 0<r<\mathrm{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r)\cap\partial\Omega)} \iint_{B(x,r)\cap\Omega} \frac{a(X)^2}{\delta(X)} dX < \infty. \tag{4.61}$$

(ii) *$D \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is antisymmetric and its divergence acting on columns $\mathrm{div}_C D$ satisfies the Carleson measure estimate*

$$\sup_{\substack{x\in\partial\Omega\\ 0<r<\mathrm{diam}(\partial\Omega)}} \frac{1}{\sigma(B(x,r)\cap\partial\Omega)} \iint_{B(x,r)\cap\Omega} |\mathrm{div}_C D(X)|^2 \delta(X) dX < \infty. \tag{4.62}$$

Then, ω_{L_0} admits a (strong) corona decomposition (equivalently, G_{L_0} is comparable to the distance to the boundary in the corona sense or L_0 satisfies partial/weak Carleson measure estimates) if and only if ω_{L_1} admits a (strong) corona decomposition (equivalently, G_{L_1} is comparable to the distance to the boundary in the corona sense or L_1 satisfies partial/weak Carleson measure estimates).

Proof. By symmetry and Theorem 4.2, it suffices to show that if G_{L_0} is comparable to the distance to the boundary in the corona sense. then L_1 satisfies partial/weak Carleson measure estimates. We are going to use the ideas from Section 4.5.2. We fix $u \in W^{1,2}_{\mathrm{loc}}(\Omega)\cap L^\infty(\Omega)$ a non-trivial weak solution of $L_1 u = 0$ in Ω. By homogeneity we may assume that $\|u\|_{L^\infty(\Omega)} = 1$. Fix also $\tau \in (0, \frac{1}{2})$, the parameter which appears in the partial/weak Carleson measure estimate (see Definition 4.15). Assume that $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$ is the assumed corona decomposition associated with G_{L_0}.

Step 1: $\delta \lesssim G_{L_0}$ in carefully chosen regions. Given $N \geqslant 0$, large enough to be chosen momentarily (depending on τ), we may proceed as in Remark 4.12 and pick $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ the 2^N-refinement of $(\mathbb{B}_0, \mathbb{G}_0, \mathbb{F}_0)$. This implies that for every $\mathbf{S} \in \mathbb{F}$ there exist $Q_{\mathbf{S}} \in \mathbb{D}$ and $X_{\mathbf{S}} \in \Omega$ so that (4.28) and (4.29) hold with $L = L_0$. Write $G_{\mathbf{S}} := G_{L_0}(X_{\mathbf{S}}, \cdot)$. For each $Q \in \mathbf{S} \subset \mathbf{S}(N)$, we can find $P_Q \in 2\widetilde{B}_Q \cap \Omega$ with $\delta(P_Q) \geqslant c_0 \ell(Q)$ such that by (4.29) and Harnack's inequality there holds

$$\Lambda_{\mathbf{S}} \lesssim \frac{G_{\mathbf{S}}(P_Q)}{\delta(P_Q)} \approx_\tau \frac{G_{\mathbf{S}}(X)}{\delta(X)}, \quad \forall X \in V_Q := B(P_Q, (1-\tau)\delta(P_Q)).$$

This defines P_Q for every $Q \in \mathbb{G} = \bigsqcup_{\mathbf{S}\in\mathbb{F}} \mathbf{S}$. On the other hand, given $Q \in \mathbb{B} = \mathbb{D}\backslash\mathbb{G}$ we let $P_Q := X_Q$ be the corkscrew point relative to Q, in particular, $X_Q \in B_Q \cap \Omega \subset 2\,\widetilde{B}_Q \cap \Omega$ with $\delta(P_Q) \gtrsim \ell(Q)$.

Step 2: reduction to an integral estimate in a sawtooth region. Set next

$$\alpha_Q := \iint_{V_Q} |\nabla u(X)|^2 \delta(X)\, dX, \qquad Q \in \mathbb{D},$$

which, much as before, satisfies (4.31). Following the argument in Section 4.5.2 it is then easy to see that we can reduce matters to proving the following estimate:

$$\mathcal{J}_M := \iint_{\Omega^{\vartheta}_{\mathcal{F}_M,Q_0}} |\nabla u|^2\, G_{\mathbf{S}}\, dX \lesssim \Lambda_{\mathbf{S}}\, \sigma(Q_0). \tag{4.63}$$

with implicit constant independent of $M \geqslant 1$, $\mathbf{S}$, and Q_0. We warn the reader that the difference of the present case with Section 4.5.2 is that before, u and $G_{\mathbf{S}}$ were associated with the same operator L, while now u is a associated with L_1 and $G_{\mathbf{S}}$ is associated with L_0.

Step 3: integration by parts argument. To show (4.63), we pick Ψ_M from Lemma 4.20 and use Leibniz's rule to get

$$\begin{aligned} A_1\nabla u\cdot\nabla u\, G_{\mathbf{S}}\Psi_M^2 &= A_1\nabla u\cdot\nabla(uG_{\mathbf{S}}\Psi_M^2) - \frac{1}{2}A_0\nabla(u^2\Psi_M^2)\cdot\nabla G_{\mathbf{S}} \\ &\quad + \frac{1}{2}A_0\nabla(\Psi_M^2)\cdot\nabla G_{\mathbf{S}}\, u^2 - \frac{1}{2}A_0\nabla(u^2)\cdot\nabla(\Psi_M^2)G_{\mathbf{S}} + \frac{1}{2}\mathcal{E}\nabla(u^2)\cdot\nabla(G_{\mathbf{S}}\Psi_M^2), \end{aligned} \tag{4.64}$$

where $\mathcal{E}(X) := A_0(X) - A_1(X)$. Note that $u \in W^{1,2}_{\rm loc}(\Omega)\cap L^\infty(\Omega)$, $G_{\mathbf{S}} \in W^{1,2}_{\rm loc}(\Omega\backslash\{X_{\mathbf{S}}\})$, and that $\overline{\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0}}$ is a compact subset of Ω away from $X_{\mathbf{S}}$. Hence, $u \in W^{1,2}(\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0})$ and $uG_{\mathbf{S}}\Psi_M^2 \in W^{1,2}_0(\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0})$. Since $L_1 u = 0$ in the weak sense in Ω, these lead to

$$\iint_{\Omega} A_1\nabla u\cdot\nabla(uG_{\mathbf{S}}\Psi_M^2)dX = \iint_{\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0}} A_1\nabla u\cdot\nabla(uG_{\mathbf{S}}\Psi_M^2)dX = 0.$$

On the other hand, $G_{\mathbf{S}} \in W^{1,2}(\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0})$ and $L_0^\top G_{\mathbf{S}} = 0$ in the weak sense in $\Omega\backslash\{X_{\mathbf{S}}\}$. Thanks to the fact that $u^2\Psi_M^2 \in W^{1,2}_0(\Omega^{\vartheta,**}_{\mathcal{F}_M,Q_0})$, we then obtain

$$\iint_{\Omega} A_0\nabla(u^2\Psi_M^2)\cdot\nabla G_{\mathbf{S}}\, dX = \iint_{\Omega^{\vartheta,*}_{\mathcal{F}_M,Q_0}} A_0^\top\nabla G_{\mathbf{S}}\cdot\nabla(u^2\Psi_M^2)\, dX = 0. \tag{4.65}$$

Using Lemma 4.20, the ellipticity of A, (4.64)–(4.65), and the normalization $\|u\|_{L^\infty(\Omega)} = 1$, we then arrive at

$$\begin{aligned} \widetilde{\mathcal{J}}_M := \iint_{\Omega} |\nabla u|^2 G_{\mathbf{S}}\Psi_M^2\, dX &\lesssim \iint_{\Omega} A_1\nabla u\cdot\nabla u\, G_{\mathbf{S}}\Psi_M^2\, dX \\ &\lesssim \iint_{\Omega} \big(|\nabla G_{\mathbf{S}}| + |\nabla u| G_{\mathbf{S}}\big)|\nabla\Psi_M|\, dX + \left|\iint_{\Omega} \mathcal{E}\nabla(u^2)\cdot\nabla(G_{\mathbf{S}}\Psi_M^2)\, dX\right| \end{aligned}$$

$$\lesssim \sigma(Q_0) + \left| \iint_\Omega \mathcal{E}\nabla(u^2)\cdot\nabla(G_{\mathbf{S}}\Psi_M^2)\,dX \right|, \tag{4.66}$$

where the last estimate follows as in (4.43) and (4.44) with the help of (4.38), and where the implicit constant does not depend on M.

Step 4: estimates for the new terms, with the discrepancy between A_0 and A_1. To bound the last term in (4.66), we use that $\mathcal{E} = A_0 - A_1 = A + D$ to get

$$\left| \iint_\Omega \mathcal{E}\nabla(u^2)\cdot\nabla(G_{\mathbf{S}}\Psi_M^2)\,dX \right| \leqslant \left| \iint_\Omega A\nabla(u^2)\cdot\nabla(G_{\mathbf{S}}\Psi_M^2)\,dX \right| + \left| \iint_\Omega D\nabla(u^2)\cdot\nabla(G_{\mathbf{S}}\Psi_M^2)\,dX \right| =: \widetilde{\mathcal{J}}_M^1 + \widetilde{\mathcal{J}}_M^2.$$

To control $\widetilde{\mathcal{J}}_M^1$, we write

$$\widetilde{\mathcal{J}}_M^1 \lesssim \iint_\Omega |A||\nabla u||\nabla G_{\mathbf{S}}|\Psi_M^2\,dX + \iint_\Omega |\nabla u||\nabla\Psi_M|\Psi_M G_{\mathbf{S}}\,dX =: \widetilde{\mathcal{J}}_M^{1,1} + \widetilde{\mathcal{J}}_M^{1,2}.$$

Considering the first term, we observe that $\sup_{I^{**}}|A| \leqslant \inf_{I^*} a$ for any $I \in \mathcal{W}$ since $I^{**} \subset \{Y \in \Omega : |Y - X| < \delta(X)/2\}$ for each $X \in I^*$. By Caccioppoli's and Harnack's inequality, we obtain

$$\begin{aligned}
\widetilde{\mathcal{J}}_M^{1,1} &\lesssim \sum_{I\in\mathcal{W}_M^\vartheta} \Big(\sup_{I^{**}}|A|\Big)\Big(\iint_{I^{**}} |\nabla u|^2\Psi_M^2\,dX\Big)^{\frac12}\Big(\iint_{I^{**}} |\nabla G_{\mathbf{S}}|^2\,dX\Big)^{\frac12}\\
&\lesssim \sum_{I\in\mathcal{W}_M^\vartheta} \ell(I)^{-1}\Big(\sup_{I^{**}}|A|\Big)\Big(\iint_{I^{**}} |\nabla u|^2\Psi_M^2\,dX\Big)^{\frac12}\Big(\iint_{I^{***}} |G_{\mathbf{S}}|^2\,dX\Big)^{\frac12}\\
&\lesssim \sum_{I\in\mathcal{W}_M^\vartheta} \Big(\sup_{I^{**}}|A|^2\ell(I)^{n-1}G_{\mathbf{S}}(X_I)^2\Big)^{\frac12}\Big(\iint_{I^{**}} |\nabla u|^2\Psi_M^2\,dX\Big)^{\frac12}\\
&\lesssim \Lambda_{\mathbf{S}}^{\frac12}\sum_{I\in\mathcal{W}_M^\vartheta}\Big(\iint_{I^*}\frac{a(X)^2}{\delta(X)}dX\Big)^{\frac12}\Big(\iint_{I^{**}} |\nabla u|^2 G_{\mathbf{S}}\Psi_M^2\,dX\Big)^{\frac12}\\
&\lesssim \Lambda_{\mathbf{S}}^{\frac12}\Big(\iint_{B^*_{Q_0}\cap\Omega}\frac{a(X)^2}{\delta(X)}dX\Big)^{\frac12}\Big(\iint_{\Omega} |\nabla u|^2 G_{\mathbf{S}}\Psi_M^2\,dX\Big)^{\frac12}\\
&\lesssim \big(\widetilde{\mathcal{J}}_M\,\Lambda_{\mathbf{S}}\,\sigma(Q_0)\big)^{\frac12},
\end{aligned}$$

where we have used (4.37) in the fourth inequality and (4.61) in the last one. On the other hand, (4.37) and (4.44) imply

$$\widetilde{\mathcal{J}}_M^{1,2} \leqslant \Big(\iint_\Omega |\nabla\Psi_M|^2 G_{\mathbf{S}}\,dX\Big)^{\frac12}\Big(\iint_\Omega |\nabla u|^2 G_{\mathbf{S}}\Psi_M^2\,dX\Big)^{\frac12}$$

$$\begin{aligned}
&\lesssim \Big(\sum_{I\in\mathcal{W}_M^\vartheta} \ell(I)^{n-1} G_{\mathbf{S}}(X(I)) \Big)^{\frac12} \Big(\iint_\Omega |\nabla u|^2 G_{\mathbf{S}} \Psi_M^2 \, dX \Big)^{\frac12} \\
&\lesssim \Lambda_{\mathbf{S}}^{\frac12} \Big(\sum_{I\in\mathcal{W}_M^\vartheta} \ell(I)^n \Big)^{\frac12} \Big(\iint_\Omega |\nabla u|^2 G_{\mathbf{S}} \Psi_M^2 \, dX \Big)^{\frac12} \\
&\lesssim \big(\widetilde{\mathcal{J}}_M \, \Lambda_{\mathbf{S}} \, \sigma(Q_0) \big)^{\frac12}.
\end{aligned}$$

Moreover, it follows from [CHMT20, Lemma 4.1] (whose proof works to the present scenario with no change, taking into account the antisymmetry of D), (4.37), and (4.62) that

$$\begin{aligned}
\widetilde{\mathcal{J}}_M^2 &= \Big| \iint_\Omega \operatorname{div}_C D \cdot \nabla(u^2) G_{\mathbf{S}} \Psi_M^2 \, dX \Big| \\
&\lesssim \Big(\iint_\Omega |\operatorname{div}_C D|^2 G_{\mathbf{S}} \Psi_M^2 \, dX \Big)^{\frac12} \Big(\iint_\Omega |\nabla u|^2 G_{\mathbf{S}} \Psi_M^2 \, dX \Big)^{\frac12} \\
&\lesssim \widetilde{\mathcal{J}}_M^{\frac12} \Big(\sum_{I\in\mathcal{W}_M^\vartheta} G_{\mathbf{S}}(X(I)) \iint_{I^{**}} |\operatorname{div}_C D|^2 dX \Big)^{\frac12} \\
&\lesssim \Lambda_{\mathbf{S}}^{\frac12} \widetilde{\mathcal{J}}_M^{\frac12} \Big(\iint_{B^*_{Q_0}\cap\Omega} |\operatorname{div}_C D|^2 \delta(X) \, dX \Big)^{\frac12} \\
&\lesssim \big(\widetilde{\mathcal{J}}_M \, \Lambda_{\mathbf{S}} \, \sigma(Q_0) \big)^{\frac12}.
\end{aligned} \tag{4.67}$$

Now gathering (4.66)–(4.67) and using Young's inequality, we obtain

$$\widetilde{\mathcal{J}}_M \lesssim \Lambda_{\mathbf{S}} \, \sigma(Q_0) + \big(\widetilde{\mathcal{J}}_M \, \Lambda_{\mathbf{S}} \, \sigma(Q_0) \big)^{\frac12} \leqslant \frac12 \, \widetilde{\mathcal{J}}_M + C_0 \, \Lambda_{\mathbf{S}} \, \sigma(Q_0), \tag{4.68}$$

where the implicit constant and C_0 are independent of M. Observe that $\widetilde{\mathcal{J}}_M < \infty$ since $\operatorname{supp} \Psi_M \subset \overline{\Omega^{\vartheta,*}_{\mathcal{F}_M,Q_0}}$, which is a compact subset of Ω and $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Thus, we hide the last term in (4.68) and obtain

$$\mathcal{J}_M \lesssim \widetilde{\mathcal{J}}_M \lesssim \Lambda_{\mathbf{S}} \, \sigma(Q_0).$$

This shows (4.63) and completes the proof. □

We next see how Theorems 4.3 and 4.4 are deduced from Theorem 4.30.

Proof of Theorem 4.3. Let L_0 and L_1 be the elliptic operators given in Theorem 4.3. Setting $A = A_0 - A_1$ and $D = 0$ in Theorem 4.30, we see that (4.61) coincides with the Carleson measure condition (4.1) in Theorem 4.3, and (4.62) holds automatically. Therefore, Theorem 4.3 immediately follows from Theorem 4.30. □

Proof of Theorem 4.4. Let A be the matrix from Theorem 4.4. If we take $A_0 = A$, $A_1 = A^\top$, $\widetilde{A} = 0$ and $D = A - A^\top$ in Theorem 4.30, then $A_0 - A_1 = \widetilde{A} + D$ with $D \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ antisymmetric, (4.61) holds trivially and (4.62) agrees with the Carleson measure estimate (4.2) in Theorem 4.4. Thus, Theorem 4.30 implies that ω_L admits a corona decomposition if and only if so does $\omega_{L^\top}$.

Similarly, if we pick $A_0 = A$, $A_1 = (A + A^\top)/2$, $\widetilde{A} = 0$ and $D = (A - A^\top)/2$, then ω_L admits a corona decomposition if and only if so does $\omega_{L^{\mathrm{sym}}}$. □

4.7 Proof of Theorem 4.5: equivalences with Uniform Rectifiability

In this section we prove Theorem 4.5. Assume first that $\partial\Omega$ is uniformly rectifiable (here we do not need to impose the corkscrew condition). Invoking [HMM16, Theorem 1.1] with $E = \partial\Omega$ we obtain that any bounded harmonic function in $\mathbb{R}^{n+1}\backslash E$ satisfies (full) Carleson measure estimates. Hence, clearly bounded harmonic functions in Ω satisfy (full and hence partial) Carleson measure estimates in Ω. This together with Theorem 4.2, implies that the harmonic measure admits a corona a decomposition. The same can be done with the operators in (a), because they are a subclass of the Kenig-Pipher operators (see [HMM24, Theorem 7.5]), or similarly to those in (b) much as in [HMMTZ21, Corollary 10.3].

4.7.1 Proof of Theorem 4.5 for the Laplacian

The argument to show that UR follows from the corona decomposition associated with the harmonic measure is somehow implicit in [HLMN17, Section 5] and we sketch the main changes. Throughout the section, we will denote the Laplacian by $\mathcal{L} = -\Delta$, avoiding the latter notation because it is already used for surface balls. For starters we need some preliminaries. We first recall [HLMN17, Definitions 2.17 and 2.19]:

Definition 4.31. Let $E \subset \mathbb{R}^{n+1}$ be an ADR set.

(i) Given $\varepsilon > 0$, we say that $Q \in \mathbb{D}(E)$ satisfies the ε-local weak half-space approximation condition (ε-local WHSA with parameter K_0) if there is a half-space $H = H(Q)$, a hyperplane $P = P(Q) = \partial H$, and a fixed positive number K_0 satisfying

(a) $\mathrm{dist}(X, E) \leqslant \varepsilon\ell(Q)$, for every $X \in P \cap B_Q^{**}(\varepsilon)$, where $B_Q^{**}(\varepsilon) = B(x_Q, \varepsilon^{-2}\ell(Q))$.

(b) $\mathrm{dist}(Q, P) \leqslant K_0^{3/2}\ell(Q)$.

(c) $H \cap B_Q^{**}(\varepsilon) \cap E = \varnothing$.

(ii) We say that E satisfies the weak half-space approximation property (WHSA) if for some pair of positive constants ε_0 and K_0, and for every $0 < \varepsilon < \varepsilon_0$, there is a constant C_ε such that the set $\mathcal{B}$ of bad cubes in $\mathbb{D}(E)$, for which the ε-local WHSA condition with parameter K_0 fails, satisfies the packing condition

$$\sum_{Q \subset Q_0 : Q \in \mathcal{B}} \sigma(Q) \leqslant C_\varepsilon \sigma(Q_0), \qquad \forall Q_0 \in \mathbb{D}(E).$$

Step 1: enlarge Whitney regions using Harnack chains. Recall next the definition of the Whitney regions U_Q^ϑ where $\vartheta \geqslant \vartheta_0$ is some parameter large enough to be chosen momentarily. We observe that

these Whitney regions may have more than one connected component, but that the number of distinct components is uniformly bounded, depending only upon ϑ and dimension. We enumerate the connected components of U_Q^ϑ as $\{U_Q^{\vartheta,i}\}_i$. As in [HLMN17, Definition 2.26] we enlarge the Whitney regions as follows. For small enough $\varepsilon > 0$, we write $X \approx_{\varepsilon,Q} Y$ if X may be connected to Y by a chain of at most ε^{-1} balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3 \ell(Q) \leqslant \delta(Y_k) \leqslant \varepsilon^{-3}\ell(Q)$. We then set

$$\widetilde{U}_Q^{\vartheta,i} := \left\{X \in \Omega : \, X \approx_{\varepsilon,Q} Y \text{ for some } Y \in U_Q^{\vartheta,i}\right\}. \tag{4.69}$$

Given $M \gg 1$ large enough to be chosen and some small enough $\varepsilon > 0$, for $Y \in \Omega$, we set

$$B_Y := \overline{B(Y, (1-\varepsilon^{2M/\gamma})\delta(Y))},$$

where $\gamma \in (0,1)$ is the exponent appearing in Lemma 4.22. We augment $\widetilde{U}_Q^{\vartheta,i}$ (cf. (4.69)) as follows. Set

$$\mathcal{W}_Q^{\vartheta,i,*} := \left\{I \in \mathcal{W} : I^* \text{ meets } B_Y \text{ for some } Y \in \bigcup_{X \in \widetilde{U}_Q^{\vartheta,i}} B_X\right\} \tag{4.70}$$

and

$$U_Q^{\vartheta,i,*} := \bigcup_{I \in \mathcal{W}_Q^{\vartheta,i,*}} I^{**}, \qquad U_Q^* := \bigcup_i U_Q^{\vartheta,i,*}, \tag{4.71}$$

where we recall that $I^{**} = (1+2\lambda)I$. By construction,

$$\widetilde{U}_Q^{\vartheta,i} \subset \bigcup_{X \in \widetilde{U}_Q^{\vartheta,i}} B_X \subset \bigcup_{Y \in \bigcup\limits_{X \in \widetilde{U}_Q^{\vartheta,i}} B_X} B_Y \subset U_Q^{\vartheta,i,*},$$

and for all $Y \in U_Q^{\vartheta,i,*}$, we have that $\delta(Y) \approx \ell(Q)$ (depending of course on ε). Moreover, also by construction, for every $I \in \mathcal{W}_Q^{\vartheta,i,*}$

$$\varepsilon^s \, \ell(Q) \lesssim \ell(I) \lesssim \varepsilon^{-3} \, \ell(Q), \qquad \operatorname{dist}(I, Q) \lesssim \varepsilon^{-4} \, \ell(Q),$$

where $s = s(M,\gamma) > 0$.

Step 2: the WHSA property (which implies UR) is reduced to showing a packing condition within a corona regime. We are now ready to establish that if $\omega_{\mathcal{L}}$ admits a semi-coherent corona decomposition, then $\partial\Omega$ is uniformly rectifiable. In view of Theorem 4.2 and Remark 4.29 we may assume that $\omega_{\mathcal{L}}$ admits a semi-coherent corona decomposition $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ with the property that $Q_{\mathbf{S}} = \operatorname{Top}(\mathbf{S})$ and $\omega_{\mathcal{L}}^{X_{\mathbf{S}}}(Q_{\mathbf{S}}) \approx 1$ for each $\mathbf{S} \in \mathbb{F}$. Our goal is then to obtain that $\partial\Omega$ is uniformly rectifiable, and by [HLMN17, Proposition 1.17] it suffices to see that $\partial\Omega$ it satisfies the WHSA property.

Let K_0 be large enough to be chosen and let $\varepsilon_0 \ll K_0^{-6}$. Fix an arbitrary $\varepsilon \in (0, \varepsilon_0)$, and let $\mathcal{B}$ be the collection of cubes in $\mathbb{D} = \mathbb{D}(\partial\Omega)$ for which the ε-local WHSA condition with parameter K_0 fails. Then, we are reduced to showing that $\mathcal{B}$ satisfies a Carleson packing condition.

With this goal in mind we take an arbitrary cube $Q_0 \in \mathbb{D}$ and write

$$\sum_{Q \in \mathcal{B} \cap \mathbb{D}_{Q_0}} \sigma(Q) = \sum_{Q \in \mathbb{B} \cap \mathcal{B} \cap \mathbb{D}_{Q_0}} \sigma(Q) + \sum_{Q \in \mathbb{G} \cap \mathcal{B} \cap \mathbb{D}_{Q_0}} \sigma(Q).$$

Since the bad cubes $\mathbb{B}$ satisfy a packing condition, we have

$$\sum_{Q\in\mathbb{B}\cap\mathcal{B}\cap\mathbb{D}_{Q_0}}\sigma(Q)\leqslant\sum_{Q\in\mathbb{B}\cap\mathbb{D}_{Q_0}}\sigma(Q)\lesssim\sigma(Q_0).$$

Much as we did in (4.33)–(4.34), and recalling that $Q_{\mathbf{S}}=\mathrm{Top}(\mathbf{S})$ one can then see that it suffices to prove

$$\sum_{Q\in\mathbf{S}\cap\mathcal{B}\cap\mathbb{D}_{Q_0}}\sigma(Q)\lesssim\sigma(Q_{\mathbf{S}}\cap Q_0),\qquad\forall\mathbf{S}\in\mathbb{F}. \tag{4.72}$$

Step 3: some preliminary conclusions inside the corona regime where we want to prove the packing condition. To show (4.72), we take an arbitrary $\mathbf{S}\in\mathbb{F}$ and perform a stopping time argument to extract $\mathcal{F}_{\mathbf{S}}$, a family of dyadic subcubes of $Q_{\mathbf{S}}$ which are maximal (hence pairwise-disjoint) with respect to the property $Q\notin\mathbf{S}$. The semi-coherency of $\mathbf{S}$ clearly implies that $\mathbf{S}=\mathbb{D}_{\mathcal{F}_{\mathbf{S}},Q_{\mathbf{S}}}$. Set

$$\mu:=\sigma(Q_{\mathbf{S}})\omega_{\mathcal{L}}^{X_{\mathbf{S}}}\quad\text{ and }\quad\mathcal{G}:=\sigma(Q_{\mathbf{S}})G_{\mathcal{L}}(X_{\mathbf{S}},\cdot), \tag{4.73}$$

and note that

$$\frac{\mu(Q_{\mathbf{S}})}{\sigma(Q_{\mathbf{S}})}=\omega_{\mathcal{L}}^{X_{\mathbf{S}}}(Q_{\mathbf{S}})\approx 1.$$

Thus, (4.6) becomes

$$1\lesssim\frac{\mu(Q)}{\sigma(Q)}\lesssim\left(\fint_Q(\mathcal{M}\mu)^{\frac12}d\sigma\right)^2\lesssim 1,\qquad\forall Q\in\mathbb{D}_{\mathcal{F}_{\mathbf{S}},Q_{\mathbf{S}}}. \tag{4.74}$$

Apply Remark 4.26, fixing N large enough. Observing that in this case $\nabla\mathcal{G}$ is continuous (away from $X_{\mathbf{S}}$), we obtain for every $Q\in\mathbb{D}_{\mathcal{F}_{\mathbf{S}},Q_{\mathbf{S}}}$ with $\ell(Q)\leqslant 2^{-N}\ell(Q_{\mathbf{S}})$, that there exists $Y_Q\in 2\widetilde{B}_Q\cap\Omega$ with $\delta(Y_Q)\geqslant 2^{-N}\ell(Q)$ for which

$$|\nabla\mathcal{G}(Y_Q)|\gtrsim 1\qquad\text{and}\qquad\frac{\mathcal{G}(Y_Q)}{\delta(Y_Q)}\gtrsim 1. \tag{4.75}$$

Note also that if $Q\in\mathbf{S}$ is so that $\ell(Q)\leqslant 2^{-N'}\ell(Q_{\mathbf{S}})$, with N' large enough, we have

$$\begin{aligned}\delta(X_{\mathbf{S}})&\leqslant|X_{\mathbf{S}}-Y_Q|+\delta(Y_Q)\leqslant|X_{\mathbf{S}}-Y_Q|+C\,\ell(Q)\leqslant|X_{\mathbf{S}}-Y_Q|+C\,2^{-N'}\ell(Q_{\mathbf{S}})\\&\leqslant|X_{\mathbf{S}}-Y_Q|+C'\,2^{-N'}\delta(X_{\mathbf{S}})<|X_{\mathbf{S}}-Y_Q|+\frac12\delta(X_{\mathbf{S}}),\end{aligned}$$

thus

$$|X_{\mathbf{S}}-Y_Q|\geqslant\frac12\delta(X_{\mathbf{S}})\approx\ell(Q_{\mathbf{S}})\geqslant 2^{N'}\ell(Q)\gtrsim 2^{N'}\delta(Y_Q)>\frac12\delta(Y_Q).$$

We can then invoke (4.73), (4.20) and (4.74) to obtain

$$\frac{\mathcal{G}(Y_Q)}{\delta(Y_Q)}\lesssim\frac{\mu(\Delta(x_Q,C\delta(Y_Q)))}{\delta(Y_Q)^n}\lesssim\frac{\mu(\Delta(x_Q,C\ell(Q)))}{\sigma(\Delta(x_Q,C\ell(Q)))}\leqslant\left(\fint_Q(\mathcal{M}\mu)^{\frac12}d\sigma\right)^2\lesssim 1. \tag{4.76}$$

This, (4.75), the mean value property for harmonic functions, and Caccioppoli's and Harnack's inequalities yield

$$1\lesssim|\nabla\mathcal{G}(Y_Q)|\lesssim\frac{\mathcal{G}(Y_Q)}{\delta(Y_Q)}\lesssim 1, \tag{4.77}$$

for each $Q \in \mathbb{D}_{\mathcal{F}_{\mathbf{S}},Q_{\mathbf{S}}}$ with $\ell(Q) \leqslant 2^{-(N+N')}\ell(Q_{\mathbf{S}})$.

Since $Y_Q \in \Omega$, there exists $I_Q \in \mathcal{W}$ so that $Y_Q \in I_Q$. Note that $\ell(I_Q) \approx \delta(Y_Q) \approx_N \ell(Q)$ and $\operatorname{dist}(I_Q, Q) \leqslant |Y_Q - x_Q| \lesssim \ell(Q)$. As a result, by taking $\vartheta \geqslant 10(N+N')$ large enough we clearly have that $I_Q \in \mathcal{W}_Q^{\vartheta}$, whence $Y_Q \in I_Q \subset U_Q^{\vartheta}$. From this point ϑ remains fixed and we set $K_0 = 2^{\vartheta}$. To simplify the notation, in what follows we drop the superindex ϑ and simply write U_Q with connected components $\{U_Q^{\vartheta,i}\}_i$. Since $Y_Q \in U_Q$, then it belongs to some U_Q^i, by relabeling if needed we assume that $Y_Q \in U_Q^0$.

Note that by construction (see (4.70) and (4.71)), there is a Harnack path connecting any pair of points in $U_Q^{0,*}$ (depending on ε), thus, by Harnack's inequality and (4.77), and recalling that $\varepsilon \ll K_0^{-6}$

$$C^{-1}\delta(Y) \leqslant \mathcal{G}(Y) \leqslant C\delta(Y), \qquad \forall\, Y \in U_Q^{0,*}, \quad \ell(Q) \leqslant \varepsilon\ell(Q_{\mathbf{S}}), \tag{4.78}$$

with $C = C(K_0, \varepsilon, M)$ (depending also on the allowable parameters). Moreover, much as in (4.76),

$$\mathcal{G}(Y) \leqslant C\delta(Y)\,, \qquad \forall\, Y \in U_Q^{*}, \quad \ell(Q) \leqslant \varepsilon\ell(Q_{\mathbf{S}}), \tag{4.79}$$

by (4.73), (4.20), and (4.74), where again $C = C(K_0, \varepsilon, M)$.

Step 4: proof of the packing condition within the corona regime, by a distinction of cubes where the Green's function oscillates. We are now ready to establish (4.72). We may assume that $\mathbf{S} \cap \mathbb{D}_{Q_0} \neq \varnothing$, in which case the semi-coherency of $\mathbf{S}$ gives $R_0 := Q_{\mathbf{S}} \cap Q_0 \in \mathbf{S}$ and $\mathbf{S} \cap \mathbb{D}_{Q_0} = \mathbb{D}_{\mathcal{F}_{\mathbf{S}},R_0}$. At this point we proceed as in [HLMN17, Section 5] consider three cases:

- **Case 0:** $Q \in \mathbb{D}_{\mathcal{F}_{\mathbf{S}},R_0}$ with $\ell(Q) > \varepsilon^{10}\ell(R_0)$.
- **Case 1:** $Q \in \mathbb{D}_{\mathcal{F}_{\mathbf{S}},R_0}$ with $\ell(Q) \leqslant \varepsilon^{10}\ell(R_0)$ and
$$\sup_{X \in \widetilde{U}_Q^i} \sup_{Y \in B_X} |\nabla\mathcal{G}(Y) - \nabla\mathcal{G}(Y_Q)| > \varepsilon^{2M}.$$
- **Case 2:** $Q \in \mathbb{D}_{\mathcal{F}_{\mathbf{S}},R_0}$ with $\ell(Q) \leqslant \varepsilon^{10}\ell(R_0)$ and
$$\sup_{X \in \widetilde{U}_Q^i} \sup_{Y \in B_X} |\nabla\mathcal{G}(Y) - \nabla\mathcal{G}(Y_Q)| \leqslant \varepsilon^{2M}.$$

Note that one trivially has

$$\sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}_{\mathbf{S}},R_0} \\ \textbf{Case 0} \text{ holds}}} \sigma(Q) \leqslant \sum_{\substack{Q \in \mathbb{D}_{R_0} \\ \varepsilon^{10}\ell(R_0) < \ell(Q) \leqslant \ell(R_0)}} \sigma(Q) \lesssim (\log \varepsilon^{-1})\,\sigma(R_0).$$

For the cubes in **Case 1** we can use (4.78) and (4.79) so that [HLMN17, Lemma 5.8] yields

$$\sum_{\substack{Q \in \mathbb{D}_{\mathcal{F}_{\mathbf{S}},R_0} \\ \textbf{Case 1} \text{ holds}}} \sigma(Q) \leqslant C(\varepsilon, K_0, M)\,\sigma(R_0).$$

Similarly, [HLMN17, Lemma 5.10] implies that if M is large enough then all cubes in **Case 2** satisfy the ε-local WHSA with parameter K_0. All these together eventually imply (4.72):

$$\sum_{Q \in \mathbf{S} \cap \mathcal{B} \cap \mathbb{D}_{Q_0}} \sigma(Q) = \sum_{Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}_{\mathbf{S}},R_0}} \sigma(Q) = \sum_{\substack{Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}_{\mathbf{S}},R_0} \\ \textbf{Case 0} \text{ holds}}} \sigma(Q) + \sum_{\substack{Q \in \mathcal{B} \cap \mathbb{D}_{\mathcal{F}_{\mathbf{S}},R_0} \\ \textbf{Case 1} \text{ holds}}} \sigma(Q) \lesssim \sigma(R_0).$$

This completes the proof of Theorem 4.5 for the Laplacian. □

Remark 4.32. In Theorem 4.5 the corkscrew condition cannot be removed. Much as in [AHMMT20, Example 3, Appendix A], consider $\Omega := \bigcup_{k\geqslant 0}\Omega_k \subset \mathbb{R}^2$, where for each $k \geqslant 0$, we let Ω_k be the k-th stage of Garnett's 4-corners construction (see, e.g., [DS93, Chapter 1]), positioned inside the unit square whose lower left corner is at the point $(2k, 0)$ on the x-axis. As observed in [AHMMT20, Example 3, Appendix A] one can easily see that Ω fails to satisfy the interior corkscrew condition and its boundary is ADR but not UR. However, bounded harmonic functions satisfy (full) Carleson measure estimates. To see this we fix u harmonic in Ω with $\|u\|_{L^\infty(\Omega)} \leqslant 1$, and let $x_0 \in \partial\Omega$, $0 < r_0 < \infty$. Observing that Ω is comprised of a countable number of open squares, we let $\{R_j\}_j$ be the collection of those such cubes meeting $B_0 = B(x_0, r_0)$. One can find $y_j \in B_0 \cap \partial R_j \subset \partial\Omega$. Setting $\rho_j := \min(r_0, \operatorname{diam}(\partial R_j))$, we note that $B_0 \cap R_j \subset B(y_j, 2\rho_j) \cap R_j$ and

$$\iint\limits_{B_0\cap\Omega} |\nabla u|^2\,\delta\,dX = \sum_j \iint\limits_{B_0\cap R_j} |\nabla u|^2\,\delta\,dX \leqslant \sum_j \iint\limits_{B(y_j,2\rho_j)\cap R_j} |\nabla u|^2\,\delta\,dX.$$

Recall that u is harmonic in Ω and bounded by 1, hence it satisfies the same properties in R_j. Additionally, since R_j is a square (a Lipschitz domain), bounded harmonic functions satisfy Carleson measure estimates, with implicit constant that do not depend on the particular j (the Carleson measure estimates property is scale-invariant and in that regard the R_j's are all equivalent to the unit square). Moreover, the R_j's all have ADR boundaries with uniform constant (again ADR is a scale-invariant property). Hence, we can use that $B(y_j, \rho_j) \subset 2B_0$ to obtain

$$\iint\limits_{B_0\cap\Omega} |\nabla u|^2\,\delta\,dX \lesssim \sum_j \rho_j \lesssim \sum_j \mathcal{H}^1(B(y_j,\rho_j)\cap\partial R_j) \leqslant \sum_j \mathcal{H}^1(2B_0\cap\partial R_j) \leqslant \mathcal{H}^1(2B_0\cap\partial\Omega) \lesssim r_0,$$

where we have employed that sets $\{\partial R_j\}_j$ are pairwise disjoint and contained in $\partial\Omega$, and that $\partial\Omega$ is also ADR. We have then shown that bounded harmonic functions satisfy Carleson measure estimates in Ω. One can produce a similar construction in 3 dimensions. Details are left to the interested reader.

4.7.2 Proof of Theorem 4.5 for the operators in (a)

The argument is very similar to the one for the Laplacian but it requires some changes. First of all by Theorem 4.4 we can reduce matters to the case on which L is symmetric. Writing $K := \big\||\nabla A|\,\delta(\cdot)\big\|_{L^\infty(\Omega)} < \infty$ we have by [HMT17, Lemma 4.39] that $|\nabla u(X)|\delta(X) \lesssim_K u(X)$ for every $X \in \Omega$ and for every $0 \leqslant u \in W^{1,2}_{\text{loc}}(\Omega)$ so that $Lu = 0$ in Ω in the weak sense. Additionally, if $0 \leqslant u \in W^{1,2}(6I)$ is so that $Lu = 0$ in $6I$, with I being an $(n+1)$-dimensional cube with $6I \subset \Omega$ then $\|\nabla^2 u\|_{L^2(I)} \lesssim_K \ell(I)^{-1}\|\nabla u\|_{L^2(2I)}$ ([HMT17, Lemma 4.39]) and

$$|\nabla u(X) - \nabla u(Y)| \lesssim_K \left(\frac{|X-Y|}{\ell(I)}\right)^{\frac{1}{2}} \sup_{2I} |\nabla u|, \qquad X, Y \in I. \tag{4.80}$$

The latter estimate can be derived from [GM12, Theorem 5.19] applied in the unit cube and by translation and rescaling much as in [HMT17, Lemma 4.39]. With this estimate we can proceed as in [HLMN17, Section 5] and much as before there are three cases. The cubes in **Case 0** are trivial as before. For **Case 2** one has to inspect the proof of [HLMN17, Lemma 5.10] and see that the argument follows *mutatis mutandis* in account of the previous estimates. For the cubes in **Case 1** we need to make some changes. Rather than using the argument in [HLMN17, Section 5A] we initially follow [HLMN17, Section 5B],

using (4.80) in place of [HLMN17, (3.38)], up to the [HLMN17, (5.35)] with $p = 2$ and $F\gamma \equiv 1$, that is,

$$\sum_{\substack{Q\in\mathbb{D}_{\mathcal{T}_{\mathbf{S}},R_0}\\ \textbf{Case 1 holds}}} \sigma(Q) \lesssim \iint_{\Omega^*_{\mathcal{T}_{\mathbf{S}},R_0}} |\nabla^2\mathcal{G}|^2\,\mathcal{G}\,dX.$$

From this point we integrate by parts as in [HMT17] (see also Section 4.5.2) using that we have assumed that L is symmetric. With that one can obtain the desired estimate for the cubes in **Case 1** and the proof is then complete. Further details are left to the reader. □

4.7.3 Proof of Theorem 4.5 for the operators in (b)

Much as in [HMMTZ21, Corollary 10.3] one can regularize A so that the new matrix $\widetilde{A}$ is one of the operators considered in (a) and, moreover, $\widetilde{A}$ is a Fefferman-Kenig-Pipher perturbation of A. Thus, Theorem 4.3 and the fact that we have already taken care of the operators in (a) give the desired equivalences. □

4.7.4 Proof of Theorem 4.5 for the operators in (c)

This follows at once from Theorem 4.3 and the previous cases. □

4.8 Proof of Theorem 4.6: our results extend previous ones

We introduce some notation and definitions, some of which were already introduced in Section 2.3.3. Anyway, let us recall them here.

Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geqslant 2$, be an open set with ADR boundary and satisfying the corkscrew condition. We say that $\omega_L \in A_\infty^{\rm weak}(\sigma)$, if there are a positive constant $C > 1$ and an exponent $q > 1$, such that for every surface ball $\Delta = \Delta(x,r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there exists $X_\Delta \in B(x,r)\cap\Omega$, with $\operatorname{dist}(X_\Delta,\partial\Omega) \geqslant C^{-1}r$, satisfying $\omega_L^{X_\Delta} \ll \sigma$ in 2Δ, and $k_L^{X_\Delta} := \frac{d\omega_L^{X_\Delta}}{d\sigma}$ verifies

$$\int_{2\Delta} k_L^{X_\Delta}(y)^q\,d\sigma(y) \leqslant C\,\sigma(\Delta)^{1-q}.$$

We say that an open set Ω satisfies the Harnack chain condition if there is a uniform constant C such that for every $\rho > 0$, $\Lambda \geqslant 1$, and every pair of points $X, X' \in \Omega$ with $\min\{\delta(X),\delta(X')\} \geqslant \rho$ and $|X - X'| < \Lambda\rho$, there is a chain of open balls $B_1,\dots,B_N \subset \Omega$, $N \leqslant C(\Lambda)$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \varnothing$, $C^{-1}\operatorname{diam}(B_k) \leqslant \operatorname{dist}(B_k,\partial\Omega) \leqslant C\operatorname{diam}(B_k)$. We observe that the Harnack chain condition is a scale-invariant version of path connectedness.

We say that Ω is a 1-sided NTA (non-tangentially accessible) domain if Ω satisfies both the corkscrew and Harnack chain conditions. Furthermore, we say that Ω is an NTA domain if it is a 1-sided NTA domain and if, in addition, $\mathbb{R}^{n+1}\setminus\overline{\Omega}$ satisfies the corkscrew condition. If a 1-sided NTA domain, or an NTA domain, has an ADR boundary, then it is called a 1-sided CAD (chord-arc domain) or a CAD, respectively.

In 1-sided CAD domains the elliptic measure is doubling, hence $\omega_L \in A_\infty^{\rm weak}(\sigma)$ becomes $\omega_L \in A_\infty(\sigma)$, condition that can be equivalently written as follows: there exist constants $0 < \alpha,\beta < 1$ such

that given an arbitrary surface ball $\Delta_0 := B_0 \cap \partial\Omega$, with $B_0 := B(x_0, r_0)$, $x_0 \in \partial\Omega$, $0 < r_0 < \operatorname{diam}(\partial\Omega)$, and for every surface ball $\Delta := B \cap \partial\Omega$ centered at $\partial\Omega$ with $B \subset B_0$, and for every Borel set $F \subset \Delta$, we have that

$$\frac{\omega_L^{X_{\Delta_0}}(F)}{\omega_L^{X_{\Delta_0}}(\Delta)} \leqslant \alpha \quad \Longrightarrow \quad \frac{\sigma(F)}{\sigma(\Delta)} \leqslant \beta.$$

4.8.1 Proof of Theorem 4.6, part (i)

As discussed in [HLMN17, Section 4], by slightly changing the constant C in the definition of $\omega_L \in A_\infty^{\text{weak}}(\sigma)$ we may assume that $\omega_L^{X_\Delta}(\Delta) \geqslant C_1^{-1}$ for some constants $C_1 > 1$ on n, ADR, and ellipticity. In turn, it was also shown that there exists $\beta, \eta \in (0,1)$ so that for every $Q \in \mathbb{D} = \mathbb{D}(\partial\Omega)$ one can find $X_Q \in B_Q \cap \Omega$ with $\delta(X_Q) \gtrsim \ell(Q)$ for which $\omega_L^{X_Q}(Q) \geqslant C_1^{-1}$ and

$$\sigma(A) \geqslant (1-\eta)\sigma(Q) \implies \omega_L^{X_Q}(A) \geqslant \beta\omega_L^{X_Q}(Q), \tag{4.81}$$

for any Borel set $A \subset Q$.

Step 1: the basic step, the construction of a single stopping-time regime. We need the following auxiliary result:

Lemma 4.33 ([HLMN17, Lemma 4.12]). *Let $Q \in \mathbb{D}$ and let μ be a Borel measure on $\partial\Omega$. Assume that there exist $K_1 \geqslant 1$ and $\beta, \eta \in (0,1)$ such that*

$$1 \leqslant \frac{\mu(Q)}{\sigma(Q)} \leqslant \frac{\mu(\partial\Omega)}{\sigma(Q)} \leqslant K_1,$$

and for any Borel set $A \subset Q$,

$$\sigma(A) \geqslant (1-\eta)\sigma(Q) \quad \Longrightarrow \quad \mu(A) \geqslant \beta\mu(Q).$$

Then there exists a pairwise disjoint family $\mathcal{F}_Q = \{Q_j\}_j \subset \mathbb{D}_Q \backslash \{Q\}$ such that

$$\sigma\Big(Q \backslash \bigcup_{Q_j \in \mathcal{F}_Q} Q_j\Big) \geqslant (1-\alpha)\sigma(Q), \tag{4.82}$$

and

$$\frac{\beta}{2} \leqslant \frac{\mu(Q')}{\sigma(Q')} \leqslant \Big(\fint_{Q'} (\mathcal{M}\mu)^{\frac{1}{2}} d\sigma \Big)^2 \leqslant K_2, \qquad \forall\, Q' \in \mathbb{D}_{\mathcal{F}_Q, Q}, \tag{4.83}$$

where $\alpha \in (0,1)$ and $K_2 > 1$ depend only on n, β, η, K_1, and the ADR constant.

We are now ready to prove that $\omega_L \in A_\infty^{\text{weak}}(\sigma)$ implies that ω_L admits a strong corona decomposition. By Proposition 4.24, it suffices to obtain a strong corona decomposition for ω_L on $\mathbb{D}_{Q_0}$ any cube $Q_0 \in \mathbb{D}(\partial\Omega)$. Fix then $Q_0 \in \mathbb{D}(\partial\Omega)$ and we will construct the corona decomposition by iterating Lemma 4.33. The 0-th generation cubes are constructed as follows. As observed above $\omega_L^{X_{Q_0}}(Q_0) \geqslant C_1^{-1}$ for some $C_1 \geqslant 1$. Write $\mu := C_1\sigma(Q_0)\omega_L^{X_{Q_0}}$ so that

$$1 \leqslant C_1\, \omega_L^{X_{Q_0}}(Q_0) = \frac{\mu(Q_0)}{\sigma(Q_0)} \leqslant \frac{\mu(\partial\Omega)}{\sigma(Q_0)} = C_1\, \omega_L^{X_{Q_0}}(\partial\Omega) \leqslant C_1.$$

Using this and (4.81) we can invoke Lemma 4.33 and find a pairwise family $\mathcal{F}_{Q_0} \subset \mathbb{D}_{Q_0}\backslash\{Q_0\}$ such that (4.82) and (4.83) hold with $Q = Q_0$. Let $\mathbf{S}_{Q_0} := \mathbb{D}_{\mathcal{F}_{Q_0},Q_0}$, which is semi-coherent with $\mathrm{Top}(\mathbf{S}_{Q_0}) = Q_0$. Set $Q\mathbf{s}_{Q_0} := \mathrm{Top}(\mathbf{S}_{Q_0}) = Q_0$ and $X_{\mathbf{S}_{Q_0}} := X_{Q_0}$ so that (4.5) holds for $\mathbf{S}_{Q_0}$. Observe that (4.83) and the fact that $C_1^{-1} \leqslant \omega_L^{X_{Q_0}}(Q_0) \leqslant 1$ give for every $Q \in \mathbf{S}_{Q_0}$

$$\frac{\beta}{2C_1}\frac{\omega_L^{X_{\mathbf{S}_{Q_0}}}(Q\mathbf{s}_{Q_0})}{\sigma(Q\mathbf{s}_{Q_0})} \leqslant \frac{\beta}{2C_1\sigma(Q_0)} \leqslant \frac{\omega_L^{X_{\mathbf{S}_{Q_0}}}(Q)}{\sigma(Q)}$$
$$\leqslant \Big(\fint_Q (\mathcal{M}\omega_L^{X_{\mathbf{S}_{Q_0}}})^{\frac{1}{2}}d\sigma\Big)^2 \leqslant \frac{K_2}{C_1\sigma(Q_0)} \leqslant K_2\frac{\omega_L^{X_{\mathbf{S}_{Q_0}}}(Q\mathbf{s}_{Q_0})}{\sigma(Q\mathbf{s}_{Q_0})},$$

that is, (4.6) holds for $\mathbf{S}_{Q_0}$.

Step 2: iteration to get the corona decomposition. Defining $\mathbb{F}^0 := \{\mathbf{S}_{Q_0}\}$, $\mathbb{G}^0 := \mathbf{S}_{Q_0}$, and $\mathcal{F}^0 = \mathcal{F}_{Q_0}$ one can clearly obtain

$$\mathbb{D}_{Q_0} = \mathbb{G}^0 \bigsqcup \Big(\bigsqcup_{Q\in\mathcal{F}^0} \mathbb{D}_Q\Big)$$

and, by (4.82),

$$\sigma\Big(\bigsqcup_{Q\in\mathcal{F}^0} Q\Big) \leqslant \alpha\sigma(Q_0).$$

We now iterate, repeating this process for any $Q' \in \mathcal{F}^0$. We then obtain a pairwise family $\mathcal{F}_{Q'} \subset \mathbb{D}_{Q'}\backslash\{Q'\}$. Let $\mathbf{S}_{Q'} := \mathbb{D}_{\mathcal{F}_{Q'},Q'}$ which is semi-coherent with $\mathrm{Top}(\mathbf{S}_{Q'}) = Q'$. Set $Q\mathbf{s}_{Q'} := \mathrm{Top}(\mathbf{S}_{Q'}) = Q'$ and $X_{\mathbf{S}_{Q'}} = X_{Q'}$ so that (4.5) and (4.6) hold for $\mathbf{S}_{Q'}$. Defining $\mathbb{F}^1 := \{\mathbf{S}_{Q'} : Q' \in \mathcal{F}^0\}$, $\mathbb{G}^1 := \bigsqcup_{\mathbf{S}\in\mathbb{F}^1}\mathbf{S}$, and $\mathcal{F}^1 := \{\mathcal{F}_{Q'} : Q' \in \mathcal{F}^0\}$ we easily see that

$$\mathbb{D}_{Q_0} = \mathbb{G}^0 \bigsqcup \Big(\bigsqcup_{Q\in\mathcal{F}^0} \mathbb{D}_Q\Big) = \mathbb{G}^0 \bigsqcup \mathbb{G}^1 \bigsqcup \Big(\bigsqcup_{Q\in\mathcal{F}^1} \mathbb{D}_Q\Big)$$

and

$$\sigma\Big(\bigsqcup_{Q\in\mathcal{F}^1} Q\Big) = \sum_{Q'\in\mathcal{F}^0} \sigma\Big(\bigsqcup_{Q\in\mathcal{F}_{Q'}} Q\Big) \leqslant \alpha \sum_{Q'\in\mathcal{F}^0} \sigma(Q') = \alpha\sigma\Big(\bigsqcup_{Q'\in\mathcal{F}^0} Q'\Big) \leqslant \alpha^2\sigma(Q_0).$$

We now iterate this argument with the cubes of $\mathcal{F}^1$ and so forth so on. We then define $\mathbb{F} = \bigsqcup_{j=0}^\infty \mathbb{F}^j$, $\mathbb{G} = \bigsqcup_{j=0}^\infty \mathbb{G}^j$, and $\mathcal{F} = \bigsqcup_{j=0}^\infty \mathcal{F}^j$. Note that by construction $\mathbb{G} = \bigsqcup_{\mathbf{S}\in\mathbb{F}}\mathbf{S}$, where each $\mathbf{S}$ is semi-coherent. For each $\mathbf{S} \in \mathbb{F}$ we have that $Q_{\mathbf{S}} = \mathrm{Top}(\mathbf{S})$, $X_{\mathbf{S}} = X_{Q_{\mathbf{S}}}$, and that (4.5) and (4.6) hold. It is also clear from the construction that $\mathrm{Top}(\mathbb{F}) = \mathcal{F} \sqcup \{Q_0\}$.

Step 3: there are no bad cubes. We next show that $\mathbb{D}\backslash\mathbb{G} = \varnothing$. Assume otherwise that the exists $Q' \in \mathbb{D}\backslash\mathbb{G} = \varnothing$. Since $0 < \alpha < 1$, we can find $k \geqslant 0$ so that $\alpha^k\sigma(Q_0) < \sigma(Q')$. Note that

$$\mathbb{D}_{Q_0} = \Big(\bigsqcup_{j=0}^k \mathbb{G}^j\Big) \bigsqcup \Big(\bigsqcup_{Q\in\mathcal{F}^k} \mathbb{D}_Q\Big),$$

hence $Q' \in \mathbb{D}_Q$ for some $Q \in \mathcal{F}^k$. This gives a contradiction:

$$\alpha^k \sigma(Q_0) < \sigma(Q') \leqslant \sigma\Big(\bigsqcup_{Q\in\mathcal{F}^k} Q \Big) \leqslant \alpha^{k+1}\sigma(Q_0) < \alpha^k \sigma(Q_0).$$

Step 4: the maximal cubes pack. We next show that the cubes in $\mathrm{Top}(\mathbb{F}) = \mathcal{F} \sqcup \{Q_0\}$ satisfy a Carleson packing condition. Assuming this momentarily, and if we set $\mathbb{B} = \varnothing$ we have then shown that $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ is a semi-coherent corona decomposition so that (4.5) and (4.6) hold, which completes the proof.

Let us then show that $\mathrm{Top}(\mathbb{F}) = \mathcal{F} \sqcup \{Q_0\}$ satisfy a Carleson packing condition. Take an arbitrary $Q_0' \in \mathrm{Top}(\mathbb{F})$, that is, $Q_0' \in \mathcal{F}^{j_0}$ for some j_0. Let $j \geqslant j_0$ and note that $\mathcal{F}^{j+1} = \{\mathcal{F}_{Q'} : Q' \in \mathcal{F}^j\}$, hence

$$\begin{aligned}\Sigma_{j+1} := \sum_{Q\in\mathcal{F}^{j+1}\cap\mathbb{D}_{Q_0'}} \sigma(Q) &= \sum_{Q'\in\mathcal{F}^j} \sum_{Q\in\mathcal{F}_{Q'}\cap\mathbb{D}_{Q_0'}} \sigma(Q) \\ &= \sum_{Q'\in\mathcal{F}^j\cap\mathbb{D}_{Q_0'}} \sigma\Big(\bigsqcup_{Q\in\mathcal{F}_{Q'}} Q \Big) \leqslant \alpha \sum_{Q'\in\mathcal{F}^j\cap\mathbb{D}_{Q_0'}} \sigma(Q') = \alpha\Sigma_j.\end{aligned}$$

Iterating this we obtain $\Sigma_j \leqslant \alpha^{j-j_0}\,\Sigma_{j_0} = \alpha^{j-j_0}\,\sigma(Q_0')$, for every $j \geqslant j_0$. As a result,

$$\sum_{Q\in\mathrm{Top}(\mathbb{F})\cap\mathbb{D}_{Q_0'}} \sigma(Q) = \sum_{j=j_0}^{\infty} \Sigma_j \leqslant \sigma(Q_0') \sum_{j=j_0}^{\infty} \alpha^{j-j_0} = (1-\alpha)^{-1}\,\sigma(Q_0'), \tag{4.84}$$

which is the desired estimate for any arbitrary cube $Q_0' \in \mathrm{Top}(\mathbb{F})$.

Consider next the general case $Q \in \mathbb{D}_{Q_0}$ and let $\{Q_k\}_k$ be the collection of maximal cubes (hence pairwise disjoint) in $\mathrm{Top}(\mathbb{F})$ contained in Q. Then, using (4.84) with Q_k

$$\begin{aligned}\sum_{Q'\in\mathrm{Top}(\mathbb{F})\cap\mathbb{D}_Q} \sigma(Q') = \sum_k \sum_{Q'\in\mathrm{Top}(\mathbb{F})\cap\mathbb{D}_{Q_k}} \sigma(Q') &\leqslant (1-\alpha)^{-1} \sum_k \sigma(Q_k) \\ &= (1-\alpha)^{-1}\,\sigma\Big(\bigsqcup_k Q_k \Big) \leqslant (1-\alpha)^{-1}\sigma(Q),\end{aligned}$$

and this shows that $\mathrm{Top}(\mathbb{F}) = \mathcal{F} \sqcup \{Q_0\}$ satisfy a Carleson packing condition, as desired. □

4.8.2 Proof of Theorem 4.6, part (ii)

We assume that Ω is a 1-sided CAD. Part (i) shows that (a) $\Longrightarrow$ (b), since ω_L is doubling in 1-sided CAD. Below we show that (b) $\Longrightarrow$ (a). Assuming this momentarily, [CHMT20, Theorem 1.1] gives that (a) $\Longleftrightarrow$ (e) and Theorem 4.2 yields the other equivalences.

Step 1: preliminaries on discrete Carleson measures and extrapolation. We introduce some notation. Let $E \subset \mathbb{R}^{n+1}$ be a an ADR set and let $\mathbb{D} = \mathbb{D}(E)$ be its associated family of dyadic cubes. Given sequence of non-negative numbers $\{\alpha_Q\}_{Q\in\mathbb{D}}$, we define the "measure" $\mathfrak{m}$ (acting on collection of dyadic cubes) by

$$\mathfrak{m}(\mathbb{D}') := \sum_{Q\in\mathbb{D}'} \alpha_Q, \qquad \text{for } \mathbb{D}' \subset \mathbb{D}. \tag{4.85}$$

For a fixed $Q_0 \in \mathbb{D}$, we say that $\mathfrak{m}$ is a discrete "Carleson measure" on $\mathbb{D}_{Q_0}$ (with respect to σ), and write $\mathfrak{m} \in C(Q_0)$, if

$$\|\mathfrak{m}\|_{C(Q_0)} := \sup_{Q \in \mathbb{D}_{Q_0}} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} < \infty.$$

We also set

$$\|\mathfrak{m}\|_{C} := \sup_{Q \in \mathbb{D}} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} < \infty$$

to denote the global Carleson norm on $\mathbb{D}$. Given a family $\mathcal{F} \subset \mathbb{D}$ of pairwise disjoint cubes, we define the restriction of $\mathfrak{m}$ to the sawtooth $\mathbb{D}_{\mathcal{F}}$ by

$$\mathfrak{m}_{\mathcal{F}}(\mathbb{D}') := \mathfrak{m}(\mathbb{D}' \cap \mathbb{D}_{\mathcal{F}}) = \sum_{Q \in \mathbb{D}' \setminus (\bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j})} \alpha_Q.$$

For a pairwise disjoint family $\mathcal{F} = \{Q_j\} \subset \mathbb{D}$ and a non-negative Borel measure μ on $\partial\Omega$ we define the projection measure $\mathcal{P}_{\mathcal{F}}\mu$ as

$$\mathcal{P}_{\mathcal{F}}\mu(A) := \mu\Big(A \setminus \bigcup_{Q_j \in \mathcal{F}} Q_j\Big) + \sum_{Q_j \in \mathcal{F}} \frac{\sigma(A \cap Q_j)}{\sigma(Q_j)} \mu(Q_j),$$

for any Borel set $A \subset \partial\Omega$.

Lemma 4.34 ([HM14, Lemma 8.5]). *Suppose that $E \subset \mathbb{R}^{n+1}$ is an ADR set. Fix $Q^0 \in \mathbb{D}(E)$, let σ and ω be a pair of dyadically doubling Borel measures on Q^0, and let $\mathfrak{m}$ be a discrete Carleson measure with respect to σ with*

$$\|\mathfrak{m}\|_{C(Q^0)} \leqslant M_0 < \infty.$$

Suppose that there esixts $\gamma > 0$ such that for every $Q_0 \in \mathbb{D}_{Q^0}$ and every family of pairwise disjoint dyadic cubes $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_{Q_0}$ verifying

$$\|\mathfrak{m}_{\mathcal{F}}\|_{C(Q_0)} \leqslant \gamma, \tag{4.86}$$

we have that $\mathcal{P}_{\mathcal{F}}\omega$ satisfies the following property: for all $\varepsilon \in (0,1)$ there exists $C_\varepsilon > 1$ such that

$$F \subset Q_0, \frac{\sigma(F)}{\sigma(Q_0)} \geqslant \varepsilon \quad \Longrightarrow \quad \frac{\mathcal{P}_{\mathcal{F}}\,\omega(F)}{\mathcal{P}_{\mathcal{F}}\,\omega(Q_0)} \geqslant \frac{1}{C_\varepsilon}. \tag{4.87}$$

Then, there exist $\alpha, \beta \in (0,1)$ such that for every $Q_0 \in \mathbb{D}_{Q^0}$,

$$F \subset Q_0, \quad \frac{\sigma(F)}{\sigma(Q_0)} \geqslant \alpha \quad \Longrightarrow \quad \frac{\omega(F)}{\omega(Q_0)} \geqslant \beta,$$

that is, $\omega \in A_\infty^{\mathrm{dyadic}}(Q^0, \sigma)$.

Step 2: setting up the stage to prove the result by extrapolation. We are now ready to prove that (b) $\Longrightarrow$ (a). A more restrictive version of this result appears in [GMT18, Section 4.2] using different methods. Let $(\mathbb{B}, \mathbb{G}, \mathbb{F})$ be the assumed corona decomposition associated with ω_L. Set for any $Q \in \mathbb{D}$

$$\alpha_Q := \begin{cases} \sigma(Q), & Q \in \mathbb{B} \cup \mathrm{Top}(\mathbb{F}), \\ 0, & \text{otherwise.} \end{cases} \tag{4.88}$$

We define the associated discrete measure $\mathfrak{m}$ as in (4.85). Fix $Q^0 \in \mathbb{D}$ and write $\omega := \omega_L^{X_{Q^0}}$. Note that both ω and σ are dyadically doubling on Q^0, since Ω is a 1-sided CAD. In view of the Carleson packing condition for bad and top cubes, we have

$$\|\mathfrak{m}\|_{C(Q^0)} \leqslant \|\mathfrak{m}\|_C \leqslant M_0 < \infty.$$

Therefore, Lemma 4.34 implies that $\omega_L^{X_{Q^0}} \in A_\infty^{\rm dyadic}(Q^0)$ provided we show that (4.86) for some small $\gamma \in (0,1)$ to be found implies (4.87). Since Q^0 is arbitrary and the same constants α and β are valid for all Q^0, the fact that the elliptic measure is doubling in the present scenario easily yields that $\omega \in A_\infty(\sigma)$, details are left to the interested reader.

To complete the proof, it suffices to show that (4.86) implies (4.87). To this end, fix $Q_0 \in \mathbb{D}_{Q^0}$ and a family of pairwise disjoint dyadic subcubes $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_{Q_0}$ satisfying (4.86) with $\gamma \in (0,1)$ to be chosen momentarily. Let $F \subset Q_0$ be an arbitrary Borel set.

The case $\mathcal{F} = \{Q_0\}$ is trivial since $\frac{\mathcal{P}_\mathcal{F}\,\omega(F)}{\mathcal{P}_\mathcal{F}\,\omega(Q_0)} = \frac{\sigma(F)}{\sigma(Q_0)}$, and (4.87) clearly holds. Thus we may assume that $\mathcal{F} \subset \mathbb{D}_{Q_0}\backslash\{Q_0\}$. Write $E_0 := Q_0\backslash(\bigcup_{Q_j\in\mathcal{F}} Q_j)$. We claim that there exists $\mathbf{S}_0 \in \mathbb{G}$ such that $Q_0 \subsetneq \mathrm{Top}(\mathbf{S}_0)$, $\mathbb{D}_{\mathcal{F},Q_0} \subset \mathbf{S}_0$, and

$$\frac{\omega(Q)}{\sigma(Q)} \approx \frac{\omega(Q^0 \cap \mathrm{Top}(\mathbf{S}_0))}{\sigma(Q^0 \cap \mathrm{Top}(\mathbf{S}_0))} =: \Lambda_0, \quad \forall Q \in \mathbb{D}_{\mathcal{F},Q_0}. \tag{4.89}$$

Step 3: assuming that we may change scales as in (4.89), we can make the extrapolation work. Assuming (4.89) momentarily, we conclude (4.87) as follows. By definition,

$$\mathcal{P}_\mathcal{F}\omega(F) = \omega(F\cap E_0) + \sum_{Q_j\in\mathcal{F}} \frac{\omega(Q_j)}{\sigma(Q_j)}\sigma(F\cap Q_j) := \mathcal{I} + \mathcal{II}. \tag{4.90}$$

We first deal with the second term. If we write $\widehat{Q}_j$ for the dyadic parent of $Q_j \in \mathcal{F}$, it is clear than $\widehat{Q}_j \in \mathbb{D}_{\mathcal{F},Q_0}$. Since both ω and σ are dyadically doubling, it follows from (4.89) that

$$\mathcal{II} \approx \sum_{Q_j\in\mathcal{F}} \frac{\omega(\widehat{Q}_j)}{\sigma(\widehat{Q}_j)}\sigma(F\cap Q_j) \approx \Lambda_0 \sum_{Q_j\in\mathcal{F}} \sigma(F\cap Q_j) = \Lambda_0\,\sigma(F\backslash E_0). \tag{4.91}$$

To bound the first term, let $\eta > 0$ be an arbitrary number. Since ω is a regular Borel measure, one can find an open set $U_\eta \supset F\cap E_0$ such that $\omega(U_\eta\backslash(F\cap E_0)) < \eta$. Let $x \in F\cap E_0 \subset U_\eta$. Then $\Delta(x,r_x) \subset U_\eta$ for some $0 < r_x < \infty$. Pick $Q_x \in \mathbb{D}_{Q_0}$ containing x such that $\mathrm{diam}(Q_x) < r_x$ and $\ell(Q_x) < \ell(Q_0)$. This and the fact $x \in Q_0$ give $Q_x \subset \Delta(x,r_x)\cap Q_0 \subset U_\eta \cap Q_0$. Let $Q_x^{\max} \in \mathbb{D}_{Q_0}$ be the maximal cube with $Q_x^{\max} \supset Q_x$ such that $Q_x^{\max} \subset U_\eta \cap Q_0$. Denote by $\widetilde{\mathcal{F}}$ be the collection of the maximal cubes $Q_x^{\max}$ for $x \in F\cap E_0$. Note that $\widetilde{\mathcal{F}}$ is pairwise disjoint,

$$F\cap E_0 \subset \bigcup_{Q\in\widetilde{\mathcal{F}}} Q \subset U_\eta \cap Q_0, \quad \text{and} \quad F\cap E_0 \cap Q \neq \varnothing, \quad \forall Q \in \widetilde{\mathcal{F}}. \tag{4.92}$$

On the other hand,

$$\widetilde{\mathcal{F}} \subset \mathbb{D}_{\mathcal{F},Q_0}. \tag{4.93}$$

Otherwise, there are $Q \in \widetilde{\mathcal{F}}$ and $Q_j \in \mathcal{F}$ such that $Q \subset Q_j$. Then $F \cap E_0 \cap Q \subset E_0 \cap Q_j = \varnothing$, which contradicts (4.92). Then, we use (4.89), (4.92) and (4.93) to obtain

$$\begin{aligned}\Lambda_0\sigma(F \cap E_0) \leqslant \sum_{Q\in\widetilde{\mathcal{F}}} \Lambda_0\sigma(Q) \approx \sum_{Q\in\widetilde{\mathcal{F}}} \omega(Q) &\leqslant \omega(U_\eta \cap Q_0)\\ &\leqslant \omega(U_\eta\backslash(F \cap E_0)) + \omega(F \cap E_0) \leqslant \eta + \omega(F \cap E_0),\end{aligned}$$

where the implicit constant is independent of η. Letting $\eta \to 0$, we arrive at

$$\mathcal{I} = \omega(F \cap E_0) \gtrsim \Lambda_0\, \sigma(F \cap E_0). \tag{4.94}$$

Now collecting (4.90), (4.91), and (4.94) we get

$$\mathcal{P}_{\mathcal{F}}\omega(F) \gtrsim \Lambda_0\, \sigma(F).$$

This and the fact that $\mathcal{P}_{\mathcal{F}}\omega(Q_0) = \omega(Q_0) \approx \Lambda_0\sigma(Q_0)$ easily yield

$$\frac{\mathcal{P}_{\mathcal{F}}\,\omega(F)}{\mathcal{P}_{\mathcal{F}}\,\omega(Q_0)} \gtrsim \frac{\sigma(F)}{\sigma(Q_0)},$$

which immediately gives (4.87).

Step 4: proof of (4.89) using the change of poles formula, because the extrapolation procedure allows us to work in a single corona regime. It remains to show (4.89). Note that if $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, then

$$\alpha_Q \leqslant \sum_{Q'\in\mathbb{D}_{\mathcal{F},Q}} \alpha_{Q'} \leqslant \|\mathfrak{m}_{\mathcal{F}}\|_{C(Q_0)}\sigma(Q) \leqslant \gamma\,\sigma(Q).$$

By (4.88), if we simply take $\gamma = \frac{1}{2}$ definition, we conclude that $Q \notin \mathbb{B} \cup \mathrm{Top}(\mathbb{F})$. That is, our choice of γ guarantees that $\mathbb{D}_{\mathcal{F},Q_0} \subset \mathbb{D}\backslash(\mathbb{B} \cup \mathrm{Top}(\mathbb{F}))$. This and the semi-coherency of the regimes in $\mathbb{F}$ immediately imply that there exists a unique $\mathbf{S}_0 \in \mathbb{G}$ such that $\mathbb{D}_{\mathcal{F},Q_0} \subset \mathbf{S}_0$ and $Q_0 \subsetneq \mathrm{Top}(\mathbf{S}_0)$. Note that $Q^0 \cap \mathrm{Top}(\mathbf{S}_0) \in \mathbf{S}_0$ since $\mathbf{S}_0$ is semi-coherent and $\mathbf{S}_0 \ni Q_0 \subset Q^0 \cap \mathrm{Top}(\mathbf{S}_0) \subset \mathrm{Top}(\mathbf{S}_0)$. Thus, (4.8) implies

$$\frac{\omega_L^{X_{\mathbf{S}_0}}(Q)}{\sigma(Q)} \approx \frac{\omega_L^{X_{\mathbf{S}_0}}(Q_{\mathbf{S}_0})}{\sigma(Q_{\mathbf{S}_0})} \approx \frac{\omega_L^{X_{\mathbf{S}_0}}(Q^0 \cap \mathrm{Top}(\mathbf{S}_0))}{\sigma(Q^0 \cap \mathrm{Top}(\mathbf{S}_0))}, \qquad \forall Q \in \mathbb{D}_{\mathcal{F},Q_0}. \tag{4.95}$$

Note that $Q_0 \subsetneq \mathrm{Top}(\mathbf{S}_0) \cap Q^0 \subset Q_{\mathbf{S}_0} \cap Q^0$, hence $Q_{\mathbf{S}_0} \cap Q^0 \neq \varnothing$. If $Q^0 \subset Q_{\mathbf{S}_0}$, then the change of pole formula gives

$$\frac{d\omega_L^{X_{Q^0}}}{d\omega_L^{X_{\mathbf{S}_0}}}(y) \approx \frac{1}{\omega_L^{X_{\mathbf{S}_0}}(Q^0)}, \qquad \text{for } \omega_L^{X_{\mathbf{S}_0}}\text{-a.e. } y \in Q^0.$$

As a result, if $Q, Q' \subset Q^0$ we obtain

$$\frac{\omega(Q)}{\omega(Q')} = \frac{\omega_L^{X_{Q^0}}(Q)}{\omega_L^{X_{Q^0}}(Q')} \approx \frac{\omega_L^{X_{\mathbf{S}_0}}(Q)}{\omega_L^{X_{\mathbf{S}_0}}(Q')}.$$

Analogously, if $Q_{\mathbf{S}_0} \subset Q^0$, then the change of pole formula gives

$$\frac{d\omega_L^{X_{\mathbf{S}_0}}}{d\omega_L^{X_{Q^0}}}(y) \approx \frac{1}{\omega_L^{X_{Q^0}}(Q_{\mathbf{S}_0})}, \qquad \text{for } \omega_L^{X_{Q^0}}\text{-a.e. } y \in Q_{\mathbf{S}_0}.$$

Thus, if $Q, Q' \subset Q_{\mathbf{S}_0}$, then

$$\frac{\omega_L^{X_{\mathbf{S}_0}}(Q)}{\omega_L^{X_{\mathbf{S}_0}}(Q')} \approx \frac{\omega_L^{X_{Q^0}}(Q)}{\omega_L^{X_{Q^0}}(Q')} = \frac{\omega(Q)}{\omega(Q')}.$$

This means that in either scenario if $Q, Q' \subset Q_{\mathbf{S}_0} \cap Q^0$ we obtain

$$\frac{\omega(Q)}{\omega(Q')} \approx \frac{\omega_L^{X_{\mathbf{S}_0}}(Q)}{\omega_L^{X_{\mathbf{S}_0}}(Q')}.$$

Recalling that $\mathbb{D}_{\mathcal{F},Q_0} \subset \mathbf{S}_0 \cap \mathbb{D}_{Q_0} \subset \mathbb{D}_{Q_{\mathbf{S}_0}} \cap \mathbb{D}_{Q^0}$, the previous estimate and (4.95) give as desired (4.89):

$$\frac{\omega(Q)}{\omega(Q^0 \cap \mathrm{Top}(\mathbf{S}_0))} \approx \frac{\omega_L^{X_{\mathbf{S}_0}}(Q)}{\omega^{X_{\mathbf{S}_0}}(Q^0 \cap \mathrm{Top}(\mathbf{S}_0))} \approx \frac{\sigma(Q)}{\sigma(Q^0 \cap \mathrm{Top}(\mathbf{S}_0))},$$

for all $Q \in \mathbb{D}_{\mathcal{F},Q_0}$. The proof is then complete. □

4.8.3 Proof of Theorem 4.6, part (iii)

Assuming that Ω is a 1-sided CAD, we just need to apply [AHMNT17, Theorem 1.2] for the Laplacian, [HMMTZ21, Theorem 1.6] for the Kenig-Piper operators, [HMMTZ21, Corollary 10.3] for the analog class using oscillations, and [CHMT20, Theorem 1.4] for the perturbations. Details are left to the interested reader. □

CHAPTER 5

The parabolic Dirichlet problem with singular boundary data

As seen in Chapter 4, the solvability of the Dirichlet problem with singular data is a delicate matter, and it has been shown already that there are very strong interactions between geometry and analysis in this problem. In this chapter, we aim to show that the same strong relationship between geometry and analysis is true when one considers parabolic equations (i.e. time dependent). Even more: the problem becomes substantially more difficult to analyze as compared to the elliptic one studied in Chapter 4.

In the realm of elliptic PDE with singular data, Dahlberg's result [Da77] was the first breakthrough. We recall that he showed that the boundary value problem associated to the Laplace equation, with L^2 boundary data, is well-posed in Lipschitz domains. The difficulty in the parabolic world rapidly becomes apparent because Dahlberg's cannot be naively adapted to the heat equation: Kaufmann and Wu [KW80] showed that there are (parabolic) Lipschitz graphs for which the associated caloric measure is not absolutely continuous with respect to the surface measure (so definitely is not an A_∞ weight), which makes it impossible that the L^p Dirichlet problem is solvable. Thus, Lipschitz domains were recognized not to be the correct sets in which to pose the parabolic Dirichlet problem with singular data.

It took around 20 years to find the appropriate framework to generalize appropriately Dahlberg's result to the heat equation: after several preliminary studies (that we will summarize later), Lewis and Murray [LM95] found a sufficient condition for domains in which the L^p Dirichlet problem for the heat equation is solvable, and shortly after, Hofmann and Lewis [HL96] showed that in such domains one can actually take $p = 2$ (which was the original Dahlberg's result). The domains found by Lewis and Murray are given locally by graphs of (parabolic) Lipschitz functions which satisfy the additional property that their half-order derivative in the time direction belongs to the (parabolic) BMO space (we will explain this in more detail later). Naturally, this very convolved definition made all the arguments more difficult than their elliptic counterparts. Very recently, Bortz, Hofmann, Martell and Nyström [BHMN25] have shown that these are *exactly* the sets which are the appropriate analogues of Lipschitz domains in the elliptic setting.

Theorem 5.1 (Lewis, Murray [LM95]; Bortz, Hofmann, Martell, Nyström [BHMN25])**.** *Let $n \geqslant 2$. Let $\Omega \subset \mathbb{R}^{n+1}$ be the region above a parabolic Lipschitz graph, i.e.*

$$\Omega = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : \ x_0 > \psi(x,t)\},$$

where ψ is a $\mathrm{Lip}(1,1/2)$ *function (see Section 2.5). Then the Dirichlet problem for the heat equation in Ω is well-posed for boundary data in L^p for some $p < \infty$ (see Section 5.6) if and only if $\partial_t^{1/2}\psi$ belongs to the parabolic BMO space (and we call ψ a regular* $\mathrm{Lip}(1,1/2)$ *graph in such case, see Section 5.4).*

These results tell us that, although at first sight artificial, the condition about the oscillations of the half-order time derivatives of the parametrization of the boundary is really the right condition to have in mind when solving parabolic Dirichlet problems with singular boundary data. As a consequence, it becomes clear that parabolic boundary value problems are inherently more difficult than their elliptic counterparts, because one will always need to tackle these oscillation conditions, which are very subtle to handle.

In this chapter, we show an extension of [BHMN25] to more general equations, beyond the heat equation. Actually, we show that for a subfamily of the (parabolic analogues of) operators from [KP01] (that were already considered in Chapter 4), the result of [BHMN25] is still true: Lipschitz boundaries need to be regular in order for the L^p Dirichlet problem to be solvable.

The novelty of the work resides on a careful study of the properties of the Green's functions (which are the central objects in the proof of [BHMN25]), which for general equations are not as well-behaved as for the heat equation. Concretely, the fact that the heat equation has constant coefficients allowed the authors of [BHMN25] to show some monotonicity of the Green's functions when approaching the boundary, which is not necessarily true for the general equations that we will consider in this chapter. Hence, we need to carry out a more detailed study and perform several auxiliary geometrical constructions to overcome the lack of monotonicity of the Green's function, and still be able to complete the method of [BHMN25].

This chapter is based on the paper [BFHH25], done in collaboration with Simon Bortz, Sandra Ferris and Steve Hofmann.

We suggest the reader to have Section 5.3 handy while reading the chapter, for it has the key ideas and provides a simplified and understandable sketch of the proofs that will be carried out in all detail along the rest of the chapter.

5.1 Main results . . . 159
5.2 Further historical context . . . 160
5.3 Main ideas and outline of the chapter . . . 162
5.4 Preliminaries . . . 164
5.4.1 Notation . . . 164
5.4.2 Parabolic Hausdorff measure . . . 166
5.4.3 Lip(1,1/2) graph domains . . . 166
5.4.4 Surface cubes and reference points . . . 167
5.4.5 (Parabolic) BMO and fractional integral operators . . . 168
5.4.6 Regular Lip(1,1/2) graph domains . . . 169
5.4.7 Dyadic cubes and a criterion for a Lip(1,1/2) graph to be regular . . . 170
5.4.8 Whitney regions and sawtooths . . . 171
5.4.9 Convention concerning constants . . . 172
5.5 Boundary estimates in parabolic Lipschitz domains . . . 172
5.5.1 Green's functions and parabolic measure . . . 173

5.5.2 PDE estimates . . . 174
5.6 The assumptions in our main results . . . 175
5.7 Step 1: Initial stopping time regimes and the preliminary estimates . . . 177
5.7.1 Control of the parabolic measure in "ample" stopping-time regimes . . . 178
5.7.2 Level lines and estimates for the Green function in Whitney regions . . . 180
5.8 Step 2: A local square function estimate and refined corona decomposition . . . 184
5.8.1 Local square function estimate via integration by parts . . . 185
5.8.2 Refinement of the corona decomposition to obtain vertical growth of u . . . 192
5.8.3 Reduction to finding regular Lip(1,1/2) approximating graphs in each stopping-time . . . 194
5.9 Step 3: Construction of approximating graphs via a stopping-time distance . . . 194
5.9.1 Construction of the approximating Lip(1,1/2) graphs . . . 195
5.9.2 Pointwise estimates at level sets, and Lipschitzianity of the approximating graph 198
5.10 Step 4: Regularity of the approximating graphs . . . 201
5.10.1 Handling Case 4 cubes, assuming Cases 1, 2 and 3 . . . 202
5.10.2 Localization of $\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y})$ . . . 204
5.10.3 Case 3 cubes . . . 206
5.10.4 Proving the localized estimate for Case 2 cubes . . . 206
5.10.5 Modified level sets and square function estimates . . . 207
5.10.6 Treating the Case 1 cubes with methods from [BHMN25] . . . 208
5.10.7 Estimating the boundary terms b_1, b_2, and proof of Lemma 5.53. . . . 211
5.11 Cut-off function construction . . . 213
5.12 Comments on smoothness assumptions . . . 217
5.13 A linear algebra fact about elliptic matrices . . . 218

5.1 Main results

In this chapter we extend the results of [BHMN25] to the setting of variable coefficient operators satisfying a Carleson condition on their oscillation. In particular, we consider parabolic operators in divergence form

$$Lf := \partial_t f - \operatorname{div}_X(A(X,t)\nabla_X f), \qquad L^*f := -\partial_t f - \operatorname{div}_X(A^*(X,t)\nabla_X f), \tag{5.1}$$

acting in $\mathbb{R}^{n+1} = \{(X,t) : X \in \mathbb{R}^n, t \in \mathbb{R}\}$, where A is a real $n \times n$ matrix with transpose A^*, satisfying the point-wise ellipticity bounds for some $\Lambda \geqslant 1$

$$\|A\|_{L^\infty} \leqslant \Lambda, \quad \text{and} \quad \langle A(X,t)\xi,\xi\rangle \geqslant \Lambda^{-1}|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^n, \text{ a.e. } (X,t) \in \mathbb{R}^{n+1}. \tag{5.2}$$

Our results concern how the L^p-solvability of the Dirichlet problem for L and L^* imposes additional regularity on the boundary of certain domains. In particular we prove the following theorems.

Theorem 5.2. *Let $n \geqslant 2$. Suppose that $\Omega \subseteq \mathbb{R}^{n+1}$ is the region above the graph of a* Lip$(1,1/2)$ *function ψ (see Definition 5.4.3), and A is a* ***symmetric*** *elliptic matrix satisfying (5.2) and the L^1-Carleson oscillation condition in Definition 5.24. Assume further that there exists some $1 < p < \infty$ such that the L^p Dirichlet problem is solvable for L (see Definition 5.20), where*

$$Lf := \partial_t f - \operatorname{div}_X(A(X,t)\nabla_X f).$$

Then $D_t^{1/2}\psi \in$ BMO (the parabolic BMO *space, see Section 5.4.5), with norm controlled uniformly depending only on the constants appearing in the hypotheses. Equivalently, the L^p solvability of the Dirichlet problem for L implies that the graph of ψ is parabolic uniformly rectifiable.*

Theorem 5.3. *Let $n \geqslant 2$. Suppose that $\Omega \subseteq \mathbb{R}^{n+1}$ is the region above the graph of a $\mathrm{Lip}(1, 1/2)$ function ψ (see Definition 5.4.3), and A is a (not necessarily symmetric) elliptic matrix satisfying (5.2) and the L^1-Carleson oscillation condition in Definition 5.24. Assume further that there exists some $1 < p < \infty$ such that the L^p Dirichlet problem is solvable for both L and L^* (see Definition 5.20), where*

$$Lf := \partial_t f - \mathrm{div}_X(A(X,t)\nabla_X f), \qquad L^* f := -\partial_t f - \mathrm{div}_X(A^*(X,t)\nabla_X f).$$

Then $D_t^{1/2}\psi \in$ BMO (the parabolic BMO *space, see Section 5.4.5), with norm controlled uniformly depending only on the constants appearing in the hypotheses. Equivalently, the L^p solvability of the Dirichlet problem for L and L^* implies that the graph of ψ is parabolic uniformly rectifiable.*

The converse to Theorem 5.3 (hence also to Theorem 5.2) is known: assuming that Ω is the region above the graph of a $\mathrm{Lip}(1, 1/2)$ function ψ with $D_t^{1/2}\psi \in$ BMO, and given that A is an elliptic matrix satisfying (5.2), and also that the L^1 Carleson oscillation condition holds, one obtains that there exists $p \in (1, \infty)$ such that the L^p Dirichlet problem is solvable for L and L^*. In fact, this converse is known under a somewhat weaker hypothesis on the coefficients, namely an L^2 Carleson measure condition on ∇A; see [Ri03], which adapts the methods of [KKPT00] and [KP01] to the parabolic setting. One may give an easier proof, using more recently developed techniques, by applying the criterion in [DPP17][1], and using the local square function estimate in [HL01, Lemma A.2 (i)][2] (plus a pullback argument, using a parabolic version of the Dahlberg-Kenig-Stein mapping). A slightly weaker version of the result (i.e., with a more restrictive Carleson measure condition on the coefficients) appeared earlier in [HL01]. Under the weaker L^2 Carleson measure assumption, at present we lack certain tools used in Theorems 5.2 and 5.3, in particular, the local square function estimates of Lemma 5.34, for the time derivative and second spatial partial derivatives of the Green function. Thus we require the stronger L^1 condition in Definition 5.24. In fact, under the L^2 Carleson measure assumption, it remains an open problem to establish even the elliptic analogue of such square function estimates (we note that a variant of these estimates, in the elliptic setting, has been obtained in work of Feneuil and Li [FL23], but it is not clear whether the estimates of [FL23] would suffice for our purposes in the present chapter).

5.2 Further historical context

Let us complement the discussion in the previous pages and Section 1.3 with some further and more detailed data about the history of parabolic PDEs with singular data, to put our result in context.

It is known that the solvability of the L^p Dirichlet problem for L (for some $p > 1$) is equivalent (in the presence of the doubling property of parabolic measure) to a certain scale-invariant mutual absolute continuity condition (of parabolic measure with respect to parabolic surface measure) known as the parabolic (Muckenhoupt) A_∞ condition[3]. This condition has been extensively studied in its connection with the boundedness of classical objects in harmonic analysis (maximal functions and SIOs) since the early 70's [Mu72]. In 1977, Dahlberg [Da77] resolved a major open conjecture concerning the absolute

[1] which in turn is a parabolic version of [KKPT16]

[2] which in turn is a parabolic version of analogous elliptic bounds in [KP01]

[3] In the absence of the doubling property, L^p solvability is equivalent to the "weak-A_∞" condition; see, e.g., [GH20].

continuity of *harmonic* measure in Lipschitz domains by (implicitly) showing that the harmonic measure is an A_∞ weight. It was also shown that the A_∞ condition gives L^p non-tangential maximal function estimates for solutions with L^p data, for some $p < \infty$. This led R. Hunt to conjecture that if the lateral boundary Σ of a parabolic domain is given by the graph of a $\mathrm{Lip}(1,1/2)$ function, then *caloric* measure should be mutually absolutely continuous with respect to the parabolic surface measure, but this conjecture was ultimately disproven by Kaufman and Wu [KW80]. In [LS88], J. Lewis and his student J. Silver pioneered the search for the "correct" regularity condition on Σ that would allow solvability of the parabolic L^p Dirichlet problem in domains with time-varying rough (lateral) boundary. In particular, the work [LS88] showed that if $\Omega \subseteq \mathbb{R}^2$ was the region above a the graph of a $\mathrm{Lip}(1/2)$ function ψ, then a (stronger) condition, similar to (but slightly stronger than) $D_t^{1/2}\psi \in \mathrm{BMO}$, is sufficient for the mutual absolute continuity of caloric measure and (parabolic) surface measure.

The condition that $D_t^{1/2}\psi \in \mathrm{BMO}$ was seen to be natural, from the standpoint of singular integrals[4], in light of the work of Murray [Mu85]. Motivated by these considerations, Lewis and Murray [LM95] showed that the additional assumption $D_t^{1/2}\psi \in \mathrm{BMO}$, is sufficient for the L^p solvability of the Dirichlet problem for the *heat equation* above the graph of ψ (for some $p > 1$). Led by the interesting papers of Lewis, Murray and Silver, Hofmann began to work on parabolic singular integrals [Ho97], and together with Lewis [HL96] showed that if $D_t^{1/2}\psi$ has sufficiently small BMO norm, then one has L^2 solvability of the Dirichlet, Neumann, and Regularity problems, via the method of layer potentials, for the heat equation in the domain above the graph of ψ. Later works extended these results beyond the graphical setting and/or to the variable coefficient setting [HL01, HL05, DPP17, NS17, BHHLN22b, GH20, GH21], yielding in particular the aforementioned converse of Theorems 5.2 and 5.3.

The work [BHMN25] shows that the results in [LM95] are sharp, that is, if Ω is the region above the graph of a $\mathrm{Lip}(1,1/2)$ function ψ, the solvability of the L^p Dirichlet problem for some $p > 1$ is *equivalent* to $D_t^{1/2}\psi \in \mathrm{BMO}$. Our work here concerns an extension of [BHMN25], using ideas from [HMT17]. In particular, we use an integration by parts scheme, analogous to that of [HMT17] (but significantly more delicate in the parabolic setting), in order to establish an essential ingredient in [BHMN25], namely, a local square function estimate for

$$(|\partial_t G|^2 + |\nabla^2 G|^2)\,\mathrm{dist}(\cdot, \partial\Omega).$$

On the other hand, in contrast to the situation in the constant coefficient case treated in [BHMN25], it is not clear whether the level sets of the Green function locally stratify the region near $\partial\Omega$ graphically, in the sense that the sets $E_\lambda = \{Y : G(X,Y) = \lambda\}$ are 'nice' graphs for small λ, and $\cup_{\lambda\in(0,\lambda_0)} E_\lambda$ is a sufficiently 'fat' strip along the boundary. This local stratification by graphical level sets is a crucial ingredient in [BHMN25], and the way we circumvent this obstacle casts new light on [BHMN25], which may be useful in other settings. In particular, our strategy involves using the estimates on the time and second spatial partial derivatives of G to prove that it is "often" the case the $\partial_{x_0} G \gtrsim 1$ (here the domain is given by $\Omega := \{(x_0, x', t) : x_0 > \psi(x,t)\}$), which, in turn, allows us to construct graphs in certain "sawtooth regions". Then we can refine the estimates in [BHMN25] to prove a certain "corona" decomposition of the graph of ψ, by graphs of *regular* $\mathrm{Lip}(1,1/2)$ functions, that is, Lip(1,1/2) functions that have a half-order time derivative in (parabolic) BMO. In [BHHLN23b], it is shown that such a corona decomposition implies that the graph of ψ is parabolic uniformly rectifiable, which in turn, in the graph setting, is equivalent to $D_t^{1/2}\psi \in \mathrm{BMO}$ (since ψ is $\mathrm{Lip}(1,1/2)$ by hypothesis).

[4] It is also possible to recover Murray's result from the $T1$ theorem [DJ84].

We mention again that it would be interesting to know whether our results hold under the weaker L^2-Carleson oscillation condition. In the elliptic setting this is accomplished in [HMMTZ21], by using both compactness methods and the technique of extrapolation of Carleson measures; however, in the context of [HMMTZ21] the authors merely need to show an exterior corkscrew condition. Indeed, it is known that, in the elliptic setting, if a set satisfies the two-sided corkscrew condition then it must be uniformly rectifiable, but this is not the case in the parabolic setting[5]. Moreover, it is known that weak flatness conditions (e.g. WHSA [HLMN17], BWGL [DS93]) that characterize (elliptic) uniformly rectifiable sets fail to characterize parabolic uniform rectifiability. It would be interesting if a more direct, "constructive" (no compactness) argument, via a square function estimate, could be used to recover the result in [HMMTZ21], even in the elliptic setting, as this might provide a path to prove the results here under the L^2-Carleson oscillation condition. Perhaps the ideas in [FL23] could serve as a first step on that path.

Remark 5.4. To conclude we make a remark for the careful reader, who may notice that in the L^1-Carleson oscillation condition in Definition 5.24, we assume that the matrix function A is locally smooth. In fact, at least in the setting of Theorem 5.2, this is merely qualitative and inessential, as one can assume a weaker oscillation condition and smooth the coefficients, maintaining the A_∞ property for the parabolic measure of the smoothed operator. This is known for the case that A is symmetric [Ny97] and it could be that a straightforward modification of [Ny97] would give the non-symmetric case. These topics are discussed in Section 5.12.

5.3 Main ideas and outline of the chapter

In Section 5.4, we discuss definitions and background needed to understand the rest of the chapter. We will revisit many of the geometric and analytical properties exposed in Chapter 2, but we will emphasize how everything adapts to the parabolic world. We already gave a quick overview of the necessary changes in Section 2.5 (changes in the metric and all its consequences, half-time derivatives...), and we will make the discussion more detailed. We will also introduce some dyadic constructions (like enlarged Whitney regions) that resemble those in Chapter 4.

Similarly, in Section 5.5, we review the necessary PDE background that will be needed throughout the chapter, as sketched already in Section 2. The main difference with respect to the elliptic setting the time-directed nature of all the estimates: the poles of the corresponding Green's functions or parabolic measures must be placed correctly in the past or future, depending on the situation.

Later, in Section 5.6, we explain more deeply two of the main definitions of the chapter. First, the L^p solvability of the Dirichlet problem or, equivalently, the A_∞ property of the parabolic measure: these are notions that were already used in Chapters 2 and 4 for elliptic equations, the only difference in this setting being again the time-directedness of the estimates (again, poles must be correctly situated in the past/future). Second, we define the class of coefficients with which we will be able to work: these are defined via Carleson measures as their elliptic counterparts (see Theorem 4.5). Again, the key difference is the special role of time: we will perceive that by means of the different homogeneities that we expect for spatial or time derivatives. The reader will also note that we assume the coefficients to be smooth: this assumption will be of qualitative nature and will simplify some of the forthcoming arguments (it ensures that, at least qualitatively, solutions are regular, too). Moreover, it turns out that the assumption

[5]Otherwise every $\mathrm{Lip}(1,1/2)$ function would have a half-order time derivative in (parabolic) BMO, which is not true, see [BHHLN22a].

on smoothness can be somehow removed via abstract approximation schemes, as we will explain in Section 5.12.

The remaining part of the chapter is devoted to the proof of our main results (Theorems 5.2 and 5.3). Concretely, the proof will be in Sections 5.7–5.10, and in Sections 5.11 and 5.13 we will prove a couple of auxiliary results that will be useful, too. Let us give a brief overview of the ideas of the proof, because it is fairly long and technical.

The goal in our main theorems is to show that the graph defining the boundary of the domain is regular (see Subsection 5.4.6). Instead of showing this directly, we show that at most scales, there are regular graphs approximating our original graph: if we achieve this in a corona decomposition sense, our original graph will be regular by Lemma 5.9 (see also Proposition 5.39).

We will construct the approximating graphs in Section 5.9 (see (5.81)) using level lines of the Green's function, because these are expected to follow the shape of the boundary. But for that to be true, the Green's function should behave linearly near the boundary of our region of interest. For variable coefficient operators (as opposed to the situation in [BHMN25]), this linear behavior is not always ensured: we will show that it happens at most scales (in a corona decomposition sense). Indeed, via the connection between Green's functions and parabolic measure (CFMS estimate, see Subsection 5.5.2), we will be able to show (see Subsection 5.7.2) that there are many points where the Green's function is comparable to the distance to the boundary. For that to work, we need the parabolic measure to behave correctly, too: we will show that, as a consequence of the A_∞ property (our main hypothesis), it is in fact comparable to the surface measure in ample regions in Subsection 5.7.1 (which should remind the reader of the corona decompositions appearing in Chapter 4).

Later in Subsection 5.7.2, we will show that, at most scales the Green's function grows in the vertical direction, which gives us the conclusion that its level lines are stratified (hence they intuitively follow the shape of the boundary). We will actually be able to show (in Lemma 5.37) that this vertical growth is true in *complete* Whitney regions (not only at a certain point as in Lemma 5.31). For that, we will use the fact that the derivatives of the Green's function often do not oscillate too much: indeed, if they oscillate, they contribute to certain square functions (see (5.72)), but these square functions are overall not too large. The latter fact is shown in Subsection 5.8.1 via an integration by parts argument, that extends the one in [HMT17] to parabolic equations (and includes some relevant novelties when the coefficients are symmetric, inspired by [LN07]). To carry out the integration by parts in our sawtooth regions (unions of Whitney boxes), we will use cut-off functions which are specially designed for this dyadic setting: we construct them carefully in Section 5.11. Moreover, at the beginning of the integration by parts scheme, we will need an auxiliary estimate that we show in Section 5.13.

With the tools developed in Sections 5.7 and 5.8, we finally construct the approximating graphs in Section 5.9, concretely in Subsection 5.9.1: they will be level lines of the Green's function appropriately localized to the scale in which we are working via a stopping-time distance (actually, a regularization of it, which will satisfy better estimates that will prove useful later).

The main advantage (which is the key idea in [BHMN25]) of working with these approximating graphs (level sets of Green's functions) instead of with the original graph is that we may obtain information of the derivatives of the approximating graphs from information about the derivatives of the Green's function (this is the content of Lemma 5.44). As a consequence, we can use all the knowledge about the Green's function that we already have, to prove (in the remaining part of Subsection 5.9.2) that the constructed graphs really approximate our original graph, and that they are Lipschitz (in the parabolic

sense).

Lastly, in Section 5.10 we show that the approximating graphs are regular, which finishes the whole argument. For that, we follow closely [BHMN25], although some changes are needed because we are forced to consider several scales (as opposed [BHMN25]): we introduce a distinction of cases below each top cube Q_0 for that, where Case 1 cubes will be very similar in spirit to Q_0 and hence will be treated as in [BHMN25].

First, we show (in Subsection 5.10.2) that, although we have to evaluate a (non-local) half-order derivative, we may truncate to localize estimates. This automatically takes care of Case 3 cubes, which are very far from our region of interest, whence the localization just makes everything trivial (see Subsection 5.10.3). In turn, Case 2 cubes are so small that the approximating graphs at their scale are very well inside our domain, and hence they are appropriate level sets of Green's functions, and the required estimates follow (see Subsection 5.10.4).

Next, in Subsections 5.10.5, 5.10.6 and 5.10.7, we handle Case 1 cubes (these are the difficult ones) following the method of [BHMN25]. The main idea is to estimate the half-order time derivative via an integration by parts (see (5.118)): this way, the problem of obtaining estimates for the boundary (the 0-level set in $\tilde{\psi}(0;\mathbf{x})$) gets transferred to the interior of the domain (the r-level set for $r > 0$). This is of key importance, because for level sets in the interior of the domain we can prove many useful estimates: that is the content of Subsection 5.10.5, where we again take advantage of the relationship between derivatives of the approximating graphs and derivatives of Green's functions.

For technical reasons, we need to regularize our approximating graphs even more (see (5.114)): we take care of the difference between the approximating graphs and their regularized versions in Subsection 5.10.7 using Littlewood-Paley theory. In this subsection, we also take care of the boundary terms that were introduced along the integration by parts process sketched in last paragraph.

With Cases 1, 2 and 3 already solved, we can take care of Case 4 cubes (see Subsection 5.10.1): these are cubes which lie somehow around the boundary of the top scale $Q(\mathbf{S})$. After a subdivision in smaller cubes, every one of these subcubes will turn out to be in Cases 1, 2, 3, or at least connected to one of those by a chain of subcubes of uniform length. This will allow us to reduce the estimates for Case 4 cubes to those of Cases 1, 2, 3.

5.4 Preliminaries

Although we will try to follow the principles introduced in Chapter 2, the parabolic setting has substantial differences with respect to the elliptic setting which is addressed in the rest of chapters, as we saw in Section 2.5. Therefore, we will need specific notation along this chapter, that we will present next. Later, we will convey a number of auxiliary definitions, constructions and results that have been partially introduced in Section 2.5, but we prefer to state them here again to be able to get to more concrete details that will be used throughout the chapter.

5.4.1 Notation

- Throughout the chapter, $n \geqslant 2$ is a natural number and we let $d := n + 1$ denote the natural parabolic homogeneous dimension of space-time $\mathbb{R}^n$:

$$\mathbb{R}^n = \big\{\mathbf{x} = (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}\big\}.$$

- The ambient space we work in is $\mathbb{R}^{n+1} := \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$,

$$\mathbb{R}^{n+1} = \big\{\mathbf{X} = (X,t) = (x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}\big\}.$$

  Here we have distinguished the last coordinate as the time coordinate and the first spatial coordinate as the graph coordinate.

- To help the reader identify the nature of points used in the chapter, we use the notation employed above, which we here describe in detail. We use lower case letters (e.g. x, y, z) to denote spatial points in $\mathbb{R}^{n-1}$, and capital letters (e.g. $X = (x_0, x)$, $Y = (y_0, y)$, $Z = (z_0, z)$), to denote spatial points in $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$. We also use boldface capital letters (e.g. $\mathbf{X} = (X,t)$, $\mathbf{Y} = (Y,s)$, $\mathbf{Z} = (Z,\tau)$) to denote points in space-time $\mathbb{R}^{n+1}$, and boldface lowercase letters (e.g. $\mathbf{x} = (x,t)$, $\mathbf{y} = (y,s)$, $\mathbf{z} = (z,\tau)$) to denote points in space-time $\mathbb{R}^n$. In accordance with this notation, given $\mathbf{X} = (X,t) \in \mathbb{R}^{n+1}$ (resp. $\mathbf{x} = (x,t) \in \mathbb{R}^n$) we use the notation $t(\mathbf{X})$ (resp. $t(\mathbf{x})$) to denote its time component, that is, $t(\mathbf{X}) = t$ (resp. $t(\mathbf{x}) = t$).

- We also denote 1-dimensional integrals by $\int$, integrals in space-time $\mathbb{R}^n$ by $\iint$, and integrals in space-time $\mathbb{R}^{n+1}$ by $\iiint$. Averages are denoted, in each case, by $⨍$, $⨌$ and $⨍⨍⨍$.

- We denote the parabolic length by

$$\|\mathbf{X}\| = \|(X,t)\| := |X| + |t|^{\frac{1}{2}}, \quad \mathbf{X} = (X,t) \in \mathbb{R}^{n+1},$$
$$\|\mathbf{x}\| = \|(x,t)\| := |x| + |t|^{\frac{1}{2}}, \quad \mathbf{x} = (x,t) \in \mathbb{R}^{n},$$

  and accordingly, all distances will be measured with respect to the parabolic metric

$$\operatorname{dist}(\mathbf{X},\mathbf{Y}) := \|\mathbf{X} - \mathbf{Y}\| := |X - Y| + |t - s|^{\frac{1}{2}}, \qquad \mathbf{X} = (X,t),\ \mathbf{Y} = (Y,s) \in \mathbb{R}^{n+1},$$
$$\operatorname{dist}(\mathbf{x},\mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| := |x - y| + |t - s|^{\frac{1}{2}}, \qquad \mathbf{x} = (x,t),\ \mathbf{y} = (y,s) \in \mathbb{R}^{n}.$$

- It is sometimes convenient to use a different (smooth) parabolic distance. Given $\mathbf{x} = (x,t) \in \mathbb{R}^n \backslash \{0\}$, we let $|||\mathbf{x}|||$ be defined as the unique positive solution of the equation

$$\frac{|x|}{|||\mathbf{x}|||^2} + \frac{t^2}{|||\mathbf{x}|||^4} = 1,$$

  In particular, $|||\mathbf{x}||| \approx \|\mathbf{x}\|$ with implicit constants depending only on n.

- Given $\mathbf{x} = (x,t) \in \mathbb{R}^n$ and $R > 0$, we define the parabolic **cube in** $\mathbb{R}^n$ as

$$Q_R(\mathbf{x}) := \big\{\mathbf{y} = (y,s) \in \mathbb{R}^n : |y_i - x_i| < R,\ 1 \leqslant i \leqslant n-1,\ \ |t - s| < R^2\big\}.$$

- Given $\mathbf{x} = (x,t) \in \mathbb{R}^n$ and $R > 0$, we denote a closed parabolic **cube in** $\mathbb{R}^{n+1}$ by

$$\mathcal{J}_R(\mathbf{X}) = \mathcal{J}_R(x_0, x, t) := \big[x_0 - R, x_0 + R\big] \times \overline{Q_R(\mathbf{x})}\,. \tag{5.3}$$

  For $\mathcal{J} = \mathcal{J}_R(\mathbf{X})$, we let $\ell(\mathcal{J}) := \ell(\mathcal{J}_R(\mathbf{X})) = 2R$ denote the parabolic side length of $\mathcal{J}$.

- We let $\mathbb{D}$ denote the standard grid of parabolic dyadic cubes on $\mathbb{R}^n$, and $\ell(Q)$ denote the side length of $Q \in \mathbb{D}$. Also, for a given $Q_0 \in \mathbb{D}$, we denote $\mathbb{D}(Q_0) := \{Q \in \mathbb{D} : Q \subseteq Q_0\}$.

5.4.2 Parabolic Hausdorff measure

Given $\eta \geqslant 0$, we let $\mathcal{H}^\eta$ denote standard η-dimensional Hausdorff measure (recall Section 2.3.1). We also define a parabolic Hausdorff measure of homogeneous dimension η, denoted $\mathcal{H}^\eta_{\mathrm{p}}$, in the same way that one defines standard Hausdorff measure, but instead using coverings by parabolic cubes. I.e., for $\delta > 0$, and for $E \subseteq \mathbb{R}^{n+1}$, we set

$$\mathcal{H}^\eta_{\mathrm{p},\delta}(E) := \inf \sum_k \operatorname{diam}(E_k)^\eta \,,$$

where the infimum runs over all countable such coverings of E, $\{E_k\}_k$, with $\operatorname{diam}(E_k) \leqslant \delta$ for all k. Of course, the diameter is measured in the parabolic metric. We then define

$$\mathcal{H}^\eta_{\mathrm{p}}(E) := \lim_{\delta \to 0^+} \mathcal{H}^\eta_{\mathrm{p},\delta}(E) \,.$$

As for classical Hausdorff measure, $\mathcal{H}^\eta_{\mathrm{p}}$ is a Borel regular measure. We refer the reader to [EG92, Chapter 2] for a discussion of the basic properties of standard Hausdorff measure, which adapt readily to treat $\mathcal{H}^\eta_{\mathrm{p}}$. In particular, one obtains a measure equivalent to $\mathcal{H}^\eta_{\mathrm{p}}$ if one defines $\mathcal{H}^\eta_{\mathrm{p},\delta}$ in terms of coverings by arbitrary sets of parabolic diameter at most δ, rather than cubes.

We also recall that we explored some particularities of H_{p} in Section 2.5.

5.4.3 Lip(1,1/2) graph domains

A function $\psi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ is Lip(1,1/2) with constant C, if

$$|\psi(x,t) - \psi(y,s)| \leqslant C(|x-y| + |t-s|^{1/2}) = C\|(x,t) - (y,s)\|, \quad \forall (x,t),(y,s) \in \mathbb{R}^n. \tag{5.4}$$

We define $\|\psi\|_{\mathrm{Lip}(1,1/2)}$ to be the infimum of all constants C as in (5.4). If we set

$$\Sigma := \big\{(\psi(x,t),x,t) : (x,t) \in \mathbb{R}^n\big\} = \big\{(\psi(\mathbf{x}),\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\big\} =: \big\{\mathbf{\Psi}(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\big\}, \tag{5.5}$$

then we say that Σ is a Lip(1,1/2) graph. The set Σ is the boundary of the domain

$$\Omega := \big\{\mathbf{X} = (x_0,x,t) \in \mathbb{R}^{n+1} : x_0 > \psi(x,t)\big\}, \tag{5.6}$$

We refer to $\Omega \subseteq \mathbb{R}^{n+1}$ as an (unbounded) Lip$(1,1/2)$ **graph domain** with constant $\|\psi\|_{\mathrm{Lip}(1,1/2)}$. Given the closed set $\Sigma \subseteq \mathbb{R}^{n+1}$ of homogeneous dimension $\dim_{\mathcal{H}_{\mathrm{p}}}(\Sigma) = n+1$, we define a surface measure on Σ as the restriction of $\mathcal{H}^{n+1}_{\mathrm{p}}$ to Σ, i.e.,

$$\sigma := \sigma_\Sigma := \mathcal{H}^{n+1}_{\mathrm{p}}|_\Sigma \,. \tag{5.7}$$

We remark that for Lip(1,1/2) graphs, σ as defined in (5.7) is equivalent to $d\sigma^{\mathbf{s}} := d\sigma_t\, dt$, where $d\sigma_t$ is standard $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$, restricted to the cross section $\Sigma_t := \{x : (x,t) \in \Sigma\}$. We refer the reader to [BHHLN23b, Remark 2.8 and Appendix B] for details.

5.4.4 Surface cubes and reference points

We let

$$M_0 := 2 + \|\psi\|_{\mathrm{Lip}(1,1/2)}. \tag{5.8}$$

For every $\mathbf{X} = (x_0, x, t) \in \mathbb{R}^{n+1}$ and $R > 0$, we introduce **vertically elongated open "cubes"**

$$I_R(\mathbf{X}) := (x_0 - 3M_0\sqrt{n}R, x_0 + 3M_0\sqrt{n}R) \times Q_R(\mathbf{x})$$

and set

$$\Delta_R(\mathbf{X}) := I_R(\mathbf{X}) \cap \Sigma.$$

We will refer to $\Delta_R(\mathbf{X})$ as a surface box or cube of size $R > 0$ and centered at $\mathbf{X}$. Unless otherwise specified, we implicitly assume that the center $\mathbf{X} = (x_0, \mathbf{x}) = (x_0, x, t)$, of any surface box $\Delta_R(\mathbf{X})$, is in Σ, that is, $x_0 = \psi(x,t)$.

Note the crude estimate

$$\mathbf{Y} \in I_R(\mathbf{X}) \implies \|\mathbf{Y} - \mathbf{X}\| \leqslant 5M_0\sqrt{n}R,$$

and that by construction,

$$\Delta_R(\mathbf{X}) = \mathbf{\Psi}\big(Q_R(\mathbf{x})\big) = \big\{\mathbf{\Psi}(\mathbf{y}) : \mathbf{y} \in Q_R(\mathbf{x})\big\}, \qquad \forall\, \mathbf{X} = (x_0, \mathbf{x}) \in \Sigma, \tag{5.9}$$

where we recall that $\mathbf{y} \mapsto \mathbf{\Psi}(\mathbf{y}) := (\psi(\mathbf{y}), \mathbf{y})$ is the graph parametrization of Σ (see (5.5)). Indeed, if $\mathbf{y} \in Q_R(\mathbf{x})$, then $\|\mathbf{x} - \mathbf{y}\| \leqslant (\sqrt{n}R + R)$ and hence

$$|\psi(\mathbf{x}) - \psi(\mathbf{y})| \leqslant M_0(\sqrt{n}R + R) \leqslant 2\sqrt{n}M_0R.$$

We also note, by the same reasoning, that if $R > 0$, $\mathbf{X} = (x_0, \mathbf{x}) \in \Sigma$ and $\mathbf{y} \in Q_R(\mathbf{x})$, then

$$\psi(\mathbf{y}) + s \in I_R(\mathbf{X}) \quad \forall s \in (-M_0\sqrt{n}R, M_0\sqrt{n}R).$$

Given $\tau > 0$, we define the parabolic dilation $\tau\Delta_R(\mathbf{X}) := \Delta_{\tau R}(\mathbf{X})$.

We introduce time forward and time backwards **corkscrew points** relative to $\Delta_R(\mathbf{X})$,

$$\mathcal{A}_R^{\pm}(\mathbf{X}) := \left(x_0 + 2M_0R, x, t \pm 2R^2\right), \quad \mathbf{X} = (x_0, x, t) = (x_0, \mathbf{x}) = (\psi(\mathbf{x}), \mathbf{x}) \in \Sigma. \tag{5.10}$$

Note that for $\mathbf{X} \in \Sigma$,

$$\mathcal{A}_R^{\pm}(\mathbf{X}) \in \Omega_{2R}(\mathbf{X}) := I_{2R}(\mathbf{X}) \cap \Omega,$$

and that

$$\mathrm{dist}\left(\mathcal{A}_R^{\pm}(\mathbf{X}), \partial\Omega_{2R}\right) \approx \mathrm{dist}\left(\mathcal{A}_R^{\pm}(\mathbf{X}), \partial\Omega_{3R}\right) \approx R,$$

where the implicit constants depend only on n and $\|\psi\|_{\mathrm{Lip}(1,1/2)}$. Furthermore,

$$|\psi(x_0, x, t \pm 2R^2) - \psi(x_0, x, t)| \leqslant \sqrt{2}\|\psi\|_{\mathrm{Lip}(1,1/2)}R,$$

and hence

$$\mathrm{dist}(\mathcal{A}_R^{\pm}(\mathbf{X}), \partial\Omega) \geqslant 2M_0R - \sqrt{2}\|\psi\|_{\mathrm{Lip}(1,1/2)}R \geqslant 2R.$$

We frequently use (sometimes without mention) the following fact: the distance to the Lipschitz graph is, up to constants, attained in the vertical direction (in which we parametrize the graph).

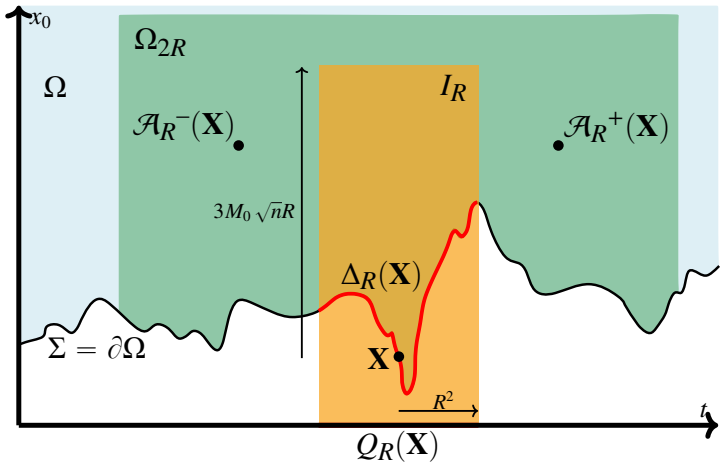


Figure 5.1: A (not-to-scale) depiction of elongated cubes and corkscrews associated to a surface ball $\Delta_R(\mathbf{X}) = \mathbf{\Psi}(Q_R(\mathbf{X}))$. We always think that time flows from left to right, and x_0 is the vertical direction. One should imagine the other $n-1$ spatial directions as being perpendicular to both x_0 and t, but we will not reflect those in our 2D pictures for simplicity.

Lemma 5.5. *Assume that* $\mathbf{X} = (x_0, \mathbf{x}) \in \Omega$. *Then*

$$M_0^{-1}(x_0 - \psi(\mathbf{x})) \leqslant \operatorname{dist}(\mathbf{X}, \Sigma) \leqslant x_0 - \psi(\mathbf{x}).$$

Proof. Let $\mathcal{L} := \|\psi\|_{\mathrm{Lip}(1,1/2)}$. Find $\mathbf{Y} = (\psi(\mathbf{y}), \mathbf{y}) \in \Sigma$ be such that $\operatorname{dist}(\mathbf{X}, \Sigma) = \|\mathbf{Y} - \mathbf{X}\|$. Then

$$\|(\psi(\mathbf{x}), \mathbf{x}) - \mathbf{Y}\| = |\psi(\mathbf{x}) - \psi(\mathbf{y})| + \|\mathbf{x} - \mathbf{y}\| \leqslant (\mathcal{L}+1)\|\mathbf{x} - \mathbf{y}\| \leqslant (\mathcal{L}+1)\|\mathbf{X} - \mathbf{Y}\|.$$

Therefore, using the (reverse) triangle inequality we obtain

$$\|\mathbf{X} - \mathbf{Y}\| \geqslant \|\mathbf{X} - (\psi(\mathbf{x}), \mathbf{x})\| - \|(\psi(\mathbf{x}), \mathbf{x}) - \mathbf{Y}\| \geqslant (x_0 - \psi(\mathbf{x})) - (\mathcal{L}+1)\|\mathbf{X} - \mathbf{Y}\|,$$

which yields the desired result after rearranging the terms.

□

5.4.5 (Parabolic) BMO and fractional integral operators

Given a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, we say that $f \in \mathrm{BMO}(\mathbb{R}^n)$, the **parabolic BMO-space**, if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{Q \subseteq \mathbb{R}^n} \fint\!\!\!\fint_Q |f(x,t) - f_Q| \, dx\, dt < \infty,$$

where the supremum runs over all parabolic cubes $Q = Q_R(\mathbf{y})$, with $\mathbf{y} \in \mathbb{R}^n$ and $R > 0$. Here f_Q denotes the average of f on Q.

We introduce the **fractional integral operator** of parabolic order 1 on $\mathbb{R}^n$, via Fourier transform,

$$\widehat{(\mathrm{I}_{\mathrm{P}}\psi)}(\xi,\tau) := |||(\xi,\tau)|||^{-1} \, \widehat{\psi}(\xi,\tau), \qquad (\xi,\tau) \in \mathbb{R}^n.$$

Then, we can represent it by a kernel V as

$$\mathrm{I}_{\mathrm{P}}\psi(\mathbf{x}) = \iint_{\mathbb{R}^n} V(\mathbf{x}-\mathbf{y})\,\psi(\mathbf{y})\,d\mathbf{y}, \qquad \mathbf{x}\in\mathbb{R}^n, \tag{5.11}$$

where, recalling that $n+1$ is the homogeneous dimension of parabolic $\mathbb{R}^n$, it additionally holds

$$0 \leqslant V(\mathbf{y}) \lesssim \|\mathbf{y}\|^{-n}. \tag{5.12}$$

Using I_{P}, following [FR22], we introduce a (parabolic) **half-order time derivative** as

$$\mathcal{D}_t\psi(\mathbf{x}) := \partial_t \circ \mathrm{I}_{\mathrm{P}}\psi(\mathbf{x}) = \iint_{\mathbb{R}^n} \partial_t\big(V(\mathbf{x}-\mathbf{y})\big)\,\psi(\mathbf{y})\,d\mathbf{y}, \qquad \mathbf{x}=(x,t)\in\mathbb{R}^n.$$

This operator should be viewed as a principal value operator, or one should consider ∂_t in the weak sense. Note that the Fourier symbol for $\mathcal{D}_t$ is $2\pi i\tau/|||(\xi,\tau)|||$.

Another half-order time derivative, $D^t_{1/2}$, can be introduced via the multiplier $|\tau|^{1/2}$, or by

$$D^t_{1/2}\psi(\mathbf{x}) = D^t_{1/2}\psi(x,t) := c\int_{\mathbb{R}} \frac{\psi(x,s)-\psi(x,t)}{|s-t|^{3/2}}\,ds,$$

for properly chosen c.

5.4.6 Regular Lip(1,1/2) graph domains

Let $\psi : \mathbb{R}^{n-1}\times\mathbb{R}\to\mathbb{R}$ be a $\mathrm{Lip}(1,1/2)$ function (see (5.4)). Such a function is said to be a **regular Lip(1,1/2)** function if, in addition,

$$\mathcal{D}_t\psi \in \mathrm{BMO}(\mathbb{R}^n).$$

If ψ is a regular $\mathrm{Lip}(1,1/2)$ function, then we say that Σ as in (5.5) is a *regular* $\mathrm{Lip}(1,1/2)$ graph, and that $\Omega\subseteq\mathbb{R}^{n+1}$ as in (5.6) is an (unbounded) regular $\mathrm{Lip}(1,1/2)$ graph domain. In both cases the regularity is determined by $\|\psi\|_{\mathrm{Lip}(1,1/2)}$ and $\|\mathcal{D}_t\psi\|_{\mathrm{BMO}(\mathbb{R}^n)}$. In [HL96] it is proved that

$$\|\psi\|_{\text{R-Lip}} := \|\mathcal{D}_t\psi\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|\nabla_x\psi\|_\infty \approx \|D^t_{1/2}\psi\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|\nabla_x\psi\|_\infty\,, \tag{5.13}$$

where $\mathcal{D}_t, D^t_{1/2}$ were defined in subsection 5.4.5. In particular, that a function is a regular $\mathrm{Lip}(1,1/2)$ function can be equivalently formulated using $D^t_{1/2}$ instead of $\mathcal{D}_t$, but the latter will be considerably more convenient for us to work with in this chapter.

Remark 5.6. One can prove that, in general, the class of regular Lip(1,1/2) functions is strictly contained in Lip(1,1/2), i.e., there are examples of functions ψ which are Lip(1,1/2) but not regular Lip(1,1/2), see [LS88], [KW80]. Moreover, it follows from the arguments of Strichartz [St80] that for a regular Lip(1,1/2) function ψ, the assumption that ψ is Lip(1/2) in the time variable is redundant: it follows from the finiteness of the R-Lip norm in (5.13).

Remark 5.7. One can prove, in the context of Lip(1,1/2) graphs, that the graph being regular Lip(1,1/2) is equivalent to the graph being parabolic uniformly rectifiable. The notion of parabolic uniform rectifiability was introduced in [HLN03, HLN04][6], in connection with the study of parabolic versions of the free boundary results of Kenig and Toro [KT97, KT99, KT03][7]. The results in [HLN04, En17] are in some sense "small constant" versions of the results presented here; we refer the interested reader to the introduction of [BHMN25] for a discussion of the latter point, and to [HLN03, HLN04] (or for that matter to [BHHLN23a, BHHLN23b]) for the precise definition of parabolic uniform rectifiability. For our purposes, we simply remark that for the graph of a Lip(1,1/2) function ψ, the definition of parabolic uniform rectifiability reduces to (5.15) below, with β defined as in (5.14), and that for such ψ and β, (5.15) is equivalent to the *regular* Lip(1,1/2) condition (see [HLN04] for details). To be precise, set

$$\beta(r,x,t) := \inf_L \left[\iint_{Q_r(x,t)} \left(\frac{\psi(y,s) - L(y)}{r} \right)^2 d\sigma(y,s) \right]^{1/2}, \quad (r,x,t) \in \mathbb{R}^{n+1}_+, \tag{5.14}$$

where the infimum is taken over all affine functions L of y only, and let

$$d\nu := d\nu_\psi := \beta^2(r,x,t) \frac{dr\, dx\, dt}{r}.$$

If $\psi(x,t)$ is Lipschitz in the space variable x, uniformly in t, then the condition that $\mathcal{D}_t\psi \in \mathrm{BMO}(\mathbb{R}^n)$ is equivalent to saying that $d\nu$ is a Carleson measure on $\mathbb{R}^{n+1}_+$, i.e., there exists a finite constant $\|\nu\|$ such that for all $(z,\tau) \in \mathbb{R}^n$ and for all $R > 0$, we have the uniform bound

$$\int_0^R \iint_{Q_R(z,\tau)} \beta^2(r,x,t) \frac{dr\, dx\, dt}{r} \leqslant \|\nu\| < \infty. \tag{5.15}$$

5.4.7 Dyadic cubes and a criterion for a Lip(1,1/2) graph to be regular

Recall that $\mathbb{D}$ denotes the standard grid of parabolic dyadic cubes on $\mathbb{R}^n$, and $\ell(Q)$ denotes the side length of any $Q \in \mathbb{D}$.

Definition 5.8 ((Semi-)coherent stopping times)**.** A collection of dyadic cubes $\mathbf{S} \subseteq \mathbb{D}$ is called **semi-coherent stopping time** if there exists a maximal cube $Q(\mathbf{S})$ (with respect to containment) in $\mathbf{S}$ and for any $Q', Q, Q^* \in \mathbb{D}$ we have

$$\mathbf{S} \ni Q' \subseteq Q \subseteq Q^* \in \mathbf{S} \quad \Longrightarrow \quad Q \in \mathbf{S}.$$

A stopping time regime is called **coherent** if it is semi-coherent and whenever $Q \in \mathbf{S}$, then either all or none of its children are in $\mathbf{S}$.

We are going to prove Theorem 5.2 (and Theorem 5.3) using the following "Corona decomposition" characterization of regular Lip$(1,1/2)$ functions. In fact, it is a characterization, more generally, of parabolic uniformly rectifiable (PUR) sets, but as noted above, for a Lip(1,1/2) graph, PUR is equivalent to regularity. The use of the Corona decomposition is a novelty of the present work, as compared to [BHMN25], and as noted in the introduction, we use this approach in order to overcome the fact that

[6] based of course on the work of David and Semmes in the elliptic setting [DS91, DS93]

[7] See also [En17], for an improved version of the result in [HLN04].

in the case of non-constant coefficients, we are able to obtain only a weaker version of stratification by the level sets of the Green function, namely, we obtain an appropriate stratification only in a Corona sense, rather than in a "big pieces" sense as in [BHMN25].

Lemma 5.9 ([BHHLN23b, Theorem 1.2]). *Suppose that ψ is a* $\mathrm{Lip}(1,1/2)$ *function, with associated graph Σ. Then ψ is a regular* $\mathrm{Lip}(1,1/2)$ *function if and only if the following holds. There exist constants $C_1, b_1, b_2 > 1$ such that we can form the disjoint decomposition $\mathbb{D} = \mathcal{G} \sqcup \mathcal{B}$ so that:*

- *$\mathcal{G}$ is further divided into semi-coherent stopping time regimes $\{\mathbf{S}\}$,*
- *the bad cubes $\mathcal{B}$ and the maximal cubes $\{Q(\mathbf{S})\}$ pack, that is,*

$$\sum_{Q' \in \mathcal{B} \cap \mathbb{D}(Q)} |Q'| + \sum_{\mathbf{S}: Q(\mathbf{S}) \subseteq Q} |Q(\mathbf{S})| \leqslant C(C_1)|Q|, \quad \forall Q \in \mathbb{D},$$

- *for each $\mathbf{S}$, there exists a **regular** $\mathrm{Lip}(1,1/2)$ graph $\Sigma_{\mathbf{S}}$ such that*

$$\sup_{\mathbf{X} \in \mathbf{\Psi}(2Q)} \operatorname{dist}(\mathbf{X}, \Sigma_{\mathbf{S}}) \leqslant C_1 \operatorname{diam}(Q), \quad \forall Q \in \mathbf{S}.$$

For future reference, we state the following lemma, which we intend to employ later: it will allow us to work below the scale of a given cube in $\mathbb{D}$, since the passage to the whole $\mathbb{D}$ is of very abstract nature.

Lemma 5.10 ([DS91, Section 7]). *There exists a collection of disjoint cubes $C \subseteq \mathbb{D}$ such that*

$$\Sigma = \bigcup_{Q' \in C} \mathbf{\Psi}(Q'),$$

and such that the collection

$$\widetilde{\mathcal{B}} := \{Q \in \mathbb{D} : Q \nsubseteq Q', \forall Q' \in C\}$$

satisfies the packing condition

$$\sum_{Q \in \widetilde{\mathcal{B}}: Q \subseteq R} \sigma(\mathbf{\Psi}(Q)) \leqslant C\sigma(\mathbf{\Psi}(R)), \quad \forall R \in \mathbb{D}.$$

Here the constant C depends only on n and $\|\psi\|_{\mathrm{Lip}(1,1/2)}$.

For a proof of the lemma, we refer the reader to the discussion in [DS91, Section 7, p. 38]; note that the same argument works in the parabolic setting treated here.

5.4.8 Whitney regions and sawtooths

For any dyadic cube $Q \in \mathbb{D}$ and $K > 4$, we define the **Whitney regions** $U_Q(K)$ as

$$U_Q(K) := \{(x_0, x, t) \in \mathbb{R}^{n+1} : \ (x,t) \in 10^5 Q, \ \ x_0 - \psi(x,t) \in (K^{-1}\ell(Q), K\ell(Q))\}. \tag{5.16}$$

By construction, for any $K > 4$, $\Omega = \cup_{Q \in \mathbb{D}} U_Q(K)$.

We also define a few fattened versions of these:

$$U_Q^*(K) := \{(x_0, x, t) \in \mathbb{R}^{n+1} : \ (x,t) \in 10^6 Q, \ \ x_0 - \psi(x,t) \in ((2K)^{-1}\ell(Q), 2K\ell(Q))\},$$

$$U_Q^{**}(K) := \big\{(x_0,x,t)\in\mathbb{R}^{n+1}:\ (x,t)\in 10^7 Q,\ \ x_0-\psi(x,t)\in((4K)^{-1}\ell(Q),4K\ell(Q))\big\},$$

and

$$U_Q^{***}(K) := \bigcup_{Q'\in\mathfrak{U}_Q} U_{Q'}^{**},\quad \text{where}\quad \mathfrak{U}_Q := \big\{Q'\in\mathbb{D}:\ U_{Q'}^{**}\cap U_Q^{**}\neq\varnothing\big\}.$$

Note that it holds that (see Lemma 5.5)

$$\operatorname{dist}(\mathbf{X},\partial\Omega)\approx_K \ell(Q)\approx_K \operatorname{dist}(\mathbf{X},\mathbf{\Psi}(Q)),\qquad \forall \mathbf{X}\in U_Q^{***}(K)$$

(and by extension any $\mathbf{X}$ in $U_Q(K)$, $U_Q^*(K)$ or $U_Q^{**}(K)$).

Moreover, it is easy to show that, for any $Q\in\mathbb{D}$, it holds (the constants are in no way optimal)

$$U_Q^{**}\subseteq Q_{10^{10}KM_0\operatorname{diam}(Q)}(\psi(\mathbf{x}_Q),\mathbf{x}_Q),\quad \text{and}\quad U_Q^{***}\subseteq Q_{10^{13}K^3M_0\operatorname{diam}(Q)}(\psi(\mathbf{x}_Q),\mathbf{x}_Q). \tag{5.17}$$

Definition 5.11 (Sawtooth regions)**.** Given a collection[8] of cubes $\mathbf{S}\subseteq\mathbb{D}$ we define the following (dyadic) **"sawtooth regions"**:

$$\Omega_{\mathbf{S}} := \Omega_{\mathbf{S}}(K) = \bigcup_{Q\in\mathbf{S}} U_Q(K),\qquad \Omega_{\mathbf{S}}^* := \Omega_{\mathbf{S}}^*(K) = \bigcup_{Q\in\mathbf{S}} U_Q^*(K),$$

$$\Omega_{\mathbf{S}}^{**} := \Omega_{\mathbf{S}}^{**}(K) = \bigcup_{Q\in\mathbf{S}} U_Q^{**}(K),\qquad \Omega_{\mathbf{S}}^{***} := \Omega_{\mathbf{S}}^{***}(K) = \bigcup_{Q\in\mathbf{S}} U_Q^{***}(K).$$

See a (not-to-scale) depiction of the constructions in Figure 5.2.

5.4.9 Convention concerning constants

We refer to n, Λ, $\|\psi\|_{\mathrm{Lip}(1,1/2)}$ (see (5.4)), and the constants C_* and $q>1$ appearing in Definition 5.21 below, as the *structural constants*. For all constants $A,B>0$, the notation $A\lesssim B$ means, unless otherwise stated, that A/B is bounded from above by a positive constant depending at most on the structural constants; $A\gtrsim B$ of course means $B\lesssim A$. We write $A\approx B$ if $A\lesssim B$ and $B\lesssim A$, while for a given constant η, $A\lesssim_\eta B$ means, unless otherwise stated, that the implicit constant depends at most on the structural constants and η.

5.5 Boundary estimates in parabolic Lipschitz domains

In the sequel, we will assume that A is an elliptic $n\times n$ matrix (see (5.2)) and Ω is the region above a $\mathrm{Lip}(1,1/2)$ graph as in (5.6), with boundary Σ.

The Dirichlet problem, parabolic measure, and the boundary behavior of non-negative solutions, for the heat equation but also for more general linear uniformly parabolic equations with space and time dependent coefficients, have been studied intensively in Lipschitz cylinders and in Lip(1,1/2) domains over the years, see, e.g., [FGS84, FGS86, FS97, FSY99, LM95, Ny97]. Apart from the discussion in Section 2.5. Results include Carleson type estimates, the relation between the associated parabolic measure and the Green function, the backward in time Harnack inequality, the doubling property of

[8] Here we do not insist on $\mathbf{S}$ being a stopping time regime. We will use these objects with $\mathbf{S}$ a doubly truncated stopping time regime, which will destroy the property that $\mathbf{S}$ has a unique maximal cube.

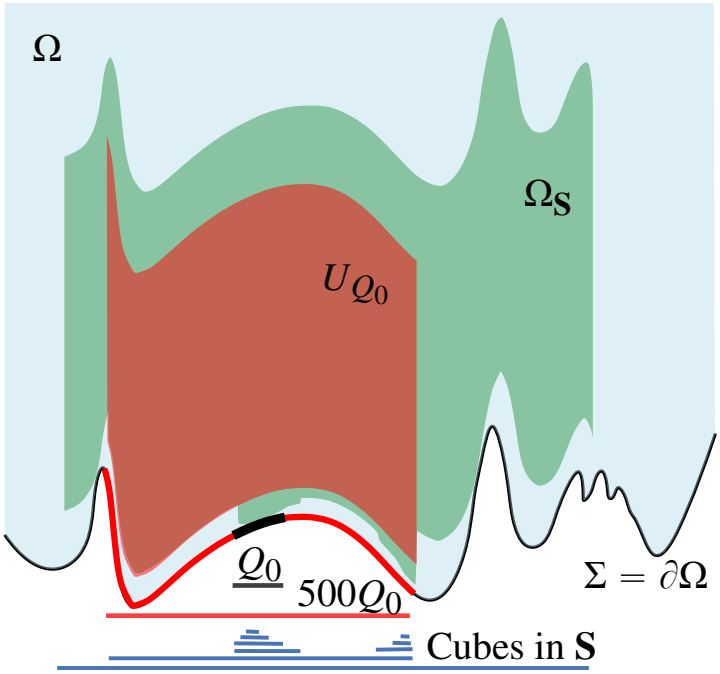


Figure 5.2: A sketch of how a sawtooth $\Omega_{\mathbf{S}}$ is comprised of several (fattened) Whitney regions U_{Q_0}. If the cubes in $\mathbf{S}$ get smaller, the sawtooth region $\Omega_{\mathbf{S}}$ gets closer to the boundary.

parabolic measure, boundary Harnack principles (local and global) and Hölder continuity up to the boundary of quotients of non-negative solutions vanishing on the lateral boundary. All estimates stated in this section are known, and also apply for solutions to the adjoint equation subject to the appropriate changes (typically just exchanging $\mathcal{A}_R^+$ with $\mathcal{A}_R^-$) induced by the change of variables $t \mapsto -t$.

5.5.1 Green's functions and parabolic measure

We refer the reader to [Ny97] for the results stated in this subsection and the next (Subsections 5.5.1 and 5.5.2), in the setting of a Lip(1,1/2) domain, or also [DK87, GH20] in very general frameworks. In particular, for the parabolic equations that we consider in this chapter, the Dirichlet problem with bounded and continuous data has a unique solution in Ω. Given $\mathbf{Y} \in \Omega$ we let $G(\cdot) = G(\cdot, \mathbf{Y})$ denote **Green's function** for L in Ω with pole at $\mathbf{Y}$, i.e.

$$(\partial_t - \operatorname{div}_X A\nabla_X)\, G(\mathbf{X}, \mathbf{Y}) = \delta_{\mathbf{Y}}(\mathbf{X}) \quad \text{for } \mathbf{X} \in \Omega, \qquad \text{and} \qquad G(\cdot, \mathbf{Y}) \equiv 0 \quad \text{on } \Sigma.$$

Moreover, we note that $G(\mathbf{Y}, \cdot)$ is the Green's function for the adjoint equation with pole at $\mathbf{Y}$, i.e.,

$$(-\partial_t - \operatorname{div}_X A^*\nabla_X)\, G(\mathbf{Y}, \mathbf{X}) = \delta_{\mathbf{Y}}(\mathbf{X}) \quad \text{for } \mathbf{X} \in \Omega, \qquad \text{and} \qquad G(\mathbf{Y}, \cdot) \equiv 0 \quad \text{on } \Sigma.$$

We let $\omega^{\mathbf{Y}}(\cdot)$ and $\widetilde{\omega}^{\mathbf{Y}}(\cdot)$ be the **parabolic and adjoint parabolic measures**, with pole at $\mathbf{Y} \in \Omega$, associated to the equation for L and the adjoint one for L^*, in Ω. Then, given $\mathbf{Y} \in \Omega$, if we extend our Green's function G by 0 outside Ω, the following identities hold for every $\phi \in C_0^\infty(\mathbb{R}^{n+1} \setminus \{\mathbf{Y}\})$:

$$\iiint (A^*\nabla_X G(\mathbf{Y}, \mathbf{X}) \cdot \nabla\phi - G(\mathbf{Y}, \mathbf{X})\, \partial_t\phi)\, d\mathbf{X} = \iint \phi\, d\omega^{\mathbf{Y}},$$

$$\iiint \left(A\nabla_X G(\mathbf{Y},\mathbf{X})\cdot\nabla\phi + G(\mathbf{Y},\mathbf{X})\,\partial_t\phi\right)\,d\mathbf{X} = \iint \phi\,d\widetilde{\omega}^{\mathbf{Y}}.$$

5.5.2 PDE estimates

As in the preceding subsection, we refer to [Ny97] for the results stated in Lemmas 5.12, 5.13, 5.16, 5.17, and 5.18 below (we also mention [FGS84, FGS86, FS97, FSY99, LM95] where one may find analogous results specifically for the heat equation, or for variable coefficient parabolic equations in Lipschitz cylinders). In particular, the proofs of the backwards Harnack inequality 5.16 and the doubling property of parabolic measure 5.18 may be found in [FS97, FSY99]; in [Ny97], the author simply observes that the proofs in [FS97, FSY99] may be carried over to the Lip(1,1/2) setting, *mutatis mutandis*.

Note that all implicit constants in this subsection depend only on n, Λ and $\|\psi\|_{\mathrm{Lip}(1,1/2)}$.

Lemma 5.12 (Boundary Hölder continuity)**.** *Let $\mathbf{X}\in\Sigma$ and $R>0$. Assume that $0\leqslant u\in C(\overline{I_{2R}(\mathbf{X})\cap\Omega})$ satisfies $Lu=0$ in $I_{2R}(\mathbf{X})\cap\Omega$, with $u=0$ in $\Delta_{2R}(\mathbf{X})$. Then there exists $\alpha\in(0,1/2)$ such that*

$$u(\mathbf{Y}) \lesssim \left(\frac{\mathrm{dist}(\mathbf{Y},\Sigma)}{R}\right)^{\alpha} \sup_{\mathbf{Z}\in I_{2R}(\mathbf{X})\cap\Omega} u(\mathbf{Z}), \qquad \mathbf{Y}\in I_R(\mathbf{X})\cap\Omega.$$

Lemma 5.13 (Carleson's estimate)**.** *Let $\mathbf{X}$, R, u, and α be as in the statement of Lemma 5.12. Then*

$$u(\mathbf{Y}) \lesssim u(\mathcal{A}_R^+(\mathbf{X})), \qquad \mathbf{Y}\in I_R(\mathbf{X})\cap\Omega,$$

where $\mathcal{A}_R^+(\mathbf{X})$, is the time-forward corkscrew point, defined in (5.10), relative to the "surface ball" $\Delta_R(\mathbf{X})$. In particular,

$$u(\mathbf{Y}) \lesssim \left(\frac{\mathrm{dist}(\mathbf{Y},\Sigma)}{R}\right)^{\alpha} u(\mathcal{A}_R^+(\mathbf{X})), \qquad \mathbf{Y}\in I_{R/2}(\mathbf{X})\cap\Omega. \tag{5.18}$$

Corollary 5.14 (Bourgain's estimate)**.** *There is a uniform constant $c>0$ such that for all $\mathbf{X}=(\psi(\mathbf{x}),\mathbf{x})\in\Sigma$, and every $R>0$,*

$$\omega^{\mathcal{A}_R^+(\mathbf{X})}\big(\Delta_R(\mathbf{X})\big)\geqslant c\,.$$

Sketch of proof. First apply Lemma 5.12 to the solution $u(\mathbf{Y}):=1-\omega^{\mathbf{Y}}\big(\Delta_R(\mathbf{X})\big)$, with $\mathbf{Y}$ sufficiently close to the boundary, and then use Harnack's inequality to connect to $\mathcal{A}_R^+(\mathbf{X})$. We omit the standard details. □

Remark 5.15. It is well-known that, in fact, the conclusion of Corollary 5.14 is actually equivalent to that of Lemma 5.12; see, e.g., [GH20, Lemma 2.5] for a proof of the converse implication in a much more general setting.

For $\mathbf{X}=(x_0,x,t)\in\Sigma$, $r>0$, and $\kappa=\kappa(n,M_0)$, a sufficiently large constant to be fixed, we introduce the **space-time parabolas** by

$$\mathcal{P}^{\pm}_{\kappa,r}(\mathbf{X}) := \big\{(y_0,y,s)\in\Omega:\ \ |(x_0,x)-(y_0,y)|\leqslant\kappa\,|t-s|^{1/2},\ \ \pm(s-t)\geqslant 16r^2\big\}. \tag{5.19}$$

Recall that $M_0=2+\|\psi\|_{\mathrm{Lip}(1,1/2)}$ (see (5.8)). The parabola $\mathcal{P}^+_{\kappa,r}$ is the forward in time parabola and $\mathcal{P}^-_{\kappa,r}$ is the backward in time parabola. Note that if $\mathbf{Y}\in\mathcal{P}^{\pm}_{\kappa,r}(\mathbf{X})$, then $\mathbf{Y}\in\mathcal{P}^{\pm}_{\kappa,r'}(\mathbf{X})$ for all $r'\in(0,r)$. Concerning the "aperture" κ, we may take this constant as large as we like, but we will choose

$$\kappa := 40M_0\sqrt{n}. \tag{5.20}$$

Lemma 5.16 (Backwards/Strong Harnack inequality)**.** *If* $\mathbf{X} \in \Sigma$ *and* $\mathbf{Y} \in \mathcal{P}^+_{\kappa,R}(\mathbf{X})$*, then*

$$G(\mathbf{Y}, \mathcal{A}^-_R(\mathbf{X})) \approx G(\mathbf{Y}, \mathcal{A}^+_R(\mathbf{X})).$$

Similarly, if $\mathbf{X} \in \Sigma$ *and* $\mathbf{Y} \in \mathcal{P}^-_{\kappa,R}(\mathbf{X})$*, then*

$$G(\mathcal{A}^+_R(\mathbf{X}), \mathbf{Y}) \approx G(\mathcal{A}^-_R(\mathbf{X}), \mathbf{Y}).$$

Lemma 5.17 ("CFMS estimate")**.** *If* $\mathbf{X} \in \Sigma$ *and* $\mathbf{Y} \in \mathcal{P}^+_{\kappa,R}(\mathbf{X})$*, then*

$$R^n\, G(\mathbf{Y}, \mathcal{A}^+_R(\mathbf{X})) \approx \omega^{\mathbf{Y}}(\Delta_R(\mathbf{X})) \approx R^n\, G(\mathbf{Y}, \mathcal{A}^-_R(\mathbf{X})).$$

Similarly, if $\mathbf{X} \in \Sigma$ *and* $\mathbf{Y} \in \mathcal{P}^-_{\kappa,R}(\mathbf{X})$*, then*

$$R^n\, G(\mathcal{A}^-_R(\mathbf{X}), \mathbf{Y}) \approx \widetilde{\omega}^{\mathbf{Y}}(\Delta_R(\mathbf{X})) \approx R^n\, G(\mathcal{A}^+_R(\mathbf{X}), \mathbf{Y}).$$

Lemma 5.18 (Doubling property)**.** *If* $\mathbf{X} \in \Sigma$ *and* $\mathbf{Y} \in \mathcal{P}^+_{\kappa,R}(\mathbf{X})$*, then*

$$\omega^{\mathbf{Y}}(\Delta_R(\mathbf{X})) \approx \omega^{\mathbf{Y}}(\Delta_{R/2}(\mathbf{X})).$$

Similarly, if $\mathbf{X} \in \Sigma$ *and* $\mathbf{Y} \in \mathcal{P}^-_{\kappa,R}(\mathbf{X})$*, then*

$$\widetilde{\omega}^{\mathbf{Y}}(\Delta_R(\mathbf{X})) \approx \widetilde{\omega}^{\mathbf{Y}}(\Delta_{R/2}(\mathbf{X})).$$

Actually, we will use the following corollary of Lemma 5.18.

Lemma 5.19 ([BHMN25, Lemma 3.12])**.** *Let* κ *be as in* (5.20) *and consider* $\mathbf{X} \in \Sigma$ *and* $r > 0$*. Then,*

$$\mathcal{A}^+_{4r}(\mathbf{X}) \in \mathcal{P}^+_{\kappa,2\rho}(\mathbf{Z})$$

for all $\mathbf{Z} \in \Sigma$ *and* $\rho > 0$ *such that* $\Delta_{2\rho}(\mathbf{Z}) \subseteq \Delta_r(\mathbf{X})$*. In particular, for such* $\mathbf{Z}$ *and* ρ*, it holds*

$$\omega^{\mathcal{A}^+_{4r}(\mathbf{X})}(\Delta_{2\rho}(\mathbf{Z})) \approx \omega^{\mathcal{A}^+_{4r}(\mathbf{X})}(\Delta_{\rho}(\mathbf{Z})).$$

5.6 The assumptions in our main results

Let us first define what we mean by L^p solvability of the Dirichlet problem, the assumption on our main theorems. Actually, we are not going to use it directly, so we refer the interested reader to [GH20] for further details.

Definition 5.20 (L^p solvability of the Dirichlet problem)**.** Let $\Omega \subseteq \mathbb{R}^{n+1}$ be the region above a $\mathrm{Lip}(1, 1/2)$ function, and A an elliptic matrix satisfying (5.2). We say that the L^p Dirichlet problem for L as in (5.1) if there exists some $p < \infty$ such that the following holds: for any $f \in L^p(\Sigma)$, the solution u of the Dirichlet boundary value problem $Lu = 0$ in Ω, $u = f$ on Σ (understood in the sense of parabolic non-tangential convergence), provided by the parabolic measure, satisfies the estimate $\|N_* u\|_{L^p(\Sigma)} \lesssim \|f\|_{L^p(\Sigma)}$ (with constants uniform on f). Here we denote by $N_* u$ the nontangential maximal function of u (for the precise definition, see [GH20, Section 4]).

As commented above, we will not use this property directly, but rather another condition which is easier to deal with, and we define next. We already considered this condition about weights in Chapters 2 and 4: note that the difference is essentially just the position of the poles.

Definition 5.21. (A_∞ property) Recall that the parabolic and adjoint parabolic measures satisfy appropriate doubling properties, by virtue of Lemma 5.18. We therefore say that ω is in A_∞ (or that its density is a parabolic A_∞ weight) if there exist $C_* \geqslant 1$ and $q > 1$ such that the following holds: if $\mathbf{X} \in \Sigma$, $r > 0$, and $\mathbf{Y} \in \mathcal{P}^+_{\kappa,2r}(\mathbf{X})$, then $\omega^{\mathbf{Y}} \ll \sigma$ on $\Delta_{2r}(\mathbf{X})$ and $k^{\mathbf{Y}} := d\omega^{\mathbf{Y}}/d\sigma$ satisfies the reverse-Hölder inequality

$$\left(\fint\!\!\!\fint_{\Delta_r(\mathbf{X})} (k^{\mathbf{Y}})^q \, d\sigma\right)^{1/q} \leqslant C_* \fint\!\!\!\fint_{\Delta_r(\mathbf{X})} k^{\mathbf{Y}} \, d\sigma. \tag{5.21}$$

Similarly, we say $\widetilde{\omega}$ is in A_∞ (or that its density is a parabolic A_∞ weight) if there exist $C_* \geqslant 1$ and $q > 1$ such that the following holds. If $\mathbf{X} \in \Sigma$, $r > 0$, and $\mathbf{Y} \in \mathcal{P}^-_{\kappa,2r}(\mathbf{X})$, then $\widetilde{\omega}^{\mathbf{Y}} \ll \sigma$ on $\Delta_{2r}(\mathbf{X})$ and $\tilde{k}^{\mathbf{Y}} := d\omega^{\mathbf{Y}}/d\sigma$ satisfies the reverse-Hölder inequality

$$\left(\fint\!\!\!\fint_{\Delta_r(\mathbf{X})} (\tilde{k}^{\mathbf{Y}})^q \, d\sigma\right)^{1/q} \leqslant C_* \fint\!\!\!\fint_{\Delta_r(\mathbf{X})} \tilde{k}^{\mathbf{Y}} \, d\sigma$$

Remark 5.22. Consider $\mathbf{X}^0 \in \Sigma$ and $R_* > 0$, Using Lemma 5.19 we have that

$$\mathcal{A}^+_{4R_*}(\mathbf{X}^0) \in \mathcal{P}^+_{\kappa,2\rho}(\mathbf{Z}),$$

for all $\mathbf{Z} \in \Sigma$ and $\rho > 0$ such that $\Delta_{2\rho}(\mathbf{Z}) \subseteq \Delta_{R_*}(\mathbf{X}^0)$. Thus (5.21) is valid with $\mathbf{Y} = \mathcal{A}^+_{4R_*}(\mathbf{X}^0)$, i.e.,

$$\left(\fint\!\!\!\fint_{\Delta_r(\mathbf{Z})} (k^{\mathcal{A}^+_{4R_*}(\mathbf{X}^0)})^q \, d\sigma\right)^{1/q} \leqslant C_* \fint\!\!\!\fint_{\Delta_r(\mathbf{Z})} k^{\mathcal{A}^+_{4R_*}(\mathbf{X}^0)} \, d\sigma, \qquad \mathbf{Z} \in \Delta_{R_*/2}(\mathbf{X}^0), r < R_*/2.$$

As anticipated above, we can actually use the handier A_∞ condition instead of the L^p solvability of the Dirichlet problem. Therefore, from here on, we will just use the A_∞ condition as hypothesis for Theorems 5.2 and 5.3. This is justified by the following result.

Theorem 5.23. *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be the region above a* $\mathrm{Lip}(1,1/2)$ *function, and A an elliptic matrix satisfying* (5.2)*. Then, the parabolic measure associated to the operator L with coefficients A (as in* (5.1)*) is in A_∞ (with exponent $q > 1$) if and only if the $L^{q'}$ Dirichlet problem for L is solvable in Ω (where $1/q + 1/q' = 1$, so concretely $q' < \infty$).*

The equivalence stated in Theorem 5.23 is well known, but see, e.g., [LM95] and [Ny97], and also [GH20, Theorem 2.10] for a proof in a much more general setting.

Now we give the definition of our extra condition on the matrix of coefficients A of our PDE. Note also its relationship with the matrices in Theorem 4.5 from the previous chapter.

Definition 5.24 (Smoothness and L^1-Carleson oscillation condition)**.** Let A be an $n \times n$ elliptic matrix on $\mathbb{R}^{n+1}$ as in (5.2). We say A satisfies an **L^1-Carleson oscillation condition** if A is $C^\infty(\Omega)$ and there exists a constant $C_A > 0$ such that for every $(X,t) \in \Omega$ and $k = 1, 2$ it holds

$$\mathrm{dist}((X,t),\partial\Omega)^k \, |\nabla_X^k A(X,t)| + \mathrm{dist}((X,t),\partial\Omega)^{2k} \, |\partial_t^k A(X,t)| + \mathrm{dist}((X,t),\partial\Omega)^3 \, |\partial_t \nabla_X A(X,t)| \leqslant C_A, \tag{5.22}$$

and μ is a Carleson measure relative to Ω with constant bounded by C_A, where

$$d\mu(X,t) := \big(|\nabla_X A(X,t)| \, + \, |\partial_t A(X,t)| \, \operatorname{dist}((X,t), \partial\Omega) \big) \, dX \, dt. \tag{5.23}$$

Here μ being a Carleson measure relative to Ω means for every $\mathbf{X} \in \partial\Omega$ and $\rho > 0$ it holds

$$\mu(\mathcal{J}_\rho(\mathbf{X}) \cap \Omega) \leqslant C_A \, \rho^{n+1}. \tag{5.24}$$

Note that if

$$d\tilde{\mu}(X,t) := \big(|\nabla_X A(X,t)|^2 \operatorname{dist}((X,t), \partial\Omega) + |\partial_t A(X,t)|^2 \operatorname{dist}((X,t), \partial\Omega)^3 \big) \, dX \, dt,$$

then it still holds, simply by using (5.22) and the Carleson condition on μ,

$$\tilde{\mu}(\mathcal{J}_\rho(\mathbf{X}) \cap \Omega) \lesssim \rho^{n+1}, \qquad \mathbf{X} \in \partial\Omega, \ \rho > 0. \tag{5.25}$$

This means that, as is well known, our L^1 Carleson condition is stronger than the usual L^2 Carleson condition frequently considered in the literature. As remarked in the Introduction, the essential obstacle to working with the weaker L^2 Carleson condition is that at present, we are unable to prove the local square function estimates of Lemma 5.34 without the L^1 hypothesis.

We recall that the smoothness assumption in (5.22) can be relaxed, at least in the context of Theorem 5.2: see Remark 5.4, and Section 5.12.

5.7 Step 1: Initial stopping time regimes and the preliminary estimates

In the remaining part of the chapter, we prove Theorems 5.2 and 5.3. For that, we will follow a sequence of steps, some of which are similar to those in [BHMN25]. Nevertheless, there will be, at some points, substantial differences to be able to deal with our variable coefficient matrices.

In this first step, we start by exploiting (in Subsection 5.7.1) the A_∞ assumption for ω (and $\widetilde{\omega}$ if proving Theorem 5.3) to create ample (sawtooth) regions where ω is "comparable" to σ. Later, in Subsection 5.7.2, we will analyze the behavior of the Green function. Here we will find the most challenging difficulty as compared to [BHMN25]: in our setting, we are only able to show that the Green function grows in the vertical direction x_0 (which is essential to run the forthcoming arguments) only in Whitney regions associated to the "A_∞ sawtooths", rather than in whole strips surrounding the boundary as in [BHMN25]. This will have a strong influence in the rest of the chapter, because it forces us to use coronizations through the whole argument.

Remark 5.25. In the sequel, all constants (explicit and implicit) may depend on the *structural constants* $n, \Lambda, \|\psi\|_{\mathrm{Lip}(1,1/2)}$, and the A_∞ constants of ω (namely, C_* and q); and also the A_∞ constants of $\widetilde{\omega}$ in the case of Theorem 5.3.

For the time being, we will keep track of dependencies on the constant K defining the Whitney regions (see Subsection 5.4.8), but later we will choose it depending only on the structural constants.

5.7.1 Control of the parabolic measure in "ample" stopping-time regimes

The following lemma will be the primary tool to produce our initial corona decomposition (one for the parabolic measure). We remind the reader that $\mathbf{\Psi}(Q)$ is the "surface ball" on the graph Σ, obtained by "lifting" the parabolic cube $Q \subset \mathbb{R}^n$ to the graph; see (5.9).

We now present two Lemmata that will be useful in the proof of Theorem 5.2 (i.e., in the case that A is symmetric). For the general case considered in Theorem 5.3, we will require Lemma 5.28 below.

Lemma 5.26. *Given $Q_0 = Q_R(\mathbf{x}_{Q_0}) \in \mathbb{D}$ (i.e. it has center $\mathbf{x}_{Q_0}$ and 'radius' R), we define*

$$\mathbf{X}^{\pm}_{Q_0} := \mathcal{A}^{\pm}_{M_\star R}(\psi(\mathbf{x}_{Q_0}), \mathbf{x}_{Q_0}), \qquad \text{where } M_\star := 10^{100} M_0^2 n. \tag{5.26}$$

Then, under the hypotheses of Theorem 5.2, there exists a constant $M' \geqslant 1$ and a collection $\mathcal{F} = \{Q_j\}_j$ of pairwise disjoint subcubes of Q_0 satisfying

$$\left| \bigcup_{Q_j \in \mathcal{F}} Q_j \right| \leqslant \frac{1}{4}|Q_0|, \tag{5.27}$$

and also

$$\frac{1}{M'} \frac{\omega^{\mathbf{X}^+_{Q_0}}(\mathbf{\Psi}(Q_0))}{\sigma(\mathbf{\Psi}(Q_0))} \leqslant \frac{\omega^{\mathbf{X}^+_{Q_0}}(\mathbf{\Psi}(Q))}{\sigma(\mathbf{\Psi}(Q))} \leqslant M' \frac{\omega^{\mathbf{X}^+_{Q_0}}(\mathbf{\Psi}(Q_0))}{\sigma(\mathbf{\Psi}(Q_0))}, \quad \forall Q \in \mathbb{D}(Q_0), Q \nsubseteq Q_j \ \forall Q_j \in \mathcal{F}. \tag{5.28}$$

The proof of Lemma 5.26 is a fairly routine consequence of the A_∞ property of ω, see [BHMN25, Lemma 4.9]. Now, given Q_0 as in Lemma 5.26 we may define the semi-coherent stopping time regime by

$$\mathbf{S}^* := \mathbf{S}^*_{Q_0} := \mathbb{D}_{\mathcal{F},Q_0} := \{Q \in \mathbb{D}(Q_0) : Q \nsubseteq Q_j, \ \forall Q_j \in \mathcal{F}\}. \tag{5.29}$$

Note that (5.27) gives the following estimate on the "contact region" for the discrete sawtooth

$$|E_{Q_0}| \geqslant (3/4)|Q_0|, \quad \text{where} \quad E_{Q_0} := Q_0 \setminus \cup_{Q_j \in \mathcal{F}} Q_j. \tag{5.30}$$

Now, producing our "corona decomposition for the parabolic measure" is a standard argument: it is just a matter of iterating Lemma 5.26.

Lemma 5.27. *Suppose that the hypotheses of Theorem 5.2 hold. Then, there exist constants $C', M' > 1$ such that following holds. We can decompose the dyadic grid as*

$$\mathbb{D} = \mathcal{G}^* \sqcup \mathcal{B}^*$$

where the 'good cubes' $\mathcal{G}^*$ are further decomposed into disjoint semi-coherent stopping time regimes $\{\mathbf{S}^*\}$ (i.e. $\mathcal{G}^* = \sqcup \mathbf{S}^*$) so that the maximal cubes $\{Q(\mathbf{S}^*)\}_{\mathbf{S}^*}$ and the bad cubes pack, that is,

$$\sum_{Q' \in \mathcal{B}^* \cap \mathbb{D}(Q)} |Q'| + \sum_{\mathbf{S}: Q(\mathbf{S}^*) \subseteq Q} |Q(\mathbf{S}^*)| \leqslant C'|Q|, \quad \forall Q \in \mathbb{D},$$

and for each $\mathbf{S}^*$ it holds

$$\frac{1}{M'} \frac{\omega^{\mathbf{X}^+_{Q(\mathbf{S}^*)}}(\mathbf{\Psi}(Q(\mathbf{S}^*)))}{\sigma(\mathbf{\Psi}(Q(\mathbf{S}^*)))} \leqslant \frac{\omega^{\mathbf{X}^+_{Q(\mathbf{S}^*)}}(\mathbf{\Psi}(Q))}{\sigma(\mathbf{\Psi}(Q))} \leqslant M' \frac{\omega^{\mathbf{X}^+_{Q(\mathbf{S}^*)}}(\mathbf{\Psi}(Q(\mathbf{S}^*)))}{\sigma(\mathbf{\Psi}(Q(\mathbf{S}^*)))}, \quad \forall Q \in \mathbf{S}^*. \tag{5.31}$$

Given Lemma 5.26, we observe that Lemma 5.27 is simply an illustration of the well-known fact that "Big Pieces implies Corona". Since the proof is standard, let us only sketch it. First, using Lemma 5.10, it is enough to produce a corona decomposition for $\mathbb{D}(Q')$ for each $Q' \in C$ (with constants independent of Q'). With this in mind, we fix $Q' \in C$, and apply Lemma 5.26 with $Q_0 := Q'$, hence obtaining $\mathbf{S}^*_{Q'} = \mathbb{D}_{\mathcal{F},Q'}$, a semi-coherent stopping time regime. Applying again Lemma 5.10 wherever we stopped before, i.e., at each $Q_j \in \mathcal{F}$, we obtain new stopping time regimes $\mathbf{S}^*_{Q_j}$. Iterating this, we end up exhausting $\mathbb{D}(Q')$ by stopping-time regimes with the estimate (5.31). Indeed, this produces a 'coronization' of $\mathbb{D}(Q')$ with no bad cubes, where the estimates (5.31) hold within each stopping time $\mathbf{S}^*$. In order to show that the maximal cubes $\{Q(\mathbf{S}^*)\}_{\mathbf{S}^*}$ satisfy the necessary packing condition, we use (5.30). Indeed, by construction the sets $\{E_{Q(\mathbf{S}^*)}\}_{\mathbf{S}^*}$ are disjoint (up to a set of measure zero) and fixing $Q \in \mathbb{D}(Q')$ it holds, using (5.30), that

$$\sum_{\mathbf{S}^*:Q(\mathbf{S}^*)\subseteq Q} |Q(\mathbf{S}^*)| \leqslant \frac{4}{3} \sum_{\mathbf{S}^*:Q(\mathbf{S}^*)\subseteq Q} |E_{Q(\mathbf{S}^*)}| \leqslant \frac{4}{3}|Q|. \tag{5.32}$$

In the case of Theorem 5.3, almost identical techniques yield the following result.

Lemma 5.28. *Suppose that the hypotheses of Theorem 5.3 hold. Then there exists constants $C', M' > 1$ such that following holds. The dyadic grid $\mathbb{D}$ can be decomposed as*

$$\mathbb{D} = \mathcal{G}^* \sqcup \mathcal{B}^*,$$

where the 'good cubes' $\mathcal{G}^*$ are further decomposed into disjoint semi-coherent stopping time regimes $\{\mathbf{S}^*\}$ (i.e. $\mathcal{G}^* = \sqcup\, \mathbf{S}^*$) so that the maximal cubes $\{Q(\mathbf{S}^*)\}_{\mathbf{S}^*}$ and the bad cubes pack, that is,

$$\sum_{Q'\in\mathbb{D}_Q\cap\mathcal{B}^*} |Q'| + \sum_{\mathbf{S}:Q(\mathbf{S}^*)\subseteq Q} |Q(\mathbf{S}^*)| \leqslant C'|Q|, \quad \forall Q \in \mathbb{D},$$

and for each $\mathbf{S}^*$ it holds

$$\frac{1}{M'}\frac{\omega^{\mathbf{X}^+_{Q(\mathbf{S}^*)}}(\mathbf{\Psi}(Q(\mathbf{S}^*)))}{\sigma(\mathbf{\Psi}(Q(\mathbf{S}^*)))} \leqslant \frac{\omega^{\mathbf{X}^+_{Q(\mathbf{S}^*)}}(\mathbf{\Psi}(Q))}{\sigma(\mathbf{\Psi}(Q))} \leqslant M'\frac{\omega^{\mathbf{X}^+_{Q(\mathbf{S}^*)}}(\mathbf{\Psi}(Q(\mathbf{S}^*)))}{\sigma(\mathbf{\Psi}(Q(\mathbf{S}^*)))}, \quad \forall Q \in \mathbf{S}^*, \tag{5.33}$$

and additionally

$$\frac{1}{M'}\frac{\widetilde{\omega}^{\mathbf{X}^-_{Q(\mathbf{S}^*)}}(\mathbf{\Psi}(Q(\mathbf{S}^*)))}{\sigma(\mathbf{\Psi}(Q(\mathbf{S}^*)))} \leqslant \frac{\widetilde{\omega}^{\mathbf{X}^-_{Q(\mathbf{S}^*)}}(\mathbf{\Psi}(Q))}{\sigma(\mathbf{\Psi}(Q))} \leqslant M'\frac{\widetilde{\omega}^{\mathbf{X}^-_{Q(\mathbf{S}^*)}}(\mathbf{\Psi}(Q(\mathbf{S}^*)))}{\sigma(\mathbf{\Psi}(Q(\mathbf{S}^*)))}, \quad \forall Q \in \mathbf{S}^*. \tag{5.34}$$

The only difference in proving Lemma 5.28, in comparison to Lemma 5.27, is using a slightly modified version of Lemma 5.26. Indeed, we should run the argument in Lemma 5.26 for both ω and $\widetilde{\omega}$ separately, hence obtaining $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ so that (5.27) holds for $\mathcal{F}$ and also for $\widetilde{\mathcal{F}}$,

$$\frac{1}{M'}\frac{\widetilde{\omega}^{\mathbf{X}^-_{Q_0}}(\mathbf{\Psi}(Q_0))}{\sigma(\mathbf{\Psi}(Q_0))} \leqslant \frac{\widetilde{\omega}^{\mathbf{X}^-_{Q_0}}(\mathbf{\Psi}(Q))}{\sigma(\mathbf{\Psi}(Q))} \leqslant M'\frac{\widetilde{\omega}^{\mathbf{X}^-_{Q_0}}(\mathbf{\Psi}(Q_0))}{\sigma(\mathbf{\Psi}(Q_0))}, \quad \forall Q \in \mathbb{D}(Q_0), Q \notin \widetilde{Q}_j\, \forall \widetilde{Q}_j \in \widetilde{\mathcal{F}}.$$

Then, letting $\mathcal{F}'$ denote the maximal cubes (with respect to containment) in $\mathcal{F} \cup \widetilde{\mathcal{F}}$, we have

$$|\cup_{Q_j\in\mathcal{F}'} Q_j| \leqslant (1/2)|Q_0| \tag{5.35}$$

and the estimates (5.33) and (5.34) hold for $\mathbf{S}^* := \mathbf{S}^*(Q_0)$ now defined by

$$\mathbf{S}^*(Q_0) := \mathbb{D}_{\mathcal{F}',Q_0} := \{Q \in \mathbb{D}(Q_0) : Q \nsubseteq Q'_j,\ \forall Q'_j \in \mathcal{F}'\}.$$

Then all of the arguments that follow are identical, since (5.35) can be used for the packing of the maximal cubes with the only difference being that the constant in (5.32) changes from $4/3$ to 2.

5.7.2 Level lines and estimates for the Green function in Whitney regions

In the sequel we fix a semi-coherent stopping time regime $\mathbf{S}^*$ from Lemma 5.27 (or Lemma 5.28 if we are proving Theorem 5.3). We are going to further decompose $\mathbf{S}^*$ into stopping times where nice estimates on the Green function hold (and therefore, there will be "bad cubes" where these estimates fail).

Just as in [BHMN25], we need to transfer information about the parabolic measure to the Green function. We define the normalized Green function

$$u(\mathbf{Y}) := \sigma(Q(\mathbf{S}^*))\, G(\mathbf{X}^+_{Q(\mathbf{S}^*)}, \mathbf{Y}), \tag{5.36}$$

which solves $L^*u = 0$ in $\{(X,t) \in \Omega : t < t(\mathbf{X}^+_{Q(\mathbf{S}^*)})\}$. In the case we are proving Theorem 5.3, we also define the normalized adjoint Green function (solving $Lv = 0$ in $\{(X,t) \in \Omega : t > t(\mathbf{X}^-_{Q(\mathbf{S}^*)})\}$)

$$v(\mathbf{X}) := \sigma(Q(\mathbf{S}^*))\, G(\mathbf{X}, \mathbf{X}^-_{Q(\mathbf{S}^*)}).$$

We note that, since the coefficients A of our equation are locally Hölder continuous by (5.2) and (5.22), both u and v are locally $C^{2,\alpha}$ for $\alpha > 0$ (in a parabolic sense, see [Li96, Chapter 4]) inside Ω, away from their poles, by standard regularity results (like Nash's theorem and Schauder regularity, see [Li96, Chapters 5 and 6]). Concretely, we will use (without further mention) the fact that u and v are (qualitatively) twice differentiable inside Ω (away from the poles).

To produce our initial estimates on the Green function we use Section 5.5. Actually, we want to obtain estimates on our Whitney regions U_Q (from subsection 5.4.8). Recall that these constructions included a constant K, that we will take sufficiently large below. In any case, we will insist that

$$K = 2^N, \quad N > 2000, \tag{5.37}$$

so that $K > 10^{500}$. With this, we obtain a consequence of the results in Section 5.5.2 and Lemma 5.27.

Lemma 5.29. *Suppose that the hypotheses of Theorem 5.2 hold, and that $K = 2^N, N > 2000$. Let $Q \in \mathbf{S}^*$ (Lemma 5.27). Then there exist two constants $M_1, M_2 > 1$ (M_2 depending on K) such that the following holds. For any $Q \in \mathbf{S}^*$, without restriction on its side-length,*

$$(M_1)^{-1} \leqslant \frac{u(\mathbf{Y})}{\operatorname{dist}(\mathbf{Y}, \partial\Omega)} \leqslant M_1, \tag{5.38}$$

whenever (recall that $\mathbf{x}_Q$ denotes the center of Q)

$$\mathbf{Y} \in \big\{(y_0, y, s) \in \mathbb{R}^{n+1} :\ (y,s) \in 10^{10}Q,\ \ y_0 = \psi(\mathbf{x}_Q) + 2M_0 10^8 \ell(Q)\big\}. \tag{5.39}$$

Moreover, if Q satisfies the additional property that $\ell(Q) \leqslant K^{-10}\ell(Q(\mathbf{S}^*))$, then for all $\mathbf{X} \in U_Q^{***}(K)$

$$(M_2)^{-1} \leqslant \frac{u(\mathbf{Y})}{\operatorname{dist}(\mathbf{Y}, \partial\Omega)} \leqslant M_2, \qquad \forall\, \mathbf{Y} \in B\left(\mathbf{X}, \frac{\operatorname{dist}(\mathbf{X}, \partial\Omega)}{10K}\right). \tag{5.40}$$

Proof. Let us rapidly sketch how to obtain (5.38). Indeed, the upper bound follows like this:

$$\begin{aligned} u(\mathbf{Y}) \;&=\; \sigma(Q(\mathbf{S}^*))\, G(\mathbf{X}^+_{Q(\mathbf{S}^*)}, \mathbf{Y}) \;\lesssim\; \sigma(Q(\mathbf{S}^*))\, G\big(\mathbf{X}^+_{Q(\mathbf{S}^*)}, \mathcal{A}^-_{10^{12}\ell(Q)}(\psi(\mathbf{x}_Q), \mathbf{x}_Q)\big) \\ &\approx\; \sigma(Q(\mathbf{S}^*))\, \ell(Q)^{-n}\, \omega^{\mathbf{X}^+_{Q(\mathbf{S}^*)}}\big(\Delta_{10^{12}\ell(Q)}(\psi(\mathbf{x}_Q), \mathbf{x}_Q)\big) \;\approx\; \sigma(Q(\mathbf{S}^*))\, \ell(Q)^{-n}\, \omega^{\mathbf{X}^+_{Q(\mathbf{S}^*)}}(\boldsymbol{\Psi}(Q)) \\ &\approx\; \sigma(\boldsymbol{\Psi}(Q))\, \ell(Q)^{-n}\, \omega^{\mathbf{X}^+_{Q(\mathbf{S}^*)}}(\boldsymbol{\Psi}(Q(\mathbf{S}^*))) \;\approx\; \ell(Q) \;\approx\; \operatorname{dist}(\mathbf{Y}, \Sigma). \end{aligned}$$

Let us explain more carefully each step. The first step is the definition of u in (5.36). The second follows from the parabolic Harnack's inequality for adjoint solutions like u (see [FGS86, Theorem 0.2] for a precise statement). The third follows from the CFMS estimate in Lemma 5.17 because $\mathbf{X}^+_{Q(\mathbf{S}^*)} = \mathcal{A}^+_{M_\star \operatorname{diam}(Q(\mathbf{S}^*))}(\psi(\mathbf{x}_{Q(\mathbf{S}^*)}), \mathbf{x}_{Q(\mathbf{S}^*)}) \in \mathcal{P}^+_{\kappa, 10^{12}\ell(Q)}(\psi(\mathbf{x}_Q), \mathbf{x}_Q)$ (see the definitions of all the objects in (5.19), (5.20) and (5.26)). The fourth follows from the doubling property of the parabolic measure in Lemma 5.18. The fifth comes from Lemma 5.27. The second-to-last estimate follows from Corollary 5.14, and the fact that $\sigma(\boldsymbol{\Psi}(Q)) \approx \ell(Q)^{n+1}$ because ψ is Lip(1,1/2), and the last estimate from Lemma 5.5 and (5.39).

In turn, to obtain the lower bound in (5.38), we can repeat the computations with the corkscrew point $\mathcal{A}^+_{10^{12}\ell(Q)}(\psi(\mathbf{x}_Q), \mathbf{x}_Q)$ replacing $\mathcal{A}^-_{10^{12}\ell(Q)}(\psi(\mathbf{x}_Q), \mathbf{x}_Q)$, hence reversing Harnack's inequality.

Lastly, to show (5.40), the argument is similar to the above using massive dilates of the cube Q (having (5.17) in mind), depending on K, so that we obtain the estimates for u in the whole region $U_Q^{***}(K)$ (therefore the constants will depend on K, too). The assumption $\ell(Q) \leqslant K^{-10}\ell(Q(\mathbf{S}^*))$ ensures that the pole $\mathbf{X}^+_{Q(\mathbf{S}^*)}$ is still far to the right of these dilations, so that all the estimates from Section 5.5.2 continue to hold. We leave further details to the interested reader. □

In the case that we are proving Theorem 5.3, the analogous result reads as follows.

Lemma 5.30. *Suppose that the hypotheses of Theorem 5.3 hold and that $K = 2^N, N > 2000$. Let $Q \in \mathbf{S}^*$ (Lemma 5.28). Then there exist two constants $M_1, M_2 > 1$ (M_2 depending on K) such that the following holds. For any $Q \in \mathbf{S}^*$, without restriction on its side-length,*

$$(M_1)^{-1} \leqslant \frac{u(\mathbf{Y})}{\operatorname{dist}(\mathbf{Y}, \partial\Omega)} \leqslant M_1,$$

whenever (recalling that $\mathbf{x}_Q$ is the center of Q)

$$\mathbf{Y} \in \big\{(y_0, y, s) :\; (y, s) \in 10^{10} Q,\;\; y_0 = \psi(\mathbf{x}_Q) + 2M_0 10^8 \ell(Q)\big\}.$$

Moreover, if Q satisfies the additional property that $\ell(Q) \leqslant K^{-10}\ell(Q(\mathbf{S}^*))$, then for all $\mathbf{X} \in U_Q^{***}(K)$

$$(M_2)^{-1} \leqslant \frac{u(\mathbf{Y})}{\operatorname{dist}(\mathbf{Y}, \partial\Omega)} \leqslant M_2, \qquad \forall\, \mathbf{Y} \in B\left(\mathbf{X}, \frac{\operatorname{dist}(\mathbf{X}, \partial\Omega)}{10K}\right), \tag{5.41}$$

and also

$$(M_2)^{-1} \leqslant \frac{v(\mathbf{Y})}{\operatorname{dist}(\mathbf{Y}, \partial\Omega)} \leqslant M_2, \qquad \forall\, \mathbf{Y} \in B\left(\mathbf{X}, \frac{\operatorname{dist}(\mathbf{X}, \partial\Omega)}{10K}\right). \tag{5.42}$$

For some of the forthcoming arguments, the estimates in Lemmas 5.29 and 5.30 in U_Q^{***} will be enough. However, it will be crucial for us later to be able to use level sets of these Green functions, too. The following lemma makes it explicit, how these level sets traverse our Whitney regions.

Lemma 5.31. *Suppose that the hypotheses of Theorem 5.2 or Theorem 5.3 hold. Then there exists $N_0 \geqslant 2000$ such that: if $K = 2^N$ for $N \geqslant N_0$, and $Q \in \mathbf{S}^*$ with $\ell(Q) \leqslant K^{-10}\ell(Q(\mathbf{S}^*))$, then*

$$\mathbf{y} \in 10Q \quad \Longrightarrow \quad \exists\, y_0 \in \mathbb{R} \ \text{ with } \ (y_0, \mathbf{y}) \in U_Q(K) \quad \text{and} \quad \partial_{y_0} u(y_0, \mathbf{y}) \geqslant c_1, \tag{5.43}$$

for some $c_1 > 0$ which is independent of K.

Moreover, for every $\delta \in (0, 1/2]$, there exists $N = N(\delta) \geqslant 2000$ such that if $K = 2^N$, then for each $Q \in \mathbf{S}^*$ with $\ell(Q) \leqslant K^{-10}\ell(Q(\mathbf{S}^*))$ we have

$$\mathbf{y} \in 10^5 Q, \ \ r \in [\delta\ell(Q), \delta^{-1}\ell(Q)] \quad \Longrightarrow \quad \exists\, y_r \in \mathbb{R} \ \text{ so that } \ (y_r, \mathbf{y}) \in U_Q \quad \text{and} \quad u(y_r, \mathbf{y}) = r.$$

Proof. Let us start by proving the second half of the lemma. Before that, note first that we can make $N(\delta)$ to be a decreasing function in δ.

Fix $\delta \in (0, 1/2]$ and $\mathbf{y} \in 10^5 Q$. Recalling the definition of U_Q in (5.16), and using the intermediate value theorem, it suffices to show that for $K = 2^N$ with $N = N(\delta)$ large enough

$$u(\psi(\mathbf{y}) + K^{-1}\ell(Q), \mathbf{y}) < \delta\ell(Q), \tag{5.44}$$

and that there exists y_0 such that $(y_0, \mathbf{y}) \in U_Q(K)$ with

$$u(y_0, \mathbf{y}) > \delta^{-1}\ell(Q). \tag{5.45}$$

To show (5.44), we use the Lemma 5.13. Letting $\mathbf{X} := \mathbf{\Psi}(\mathbf{x}_Q)$ and $R := 10^8\ell(Q)$, we clearly have

$$\mathcal{A}_R^-(\mathbf{X}) \in \big\{(y_0, y, s) : \ (y, s) \in 10^{10} Q, \ \ y_0 = \psi(\mathbf{x}_Q) + 2M_0 10^8 \ell(Q)\big\}.$$

Then, by (5.18), and later (5.38) (along with Lemma 5.5 repeatedly), we obtain

$$u(\psi(\mathbf{y}) + K^{-1}\ell(Q), \mathbf{y}) \lesssim_{M_0} \left(\frac{K^{-1}\ell(Q)}{10^8\ell(Q)}\right)^{\alpha} u(\mathcal{A}_R^-(\mathbf{X})) \lesssim_{M_0, M_1} \left(\frac{K^{-1}\ell(Q)}{R}\right)^{\alpha} R \approx K^{-\alpha}\ell(Q),$$

from where (5.44) simply follows after choosing K sufficiently large, depending on δ.

Let us now show (5.45). Here we are going to use (5.38) again, but for an ancestor of Q. Let Q^* be the unique cube in $\mathbf{S}^*$ which contains Q with $\ell(Q^*) = 2^{\lfloor N/2 \rfloor + 1}\ell(Q) \approx K^{1/2}\ell(Q)$, which is a possible choice since $Q \in \mathbf{S}^*$ with $\ell(Q) \leqslant K^{-10}\ell(Q(\mathbf{S}^*))$. Now consider the point

$$\mathbf{Y} := (\psi(\mathbf{x}_{Q^*}) + 2M_0 10^8 \ell(Q^*), \mathbf{y}),$$

that is, the point over $\mathbf{y}$ in the set (5.39) for Q^*. By (5.38) (and Lemma 5.5) it holds that

$$u(\mathbf{Y}) \geqslant M_1 \operatorname{dist}(\mathbf{Y}, \partial\Omega) \gtrsim M_1 \ell(Q^*) \geqslant cK^{1/2}\ell(Q), \tag{5.46}$$

where c is independent of K. Moreover, by definition of $\mathbf{Y}$ (and Lemma 5.5) it holds that

$$\|\mathbf{Y} - \mathbf{\Psi}(\mathbf{y})\| \approx K^{1/2}\ell(Q), \tag{5.47}$$

where the implicit constant is independent of K. Hence, by choosing K large enough (depending on δ), (5.46) implies that $u(\mathbf{Y}) > \delta^{-1}\ell(Q)$, and (5.47) ensures $(y_0, \mathbf{y}) := \mathbf{Y} \in U_Q(K)$ (see (5.16)). This finishes the proof of the second half of the lemma.

Finally, let us prove (5.43). Fix $N_0 := N(1/2)$ (i.e., put $\delta = 1/2$ in the above argument), and denote $K_0 := 2^{N_0}$ (so it is clear that both N_0 and K_0 only depend on n, $\|\psi\|_{\mathrm{Lip}(1,1/2)}$, and the A_∞ constants for $\omega^{\mathbf{X}^+_{Q_0}}$). Then, for a given $\mathbf{y} \in 10Q$, we repeat the argument above to find some y_0 such that $(y_0, \mathbf{y}) \in U_Q(K_0)$ (therefore, by (5.16), $y_0 \in (\psi(\mathbf{y}) + K_0^{-1}\ell(Q), \psi(\mathbf{y}) + K_0\ell(Q))$) such that

$$u(\psi(\mathbf{y}) + K_0^{-1}\ell(Q), \mathbf{y}) < (1/2)\ell(Q) < 2\ell(Q) < u(y_0, \mathbf{y}).$$

Therefore, by the mean value theorem, there exists $x_0 \in (\psi(\mathbf{y}) + K_0^{-1}\ell(Q), y_0)$ (so that $(x_0, \mathbf{y}) \in U_Q(K_0) \subseteq U_Q(K)$ for any $K \geqslant K_0$, i.e., $N \geqslant N_0$) such that

$$\partial_{x_0} u(x_0, \mathbf{y}) = \frac{u(y_0, \mathbf{y}) - u(\psi(\mathbf{y}) + K_0^{-1}\ell(Q), \mathbf{y})}{y_0 - (\psi(\mathbf{y}) + K_0^{-1}\ell(Q))} \geqslant \frac{2\ell(Q) - (1/2)\ell(Q)}{(\psi(\mathbf{y}) + K_0\ell(Q)) - (\psi(\mathbf{y}) + K_0^{-1}\ell(Q))} = C(K_0),$$

where $C(K_0) > 0$. This is indeed the result stated in the first half of the lemma. □

The conditions for our matrix A, allow us to obtain the following.

Lemma 5.32. *Assume the hypotheses of Theorem 5.2 (or Theorem 5.3). Let $\mathbf{S}^*$ be as in Lemma 5.27 (or 5.28). Suppose $K = 2^N, N > 2000$. If $Q \in \mathbf{S}^*$ satisfies $\ell(Q) \leqslant K^{-10}\ell(Q(\mathbf{S}^*))$, then*

$$|\nabla_Y^k u(\mathbf{Y})| \lesssim \mathrm{dist}(\mathbf{Y}, \partial\Omega)^{1-k}, \quad |\partial_t u(\mathbf{Y})| \lesssim \mathrm{dist}(\mathbf{Y}, \partial\Omega)^{-1} \quad \forall\, \mathbf{Y} \in U_Q^{***},\ k = 0, 1, 2, \tag{5.48}$$

$$|\partial_t \nabla_Y u(\mathbf{Y})| \lesssim \mathrm{dist}(\mathbf{X}, \partial\Omega)^{-2}, \quad \forall\, \mathbf{Y} \in U_Q^{***}. \tag{5.49}$$

Moreover, in the case that we assume the hypotheses of Theorem 5.3, then (5.48) and (5.49) also hold with v in place of u.

Proof. These estimates are a consequence of standard interior regularity for solutions of linear parabolic PDE with twice differentiable coefficients. Let us include a complete proof for completeness.

Fix $\mathbf{X} \in U_Q^{***}$, and denote by $\mathcal{J}$ a Whitney box containing $\mathbf{X}$, for instance $\mathcal{J} := \mathcal{J}_{\mathrm{dist}(\mathbf{X},\partial\Omega)/100}(\mathbf{X})$. By [Li96, Theorem 4.9] applied in $4\mathcal{J} \subset \Omega$ (check the definitions in [Li96, Chapter IV]),

$$\sup_{\mathbf{Y}\in 4\mathcal{J}} \Big[\mathrm{dist}(\mathbf{Y}, \partial(4\mathcal{J}))\, |\nabla u(\mathbf{Y})| + \mathrm{dist}(\mathbf{Y}, \partial(4\mathcal{J}))^2\, |\nabla^2 u(\mathbf{Y})| + \mathrm{dist}(\mathbf{Y}, \partial(4\mathcal{J}))^2\, |\partial_t u(\mathbf{Y})| \Big] \lesssim \|u\|_{L^\infty(4\mathcal{J})},$$

(where, across the proof, ∇ denotes the spatial gradient) and also, for $\alpha > 0$,

$$\sup_{\substack{(Y,t),(Y,s)\in 4\mathcal{J} \\ t\neq s}} \min\big\{\mathrm{dist}((Y,t), \partial(4\mathcal{J})), \mathrm{dist}((Y,s), \partial(4\mathcal{J}))\big\}^{2+\alpha} \frac{|\nabla u(Y,t) - \nabla u(Y,s)|}{|(Y,t) - (Y,s)|^{1+\alpha}} \lesssim \|u\|_{L^\infty(4\mathcal{J})}. \tag{5.50}$$

The implicit constants are uniform because the C^α norm of A is controlled by (5.2) and (5.22).

Note that, since $\mathcal{J}$ is a Whitney box and $\mathbf{X} \in \mathcal{J}$, it holds $\mathrm{dist}(\mathbf{X}, \partial(4\mathcal{J})) \approx \mathrm{diam}(\mathcal{J}) \approx \mathrm{dist}(\mathbf{X}, \partial\Omega)$. Thus, the estimate in the first display yields (5.48) because $\|u\|_{L^\infty(4\mathcal{J})} \lesssim \sup_{\mathbf{Y}\in 4\mathcal{J}} \mathrm{dist}(\mathbf{Y}, \partial\Omega) \approx \mathrm{dist}(\mathbf{X}, \partial\Omega)$ by Lemma 5.29 (and again properties of Whitney boxes).

The argument to obtain (5.49) is similar. Differentiating the equation that u solves, we get

$$\partial_t(\partial_t u) - \operatorname{div}(A\nabla(\partial_t u)) = \operatorname{div}((\partial_t A)\nabla u).$$

Hence, applying [Li96, Theorem 4.8] in $2\mathcal{J}$, we obtain, after some simplifications as before,

$$\begin{aligned}\operatorname{dist}(\mathbf{X},\partial\Omega)\,|\nabla(\partial_t u(\mathbf{X}))| &\lesssim \|\partial_t u\|_{L^\infty(2\mathcal{J})} + \operatorname{dist}(\mathbf{X},\partial\Omega)\,\|(\partial_t A)\nabla u\|_{L^\infty(2\mathcal{J})}\\ &\quad + \sup_{\mathbf{Y}\neq\mathbf{Z}\in 2\mathcal{J}} \min\big\{\operatorname{dist}(\mathbf{Y},\partial(2\mathcal{J})),\operatorname{dist}(\mathbf{Z},\partial(2\mathcal{J}))\big\}^{1+\alpha}\frac{|\partial_t A(\mathbf{Y})\nabla u(\mathbf{Y}) - \partial_t A(\mathbf{Z})\nabla u(\mathbf{Z})|}{|\mathbf{Y}-\mathbf{Z}|^\alpha}. \qquad (5.51)\end{aligned}$$

The first term in the right-hand-side is controlled (up to constants) by $\operatorname{dist}(\mathbf{X},\partial\Omega)^{-1}$ by (5.48). So is the second one by using (5.48) and also (5.22). In turn, to control the last term, we compute

$$\begin{aligned}|\partial_t A(\mathbf{Y})\nabla u(\mathbf{Y}) - \partial_t A(\mathbf{Z})\nabla u(\mathbf{Z})| &\leqslant |\partial_t A(\mathbf{Y}) - \partial_t A(\mathbf{Z})|\,|\nabla u(\mathbf{Y})| + |\partial_t A(\mathbf{Z})|\,|\nabla u(\mathbf{Y}) - \nabla u(\mathbf{Z})|\\ &\lesssim \operatorname{dist}(\mathbf{X},\partial\Omega)^{-4}\,|\mathbf{Y}-\mathbf{Z}|^2 + \operatorname{dist}(\mathbf{X},\partial\Omega)^{-3}\,|\mathbf{Y}-\mathbf{Z}| + \operatorname{dist}(\mathbf{X},\partial\Omega)^{-2}\,|\nabla u(\mathbf{Y}) - \nabla u(\mathbf{Z})|,\end{aligned}$$

where in the last estimate we have used the mean value theorem along with (5.22), and also (5.48). Noting that $\operatorname{dist}(\mathbf{Y},\partial(2\mathcal{J})), |\mathbf{Y}-\mathbf{Z}| \lesssim \operatorname{diam}(\mathcal{J}) \approx \operatorname{dist}(\mathbf{X},\partial\Omega)$ if $\mathbf{Y},\mathbf{Z}\in 2\mathcal{J}$, the last display yields

$$\begin{aligned}&\sup_{\mathbf{Y}\neq\mathbf{Z}\in 2\mathcal{J}} \min\big\{\operatorname{dist}(\mathbf{Y},\partial(2\mathcal{J})),\operatorname{dist}(\mathbf{Z},\partial(2\mathcal{J}))\big\}^{1+\alpha}\frac{|\partial_t A(\mathbf{Y})\nabla u(\mathbf{Y}) - \partial_t A(\mathbf{Z})\nabla u(\mathbf{Z})|}{|\mathbf{Y}-\mathbf{Z}|^\alpha} \qquad (5.52)\\ &\lesssim \operatorname{dist}(\mathbf{X},\partial\Omega)^{-1} + \operatorname{dist}(\mathbf{X},\partial\Omega)^{-2}\sup_{\mathbf{Y}\neq\mathbf{Z}\in 2\mathcal{J}} \min\big\{\operatorname{dist}(\mathbf{Y},\partial(2\mathcal{J})),\operatorname{dist}(\mathbf{Z},\partial(2\mathcal{J}))\big\}^{1+\alpha}\frac{|\nabla u(\mathbf{Y}) - \nabla u(\mathbf{Z})|}{|\mathbf{Y}-\mathbf{Z}|^\alpha}.\end{aligned}$$

Let us estimate the last term using the mean value theorem and (5.48), where $\mathbf{Y} = (Y,t), \mathbf{Z} = (Z,s)$:

$$\begin{aligned}|\nabla u(\mathbf{Y}) - \nabla u(\mathbf{Z})| &\leqslant |\nabla u(Y,t) - \nabla u(Z,t)| + |\nabla u(Z,t) - \nabla u(Z,s)|\\ &\lesssim \operatorname{dist}(\mathbf{X},\partial\Omega)^{-1}|\mathbf{Y}-\mathbf{Z}| + |\nabla u(Z,t) - \nabla u(Z,s)|.\end{aligned}$$

Inserting this back in (5.52), we obtain

$$\begin{aligned}&\sup_{\mathbf{Y}\neq\mathbf{Z}\in 2\mathcal{J}} \min\big\{\operatorname{dist}(\mathbf{Y},\partial(2\mathcal{J})),\operatorname{dist}(\mathbf{Z},\partial(2\mathcal{J}))\big\}^{1+\alpha}\frac{|\partial_t A(\mathbf{Y})\nabla u(\mathbf{Y}) - \partial_t A(\mathbf{Z})\nabla u(\mathbf{Z})|}{|\mathbf{Y}-\mathbf{Z}|^\alpha}\\ &\lesssim \operatorname{dist}(\mathbf{X},\partial\Omega)^{-1}\\ &\quad + \operatorname{dist}(\mathbf{X},\partial\Omega)^{-2}\sup_{\substack{(Z,t),(Z,s)\in 2\mathcal{J}\\ t\neq s}} \min\big\{\operatorname{dist}((Z,t),\partial(4\mathcal{J})),\operatorname{dist}((Z,s),\partial(4\mathcal{J}))\big\}^{2+\alpha}\frac{|\nabla u(Z,t) - \nabla u(Z,s)|}{|(Z,t)-(Z,s)|^{1+\alpha}}\\ &\lesssim \operatorname{dist}(\mathbf{X},\partial\Omega)^{-1},\end{aligned}$$

where the last estimate is due to (5.50) (and in the first step, we also used that $\operatorname{dist}(\mathbf{Y},\partial(4\mathcal{J})) \approx \operatorname{diam}(\mathcal{J}) \approx \operatorname{dist}(\mathbf{X},\partial\Omega)$ because $\mathbf{Y}\in 2\mathcal{J}$). Back in (5.51), this finishes the proof of (5.49). □

5.8 Step 2: A local square function estimate and refined corona decomposition

In this section, we will reduce the proofs of Theorems 5.2 and 5.3 to constructing some regular $\mathrm{Lip}(1,1/2)$ graphs that approximate our original graph Σ in a corona decomposition sense (concretely, see Subsection 5.8.3). To construct those graphs (which we will do in the next sections, we will use level lines

of the Green's function u. And actually, it will be convenient that this level lines "follow the shape" of Σ: concretely, we would like that $\partial_{x_0} u > 0$ (uniformly). Indeed, we will prove in Subsection 5.8.2 that this property holds in our Whitney regions. The price we need to pay is that we need to refine our initial corona decomposition for the parabolic measure. And to prove the packing condition for the "bad cubes", we will need estimates for the regions where the Green's function behaves poorly: that will be the content of Subsection 5.8.1.

In fact, the integration by parts scheme that we will develop in Subsection 5.8.1 to obtain square function estimates for the Green's function, is one of the main novelties of this chapter. Inspired by the work of [LN07] for the heat equation, we develop an integration by parts method that allows us to deal with symmetric matrices without needing to simultaneously use solutions to the forward equation (that for the operator L) and the adjoint equation (for L^*). Concretely, that allows us to only require the L^p solvability for L (and not for L^*) when proving Theorem 5.2.

5.8.1 Local square function estimate via integration by parts

In order to later rigorously integrate by parts, we need a smooth cutoff function adapted to sawtooth regions like $\Omega_{\mathbf{S}}$. The following lemma is sufficient for our purposes and we have put its proof in Section 5.11.

Lemma 5.33. *Let $K \geqslant 4$ and $\mathbf{S} \subseteq \mathbb{D}$ a semi-coherent stopping time, with maximal cube $Q(\mathbf{S})$. For every $N_1, N_2 \in \mathbb{N} \cup \{0\}$, $N_1 \leqslant N_2$, define the following truncation of $\mathbf{S}$ as*

$$\mathbf{S}_{N_1,N_2} := \{Q \in \mathbf{S} : 2^{-N_2}\ell(Q(\mathbf{S})) \leqslant \ell(Q) \leqslant 2^{-N_1}\ell(Q(\mathbf{S}))\}$$

Then, for each N, there exists a cut off function $\eta = \eta_N$ with the following properties:

$$\mathbf{1}_{\Omega^{**}_{\mathbf{S}_{N_1,N_2}}} \leqslant \eta \leqslant \mathbf{1}_{\Omega^{***}_{\mathbf{S}_{N_1,N_2}}}, \tag{5.53}$$

$$|\nabla_X \eta(X,t)|\,\mathrm{dist}((X,t),\partial\Omega) + |\partial_t \eta(X,t)|\,\mathrm{dist}((X,t),\partial\Omega)^2 \lesssim 1, \tag{5.54}$$

and if we define the "boundary" of the region where $\eta \equiv 1$ by

$$\mathcal{F}_b := \big\{Q \in \mathbb{D} : \exists\, \mathbf{X} \in U_Q^{***} \text{ with } \nabla_{\mathbf{X}} \eta(\mathbf{X}) \neq 0\big\},$$

then the "perimeter" of $\{\eta \equiv 1\}$ behaves well in the sense that

$$\sum_{Q \in \mathcal{F}_b} |Q| \lesssim |Q(\mathbf{S})|. \tag{5.55}$$

In particular, it holds

$$\iiint_{\mathbb{R}^{n+1}} \big(|\nabla_X \eta(X,t)| + |\partial_t \eta(X,t)|\,\mathrm{dist}((X,t),\partial\Omega)\big)\, dX\, dt \lesssim |Q(\mathbf{S})|. \tag{5.56}$$

Here all the implicit constants depend only on the structural constants and K (defining the Whitney regions in Subsection 5.4.8).

The following square function estimate is a crucial ingredient in our proof of Theorems 5.2 and 5.3. Indeed, after proving Lemma 5.34, we will no longer have to consider whether we are proving Theorem 5.2 or Theorem 5.3.

We recall that K is the large (see (5.37)) parameter in the definitions of the various fattened Whitney regions U_Q, U_Q^*, etc. (see Subsection 5.4.8).

Lemma 5.34. *Suppose that the hypotheses of Theorem 5.2 or Theorem 5.3 hold. If $K = 2^N$, with $N \geqslant 2000$ is sufficiently large and $N_1 := 10N$ then for all $N_2 \geqslant N_1$ it holds*

$$\iiint_{\Omega^{**}_{\mathbf{S}^*_{N_1,N_2}}} \left(|\partial_t u(\mathbf{X})|^2 + |\nabla_X^2 u(\mathbf{X})|^2\right) \operatorname{dist}(\mathbf{X}, \partial\Omega)\, d\mathbf{X} \lesssim |Q(\mathbf{S}^*)|, \tag{5.57}$$

with implicit constants depending on the structural constants and K, and independent of N_2.

Moreover, if $\mathbf{S} \subseteq \mathbb{D}$ is any semi-coherent stopping time with $\mathbf{S} \subseteq \mathbf{S}^*$ and $\ell(Q(\mathbf{S})) \leqslant 2^{-10N}\ell(Q(\mathbf{S}^*))$, then for any $N_3 > 0$ we have

$$\iiint_{\Omega^{**}_{\mathbf{S}_{0,N_3}}} \left(|\partial_t u(\mathbf{X})|^2 + |\nabla_X^2 u(\mathbf{X})|^2\right) \operatorname{dist}(\mathbf{X}, \partial\Omega)\, d\mathbf{X} \lesssim |Q(\mathbf{S})|, \tag{5.58}$$

with implicit constants depending on the structural constants and K.

Remark 5.35. For the proof of Lemma 5.34, it is crucial that u is an adjoint solution wherever $\eta \neq 0$. And that is true (see the comment right after (5.36)) because our choice of pole $\mathbf{X}^+_{Q(\mathbf{S}^*)}$ in (5.26) was made with $M_\star$ so large that $\{\eta \neq 0\}$ lies way back in time from $t(\mathbf{X}^+_{Q(\mathbf{S}^*)})$. Indeed, since in Lemma 5.34 the cubes Q that compose the sets $\mathbf{S}^*_{N_1,N_2}$ and $\mathbf{S}_{0,N_3}$ always satisfy $\ell(Q) \leqslant 2^{-10N}\ell(Q(\mathbf{S}^*)) = K^{-10}\ell(Q(\mathbf{S}^*))$, it is not difficult to conclude from (5.17) that $\Omega^{**}_{\mathbf{S}^*_{N_1,N_2}}, \Omega^{**}_{\mathbf{S}_{0,N_3}} \subseteq Q_{K^{-7}M_\star \operatorname{diam}(Q(\mathbf{S}^*))}(\psi(\mathbf{x}_Q), \mathbf{x}_Q)$, which implies that $\mathbf{X}^+_{Q(\mathbf{S}^*)}$ is indeed always far enough to the future.

The proof of Lemma 5.34 uses an integration by parts argument, which is different whether we assume the hypotheses of Theorem 5.2 or Theorem 5.3. So let us separate them. Throughout the proof, we will frequently abbreviate $\nabla := \nabla_X$ and forget about the $d\mathbf{X}$, simply for brevity.

Proof of Lemma 5.34 under the hypotheses of Theorem 5.2. Let us prove (5.57) (indeed, (5.58) follows by picking the cut-off function in the next line adapted to $\mathbf{S}_{0,N_3}$ instead of $\mathbf{S}^*_{N_1,N_2}$). Let η be the cut-off function for $\mathbf{S}^*_{N_1,N_2}$ provided by Lemma 5.33. By Lemma 5.33 and (5.40) it suffices to show

$$\iiint \left(|\partial_t u(\mathbf{X})|^2 + |\nabla_X^2 u(\mathbf{X})|^2\right) u\eta\, d\mathbf{X} \lesssim |Q(\mathbf{S}^*)|. \tag{5.59}$$

Before continuing, we let $\mathtt{E}$ denote a generic term that enjoys the estimate $|\mathtt{E}| \lesssim |Q(\mathbf{S}^*)|$. In particular, we denote by $\mathtt{E}_i$ ($i = 1, 2, \ldots, 9$) any term with the following bounds

$$|\mathtt{E}_1| \lesssim \iiint |\nabla u|^2 u\, |\partial_t \eta|\, d\mathbf{X}, \qquad |\mathtt{E}_2| \lesssim \iiint |\partial_t u|\, |\nabla u|\, u\, |\nabla \eta|\, d\mathbf{X}, \qquad |\mathtt{E}_3| \lesssim \iiint |\nabla u|^3 |\nabla \eta|\, d\mathbf{X},$$

$$|\mathtt{E}_4| \lesssim \iiint u\,|\nabla A|\,|\nabla u|^2|\nabla\eta|\,d\mathbf{X}, \qquad |\mathtt{E}_5| \lesssim \iiint u\,|\nabla u|\,|\nabla^2 u|\,|\nabla\eta|\,d\mathbf{X}, \qquad |\mathtt{E}_6| \lesssim \iiint |\partial_t A|\,|\nabla u|^2 u\eta\,d\mathbf{X},$$
$$|\mathtt{E}_7| \lesssim \iiint |\nabla A|\,|\nabla^2 u|\,|\nabla u|\,u\eta\,d\mathbf{X}, \qquad |\mathtt{E}_8| \lesssim \iiint |\nabla A|^2|\nabla u|^2\,u\eta\,d\mathbf{X}, \qquad |\mathtt{E}_9| \lesssim \iiint |\nabla u|^3|\nabla A|\,\eta\,d\mathbf{X}.$$

Indeed, each term has the control $|\mathtt{E}_i| \lesssim |Q(\mathbf{S}^*)|$. To show that, first use point-wise bounds from Lemma 5.32, (5.22) and (5.53), and then finish by integrating with the nice properties of the derivatives of η in (5.56) for the terms $\mathtt{E}_1$–$\mathtt{E}_5$, and the Carleson bounds for A in (5.24) for the terms $\mathtt{E}_6$–$\mathtt{E}_9$ (to use this, note that the region $\Omega^{***}_{\mathbf{S}^*}$ can be contained in a cube of the form $\mathcal{J}_{C\ell(Q(\mathbf{S}^*))}(\mathbf{X}_{\mathbf{S}^*})$, where $C = C(K)$ is a large constant, and $\mathbf{X}_{\mathbf{S}^*}$ is any point in $U^{**}_{Q(\mathbf{S}^*)}$).

We are going to modify the argument in [LN07] to our variable coefficient setting. In particular we introduce two quantities (where subscripts of u mean derivatives)

$$\alpha := \iiint u\,(a_{i,j}u_{x_i,x_k})\,(a_{k,\ell}u_{x_j,x_\ell})\,\eta, \qquad \beta := \iiint u\,(u_t)^2\eta.$$

Here we have adopted Einstein summation notation for convenience. The ellipticity of A yields (this requires some elementary linear algebra work, which we defer to Section 5.13)

$$\alpha \approx \iiint u\,|\nabla^2 u|^2\eta. \tag{5.60}$$

Thus, since $\alpha,\beta \geqslant 0$, to prove (5.59), it is enough to show that

$$\alpha + 2\beta = \mathtt{E}, \tag{5.61}$$

where $\mathtt{E}$ satisfies the estimate $|\mathtt{E}| \lesssim |Q(\mathbf{S}^*)|$.

Let us begin by treating β. First using the equation $L^*u = 0$, then integrating by parts, we obtain

$$\beta = \iiint -(\operatorname{div} A\nabla u)\,u_t u\eta = \iiint A\nabla u\cdot\nabla(u_t u\eta) = \mathtt{E} + \iiint A\nabla u\cdot\nabla u_t\,u\eta + \iiint A\nabla u\cdot\nabla u\,u_t\eta,$$

where, after using the product rule, the error term is of type $\mathtt{E}_2$. Next, we will use this fact:

$$(1/2)\,\partial_t(A\nabla u\cdot\nabla u) = A\nabla u\cdot\nabla u_t + (1/2)(\partial_t A)\nabla u\cdot\nabla u. \tag{5.62}$$

To verify it, just expand and remember that A **is symmetric**. Using this identity to continue rewriting β, it turns out that, up to a new error term of type $\mathtt{E}_6$, we have

$$\beta = E + \iiint (1/2)\,\partial_t(A\nabla u\cdot\nabla u)\,u\eta + \iiint A\nabla u\cdot\nabla u\,u_t\eta.$$

Finally, integrating by parts in t on the first term we have, up to a new "error term" of type $\mathtt{E}_1$,

$$\beta = \mathtt{E} - \frac{1}{2}\iiint A\nabla u\cdot\nabla u\,u_t\eta + \iiint A\nabla u\cdot\nabla u\,u_t\eta = \mathtt{E} + \frac{1}{2}\iiint A\nabla u\cdot\nabla u\,u_t\eta. \tag{5.63}$$

Next, our goal is to show that α is nearly of the same form as β, to get cancellations. First, using the product rule both for $\partial_{x_k}(a_{i,j}u_{x_i})$ and $\partial_{x_j}(a_{k,\ell}u_{x_\ell})$ simultaneously, we obtain

$$\alpha = \iiint u\,\partial_{x_k}(a_{i,j}u_{x_i})\,\partial_{x_j}(a_{k,\ell}u_{x_\ell})\,\eta - \mathtt{E}_8 - \mathtt{E}_7 - \mathtt{E}_7,$$

and we will abbreviate the errors by $\mathtt{E}$. Next, we integrate by parts in x_k and use the product rule to obtain, up to a new error term of type $-\mathtt{E}_4 - \mathtt{E}_5$ (when ∂_{x_k} lands on η),

$$\alpha = E - \iiint u_{x_k}(a_{i,j}u_{x_i})\,\partial_{x_j}(a_{k,\ell}u_{x_\ell})\,\eta - \iiint u\,(a_{i,j}u_{x_i})\,\partial_{x_k}\partial_{x_j}(a_{k,\ell}u_{x_\ell})\,\eta =: \mathtt{E} + I + II.$$

We estimate I and II separately, starting with II. Using the equation $L^*u = 0$ (and exchanging the order of derivatives when needed) and rewriting the resulting expression using that A is symmetric, then invoking (5.62) (and hence introducing an error term of type $\mathtt{E}_6$), and lastly integrating by parts in t (introducing an error of type $\mathtt{E}_1$), we have

$$\begin{aligned} II = -\iiint u\,(a_{i,j}u_{x_i})\,\partial_{x_j}\partial_{x_k}(a_{k,\ell}u_{x_\ell})\,\eta = \iiint u\,(a_{i,j}u_{x_i})\,\partial_{x_j}u_t\,\eta = \iiint u\,A\nabla u\cdot\nabla u_t\,\eta \\ = \mathtt{E} + \frac{1}{2}\iiint u\,\partial_t(A\nabla u\cdot\nabla u)\,\eta = \mathtt{E} - \frac{1}{2}\iiint A\nabla u\cdot\nabla u\,u_t\eta. \end{aligned}$$

We have a desirable estimate for II and now turn to I. We start by integrating by parts in x_j, to obtain, up to a new error term of type $\mathtt{E}_3$,

$$I = \mathtt{E} + \iiint u_{x_k,x_j}(a_{i,j}u_{x_i})(a_{k,\ell}u_{x_\ell})\,\eta + \iiint u_{x_k}\partial_{x_j}(a_{i,j}u_{x_i})\,a_{k,\ell}u_{x_\ell}\,\eta =: \mathtt{E} + I_1 + I_2,$$

By the equation $L^*u = 0$ and symmetry of A, we simply compute

$$I_2 = -\iiint A\nabla u\cdot\nabla u\,u_t\eta.$$

To handle I_1 and get some cancellation with I_2 in this form, we use a similar identity to (5.62),

$$(1/2)\,\partial_{x_k}(A\nabla u\cdot\nabla u) = A\nabla u\cdot\nabla u_{x_k} + (1/2)\,(\partial_{x_k}A)\nabla u\cdot\nabla u.$$

Now, rewrite I_1, then use this identity (introducing an error term of type $\mathtt{E}_9$), next integrate by parts in ∂_{x_k} (introducing a new error term of type $\mathtt{E}_3$), and lastly use the equation $L^*u = 0$, we obtain

$$\begin{aligned} I_1 = \iiint A\nabla u_{x_k}\cdot\nabla u\,(a_{k,\ell}u_{x_\ell})\,\eta = \mathtt{E} + \frac{1}{2}\iiint \partial_{x_k}(A\nabla u\cdot\nabla u)\,(a_{k,\ell}u_{x_\ell})\,\eta \\ = \mathtt{E} - \frac{1}{2}\iiint A\nabla u\cdot\nabla u\,\partial_{x_k}(a_{k,\ell}u_{x_\ell})\,\eta = \mathtt{E} + \frac{1}{2}\iiint A\nabla u\cdot\nabla u\,u_t\eta. \end{aligned}$$

Joining all the previous computations, we obtain

$$I = \mathtt{E} - \frac{1}{2}\iiint A\nabla u\cdot\nabla u\,u_t\eta.$$

and combining this with our estimate for II before, we end up with

$$\alpha = \mathtt{E} - \iiint A\nabla u\cdot\nabla u\,u_t\eta,$$

which together with (5.63) yield the cancellation needed to obtain (5.61) and finish the proof. □

Proof of Lemma 5.34 under the hypotheses of Theorem 5.3. In this proof we are going to follow more closely the ideas in [HMT17, Proposition 4.36]. Again, (5.58) and (5.57) only differ in which cut-off function we use in Lemma 5.33, so we only prove (5.57). Let η be the cut-off function for $\mathbf{S}^*_{N_1,N_2}$ introduced in Lemma 5.33.

First, let us show that only the spatial derivatives are relevant. Indeed, using the equation $L^*u = 0$ (which we can use in a pointwise sense because the pole of u is far from the region we are working in, see Remark 5.35) and the product rule (along with the boundedness of A), we obtain

$$\begin{aligned}\iiint\limits_{\substack{\Omega^{**}\\ \mathbf{S}^*_{N_1,N_2}}} |\partial_t u(\mathbf{X})|^2 \operatorname{dist}(\mathbf{X},\partial\Omega)\, d\mathbf{X} &= \iiint\limits_{\substack{\Omega^{**}\\ \mathbf{S}^*_{N_1,N_2}}} |\operatorname{div}(A^*\nabla u)|^2 \operatorname{dist}(\mathbf{X},\partial\Omega)\, d\mathbf{X}\\ &\lesssim \iiint\limits_{\substack{\Omega^{**}\\ \mathbf{S}^*_{N_1,N_2}}} |\nabla A|^2|\nabla u|^2 \operatorname{dist}(\mathbf{X},\partial\Omega)\, d\mathbf{X} + \iiint\limits_{\substack{\Omega^{**}\\ \mathbf{S}^*_{N_1,N_2}}} |\nabla^2 u|^2 \operatorname{dist}(\mathbf{X},\partial\Omega)\, d\mathbf{X},\end{aligned}$$

and the first term is $\lesssim |Q(\mathbf{S}^*)|$ by (5.32) and our hypothesis on A, namely (5.25). Therefore, it suffices to deal with the spatial derivatives. Moreover, by the properties of η (concretely (5.53)) and (5.42) (recalling that $N_1 = 10N$ to use that estimate) it suffices to show

$$\iiint |\nabla^2 u(\mathbf{X})|^2\, v(\mathbf{X})\, \eta\, d\mathbf{X} \lesssim |Q(\mathbf{S}^*)|.$$

In fact, denoting by $\partial u := \partial_{x_k} u$ an arbitrary spatial derivative of u, this reduces to

$$\iiint |\nabla \partial u(\mathbf{X})|^2\, v(\mathbf{X})\, \eta\, d\mathbf{X} \lesssim |Q(\mathbf{S}^*)|. \tag{5.64}$$

Using the ellipticity of A and the product rule, we first obtain

$$\begin{aligned}\iiint |\nabla\partial u|^2\, v\eta &\approx \iiint A^*\nabla(\partial u)\cdot\nabla(\partial u)\, v\eta\\ &= \iiint A^*\nabla(\partial u)\cdot\nabla(\partial u v\eta) - \frac{1}{2}\iiint A^*\nabla((\partial u)^2)\cdot\nabla(v\eta) =: I + II. \end{aligned}\tag{5.65}$$

Below we let $\mathtt{E}$ be a generic error term (i.e. $|\mathtt{E}| \lesssim |Q(\mathbf{S}^*)|$), as was in the proof of Lemma 5.34 under the hypotheses of Theorem 5.2. We note that the terms $\mathtt{E}_i$ ($i = 1, 2, \dots 9$) are still under control if we replace any occurrence of u (including its derivatives) with v, as these terms are controlled using the Carleson/size conditions on A (Definition 5.24), properties of η (Lemma 5.33), and Lemma 5.32.

Now we continue by handling term I. First, we compute

$$I = \iiint \partial(A^*(\nabla u))\cdot\nabla(\partial u v\eta) - \iiint \partial(A^*)(\nabla u)\cdot\nabla(\partial u v\eta) = -\iiint A^*\nabla u\cdot\nabla\partial(\partial u v\eta) + \mathtt{E},$$

where in the first equality we have used the product rule; and in the second equality, we have integrated the first term by parts, and we have expanded the gradient in the second term to identify an error term of type $\mathtt{E}_4 + \mathtt{E}_7 + \mathtt{E}_9$. Then, using the equation $L^*u = 0$ and integrating by parts again:

$$I = -\iiint \partial_t u\, \partial(\partial u v\eta) + \mathtt{E} = \iiint \partial_t(\partial u)\, \partial u\, v\eta + \mathtt{E}.$$

Note the difference with respect to the computation in [HMT17]: we get an extra term, apart from the error term. Our goal now is to cancel it with II.

Now we treat II. Start by using the duality between A and A^*, and then the product rule to get

$$II = -\frac{1}{2}\iiint A\nabla(v\eta)\cdot\nabla((\partial u)^2) = -\frac{1}{2}\iiint A\nabla v\cdot\nabla((\partial u)^2\eta) + \frac{1}{2}\iiint A\nabla v\cdot\nabla v(\partial u)^2 \\ - \frac{1}{2}\iiint A\nabla\eta\cdot\nabla((\partial u)^2)\,v = \frac{1}{2}\iiint \partial_t v\,(\partial u)^2\eta + \mathtt{E},$$

where in the last equality we have used the equation $Lv = 0$ in the first term, and we have noticed that the last two terms are $\mathtt{E}_3 + \mathtt{E}_5$. Then, integrate by parts in t and use the product rule

$$II = -\frac{1}{2}\iiint v\,\partial_t((\partial u)^2\eta) + \mathtt{E} = \iiint v\,\partial_t(\partial u)\,\partial u\,\eta - \frac{1}{2}\iiint v\,\partial_t v\,(\partial u)^2 + \mathtt{E},$$

where we notice the last term is an error of type $\mathtt{E}_1$. Therefore, putting I and II back in (5.65), we obtain the desired cancellation which yields

$$I + II = \mathtt{E},$$

which shows (5.64) and finishes the proof. □

In the sequel, we make no distinction between whether we assume Theorem 5.2 or Theorem 5.3.

Next, let us obtain a consequence of Lemma 5.32, where the square function estimates take exactly the form that we will need later in Subsection 5.10.5.

Lemma 5.36. *Suppose that the hypotheses of Theorem 5.2 or Theorem 5.3 hold. Let $K = 2^N$, with $N \geqslant 2000$ being sufficiently large, and $N_1 := 10N$. Then, for all $N_2 \geqslant N_1$, it holds*

$$\iiint_{\Omega^*_{\mathbf{S}^*_{N_1,N_2}}} \left(|\operatorname{dist}(\cdot,\partial\Omega)\,\partial_t u|^2 + |\operatorname{dist}(\cdot,\partial\Omega)\,\nabla_X^2 u|^2\right)\frac{d\mathbf{X}}{\operatorname{dist}(\cdot,\partial\Omega)(\mathbf{X})} \\ + \iiint_{\Omega^*_{\mathbf{S}^*_{N_1,N_2}}} \left(|\operatorname{dist}(\cdot,\partial\Omega)^2\,\partial_t u\,\nabla_X^2 u|^2 + |\operatorname{dist}(\cdot,\partial\Omega)^2\,\nabla_X\partial_t u|^2\right)\frac{d\mathbf{X}}{\operatorname{dist}(\cdot,\partial\Omega)(\mathbf{X})} \lesssim |Q(\mathbf{S}^*)|, \quad (5.66)$$

with implicit constants depending on the structural constants and K, and independent of N_2.

Moreover, if $\mathbf{S} \subseteq \mathbf{S}^*$ is semi-coherent and $\ell(Q(\mathbf{S})) \leqslant 2^{-10N}\ell(Q(\mathbf{S}^*))$, then for any $N_3 > 0$

$$\iiint_{\Omega^*_{\mathbf{S}_{0,N_3}}} \left(|\operatorname{dist}(\cdot,\partial\Omega)\,\partial_t u|^2 + |\operatorname{dist}(\cdot,\partial\Omega)\,\nabla_X^2 u|^2\right)\frac{d\mathbf{X}}{\operatorname{dist}(\cdot,\partial\Omega)(\mathbf{X})} \\ + \iiint_{\Omega^*_{\mathbf{S}_{0,N_3}}} \left(|\operatorname{dist}(\cdot,\partial\Omega)^2\,\partial_t u\,\nabla_X^2 u|^2 + |\operatorname{dist}(\cdot,\partial\Omega)^2\,\nabla_X\partial_t u|^2\right)\frac{d\mathbf{X}}{\operatorname{dist}(\cdot,\partial\Omega)(\mathbf{X})} \lesssim |Q(\mathbf{S})|, \quad (5.67)$$

with implicit constants depending on the structural constants and K, and independent of N_3.

Proof. Let us only show (5.66), because (5.67) is analogous in view of Lemma 5.34. In turn, the first two terms in (5.66) are exactly those in Lemma 5.34. For the third one, we use the pointwise interior estimates from Lemma 5.32 (along with Lemma 5.34)

$$\iiint\limits_{\Omega^*_{\mathbf{s}^*_{N_1,N_2}}} |\operatorname{dist}(\cdot,\partial\Omega)^2\,\partial_t u\,\nabla_X^2 u|^2\,\frac{d\mathbf{X}}{\operatorname{dist}(\cdot,\partial\Omega)(\mathbf{X})} \lesssim \iiint\limits_{\Omega^*_{\mathbf{s}^*_{N_1,N_2}}} |\nabla_X^2 u|^2\operatorname{dist}(\cdot,\partial\Omega)\,d\mathbf{X} \lesssim |Q(\mathbf{S}^*)|.$$

The fourth one is more elaborate, and will follow from a Caccioppoli-type inequality for $\partial_t u$. However, since our PDE have variable coefficients, $\partial_t u$ does not solve a homogeneous PDE, and we will need to use our hypothesis on A (see Definition 5.24) to finish the argument. Let us include the details. Using a (parabolic) Whitney decomposition (with cubes as in (5.3)), we estimate

$$\iiint\limits_{\Omega^*_{\mathbf{s}^*_{N_1,N_2}}} |\operatorname{dist}(\mathbf{X},\partial\Omega)^2\nabla_X\partial_t u(\mathbf{X})|^2\frac{d\mathbf{X}}{\operatorname{dist}(\mathbf{X},\partial\Omega)} \lesssim \sum_{\mathcal{J}:\mathcal{J}\cap\Omega^*_{\mathbf{s}^*_{N_1,N_2}}\neq\varnothing} \ell(\mathcal{J})^3 \iiint\limits_{\mathcal{J}} |\nabla_X\partial_t u|^2. \tag{5.68}$$

Now write $v := \partial_t u$, and fix one such Whitney cube $\mathcal{J}$. We proceed to prove a Caccioppoli inequality. Find a cut-off $\phi = \phi_{\mathcal{J}} \in C_c^\infty(\mathbb{R}^n)$ satisfying $\mathbf{1}_{\mathcal{J}} \leqslant \phi \leqslant \mathbf{1}_{2\mathcal{J}}$ and $\ell(\mathcal{J})|\nabla_X\phi| + \ell(\mathcal{J})^2|\partial_t\phi| \lesssim 1$. Then compute

$$\iiint\limits_{2\mathcal{J}} |\nabla_X v|^2\phi^2 \lesssim \iiint\limits_{2\mathcal{J}} A\nabla_X v\cdot\nabla_X v\phi^2 = \iiint\limits_{2\mathcal{J}} A\nabla_X v\cdot\nabla_X(v\phi^2) - \iiint\limits_{2\mathcal{J}} A\nabla_X v\cdot\nabla_X(\phi^2)v =: I + II.$$

To deal with II, use Young's inequality with $\varepsilon_1 > 0$

$$|II| \lesssim \varepsilon_1 \iiint\limits_{2\mathcal{J}} |\nabla_X v|^2\phi^2 + \varepsilon_1^{-1}\iiint\limits_{2\mathcal{J}} v^2|\nabla_X\phi|^2 \lesssim \varepsilon \iiint\limits_{2\mathcal{J}} |\nabla_X v|^2\phi^2 + \varepsilon^{-1}\ell(\mathcal{J})^{-2}\iiint\limits_{2\mathcal{J}} v^2,$$

and for ε_1 small enough we will be able to later hide the first term. For the term I, note that differentiating the equation that u solves, we obtain that v solves the (inhomogeneous) equation $\partial_t v - \operatorname{div}(A\nabla_X v) = \operatorname{div}((\partial_t A)\nabla_X u)$. Therefore, interpreting that equation in the weak sense, we get

$$I = -\iiint\limits_{2\mathcal{J}} v\partial_t(v\phi^2) + \iiint\limits_{2\mathcal{J}} (\partial_t A)\nabla_X u\cdot\nabla_X(v\phi^2) =: I_1 + I_2.$$

The term I_1 is easy to handle. Using the product rule and an integration by parts, we obtain

$$|I_1| \leqslant |\frac{1}{2}\iiint\limits_{2\mathcal{J}} \partial_t(v^2)\phi^2| + |\iiint\limits_{2\mathcal{J}} v^2\partial_t(\phi^2)| = \frac{3}{2}|\iiint\limits_{2\mathcal{J}} v^2\partial_t(\phi^2)| \lesssim \ell(\mathcal{J})^{-2}\iiint\limits_{2\mathcal{J}} v^2,$$

In turn, for I_2, using the product rule and Young's inequality with $\varepsilon_2 > 0$, we get

$$I_2 = \iiint\limits_{2\mathcal{J}} (\partial_t A)\nabla_X u\cdot\nabla_X v\phi^2 + 2\iiint\limits_{2\mathcal{J}} (\partial_t A)\nabla_X u\cdot\nabla_X\phi v\phi$$

$$\lesssim \varepsilon_2 \iiint_{2\mathcal{J}} |\nabla_X v|^2\phi^2 + \varepsilon_2^{-1} \iiint_{2\mathcal{J}} |\partial_t A|^2|\nabla_X u|^2\phi^2 + \iiint_{2\mathcal{J}} v^2|\nabla_X \phi|^2 + \iiint_{2\mathcal{J}} |\partial_t A|^2|\nabla_X u|^2\phi^2$$
$$\leqslant \varepsilon_2 \iiint_{2\mathcal{J}} |\nabla_X v|^2\phi^2 + \ell(\mathcal{J})^{-2} \iiint_{2\mathcal{J}} v^2 + C(\varepsilon_2) \iiint_{2\mathcal{J}} |\partial_t A|^2,$$

where in the last estimate we have used the estimates for u from Lemma 5.32 (because points in $\mathcal{J}$ fall into the assumptions of this lemma because $\mathcal{J} \cap \Omega^*_{\mathbf{S}^*_{N_1,N_2}} \neq \varnothing$ with N_1 large enough). Then, for ε_2 small enough, we will hide the first term.

Putting all the above estimates together back in (5.68) (and hiding the appropriate terms as mentioned before), and recalling that $\phi_{\mathcal{J}} \equiv 1$ in $\mathcal{J}$ and that $v = \partial_t u$, we obtain

$$\iiint_{\Omega^*_{\mathbf{S}^*_{N_1,N_2}}} |\operatorname{dist}(\mathbf{X}, \partial\Omega)^2 \nabla_X \partial_t u(\mathbf{X})|^2 \frac{d\mathbf{X}}{\operatorname{dist}(\mathbf{X}, \partial\Omega)} \lesssim \sum_{\mathcal{J}:\mathcal{J}\cap\Omega^*_{\mathbf{S}^*_{N_1,N_2}} \neq \varnothing} \ell(\mathcal{J})^3 \iiint_{2\mathcal{J}} |\nabla_X \partial_t u|^2 \phi_{\mathcal{J}}^2$$
$$\lesssim \sum_{\mathcal{J}:\mathcal{J}\cap\Omega^*_{\mathbf{S}^*_{N_1,N_2}} \neq \varnothing} \ell(\mathcal{J})^3 \left(\ell(\mathcal{J})^{-2} \iiint_{2\mathcal{J}} |\partial_t u|^2 + \iiint_{2\mathcal{J}} |\partial_t A|^2 \right)$$
$$\lesssim \iiint_{\Omega^{**}_{\mathbf{S}^*_{N_1,N_2}}} |\partial_t u|^2 \operatorname{dist}(\cdot, \partial\Omega)\, d\mathbf{X} + \iiint_{\Omega^{**}_{\mathbf{S}^*_{N_1,N_2}}} |\partial_t A|^2 \operatorname{dist}(\cdot, \partial\Omega)^3\, d\mathbf{X} \lesssim |Q(\mathbf{S}^*)|,$$

where in the next-to-last step, we used that, for a sufficiently large choice of the parameter K, all the doubled Whitney boxes $2\mathcal{J}$, appearing in the sum, are contained in $\Omega^{**}_{\mathbf{S}^*_{N_1,N_2}}$, and then in the last step we used Lemma 5.34 and our hypothesis on A (namely (5.25)). □

5.8.2 Refinement of the corona decomposition to obtain vertical growth of u

We also deduce the following from Lemma 5.34.

Lemma 5.37. *Let $\mathbf{S}^*$ from Lemma 5.27 (or 5.28). Suppose $K = 2^N$, with $N \geqslant 2000$ sufficiently large. Then there exists a constant $C'' \geqslant 1$ (depending on K) such the following holds. The stopping time regime $\mathbf{S}^*$ can be decomposed as $\mathbf{S}^* = \mathcal{G}_{\mathbf{S}^*} \sqcup \mathcal{B}_{\mathbf{S}^*}$, where the "bad cubes" $\mathcal{B}_{\mathbf{S}^*}$ contain all the cubes $Q \in \mathbf{S}^*$ with $\ell(Q) \geqslant 2^{-10N}\ell(Q(\mathbf{S}^*))$ and moreover satisfy a packing condition*

$$\sum_{Q\in\mathcal{B}_{\mathbf{S}^*}, Q\subseteq P} |Q| \leqslant C''|P|, \quad \forall P \in \mathbb{D} \tag{5.69}$$

and for the "good cubes" $Q \in \mathcal{G}_{\mathbf{S}^*}$ it holds (for c_1 from (5.43))

$$\partial_{y_0} u(\mathbf{Y}) \geqslant c_1/8, \qquad \forall \mathbf{Y} \in U_Q. \tag{5.70}$$

Proof. Let $\varepsilon > 0$ be small to be chosen, possibly depending on the structural constants and K. We define $\mathcal{B}_{\mathbf{S}^*}$ as the collection of cubes $Q \in \mathbf{S}^*$ that are either "large"

$$\ell(Q) \geqslant 2^{-10N}\ell(Q(\mathbf{S}^*)) = K^{-10}\ell(Q(\mathbf{S}^*)) \tag{5.71}$$

or the Green's function u has some (undesired) oscillation at their scale, in the form

$$\iiint_{U_Q^*} (|\nabla^2 u|^2 + |\partial_t u|^2)\,\mathrm{dist}(\mathbf{X}, \partial\Omega)\; d\mathbf{X} \geqslant \varepsilon |Q|. \tag{5.72}$$

The cubes in $\mathcal{B}_{\mathbf{S}*}$ satisfying (5.71) pack trivially. Then, if we call $\mathcal{B}'_{\mathbf{S}*}$ the cubes for which (5.72) holds, we may use the bounded overlap of the U_Q^* regions and (5.58) to obtain

$$\sum_{Q\in\mathcal{B}'_{\mathbf{S}*}, Q\subseteq P} |Q| \leqslant \varepsilon^{-1} \sum_{Q\in\mathcal{B}'_{\mathbf{S}*}, Q\subseteq P} \iiint_{U_Q^*} (|\nabla^2 u|^2 + |\partial_t u|^2)\,\mathrm{dist}(\mathbf{X}, \partial\Omega)\, d\mathbf{X} \lesssim \varepsilon^{-1}|P|, \quad \forall P \in \mathbb{D}, \tag{5.73}$$

Indeed, to use (5.58), we may basically assume that $P \in \mathbf{S}^*$ with $\ell(P) \leqslant 2^{-10N}\ell(Q(\mathbf{S}^*))$; for if this were not the case (let us quickly sketch the standard argument) then either $\{Q \in \mathcal{B}'_{\mathbf{S}*}, Q \subseteq P\} = \varnothing$, or else we could apply the estimate to the collection $\{P'\}$ of maximal cubes in $\{Q \in \mathcal{B}'_{\mathbf{S}*}, Q \subseteq P\}$ and sum over P', which are disjoint subcubes of P.

In any case, (5.73) above shows the packing of $\mathcal{B}_{\mathbf{S}*}$ with constant depending on ε. Therefore, we need to show that (5.70) holds for $Q \in \mathbf{S}^*\backslash\mathcal{B}_{\mathbf{S}*} =: \mathcal{G}_{\mathbf{S}*}$, provided ε is chosen sufficiently small.

To this end, for the sake of a contradiction, suppose there is $\mathbf{X} = (x_0, \mathbf{x}) \in U_Q$ for $Q \in \mathcal{G}_{\mathbf{S}*}$ with

$$\partial_{x_0} u(\mathbf{X}) \leqslant c_1/8.$$

Then, by construction of U_Q (see (5.16)), it holds $\mathbf{x} \in 10^5 Q$. Therefore, Lemma 5.31 gives the existence of some point $\mathbf{Y} = (y_0, \mathbf{x}) \in U_Q$ such that

$$\partial_{x_0} u(\mathbf{Y}) \geqslant c_1.$$

Now, by the estimates for the second derivatives of u shown in Lemma 5.32 (note that, since $Q \in \mathcal{G}_{\mathbf{S}*}$, it holds $\ell(Q) \leqslant K^{-10}\ell(Q(\mathbf{S}^*))$), there exists a small enough $c = c(K) > 0$ so that

$$\partial_{x_0} u(\mathbf{X}') \leqslant c_1/4, \quad \forall \mathbf{X}' \in \mathcal{J}_{c\ell(Q)}(\mathbf{X}), \quad \text{and} \quad \mathcal{J}_{c\ell(Q)}(\mathbf{X}) \subseteq U_Q^*, \tag{5.74}$$

and similarly

$$\partial_{x_0} u(\mathbf{Y}') \geqslant c_1/2, \quad \forall \mathbf{Y}' \in \mathcal{J}_{c\ell(Q)}(\mathbf{Y}), \quad \text{and} \quad \mathcal{J}_{c\ell(Q)}(\mathbf{Y}) \subseteq U_Q^*. \tag{5.75}$$

Now, recalling that $\mathcal{Q}_{c\ell(Q)}(\mathbf{x})$ is the projection of $\mathcal{J}_{c\ell(Q)}$ into space-time $\mathbb{R}^n$, define $\theta_0 := y_0 - x_0$, so that the fundamental theorem of calculus and the estimates (5.74) and (5.75) yield

$$c_1/4 \leqslant \frac{1}{|\mathcal{Q}_{c\ell(Q)}(\mathbf{x})|} \iint_{\mathcal{Q}_{c\ell(Q)}(\mathbf{x})} \int_0^{\theta_0} \partial_{z_0}^2 u(x_0 + z_0, \mathbf{z})\; dz_0\; d\mathbf{z} \lesssim_K |Q|^{-1} \iiint_{U_Q^*} |\nabla^2 u(\mathbf{Z})|\; d\mathbf{Z} \lesssim_K \varepsilon^{1/2},$$

where in the last estimate we have used Cauchy-Schwarz followed by the opposite inequality to (5.72), because this time $Q \in \mathcal{G}_{\mathbf{S}*}$ (and also the easy fact that $\mathrm{dist}(\mathbf{Z}, \partial\Omega) \approx_K \ell(Q)$ for $\mathbf{Z} \in U_Q^*$, coming from Lemma 5.5). Therefore, for $\varepsilon = \varepsilon(c_1, K)$ sufficiently small, we get the contradiction that finishes the proof of the lemma. □

Now let us break up $\mathcal{G}_{\mathbf{S}*}$ into stopping time regimes, so that we obtain a coronization.

Lemma 5.38. *In the conditions of Lemma 5.37, we can write*

$$\mathcal{G}_{\mathbf{S}^*} = \sqcup_j \mathbf{S}_j,$$

where each $\mathbf{S}_j$ is a semi-coherent stopping time regime and the maximal cubes $\{Q(\mathbf{S}_j)\}$ pack, i.e.,

$$\sum_{Q(\mathbf{S}_j):Q(\mathbf{S}_j)\subseteq P} |Q(\mathbf{S}_j)| \lesssim |P|, \quad \forall P \in \mathbb{D},$$

with implicit constants depending on the structural constants and K.

Proof. This proof is standard, too. Let us sketch it quickly. Starting at $Q(\mathbf{S}^*)$, we call $\mathcal{F} = \{Q_j\}_j$ the family of maximal cubes (with respect to containment) in $\mathcal{B}_{\mathbf{S}^*} \cap \mathbb{D}(Q(\mathbf{S}^*))$. Then, if we call the first stopping-time regime $\mathbf{S} := \mathbb{D}_{\mathcal{F},Q(\mathbf{S}^*)}$ as in (5.29), it is easy to see that $\mathbf{S}$ is semi-coherent (follows directly from maximality of $\mathcal{F}$). After this first step, we iterate the same stopping-time procedure below each Q_j, so that every time we find a new stopping-time regime $\mathbf{S}$, its top cube $Q(\mathbf{S})$ is right below some cube in $\mathcal{B}_{\mathbf{S}^*}$ –with the exception of the first level $Q(\mathbf{S}^*)$–. In this way, after iterating the procedure and exhausting $\mathbf{S}^*$, the top cubes $Q(\mathbf{S})$ pack because their measure is controlled by that of their parents, which are $\mathcal{B}_{\mathbf{S}^*}$ cubes which already pack by (5.69). □

5.8.3 Reduction to finding regular Lip(1,1/2) approximating graphs in each stopping-time

To conclude this section, we set the stage for the remainder of the proof of Theorems 5.2 and 5.3. In particular, Steps 3 and 4 are aimed at showing the following (concretely, it will follow from Lemmas 5.45 and 5.47, and Proposition 5.48).

Proposition 5.39. *Suppose that $K = 2^N$ with $N \geqslant 2000$ sufficiently large. Then there exist constants b_1, b_2 and C_1 (depending on K) such that the following holds. For each $\mathbf{S}$ produced in Lemma 5.38, there exists a regular* $\mathrm{Lip}(1,1/2)$ *function $\psi_{\mathbf{S}}$, with graph $\Sigma_{\mathbf{S}}$, such that*

$$\sup_{\mathbf{X}\in\mathbf{\Psi}(2Q)} \operatorname{dist}(\mathbf{X},\Sigma_{\mathbf{S}}) \leqslant C_1 \operatorname{diam}(Q), \quad \forall Q \in \mathbf{S}. \tag{5.76}$$

We claim that to prove Theorem 5.2 and 5.3 it suffices to prove Proposition 5.39. Indeed, by Lemma 5.9, we just need to see that we can take $\mathcal{B} := \cup_{\mathbf{S}^*}\mathcal{B}_{\mathbf{S}^*}$ and $\mathcal{G} := \cup_{\mathbf{S}^*}\mathcal{G}_{\mathbf{S}^*}$ and then use Lemma 5.38 in concert with either Lemma 5.29 or Lemma 5.30 to pack the maximal cubes in the stopping time (sub-)regimes $\{\mathbf{S}\}$.

5.9 Step 3: Construction of approximating graphs via a stopping-time distance

Having reduced matters to showing Proposition 5.39, we now fix $\mathbf{S}$ a "graph stopping time regime" from Lemma 5.38, aiming to produce the regular $\mathrm{Lip}(1,1/2)$ function $\psi_{\mathbf{S}}$. Note that we have still left $K = 2^N$ with $N \geqslant 2000$ as a parameter at our disposal (which will be fixed shortly): recall (see Lemma 5.37) that $\mathbf{S}$ contains only cubes Q with $\ell(Q) \leqslant 2^{-10N}\ell(Q(\mathbf{S}^*))$, so the parameter N will help us obtaining some "space" between our region $\mathbf{S}$ and the top scale $Q(\mathbf{S}^*)$ which will help us later when obtaining estimates.

In this section, we will first (along Subsection 5.9.1) focus on analyzing some geometrical properties of our Whitney regions to construct $\psi_{\mathbf{S}}$ using level lines of the Green's function u. The fact that those level lines only behave well on Whitney regions associated to each $\mathbf{S}$ introduces several technical difficulties, like the need to use stopping-time distances and cut-offs. In the second part (Subsection 5.9.2) we take advantage of the connection between the Green's function u and the graphs $\psi_{\mathbf{S}}$ that we just constructed, to transfer estimates (obtained from the PDE) from u to $\psi_{\mathbf{S}}$. From these, we will deduce that $\psi_{\mathbf{S}}$ is indeed Lipschitz, and approximates the original graph Σ at the scale of $\mathbf{S}$.

5.9.1 Construction of the approximating Lip(1,1/2) graphs

In contrast to [BHMN25], our sawtooth domains will not be defined in terms of a (regularized) distance to a 'contact set', instead we use a regularized version of a stopping time distance from [DS91]. For our fixed $\mathbf{S}$ we define

$$d(\mathbf{x}) := \inf_{Q\in\mathbf{S}} \big(\operatorname{dist}(\mathbf{x}, Q) + \operatorname{diam}(Q)\big), \qquad \mathbf{x}\in\mathbb{R}^n.$$

Then the function $d(\mathbf{x})$ is clearly $\mathrm{Lip}(1,1/2)$ with constant at most 1 since it is the point-wise infimum over such functions. We set also

$$F := \big\{\mathbf{x}\in\mathbb{R}^n : d(\mathbf{x}) = 0\big\}.$$

Unfortunately, he function $d(\mathbf{x})$ is not regular enough for us to carry out our computations later, so we prefer to work with the following regularization of d.

Lemma 5.40 ([BHHLN23a, Lemma 3.24]). *There exists a function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying*

$$d(\mathbf{x})/10 \leqslant h(\mathbf{x}) \leqslant C_n d(\mathbf{x}), \qquad \|h\|_{\mathrm{Lip}(1,1/2)} \lesssim 1, \qquad \|\mathcal{D}_t h\|_{\mathrm{BMO}(\mathbb{R})} \lesssim 1, \tag{5.77}$$

for a constant $C_n > 1$ (so concretely, $h(\mathbf{x}) = 0$ on F), and

$$d(\mathbf{x})^{2k-1}\,|\partial_t^k h(\mathbf{x})| + d(\mathbf{x})^{k-1}\,|\nabla_x^k h(\mathbf{x})| \lesssim_k 1, \qquad \forall\,\mathbf{x}\notin F,\ k\in\mathbb{N},$$

where the implicit constants and C_n depend only on n (and on k in the last estimate).

The purpose of the function $h(\mathbf{x})$ is to create a sawtooth (above its graph) where we have the non-degeneracy estimates for u. Roughly speaking, for each $\mathbf{x}$, we want to work with level sets of u that are roughly above $h(\mathbf{x})$, and ensure these points lie in our U_Q regions, because u behaves well there. Let us first use the stopping time distance to identify the cube with the correct scale.

Lemma 5.41. *Let $Q_0 \in \mathbf{S}$, and suppose that $\mathbf{x}\in 500Q_0$ and $\rho \in (h(\mathbf{x})/10, 10^{10}C_n \operatorname{diam}(Q_0))$. Then there exists a cube $Q\subseteq Q_0$, $Q\in\mathbf{S}$ such that*

$$\mathbf{x}\in 500Q \quad \text{and} \quad 10^{-10}\operatorname{diam}(Q) \leqslant \rho \leqslant 10^{10}C_n\operatorname{diam}(Q).$$

Proof. If $\rho \geqslant 10^{-10}\operatorname{diam}(Q_0)$, then we may just select $Q = Q_0$. Hence, assume $\rho < 10^{-10}\operatorname{diam}(Q_0)$.

To begin, since $\rho > h(\mathbf{x})/10$, Lemma 5.40 implies that $\rho > d(\mathbf{x})/100$. So by definition of $d(\mathbf{x})$ there exists some $Q'\in\mathbf{S}$ such that

$$\operatorname{dist}(\mathbf{x}, Q') + \operatorname{diam}(Q') < 100\rho.$$

Then, we may traverse one by one the ancestors of Q' until we find the minimal Q with $10^5\rho \leqslant \operatorname{diam}(Q)$, which will then satisfy

$$10^5\rho \leqslant \operatorname{diam}(Q) \leqslant 2\cdot 10^5\rho \leqslant 10^{-4}\operatorname{diam}(Q_0). \tag{5.78}$$

Therefore, $Q' \subseteq Q \subseteq Q_0$, whence $Q \in \mathbf{S}$ by semi-coherency. Also, since $Q' \subseteq Q$, we also have

$$\operatorname{dist}(\mathbf{x}, Q) \leqslant \operatorname{dist}(\mathbf{x}, Q') < 100\rho \leqslant 10^{-3} \operatorname{diam}(Q),$$

so $\mathbf{x} \in 500Q$. (In fact, (5.78) gives a better bound than when $\rho \geqslant 10^{-10} \operatorname{diam}(Q_0)$ and $Q = Q_0$.) □

Next, with regards to later putting together Lemmas 5.31 and 5.41, we choose $\delta \in (0, 1/2)$ with

$$\big(10^{-10} \operatorname{diam}(Q), 10^{10} C_n \operatorname{diam}(Q)\big) \subseteq \big(\delta \ell(Q), \delta^{-1} \ell(Q)\big). \tag{5.79}$$

And then, for this δ, we fix $N = N(\delta)$ as given by Lemma 5.31. This is our **final choice of N, which defines $K = 2^N$**. Clearly, N and K only depend on the structural constants; therefore we will not make dependencies on N or K explicit anymore.

Having fixed K, we collect some the information we have produced in a more concise format. First we have the following lemma.

Lemma 5.42 (Definition of $\psi(r;\cdot)$)**.** *Define the projections $\pi^{\perp} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ and $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ by*

$$\pi^{\perp}(x_0, \mathbf{x}) := x_0, \qquad \pi(x_0, \mathbf{x}) = \mathbf{x}.$$

Then, for any fixed $\mathbf{x} \in 10^5 Q(\mathbf{S})$,

$$\text{the set } \ \pi^{-1}(\mathbf{x}) \cap \Omega_{\mathbf{S}} \ \text{ is a line segment, } \quad \text{and} \quad \mathcal{I}_{\mathbf{x}} := \overline{\pi^{\perp}(\pi^{-1}(\mathbf{x}) \cap \Omega_{\mathbf{S}})} \ \text{ is an interval.}$$

Moreover, the function (of x_0) $u(x_0, \mathbf{x}) : \mathcal{I}_{\mathbf{x}} \to \mathbb{R}$ is strictly increasing. Therefore, we define, for $r \in u(\mathcal{I}_{\mathbf{x}}, \mathbf{x})$, the function $\psi(r;\mathbf{x})$ implicitly as the unique element in $\mathcal{I}_{\mathbf{x}}$ such that

$$u(\psi(r;\mathbf{x}), \mathbf{x}) = r.$$

For brevity in the notation, we also define

$$\mathbf{\Psi}(r;\mathbf{x}) := (\psi(r;\mathbf{x}), \mathbf{x}).$$

Proof. First, recall that $\Omega_{\mathbf{S}}$ is the union of U_Q with $Q \in \mathbf{S}$, and $\mathbf{S} \subseteq \mathcal{G}_{\mathbf{S}^*}$ (see Lemma 5.38). Therefore, Lemma 5.37 (specifically (5.70)) yields that $\partial_{y_0} u(y_0, \mathbf{y}) > 0$ for any $(y_0, \mathbf{y}) \in \Omega_{\mathbf{S}}$.

Now fix $\mathbf{x} \in 10^5 Q(\mathbf{S})$. We are only left with showing that $\pi^{-1}(\mathbf{x}) \cap \Omega_{\mathbf{S}}$ is a line segment or, equivalently, that $\mathcal{I}_{\mathbf{x}}$ is an interval. Fix $Q \in \mathbf{S}$. First note that we have (see (5.16))

$$\pi^{\perp}(\pi^{-1}(\mathbf{x}) \cap U_Q) = (\psi(\mathbf{x}) + K^{-1}\ell(Q), \psi(\mathbf{x}) + K\ell(Q)),$$

which is indeed an interval. Next we note that if Q^* is the parent of Q in $\mathbb{D}$, then their associated intervals (as in the last display) intersect because $\big(K^{-1}\ell(Q), K\ell(Q)\big) \cap \big(K^{-1}\ell(Q^*), K\ell(Q^*)\big) \neq \varnothing$ if K is large enough (depending on n). So going all the way up from Q to $Q(\mathbf{S})$ (by semi-coherency):

$$\bigcup_{Q' : Q \subseteq Q' \subseteq Q(\mathbf{S})} \pi^{\perp}(\pi^{-1}(\mathbf{x}) \cap U_{Q'}) = (\psi(\mathbf{x}) + K^{-1}\ell(Q), \psi(\mathbf{x}) + K\ell(Q(\mathbf{S}))),$$

which is an interval. Since the right endpoint of these intervals does not depend on Q, the arbitrariness of $Q \in \mathbf{S}$ in the last display implies that $\mathcal{I}_{\mathbf{x}}$ is an interval, hence finishing the proof. □

Without loss of generality (by a translation) we assume that, if $\mathbf{x}_{Q(\mathbf{S})}$ is the center of $Q(\mathbf{S})$,

$$\psi(h(\mathbf{x}_{Q(\mathbf{S})}); \mathbf{x}_{Q(\mathbf{S})}) = 0. \tag{5.80}$$

With that, we finally define the (candidate to) **approximating graph** using $\psi(r;\cdot)$ from Lemma 5.42:

$$\psi_{\mathbf{S}}(\mathbf{x}) := \psi^*_{\mathbf{S}}(\mathbf{x})\,\phi(\mathbf{x}), \qquad \text{where} \qquad \psi^*_{\mathbf{S}}(\mathbf{x}) := \psi(h(\mathbf{x}); \mathbf{x}), \tag{5.81}$$

where $\phi \in C^\infty_c(\mathbb{R}^n)$ is a cut-off function adapted to $Q(\mathbf{S})$, in the sense that

$$\mathbf{1}_{2Q(\mathbf{S})} \leqslant \phi \leqslant \mathbf{1}_{4Q(\mathbf{S})}, \quad \|\phi\|_{\mathrm{Lip}(1,1/2)} \lesssim \ell(Q(\mathbf{S}))^{-1}. \tag{5.82}$$

Actually, we may choose ϕ to satisfy

$$|\nabla^k_x \phi|\,\ell(Q(\mathbf{S}))^k + |\partial^k_t \phi|\,\ell(Q(\mathbf{S}))^{2k} \lesssim_k 1 \qquad \text{for } k = 1, 2, \dots \tag{5.83}$$

Now that we have defined $\psi_{\mathbf{S}}$, we need to show that it is regular $\mathrm{Lip}(1, 1/2)$, and that it approximates the original graph Σ in the sense of (5.76). As anticipated at the beginning of the section, it will be key to exploit the connection between $\psi_{\mathbf{S}}$ and u later. But since u only behaves well on sawtooth/Whitney regions, we need to check that our construction also works on those.

Indeed, for any given $Q_0 \in \mathbf{S}$, we define a localized stopping time by

$$\mathbf{S}(Q_0) := \{Q \in \mathbf{S} : Q \subseteq Q_0\}$$

Then we associate two sawtooths to $\mathbf{S}(Q_0)$ (see Figure 5.3): the 'usual' (dyadic) one, adapted to Ω,

$$\Omega_{\mathbf{S}(Q_0)} := \bigcup_{Q \in \mathbf{S}(Q_0)} U_Q \quad (\subseteq \Omega);$$

and the other is a continuous version, in the upper half space

$$\mathcal{S}(Q_0) := \big\{(r, \mathbf{x}) :\ x \in 500Q_0,\ \ r \in \big(h(\mathbf{x})/10, 10^{10} C_n \operatorname{diam}(Q_0)\big)\big\} \quad \big(\subseteq \mathbb{R}^{n+1}{}_+\big). \tag{5.84}$$

In the case that $Q_0 = Q(\mathbf{S})$ we omit Q_0 from our notation, e.g. $\mathcal{S} = \mathcal{S}(Q(\mathbf{S}))$. The reason for introducing these objects (and also for our choice of N and K in (5.79)) is apparent in the following lemma: $\psi(r;\cdot)$ "maps" the sawtooth $\mathcal{S}(Q_0)$ to the other sawtooth $\Omega_{\mathbf{S}(Q_0)}$.

Lemma 5.43. *The function $\psi(r;\mathbf{x})$ from Lemma 5.42 is well-defined for $(r,\mathbf{x}) \in \mathcal{S}$. Moreover, for any $Q_0 \in \mathbf{S}$, it holds*

$$(r, \mathbf{x}) \in \mathcal{S}(Q_0) \quad \Longrightarrow \quad (\psi(r;\mathbf{x}), \mathbf{x}) \in \Omega_{\mathbf{S}(Q_0)}.$$

Proof. Let $(r,\mathbf{x}) \in \mathcal{S}(Q_0)$. By definition of $\mathcal{S}(Q_0)$ and Lemma 5.41 it holds that there exists $Q \subseteq Q_0$ with $Q \in \mathbf{S}$ such that

$$\mathbf{x} \in 500Q \quad \text{and} \quad 10^{-10} \operatorname{diam}(Q) \leqslant r \leqslant 10^{10} C_n \operatorname{diam}(Q).$$

By Lemma 5.31 and the fact we have chosen K so that (5.79) holds, there exists $(x_0, \mathbf{x})$ such that

$$u(x_0, \mathbf{x}) = r, \quad \text{and} \quad (x_0, \mathbf{x}) \in U_Q.$$

This of course gives that $x_0 \in \mathcal{I}_{\mathbf{x}}$, as defined in Lemma 5.42. By the same Lemma 5.42, $\psi(r;\mathbf{x})$ is well-defined and it must be $\psi(r;\mathbf{x}) = x_0$. Hence $(\psi(r;\mathbf{x}), \mathbf{x}) = (x_0, \mathbf{x}) \in U_Q \subseteq \Omega_{\mathbf{S}(Q_0)}$, as desired. □

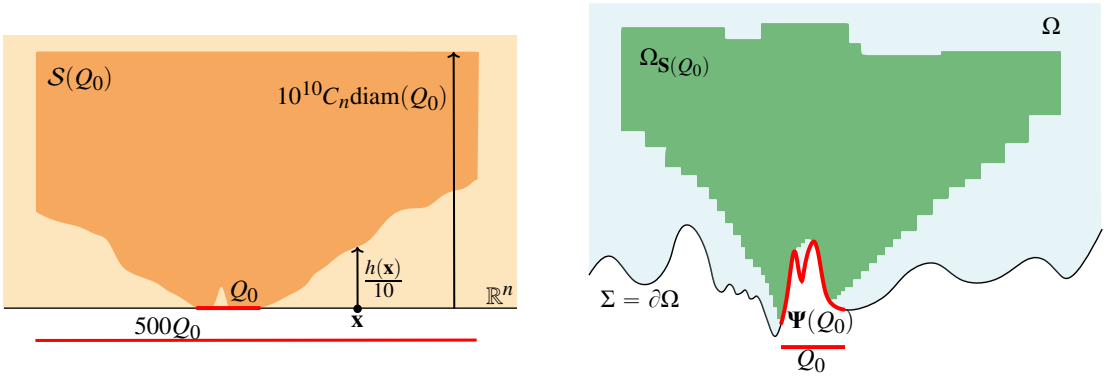


Figure 5.3: A (not-to-scale) depiction of a "continuous" sawtooth $S(Q_0)$ in $\mathbb{R}^{n+1}$ (the left figure) associated with the dyadic sawtooth $\Omega_{\mathbf{S}(Q_0)}$ inside Ω (the right figure). The map $\psi(r;\cdot)$ takes $S(Q_0)$ inside $\Omega_{\mathbf{S}(Q_0)}$, twisting it to adapt to the shape of Ω.

5.9.2 Pointwise estimates at level sets, and Lipschitzianity of the approximating graph

Next we want to look closer at the properties of $\psi(r,\mathbf{x})$ over $\mathcal{S}$. In fact, it will be crucial to exploit that $\psi(r;\cdot)$ has been defined in terms of level sets of the Green function, for which we have nice estimates in sawtooth/Whitney regions. Let us confirm that the estimates also apply to $\psi(r;\cdot)$.

Lemma 5.44 (Transference of estimates from u to $\psi(r;\cdot)$)**.** *Let $(r,y,s)\in\mathcal{S}$. Then,*

$$|\partial_s\psi(r;y,s)| \lesssim |(\partial_s u)(\mathbf{\Psi}(r;y,s))|, \quad |\nabla^2_{y,r}\psi(r;y,s)| \lesssim |(\nabla^2_Y u)(\mathbf{\Psi}(r;y,s))|, \tag{5.85}$$

$$|\nabla_{y,r}\partial_s\psi(r;y,s)| \lesssim |(\nabla_Y\partial_s u)(\mathbf{\Psi}(r;y,s))| + |(\partial_s u)(\mathbf{\Psi}(r;y,s))|\,|(\nabla^2_Y u)(\mathbf{\Psi}(r;y,s))|, \tag{5.86}$$

$$|\nabla_{y,r}\psi(r;y,s)| + r\,|\partial_s\psi(r;y,s)| + r\,|\nabla^2_{y,r}\psi(r;y,s)| + r^2\,|\nabla_{y,r}\partial_s\psi(r;y,s)| \lesssim 1. \tag{5.87}$$

Proof. Let us include the proof from [BHMN25, Lemma 5.18]. By Lemma 5.43, it holds

$$u(\psi(r;y,s),y,s) = r, \qquad (r,y,s)\in\mathcal{S}.$$

Differentiating implicitly this equation with respect to r yields

$$(\partial_{y_0}u)(\mathbf{\Psi}(r;y,s))\,\partial_r\psi(r;y,s) = 1, \quad \text{i.e.} \quad \partial_r\psi(r;y,s) = \frac{1}{(\partial_{y_0}u)(\mathbf{\Psi}(r;y,s))}, \tag{5.88}$$

and similarly, differentiating with respect to s and y_i ($i=1,\dots,n$), we obtain

$$\partial_s\psi(r;y,s) = -\frac{(\partial_s u)(\mathbf{\Psi}(r;y,s))}{(\partial_{y_0}u)(\mathbf{\Psi}(r;y,s))} \quad \text{and} \quad \partial_{y_i}\psi(r;y,s) = -\frac{(\partial_{y_i}u)(\mathbf{\Psi}(r;y,s))}{(\partial_{y_0}u)(\mathbf{\Psi}(r;y,s))}.$$

Recalling that $\partial_{y_0}u(\mathbf{\Psi}(r;y,s)) \approx 1$ for $(r,y,s)\in\mathcal{S}$ (a consequence of Lemma 5.37, which can be applied for points in $\mathcal{S}$ because of Lemma 5.43–, the estimates in (5.85) and (5.86) follow from the equations in the previous displays, differentiating them again when needed, and using the estimates in (5.48). The estimates in (5.85) follow similarly, using Lemma 5.32 and the fact that $\mathrm{dist}(\mathbf{\Psi}(r;y,s)) \approx u(\mathbf{\Psi}(r;y,s)) = r$ by (5.41). □

These pointwise estimates will be the key to confirm that $\psi_{\mathbf{S}}$ defined in (5.81) is $\mathrm{Lip}(1,1/2)$ and has approximates the original graph Σ in the required way (5.76). Let us start by the latter.

Lemma 5.45 ($\psi_{\mathbf{S}}$ approximates Σ)**.** *For every $\mathbf{x} \in 4Q(\mathbf{S})$ it holds that*

$$0 \leqslant \psi_{\mathbf{S}}(\mathbf{x}) - \psi(\mathbf{x}) \leqslant M_0 M_2 h(\mathbf{x}), \tag{5.89}$$

where M_2 is from Lemma 5.29 (or 5.30). In particular, (5.76) holds with $C_1 := 2C_n M_0 M_2$.

Proof. The fact that (5.76) follows from (5.89) is because if $Q \in \mathbf{S}$ and $\mathbf{x} \in 2Q$ (using Lemma 5.40)

$$(C_n)^{-1} h(\mathbf{x}) \leqslant d(\mathbf{x}) \leqslant \operatorname{dist}(\mathbf{x}, Q) + \operatorname{diam}(Q) \leqslant 2\operatorname{diam}(Q).$$

Next, let us show (5.89). Given $\mathbf{x} \in 4Q(\mathbf{S})$, Lemma 5.43 yields

$$u(\psi^*_{\mathbf{S}}(\mathbf{x}), \mathbf{x}) = h(\mathbf{x}), \quad \text{and} \quad (\psi^*_{\mathbf{S}}(\mathbf{x}), \mathbf{x}) \in \Omega_{\mathbf{S}}.$$

Then, by Lemma 5.5 and (5.40) (and also $\psi_{\mathbf{S}}(\mathbf{x}) \leqslant \psi^*_{\mathbf{S}}(\mathbf{x})$, see (5.81)), we get

$$\psi_{\mathbf{S}}(\mathbf{x}) - \psi(\mathbf{x}) \leqslant \psi^*_{\mathbf{S}}(\mathbf{x}) - \psi(\mathbf{x}) \leqslant M_0 \operatorname{dist}((\psi^*_{\mathbf{S}}(\mathbf{x}), \mathbf{x}), \Sigma) \leqslant M_0 M_2 u(\psi^*_{\mathbf{S}}(\mathbf{x}), \mathbf{x}) = M_0 M_2 h(\mathbf{x}),$$

which is the upper bound in (5.89); and the lower bound by 0 comes from $(\psi_{\mathbf{S}}(\mathbf{x}), \mathbf{x}) \in \Omega_{\mathbf{S}} \subseteq \Omega$. □

Now, turning to show $\psi_{\mathbf{S}}$ is $\mathrm{Lip}(1,1/2)$, we make an observation following from Lemma 5.44.

Lemma 5.46. *If $(x,t) \in \mathbb{R}^n$ and $h(x,t) > 0$, then*

$$|\partial_t \psi_{\mathbf{S}}(x,t)| \lesssim h(x,t)^{-1}.$$

Proof. From the definition of $\psi_{\mathbf{S}}$ in (5.81), we see that the estimate is trivial if $(x,t) \notin 4Q(\mathbf{S})$. Again recalling the definition of $\psi_{\mathbf{S}}$ and $\psi^*_{\mathbf{S}}$ (see (5.81)), Lemma 5.44 gives the estimate if $(x,t) \in 2Q(\mathbf{S})$.

Lastly, fix $\mathbf{x} = (x,t) \in 4Q(\mathbf{S}) \backslash 2Q(\mathbf{S})$. Then, it is easy to check that (having in mind Lemma 5.40 and the definition of d) $h(\mathbf{x}) \approx d(\mathbf{x}) \approx \operatorname{dist}(\mathbf{x}, Q(\mathbf{S})) \approx \ell(Q(\mathbf{S}))$. Also, Lemma 5.40 yields

$$|\partial_t h(x,t)| \lesssim d(x,t)^{-1} \approx h(x,t)^{-1}. \tag{5.90}$$

Then, by Lemma 5.44, the properties of ϕ, and that h satisfies (5.90), it holds

$$\begin{aligned}
|\partial_t \psi_{\mathbf{S}}(x,t)| &\leqslant |\partial_t \psi(h(x,t);x,t)| + |\psi(h(x,t);x,t)\, \partial_t \phi(x,t)| \\
&\lesssim |(\partial_{x_0}\psi)(h(x,t);x,t)\, \partial_t h(x,t)| + |(\partial_t \psi)(h(x,t);x,t)| + |\psi(h(x,t);x,t)\, \partial_t \phi(x,t)| \\
&\lesssim \ell(Q(\mathbf{S}))^{-1} + \ell(Q(\mathbf{S}))^{-1} + \ell(Q(\mathbf{S}))\ell(Q(\mathbf{S}))^{-2} \lesssim \ell(Q(\mathbf{S})) \approx h(x,t)^{-1},
\end{aligned}$$

where we used our assumption that $\psi^*_{\mathbf{S}}(\mathbf{x}_{Q(\mathbf{S})}) = 0$ in (5.80) (along with the computations in Lemma 5.45, and also $h(\mathbf{y}) \lesssim d(\mathbf{y}) \lesssim \ell(Q(\mathbf{S}))$ for $\mathbf{y} \in 4Q(\mathbf{S})$) to obtain

$$|\psi(h(\mathbf{x});\mathbf{x})| \leqslant |\psi^*_{\mathbf{S}}(\mathbf{x}) - \psi(\mathbf{x})| + |\psi(\mathbf{x}) - \psi(\mathbf{x}_{Q(\mathbf{S})})| + |\psi(\mathbf{x}_{Q(\mathbf{S})}) - \psi^*_{\mathbf{S}}(\mathbf{x}_{Q(\mathbf{S})})| \lesssim \ell(Q(\mathbf{S})).$$

□

Lemma 5.47 ($\psi_{\mathbf{S}}$ is Lipschitz)**.** *The function $\psi_{\mathbf{S}}$, as defined in (5.81), is* $\mathrm{Lip}(1,1/2)$.

Proof. We first show that $\psi_{\mathbf{S}}^*$ is Lip(1,1/2) when restricted to $4Q(\mathbf{S})$. To this end, fix $\varepsilon' \in (0,1)$ be small, to be chosen later only depending on n. Fix also $\mathbf{x}, \mathbf{y} \in 4Q(\mathbf{S})$.

First, if $\|\mathbf{x}-\mathbf{y}\| \geqslant \varepsilon' \max\{h(\mathbf{x}), h(\mathbf{y})\}$ then from (5.89) and the fact that ψ is Lip(1,1/2) we have

$$|\psi_{\mathbf{S}}^*(\mathbf{x}) - \psi_{\mathbf{S}}^*(\mathbf{y})| \lesssim \|\mathbf{x}-\mathbf{y}\| + h(\mathbf{x}) + h(\mathbf{y}) \lesssim_{\varepsilon'} \|\mathbf{x}-\mathbf{y}\|.$$

Therefore it suffices to treat the case $\|\mathbf{x}-\mathbf{y}\| < \varepsilon' \max\{h(\mathbf{x}), h(\mathbf{y})\}$. Without loss of generality we assume that $\max\{h(\mathbf{x}), h(\mathbf{y})\} = h(\mathbf{x})$.

We observe that since $h(\mathbf{x}) > 0$ and h is Lip(1,1/2), it is trivial to obtain

$$h(\mathbf{z}) \geqslant h(\mathbf{x})/2, \quad \forall \mathbf{z} : \|\mathbf{z}-\mathbf{x}\| \leqslant 10\,\varepsilon' h(\mathbf{x}) \tag{5.91}$$

if ε' is sufficiently small, merely depending on the Lip$(1,1/2)$ constant of h, which in turn only depends on n by Lemma 5.40. Actually, note that $\mathbf{y}$ lies well inside this set.

Now, using our estimates on $\psi(r;\mathbf{z})$ from Lemma 5.44, and our estimates on h from Lemma 5.40 it holds for all $\mathbf{z} = (z,\tau)$ with $\|\mathbf{z}-\mathbf{x}\| \leqslant 10\varepsilon' h(x)$ that

$$|\nabla_z \psi_{\mathbf{S}}^*(z,\tau)| = |\nabla_z \psi(h(z,\tau);z,\tau)| \leqslant |(\partial_r \psi)(h(z,\tau);z,\tau)|\,|\nabla_z h(z,\tau)| + |(\partial_z \psi)(h(z,\tau);z,\tau)| \lesssim 1,$$

and similarly (see Lemma 5.46), recalling also (5.91),

$$|\partial_\tau \psi_{\mathbf{S}}^*(z,\tau)| \lesssim h(z,\tau)^{-1} \lesssim h(\mathbf{x})^{-1}.$$

Then connecting $\mathbf{x}$ to $\mathbf{y}$ via a polygonal path made up of purely spatial and purely temporal line segments in the set $\{\mathbf{z} : \|\mathbf{z}-\mathbf{x}\| \leqslant 10\varepsilon' h(x)\}$ and using the previous two estimates gives that

$$|\psi_{\mathbf{S}}^*(\mathbf{x}) - \psi_{\mathbf{S}}^*(\mathbf{y})| \lesssim \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{x}-\mathbf{y}\|^2 h(\mathbf{x})^{-1} \lesssim_{\varepsilon'} \|\mathbf{x}-\mathbf{y}\|$$

since $h(\mathbf{x})^{-1} \lesssim_{\varepsilon'} \|\mathbf{x}-\mathbf{y}\|^{-1}$. So we have checked that $\psi_{\mathbf{S}}^*$ is Lip(1,1/2) when restricted to $4Q(\mathbf{S})$.

To continue, recall that we assumed $\psi(h(\mathbf{x}_{Q(\mathbf{S})});\mathbf{x}_{Q(\mathbf{S})}) = 0$ in (5.80). Therefore, since we just showed that $\psi_{\mathbf{S}}^*$ is Lip(1,1/2) in $4Q(\mathbf{S})$ (actually, the computations in Lemma 5.45 suffice), we have

$$|\psi_{\mathbf{S}}^*(\mathbf{x})| \lesssim \operatorname{diam}(Q(\mathbf{S})), \quad \forall \mathbf{x} \in 4Q(\mathbf{S}). \tag{5.92}$$

Now we need to show that $\psi_{\mathbf{S}}$ is Lip(1,1/2), that is,

$$|\psi_{\mathbf{S}}(\mathbf{x}) - \psi_{\mathbf{S}}(\mathbf{y})| \lesssim \|\mathbf{x}-\mathbf{y}\|. \tag{5.93}$$

By definition, if $\psi_{\mathbf{S}} \equiv 0$ in $(4Q(\mathbf{S}))^c$, so that (5.93) holds whenever $\mathbf{x}, \mathbf{y} \in (4Q(\mathbf{S}))^c$. When $\mathbf{x}, \mathbf{y} \in 4Q(\mathbf{S})$, we use that $\psi_{\mathbf{S}}^*$ is Lip(1,1/2), (5.92) and (5.82) to also obtain (5.93)

$$\begin{aligned}
|\psi_{\mathbf{S}}(\mathbf{x}) - \psi_{\mathbf{S}}(\mathbf{y})| &= |\psi_{\mathbf{S}}^*(\mathbf{x})\phi(\mathbf{x}) - \psi_{\mathbf{S}}^*(\mathbf{y})\phi(\mathbf{y})| \\
&\leqslant |\psi_{\mathbf{S}}^*(\mathbf{x})\phi(\mathbf{y}) - \psi_{\mathbf{S}}^*(\mathbf{y})\phi(\mathbf{y})| + |\psi_{\mathbf{S}}^*(\mathbf{x})\phi(\mathbf{x}) - \psi_{\mathbf{S}}^*(\mathbf{x})\phi(\mathbf{y})| \\
&\lesssim \|\mathbf{x}-\mathbf{y}\| + \operatorname{diam}(Q(\mathbf{S}))\operatorname{diam}(Q(\mathbf{S}))^{-1}\|\mathbf{x}-\mathbf{y}\| \lesssim \|\mathbf{x}-\mathbf{y}\|.
\end{aligned}$$

Finally, we are left treating when $\mathbf{x} \in 4Q(\mathbf{S})$ and $\mathbf{y} \in (4Q(\mathbf{S}))^c$, so $\psi_{\mathbf{S}}(\mathbf{y}) = 0$. In this case, find $\hat{\mathbf{y}}$ to be the point in the line segment joining $\mathbf{x}$ to $\mathbf{y}$ which is in $\partial(4Q(\mathbf{S}))$. Therefore, by the estimates above and the continuity of $\psi_{\mathbf{S}}$, we are also able to verify (5.93) in this case simply by computing

$$|\psi_{\mathbf{S}}(\mathbf{x}) - \psi_{\mathbf{S}}(\mathbf{y})| = |\psi_{\mathbf{S}}(\mathbf{x})| = |\psi_{\mathbf{S}}(\mathbf{x}) - \psi_{\mathbf{S}}(\hat{\mathbf{y}})| \lesssim \|\mathbf{x}-\hat{\mathbf{y}}\| \leqslant \|\mathbf{x}-\mathbf{y}\|.$$

□

5.10 Step 4: Regularity of the approximating graphs

The goal of this section is to show the following result.

Proposition 5.48 ($\psi_{\mathbf{S}}$ is regular)**.** *Let $\psi_{\mathbf{S}}$ be constructed by* (5.81)*. Then $\mathcal{D}_t\psi_{\mathbf{S}}$ is in* BMO*, that is, $\psi_{\mathbf{S}}$ is a regular* $\mathrm{Lip}(1,1/2)$ *function.*

Indeed, once we show this, the proof of Proposition 5.39 will be finished (because we already established Lemmas 5.45 and 5.47), and hence Theorems 5.2 and 5.3 will have been proved. For that purpose, it will be crucial to use square function estimates developed in the previous sections.

We will break Step 4 up into smaller steps, mostly using the ideas of [BHMN25] with one large difference. In [BHMN25] only a BMO estimate on the 'big cube' $Q(\mathbf{S})$ is obtained (and then passed on to the original graph function). It will turn out that if we are trying to get an estimate on the BMO norm of $\mathcal{D}_t\psi_{\mathbf{S}}$ when the cube Q (in the definition of BMO) is in the stopping time $\mathbf{S}$, then the proof will be almost identical to that of [BHMN25]. In fact, the same analysis extends to the case that $Q \subseteq 50Q_0$, for $Q_0 \in \mathbf{S}$ with similar size to Q. On the other hand, if this is not the case (so deviating from [BHMN25]), we will see that $\psi_{\mathbf{S}}$ is even more regular over Q and we can still establish the BMO estimates. In any case, the BMO norm of $\mathcal{D}_t\psi_{\mathbf{S}}$ will be under control within $50Q(\mathbf{S})$. Finally, to handle the part outside $50Q(\mathbf{S})$ we use cut-off functions and the John-Strömberg inequality.

Getting to the details, as in [BHMN25], using the John-Strömberg inequality [Jo65, St79], to show that $\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) \in$ BMO it suffices that there exists a (large) constant $M_\star$ such that

$$\inf_{C_Q\in\mathbb{R}} \left|\left\{\mathbf{y}\in Q : |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_Q| > M_\star\right\}\right| \leqslant \frac{1}{3}\,|Q|, \qquad \forall Q\in\mathbb{D}. \tag{5.94}$$

Of course, by Chebyshev's inequality, if there exists $C_\star \geqslant 0$ (independent of Q) such that

$$\inf_{C_Q\in\mathbb{R}} \frac{1}{|Q|}\iint_Q |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_Q|\,d\mathbf{y} \leqslant C_\star, \qquad \forall Q\in\mathbb{D}, \tag{5.95}$$

then (5.94) holds with $M_\star = 3C_\star$. Moreover, the following stronger estimate implies (5.95) and (5.94):

$$\inf_{C_Q\in\mathbb{R}} |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_Q| \leqslant C_\star, \qquad \forall\,\mathbf{y}\in Q,\ Q\in\mathbb{D}. \tag{5.96}$$

We are going to break proving (5.94) into a few cases. In the "interesting" cases (Case 1, 2 and 3), we will actually show one of the stronger estimates (5.95) or (5.96), and we will later use these cases to prove (5.94) in the remaining case (Case 4). We organize these cases now.

- **Case 1**: $Q \subseteq 50Q(\mathbf{S})$ and there exists $Q_0 \in \mathbf{S}$ satisfying

  $$Q\subseteq 50Q_0, \qquad \operatorname{diam}(Q) \geqslant \varepsilon_0 \operatorname{diam}(Q_0),$$

  where $\varepsilon_0 > 0$ is to be chosen below.

- **Case 2**: $Q \subseteq 50Q(\mathbf{S})$, and whenever $Q_0 \in \mathbf{S}$ is such that $Q \subseteq 50Q_0$, then it holds

  $$\operatorname{diam}(Q) < \varepsilon_0 \operatorname{diam}(Q_0).$$

- **Case 3**: $Q \not\subseteq 50Q(\mathbf{S})$ and
$$\operatorname{dist}(Q, 4Q(\mathbf{S})) \geqslant 6 \operatorname{diam}(Q).$$
- **Case 4:** Q is not in Cases 1-3.

For a comparison, in [BHMN25] the authors only had to deal with $Q(\mathbf{S})$ (we need to consider all the other cases by the complexity of our construction). Actually, we will follow their strategy closely to deal with not only $Q(\mathbf{S})$, but all cubes in Case 1 (and we postpone the proof for that reason, to focus in the more novel cases). Furthermore, it will turn out that the localization technique already introduced in [BHMN25] helps when dealing with Case 2 and Case 3 cubes. Lastly, Case 4 cubes must be treated in a new different way, which we address right now.

5.10.1 Handling Case 4 cubes, assuming Cases 1, 2 and 3

As mentioned above, we will (in the next subsections) prove (5.95) or (5.96) for cubes in Cases 1-3. Assuming this, let us handle Case 4 cubes.

Lemma 5.49. *Suppose that* (5.95) *holds for cubes in Cases 1-3. Then* (5.94) *holds for Case 4 cubes.*

Proof. Fix $Q \in \mathbb{D}$ a cube in Case 4. This means that

$$Q \not\subseteq 50Q(\mathbf{S}), \quad \text{and} \quad \operatorname{dist}(Q, 4Q(\mathbf{S})) < 6 \operatorname{diam}(Q).$$

The idea now is to cover Q by subcubes which are in Cases 1-3. For that purpose, it will be notationally convenient to "shift" the generations in our dyadic structure $\mathbb{D}$ to a new dyadic grid $\widetilde{\mathbb{D}}$, with exactly the same cubes, but where the generations are relabeled so that $\widetilde{\mathbb{D}}_k$ consists of cubes with side length $2^{-k}\ell(Q)$ (so concretely $\widetilde{\mathbb{D}}_0$ contains Q). Writing $k_0 := 10^5$, we set

$$\mathcal{F}_1 := \big\{Q' \in \widetilde{\mathbb{D}}_{k_0} : \operatorname{dist}(10Q', 4Q(\mathbf{S})) < 60 \operatorname{diam}(Q')\big\}.$$

The point of this definition is that we should focus mostly on $\mathcal{F}_1$ cubes, because it holds that

$$Q' \in \widetilde{\mathbb{D}}_{k_0} \backslash \mathcal{F}_1 \quad \Longrightarrow \quad 10Q' \text{ is in Cases 1-3.} \tag{5.97}$$

Indeed, if $10Q' \subseteq 50Q(\mathbf{S})$, then $10Q'$ is in Cases 1-2. Otherwise, if $10Q'$ did not belong to Case 3, we would have $\operatorname{dist}(10Q', 4Q(\mathbf{S})) < 6 \operatorname{diam}(10Q') = 60 \operatorname{diam}(Q')$, which would contradict $Q' \notin \mathcal{F}_1$.

Let us consider two cases depending on the size of Q:

Case I: Q is "not too big", say $\operatorname{diam}(Q) \leqslant 2^{200} \operatorname{diam}(Q(\mathbf{S}))$. Then, for $Q' \in \mathcal{F}_1$ and $\mathbf{x} \in 10Q'$:

$$\begin{aligned}\|\mathbf{x}_{Q(\mathbf{S})} - \mathbf{x}\| &\leqslant \operatorname{diam}(10Q') + \operatorname{dist}(10Q', Q(\mathbf{S})) + \operatorname{diam}(Q(\mathbf{S})) \leqslant 70 \operatorname{diam}(Q') + \operatorname{diam}(Q(\mathbf{S})) \\ &\leqslant 70 \cdot 2^{200} \cdot 2^{-10^5} \operatorname{diam}(Q(\mathbf{S})) + \operatorname{diam}(Q(\mathbf{S})) \leqslant 2 \operatorname{diam}(Q(\mathbf{S})).\end{aligned}$$

Therefore, $10Q' \subseteq 50Q(\mathbf{S})$, so $10Q'$ falls in Cases 1-2. And if $Q' \in \widetilde{\mathbb{D}}_{k_0} \backslash \mathcal{F}_1$, we already showed in (5.97) that $10Q'$ is in Cases 1-3. Putting these together and using the hypothesis of the lemma, we have shown that for every $Q' \in \widetilde{\mathbb{D}}_{k_0}$ there exists some $C_{Q'} \in \mathbb{R}$ such that

$$\frac{1}{|10Q'|} \iint_{10Q'} |\mathcal{D}_t \psi_{\mathbf{S}}(\mathbf{y}) - C_{Q'}| \, d\mathbf{y} \leqslant C_\star,$$

which of course this implies that

$$\frac{1}{|Q'|}\iint_{Q'} |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_{Q'}|\,d\mathbf{y} \leqslant 10^{n+1}C_\star. \tag{5.98}$$

To finish the proof in Case I, it remains to obtain a common constant C_Q that works well for all $Q' \in \widetilde{\mathbb{D}}_{k_0}(Q) := \{Q' \in \widetilde{\mathbb{D}}_{k_0} : Q' \subseteq Q\}$. For that purpose, first note that for adjacent cubes $Q', Q'' \in \widetilde{\mathbb{D}}_{k_0}(Q)$ (so that $Q' \subseteq 10Q''$) we have, using the previous estimates, that

$$\begin{aligned}|C_{Q'} - C_{Q''}| &\leqslant \iint_{Q'} |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_{Q''}|\,d\mathbf{y} + \iint_{Q'} |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_{Q'}|\,d\mathbf{y}\\ &\leqslant 10^{n+1}\iint_{10Q''} |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_{Q''}|\,d\mathbf{y} + \iint_{Q'} |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_{Q'}|\,d\mathbf{y} \leqslant 2\cdot 10^{n+1}C_\star. \end{aligned}\tag{5.99}$$

Hence, if we pick any $\widetilde{Q} \in \widetilde{\mathbb{D}}_{k_0}(Q)$ and set $C_Q := C_{\widetilde{Q}}$, we can "connect" $\widetilde{Q}$ to any other cube $Q' \in \widetilde{\mathbb{D}}_{k_0}(Q)$ by a chain of at most $L_n \lesssim (2^{10^5})^{n+1}$ adjacent cubes (the amount of cubes in $\widetilde{\mathbb{D}}_{k_0}(Q)$, see Figure 5.4) so that iterating the estimate in the previous display all along the chain yields

$$|C_Q - C_{Q'}| \leqslant 2\cdot 10^{n+1}L_nC_\star, \qquad \forall Q' \in \widetilde{\mathbb{D}}_{k_0}(Q).$$

Therefore, if we define $M_\star := (2\cdot 10^{n+1}L_n + 100^{n+1})C_\star$, we obtain that, for any $Q' \in \widetilde{\mathbb{D}}_{k_0}(Q)$,

$$\big\{\mathbf{y} \in Q' : |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_Q| > M_\star\big\} \subseteq \big\{\mathbf{y} \in Q' : |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_{Q'}| > 100^{n+1}C_\star\big\},$$

so that using Chebyshev and (5.98) it holds

$$\big|\big\{\mathbf{y} \in Q' : |\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_Q| > M_\star\big\}\big| \leqslant 10^{-(n+1)}\,|Q'| \leqslant (1/10)\,|Q'|. \tag{5.100}$$

Summing over $Q' \in \mathbb{D}_{k_0}(Q)$ (they are disjoint and cover Q) we finally get (5.94) in Case I.

Case II: Q is very big, say $\operatorname{diam}(Q) > 2^{200}\operatorname{diam}(Q(\mathbf{S}))$. In this case we are going to be able to ignore the cubes in $\mathcal{F}_1$. Indeed, if $Q' \in \mathcal{F}_1$ and $\mathbf{x} \in 10Q'$, we may compute, similarly as in Case I,

$$\|\mathbf{x}_{Q(\mathbf{S})} - \mathbf{x}\| \leqslant 70\operatorname{diam}(Q') + \operatorname{diam}(Q(\mathbf{S})) \leqslant 70\cdot 2^{10^{-5}}\operatorname{diam}(Q) + 2^{-200}\operatorname{diam}(Q) \leqslant 2^{-100}\operatorname{diam}(Q),$$

from which we readily infer the smallness of $\mathcal{F}_1$, in the form of

$$\Big|\bigcup_{Q'\in\mathcal{F}_1} Q'\Big| \leqslant \frac{1}{100}\,|Q|. \tag{5.101}$$

On the other hand, for each $Q' \in \widetilde{\mathbb{D}}_{k_0}\backslash\mathcal{F}_1$, (5.97) implies that (5.98) holds, as in Case I. Again, the problem resides on using a common C_Q. For that, we argue as in Case I: define $C_Q := C_{\widetilde{Q}}$, where $\widetilde{Q}$ is any cube in $\widetilde{\mathbb{D}}_{k_0}(Q)\backslash\mathcal{F}_1$. Then, the point is to connect any given $Q' \in \widetilde{\mathbb{D}}_{k_0}(Q)\backslash\mathcal{F}_1$ to $\widetilde{Q}$ using a chain of at most L_n cubes, but this time all of the cubes in the chain belong to $\widetilde{\mathbb{D}}_{k_0}\backslash\mathcal{F}_1$ (so that we may apply (5.99) at each step). To construct such chain from Q' to $\widetilde{Q}$, we may have problems avoiding cubes which are simultaneously in $\mathcal{F}_1$ and below Q (see Figure 5.4). However, if we allow ourselves to use all the cubes

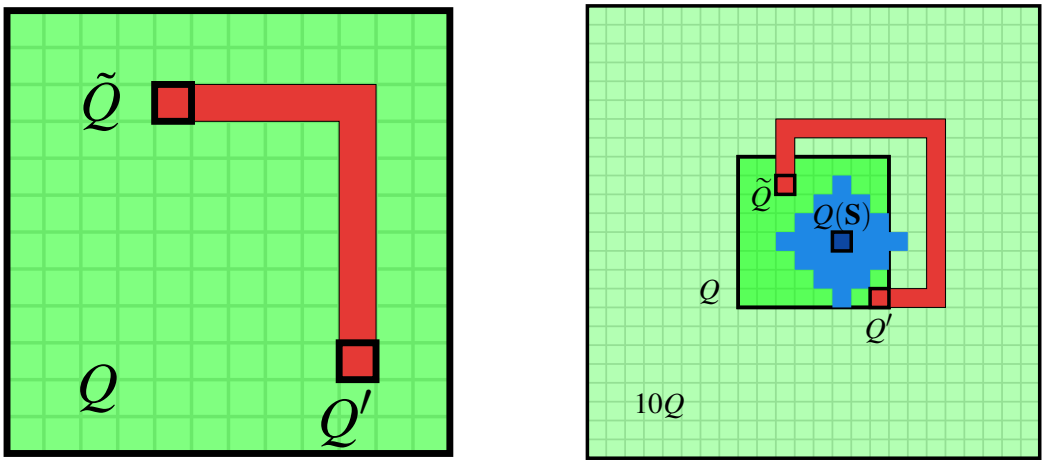


Figure 5.4: The picture in the left represents Case I: we show (in red) a possible chain of cubes in $\widetilde{D}_{k_0}(Q)$ connecting the fixed $\tilde{Q}$ with some other $Q' \in \widetilde{D}_{k_0}(Q)$. The picture in the right represents Case II: to connect Q' to $\tilde{Q}$, we need to avoid the cubes in $\mathcal{F}_1$ (in purple), which surround $Q(\mathbf{S})$, and may "disconnect" Q' and $\tilde{Q}$ within Q; but there will always be space for a chain within $10Q$.

in the larger region $\{Q'' \in \widetilde{\mathbb{D}}_{k_0} \backslash \mathcal{F}_1 : Q'' \cap 10Q \neq \varnothing\}$, we will have enough space (and this dilation preserves $L_n \lesssim (2^{10^5})^{n+1}$): indeed, if $\mathbf{x} \in Q'' \in \mathcal{F}_1$, then

$$\|\mathbf{x} - \mathbf{x}_{Q(\mathbf{S})}\| \leqslant \operatorname{diam}(Q'') + \operatorname{dist}(Q'', Q(\mathbf{S})) + \operatorname{diam}(Q(\mathbf{S})) \leqslant (61 \cdot 10^{-5} + 2^{-200}) \operatorname{diam}(Q) \leqslant \frac{1}{100} \operatorname{diam}(Q),$$

which says that the "diameter" of $\mathcal{F}_1$ is much smaller than that of $10Q$, so that if connecting Q' to $\tilde{Q}$ was impossible within Q (avoiding $\mathcal{F}_1$), there will always be room to first escape to $10Q \backslash Q$, and then find an alternative path towards $\tilde{Q}$ (because $\mathcal{F}_1$ is too small to block any section of $10Q \backslash Q$). Summing up, we also obtain (5.100) for $Q' \in \widetilde{\mathbb{D}}_{k_0}(Q) \backslash \mathcal{F}_1$ in Case II.

Therefore, separating the cubes in $\mathcal{F}_1$ (for which we use (5.101)) and in $\widetilde{\mathbb{D}}_{k_0}(Q) \backslash \mathcal{F}_1$ (for which we just argued that we can obtain the same estimates as in Case I), we obtain (5.94) also in Case II (and with the same $M_\star$ as in Case I):

$$\left|\left\{\mathbf{y} \in Q : |\mathcal{D}_t \psi_{\mathbf{S}}(\mathbf{y}) - C_Q| > M_\star\right\}\right| \leqslant \frac{1}{100} |Q| + \sum_{Q' \in \widetilde{\mathbb{D}}_{k_0}(Q) \backslash \mathcal{F}_1} \frac{1}{10} |Q'| \leqslant \frac{1}{3} |Q|.$$

□

5.10.2 Localization of $\mathcal{D}_t \psi_{\mathbf{S}}(\mathbf{y})$

In this (sub)section we localize $\mathcal{D}_t \psi_{\mathbf{S}}(\mathbf{y})$, as was done in [BHMN25]. Heuristically (and realistically) we can only expect to gain information about $\psi_{\mathbf{S}}$ from our hypotheses on the parabolic measure for cubes in or near the stopping time $\mathbf{S}$; therefore the BMO estimates on $\mathcal{D}_t \psi_{\mathbf{S}}(\mathbf{y})$ for cubes that are in Case 2 and 3 should come from somewhere else. This localization will not only help us in Case 1; but it will trivialize Case 3, and make Case 2 easier to handle.

Let $\varphi \in C_c^\infty(\mathbb{R})$ be an even function with $\mathbf{1}_{(-1,1)} \leqslant \varphi \leqslant \mathbf{1}_{(-2,2)}$ and set

$$\Phi(x,t) := \varphi(x_1)\,\varphi(x_2) \ldots \varphi(x_{n-1})\,\varphi(t/2), \qquad \mathbf{x} = (x,t) = (x_1, \ldots x_{n-1}, t) \in \mathbb{R}^n.$$

We set $\Phi_R(\mathbf{x}) := \Phi(x/R, t/R^2)$ for $\mathbf{x} = (x,t) \in \mathbb{R}^n$. Recalling the definition of I_{P} in (5.11), we write $V_R := \Phi_R\, V$ for the locally truncated kernel, and we consider the localized parabolic fractional integral

$$\mathrm{I}_{\mathrm{P}}^R h(\mathbf{x}) := \iint\limits_{\mathbb{R}^n} V_R(\mathbf{x}-\mathbf{y})\, h(\mathbf{y})\, d\mathbf{y} = \iint\limits_{\mathbb{R}^n} \Phi_R(\mathbf{x}-\mathbf{y})\, V(\mathbf{x}-\mathbf{y})\, h(\mathbf{y})\, d\mathbf{y}. \tag{5.102}$$

We can then define the localized half-order time derivative

$$\begin{aligned}\mathcal{D}_t^R \psi_{\mathbf{S}}(\mathbf{x}) := \partial_t \circ \mathrm{I}_{\mathrm{P}}^R \psi_{\mathbf{S}}(\mathbf{x}) &= \iint\limits_{\mathbb{R}^n} K^R(\mathbf{x}-\mathbf{y})\, \psi_{\mathbf{S}}(\mathbf{y})\, d\mathbf{y} := \iint\limits_{\mathbb{R}^n} \partial_t\big(V_R(\mathbf{x}-\mathbf{y})\big)\, \psi_{\mathbf{S}}(\mathbf{y})\, d\mathbf{y} \\ &= \iint\limits_{\mathbb{R}^n} \partial_t\big(V(\mathbf{x}-\mathbf{y})\, \Phi_R(\mathbf{x}-\mathbf{y})\big)\, \psi_{\mathbf{S}}(\mathbf{y})\, d\mathbf{y}, \qquad \mathbf{x} \in \mathbb{R}^n. \end{aligned} \tag{5.103}$$

This operator should be viewed as a principal value operator, or ∂_t should be considered in the weak sense. Let $\mathcal{E}^R := \mathcal{D}_t - \mathcal{D}_t^R$ and set $K_R := \partial_t V - K^R$. Thus

$$\mathcal{E}^R \psi_{\mathbf{S}}(\mathbf{x}) = \iint\limits_{\mathbb{R}^n} \partial_t\big(V(\mathbf{x}-\mathbf{y})\,(1-\Phi_R(\mathbf{x}-\mathbf{y}))\big)\, \psi_{\mathbf{S}}(\mathbf{y})\, d\mathbf{y} = \iint\limits_{\mathbb{R}^n} K_R(\mathbf{x}-\mathbf{y})\, \psi_{\mathbf{S}}(\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n.$$

Recalling that $d = n+1$, and the (parabolic) scaling of our equations, we observe that

$$\begin{aligned} &|K_R(\mathbf{x})| \lesssim \|\mathbf{x}\|^{-d-1}\, \mathbf{1}_{(Q_R(0))^c}(\mathbf{x}), \\ &|K_R(\mathbf{x}) - K_R(\mathbf{x}')| \lesssim \frac{R}{(\|\mathbf{x}\|+R)^{d+2}}, \qquad \text{if } \|\mathbf{x}-\mathbf{x}'\| \lesssim R, \\ &\iint\limits_{\mathbb{R}^n} K_R(\mathbf{x})\, d\mathbf{x} = 0, \qquad (\text{i.e., } \mathcal{E}^R\mathbf{1} = 0). \end{aligned}$$

These estimates imply that if $\|\mathbf{x}-\mathbf{x}'\| \lesssim R$, using that $\psi_{\mathbf{S}}$ is $\mathrm{Lip}(1,1/2)$ (see Lemma 5.47), then

$$\begin{aligned} \big|\mathcal{E}^R\psi_{\mathbf{S}}(\mathbf{x}) - \mathcal{E}^R\psi_{\mathbf{S}}(\mathbf{x}')\big| &= \Big| \iint\limits_{\mathbb{R}^n} \big(K_R(\mathbf{x}-\mathbf{y}) - K_R(\mathbf{x}'-\mathbf{y})\big)\, \big(\psi_{\mathbf{S}}(\mathbf{y}) - \psi_{\mathbf{S}}(\mathbf{x})\big)\, d\mathbf{y}\Big| \\ &\lesssim \iint\limits_{\mathbb{R}^n} \frac{R}{(\|\mathbf{x}-\mathbf{y}\|+R)^{d+2}}\, \|\mathbf{y}-\mathbf{x}\|\, d\mathbf{y} \lesssim \iint\limits_{\mathbb{R}^n} \frac{R}{(\|\mathbf{x}-\mathbf{y}\|+R)^{d+1}}\, d\mathbf{y} \lesssim 1. \end{aligned}$$

Thus, if $Q = Q_R(\mathbf{x}_Q)$, we define $C_Q := \mathcal{E}^R\psi_{\mathbf{S}}(\mathbf{x}_Q)$ and $\mathcal{D}_t^Q\psi_{\mathbf{S}}(\mathbf{y}) := \mathcal{D}_t^R\psi_{\mathbf{S}}(\mathbf{y})$, we have, for $\mathbf{y} \in Q$,

$$\big|\mathcal{D}_t\psi_{\mathbf{S}}(\mathbf{y}) - C_Q\big| = \big|\mathcal{D}_t^R\psi_{\mathbf{S}}(\mathbf{y}) + \mathcal{E}^R\psi_{\mathbf{S}}(\mathbf{y}) - \mathcal{E}^R\psi_{\mathbf{S}}(\mathbf{x}_Q)\big| \lesssim \big|\mathcal{D}_t^R\psi_{\mathbf{S}}(\mathbf{y})\big| + 1 = \big|\mathcal{D}_t^Q\psi_{\mathbf{S}}(\mathbf{y})\big| + 1.$$

Therefore, to show (5.94) for some $Q \in \mathbb{D}$, it suffices to show that there exists $M_\star > 0$ such that

$$\big|\big\{\mathbf{y} \in Q : |\mathcal{D}_t^Q\psi_{\mathbf{S}}(\mathbf{y})| > M_\star\big\}\big| \leqslant (1/3)\,|Q|,$$

In fact, for Case 1 cubes we will show, for some $C_\star \leqslant M_\star/3$, the stronger condition (by Chebyshev)

$$\frac{1}{|Q|} \iint\limits_{Q} |\mathcal{D}_t^Q\psi_{\mathbf{S}}(\mathbf{y})|\, d\mathbf{y} \leqslant C_\star, \tag{5.104}$$

and for Case 2 and 3 cubes, we will show the even stronger condition

$$\operatorname*{ess\,sup}_{\mathbf{y}\in Q} |\mathcal{D}_t^Q\psi_{\mathbf{S}}(\mathbf{y})| \leqslant C_\star. \tag{5.105}$$

5.10.3 Case 3 cubes

Recall that Q falls in Case 3 when $Q \nsubseteq 50Q(\mathbf{S})$ and

$$\operatorname{dist}(Q, 4Q(\mathbf{S})) \geqslant 6 \operatorname{diam}(Q). \tag{5.106}$$

Also recall that

$$\psi_{\mathbf{S}}(\mathbf{x}) = \psi^*_{\mathbf{S}}(\mathbf{x})\, \phi(\mathbf{x}), \quad \text{where} \quad |\phi| \leqslant \mathbf{1}_{4Q(\mathbf{S})}.$$

Moreover, for $Q = Q_R(\mathbf{x}_Q)$ the object $\mathcal{D}^Q_t \psi_{\mathbf{S}}$ has kernel V_R supported in $Q_{2R}(0)$. Therefore, by (5.106) the integrand in the definition of $\mathcal{D}^Q_t \psi_{\mathbf{S}}$ is zero in Q and (5.105) holds with $C_\star = 0$.

5.10.4 Proving the localized estimate for Case 2 cubes

Here we prove that (5.105) holds in Case 2. Recall that Case 2 cubes satisfy $Q \subseteq 50Q(\mathbf{S})$; and whenever $Q_0 \in \mathbf{S}$ is such that $Q \subseteq 50Q_0$, it holds $\operatorname{diam}(Q) < \varepsilon_0 \operatorname{diam}(Q_0)$. Here ε_0 is a small parameter at our disposal. We shall use it now to conclude that in Case 2, for $\mathbf{y} \in Q$ the function h is much larger than the diameter of the cube Q and therefore we can conclude that the portion of $\psi_{\mathbf{S}}$ above Q has an estimate on its (classical, full order) time-derivative that is sufficient for proving (5.105). The first step is the following lemma.

Lemma 5.50. *If ε_0 is sufficiently small (depending only on n), and Q is a Case 2 cube, then*

$$h(\mathbf{x}) \gtrsim \ell(Q), \quad \forall \mathbf{x} \in 10Q.$$

Proof. Let $\mathbf{x} \in 10Q$ and suppose, for the sake of contradiction (using Lemma 5.40), that $d(\mathbf{x}) < \operatorname{diam}(Q)$. Then, by definition of d, there exists $Q' \in \mathbf{S}$ such that

$$\operatorname{dist}(\mathbf{x}, Q') + \operatorname{diam}(Q') \leqslant \operatorname{diam}(Q)$$

Since $\operatorname{diam}(Q) \leqslant \varepsilon_0 \operatorname{diam}(Q(\mathbf{S}))$ (just use the definition of Case 2 with $Q_0 = Q(\mathbf{S})$), with $\varepsilon_0 \ll 1$, we can find an ancestor $\widetilde{Q}$ of Q' with $\widetilde{Q} \in \mathbf{S}$ such that

$$\operatorname{diam}(Q) \leqslant \operatorname{diam}(\widetilde{Q}) \lesssim \operatorname{diam}(Q),$$

where the implicit constant in the last inequality depends only on dimension (to find such $\widetilde{Q}$, just traverse one by one the ancestors of Q' until the first time that $\ell(\widetilde{Q}) \geqslant \ell(Q)$). Note that since $Q' \subseteq \widetilde{Q}$, we also have $\operatorname{dist}(\mathbf{x}, \widetilde{Q}) \leqslant \operatorname{dist}(\mathbf{x}, Q') \leqslant \operatorname{diam}(Q)$. Then it is clear that $Q \subseteq 50\widetilde{Q}$, and if we choose ε_0 sufficiently small, $\operatorname{diam}(Q) \geqslant \varepsilon_0 \operatorname{diam}(\widetilde{Q})$ (see the last display), which contradicts the fact that Q is in Case 2 (using $Q_0 = \widetilde{Q} \in \mathbf{S}$). □

Therefore, if we are given a Case 2 cube $Q = Q_R(\mathbf{x}_Q)$, we can successfully control the localized half-order time derivative $\mathcal{D}^Q_t \psi_{\mathbf{S}} = \mathcal{D}^R_t \psi_{\mathbf{S}}$ in (5.103). Indeed, using Lemmas 5.46 and 5.50, and the support properties of Φ_R, we may integrate by parts in (5.103) to obtain for $\mathbf{x} \in Q$

$$\begin{aligned} |\mathcal{D}^R_t \psi_{\mathbf{S}}(\mathbf{x})| &= \left| \iint_{\mathbb{R}^n} \big(V(\mathbf{x}-\mathbf{y})\, \Phi_R(\mathbf{x}-\mathbf{y}) \big)\, \partial_t \psi(\mathbf{y})\, d\mathbf{y} \right| \\ &\lesssim \frac{1}{\operatorname{diam}(Q)} \iint_{Q_{2R}(\mathbf{x})} |V(\mathbf{x}-\mathbf{y})|\, d\mathbf{y} \lesssim \frac{1}{\operatorname{diam}(Q)} \iint_{Q_{2R}(0)} \|\mathbf{x}\|^{-n}\, d\mathbf{x} \lesssim 1, \end{aligned}$$

where we used (5.12)) and the fact that $\|\cdot\|$ is the parabolic metric. This shows (5.105) in Case 2.

5.10.5 Modified level sets and square function estimates

After all the previous (sub)sections, to finish the proof of Proposition 5.48, we are only left with obtaining (5.104) for Case 1 cubes. In fact, this is the most challenging case, but luckily it is also the closest to [BHMN25]. So before obtaining (5.104), let us put together some square function estimates that we essentially obtained in the previous sections.

Proposition 5.51. *Let $Q_0 \in \mathbf{S}$. Then it holds*

$$\iiint_{\mathcal{S}(Q_0)} \left(|r\,\partial_s\psi(r;y,s)|^2 + |r\,\nabla^2_{y,r}\psi(r;y,s)|^2 + |r^2\,\nabla_{y,r}\partial_s\psi(r;y,s)|^2 \right) \frac{dr}{r}\,dy\,ds \lesssim |Q_0|.$$

Proof. With all the work that we have already done in the previous sections, the proof of this result is just a tiny fraction of the proof of [BHMN25, Proposition 5.1]. Let us include the argument for completeness. For the time being, consider the first term. Using Lemma 5.44, we obtain

$$\iiint_{\mathcal{S}(Q_0)} |r\,\partial_s\psi(r;y,s)|^2\,\frac{dr}{r}\,dy\,ds \lesssim \iiint_{\mathcal{S}(Q_0)} |r\,(\partial_s u)(\mathbf{\Psi}(r;y,s))|^2\,\frac{dr}{r}\,dy\,ds.$$

Now perform the change of variables $(y_0,y,s) = \mathbf{Y} = \mathbf{\Psi}(r;y,s) = (\psi(r;y,s),y,s)$. The domain of integration changes to $\Omega_{\mathbf{S}(Q_0)}$ by Lemma 5.43. The Jacobian of the transformation, which is $|\partial_r\psi(r;y,s)|^{-1}$, which by (5.88) equals $|(\partial_{y_0}u)(\mathbf{\Psi}(r;y,s))| \lesssim 1$ after using Lemma 5.32. Moreover, $r = u(\mathbf{\Psi}(r;y,s)) = u(\mathbf{Y}) \approx_{M_2} \operatorname{dist}(\mathbf{Y},\partial\Omega)$ because $\mathbf{Y} \in \Omega_{\mathbf{S}(Q_0)} \subset \Omega_{\mathbf{S}}$, so we can apply (5.41) since $\mathbf{S}$ only contains cubes with side-length smaller than $2^{-10N}\ell(Q(\mathbf{S}^*)) = K^{-10}\ell(Q(\mathbf{S}^*))$ (see Lemma 5.37 or the introduction of Section 5.9). All these considerations yield

$$\iiint_{\mathcal{S}(Q_0)} |r\,\partial_s\psi(r;y,s)|^2\,\frac{dr}{r}\,dy\,ds \lesssim \iiint_{\Omega_{\mathbf{S}(Q_0)}} |\operatorname{dist}(\mathbf{Y},\partial\Omega)\,\partial_s u(\mathbf{Y})|^2\,\frac{d\mathbf{Y}}{\operatorname{dist}(\mathbf{Y},\partial\Omega)} \lesssim |Q(\mathbf{S})|,$$

where in the last estimate we have used Lemma 5.36 (letting $N_3 \to \infty$ using the monotone convergence theorem). This ends the proof for the first term of the statement, and the same procedure and all the estimates in Lemma 5.36 yield the result for the remaining terms. □

Note here a difference with [BHMN25]: we need to obtain Proposition 5.51 in $\mathcal{S}(Q_0)$ for every $Q_0 \in \mathbf{S}$ since our construction of the approximating graphs $\psi_{\mathbf{S}}$ is more intricate, and we need to consider many cubes in Case 1, not only $Q(\mathbf{S})$ as in [BHMN25]. Hence the importance of working in Section 5.8 with arbitrary subregimes (which we can now localize to be below any given Q_0) like in (5.58), as opposed to only dealing with the original scale like in (5.57).

Actually, we will use a slightly modified version of the previous proposition. First we define

$$\psi^{\blacktriangledown}(r;x,t) := \psi(r;x,t)\,\phi(x,t), \tag{5.107}$$

where $\phi \in C_c^\infty(\mathbb{R}^n)$ is a cut-off adapted to $Q(\mathbf{S})$ satisfying (5.82) and (5.83). Note that

$$\psi^{\blacktriangledown}(h(x,t);x,t) = \psi_{\mathbf{S}}(x,t),$$

so $\psi^{\blacktriangledown}$ somehow "interpolates" between ψ (with $r=0$) and $\psi_{\mathbf{S}}$ (with $r = h(x,t)$).

Proposition 5.52. *Let $Q_0 \in \mathbf{S}$. Then it holds*

$$\iiint_{\mathcal{S}(Q_0)} \left(|r\,\partial_s\psi^\blacktriangledown(r;y,s)|^2 + |r\,\nabla^2_{y,r}\psi^\blacktriangledown(r;y,s)|^2 + |r^2\,\nabla_{y,r}\partial_s\psi^\blacktriangledown(r;y,s)|^2\right)\frac{dr}{r}\,dy\,ds \lesssim |Q_0|. \tag{5.108}$$

Proof. We are going to leverage Proposition 5.51 heavily. Let us note that since $0 \leqslant \phi \leqslant 1$, when we expand the terms in (5.108) out using the product rule, the only terms not handled by Proposition 5.51 are those where at least one derivative lands on ϕ. Because $\mathbf{1}_{2Q(\mathbf{S})} \leqslant \phi \leqslant \mathbf{1}_{4Q(\mathbf{S})}$ these additional terms are zero unless $(y,s) \in 4Q(\mathbf{S})\backslash 2Q(\mathbf{S})$. In such case, by definition of d,

$$h(y,s) \approx d(y,s) \approx \operatorname{diam}(Q(\mathbf{S})), \quad \forall (y,s) \in 4Q(\mathbf{S})\backslash 2Q(\mathbf{S}). \tag{5.109}$$

Therefore, recalling the definition of $\mathcal{S}(Q_0)$ in (5.84), to control the terms not handled by Proposition 5.51, it is enough to obtain a bound of the form

$$\iint_{500Q_0}\int_{c\operatorname{diam}(Q(\mathbf{S}))}^{10^{10}C_n\operatorname{diam}(Q_0)} \left(|r\,\partial_s\psi^\blacktriangledown(r;y,s)|^2 + |r\,\nabla^2_{y,r}\psi^\blacktriangledown(r;y,s)|^2 + |r^2\,\nabla_{y,r}\partial_s\psi^\blacktriangledown(r;y,s)|^2\right)\frac{dr}{r}\,dy\,ds \lesssim |Q_0|,$$

where c is implicitly defined by (5.109). Now, using Lemma 5.44, the estimates on the derivatives of ϕ (see (5.83)), and the fact that $|\psi(y,s)| \lesssim \operatorname{diam}(Q(\mathbf{S}))$ for $(y,s) \in 500Q(\mathbf{S})$ (recall the normalization in (5.81), and the bounds for our regions in Lemma 5.43 and (5.17)), we get

$$|r\,\partial_s\psi^\blacktriangledown(r;y,s)|^2 + |r\,\nabla^2_{y,r}\psi^\blacktriangledown(r;y,s)|^2 + |r^2\,\nabla_{y,r}\partial_s\psi^\blacktriangledown(r;y,s)|^2 \lesssim 1$$

for every (r,y,s) in the domain of integration in the previous display, which readily gives the desired estimate (recalling that $\operatorname{diam}(Q_0) \leqslant \operatorname{diam}(Q(\mathbf{S}))$ because $Q_0 \in \mathbf{S}$). □

5.10.6 Treating the Case 1 cubes with methods from [BHMN25]

To finish handling the Case 1 cubes, we will largely follow the ideas in [BHMN25]. In fact, in some sense we have an easier task, because in [BHMN25] the authors needed to more closely relate the object $\mathcal{D}^R_t\psi_{\mathbf{S}}$ to $\mathcal{D}^R_t\psi$ (see e.g. [BHMN25, "Step 2"]), whereas we are proving a Coronization and using Lemma 5.9.

So fix a Case 1 cube $Q = Q_R(\mathbf{x}_Q)$. Our goal is to show (5.104). Since $Q \subseteq 50Q_0$ for some $Q_0 \in \mathbf{S}$ with $\operatorname{diam}(Q) \approx_{\varepsilon_0} \operatorname{diam}(Q_0)$ we may (and do) assume that $Q = 50Q_0$ for $Q_0 \in \mathbf{S}$. We are going to work with a smoothed, cut-off version of $\psi(r;x,t)$. To begin, we let $\zeta \in C^\infty_0([-1,1])$ be an even function with $\mathbf{1}_{[-1/2,1/2]} \leqslant \zeta \leqslant \mathbf{1}_{[-1,1]}$, and set

$$p(x,t) := c_n\,\zeta(x_1)\,\zeta(x_2)\ldots\zeta(x_{n-1})\,\zeta(t), \quad \forall\,\mathbf{x} = (x,t) = (x_1,\ldots,x_{n-1},t) \in \mathbb{R}^n, \tag{5.110}$$

where c_n is chosen so that $\iint_{\mathbb{R}^n} p(x,t)\,dx\,dt = 1$. We define, for $(x,t) \in \mathbb{R}^n$,

$$P_r f(x,t) := (p_r * f)(x,t), \tag{5.111}$$

where $p_r(x,t) := r^{-n-1}p(x/r, t/r^2)$, Thus, P_r is a nice parabolic approximation to the identity on $\mathbb{R}^n$.

Since h is Lip(1,1/2) with $\|h\|_{\operatorname{Lip}(1,1/2)} \lesssim_n 1$ (Lemma 5.40), there exists $\gamma \ll_n 1/100$ such that

$$|h(\mathbf{x}) - P_{\gamma r}h(\mathbf{x})| \leqslant r/4, \quad \forall r > 0,\ \mathbf{x} \in \mathbb{R}^n. \tag{5.112}$$

The estimate (5.112) implies

$$r + P_{\gamma r} h(\mathbf{x}) \geqslant 3r/4 + h(\mathbf{x}) > h(\mathbf{x}), \quad \forall\, r > 0,\ \mathbf{x} \in \mathbb{R}^n. \tag{5.113}$$

Now we define the smoothed cut-off version of $\psi(r; x, t)$ (with $\psi^{\blacktriangledown}$ defined in (5.107))

$$\widetilde{\psi}_r(\mathbf{x}) := \widetilde{\psi}(r; \mathbf{x}) := \psi^{\blacktriangledown}(r + P_{\gamma r} h(\mathbf{x}); \mathbf{x}), \qquad \mathbf{x} \in 500 Q_0, 0 \leqslant r \leqslant 10 \operatorname{diam}(Q). \tag{5.114}$$

We will leave $\gamma = \gamma(n)$ as a parameter we can adjust. Let us make two observations. We have that

$$\widetilde{\psi}(0; \mathbf{x}) = \psi_{\mathbf{S}}(\mathbf{x}). \tag{5.115}$$

We also want to observe that in the domain where we consider $\widetilde{\psi}_r(\mathbf{x})$, the function only pulls information for $\psi^{\blacktriangledown}$ in the region $\mathcal{S}(Q_0)$, where we have established a local square function estimate. In particular, we want to confirm that (recalling that $Q = 50 Q_0$)

$$\mathbf{x} \in 500 Q_0, \ \ 0 \leqslant r \leqslant 10 \operatorname{diam}(Q) \quad \Longrightarrow \quad (r + P_{\gamma r} h(\mathbf{x}), \mathbf{x}) \in \mathcal{S}(Q_0). \tag{5.116}$$

Indeed, inspecting the definition of $\mathcal{S}(Q_0)$ (see (5.84)), we would want to confirm that

$$h(\mathbf{x})/10 < r + P_{\gamma r} h(\mathbf{x}) < 10^{10} C_n \operatorname{diam}(Q_0).$$

The first inequality follows from (5.113). For the second, we have $d(\mathbf{x}) \leqslant 501 \operatorname{diam}(Q_0)$ for $\mathbf{x} \in 500 Q_0$ by definition of d, so by (5.112) and the properties of h (see (5.77)) it holds

$$r + P_{\gamma r} h(\mathbf{x}) \leqslant \frac{5}{4} r + h(x) \leqslant \frac{50}{4} \operatorname{diam}(Q) + 501 C_n \operatorname{diam}(Q_0) < 10^{10} C_n \operatorname{diam}(Q_0),$$

(where we again recall that $Q = 50 Q_0$) which finishes the proof of (5.116).

Recall we are trying to prove (5.104). By Hölder's inequality it suffices to show

$$\iint\limits_{Q} |\mathcal{D}_t^Q \psi_{\mathbf{S}}(\mathbf{y})|^2 \, d\mathbf{y} \leqslant C_\star |Q|^{1/2}.$$

By duality and density, to prove this, it suffices to show, for $f \in C_c^\infty(Q)$ with $\|f\|_{L^2(Q)} \leqslant 1$, that

$$\left| \iint \mathcal{D}_t^Q \psi^{\mathbf{S}}(\mathbf{x}) f(\mathbf{x}) \, d\mathbf{x} \right| =: \left| \iint \mathrm{I}_{\mathrm{P}}^Q \psi^{\mathbf{S}}(\mathbf{x}) \, \partial_t f(\mathbf{x}) \, d\mathbf{x} \right| \lesssim |Q|^{1/2}, \tag{5.117}$$

where we have interpreted the ∂_t in the definition of $\mathcal{D}_t^Q := \mathcal{D}_t^R$ (see (5.103)) in the weak sense. Now, following [BHMN25], we use (5.115), introduce an approximate identity on f and integrate several times by parts (we will explain this below) to obtain

$$\begin{aligned} -\iint \mathrm{I}_{\mathrm{P}}^Q \psi^{\mathbf{S}}(\mathbf{x}) \, \partial_t f(\mathbf{x}) \, d\mathbf{x} \ &= \ -\iint \mathrm{I}_{\mathrm{P}}^Q \tilde{\psi}(0; \mathbf{x}) \, \partial_t f(\mathbf{x}) \, d\mathbf{x} \\ &= -\int_0^{\operatorname{diam}(Q)} \iint\limits_{2Q} \partial_r [\mathcal{D}_t^Q \tilde{\psi}(r; \mathbf{x}) P_{\gamma r} f(\mathbf{x})] \, d\mathbf{x} \, dr \ + \ b_1 \end{aligned} \tag{5.118}$$

$$= \int_0^{\operatorname{diam}(Q)} \iint_{2Q} \partial_r^2 [\mathcal{D}_t^Q \tilde{\psi}(r;\mathbf{x}) P_{\gamma r} f(\mathbf{x})] \, d\mathbf{x}\, r\, dr \, + \, b_1 - b_2 \, =: \, I + b_1 - b_2,$$

where the boundary terms b_1 and b_2 are defined by

$$b_1 := -\iint_{2Q} \mathrm{I}_{\mathrm{P}}^Q \tilde{\psi}(\operatorname{diam}(Q);\mathbf{x}) \partial_t P_{\gamma \operatorname{diam}(Q)} f(\mathbf{x}) \, d\mathbf{x} = \iint_{2Q} \mathcal{D}_t^Q \tilde{\psi}(\operatorname{diam}(Q),\mathbf{x}) P_{\gamma \operatorname{diam}(Q)} f(\mathbf{x}) \, d\mathbf{x}, \quad (5.119)$$

where we have integrated by parts in t recalling that $\mathcal{D}_t^Q = \partial_t \circ \mathrm{I}_{\mathrm{P}}^Q$ (see (5.103)), and in turn

$$b_2 := \iint_{2Q} \partial_r [\mathcal{D}_t^Q \tilde{\psi}(r;\mathbf{x}) P_{\gamma r} f(\mathbf{x})]\big|_{r=\operatorname{diam}(Q)} \operatorname{diam}(Q) \, d\mathbf{x}. \quad (5.120)$$

Let us explain (5.118) more carefully. First, note the fact that all the integrals are taken over $2Q = 100Q_0$ because $\gamma \ll_n 1/100$ and f being supported in Q imply that $P_{\gamma r} f$ is supported in $2Q$ for $r < \operatorname{diam}(Q)$. Then, the first equality in (5.118) is just (5.115). In turn, in the second inequality, we first use the fundamental theorem of calculus in the variable r (so that we obtain the boundary term b_1), and then we integrate by parts in t in the main term, passing the ∂_t from f to $\mathrm{I}_{\mathrm{P}}^Q$ (recall $\mathcal{D}_t^Q = \partial_t \circ \mathrm{I}_{\mathrm{P}}^Q$). This actually makes sense because we will show in (5.125) below that $\mathcal{D}_t^Q \tilde{\psi}(r;\mathbf{x})$ is well defined for $r > 0$. Lastly, the third equality is an elementary 1D integration by parts in r.

To continue, we claim, and will prove in (sub)section 5.10.7 below, that

$$|b_1| + |b_2| \lesssim |Q|^{1/2}. \quad (5.121)$$

This leaves the contribution of the main term I, in which we distribute the r-derivatives:

$$I = \int_0^{\operatorname{diam}(Q)} \iint_{2Q} \mathcal{D}_t^Q \partial_r^2 \tilde{\psi}(r;\mathbf{x}) \, P_{\gamma r} f(\mathbf{x}) \, d\mathbf{x}\, r\, dr \, + \, 2 \int_0^{\operatorname{diam}(Q)} \iint_{2Q} \mathcal{D}_t^Q \partial_r \tilde{\psi}(r;\mathbf{x}) \, \partial_r P_{\gamma r} f(\mathbf{x}) \, d\mathbf{x}\, r\, dr$$
$$+ \int_0^{\operatorname{diam}(Q)} \iint_{2Q} \mathcal{D}_t^Q \tilde{\psi}(r;\mathbf{x}) \, \partial_r^2 P_{\gamma r} f(\mathbf{x}) \, d\mathbf{x}\, r\, dr \, =: \, I_1 + I_2 + I_3. \quad (5.122)$$

To deal with these terms, we will use Littlewood-Paley theory. So let us define $Q_r^{(j)} = Q_r^{(j,Q)}$ by

$$Q_r^{(1)} \, := \, r\, \mathcal{D}_t^Q P_{\gamma r}, \quad Q_r^{(2)} \, := \, \mathrm{I}_P^Q \, \partial_r P_{\gamma r}, \quad Q_r^{(3)} \, := \, r\, \mathrm{I}_P^Q \, \partial_r^2 P_{\gamma r}\,.$$

Then, recall that I_P^Q is the smoothly truncated fractional integral operator of order 1 (see (5.102)), and is self-adjoint. This implies (by an integration by parts in t, at least when we test against functions with compact suppport, as is our case) that $\mathcal{D}_t^Q = \partial_t \circ \mathrm{I}_P^Q$ is skew-adjoint. This implies that we can continue estimating I_1 in (5.122) as follows:

$$|I_1| = \left| \int_0^{\operatorname{diam}(Q)} \iint_{2Q} r \partial_r^2 \tilde{\psi}(r;\mathbf{x}) Q_r^{(1)} f(\mathbf{x}) \, d\mathbf{x} \, \frac{dr}{r} \right|$$

$$\leqslant \left(\int_0^{\operatorname{diam}(Q)} \iint_{2Q} |r\partial_r^2 \tilde{\psi}(r;\mathbf{x})|^2 \, d\mathbf{x} \, \frac{dr}{r} \right)^{1/2} \left(\int_0^{\operatorname{diam}(Q)} \iint_{2Q} |Q_r^{(1)} f(\mathbf{x})|^2 \, d\mathbf{x} \, \frac{dr}{r} \right)^{1/2}, \quad (5.123)$$

To get a bound for the last term, note that by parabolic Littlewood-Paley theory[9] we have

$$\int_0^\infty \iint_{\mathbb{R}^n} |Q_r^{(1)} f(\mathbf{x})|^2 + |Q_r^{(2)} f(\mathbf{x})|^2 + |Q_r^{(3)} f(\mathbf{x})|^2 \, d\mathbf{x} \, \frac{dr}{r} \lesssim \|f\|_{L^2}^2 \leqslant 1, \quad (5.124)$$

where the implicit constant is independent of Q. Note that the cancellation for $Q_r^{(1)}$ comes from this t-derivative.

Therefore, to complete the estimate for I_1 in (5.123), we only lack the following lemma.

Lemma 5.53. *Let $\tilde{\psi}$ be defined as in (5.114) and $Q_0 \in \mathbf{S}$. Then, if $Q = 50Q_0$, it holds*

$$|r\partial_r^2 \tilde{\psi}(r;\mathbf{x})| + |r\partial_t \tilde{\psi}(r;\mathbf{x})| + |r^2 \partial_t \partial_r \tilde{\psi}(r;\mathbf{x})| \lesssim 1, \quad \forall\, r \in (0, 10\operatorname{diam}(Q)),\, \mathbf{x} \in 10Q, \quad (5.125)$$

and

$$\int_0^{\operatorname{diam}(Q)} \iint_{2Q} \left(|r\partial_r^2 \tilde{\psi}(r;\mathbf{x})|^2 + |r\partial_t \tilde{\psi}(r;\mathbf{x})|^2 + |r^2 \partial_t \partial_r \tilde{\psi}(r;\mathbf{x})|^2 \right) d\mathbf{x} \frac{dr}{r} \lesssim |Q|. \quad (5.126)$$

We postpone the proof of the lemma to (sub)section 5.10.7: indeed, it is a consequence of Proposition 5.52 (and Lemma 5.54 in the next subsection). Taking it for granted, we can quickly finish the proof of (5.117). Indeed, just put together (5.123), (5.124) and (5.126) to get the desired estimate for I_1 (and similarly for I_2 and I_3):

$$|I_1| + |I_2| + |I_3| \lesssim |Q|^{1/2}.$$

Therefore, combining our estimates for I_1, I_2, I_3 here, and b_1, b_2 in (5.121), we conclude that (5.117) holds, which finally shows (5.104) for Case 1 cubes, modulo the proofs in the next subsection.

5.10.7 Estimating the boundary terms b_1, b_2, and proof of Lemma 5.53.

To finish the proof of (5.104) (and therefore of Theorems 5.2 and 5.3) we are only left with showing (5.121) and Lemma 5.53. For that, we will also follow [BHMN25] very closely in this subsection. We start with the following result, which is based on Lemma 5.40. We refer to [HL96, Lemma 2.8] for its proof.

Lemma 5.54. *Define P_r as in (5.110)-(5.111). Then we have*

$$\left| \frac{1}{r}(P_r - I)h(\mathbf{x}) \right| + \left| \partial_r P_r\, h(\mathbf{x}) \right| + \left| r^{2j+m-1} \nabla_{x,r}^m \partial_t^j P_r h(\mathbf{x}) \right| \lesssim 1,$$

where the implicit constant depends at most on (n, m, j). Furthermore, for all $Q \subseteq \mathbb{R}^n$,

$$\text{(i)} \quad \int_0^{50\ell(Q)} \iint_{50Q} \left| \frac{1}{r}(P_r - I)\, h(\mathbf{x}) \right|^2 d\mathbf{x} \, \frac{dr}{r} \lesssim |Q|,$$

[9] Estimate (5.124) may be proved via Plancherel's theorem. We omit the standard argument.

$$\text{(ii)} \quad \int_0^{50\ell(Q)} \iint_{50Q} |\partial_r P_r h(\mathbf{x})|^2 \, d\mathbf{x} \, \frac{dr}{r} \lesssim |Q|,$$

$$\text{(iii)} \quad \int_0^{50\ell(Q)} \iint_{50Q} \left| r^{2j+m-1} \nabla_{x,r}^m \partial_t^j P_r h(\mathbf{x}) \right|^2 d\mathbf{x} \, \frac{dr}{r} \lesssim |Q|,$$

where in (iii) *we require that* $j, m \geq 0$*, and that either* $j \geq 1$*, or that* $m \geq 2$*. Again the implicit constants depend at most on* (n, m, j).

With this lemma in hand, we prove Lemma 5.53.

Proof of Lemma 5.53. We start by proving (5.125). We will only handle the first term to the left in (5.125), as the rest of the terms can be handled analogously. We have

$$\partial_r \tilde{\psi}(r; \mathbf{x}) = \partial_r \left[\psi(r + P_{\gamma r} h(\mathbf{x}); \mathbf{x}) \phi(\mathbf{x}) \right] = (\partial_r \psi)(r + P_{\gamma r} h(\mathbf{x}), \mathbf{x})(1 + \partial_r P_{\gamma r} h(\mathbf{x})) \phi(\mathbf{x}),$$

and hence

$$r\partial_r^2 \tilde{\psi}(r; \mathbf{x}) = r(\partial_r^2 \psi)(r + P_{\gamma r} h(\mathbf{x}); \mathbf{x}) \, (1 + \partial_r P_{\gamma r} h(\mathbf{x}))^2 + (\partial_r \psi)(r + P_{\gamma r} h(\mathbf{x}); \mathbf{x}) \, r \left(\partial_r^2 P_{\gamma r} h(\mathbf{x}) \right) \phi(\mathbf{x}). \quad (5.127)$$

The bound for $|r\partial_r^2 \tilde{\psi}(r; \mathbf{x})|$ now follows from Lemma 5.44, specifically (5.85) and (5.87), and Lemma 5.54. In particular, (5.116) justifies the appropriate use of Lemma 5.44. Note that for the remaining terms in (5.125), there are derivatives landing on ϕ, but these contribute terms that act well since $h(\mathbf{x}) \approx \ell(Q(\mathbf{S}))$ where $\nabla_{x,t} \phi \neq 0$, as observed in the proof of Proposition 5.52.

Next we turn our attention to (5.126), which is a little more delicate. Again, we will only handle the first term (in the integral) as the others terms can be handled analogously. First, we control a closely related expression, without the smoothing factor. Indeed, we observe that

$$\int_0^{\operatorname{diam}(Q)} \iint_{2Q} |r\partial_r^2 \psi^\blacktriangledown(r + h(\mathbf{x}); \mathbf{x})|^2 \, d\mathbf{x} \frac{dr}{r} = \iint_{2Q} \int_{h(\mathbf{x})}^{h(\mathbf{x}) + \operatorname{diam}(Q)} |\partial_s^2 \psi^\blacktriangledown(s; \mathbf{x})|^2 \left(s - h(\mathbf{x}) \right) ds \, d\mathbf{x}$$
$$\leq \iiint_{\mathcal{S}_Q} |\partial_s^2 \psi^\blacktriangledown(s; \mathbf{x})|^2 s \, d\mathbf{x} \, ds \lesssim |Q|,$$

where in the last step we used Proposition 5.52, and in the previous inequality, we used (5.116) and $h \geq 0$. Now, we need to relate this to our smoothed version of ψ. With the preceding estimate in hand, we see that to obtain the bound

$$\int_0^{\operatorname{diam}(Q)} \iint_{2Q} |r\partial_r^2 \tilde{\psi}(r; \mathbf{x})|^2 \, d\mathbf{x} \, \frac{dr}{r} \lesssim |Q|$$

in (5.126), it is enough to prove

$$\int_0^{\operatorname{diam}(Q)} \iint_{2Q} |r\partial_r^2 [\tilde{\psi}(r; \mathbf{x}) - \psi^\blacktriangledown(r + h(\mathbf{x}); \mathbf{x})]|^2 \, d\mathbf{x} \, \frac{dr}{r} \lesssim |Q|. \quad (5.128)$$

To do this, we first note that by (5.127), Lemma 5.44, and Lemma 5.54,

$$r\partial_r^2\tilde{\psi}(r;\mathbf{x}) = r(\partial_r^2\psi^{\blacktriangledown})\big(r + P_{\gamma r}h(\mathbf{x});\mathbf{x}\big) \,+\, O\left(\left|\partial_r P_{\gamma r}h(\mathbf{x})\right| \,+\, r\left|\partial_r^2 P_{\gamma r}h(\mathbf{x})\right|\right),$$

and observe further that the "big-O" term may be handled via Lemma 5.54. To treat the contribution to (5.128) of the remaining term, we use the mean value theorem: for $(r,\mathbf{x}) \in (0,\operatorname{diam}(Q)) \times 2Q$, there exists $\tilde{r}$ between $r + h(\mathbf{x})$ and $r + P_{\gamma r}h(\mathbf{x})$, such that

$$\begin{aligned}\left|(\partial_r^2\psi^{\blacktriangledown})(r + P_{\gamma r}h(\mathbf{x});\mathbf{x}) \,-\, \partial_r^2\psi^{\blacktriangledown}(r + h(\mathbf{x});\mathbf{x})\right| \\ = \left|\left(P_{\gamma r} - I\right)h(\mathbf{x})\right| \left|(\partial_r^3\psi)^{\blacktriangledown}(\tilde{r};\mathbf{x})\right| \lesssim \tilde{r}^{-2}\left|\left(P_{\gamma r} - I\right)h(\mathbf{x})\right| \leqslant r^{-2}\left|\left(P_{\gamma r} - I\right)h(\mathbf{x})\right|. \quad (5.129)\end{aligned}$$

The use of (5.87) to derive the second-to-last inequality in this display may be justified by the fact that $\tilde{r}$ is between $r + h(\mathbf{x})$ and $r + P_{\gamma r}h(\mathbf{x})$; in particular, $(\tilde{r},\mathbf{x})$ lies in $\mathcal{S}(Q_0) \subseteq \mathcal{S}$. (We also remark that the cut-off function ϕ is independent of r and $|\phi| \leqslant 1$.) Also, in the last inequality we have used the fact that $\tilde{r} \geqslant \min\{r + h(\mathbf{x}), r + P_{\gamma r}h(\mathbf{x})\} \geqslant r$ because $h \geqslant 0$. Multiplying (5.129) by r, and using Lemma 5.54 (i), we obtain (5.128), thus completing the proof of the lemma. Note that in the (omitted) proofs of bounds for the second third terms in (5.126) that when derivatives land on the cut-off function these terms can be handled as in Proposition 5.52. □

We are left with handling the "boundary terms" b_1 and b_2, defined in (5.119) and (5.120). Let us estimate $|b_1|$ in (5.119). Recall that $\|f\|_{L^2} \leqslant 1$, and that, for $r > 0$, the operator P_r is bounded on L^2, uniformly in r. Hence

$$\begin{aligned}|b_1| \,&\leqslant\, \left(\iint_{2Q} |\mathcal{D}_t^Q\tilde{\psi}(\operatorname{diam}(Q);\mathbf{x})|^2 \, d\mathbf{x}\right)^{1/2} \left(\iint_{2Q} |P_{\gamma\operatorname{diam}(Q)}f(\mathbf{x})|^2 \, d\mathbf{x}\right)^{1/2} \\ &\lesssim \left(\iint_{2Q} |\mathcal{D}_t^Q\tilde{\psi}(\operatorname{diam}(Q);\mathbf{x})|^2 \, d\mathbf{x}\right)^{1/2} \lesssim \|\mathcal{D}_t^Q\tilde{\psi}(\operatorname{diam}(Q);\mathbf{x})\|_{L^\infty} \, |Q|^{1/2}. \quad (5.130)\end{aligned}$$

Next, recall that $\mathcal{D}_t^Q$ is a convolution operator with kernel $K^Q = \partial_t V_Q$, where V_Q is the truncated version of the kernel of I_P, localized at scale $\operatorname{diam}(Q)$ (see section 5.10.2). Hence, passing the t-derivative onto $\tilde{\psi}(\operatorname{diam}(Q);\mathbf{x})$ because V_Q acts by convolution, and then using (5.125), (5.12) (and properties of ϕ) we have

$$\left|\mathcal{D}_t^Q\tilde{\psi}(\operatorname{diam}(Q);\mathbf{x})\right| = \left|V_Q * \partial_t\tilde{\psi}\,(\operatorname{diam}(Q);\mathbf{x})\right| \lesssim \frac{1}{\ell(Q)} \iint_{C_{10\operatorname{diam}(Q)}(\mathbf{x})} \|\mathbf{x} - \mathbf{y}\|^{1-d} \, d\mathbf{y} \lesssim 1\,.$$

Plugging the latter bound into (5.130) gives $|b_1| \lesssim |Q|^{1/2}$, as desired. The proof in the case of b_2 in (5.120) proceeds analogously and we omit the details. Having proved Lemma 5.53 and handling the boundary terms we have tied up all of the loose ends and proved Theorem 5.2 and Theorem 5.3.

5.11 Cut-off function construction

In this section we construct the cut-off function in Lemma 5.33. Here we use ideas in a forthcoming joint work of Bortz and Hofmann, with J.M. Martell and K. Nyström [BHMN25b].

First, consider P_r a parabolic approximate identity on $\mathbb{R}^{n+1}$, that is, we take $\zeta \in C_0^\infty([-1,1])$ be an even function with $\mathbf{1}_{[-1/2,1/2]} \leqslant \zeta \leqslant \mathbf{1}_{[-1,1]}$ and set

$$p(x_0,x,t) := c_n\,\zeta(x_0)\,\zeta(x_1)\ldots\zeta(x_{n-1})\,\zeta(t), \quad \forall\,\mathbf{X}=(x_0,x,t)=(x_0,x_1,\ldots,x_{n-1},t)\in\mathbb{R}^{n+1}, \tag{5.131}$$

where c_n is chosen so that $\iint_{\mathbb{R}^n} p(x_0,x,t)\,dx_0\,dx\,dt = 1$ and we define $P_r f(\mathbf{X}) := p_r * f(\mathbf{X})$, where $p_r(x_0,x,t) := r^{-n-2}p(x_0/r, x/r, t/r^2)$.

Now, for each $Q\in\mathbb{D}$, we set, for $\varepsilon=\varepsilon(K)>0$ sufficiently small depending on K

$$\tilde\eta_Q(x) := P_{\varepsilon\ell(Q)}\mathbf{1}_{U_Q^*}(x).$$

Then it is easy to verify (simply use elementary properties of convolutions and rescalings) that

$$\mathbf{1}_{U_Q^*} \lesssim \tilde\eta_Q \leqslant \mathbf{1}_{U_Q^{**}}, \quad \text{and} \quad |\nabla\tilde\eta_Q|\,\ell(Q) + |\partial_t\tilde\eta_Q|\,\ell(Q)^2 \lesssim 1, \tag{5.132}$$

with implicit constants depending on n, K and $\|\psi\|_{\mathrm{Lip}(1,1/2)}$. Let us omit these dependencies from now on.

Then, we set

$$\eta_Q := \frac{\tilde\eta_Q}{\sum_{Q'\in\mathbb{D}}\tilde\eta_{Q'}}.$$

Note now that whenever $U_Q^{**}\cap U_{Q'}^{**}\neq\varnothing$, it holds $\ell(Q)\approx\ell(Q')$. Actually, using this and (5.17), it is easy to deduce that the U_Q^{**} regions have bounded overlap (i.e., given a cube $Q\in\mathbb{D}$, there is only a –uniformly– bounded number of $Q'\in\mathbb{D}$ with $U_Q^{**}\cap U_{Q'}^{**}\neq\varnothing$). Therefore, from (5.132) it follows

$$\mathbf{1}_{U_Q^*} \lesssim \eta_Q \leqslant \mathbf{1}_{U_Q^{**}}, \quad \text{and} \quad |\nabla\eta_Q|\,\ell(Q) + |\partial_t\eta_Q|\,\ell(Q)^2 \lesssim 1. \tag{5.133}$$

Moreover, by construction,

$$\sum_{Q\in\mathbb{D}}\eta_Q \equiv 1 \quad \text{in } \Omega.$$

Now we define the cut-off function η. For simplicity of notation, set $\mathbf{S}' := \mathbf{S}_{N_1,N_2}$ and define

$$\eta := \sum_{Q\in\widetilde{\mathbf{S}}}\eta_Q, \quad \text{where} \quad \widetilde{\mathbf{S}} := \big\{Q'\in\mathbb{D} : \exists Q\in\mathbf{S}' \text{ such that } U_Q^{**}\cap U_{Q'}^{**}\neq\varnothing\big\}. \tag{5.134}$$

Then it is easy to verify (5.53): just use (5.133) and the definitions in Subsection 5.4.8. It is even easier to obtain (5.54): simply use (5.133) and the fact that at a given point, only finitely many η_Q can be non-zero, which follows from the bounded overlap of the U_Q^{**} regions.

Therefore, it remains to show (5.55) and (5.56). By definition

$$\big\{\nabla_{\mathbf{X}}\eta\neq 0\big\} \subseteq \bigcup_{Q\in\mathcal{F}_b} U_Q^{***}.$$

Using that $|U_Q^{***}|\approx\ell(Q)^{n+2}$ and (5.54), we easily check that (5.55) implies (5.56).

In order to prove (5.55) we make an observation:

$$Q\in\mathcal{F}_b \quad\Longrightarrow\quad \exists\ Q_{\mathrm{in}}\in\mathbf{S}',\ Q_{\mathrm{out}}\in\mathbb{D}\backslash\mathbf{S}' \text{ such that } U_Q^{***}\cap U_{Q_{\mathrm{in}}}^{***}\neq\varnothing\neq U_Q^{***}\cap U_{Q_{\mathrm{out}}}^{***}. \tag{5.135}$$

Indeed, given $Q \in \mathcal{F}_b$, pick $\mathbf{X} \in U_Q^{***}$ with $\nabla_{\mathbf{X}}\eta(\mathbf{X}) \neq 0$. Then, by (5.134) and (5.133), there exists $Q_{\text{out}} \in \widetilde{S}$ with $\mathbf{X} \in U_{Q_{\text{out}}}^{**}$. Then, using the definition of $\widetilde{S}$, we may find $Q_{\text{in}} \in \mathbf{S}'$ with $U_{Q_{\text{out}}}^{**} \subseteq U_{Q_{\text{in}}}^{***}$. This gives the result for Q_{out} because $\mathbf{X} \in U_Q^{***} \cap U_{Q_{\text{out}}}^{***}$. After that, let us note that if Q_{out} belonged to $\mathbf{S}'$, then $\mathbf{X}$ would belong to $\Omega_{\mathbf{S}'}^{**}$, which is an open set where $\eta \equiv 1$ (this is (5.53)), which would lead to a contradiction because then we would have $\nabla_{\mathbf{X}}\eta(\mathbf{X}) = 0$. This gives the result for Q_{in}.

In this case, let us note that since $U_Q^{***} \cap U_{Q_{\text{in}}}^{***} \neq \varnothing \neq U_Q^{***} \cap U_{Q_{\text{out}}}^{***}$, it holds that

$$\ell(Q_{\text{in}}) \approx \ell(Q) \approx \ell(Q_{\text{out}}) \quad \text{and} \quad \operatorname{dist}(Q, Q_{\text{in}}),\, \operatorname{dist}(Q, Q_{\text{out}}) \lesssim \ell(Q). \tag{5.136}$$

Also, this implies that, given $Q' \in \mathbf{S}$, the amount of cubes for which Q' is their Q_{in} is bounded:

$$\#\big\{Q \in \mathcal{F}_b : Q_{\text{in}} = Q'\big\} \lesssim 1. \tag{5.137}$$

We are going to break up into cases, depending on the properties of the Q_{out} from (5.135).

- $\mathcal{F}_1$ is the collection of cubes $Q \in \mathcal{F}_b$ such that $\ell(Q_{\text{out}}) > 2^{-N_1}\ell(Q(\mathbf{S}))$,
- $\mathcal{F}_2$ is the collection of cubes $Q \in \mathcal{F}_b$ such that $\ell(Q_{\text{out}}) \leqslant 2^{-N_1}\ell(Q(\mathbf{S}))$ and $Q_{\text{out}} \cap Q(\mathbf{S}) \neq \varnothing$,
- $\mathcal{F}_3$ is the collection of cubes $Q \in \mathcal{F}_b$ such that $\ell(Q_{\text{out}}) \leqslant 2^{-N_1}\ell(Q(\mathbf{S}))$ and $Q_{\text{out}} \cap Q(\mathbf{S}) = \varnothing$.

Case $\mathcal{F}_1$. If we have $Q \in \mathcal{F}_1$, then, after recalling that $Q_{\text{in}} \in \mathbf{S}' = \mathbf{S}_{N_1,N_2}$, we quickly realize that $\ell(Q_{\text{in}}) \leqslant 2^{-N_1}\ell(Q(\mathbf{S}))$. Then, since $\ell(Q_{\text{in}}) \approx \ell(Q_{\text{out}})$ (see (5.136)) and $\ell(Q_{\text{out}}) > 2^{-N_1}\ell(Q(\mathbf{S}))$ (for $Q \in \mathcal{F}_1$), we obtain that

$$\ell(Q_{\text{in}}) \approx 2^{-N_1}\ell(Q(\mathbf{S})).$$

Using this, (5.137), $\ell(Q_{\text{in}}) \approx \ell(Q)$ from (5.136), and the fact that $Q_{\text{in}} \subseteq Q(\mathbf{S})$, we get

$$\sum_{Q\in\mathcal{F}_1} |Q| \lesssim \sum_{\substack{Q_{\text{in}}\in\mathbf{S}' \\ \ell(Q_{\text{in}})\approx 2^{-N_1}\ell(Q(\mathbf{S}))}} |Q_{\text{in}}| \lesssim |Q(\mathbf{S})|.$$

Next, to treat the cubes in $\mathcal{F}_2$ and $\mathcal{F}_3$, let us fix some notation. Let $\mathbf{x}_{Q_{\text{out}}}$ be the center of Q_{out} and let $Q_{\text{out}}{}^\star \in \mathbb{D}(Q_{\text{out}})$ be such that $\mathbf{x}_{Q_{\text{out}}} \in Q_{\text{out}}{}^\star$ and $\ell(Q_{\text{out}}{}^\star) = 2^{-10}\ell(Q_{\text{out}})$. Then

$$\operatorname{dist}(Q_{\text{out}}{}^\star, (Q_{\text{out}})^c) \approx \ell(Q_{\text{out}}), \qquad \ell(Q_{\text{out}}{}^\star) \approx \ell(Q_{\text{out}}). \tag{5.138}$$

Case $\mathcal{F}_2$. Let us start with $\mathcal{F}_2$. Let $\mathcal{B}'$ denote the minimal cubes in $\mathbf{S}'$. If $Q \in \mathcal{F}_2$, then $Q_{\text{out}} \subseteq \widetilde{Q}$ for some $\widetilde{Q} \in \mathcal{B}'$ (just develop the definition of $Q \in \mathcal{F}_2$), and hence $Q_{\text{out}}{}^\star \subseteq \widetilde{Q}$. We claim that

$$\ell(Q_{\text{out}}{}^\star) \lesssim \operatorname{dist}(Q_{\text{out}}{}^\star, (\widetilde{Q})^c) \lesssim \ell(Q_{\text{out}}{}^\star). \tag{5.139}$$

Taking this claim momentarily for granted, let us continue. Fix $\widetilde{Q} \in \mathcal{B}'$. Let us consider the family of cubes $Q_k \in \mathcal{F}_2$ for which $(Q_k)_{\text{out}} \subseteq \widetilde{Q}$. Then (5.139) implies that the associated $\{(Q_k)_{\text{out}}^\star\}_k$ have bounded overlap. Indeed, if $(Q_k)_{\text{out}}^\star \cap (Q_{k'})_{\text{out}}^\star \neq \varnothing$, then (5.139) (and (5.138)) would imply that $\ell(Q_k) \approx \ell(Q_k^\star) \approx \ell(Q_{k'}^\star) \approx \ell(Q_{k'})$, which would easily imply that $\operatorname{dist}((Q_k)_{\text{out}}^\star, (Q_{k'})_{\text{out}}^\star) \lesssim \ell((Q_k)_{\text{out}})$. Therefore, for a fixed k, the bounded overlap follows by the above facts:

$$\#\big\{Q_{k'}^\star : Q_{k'}^\star \cap Q_k^\star \neq \varnothing\big\} \leqslant \#\big\{Q_{k'} : \ell(Q_{k'}) \approx \ell(Q_k),\ \operatorname{dist}(Q_k, Q_{k'}) \lesssim \ell(Q_k)\big\} \lesssim 1.$$

These facts allow us to finish this case by computing

$$\sum_{Q\in\mathcal{F}_2}|Q|=\sum_{\widetilde{Q}\in\mathcal{B}'}\sum_{\substack{Q_k\in\mathcal{F}_2\\(Q_k)_{\text{out}}\subseteq\widetilde{Q}}}|Q_k|\approx\sum_{\widetilde{Q}\in\mathcal{B}'}\sum_{\substack{Q_k\in\mathcal{F}_2\\(Q_k)_{\text{out}}\subseteq\widetilde{Q}}}|(Q_k)^\star_{\text{out}}|\lesssim\sum_{\widetilde{Q}\in\mathcal{B}'}|\widetilde{Q}|\leqslant|Q(\mathbf{S})|.$$

In this line, the first equality is an elementary rearrangement since $Q\in\mathcal{F}_2$ implies $Q_{\text{out}}\subseteq\widetilde{Q}$, then we use $\ell((Q_k)^\star_{\text{out}})\approx\ell((Q_k)_{\text{out}})\approx\ell(Q)$, later we use the bounded overlap discussed in the last paragraph, and we finish by disjointness of the family $\mathcal{B}'$.

Therefore, we are left to prove the claim (5.139). First, the lower bound comes from (5.138), since $Q'\subseteq\widetilde{Q}$. In turn, for the upper bound, we break into cases:

- Case $\widetilde{Q}\cap Q_{\text{in}}=\varnothing$. Then $Q_{\text{in}}\subseteq\widetilde{Q}$, which allows us to compute (using also (5.136) and (5.138))

$$\operatorname{dist}(Q_{\text{out}}{}^\star,(\widetilde{Q})^c)\leqslant\operatorname{dist}(Q_{\text{out}}{}^\star,Q_{\text{in}})\leqslant\operatorname{diam}(Q_{\text{out}})+\operatorname{dist}(Q_{\text{out}},Q_{\text{in}})\lesssim\ell(Q_{\text{out}})\approx\ell(Q_{\text{out}}{}^\star).$$

- Case $\widetilde{Q}\subseteq Q_{\text{in}}$. Then we have, using (5.136) and (5.138),

$$Q_{\text{out}}{}^\star\subseteq\widetilde{Q}\subseteq Q_{\text{in}},\quad\text{ and }\quad\ell(Q_{\text{in}})\lesssim\ell(Q_{\text{out}})\lesssim\ell(Q_{\text{out}}{}^\star),$$

  which clearly implies that $\ell(\widetilde{Q})\approx\ell(Q_{\text{out}}{}^\star)$. With this and the fact that $\mathbf{x}_{Q_{\text{out}}}\in Q_{\text{out}}{}^\star\subseteq\widetilde{Q}$, we finish by this simple computation:

$$\operatorname{dist}(Q_{\text{out}}{}^\star,(\widetilde{Q})^c)\leqslant\operatorname{dist}(\mathbf{x}_{Q_{\text{out}}},\partial\widetilde{Q})\leqslant\operatorname{diam}(\widetilde{Q})\approx\ell(\widetilde{Q})\approx\ell(Q_{\text{out}}{}^\star)\approx\ell(Q_{\text{out}}).$$

Actually, since $Q_{\text{in}}\in\mathbf{S}'$ and $\widetilde{Q}\in\mathcal{B}'$ (minimal cube in $\mathbf{S}'$), there are no more cases to be considered. Therefore, we are done with claim (5.139), and hence with Case $\mathcal{F}_2$.

Case $\mathcal{F}_3$. Now we conclude by handling the collection $\mathcal{F}_3$, but this is done in a similar manner to $\mathcal{F}_2$. In this case, if $Q\in\mathcal{F}_3$, then we claim that

$$\ell(Q_{\text{out}}{}^\star)\lesssim\operatorname{dist}(Q_{\text{out}}{}^\star,Q(\mathbf{S}))\lesssim\ell(Q_{\text{out}}{}^\star).\tag{5.140}$$

Indeed, the lower bound follows from (5.138) and the fact that $Q(\mathbf{S})\subseteq(Q_{\text{out}})^c$. And for the upper bound, note that $Q_{\text{in}}\subseteq Q(\mathbf{S})$ because $Q_{\text{in}}\in\mathbf{S}'=\mathbf{S}_{N_1,N_2}$. Then, using also (5.136) and (5.138),

$$\operatorname{dist}(Q_{\text{out}}{}^\star,Q(\mathbf{S}))\leqslant\operatorname{dist}(Q_{\text{out}}{}^\star,Q_{\text{in}})\leqslant\operatorname{diam}(Q_{\text{out}})+\operatorname{dist}(Q_{\text{out}},Q_{\text{in}})\lesssim\ell(Q_{\text{out}})\approx\ell(Q_{\text{out}}{}^\star).$$

Now, from (5.140) and the fact that $\ell(Q_{\text{out}}{}^\star)\lesssim\ell(Q(\mathbf{S}))$ (because $Q_{\text{in}}\in\mathbf{S}$, and also (5.136) and (5.138)), it is easy to deduce that there exists a uniform $C\geqslant1$ such that $Q_{\text{out}}{}^\star\subseteq CQ(\mathbf{S})$. This and the fact that $\{Q_{\text{out}}{}^\star\}$ have bounded overlap (which can be proved in a similar manner as in Case $\mathcal{F}_2$, this time using (5.140) instead of (5.139)) readily yield the desired estimate

$$\sum_{Q\in\mathcal{F}_3}|Q|\lesssim\sum_{Q\in\mathcal{F}_3}|Q_{\text{out}}{}^\star|\lesssim|CQ(\mathbf{S})|\lesssim|Q(\mathbf{S})|.$$

This finishes Case $\mathcal{F}_3$, and ultimately the proof of (5.55), and thus of Lemma 5.33.

5.12 Comments on smoothness assumptions

In this section, we describe how one can extend the main results in this chapter to rougher matrices than the ones considered in our Definition 5.24, because concretely we have assumed them to be smooth. In fact, we will obtain a satisfactory extension of Theorem 5.2 in the case that the coefficients A are symmetric; and at the end of the section, we discuss how one might extend the results to non-symmetric matrices (for a possible extension of Theorem 5.3).

For brevity in the notation, in the setting of Theorem 5.2, we denote

$$\delta(\mathbf{X}) := \operatorname{dist}(\mathbf{X}, \partial\Omega), \qquad \text{and} \qquad \operatorname{osc}(A, \mathbf{X}) := \sup_{\mathbf{Z},\mathbf{Y}\in\mathcal{J}_{\delta(\mathbf{X})/(1000\sqrt{n+1})}(\mathbf{X})} |A(\mathbf{Y}) - A(\mathbf{Z})|.$$

Then, the following extension of Theorem 5.2 can be made.

Theorem 5.55. *Let $n \geqslant 2$. Suppose that Ω is the region above a $\operatorname{Lip}(1, 1/2)$ function ψ (see Definition 5.4.3), and A is a **symmetric** elliptic matrix satisfying (5.2) and the following Carleson condition*

$$\sup_{\mathbf{X}\in\partial\Omega} \sup_{\rho>0} \rho^{-n-1} \iiint_{\mathcal{J}_\rho(\mathbf{X})\cap\Omega} \frac{\operatorname{osc}(A, \mathbf{Y})}{\delta(\mathbf{Y})}\, d\mathbf{Y} < \infty. \tag{5.141}$$

Assume further that there exists $1 < p < \infty$ such that the L^p Dirichlet problem is solvable for L (see Definition 5.20), where

$$Lf := \partial_t f - \operatorname{div}_X(A(X,t)\nabla_X f).$$

Then $D_t^{1/2}\psi \in \operatorname{BMO}$ (the parabolic BMO space, see Section 5.4.5). Equivalently, the L^p solvability of the Dirichlet problem implies that the graph of ψ is parabolic uniformly rectifiable.

Remark 5.56. Note that (5.141) holds whenever A satisfies the Carleson condition

$$\sup_{\mathbf{X}\in\partial\Omega} \sup_{\rho>0} \rho^{-n-1} \iiint_{\mathcal{J}_\rho(\mathbf{X})\cap\Omega} \sup_{\mathbf{Y}\in\mathcal{J}_{\delta(\mathbf{X})/(1000\sqrt{n+1})}(\mathbf{X})} \big(|\nabla A(\mathbf{Y})| + |\partial_t A(\mathbf{Y})|\delta(\mathbf{Y})\big)\, d\mathbf{Y} < \infty,$$

which is actually a stronger condition than our Carleson measure assumption in Theorem 5.2, namely (5.24) (indeed, the measure in (5.23) does not have the supremum above). Therefore, Theorem 5.55 extends Theorem 5.2.

Proof. The idea is to regularize A and use some perturbative argument. For that, first let P_r be the parabolic approximate identity on $\mathbb{R}^{n+1}$ from Section 5.11 (see (5.131)). We then let $\delta'(\cdot)$ be a regularized version of $\delta(\cdot)$ (we can use [BHHLN23a, Lemma 3.24], which is based on [**?**, Chapter VI, §1 & 2]) so that

$$c_n\delta(\mathbf{X}) \leqslant \delta'(\mathbf{X}) \leqslant \delta(\mathbf{X}), \qquad \delta(\mathbf{X})^{k-1}|\nabla_X^k\delta'(\mathbf{X})| + \delta(\mathbf{X})^{2k-1}|\partial_t^k\delta'(\mathbf{X})| \lesssim_{k,n} 1, \qquad \mathbf{X}\in\Omega.$$

Then for $c := (10^5\sqrt{n+1})^{-1}$ we define

$$\widetilde{A}(\mathbf{X}) := p_{c\delta'(\mathbf{X})} * A(\mathbf{X}) := \iiint p_{c\delta'(\mathbf{X})}(\mathbf{Y})A(\mathbf{X}-\mathbf{Y})\, d\mathbf{Y},$$

where the integral is taken component-wise. Then it is not hard to show that

$$|\nabla_X \widetilde{A}(\mathbf{Y})| + |\partial_t \widetilde{A}(\mathbf{Y})|\delta(\mathbf{Y}) \lesssim \frac{\operatorname{osc}(A,\mathbf{Y})}{\delta(\mathbf{Y})}, \qquad \mathbf{Y}\in\Omega,$$

and, moreover,

$$\delta(\mathbf{Y})^k|\nabla_X^k \widetilde{A}(\mathbf{Y})| + \delta(\mathbf{Y})^{2k}|\partial_t^k \widetilde{A}(\mathbf{Y})| \lesssim_{n,k} \|A\|_{L^\infty}, \qquad k=1,2,\dots,\ \mathbf{Y}\in\Omega.$$

These (and the hypothesis (5.141)) easily imply that $\widetilde{A}$ satisfies the L^1-Carleson oscillation condition in Definition 5.24 (and it is elementary to see that it is also elliptic). Therefore, since we have not modified Ω nor Σ, to prove Theorem 5.55 it suffices to show that there exists $1<q<\infty$ such that the L^q Dirichlet problem is solvable for $\widetilde{L}$, where

$$\widetilde{L}f := \partial_t f - \operatorname{div}_X(\widetilde{A}(X,t)\nabla_X f).$$

To this end, we use the already established perturbative theory in [Ny97], which requires us to compare A to $\widetilde{A}$. For that purpose, define

$$a(\mathbf{Y}) := \sup_{\mathbf{Z}\in\mathcal{J}_{\delta(\mathbf{X})/(10^{20}\sqrt{n+1})}(\mathbf{Y})} |A(\mathbf{Z})-\widetilde{A}(\mathbf{Z})|, \qquad \mathbf{Y}\in\Omega.$$

Then, it is easy to show that

$$\|a\|_{L^\infty} \lesssim \|A\|_{L^\infty}, \quad \text{and} \quad a(\mathbf{Y}) \lesssim \operatorname{osc}(A,\mathbf{Y}), \quad \mathbf{Y}\in\Omega,$$

which imply, jointly with (5.141), the estimate

$$\sup_{\mathbf{X}\in\partial\Omega}\sup_{\rho>0}\ \rho^{-n-1}\int_{\mathcal{J}_\rho(\mathbf{X})\cap\Omega}\frac{a(\mathbf{Y})^2}{\delta(\mathbf{Y})}\,d\mathbf{Y}<\infty.$$

With this, by the perturbative theory in [Ny97, Theorem 6.4], there exists $1<q<\infty$ such that the L^q Dirichlet problem is solvable for $\widetilde{L}$, which finishes the proof. □

Finally, we close with a remark that while in this section we assumed that A was symmetric, it seems likely that the results in [Ny97] hold for non-symmetric A. Indeed, this is the case in the elliptic setting.

5.13 A linear algebra fact about elliptic matrices

In this last section, we will show that (5.60) holds because actually the stronger pointwise estimate $|\nabla^2 u|^2 \approx (a_{i,j}u_{x_i,x_k})(a_{k,\ell}u_{x_j,x_\ell})$ is true (along the section, we use the summation convention over repeated indices). In fact, we will prove a more general result for matrices.

Lemma 5.57. *Let A,B be $n\times n$ real and symmetric matrices. Suppose further that A is elliptic and bounded, i.e. $\lambda\in[\Lambda^{-1},\Lambda]$ for every eigenvalue λ of A, for some $\Lambda>0$. Then*

$$|B|^2 \approx a_{i,j}b_{i,k}a_{k,\ell}b_{j,\ell},$$

with implicit constant depending only on Λ. Here and everywhere along the section, $|X|$ denotes the Hilbert-Schmidt norm of the matrix X, i.e. $|X|^2 := \sum_{i,j=1}^n |x_{i,j}|^2$.

Indeed, specializing the previous theorem to $B = \nabla^2 u$, which is symmetric by the discussion in Subsection 5.7.2, and A is of course our coefficient matrix, we obtain (5.60). The proof of the lemma will follow from a sequence of easier intermediate results.

Lemma 5.58. *Let $\widetilde{A}$ be as in Lemma 5.57 and B be any $n \times n$ matrix (not necessarily symmetric). Then (with implicit constants depending only on Λ)*

$$|\widetilde{A}B| \approx |B| \approx |B\widetilde{A}|.$$

Proof. Let us only show the first equivalence, the other one being similar. Since $\widetilde{A}$ is symmetric, we can use the spectral theorem to diagonalize it, i.e., there exists an orthogonal matrix U and a diagonal matrix $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ such that $\widetilde{A} = U^T D U$. In fact, the assumption on the eigenvalues of $\widetilde{A}$ implies that $\lambda_i \approx_\Lambda 1$ for $i = 1, \dots, n$. This easily implies that for any vector $v \in \mathbb{R}^n$, $|Dv| \approx_\Lambda |v|$. Hence, computing the norm of any matrix X by testing against vectors (i.e. $|X| = \sup_{v \in \mathbb{R}^n : |v| \leqslant 1} |Xv|$), we also obtain $|DX| \approx_\Lambda |X|$. Using this and the fact that multiplication by orthogonal matrices preserves the Hilbert-Schmidt norm (recall that they are isometries in the euclidean space), we finish by computing

$$|B| = |UB| \approx_\Lambda |DUB| = |U^T D U B| = |\widetilde{A}B|.$$

□

Corollary 5.59. *Let A, B as in Lemma 5.57. Then (with implicit constant depending only on Λ)*

$$|B| \approx |A^{1/2} B A^{1/2}|.$$

Proof. It is an elementary linear algebra fact that symmetric elliptic matrices (i.e. those whose eigenvalues are positive) like A have a (unique) square root matrix $A^{1/2}$ which is also symmetric. It is also elementary that the eigenvalues of $A^{1/2}$ lie in $[\Lambda^{-1/2}, \Lambda^{1/2}]$. Therefore, the result follows by applying Lemma 5.58 twice with $\widetilde{A} = A^{1/2}$. □

Proof of Lemma 5.57. Let $\tilde{a}_{i,j}$ denote the entries in $A^{1/2}$. Then, clearly the entries in $A^{1/2} B A^{1/2}$ are

$$(A^{1/2} B A^{1/2})_{\alpha,\beta} = \tilde{a}_{\alpha,i} b_{i,k} \tilde{a}_{k,\beta}.$$

And using the symmetry of both B and $A^{1/2}$, we similarly obtain

$$(A^{1/2} B A^{1/2})_{\alpha,\beta} = \tilde{a}_{\beta,\ell} b_{\ell,j} \tilde{a}_{j,\alpha}.$$

Moreover, note also that since $A^{1/2} A^{1/2} = A$, then we have

$$a_{k,\ell} = \tilde{a}_{k,m} \tilde{a}_{m,\ell}, \qquad \text{for } k, \ell = 1, \dots, n.$$

Using all the above facts together and Corollary 5.58, we finish by computing

$$|B|^2 \approx_\Lambda |A^{1/2} B A^{1/2}|^2 = \sum_{\alpha,\beta=1}^{n} ((A^{1/2} B A^{1/2})_{\alpha,\beta})^2 = \tilde{a}_{\alpha,i} b_{i,k} \tilde{a}_{k,\beta} \tilde{a}_{\beta,\ell} b_{\ell,j} \tilde{a}_{j,\alpha} = a_{j,i} a_{k,\ell} b_{i,k} b_{\ell,j},$$

which finishes the proof accounting for the symmetry of both A and B. □

CHAPTER 6

Overdetermined elliptic boundary value problems

So far, we have only considered PDE with Dirichlet boundary data, i.e. prescribing the pointwise value of our solutions at the boundary. If we would desire to fix, on top of Dirichlet data, the value of the derivatives (the so-called Neumann conditions) and in this way have an overdetermined problem, we would (most probably) not be able to find solutions. The reason is simple: we discussed in Section 2.4.1 that, under very mild assumptions, there is uniqueness of solutions with Dirichlet data. Thus, there is no flexibility to impose any further boundary condition, be it Neumann or in any other form.

However, there is still some hope if we are flexible in the sense that we do not fix a priori the domain in which we work in. The problem becomes more difficult because instead of searching only for a function (the solution to a PDE), we will be looking for both a set and a function (the domain and the solution). But the seminal work of Alt and Caffarelli [AC81] showed that there are variational methods (i.e. based on minimization of energies) that allow us to find sets and functions which are prone to be domains and solutions of a prescribed PDE with both Dirichlet and Neumann boundary conditions. The power of the work of Alt and Caffarelli resides in the fact that they not only show that solutions exist, because they could be a priori very irregular or poorly behaved, but that they also manage to prove that they are (up to some extent) regular. This makes their solutions useful to be considered in applications.

Actually, the gain in flexibility of the method introduced by Alt and Caffarelli was not automatically translated into a gain in the amount of solutions to overdetermined problems that were known: the previous groundbreaking result of Serrin in [Se71] (which has been recently generalized in [FZ24]) asserted that the only domains admitting solutions to $-\Delta u = 1$, with constant Dirichlet and Neumann boundary conditions, are balls. That is, only the trivial explicit solutions exist. This means that if one wants to find non-trivial solutions to overdetermined problems, one needs to consider non-trivial equations and/or non-trivial boundary conditions.

In this chapter, we show that, in fact, there are non-trivial solutions to a very large family of overdetermined problems. Concretely, we can show that given a fairly wide family (in fact, in some senses, optimal) of semilinear elliptic PDE and periodic boundary data, we can always find solutions of the associated overdetermined problems. As a consequence of the robustness of our method, we can also find solutions to overdetermined problems in compact manifolds.

Moreover, we can find solutions which are supported in domains of arbitrary large size. This makes an important difference with other methods based on perturbation arguments from trivial solutions in very small domains like [PS09, DS15, DEP19, DEP23].

The novelty of this work resides mainly on two aspects. First, we show existence of minimizers of the Alt-Caffarelli energy with energies which are both position dependent and non-linear, without the need to prescribe the values of the functions anywhere (in contrast to the original [AC81] and most of the related follow-ups like [Le96, GS96, Ve23]). We prefer this approach (although it is substantially more difficult to develop) because prescribing the values over any auxiliary set forces the minimizers to behave in some specific way in the nearby, and probably the energies would not have favored that behavior if we had not prescribed those values; hence the minimization of the energy is somehow losing (at least partially) its meaning. Moreover, we are able to find minimizers merely under the assumption of periodicity, which largely extends previous work. All this will be made more precise at due time.

A second novelty is that we extend Alt-Caffarelli's program as a whole, and not only some specific parts. Indeed, in the last decades there has been intensive research to improve some concrete parts of the minimization program: a well-known example is the very influential work of De Silva [De11], which opened the door to a completely new way to show that flat boundaries are regular. In contrast, we handle the whole minimization program, finding some (rather small and natural) set of hypotheses under which we are able to carry out all the necessary work.

This chapter is based on the paper [EHR24], done in collaboration with Alberto Enciso and Xavier Ros-Oton.

We suggest the reader to have Section 6.3 handy while reading the chapter, for it has the key ideas and provides a simplified and understandable sketch of the proofs that will be carried out in all detail along the rest of the chapter.

6.1 Further historical context . . . 223
6.2 Main results . . . 224
6.2.1 Some comments about the proof and connection with previous work . . . 226
6.3 Main ideas and outline of the chapter . . . 227
6.4 Assumptions and auxiliary results . . . 228
6.4.1 The admissibility and periodicity assumptions . . . 228
6.4.2 An approximate mean value formula for inhomogeneous equations . . . 230
6.5 Some basic properties of minimizers . . . 232
6.6 Asymptotic minimality at small scales . . . 237
6.7 Lipschitz continuity and non-degeneracy . . . 241
6.8 Compact support of minimizers . . . 245
6.9 The free boundary . . . 246
6.9.1 Analysis of blow-up limits . . . 247
6.9.2 Overdetermined boundary condition . . . 256
6.9.3 Decomposition and properties of the free boundary . . . 259
6.10 Proof of the main result . . . 262
6.11 Overdetermined problems on manifolds . . . 265
6.12 Non-periodic or fastly growing counterexamples . . . 267

6.1 Further historical context

Before getting to the details of the main results and their proofs, let us give further context to our work in this chapter. We explain next the state of the art in the topic, complementing the explanations in the last pages and Section 1.4.

Indeed, the study of overdetermined boundary problems, that is, problems where one prescribes both Dirichlet and Neumann data, has grown into a major field of research. Although this topic had already been considered in Lord Rayleigh's classic treatise [Ra77], the breakthrough result was Serrin's symmetry result [Se71]. Under mild technical hypotheses, he showed that if a positive function satisfies a semilinear equation of the form

$$-\Delta v = f(v)$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with the boundary conditions

$$v = 0 \quad \text{and} \quad |\nabla v| = c \qquad \text{on } \partial\Omega$$

for some constant c, then Ω is a ball and v is radial.

There is a large body of literature generalizing Serrin's result. As a sample of the available results, and without attempting to be comprehensive, let us mention the development of alternative approaches to symmetry that rely on P-functions [GL89, Ka98] or Pohozaev-type integral identities [BNST08, MP20a, MP20b], quantitative estimates for the stability of these symmetry properties [ABR99], and extensions to a variety of related contexts: different constant-coefficient elliptic operators of variational form [CS09]; degenerate elliptic equations such as the p-Laplace equation [DPR99]; exterior [AB98, GS99], unbounded [FV10] or non-smooth domains [Pr98, FZ24]; and problems on the hyperbolic space and the hemisphere [KP98]. The connection between overdetermined problems on unbounded domains and constant mean curvature surfaces has been studied in [Tr14, DPW15, RRS20]. Nontrivial solutions have been shown to exist in a number of interesting situations, for instance in the case of sign-changing solutions [Ru22], for periodic unbounded domains [FMW17, FMW18, RSW22], and for problems that are either partially overdetermined or degenerate [AG89, FGK06, FG08, FV13], and have led to recent progress on various questions in fluid mechanics [DEP21, Ru23, EFR24] and spectral geometry [FMW23, EFRS23].

Even though these questions arise in a natural way in applied contexts [To96, Si02, DEP21, HHST+22, EFR24], the study of overdetermined problems for position-dependent equations has attracted comparatively little attention. In two remarkable papers, Pacard and Sicbaldi [PS09] and Delay and Sicbaldi [DS15] proved the existence of smooth extremal domains with small volume for the first eigenvalue of the Laplacian in any compact Riemannian manifold. These results were extended in [DEP19, DEP23] to the case of fairly general overdetermined semilinear problems depending on a free parameter λ. Roughly speaking, one can consider a nonlinear eigenvalue problem of the form

$$\begin{cases} -\operatorname{div}(A(X)\nabla v) = \lambda f(X, v) & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ v = 0 \quad \text{and} \quad |\nabla v|^2 = c\, q(x) & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Here q is a positive function, c is a free constant as above, A is a uniformly elliptic matrix-valued function, and λ is an additional free constant. In all these results, the key idea is to study the high-frequency regime $\lambda \gg 1$. After a careful asymptotic analysis, and under suitable technical assumptions, one can show that for all large enough λ, the overdetermined problem (6.1) admits a positive solution on

a family of smooth domains constructed as suitably small perturbations of small balls, of radius of order $\lambda^{-1/2}$ and centered at specific points depending on $f(\cdot, 0)$. Thus these are perturbation results, asserting that, in local coordinates adapted to the matrix A around a judiciously chosen point, one can perturb a positive radial solution to the limiting problem to obtain a solution to (6.1). Similar results can be proven for overdetermined nonlinear eigenvalue problems on compact manifolds.

Our objective in this chapter is to remove the additional freedom granted by the free parameter λ. That is, we aim to study the existence of positive solutions to fairly general classes of semilinear overdetermined boundary value problems that do *not* necessarily have a free parameter, and to consider domains that are *not* necessarily deformations of small balls. Since these conditions are crucially used in the aforementioned papers to frame the problem in the context of perturbation theory, our results must therefore be based on an entirely different set of ideas, coming from the theory of elliptic free boundary problems.

6.2 Main results

We will present our results for positive solutions to the model semilinear problem

$$\begin{cases} -\operatorname{div}(A(X)\nabla v) = f(X, v) & \text{in } \Omega \subset \mathbb{R}^{n+1}, \\ v = 0 \quad \text{and} \quad \nabla v\, A(x)\, \nabla v^{\mathrm{T}} = c\, q(x) & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

Note we are using the natural Neumann boundary condition for this equation, which reduces to $\partial_\nu v = -\sqrt{c\, q(x)}$ when A is the identity matrix. One can also consider similar problems on compact manifolds, but for concreteness we will relegate the discussion to Section 6.11 in the main text. In (6.2), the functions A, f and q are fixed, while the constant c is not specified a priori. We assume that these functions are *admissible and x-periodic*. Roughly speaking, this means that $f(x, v)$, $A(x)$, and $q(x)$ are sufficiently smooth and periodic in x, that $f(x, v)$ is nonnegative and grows at most linearly in v, that $q(x)$ is positive, and that the matrix $A(x)$ is uniformly elliptic. Details given in Section 6.4.

To control the size of the (possibly disconnected) open set Ω, we will fix a constant $m > 0$ and restrict our attention to sets $\Omega \subset \mathbb{R}^{n+1}$ such that

$$\operatorname{Vol}_q(\Omega) := \iint_\Omega q(X)\, dX = m\,. \tag{6.3}$$

Since q is both upper bounded and bounded away from zero, this quantity is always comparable with the volume of Ω, and coincides with it in the classical case $q \equiv 1$.

Our main result can the be stated as follows. As we will discuss later in this section, the result is new even for the Poisson equation $-\Delta v = g(X)$ with constant Neumann data.

Theorem 6.1. *Let $n + 1 \leqslant 4$. For any constant $m > 0$ and any smooth, admissible, x-periodic functions q, A and f, there exists a bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary, satisfying the q-volume constraint (6.3), and a constant $c > 0$, for which the overdetermined boundary value problem (6.2) admits a positive solution $v \in C^\infty(\overline{\Omega})$.*

To prove Theorem 6.1, we follow a variational approach, motivated by the seminal work of Alt and Caffarelli [AC81]. To this end, let us define a function $F : \mathbb{R}^{n+1} \times [0, +\infty) \to \mathbb{R}$ by

$$F(X, u) := \int_0^u f(X, t)\, dt\,, \tag{6.4}$$

so that $f(X,u) = F'(X,u)$. Here and in what follows, $F'(X,u)$ denotes the derivative of $F(X,u)$ with respect to the second variable, u. In view of the constraint (6.3), we aim to minimize the energy $\mathcal{F}_0(u)$, among the functions in the space

$$\mathbb{K}_{\leqslant m} := \big\{u \in H^1(\mathbb{R}^{n+1}) : \ u \geqslant 0, \ \ \mathrm{Vol}_q(\{u>0\}) \leqslant m\big\},$$

with the hope that the minimizer actually belongs to the smaller space

$$\mathbb{K}_{=m} := \big\{u \in \mathbb{K}_{\leqslant m} : \mathrm{Vol}_q(\{u>0\}) = m\big\}.$$

For $D \subset \mathbb{R}^{n+1}$, we are denoting by

$$\mathcal{F}_0(u,D) := \iint_D \nabla u(X)\, A(X)\, \nabla u(X)^{\mathrm{T}}\, dX - 2 \iint_D F(X,u(X))\, dX \tag{6.5}$$

the energy functional associated with Equation (6.2), and we use the shorthand notation $\mathcal{F}_0(u) := \mathcal{F}_0(u,\mathbb{R}^{n+1})$.

Setting $\Omega := \{u>0\}$ and $v := u|_\Omega$, Theorem 6.1 is proven a consequence of the following result on the existence and regularity of minimizers to the above variational problem. For the definition of the Neumann boundary trace of u in the viscosity sense, we refer to Definition 6.32 in the main text.

Theorem 6.2. *Let $m > 0$ and assume that F, A and q are admissible and x-periodic in the sense of Definition 6.3. Then there exists a Lipschitz continuous minimizer u of $\mathcal{F}_0$ in $\mathbb{K}_{\leqslant m}$, which actually belongs to $\mathbb{K}_{=m}$. The set $\{u>0\}$ is open and bounded, and the following equation is satisfied*

$$\begin{cases} -\operatorname{div}(A\nabla u) = F'(X,u) & \text{in } \{u>0\} \subset \mathbb{R}^{n+1}, \\ \nabla u\, A\, \nabla u^{\mathrm{T}} = c\, q(x) & \text{on } \partial\{u>0\} \text{ in the viscosity sense} \end{cases} \tag{6.6}$$

for some constant $c>0$. Furthermore, the free boundary can be decomposed as a disjoint union

$$\partial\{u>0\} = \mathrm{Reg}(\{u>0\}) \cup \mathrm{Sing}(\{u>0\}),$$

where:

1. *The regular part $\mathrm{Reg}(\{u>0\})$ is a $C^{1,\alpha}$ manifold of dimension n for some $\alpha>0$, it is open within $\partial\{u>0\}$, and on it the Neumann condition holds in the classical sense: $\nabla u\, A\, \nabla u^{\mathrm{T}} = c\, q$. If A, q and f are smooth, $\mathrm{Reg}(\{u>0\})$ is smooth as well.*

2. *The singular part $\mathrm{Sing}(\{u>0\})$ is a closed set within $\partial\{u>0\}$ of Hausdorff dimension at most $(n+1)-5$. Moreover, $\mathrm{Sing}(\{u>0\})$ is empty if $n+1 \leqslant 4$, and consists at most of countably many points if $n+1=5$.*

As is well known, determining the optimal dimension of the singular part of a free boundary is a major open problem in the area, and at the moment $n-5$ is the best known bound. We refer to [Ve23, Section 1.4] for an up to date discussion of the state of the art on the regularity of free boundaries.

6.2.1 Some comments about the proof and connection with previous work

To prove Theorem 6.2, we use ideas introduced by Alt and Caffarelli in [AC81], and expanded on by many authors. Velichkov's recent book [Ve23] provides an authoritative presentation of these methods, with a number of extensions and refinements.

The proof of Theorem 6.2 involves two main differences with the existing literature. These differences are both key for our objective of constructing smooth solutions to the overdetermined problem (6.2) in low dimensions, and essential in that they present challenges that cannot be overcome through straightforward modifications of previous results.

The first difference is that in [AC81] and most of the works based on it such as [Ve23], one considers the minimization problem for functions defined on a fixed regular bounded open set $D \subset \mathbb{R}^{n+1}$, and impose some boundary conditions on ∂D. These conditions typically yields uniform Poincaré inequalities on any $\Omega \subset D$, as well as bounds for the solution u in terms of the boundary values on D. And of course, every $\Omega \subset D$ is automatically bounded.

In contrast, our minimization problem is posed on the whole $\mathbb{R}^{n+1}$. Proving that both the minimizer u and the set $\{u > 0\}$ are bounded when the ambient space is unbounded, as in our case, requires rather non-trivial arguments. In fact, a key step in our proof is to obtain *uniform* estimates for $\|u\|_{L^\infty}$ and for the diameter of the set $\{u > 0\}$ which one can effectively control in terms of bounds for the functions A, f and q.

The second difference is related to the kind of functions f that we can consider[1] in the right hand side of (6.2). Indeed, there are several previous papers using a variational approach to study problems for certain kind of functions f. Most of them only deal with some concrete part of the minimization problem, like the analysis of the improvement of flatness performed in [De11].

To the best of our knowledge, there are only two cases in which the minimization problem has been carried out completely. First, in [GS96], Gustafsson and Shahgholian consider the minimization problem on $\mathbb{R}^{n+1}$ for the Poisson equation $-\Delta v = h(X)$. A key difference here is not only that the equation is linear, but also and most fundamentally the sign of h. Indeed, these authors assume that $h \leqslant -c_0 < 0$ outside a fixed ball, which basically forces minimizers to concentrate around the origin, which is crucially used to show that minimizers have compact support. In fact, the key steps of their proof rely strongly on this choice of sign. In particular, they extensively use comparisons with radially decreasing rearrangements, which decrease their energy because they concentrate the mass around the origin. These ideas cannot be adapted to handle any class of functions $h(X) \geqslant 0$, as we do in this chapter. As is well known, overdetermined problems with a nonnegative right hand side are classical in the literature, starting with Serrin's breakthrough paper [Se71], but we shall see that handling this sign condition will require much work.

The second case concerns the torsion problem, $-\Delta u = 1$, in which the right hand side has the same sign as in our problem (6.2). In this case, Lederman [Le96] carried out the full minimization program on the unbounded ambient space $\mathbb{R}^{n+1}\backslash H$, where H is a fixed "hole". One then looks for minimizers among function whose Dirichlet trace on ∂H is a fixed positive function. This boundary condition forces solutions to concentrate around ∂H, which is the key observation used to show the key property that that

[1] There are also differences in the kind of functions A and q one can consider, but we shall focus our discussion of the role of f because the effect of f on the minimization problem is usually much more dramatic than that of A and q.

the support of minimizers is uniformly bounded. As the right hand side is constant, this property then is used to equivalently formulate the minimization problem on $B\backslash H$ for some fixed ball B, so one can effectively avoid working on an unbounded space. This is certainly not the case when one considers more general functions on the right hand side, as we do in this chapter. Aguilera, Alt and Caffarelli [AAC86] addressed a problem closely related to Lederman's using a penalized energy functional.

The admissibility and periodicity conditions we impose on the functions f, A and q are easy to interpret in the context of the minimization program. First, we require that these functions are smooth and positive (which, in the case of the matrix-valued function A, corresponds to a uniform ellipticity condition). Then we assume that F grows at most quadratically in u (which is optimal, as shown in Appendix 6.12), and that it is sufficiently positive somewhere so that nontrivial minimizers exist. Lastly, we assume that these functions are periodic in x. This does not immediately imply that minimizers have uniformly bounded supports because the background space is still all of $\mathbb{R}^{n+1}$ (it only does when the whole minimization program is carried out on the compact quotient space $(\mathbb{R}/2\pi\mathbb{Z})^{n+1}$ instead, as we do when we consider overdetermined problem on compact manifolds in Section 6.11, and there the analysis is much simpler). But periodicity is crucially used to establish the uniform estimates for u and its support and, in fact, in Section 6.12 we show that without periodicity in x, minimizers with uniformly bounded supports do not necessarily exist.

6.3 Main ideas and outline of the chapter

In Section 6.4 we state the admissibility and periodicity assumptions in our main theorems. Basically, we will demand some minimal regularity for all our paramenters, some compatible signs for different quantities, and also certain growth on f. We will discuss the optimality of the assumptions in Section 6.12. Apart from that, we introduce some notation and prove some auxiliary estimates that we will use throughout the chapter: essentially some pointwise estimates for solutions of nonlinear equations that will somehow mimic Harnack's inequalities/mean value properties, and also for their gradients, that were already presented in Chapter 2.

Recall that our main goal in Theorem 6.2 is to show existence of certain solutions, which will arise as minimizers of the energy (6.5). However, it turns out that proving *a priori* that this energy admits minimizers is hard: among other minor problems, there are important compactness issues, namely that the minimizing sequence may escape to infinity. Therefore, we will prove existence of minimizers *a posteriori*, somehow. We will draw as many conclusions from minimizers as we can, along Sections 6.5 to 6.9 (assuming that minimizers exist), and then we will actually prove existence in Section 6.10.

First, in Section 6.5, we prove some basic properties of minimizers: we formalize the Euler-Lagrange equations of the energy in (6.5), and use these to show uniform boundedness of minimizers via elementary estimates from the theory of elliptic regularity.

Then, in Section 6.6, we consider a penalized functional $\mathcal{F}_\Lambda$ which incorporates the restriction on volumes (closer to the original [AC81] approach). It turns out that very abstract Calculus of Variations techniques allow us to show that minimizers of $\mathcal{F}_0$ are close to minimizing $\mathcal{F}_\Lambda$, at least at small scales. In doing this, we will largely follow [Ve23].

Afterwards, we prove that minimizers are Lipschitz and grow linearly around the boundary via comparison against certain competitors that precisely test these conditions, as is customary in the area. The novelty as compared to previous works is that our competitors will satisfy nonlinear equations, but some

appropriate linear approximations will suffice to deal with them. Another novelty is how we show the positivity of the Lagrange multiplier, via a careful local argument with Hopf's lemma. Moreover, we use the linear growth (non-degeneracy) around the boundary to show that minimizers have compact support in Section 6.8: the idea is that, since every boundary point has some interior nearby, and the total volume of the support of the minimizer is bounded, there may only be a limited amount of boundary. This boundedness will be key later to show existence of minimizers a posteriori.

We analyze deeply the behavior of minimizers at the boundary in Section 2.3.1, via blow-ups. In a first subsection, we show appropriate convergence to blow-ups (taking advantage of the variational structure), and use this and a Weiss monotonicity formula to show that blow-ups are linear (1-homogeneous): a modification must be made here because there is no real monotonicity, but still the derivative is bounded below and we can proceed with the classical argument. Since the blow-up limits satisfy an equation with constant coefficients, we can change coordinates to consider the Laplacian, for which a very strict classification of blow-up limits is known: we will use this method to show the Neumann condition at the boundary in the second subsection. Lastly, in a third subsection, we divide the boundary in a regular part (which is at least $C^{1,\alpha}$, and actually more regular if so are our parameters) and a singular part (which is of smaller dimension): the arguments will largely follow from abstract results that can be consulted e.g. in [Ve23].

After all these sections, we have gathered enough information about minimizers to show that they actually exist, what we do in Section 6.10. For that, we penalize (somehow) potential minimizers escaping towards infinity via a truncation with a very large radius. Under the extra assumption of periodicity of our parameters, we are able to "move" connected components of minimizers (as studied in Section 6.8) towards the origin, with a positive impact on the energy by the truncation that we chose. This ultimately shows that minimizers of the truncated energy are bounded, and we can transfer this to our original energy functional if the radius with which we truncate is large enough. An easy transference of our main Theorem 6.2 to manifolds is explained in Section 6.11.

To conclude, in Section 6.12 we show that, without the periodicity assumption, admissible overdetermined problems do not generally have minimizers with uniformly bounded supports (we can make an explicit construction using arbitrarily separated balls). We also show that if our nonlinearity f grows faster than linearly, minimizers may not exist (this is an easy argument with eigenfunctions of the Laplacian), either. This somehow justifies the optimality of the hypotheses on our parameters.

6.4 Assumptions and auxiliary results

Our goal in this section is to discuss the assumptions of the functions f, A and q that we use, and to prove some basic estimates that will be used throughout the chapter.

6.4.1 The admissibility and periodicity assumptions

Let us now state carefully the assumptions we need in Theorem 6.2. As we anticipated in the Introduction, if these are relaxed, several of our key estimates can fail, as we show in Section 6.12.

Before stating the assumptions, let us suggest the reader to review the notation in Section 2.1. In this chapter, we will make frequent use of matrix notation when multiplying vectors (which will be thought as row vectors) and matrices, mixing it with the notation $x \cdot y$ for the inner product in $\mathbb{R}^{n+1}$. In Sections 6.6 and 6.9, it will also appear $\nabla A \cdot x$, where A is a matrix: this is the matrix whose entries

are $\nabla a_{ij} \cdot x$, which amounts to taking the gradient componentwise. With some abuse of notation, in this definition we denote by $\mathbb{R}^{n^2}$ the space of $n \times n$ matrices.

We are ready to state the admissibility and periodicity assumptions in Theorem 6.2. Here we are introducing several constants that will appear later on in various steps of the proof.

Definition 6.3 (Admissibility and periodicity conditions)**.** Consider the functions $f, F : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$, $A : \mathbb{R}^{n+1} \to \mathbb{R}^{(n+1)^2}$ and $q : \mathbb{R}^{n+1} \to \mathbb{R}$ appearing in (6.2)-(6.3)-(6.5).

1. These functions are *admissible* if:

   (a) The functions $f(X,u)$ and $F(X,u)$, where F is defined as in (6.4), satisfy, for (almost) every $X \in \mathbb{R}^{n+1}$ and $u \geqslant 0$,

   - $F(X,u)$ is $C^{1,1}$ in both variables, (HF1)
   - $F(X,0) = 0$, (HF2)
   - $F'(X,u) = f(X,u) \geqslant 0$, (HF3)
   - $F(X,u) \leqslant N + bu^2$, (HF4)
   - $\begin{cases} F'(X,u) = f(X,u) \leqslant N' + M'u, \\ F''(X,u) = f'(X,u) \leqslant M_2, \end{cases}$ (HF5)
   - There exists $u_0 \in \mathbb{K}_{\leqslant m}$ such that $\mathcal{F}_0(u_0) < 0$. (HF6)

   Here $0 \leqslant b < \lambda \frac{\lambda_1(B^m)}{2}$ and $N, N', M', M_2 \geqslant 0$ are constants. We note that (HF2) is automatically satisfied because of (6.4), but it will useful to keep it in mind.

   (b) The function $A(X)$ satisfies, for every $X, \xi \in \mathbb{R}^{n+1}$,

   - $A \in C^{1,1}(\mathbb{R}^{n+1})$, (HA1)
   - $A(X)$ is a symmetric matrix, (HA2)
   - $\lambda |\xi|^2 \leqslant \xi A(X) \xi^{\mathrm{T}} \leqslant \lambda^{-1} |\xi|^2$ (HA3)

   for some constant $\lambda \in (0,1)$.

   (c) The function $q(x)$ satisfies, for every $x \in \mathbb{R}^{n+1}$,

   - $q \in C^{1,1}(\mathbb{R}^{n+1})$, (Hq1)
   - $\underline{q} \leqslant q(x) \leqslant \overline{q}$ (Hq2)

   for some positive constants $\underline{q}, \overline{q}$.

2. These functions are *x-periodic* if there is some $T > 0$ such that

$$F(X,u) = F(X + Te, u), \quad A(X + Te) = A(X), \quad q(x + Te) = q(x) \tag{HPer}$$

hold for every $e \in \mathbb{Z}^{n+1}$.

With some abuse, we will also say that the functions $f(X,v)$, $A(X)$, $q(x)$ appearing in (6.2) are *admissible and x-periodic* when $F(X,v) := \int_0^v f(X,t)\,dt$, $A(X)$ and $q(x)$ (which are the functions appearing in the variational formulation) are admissible and periodic in the above sense.

Remark 6.4. The hypothesis (HF6) is obviously satisfied for many reasonable functions $F(X,u)$. In fact, it is satisfied as soon as F is positive somewhere in the sense that $F(X,u) \geqslant c_0 u$ for all X a small ball B_ε, with $c_0, \varepsilon > 0$. Indeed, we can take φ the first eigenfunction of the Laplacian in the ball B_ε, and compute, for $\tau > 0$,

$$\mathcal{F}_0(\tau\varphi) \leqslant \lambda^{-1}\lambda_1(B_\varepsilon)\,\tau^2 \iint_{B_\varepsilon} \varphi^2\,dX - 2c_0\tau \iint_{B_\varepsilon} \varphi\,dX.$$

Therefore, by the positivity of φ, $\mathcal{F}_0(\tau\varphi) < 0$ provided that τ is sufficiently small.

Remark 6.5. In Assumption (HPer), one could have considered periodicity conditions defined by any other lattice on $\mathbb{R}^{n+1}$. All the arguments in the chapter work in this case too, but we have stated the results only in terms of square lattices for concreteness.

Let us discuss these assumptions. First, in order not to worry about technicalities which are not the focus of this work, we assume (HF1), (HA1) and (Hq1), where here and everywhere, ∇F and F' respectively denote the derivatives of F with respect to the first and second variables. The structural assumptions (HA2) and (HA3) are natural if we want to obtain an elliptic PDE for our solutions, and (Hq2) is natural if one desires to obtain the overdetermined Neumann boundary condition. Then, we also make the natural assumption that u only contributes to the energy on its support, which is (HF2). As explained in the Introduction, the sign condition (HF3) is a key novelty, and will require much work. The Assumptions (HF4)–(HF6) and (HPer) will arise naturally in certain estimates, so we will comment on them later.

6.4.2 An approximate mean value formula for inhomogeneous equations

Let us now establish analogues of the Harnack inequality and the mean value formula for the equation $Lv = g$, that we already discussed in Section 2.4.3, and here we will give a detailed proof. Here and in what follows, we will use the notation

$$Lv := -\operatorname{div}(A(X)\nabla v) \tag{6.7}$$

for the elliptic operator in divergence form associated with the matrix-valued function A.

Lemma 6.6. *Assume that A is admissible in the sense of Definition 6.3. Let $r \in (0,1)$ and suppose that $u \in H^1(B_r)$ satisfies $u \geqslant 0$ on ∂B_r. For any $M \geqslant 0$, the following hold:*

1. *If $Lu \leqslant M$ in B_r, then*

$$u(X) \lesssim_n \fint_{\partial B_r} u\,d\mathcal{H}^n + Mr^2, \qquad X \in B_{r/2}. \tag{6.8}$$

2. *If $|Lu| \leqslant M$ in B_r, then*

$$\left|\nabla u(0)\right| \lesssim_n \frac{1}{r}\fint_{\partial B_r} u\,d\mathcal{H}^n + Mr. \tag{6.9}$$

Proof. Let us prove this in an elementary way for the Laplacian, and then sketch the differences to obtain the result for general L.

Proof for the Laplacian, $A \equiv I$. To show (6.8), note that $v(X) := u(X) + M|X|^2/(2(n+1))$ is subharmonic in B_r, i.e. $-\Delta v \leqslant 0$ in B_r. Therefore, by the maximum principle, $v \leqslant w$ in B_r, where w is the harmonic extension of v to B_r, that is, w is the function satisfying $\Delta w = 0$ in B_r, and coincides with v at ∂B_r. By the Poisson formula,

$$w(X) = r^{n-1} \fint_{\partial B_r} \frac{r^2 - |X|^2}{|y - X|^{n+1}} \left(u(y) + M \frac{|y|^2}{2(n+1)} \right) d\mathcal{H}^n(y), \qquad X \in B_r.$$

Therefore, if we take $X \in B_{r/2}$, a simple triangle inequality (recalling also $u \geqslant 0$ on ∂B_r) shows

$$v(X) \leqslant w(X) \leqslant r^{n-1} \frac{r^2}{(r - r/2)^{n+1}} \left(\fint_{\partial B_r} u \, d\mathcal{H}^n + M \frac{r^2}{2(n+1)} \right), \tag{6.10}$$

from which (6.8) readily follows.

To obtain (6.9) as a consequence of (6.8), we can do the following. Define $\widehat{v}(X) := u(X) + \left|\nabla u(0)\right| X_1$ (where $X = (X_1, X_2, \ldots, X_{n+1})$ are just the coordinates of points in $\mathbb{R}^{n+1}$), and note that still $\left|-\Delta \widehat{v}\right| \leqslant M$ in B_r. Applying (6.8) to $\widehat{v}$ (actually, to $\widehat{v} + \left|\nabla u(0)\right| r$, which is non-negative on ∂B_r because so is u; but this extra term will cancel out momentarily), we obtain

$$u(X) + \left|\nabla u(0)\right| X_1 \lesssim \fint_{\partial B_r} u \, d\mathcal{H}^n + Mr^2, \qquad X \in B_{r/2},$$

because of the cancellation $\fint_{\partial B_r} X_1 \, d\mathcal{H}^n(X) = 0$. Then, note that $u \geqslant -Mr^2/(2(n+1))$ in B_r: if we define $\widehat{w}(X) := u(X) - M|X|^2/(2(n+1))$, it is superharmonic (i.e. $-\Delta \widehat{w} \geqslant 0$) in B_r, and $\widehat{w} \geqslant -Mr^2/(2(n+1))$ on ∂B_r (because $u \geqslant 0$ on ∂B_r); whence we finish the minimum principle for superharmonic functions. Therefore, by taking any $X \in \partial B_{r/4}$ in the previous display, we obtain (6.9).

Proof for general L. To generalize the above proof to general L, the first modification is to use $v(X) := u(X) + M\phi(X)$ (and the same for $\widehat{w}$ just changing signs) and $\widehat{v}(X) := u(X) + \left|\nabla u(0)\right| \widehat{\phi}$, where ϕ and $\widehat{\phi}$ solve

$$\begin{cases} L\phi = -1 & \text{in } B_r \\ \phi = \dfrac{|X|^2}{2(n+1)} = \dfrac{r^2}{2(n+1)} & \text{on } \partial B_r, \end{cases} \qquad \text{and} \qquad \begin{cases} L\widehat{\phi} = 0 & \text{in } B_r \\ \widehat{\phi} = X_1 & \text{on } \partial B_r. \end{cases}$$

These are the natural analogues to v and $\widehat{v}$ in this more general context. By rescaling ϕ to B_1 by $\phi_r(X) := \phi(rX)$, which satisfies $L_r\phi_r = r^2$ in B_1 (where $A_r(X) := A(rX)$, whence it has the same ellipticity constant as A) and $\phi_r = r^2/(2n)$ on ∂B_1, the a priori bounds for elliptic equations (e.g., [GT77, Th. 3.7], where we use (HA1) and $r \leqslant 1$ to have uniform bounds on the moduli of continuity of the coefficients of L_r) yield $\|\phi\|_\infty \lesssim r^2$. This is used to deduce (6.8) from the analogue estimate in our setting to (6.10), and to bound $\widehat{w}$ from below.

The second main point is to use a Poisson formula for w (which in this case, solves $Lw = 0$ in B_r). It turns out that this can be done for fairly general operators, and for that, we may use the L-harmonic

measure. Indeed, given $g \in C(\partial B_r)$, let us denote by u_g the solution of the problem $Lu_g = 0$ in B_r, $u_g = g$ on ∂B_r. Then, given $X \in B_r$, the map $g \mapsto u_g(X)$ is a linear and bounded (by the maximum principle, and also positive) operator from $C(\partial B_r)$ to $\mathbb{R}$. Therefore, by the Riesz Representation Theorem, it can be represented by a (positive) Radon measure ω_L^X, the so-called *L-harmonic measure*, or elliptic measure associated to L:

$$u_g(X) = \int_{\partial B_r} g(y)\, d\omega_L^X(y), \qquad g \in C(\partial B_r).$$

It is known that if $A(X) \in C^\alpha$, then ω_L^X is absolutely continuous with respect to $\mathcal{H}^n|_{\partial B_r}$ (at least in regular sets like the ball, see e.g. [MM80])[2]. Therefore, we can represent w in the proof above as

$$w(X) = \int_{\partial B_r} w(y)\, d\omega_L^X(y) = \int_{\partial B_r} w(y)\, \frac{d\omega_L^X}{d\mathcal{H}^n}(y)\, d\mathcal{H}^n(y).$$

Given this representation, to be able to draw the same conclusions as in the proof for the Laplacian above, we just need to show that

$$\frac{d\omega_L^X}{d\mathcal{H}^n}(y) \approx 1, \qquad X \in B_{r/2}, y \in \partial B_r. \tag{6.11}$$

To prove this, fix $X \in B_{r/2}$, $y \in \partial B_r$, and $\rho > 0$ small, see Figure 6.1 for a sketch. Then, the estimate from [CFMS81, Lemma 2.2] yields

$$\omega_L^X(B_\rho(y) \cap \partial B_r) \approx \rho^{n-1}\, G_L((1-\rho)y, X),$$

where $G_L(\cdot,\cdot)$ is the Green's function in B_r for the operator L, i.e., the function satisfying $LG_L(\cdot, X) = \delta_X$ in B_r and $G_L(\cdot, X) = 0$ on ∂B_r. Therefore, since near ∂B_r, G_L is L-harmonic, and it vanishes identically at the boundary, Hopf's lemma (for a reference under the only assumption that $A(X) \in L^\infty$, see [Sa18]) yields some linear growth, namely $G_L((1-\rho)y, X) \approx \rho$ for $\rho > 0$ small enough. Putting all the above together, we obtain

$$\frac{d\omega_L^X}{d\mathcal{H}^n}(y) = \lim_{\rho\to 0} \frac{\omega_L^X(B_\rho(y) \cap B_r)}{\mathcal{H}^n(B_\rho(y) \cap B_r)} \approx \lim_{\rho\to 0} \frac{\rho^{n-1} G_L((1-\rho)y, X)}{\rho^{n-1}} \approx 1,$$

which finally shows (6.11).

With these properties in hand, it is straightforward to follow the proof for the Laplacian, extending the result to a general L as in (6.7) with $A(X) \in C^\alpha$. The lemma then follows. □

6.5 Some basic properties of minimizers

Over the next sections we will establish the key intermediate results that we will need to prove Theorem 6.2. For this purpose, let us fix some $m > 0$.

As is well known, instead of minimizing $\mathcal{F}_0$ over $\mathbb{K}_{=m}$, it is more convenient to minimize it over the larger family $\mathbb{K}_{\leqslant m}$, which is better behaved from the perspective of the calculus of variations. However,

[2]The Hölder continuity of the coefficients is necessary, because if we only let $A(X) \in C(\overline{B_r})$, then there are counterexamples, as those in [CFK81, MM80].

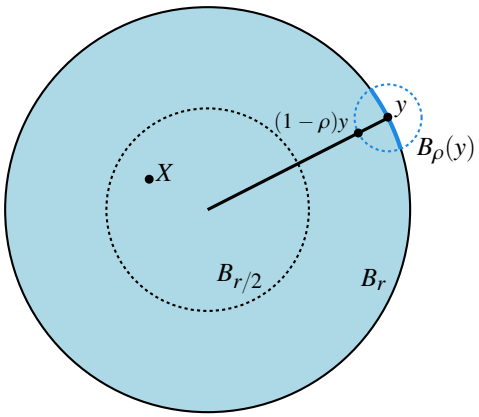


Figure 6.1: To compute the Poisson kernel at a point $X \in B_{r/2}$, one may use the Green's function at points close to the boundary.

an important first observation is that if one tries to apply the direct method in the calculus of variations (as e.g. in [Ve23, Prop 11.1]) to find a minimizer of $\mathcal{F}_0$ in $\mathbb{K}_{\leqslant m}$, one encounters a key difficulty: even with he admissibility and periodicity assumptions in Definition 6.3, it may be the case that the mass of the minimizing sequences escapes to infinity. At this stage, we still have no information about the shapes that minimize $\mathcal{F}_0$, so it would be difficult to discard that the mass escapes to infinity.

We will control this difficulty a posteriori in Section 6.10, once we have gathered enough information about possible minimizers. So for the moment, and until Section 6.10, we will work a priori, under the assumption of the existence of a minimizer, and infer as much information about it as possible. Therefore, until Section 6.10 we will make the following

Assumption 6.7. In Sections 6.5 to 6.9, we assume that:

$$\begin{cases} m > 0, \\ F, A \text{ and } q \text{ are admissible in the sense of Definition 6.3,} \\ u \text{ is a minimizer of } \mathcal{F}_0 \text{ in } \mathbb{K}_{\leqslant m}. \end{cases} \tag{H*}$$

We note that here we do not assume the periodicity condition in (HPer). We will use this in Section 6.10, but for now it is not necessary.

A first property that a minimizer u must satisfy is the following. Recall that the operator L was introduced in (6.7). Note that in this lemma we show that if

$$\Omega_u := \{u > 0\}$$

is an open set (which we have not proved yet), then the minimizer u does satisfy the PDE we want:

Lemma 6.8 (Euler–Lagrange (I))**.** *Assume (H*). Then $Lu \leqslant F'(X,u)$ in $\mathbb{R}^{n+1}$, in the weak sense. Furthermore, if $B_r(X_0) \subset \{u > 0\}$ for certain $X_0 \in \mathbb{R}^{n+1}$ and $r > 0$, then $Lu = F'(X,u)$ in $B_r(X_0)$.*

Proof. Testing the minimizing property of u against $v_\varepsilon := (u - \varepsilon\phi)_+$ for $\phi \in C_c^\infty(\mathbb{R}^{n+1})$, we obtain

(recalling that A is symmetric, elliptic, and uniformly bounded, see (HA3) and (HA2))

$$\begin{aligned} 0 \leqslant \mathcal{F}_0(v_\varepsilon) - \mathcal{F}_0(u) &\leqslant \iint\limits_{\{u>0\}} \nabla(u-\varepsilon\phi)\, A\, \nabla(u-\varepsilon\phi)^{\mathrm{T}}\, dX - \iint\limits_{\{u>0\}} \nabla u\, A\, \nabla u^{\mathrm{T}}\, dX \\ &\quad - 2 \iint\limits_{\mathbb{R}^{n+1}} \big(F(X,v_\varepsilon) - F(X,u)\big)\, dX \\ &= -2\varepsilon \iint\limits_{\mathbb{R}^{n+1}} \nabla u\, A\, \nabla\phi^{\mathrm{T}}\, dX + 2 \iint\limits_{\mathbb{R}^{n+1}} F'(X,u)(u-v_\varepsilon)\, dX + O(\varepsilon^2) \\ &\leqslant -2\varepsilon \iint\limits_{\mathbb{R}^{n+1}} \nabla u\, A\, \nabla\phi^{\mathrm{T}}\, dX + 2 \iint\limits_{\mathbb{R}^{n+1}} F'(X,u)\, \varepsilon\phi\, dX + O(\varepsilon^2). \end{aligned}$$

The third step is simply a Taylor expansion using (HF5), and in the last step we used $u - v_\varepsilon \leqslant \varepsilon\phi$ along with (HF3). Dividing by ε and letting $\varepsilon \to 0$ gives the first part of the result.

Furthermore, if $B_r(X_0) \subset \{u > 0\}$ and $0 \leqslant \phi \in C_c^\infty(B_r(X_0))$, we have that $u - \varepsilon\phi \in \mathbb{K}_{\leqslant m}$ for all $\varepsilon \in \mathbb{R}$ close enough to 0, so the above argument yields

$$0 \leqslant \mathcal{F}_0(u-\varepsilon\phi) - \mathcal{F}_0(u) = -2\varepsilon \iint\limits_{\mathbb{R}^{n+1}} \nabla u\, A\, \nabla\phi^{\mathrm{T}}\, dX + 2\varepsilon \iint\limits_{\mathbb{R}^{n+1}} F'(X,u)\, \phi\, dX + O(\varepsilon^2).$$

Therefore, $Lu = F'(X,u)$ in $B_r(X_0)$, as claimed. □

Our next goal is to show that u is uniformly bounded (that is, $\|u\|_{L^\infty(\mathbb{R}^{n+1})} < C$ for some C depending only on the admissibility constants, see Definition 6.3).

Proposition 6.9 (Boundedness of minimizers)**.** *Assume* (H*)*. Then u is uniformly bounded.*

Proof. Let us first distinguish cases depending on the shape of the fundamental solutions.

Case 1: $n+1 \geqslant 3$**.** First, write $f(X) := F'(X,u(X))\mathbf{1}_{B_{1/4}}(X)$ (one should *not* mistake this function for $f(X,u)$ in (6.2)). By the linear bounds for F' in (HF3), (HF5), one easily estimates

$$\|f\|_{L^r(\mathbb{R}^{n+1})} \lesssim_r \|u\|_{L^r(\mathbb{R}^{n+1})} + 1, \qquad \text{for any } 1 \leqslant r \leqslant +\infty. \tag{6.12}$$

On the other hand, we denote

$$w(X) := \iint\limits_{\mathbb{R}^{n+1}} \Gamma_L(X,Y)\, f(Y)\, dY \quad \text{and} \quad w_{-\Delta}(X) := \iint\limits_{\mathbb{R}^{n+1}} \Gamma_{-\Delta}(X,Y)\, f(Y)\, dY, \qquad X \in \mathbb{R}^{n+1},$$

where Γ_L and $\Gamma_{-\Delta}$ are, respectively, the fundamental solutions of the operators L and $-\Delta$ in $\mathbb{R}^{n+1}$ (see [GT77, (4.1)] for the Laplacian, and [LSW63, Section 6] for general L satisfying (HA3)). They satisfy $Lw = f$ and $-\Delta w_{-\Delta} = f$ a.e. in $B_{1/4}$ (see [GT77, Th. 9.9] for the Laplacian, or Sections 6 and 7 in [LSW63] for general operators). Moreover, the fact that $\Gamma_L \approx \Gamma_{-\Delta} < 0$ (by [LSW63, (7.9)] and [GT77, (4.1)]) and $f \geqslant 0$ (see (HF2) and (HF3)) yield $w \approx w_{-\Delta}$ pointwise, so

$$\|w\|_{L^r(B_{1/4})} \approx_r \left\|w_{-\Delta}\right\|_{L^r(B_{1/4})}, \qquad \text{for any } 1 \leqslant r \leqslant +\infty. \tag{6.13}$$

Furthermore, since $Lw = f$ a.e. in $B_{1/4}$, it holds $L(u-w) \leqslant 0$ in $B_{1/4}$ by Lemma 6.8. Note also that $w \leqslant 0$ (because we already argued that $f \geqslant 0$ and $\Gamma_L < 0$), so $0 \leqslant u \leqslant u - w$. Therefore, applying Lemma 6.6 for radii between $1/8$ and $1/4$, and then averaging, we get

$$u(0) \leqslant (u-w)(0) \lesssim \fint_{B_{1/4}\setminus B_{1/8}} (u-w)\,dX \lesssim \left(\fint_{B_{1/4}\setminus B_{1/8}} |u-w|^{2^*} dX \right)^{1/2^*} \lesssim \|u\|_{L^{2^*}(\mathbb{R}^{n+1})} + \|w\|_{L^{2^*}(B_{1/4})}.$$

First, the Sobolev embedding implies that $\|u\|_{L^{2^*}(\mathbb{R}^{n+1})} \lesssim \|u\|_{H^1(\mathbb{R}^{n+1})}$. In turn, we can estimate the second term as follows:

$$\begin{aligned}\|w\|_{L^{2^*}(B_{1/4})} \approx \left\|w_{-\Delta}\right\|_{L^{2^*}(B_{1/4})} \leqslant \left\|w_{-\Delta}\right\|_{L^{2^*}(\mathbb{R}^{n+1})} &\lesssim \left\|\nabla^2 w_{-\Delta}\right\|_{L^{2*}(\mathbb{R}^{n+1})} \\ &\lesssim \|f\|_{L^{2*}(\mathbb{R}^{n+1})} \lesssim \|u\|_{L^{2*}(\mathbb{R}^{n+1})} + 1 \lesssim \|u\|_{L^2(\mathbb{R}^{n+1})} + 1.\end{aligned}$$

Here, we have used (6.13), the Sobolev embedding twice (where 2_* satisfies $(2_*)^* = 2$, and $2_* \in (1,2)$ because $n+1 > 2$), the boundedness of the Riesz transforms in L^{2*} because $2_* > 1$ (see e.g. [GT77, Theorem 9.9]), the estimate in (6.12), and lastly Hölder's inequality (because $2_* < 2$) along with the fact that $\left|\{u>0\}\right| \leqslant \mathrm{Vol}_q(\Omega_u)/\underline{q} = m/\underline{q}$.

All the conclusions in last paragraph yield

$$u(0) \lesssim \|u\|_{H^1(\mathbb{R}^{n+1})} + \|u\|_{L^2(\mathbb{R}^{n+1})} + 1.$$

We note here that the constants involved in these estimates do not depended on the choice of our balls to be centered at the origin, so we can obtain the same estimate centering the balls at any $X \in \mathbb{R}^{n+1}$. This implies that, in this case ($n \geqslant 3$),

$$\|u\|_{L^\infty(\mathbb{R}^{n+1})} \lesssim \|u\|_{H^1(\mathbb{R}^{n+1})} + \|u\|_{L^2(\mathbb{R}^{n+1})} + 1.$$

Case 2: $n+1 = 2$**.** This case is slightly different because of the fundamental solution exhibiting a logarithmic growth (see [GT77, (4.1)]). Define w and $w_{-\Delta}$ as above. Noting that $\Gamma_L < 0$ in B_1 (see [GT77, (4.1)], and recalling $\Gamma_L \approx \Gamma_{-\Delta}$) and $f \geqslant 0$ (see (HF2) and (HF3)), we compute

$$u(0) \leqslant (u-w)(0) \lesssim \left(\fint_{B_{1/4}\setminus B_{1/8}} |u-w|^2\,dX \right)^{1/2} \lesssim \|u\|_{L^2(\mathbb{R}^{n+1})} + \|w\|_{L^2(B_{1/4})}. \tag{6.14}$$

Now, an easy computation recalling $\Gamma_L \approx \Gamma_{-\Delta}$ yields, for $Z \in B_{1/4}$,

$$\begin{aligned}w(Z) \approx w_{-\Delta}(Z) = \frac{1}{2\pi} \iint_{B_{1/4}} \log|Z-Y|\, f(Y)\,dY &= \frac{1}{2\pi} \iint_{B_{1/4}} \left|\log|Z-Y|\right| |f(Y)|\,dY \\ \lesssim \frac{1}{2\pi} \left(\iint_{B_{1/2}(Z)} \left|\log|Z-Y|\right|^2 dY \right)^{1/2} &\left(\iint_{B_{1/4}} (1+u(Y))^2\,dY \right)^{1/2} \lesssim \|u\|_{L^2(\mathbb{R}^{n+1})} + 1,\end{aligned}$$

where in the second-to-last estimate we have used Cauchy-Schwarz and (HF5). Since the constants are uniform and again do not depend on our balls being centered at the origin, back in (6.14), we obtain

$$\|u\|_{L^\infty(\mathbb{R}^{n+1})} \lesssim \|u\|_{L^2(\mathbb{R}^{n+1})} + 1,$$

and the proposition follows. □

Combining Proposition 6.9 and (HF5), we infer that

$$M_1 := \sup_{X\in\mathbb{R}^{n+1}} \left|F'(X,u(X))\right| \leqslant N' + M' \|u\|_{L^\infty(\mathbb{R}^{n+1})} < +\infty \tag{6.15}$$

is also uniformly controlled by the constants in Definition 6.3.

Corollary 6.10. *Assume (H*). Then, with M_1 defined in (6.15), it holds*

$$Lu \leqslant M_1 \qquad \text{in } \mathbb{R}^{n+1}.$$

Moreover, u has a representative defined pointwise by

$$u(Y) = \lim_{\rho\to 0} \fint_{D_\rho(Y)} u(X)\,dX, \qquad Y\in\mathbb{R}^{n+1}, \tag{6.16}$$

where the sets $D_\rho(Y)$ satisfy $B_{c_D\rho}(Y)\subset D_\rho(Y)\subset B_{C_D\rho}(Y)$ for $c_D, C_D > 0$ only depending on n and λ.

Proof. The first assertion follows from Lemma 6.8 and the definition of M_1. For the second, note that, fixed $Y\in\mathbb{R}^{n+1}$, if we find the solution to $L\phi = -1$ in $B_1(Y)$ with $\phi = 1/(2(n+1))$ on $\partial B_1(Y)$ (similarly as in Lemma 6.6, so that $\phi(X) = |X-y|^2/(2(n+1))$ for $L=-\Delta$), then $X\mapsto u(X)+M_1\phi$ and $X\mapsto M_1\phi$ are both L-subharmonic functions. Therefore, they can be defined at Y with limits of mean values as in (6.16) (for general L, see [Ca98, p.9], relying on the pointwise equivalence of any fundamental solution to that of the Laplacian, proved in [LSW63] and [BH15, Section 6]). Subtracting both formulas, we obtain (6.16) at Y, as desired. □

Remark 6.11. When $L=-\Delta$, one can also take $D_\rho(y) = \partial B_r(y)$ and integrate with respect to $d\mathcal{H}^n$ instead of dX, thereby recovering the classical mean value formulas.

The following result (and the generalization which we will prove in Lemma 6.17) will be key in many arguments: we will compare u with the function solving the exact PDE that u should solve in Ω_u (given by Lemma 6.8). It is a natural analogue in our setting of various lemmas on harmonic extensions that can be found in the literature (e.g., [AC81, Lemma 3.2] and [Ve23, Lemma 3.8]).

Lemma 6.12. *Assume (H*). Fix a small enough ball $B_r(X_0)$ and define h as the solution of $Lh = F'(X,u)$ in $B_r(X_0)$, and $h=u$ in $B_r(X_0)^c$. If $\mathrm{Vol}_q(\{h>0\})\leqslant m$, then $u\equiv h$.*

Proof. We may assume $X_0 = 0$. By Lemma 6.8, $L(u-h)\geqslant 0$ in B_r. And clearly $u-h=0$ on ∂B_r. Therefore, the maximum principle implies that $u-h\leqslant 0$ in B_r, i.e., $h\geqslant u$ $(\geqslant 0)$. Then, our assumptions imply that $h\in\mathbb{K}_{\leqslant m}$, so since u is a minimizer we infer $\mathcal{F}_0(u)\leqslant\mathcal{F}_0(h)$. With some simple computations using the definition of h (recall also (HA2) and (HA3)), this means

$$\begin{aligned}
0 &\geqslant \mathcal{F}_0(u) - \mathcal{F}_0(h)\\
&= \iint_{B_r} \nabla(u-h)\,A\,\nabla(u-h)^{\mathrm{T}}\,dX - 2\iint_{B_r}\nabla h\,A\,\nabla(h-u)^{\mathrm{T}}\,dX - 2\iint_{B_r}\big(F(X,u)-F(X,h)\big)\,dX\\
&\geqslant \lambda\iint_{B_r}\left|\nabla(u-h)\right|^2 dX - 2\iint_{B_r}\left(F(X,u)-F(X,h)-F'(X,u)(u-h)\right)dX =: \mathrm{I}+\mathrm{II}.
\end{aligned}$$

The first term is obviously nonnegative. For the second, note that, using (HF5) and Poincaré,

$$\mathrm{II} \geqslant -2M_2 \iint_{B_r} |u-h|^2\, dX \geqslant -CM_2\, r^2 \iint_{B_r} \left|\nabla(u-h)\right|^2 dX.$$

Thus, if r is small enough (so that $CM_2r^2 \leqslant \lambda/2$, say), we can absorb this contribution and get $0 \geqslant \frac{\lambda}{2}\iint_{B_r}\left|\nabla(u-h)\right|^2 dX$ from the first display, which shows that $u \equiv h$. □

Now we observe that the (very natural) assumption (HF6) ensures that $u \equiv 0$ is not a minimizer. With this, we can prove the following:

Lemma 6.13 (Saturation of the measure)**.** *Assume* (H*)*. Then* $\mathrm{Vol}_q(\Omega_u) = m$*, i.e.,* $u \in \mathbb{K}_{=m}$.

Proof. With the results already obtained, the proof follows as in [Ve23, Prop. 11.1, step 4]. Assume, for the sake of a contradiction, that $\mathrm{Vol}_q(\Omega_u) < m$. Then, using (Hq2), find $r > 0$ with $\mathrm{Vol}_q(\Omega_u) + \mathrm{Vol}_q(B_r(X)) \leqslant m$ for any $X \in \mathbb{R}^{n+1}$.

Let us show that Ω_u is closed. Indeed, let $X_j \in \Omega_u$ converge to X_0. Then, for j large, $X_j \in B_r(X_0)$. Now, Lemma 6.12 implies that $u \equiv h$ (its F'-harmonic extension) in $B_r(X_0)$. Concretely, (HF3) implies that u is L-superharmonic (i.e. $Lu \geqslant 0$) in $B_r(X_0)$. Thus, the strong minimum principle (and $u(X_j) > 0$) implies that $u > 0$ in $B_r(X_0)$. Concretely $u(X_0) > 0$.

Next, we show that Ω_u is open, too. Pick $X_0 \in \Omega_u$, i.e., $u(X_0) > 0$. Then, by (6.16), $\iint_{B_{C_D r}(X_0)} u\, dX > 0$ if $r > 0$ is small, whence $\fint_{\partial B_r(X_0)} u\, d\mathcal{H}^n > 0$ for (a possibly different) small enough $r > 0$. That means that u does not vanish identically on $\partial B_r(X_0)$. Hence, the F'-harmonic extension h of u in $B_r(X_0)$, which again coincides with u in $B_r(X_0)$ by Lemma 6.12, does not vanish in $B_r(X_0)$ by the strong minimum principle for L-superharmonic functions (as in the last paragraph). This shows that $B_r(X_0) \subset \Omega_u$.

Therefore, Ω_u is both open and closed, so either $u \equiv 0$ or $\Omega_u = \mathbb{R}^{n+1}$. Both give the desired contradiction because of (HF6) and $\mathrm{Vol}_q(\Omega_u) \leqslant m$, respectively. □

6.6 Asymptotic minimality at small scales for a penalized energy

To follow the original blueprint of [AC81], and ultimately obtain the overdetermined boundary condition, it will be convenient to use the energy functional

$$\mathcal{F}_\Lambda(v) := \mathcal{F}_0(v) + \Lambda \mathrm{Vol}_q(\{v > 0\})$$

instead of $\mathcal{F}_0$, for some $\Lambda > 0$. This corresponds to considering a Lagrange multiplier Λ for our constrained minimization problem. The basic idea in this section is to show that our minimizer u of $\mathcal{F}_0$ under the constraint (6.3) is, in a way, "almost" an unconstrained minimizer of $\mathcal{F}_\Lambda$ for some $\Lambda > 0$. This introduces some more difficulties, which were already addressed in [Ve23] in a simpler context.

Our goal here is to adapt those results to our setting. For this, we can mostly follow [Ve23, Section 11], and only few things need to be modified. There is one substantial difference, though: we have not managed to extend [Ve23, Subsection 11.3] to our setting: in the presence of $F(X,u)$, with the positive derivative (HF3), it is not clear to us how to establish the key Almgren-like monotonicity formulas as in [Ve23, Lemma 11.7] and Caccioppoli inequalities as in the proof of [Ve23, Proposition 11.4]. The

way we circumvent this difficulty is by not showing in this section that $\Lambda > 0$ (we only obtain $\Lambda \geqslant 0$). We will be able to fix this issue in later sections.

The first result we will establish is the following. Note that the first variation of $\mathcal{F}_\Lambda$ is

$$\delta\mathcal{F}_\Lambda(u)[\xi] := \delta\mathcal{F}_0(u)[\xi] + \Lambda \iint_{\Omega_u} \operatorname{div}(q\xi)\, dX,$$

where $\xi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ and

$$\delta\mathcal{F}_0(u)[\xi] := \iint_{\mathbb{R}^{n+1}} \Big(-2\nabla u\, \nabla\xi\, A\, \nabla u^{\mathrm{T}} + \nabla u\, A\, \nabla u^{\mathrm{T}}\, \operatorname{div}\xi + \nabla u\, (\nabla A\cdot\xi)\, \nabla u^{\mathrm{T}} \\ - 2\nabla F(X,u)\cdot\xi - 2F(X,u)\operatorname{div}\xi\Big)\, dX. \quad (6.17)$$

Lemma 6.14. *Assume (H*). Then, there exists $\Lambda \geqslant 0$ such that u is stationary for $\mathcal{F}_\Lambda$, i.e.*

$$\delta\mathcal{F}_\Lambda(u)[\xi] = 0 \qquad \forall\, \xi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1}). \quad (6.18)$$

Proof. First, let us deduce the expression (6.17) for the first variation of $\mathcal{F}_0$. We use the perturbations $u_t := u\circ\Phi_t(X)$, where $\Phi_t(X) := \Psi_t(X)^{-1}$ and $\Psi_t(X) := (I + t\xi)(X)$ for some $\xi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$. Note that, using the chain rule and a well-known expansion of the determinant around the identity,

$$\nabla\Psi_t(X) = I + t\nabla\xi(X), \quad \nabla\Phi_t(\Psi_t(X)) = I - t\nabla\xi(X) + o(t), \quad \det\nabla\Psi_t(X) = 1 + t\operatorname{div}\xi(X) + o(t).$$

With this, the change of variables $y = \Psi_t(X)$ (equiv. $X = \Phi_t(y)$), and some Taylor expansions using (HA1) to neglect lower order terms, we get

$$\iint_{\mathbb{R}^{n+1}} \nabla u_t(X)A(X)\nabla u_t(X)^{\mathrm{T}}dX = \iint_{\mathbb{R}^{n+1}} \nabla u(Y)\nabla\Phi_t(\Psi_t(Y))A(\Psi_t(Y))\big(\nabla u(Y)\nabla\Phi_t(\Psi_t(Y))\big)^{\mathrm{T}}\big|\det\nabla\Psi_t(Y)\big|\, dY$$
$$= \iint_{\mathbb{R}^{n+1}} \nabla u(Y)\Big(I - t\nabla\xi(Y) + o(t)\Big)\Big(A(Y) + t\nabla A(Y)\cdot\xi(Y) + o(t)\Big) \\ \cdot\Big(I - t\nabla\xi(Y) + o(t)\Big)^{\mathrm{T}}\nabla u(Y)^{\mathrm{T}}\Big(1 + t\operatorname{div}\xi(X) + o(t)\Big)\, dY$$
$$= \iint_{\mathbb{R}^{n+1}} \nabla uA\nabla u^{\mathrm{T}}\, dY + t\iint_{\mathbb{R}^{n+1}} \Big(-2\nabla u\,\nabla\xi\, A\,\nabla u^{\mathrm{T}} + \nabla u\, A\,\nabla u^{\mathrm{T}}\operatorname{div}\xi + \nabla u(\nabla A\cdot\xi)\nabla u^{\mathrm{T}}\Big)\, dY + o(t),$$

Similarly, using (HF1) to neglect lower order terms

$$\iint_{\mathbb{R}^{n+1}} F(X,u_t(X))\, dX = \iint_{\mathbb{R}^{n+1}} F(\Psi_t(Y),u(Y))\big|\det\nabla\Psi_t(Y)\big|\, dY$$
$$= \iint_{\mathbb{R}^{n+1}} \Big(F(Y,u(Y)) + \nabla F(Y,u(Y))\cdot t\xi(Y) + o(t)\Big)\Big(1 + t\operatorname{div}\xi + o(t)\Big)\, dY$$
$$= \iint_{\mathbb{R}^{n+1}} F(Y,u(Y))\, dY + t\iint_{\mathbb{R}^{n+1}} \Big(\nabla F(Y,u(Y))\cdot\xi(Y) + F(Y,u(Y))\operatorname{div}\xi\Big)\, dY + o(t),$$

Lastly, using (Hq1) to neglect lower order terms,

$$\begin{aligned}\mathrm{Vol}_q(\Omega_{u_t}) &= \iint_{\mathbb{R}^{n+1}} q(X)\,\mathbf{1}_{\Omega_u}(\Phi_t(X))\,dX = \iint_{\mathbb{R}^{n+1}} q(\Psi_t(Y))\,\mathbf{1}_{\Omega_u}(Y)\left|\det\nabla\Psi_t(Y)\right|\,dY\\ &= \iint_{\Omega_u}\Big(q(Y)+t\nabla q(Y)\cdot\xi(Y)+o(t)\Big)\Big(1+t\,\mathrm{div}\,\xi(Y)+o(t)\Big)\,dY\\ &= \mathrm{Vol}_q(\Omega_u)+t\iint_{\Omega_u}\mathrm{div}(q\xi)\,dY+o(t).\end{aligned}$$

All these give our desired formulas in the statement, like (6.17). We are left to show (6.18). One can essentially argue as in [Ve23, Section 11.2]. First, note that there is a vector field $\xi\in C^\infty_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ for which $\iint_{\Omega_u}\mathrm{div}(q\xi)\,dX=1$ (indeed, one may simply multiply by q^{-1} the vector field given by [Ve23, Lemma 11.3]). And this allows us to invoke [Ve23, Prop. 11.2] (because its proof does not rely on the explicit form of $\mathcal{F}_0$ or Vol_q) to obtain (6.18). □

Let us now show that the assumption that u is a critical point for $\mathcal{F}_\Lambda$ implies that u is almost a minimizer of $\mathcal{F}_\Lambda$ at small scales. The following lemma will be the last time that we will use (HF4) explicitly, until Section 6.10.

Lemma 6.15. *Assume (H*). Let B be a ball such that $0<|\Omega_u\cap B|<|B|$. Then, for every $\varepsilon>0$, there exists $r>0$ such that for any $B_r(X_0)\subset B$ we have*

$$\mathcal{F}_{\Lambda+\varepsilon}(u)\leqslant\mathcal{F}_{\Lambda+\varepsilon}(v)\quad\text{for every } v\in H^1(\mathbb{R}^{n+1}) \text{ such that } \begin{cases} v-u\in H^1_0(B_r(X_0)),\\ \mathrm{Vol}_q(\Omega_u)\leqslant\mathrm{Vol}_q(\Omega_v).\end{cases} \tag{6.19}$$

$$\mathcal{F}_{\Lambda-\varepsilon}(u)\leqslant\mathcal{F}_{\Lambda-\varepsilon}(v)\quad\text{for every } v\in H^1(\mathbb{R}^{n+1}) \text{ such that } \begin{cases} v-u\in H^1_0(B_r(X_0)),\\ \mathrm{Vol}_q(\Omega_u)\geqslant\mathrm{Vol}_q(\Omega_v).\end{cases} \tag{6.20}$$

Proof. This is an the analogue of [Ve23, Prop. 11.10]. The proof, given in [Ve23, Sections 11.4 and 11.5], is very robust, so most of it works for fairly general energy functionals. However, there are a couple of arguments that do not work directly in our setting because of the presence of F, A and q. We will next discuss the modifications that are needed:

1. First, at the beginning of the proof of [Ve23, Lemma 11.9], it is deduced that $\mathcal{F}_0(u_j,B)\leqslant C$ implies that u_j are uniformly bounded in $H^1(B)$ (we are adopting the notation in [Ve23] here). Since u_j are constructed as minimizers of the problem [Ve23, (11.18)], posed in the bounded domain B, the property above follows by standard arguments using the direct method of Calculus of Variations. In any case, let us make the details more explicit, re-using some computations from Step 1 in Section 6.10, where we obtain minimizers to a more difficult problem, because it is not posed in a bounded domain.

   Note that $\mathrm{Vol}_q(\Omega_{u_j})=m_j$ for some sequence $m_j\to m$. Indeed, since $\mathrm{Vol}_q(B)>m$, for j large enough we have $\mathrm{Vol}_q(B^{m_j})<\mathrm{Vol}_q(B)$ and we can prove a Poincaré inequality for u_j as in (6.48), with constant $\lambda_1(B^{\widetilde{m}_j})^{-1}$, where, following (6.28), we denote $\widetilde{m}_j:=m_j/\underline{q}$. And we can continue the proof of Step 1 in Section 6.10 noting that

$$\lambda-\frac{2b}{\lambda_1(B^{\widetilde{m}_j})}=\lambda-\frac{2b}{\lambda_1(B^{\widetilde{m}})}\frac{\lambda_1(B^{\widetilde{m}})}{\lambda_1(B^{\widetilde{m}_j})}\geqslant c>0$$

for some uniform constant c, as soon as we choose j large enough, because $2b/\lambda_1(B^{\widetilde{m}}) < \lambda$ by (HF4), and $\lambda_1(B^{\widetilde{m_j}}) \longrightarrow \lambda_1(B^{\widetilde{m}})$ because $m_j \to m$. This allows us to re-run the proof in Step 1 of Section 6.10 for the problem [Ve23, (11.18)] and deduce the existence of u_j, and get a uniform bound for $\|\nabla u_j\|_{L^2(B)}$ as in (6.50), using the uniform constant c in the previous display. With that, since $u_j - u \in H^1_0(B)$ for every j, we obtain

$$\|u_j\|_{L^2(B)} \leqslant \|u_j - u\|_{L^2(B)} + \|u\|_2 \lesssim_B \|\nabla(u_j - u)\|_2 + \|u\|_2 \leqslant \|\nabla u_j\|_2 + \|\nabla u\|_2 + \|u\|_2 \,,$$

by using Poincaré in B. Since this bound does not depend on j, this finally shows that u_j are uniformly bounded in $H^1(B)$, after passing to a subsequence.

2. In the middle of the proof of property (iii) in [Ve23, Lemma 11.9], it is asserted that the weak convergence of u_j to u_∞ in $H^1(B)$ implies $\mathcal{F}_0(u_\infty, B) \leqslant \liminf_{j\to+\infty} \mathcal{F}_0(u_j, B)$. In our setting, there appears a new problematic term to handle: we need to show

$$\iint_B \nabla u_\infty \, A \, \nabla u_\infty^{\mathrm{T}} \, dX \leqslant \liminf_{j\to+\infty} \iint_B \nabla u_j \, A \, \nabla u_j^{\mathrm{T}} \, dX. \tag{6.21}$$

But it is not hard to see that this estimate holds. Indeed, one can compute

$$\begin{aligned} &\iint_B \left(\nabla u_j \, A \, \nabla u_j^{\mathrm{T}} - \nabla u_\infty \, A \, \nabla u_\infty^{\mathrm{T}} \right) dX \\ &\qquad = \iint_B \nabla(u_j - u_\infty) \, A \, \nabla(u_j - u_\infty)^{\mathrm{T}} \, dX + 2 \iint_B \nabla(u_j - u_\infty) \, A \, \nabla u_\infty^{\mathrm{T}} \, dX, \end{aligned}$$

and the first term is non-negative by ellipticity (HA3), whereas the second converges to 0 as $j \to +\infty$ because $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2(B)$ (just test against $A\nabla u_\infty^{\mathrm{T}} \in L^2(B)$ because $u_\infty \in H^1(B_r)$ and A is bounded, see (HA3)).

3. Next, in the proof of property (iv) in [Ve23, Lemma 11.9], we should use $q\,dX$ instead of the Lebesgue measure. The proof can be readily adapted because, by (Hq2), convergence in $L^2(dx)$ is obviously equivalent to convergence in $L^2(q\,dX)$:

$$\iint_{\mathbb{R}^{n+1}} |f_j - f|^2 \, dX \underset{j\to\infty}{\longrightarrow} 0 \quad \Longleftrightarrow \quad \iint_{\mathbb{R}^{n+1}} |f_j - f|^2 \, q \, dX \underset{j\to\infty}{\longrightarrow} 0. \tag{6.22}$$

The case of weak convergence is analogous.

4. Finally, in Step 1 in the proof of [Ve23, Prop. 11.16], the author uses $\mathcal{F}_0(v_j, B) \geqslant 0$, which is not true for us because of the presence of F. To fix this, we note that actually it suffices that $\mathcal{F}_0(u, B) - \mathcal{F}_0(v_j, B)$ is bounded above uniformly on j, which does hold in our setting. Indeed, by Poincaré (because $u - v_j \in H^1_0(B_r)$)

$$\iint_{B_r} (F(X, u) - F(X, v_j)) \, dX \lesssim_{M_1} \iint_{B_r} |u - v_j| \, dX$$

$$\lesssim_r \left(\iint_{B_r} \left|\nabla(u - v_j)\right|^2 \, dX \right)^{1/2} \lesssim \|\nabla u\|_{L^2(B_r)} + \left\|\nabla v_j\right\|_{L^2(B_r)}.$$

And using this, we conclude

$$\mathcal{F}_0(u, B) - \mathcal{F}_0(v_j, B) \leqslant C(u) - \left\|\nabla v_j\right\|^2_{L^2(B_r)} + C\left\|\nabla v_j\right\|_{L^2(B_r)} \leqslant C(u) + C.$$

With these modifications, the proofs in [Ve23, Sections 11.4 and 11.5] carry over to the present setting. □

For future reference, let us record here the following immediate consequence of Lemma 6.15:

Corollary 6.16. *Assume (H*). For every $\varepsilon > 0$ there exists $r_0 > 0$ such that, for every $0 < r \leqslant r_0$,*

$$\mathcal{F}_\Lambda(u) \leqslant \mathcal{F}_\Lambda(v) + \varepsilon |B_r| \quad \text{for every } v \in H^1(\mathbb{R}^{n+1}) \text{ such that } v - u \in H^1_0(B_r(X_0)). \tag{6.23}$$

Proof. It follows directly from Lemma 6.15 and the fact that $\mathcal{F}_{\Lambda+\varepsilon}(w) = \mathcal{F}_\Lambda(w) + \varepsilon\left|\{w > 0\}\right|$. □

6.7 Lipschitz continuity and non-degeneracy

Using that our constrained minimizer u is "almost" a minimizer of some energy $\mathcal{F}_\Lambda$ by (6.23), our goal in this section is to show that u is Lipschitz and non-degenerate, which will be the key properties in the forthcoming analysis. To do so, we will extend the basic strategy used in [AC81] and [Ve23, Sections 3 and 4].

Let us start by a crucial extension of Lemma 6.12:

Lemma 6.17. *Assume (H*). Fix $\varepsilon > 0$ and a small enough ball $B_r(X_0)$ as in Lemma 6.15. Define h as the only solution of the equation $Lh = F'(X, u)$ in $B_r(X_0)$ with $h = u$ in $B_r(X_0)^c$. Then $h \geqslant u$ in $B_r(X_0)$ and*

$$\iint_{B_r(X_0)} \left|\nabla(u - h)\right|^2 dX \leqslant 2\,\overline{q}\,\frac{\Lambda + \varepsilon}{\lambda}\left|\{u = 0\} \cap B_r(X_0)\right|. \tag{6.24}$$

Proof. We may assume $X_0 = 0$, and write $\Lambda' := \Lambda + \varepsilon$. By Lemma 6.8, $L(u - h) \leqslant 0$ in B_r; and clearly $u - h = 0$ on ∂B_r. Therefore, the maximum principle implies that $h \geqslant u$. Then, by (6.19), $\mathcal{F}_{\Lambda'}(u) \leqslant \mathcal{F}_{\Lambda'}(h)$. This implies, just as in the proof of Lemma 6.12,

$$\begin{aligned} 0 \geqslant \lambda \iint_{B_r} \left|\nabla(u - h)\right|^2 dX \; - \; 2 \iint_{B_r} \big(F(X, u) - F(X, h) - F'(X, u)(u - h)\big) dX \; - \; \Lambda' \mathrm{Vol}_q(\{u = 0\} \cap B_r) \\ \geqslant \lambda \iint_{B_r} \left|\nabla(u - h)\right|^2 dX - \frac{\lambda}{2} \iint_{B_r} \left|\nabla(u - h)\right|^2 dX - \Lambda'\,\overline{q}\left|\{u = 0\} \cap B_r\right|, \end{aligned}$$

where we have used (HF5) and Poincaré to hide (for r small enough) the second term. The lemma follows. □

As a consequence, we can show some thickness of $\mathbb{R}^{n+1}\setminus\Omega_u$, where we recall the notation $\Omega_u := \{u > 0\}$:

Lemma 6.18. *Assume* (H*). *If* $x_0 \in \partial\Omega_u$, *then* $\big|\{u = 0\} \cap B_r(x_0)\big| > 0$ *for every* $r > 0$.

Proof. Fix $x_0 \in \partial\Omega_u$. Suppose, on the contrary, that $\big|\{u = 0\} \cap B_r(x_0)\big| = 0$ for some $r > 0$. Then (possibly using a smaller r), by (6.24), $\nabla u \equiv \nabla h$ a.e. in $B_r(x_0)$. Since $u = h$ on $\partial B_r(x_0)$, that translates into $u = h$ a.e. in B_r. Therefore $Lu = F'(x,u) \geqslant 0$ in the weak sense in $B_r(x_0)$ (using (HF3)), whence by elliptic regularity (e.g. Theorems 8.8 and 9.19 from [GT77], recalling (HA1), (HF1) and (6.15)) u is actually C^2 in $B_{r/2}(x_0)$, and hence L-superharmonic in a classical sense. On the other hand, since $x_0 \in \partial\Omega_u$, u is not constantly zero in $B_{r/2}(x_0)$. Thus, by the classical strong minimum principle for L-superharmonic functions, $u > 0$ in $B_{r/2}(x_0)$. But this clearly contradicts $x_0 \in \partial\Omega_u$. □

After these preliminaries, we are ready to proceed to prove the optimal regularity of u.

Proposition 6.19 (Lipschitz continuity)**.** *Assume* (H*). *Then* Ω_u *is open and*

$$\mathfrak{L} := \|\nabla u\|_{L^\infty(\mathbb{R}^{n+1})} < +\infty.$$

Proof. We can mostly follow [AC81, Th. 3.2] or [Ve23, Section 3.1]. Fix $\varepsilon > 0$ small, and set $\Lambda' := \Lambda + \varepsilon$ to use (6.19).

First, given $x_0 \in \mathbb{R}^{n+1}$ and small $r > 0$, let h be as in Lemma 6.17. Since $h \geqslant u$ and it is L-superharmonic in $B_r(x_0)$ by (HF3), we may follow the proof of [Ve23, Lemma 3.7] to obtain

$$\big|\{u = 0\ \cap B_r(x_0)\big| \left(\frac{1}{r}\fint_{\partial B_r(x_0)} u\, d\mathcal{H}^n\right)^2 \lesssim \iint\limits_{B_r(x_0)} \big|\nabla(u-h)\big|^2\, dX.$$

Dominating the right hand side with (6.24), and cancelling terms (recall Lemma 6.18), we obtain

$$\fint_{\partial B_r(x_0)} u\, d\mathcal{H}^n \leqslant C(n,\lambda,\overline{q})\,\sqrt{\Lambda'}\, r \qquad \forall\, x_0 \in \partial\Omega_u,\ r \text{ small enough.} \tag{6.25}$$

Recalling (6.16), this implies that $u \equiv 0$ on $\partial\Omega_u$ just letting $r \to 0$. Hence Ω_u is open.

It only remains to adapt [Ve23, Lemma 3.5]. Fix $X \in \Omega_u$ sufficiently close to $\partial\Omega_u$, and find $x_0 \in \partial\Omega_u$ its closest point on the boundary. Set $r := |X - x_0|$, so $r > 0$ because Ω_u is open. Then, by Lemma 6.8, $Lu = F'(X,u)$ in $B_r(X)$, whence $|Lu| \leqslant M_1$ in $B_r(X)$ (using also (HF3) and Corollary 6.10). This allows us to use Lemma 6.6 to get

$$\big|\nabla u(X)\big| \lesssim \frac{1}{r}\,\|u\|_{L^\infty(\partial B_{r/2}(X))} + r, \tag{6.26}$$

Now, for any fixed $y \in \partial B_{r/2}(X)$, again using Lemma 6.6 and also (6.25), we obtain

$$u(y) \lesssim \fint_{B_{r/4}(y)} u\, dz + r^2 \lesssim \fint_{B_{2r}(x_0)} u\, dz + r^2 \lesssim \sqrt{\Lambda'}\, r + r^2, \tag{6.27}$$

which back in (6.26) finally yields $\big|\nabla u(X)\big| \leqslant C$ if X is sufficiently close to $\partial\Omega_u$ (i.e. when r is small). Outside of a neighborhood of $\partial\Omega_u$, standard regularity for elliptic PDE yields that u is actually C^2 there, which is sufficient to finish the proof. □

Recalling Lemma 6.8, we immediately obtain the following:

Corollary 6.20 (Euler–Lagrange (II))**.** *Assume* (H*)*. Then* $Lu = F'(X,u)$ *in* Ω_u.

As a consequence of the Lipschitz estimate at the boundary (Proposition 6.19) and the exact equation that u solves in Ω_u (Corollary 6.20), we can now ensure that $\Lambda \neq 0$ using only elementary PDE techniques. It should be emphasized that, up to this moment, we never needed the strict inequality $\Lambda > 0$ in our proofs.

Proposition 6.21 (Positivity of Lagrange multiplier)**.** *Assume* (H*)*, and let* $\Lambda \geqslant 0$ *be the number obtained in Lemma 6.14. Then actually* $\Lambda > 0$.

Proof. Let us assume, on the contrary, that $\Lambda = 0$. Then, the estimates (6.26) and (6.27) that we obtained in the proof of Proposition 6.19 self-improve to $\big|\nabla u(X)\big| \leqslant Cr = C\,\mathrm{dist}(X,\partial\Omega_u)$ for X close to $\partial\Omega_u$, simply using $\Lambda = \Lambda + \varepsilon = \varepsilon$ and letting $\varepsilon \to 0$. On the other hand, by elliptic regularity, we may suppose that u is (qualitatively) regular in Ω_u, at least C^2 (one may use Theorems 8.8 and 9.19 from [GT77] taking (HF1) and (HA1) into account). Therefore, ∇u is a continuous function in Ω_u, which converges to 0 as we approach the boundary; so we can extend it by 0 in a continuous way. This means that, defining $\nabla u := 0$ on $\partial\Omega_u$, we have $u \in C^1(\overline{\Omega_u})$.

Now pick any $X \in \Omega_u$ and denote $B := B_{\mathrm{dist}(X,\partial\Omega_u)}(X)$. Clearly $B \subset \Omega_u$, so Corollary 6.20 and $F' \geqslant 0$ (see (HF3)) imply that u is L-superharmonic in B. Therefore, if $x_0 \in \partial\Omega_u$ is the closest point on the boundary to X, the classical strong minimum principle for L-superharmonic functions implies that $u(x_0) = 0 < u(y)$ for every $y \in \partial(\frac{1}{2}B)$. Moreover, recall that we already discussed in the previous paragraph that $u \in C^2(B) \cap C^1(\overline{B})$. All these allow us to apply the classical Hopf's lemma in B (because ∂B is smooth) from [Ev98, p.330] to deduce that $\partial_\nu u(x_0) < 0$, which contradicts the fact that $\nabla u = 0$ on $\partial\Omega$. □

Let us finish the section with a classical non-degeneracy estimate, which shows that around any point of the free boundary $\partial\Omega_u$, u must exhibit some linear growth. The proof uses ideas of David–Toro (see e.g. [Ve23, Lemma 4.5]):

Proposition 6.22 (Non-degeneracy)**.** *Assume* (H*)*. Then there exist small enough* $\kappa_0, r_0 > 0$ *such that, for any* $x_0 \in \mathbb{R}^{n+1}$ *and* $0 < r < r_0$*, it holds*

$$\fint_{\partial B_r(x_0)} u\,d\mathcal{H}^n \leqslant \kappa_0\, r \qquad \Longrightarrow \qquad u \equiv 0 \ \text{ in } B_{r/8}(x_0).$$

Proof. First, using h from Lemma 6.17 (adapted to $B_r(x_0)$), Lemma 6.6 (because $Lh \leqslant M_1$) implies, for any $Y \in B_{r/2}(x_0)$,

$$u(Y) \leqslant h(Y) \leqslant C \fint_{\partial B_r(x_0)} h\,d\mathcal{H}^n + Cr^2 = C \fint_{\partial B_r(x_0)} u\,d\mathcal{H}^n + Cr^2 \leqslant C(\kappa_0 + r_0)r =: \kappa_1 r.$$

Then, consider the cut-off function

$$\phi \in C_c^\infty(B_r(x_0)), \quad 0 \leqslant \phi \leqslant 1, \quad \phi \equiv 1 \text{ on } B_{r/2}(x_0), \quad |\nabla\phi| \leqslant 3r^{-1}.$$

Defining the competitor $v := (u - \kappa_1 r\phi)_+$, the computation above implies that $v \equiv 0$ in $B_{r/2}(x_0)$. Therefore, noting that $v \leqslant u$, we get

$$\begin{aligned}&\mathrm{Vol}_q(\Omega_u \cap B_{r/2}(x_0)) + \mathrm{Vol}_q(\Omega_v \cap B_r(x_0))\\ &\qquad = \mathrm{Vol}_q(\Omega_u \cap B_{r/2}(x_0)) + \mathrm{Vol}_q(\Omega_v \cap B_r(x_0)\backslash B_{r/2}(x_0)) \leqslant \mathrm{Vol}_q(\Omega_u \cap B_r(x_0)).\end{aligned}$$

This, the optimality condition (6.20) (since $\Lambda > 0$, we can use $\varepsilon := \Lambda/2 > 0$), and the fact that $|F'| \leqslant M_1$ (from (6.15) and (HF3)), yield

$$\begin{aligned}\frac{\Lambda}{2}\,\underline{q}\,\big|\Omega_u \cap B_{r/2}(x_0)\big| &\leqslant \frac{\Lambda}{2}\,\mathrm{Vol}_q(\Omega_u \cap B_{r/2}(x_0)) \leqslant \frac{\Lambda}{2}\,\mathrm{Vol}_q(\Omega_u \cap B_r(x_0)) - \frac{\Lambda}{2}\,\mathrm{Vol}_q(\Omega_v \cap B_r(x_0))\\
&\leqslant \iint\limits_{B_r(x_0)} \nabla v\, A\, \nabla v^{\mathrm{T}}\, dX - \iint\limits_{B_r(x_0)} \nabla u\, A\, \nabla u^{\mathrm{T}}\, dX - 2 \iint\limits_{B_r(x_0)} \big(F(X,v) - F(X,u)\big)\, dX\\
&\leqslant \iint\limits_{B_r(x_0)} \nabla(u - \kappa_1 r\phi)\, A\, \nabla(u - \kappa_1 r\phi)^{\mathrm{T}}\, dX - \iint\limits_{B_r(x_0)} \nabla u\, A\, \nabla u^{\mathrm{T}}\, dX + 2M_1\,|B_r|\,\|u - v\|_\infty\\
&\leqslant 2\kappa_1 r\,\|A\|_\infty \iint\limits_{B_r(x_0)} |\nabla u|\,|\nabla\phi|\; dX + \kappa_1^2 r^2\,\|A\|_\infty \iint\limits_{B_r(x_0)} |\nabla\phi|^2\; dX + 2M_1\,|B_r|\,\kappa_1 r\\
&\leqslant \big(6\kappa_1\lambda^{-1}\mathfrak{L} + 9\kappa_1^2\lambda^{-1} + 2M_1\kappa_1 r_0\big)\,|B_r|\,,\end{aligned}$$

which implies that

$$\big|\Omega_u \cap B_{r/2}(x_0)\big| \leqslant \kappa_2\,|B_r|\,, \quad \text{where } \kappa_2 := 2\kappa_1\,\frac{6\lambda^{-1}\mathfrak{L} + 9\kappa_1\lambda^{-1} + 2M_1 r_0}{\Lambda\underline{q}}.$$

To finish the proof, we may just follow the remaining part of [Ve23, Lemma 4.5] after choosing κ_0 and r_0 small enough. Let us include the details for completeness. Using the estimates already obtained, we compute

$$\iint\limits_{B_{r/2}(x_0)} u\, dX \leqslant \|u\|_{L^\infty(B_{r/2}(x_0))}\,\big|\Omega_u \cap B_{r/2}(x_0)\big| \leqslant \kappa_1\kappa_2 r\,|B_r|\,.$$

Now, given any $Y_0 \in B_{r/8}(x_0)$, it clearly holds $B_{r/4}(Y_0) \subset B_{r/2}(x_0)$. This, and an easy integration in polar coordinates, allow us to find some $\rho = \rho(Y_0) \in [r/8, r/4]$ for which

$$\iint\limits_{\partial B_\rho(Y_0)} u\, d\mathcal{H}^n \leqslant \fint_{r/8}^{r/4} \iint\limits_{\partial B_s(Y_0)} u\, d\mathcal{H}^n\, ds \leqslant \frac{1}{r/8} \iint\limits_{B_{r/2}(x_0)} u\, dX \leqslant 8\,\kappa_1\kappa_2\,|B_r|\,,$$

which trivially implies that (recalling that $\rho \geqslant r/8$ and that $n \geqslant 1$)

$$\fint_{B_\rho(Y_0)} u\, d\mathcal{H}^n \leqslant \kappa_3\,\rho, \qquad \text{where } \kappa_3 := 8^{n+2}\kappa_1\kappa_2.$$

Recalling the definitions of κ_1 and κ_2, we can choose κ_0 and r_0 small enough (depending on $n, \lambda, \Lambda, \mathfrak{L}, M_1$ and $\underline{q}$) so that $\kappa_3 \leqslant \kappa_0$. And this means that the property in the assumption of the proposition translates to smaller scales.

Let us iterate this. Indeed, fix $Y_0 \in B_{r/8}(x_0)$. Starting at $Y_0, \rho_0 := \rho \in [r/8, r/4]$ from last paragraph, use the method above (iteratively) to find a sequence of ρ_j (for $j \geqslant 1$) satisfying

$$\frac{\rho_j}{8} \leqslant \rho_{j+1} \leqslant \frac{\rho_j}{4}, \quad \text{and} \quad \fint_{\partial B_{\rho_j}(Y_0)} u\, d\mathcal{H}^n \leqslant \kappa_0\rho_j, \qquad \text{for every } j \geqslant 0.$$

Since u is Lipschitz (Proposition 6.19) and $\rho_j \to 0$, we finally obtain the desired result:

$$0 \leqslant u(Y_0) = \lim_{j\to+\infty} \fint_{\partial B_{\rho_j}(Y_0)} u\, d\mathcal{H}^n \leqslant \lim_{j\to+\infty} \kappa_0 \rho_j = 0, \qquad Y_0 \in B_{r/8}(x_0).$$

The proposition then follows. □

6.8 Compact support of minimizers

The volume of the support of the minimizer u is obviously bounded because $u \in \mathbb{K}_{\leqslant m}$, so

$$|\{u > 0\}| \leqslant m/\underline{q} =: \widetilde{m}\,. \tag{6.28}$$

But it is not clear a priori whether the support $\Omega_u = \{u > 0\}$ is bounded itself. In this section, we prove that this is actually the case: Ω_u is always bounded. Moreover, we provide uniform bounds for the number and diameter of its connected components. The fact that the support of the minimizers is not bounded a priori is one of the main differences of our work with respect to [AC81] and most of the subsequent works on this topic.

We start by partitioning Ω_u into a family of *enlarged connected components*, by which one means a union of connected components which are close to each other. More precisely, let us make the following definition, where "c.c." is short for "connected component":

Definition 6.23. Given a connected component V of Ω_u, we define the *enlarged connected component* to which it belongs as

$$\mathrm{ECC}(V) := \bigcup \left\{ U \text{ c.c. of } \Omega_u : \begin{array}{l} \exists\, r \geqslant 0 \text{ and } V_j\ (0 \leqslant j \leqslant r) \text{ c.c.'s of } \Omega_u, \\ \text{such that } V =: V_0, V_1, \dots, V_{r-1}, V_r := U, \text{ and} \\ \mathrm{dist}(\overline{V_j}, \overline{V_{j+1}}) \leqslant 1/4 \text{ for every } 0 \leqslant j \leqslant r-1 \end{array} \right\}.$$

Let us now show that Ω_u is bounded:

Proposition 6.24 (Boundedness of supports)**.** *Assume* (H*)*. Then Ω_u is a bounded set. In fact, Ω_u has at most N enlarged connected components, each of them having diameter bounded above by D, where $N, D > 0$ are uniform constants.*

Proof. This follows from the non-degeneracy estimate from Proposition 6.22 and $|\Omega_u| \leqslant \widetilde{m}$ (recall (6.28)). To spell out the details, let us fix a connected component V of Ω_u. Denote $\mathcal{J} := \left\{\mathbf{j} \in \mathbb{Z}^{n+1} : \mathrm{ECC}(V) \cap (\mathbf{j} + [0,1)^{n+1}) \neq \varnothing\right\}$. Then, for a given $\mathbf{j} \in \mathcal{J}$, find $X_\mathbf{j} \in \mathrm{ECC}(V) \cap (\mathbf{j} + [0,1)^{n+1})$.

We claim (and will prove later) that there exists some uniform $\rho > 0$, and some point $z_\mathbf{j}$ such that $B_\rho(z_\mathbf{j}) \subset \Omega_u \cap B_{1/8}(X_\mathbf{j})$. Assuming this claim for the moment, we compute

$$\widetilde{m} \geqslant |\Omega_u| \geqslant \left| \bigcup_{\mathbf{j}\in\mathcal{J}} B_\rho(z_\mathbf{j}) \right| \geqslant C(n) \sum_{\mathbf{j}\in\mathcal{J}} \left|B_\rho(z_\mathbf{j})\right| = C(n)\rho^{n+1}\,|\mathcal{J}|\,,$$

where the third inequality follows simply by bounded overlap of the $B_\rho(z_\mathbf{j})$, because each of them lives in $B_{1/8}(X_\mathbf{j})$, which are themselves balls with bounded overlap because $X_\mathbf{j} \in \mathbf{j} + [0,1)^{n+1}$. This computation

shows that $|\mathcal{J}| < +\infty$, and actually provides a uniform bound. Since $\bigcup_{\mathbf{j}\in\mathcal{J}}(\mathbf{j}+[0,1]^{n+1})$ is connected by the definition of $\mathrm{ECC}(V)$, and since it contains $\mathrm{ECC}(V)$, we conclude that $\mathrm{diam}(\mathrm{ECC}(V))$ is uniformly bounded.

Moreover, given any $X_{\mathbf{j}} \in \mathrm{ECC}(V) \cap (\mathbf{j} + [0,1)^{n+1})$ for $\mathbf{j} \in \mathcal{J}$, we have found $z_{\mathbf{j}}$ such that $B_\rho(z_{\mathbf{j}}) \subset \Omega_u \cap B_{1/8}(X_{\mathbf{j}})$. Clearly, by definition of $\mathrm{ECC}(V)$, we have $B_\rho(z_{\mathbf{j}}) \subset \mathrm{ECC}(V)$. Therefore, $|\mathrm{ECC}(V)| \gtrsim \rho^{n+1}$. Since the enlarged connected components are disjoint, this implies that the amount of them is $\lesssim \widetilde{m}/\rho^{n+1}$.

Therefore, to finish the proof of the proposition, it only remains to prove the claim. First, note that there exists a uniform $0 < \rho < 1/40$ such that for any $y \in \partial\Omega_u$, there exists $z \in \Omega_u$ with $B_\rho(z) \subset \Omega_u \cap B_{1/20}(y)$ (this is the so-called "Corkscrew condition", see Figure 6.2). This was already spotted in [AC81, Lemma 3.7]: for r small, by non-degeneracy (see Proposition 6.22) $\|u\|_{L^\infty(B_r(y))} > \kappa_0 r$, so there exists $z \in B_r(y)$ with $u(z) \geqslant \kappa_0 r$, and then by Lipschitz continuity (see Proposition 6.19) we have $u \geqslant \kappa_0 r/2$ in $B_{\kappa_0 r/(2\mathfrak{L})}(z)$. Thus, writing $\rho := \kappa_0 r/(2\mathfrak{L})$, we have $B_\rho(z) \subset \Omega_u \cap B_{1/20}(y)$ (because we can always take $r, \rho < 1/40$).

Then, fix $\mathbf{j} \in \mathcal{J}$, and $\rho > 0$ as in last paragraph. If $B_\rho(X_{\mathbf{j}}) \subset \Omega_u$, we can simply take $z_{\mathbf{j}} := X_{\mathbf{j}}$ and finish. Otherwise, there exists $y_{\mathbf{j}} \in B_\rho(X_{\mathbf{j}}) \cap \partial\Omega_u$. Applying the method in the previous paragraph, we find $z_{\mathbf{j}}$ such that $B_\rho(z_{\mathbf{j}}) \subset \Omega_u \cap B_{1/20}(y_{\mathbf{j}}) \subset \Omega_u \cap B_{1/8}(X_{\mathbf{j}})$, where the last inclusion holds because $|y_{\mathbf{j}} - X_{\mathbf{j}}| < \rho < 1/40$. This finishes the proof of the claim, and therefore of the proposition. □

Note that this proposition does *not* provide a uniform bound for the diameter of Ω_u. In fact, without the periodicity assumption (HPer), which we will not use until Section 6.10, uniform bounds do not exist in general, as discussed in Section 6.12.

6.9 The free boundary

In Section 6.7, we showed the Lipschitz continuity and the non-degeneracy estimates for our minimizer u, and also that it behaves almost like a minimizer of $\mathcal{F}_\Lambda$ for some $\Lambda > 0$ (more precisely, it is a critical point and it satisfies some "asymptotic minimality" as in (6.23)). Therefore, we are in the position to

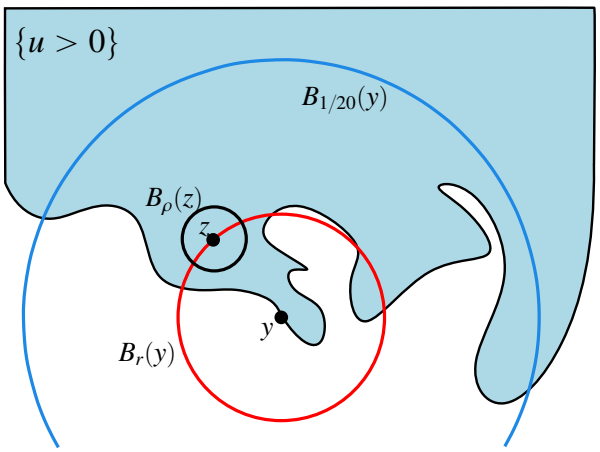


Figure 6.2: Given any free boundary point $y \in \partial\{u > 0\}$ there is always an interior ball $B_\rho(z) \subset \Omega_u \cap B_1(y)$, with $\rho > 0$ uniform.

mostly follow the classical blueprint of [AC81] or [Ve23, Sections 6–10] to show the regularity of the free boundary $\partial\Omega_u$, along with estimating the size of its singular set. The arguments will be of local nature, so Hypothesis (HPer) will not be needed.

6.9.1 Analysis of blow-up limits

In this subsection, we will construct appropriate blow-up limits at free boundary points, and will prove that they are 1-homogeneous. We will need to introduce some changes in the classical strategy to deal with the terms introduced by F; for instance, we will need to introduce a modified version of Weiss formula for it to become (almost) monotone in our setting. We will show that F basically disappears when blowing up around free boundary points, which will morally take us back to the classical setting.

In this subsection, we will fix a free boundary point. Translating the origin, we can take this point as $0 \in \partial\Omega_u$ without any loss of generality, which makes the notation less cumbersome. With this in mind, we define, for $r > 0$,

$$u_r(X) := \frac{u(rX)}{r}, \qquad F_r(X,u) := F(rX, ru), \qquad F_0(X,u) := 0,$$

where the last definition is coherent because $F(\cdot,0) \equiv 0$, see (HF2). Similarly,

$$A_r(X) := A(rX), \qquad A_0(X) := A(0), \qquad q_r(x) := q(rx), \qquad q_0(x) := q(0).$$

With all these in mind, we set, for $r \geqslant 0$ and $D \subset \mathbb{R}^{n+1}$,

$$\mathcal{F}_{\Lambda,r}(v,D) := \iint_D \nabla v\, A_r\, \nabla v^{\mathrm{T}}\, dX - 2\iint_D F_r(X,v)\, dX + \Lambda \mathrm{Vol}_{q_r}(\Omega_v \cap D),$$

and we will omit D when $D = \mathbb{R}^{n+1}$. Here and in what follows, $\Omega_v := \{v > 0\}$. Thus,

$$\mathcal{F}_{\Lambda,0}(v,D) := \iint_D \nabla v(X)\, A(0)\, \nabla v(X)^{\mathrm{T}}\, dX + \Lambda q(0)\, |\Omega_v \cap D|\,.$$

Since $u(0) = 0$ because $0 \in \partial\Omega_u$, we note that the rescalings of the minimizer u yield, for $r > 0$,

$$\|\nabla u_r\|_\infty = \|\nabla u\|_\infty = \mathfrak{L}, \quad \text{whence} \quad u_r(X) = u_r(X) - u_r(0) \leqslant \mathfrak{L}\,|X|\,, \quad X \in \mathbb{R}^{n+1}. \tag{6.29}$$

We start with the following auxiliary results:

Lemma 6.25. *Assume* (H*) *and* $0 \in \partial\Omega_u$. *Fix* $R > 0$. *Then, as* $r \to 0$, $F_r(\cdot, u_r(\cdot)) = O(r)$, $|A_r - A_0| = O(r)$, *and* $|q_r - q_0| = O(r)$. *Concretely,* $F_r(\cdot, u_r(\cdot)) \to 0$, $A_r \to A_0$ *and* $q_r \to q_0$ *in* $L^\infty(B_R)$, *as* $r \to 0$.

Proof. Using (HF2), (HF3), (6.15) and Proposition 6.19, we get, for $X \in B_R$,

$$\big|F_r(X, u_{r_j}(X))\big| = \big|F(rX, u(rX)) - F(rX,0)\big| \leqslant M_1\,\big|u(rX) - u(0)\big| \leqslant M_1 \mathfrak{L}\, rR \underset{r\to 0}{\longrightarrow} 0.$$

And then, using (HA1), we compute $\big|A_r(X) - A_0(X)\big| = \big|A(rX) - A(0)\big| \leqslant \|\nabla A\|_\infty\, rR \to 0$ as $r \to 0$. And similarly for q, this time using (Hq1). □

Lemma 6.26 (Convergence to blow-ups)**.** *Assume* (H*) *and* $0 \in \partial\Omega_u$. *Fix* $R > 0$. *Let* $r_j \to 0$. *Then, up to taking a subsequence, there exists* $u_0 \in H^1(B_R)$ *such that, for every* $0 < r < R$,

1. $u_{r_j} \to u_0$ *in* $H^1(B_r)$ *and uniformly, as* $j \to +\infty$,
2. $\mathbf{1}_{\Omega_{u_{r_j}}} \to \mathbf{1}_{\Omega_{u_0}}$ *in* $L^1(B_r)$ *and pointwise a.e., as* $j \to +\infty$.

Proof. Fix $0 < r < R$ as in the statement.

Step 1: Weak convergence. First note that (6.29) easily yields that te sequence $\{u_{r_j}\}_j$ is uniformly bounded in $H^1(B_R)$. Therefore, by standard compactness results, there exists $u_0 \in H^1(B_R)$ such that (up to taking a subsequence) as $j \to \infty$, we have $u_{r_j} \longrightarrow u_0$ both in $L^2(B_R)$ and uniformly in B_R (by Arzelà–Ascoli, because of the uniform Lipschitz constant in (6.29)), and $\nabla u_{r_j} \rightharpoonup \nabla u_0$ in $L^2(B_R)$. Clearly, that implies

$$\|\nabla u_0\|_{L^2(B_r)} \leqslant \liminf_{j\to+\infty} \left\|\nabla u_{r_j}\right\|_{L^2(B_r)}. \tag{6.30}$$

Also, similarly as in (6.21),

$$\iint_{B_r} \nabla u_0\, A_0\, \nabla u_0^{\mathrm{T}}\, dX \leqslant \liminf_{j\to+\infty} \iint_{B_r} \nabla u_{r_j}\, A_0\, \nabla u_{r_j}^{\mathrm{T}}\, dX. \tag{6.31}$$

Moreover, by the uniform pointwise convergence $u_j \to u_0$, if $X \in \Omega_{u_0}$ (that is, $u_0(X) > 0$), necessarily $u_{r_j}(X) > 0$ for sufficiently large j. This means

$$\mathbf{1}_{\Omega_{u_0}} \leqslant \liminf_{j\to+\infty} \mathbf{1}_{\Omega_{u_{r_j}}}, \quad \text{so by Fatou} \quad \mathrm{Vol}_q(\Omega_{u_0} \cap B_r) \leqslant \liminf_{j\to+\infty} \mathrm{Vol}_q(\Omega_{u_{r_j}} \cap B_r). \tag{6.32}$$

Therefore, to obtain the desired convergences, it suffices to obtain

$$\|\nabla u_0\|_{L^2(B_r)} \geqslant \liminf_{j\to+\infty} \left\|\nabla u_{r_j}\right\|_{L^2(B_r)} \quad \text{and} \quad \mathrm{Vol}_q(\Omega_{u_0} \cap B_r) \geqslant \liminf_{j\to+\infty} \mathrm{Vol}_q(\Omega_{u_{r_j}} \cap B_r). \tag{6.33}$$

Indeed, the second condition and (6.32), along with (Hq2), will imply the convergence in 2 similarly as in (6.22). Moreover, for simplicity in the notation, we may just assume that the lim inf are actually lim by passing to a further subsequence.

Step 2: Rescaling the energy. We will prove (6.33) using variational methods. Indeed, the key fact is that minimization properties of u with respect to $\mathcal{F}_\Lambda$ translate directly into minimization properties of u_{r_j} with respect to $\mathcal{F}_{\Lambda,r_j}$: an easy change of variables shows that for any $\rho > 0$ we have $\mathcal{F}_{\Lambda,r_j}(u_{r_j}, B_\rho) = r_j^{-(n+1)} \mathcal{F}_\Lambda(u, B_{r_j\rho})$. Therefore, if we fix $\varepsilon > 0$, rescaling (6.23) for $j \gg 1$ (so that $Rr_j \leqslant r_0$) yields

$$\mathcal{F}_{\Lambda,r_j}(u_{r_j}, B_R) \leqslant \mathcal{F}_{\Lambda,r_j}(v, B_R) + C(\overline{q}, R, n)\, \varepsilon, \quad \forall\, v \in H^1(\mathbb{R}^{n+1}) \text{ with } v - u_{r_j} \in H^1_0(B_R). \tag{6.34}$$

To be able to use (6.34) and still obtain convergences in B_r as in (6.33), we define the competitor

$$\widetilde{u}_{r_j} := \eta\, u_{r_j} + (1 - \eta)\, u_0,$$

where η is a cut-off function satisfying

$$\eta \in C^\infty(B_R), \quad 0 \leqslant \eta \leqslant 1, \quad \eta \equiv 0 \text{ in } B_r, \quad \eta = 1 \text{ on } \partial B_R,$$

so that $\widetilde{u}_{r_j} \equiv u_0$ in B_r and $\widetilde{u}_{r_j} - u_{r_j} \in H^1_0(B_R)$. Therefore, using (6.34) with $v = \widetilde{u}_{r_j}$, we obtain

$$\mathcal{F}_{\Lambda,r_j}(u_{r_j}, B_r) \leqslant \mathcal{F}_{\Lambda,r_j}(u_0, B_r) + C\varepsilon + \left(\mathcal{F}_{\Lambda,r_j}(\widetilde{u}_{r_j}, B_R\setminus B_r) - \mathcal{F}_{\Lambda,r_j}(u_{r_j}, B_R\setminus B_r)\right). \tag{6.35}$$

Our goal now is to disregard the last term and take limits.

Step 3: Terms with F vanish in the limit. Before that, let us show that, after taking limits, all the terms with F disappear. Indeed, if $X \in B_R$, (6.29) yields

$$\left|F_{r_j}(X, \widetilde{u}_{r_j}(X))\right| + \left|F_{r_j}(X, u_{r_j}(X))\right| \lesssim_{M_1} \left|r_j\widetilde{u}_{r_j}(X)\right| + \left|r_j u_{r_j}(X)\right| \leqslant r_j(2\mathfrak{Q}R + |u_0(X)|).$$

Therefore, the fact that $u_0 \in L^2(B_R)$ and the Dominated Convergence Theorem imply that

$$\lim_{j\to\infty} \iint_{B_R\setminus B_r} \left(F_{r_j}(X, \widetilde{u}_{r_j}) - F_{r_j}(X, u_{r_j})\right) dX = 0. \tag{6.36}$$

In fact, similar computations yield

$$\lim_{j\to\infty} \iint_{B_r} F_{r_j}(X, u_{r_j})\, dX = 0, \quad \text{and} \quad \lim_{j\to\infty} \iint_{B_r} F_{r_j}(X, u_0)\, dX = 0. \tag{6.37}$$

With this in mind, our new goal is to show that for every $\varepsilon > 0$ it holds

$$\begin{aligned} \text{LHS} := \lim_{j\to+\infty} \iint_{B_r} \nabla u_{r_j} A_0 \nabla u_{r_j}^{\mathrm{T}}\, dX + \Lambda \operatorname{Vol}_{q_0}(\Omega_{u_{r_j}} \cap B_r) \\ \leqslant \iint_{B_r} \nabla u_0 A_0 \nabla u_0^{\mathrm{T}}\, dX + \Lambda \operatorname{Vol}_{q_0}(\Omega_{u_0} \cap B_r) + C\varepsilon =: \text{RHS}. \end{aligned} \tag{6.38}$$

Indeed, assuming this momentarily, we can let $\varepsilon \to 0$. Then, noting that every term is non-negative (recall (HA3) and (Hq2)) and that (6.31) and (6.32) hold, we obtain

$$\lim_{j\to+\infty} \iint_{B_r} \nabla u_{r_j} A_0 \nabla u_{r_j}^{\mathrm{T}}\, dX = \iint_{B_r} \nabla u_0 A_0 \nabla u_0^{\mathrm{T}}\, dX, \quad \text{and} \quad \lim_{j\to+\infty} \operatorname{Vol}_{q_0}(\Omega_{u_{r_j}} \cap B_r) = \operatorname{Vol}_{q_0}(\Omega_{u_0} \cap B_r).$$

Which clearly implies (6.33) because

$$\begin{aligned} 0 &\leqslant \lim_{j\to+\infty} \iint_{B_r} \left(|\nabla u_{r_j}|^2 - |\nabla u_0|^2\right) dX = \lim_{j\to+\infty} \iint_{B_r} \left(\nabla u_{r_j} \nabla u_{r_j}^{\mathrm{T}} - \nabla u_0 \nabla u_0^{\mathrm{T}}\right) dX \\ &= \lim_{j\to+\infty} \iint_{B_r} \nabla(u_{r_j} - u_0) \nabla(u_{r_j} - u_0)^{\mathrm{T}}\, dX + 2 \lim_{j\to+\infty} \iint_{B_r} \nabla(u_{r_j} - u_0)\, \nabla u_0^{\mathrm{T}} \\ &= \lim_{j\to+\infty} \iint_{B_r} \nabla(u_{r_j} - u_0) \nabla(u_{r_j} - u_0)^{\mathrm{T}}\, dX \leqslant \lambda^{-1} \iint_{B_r} \nabla(u_{r_j} - u_0) A_0 \nabla(u_{r_j} - u_0)^{\mathrm{T}}\, dX = 0, \end{aligned}$$

where we have used (6.30) and the weak convergence $\nabla u_{r_j} \rightharpoonup \nabla u_0$ in $L^2(B_r)$.

Step 4: Analysis of the error terms. Starting from LHS in (6.38), approximating A_0 by A_{r_j} and q_0 by q_{r_j}, then applying the inequality (6.35) noting that the terms with F disappear by (6.36) and (6.37), and finally reintroducing A_0 and q_0 by approximation from A_{r_j} and q_{r_j}, we can compute

$$\begin{aligned}\text{LHS} \leqslant \text{RHS} &+ \text{Errors } A + \text{Errors } q \\ &+ \lim_{j\to+\infty} \iint_{B_R\setminus B_r} \left(\nabla\widetilde{u}_{r_j} A_{r_j} \nabla\widetilde{u}_{r_j}^{\mathrm{T}} - \nabla u_{r_j} A_{r_j} \nabla u_{r_j}^{\mathrm{T}}\right) dX \\ &+ \Lambda \lim_{j\to+\infty} \iint_{B_R\setminus B_r} q_{r_j} \left(\mathbf{1}_{\Omega_{\widetilde{u}_{r_j}}} - \mathbf{1}_{\Omega_{u_{r_j}}}\right) dX.\end{aligned} \tag{6.39}$$

Since $\left|A_{r_j}(X) - A_0(X)\right| = \left|A(r_j X) - A(0)\right| \leqslant r_j\, r \|\nabla A\|_\infty$ for $X \in B_r$ by (HA1), we can bound the errors in the approximations between A_{r_j} and A_0 as follows:

$$\begin{aligned}|\text{Errors } A| &= \left| \lim_{j\to+\infty} \iint_{B_r} \nabla u_{r_j} \left(A_{r_j} - A_0\right) \nabla u_{r_j}\, dX + \iint_{B_r} \nabla u_0 \left(A_{r_j} - A_0\right) \nabla u_0\, dX \right| \\ &\leqslant \lim_{j\to+\infty} r_j\, r \|\nabla A\|_\infty \left(\left\|\nabla u_{r_j}\right\|_{L^2(B_r)} + \|\nabla u_0\|_{L^2(B_r)} \right) = 0.\end{aligned}$$

We have used that u_{r_j} have a uniform bound in $H^1(B_R)$, and $u_0 \in H^1(B_R)$. Similarly, using (Hq1),

$$|\text{Errors } q| = \Lambda \left| \lim_{j\to+\infty} \iint_{B_r} (q_{r_j} - q_0) \mathbf{1}_{\Omega_{u_{r_j}}}\, dX + \iint_{B_r} (q_{r_j} - q_0) \mathbf{1}_{\Omega_{u_0}}\, dX \right| \leqslant 2\Lambda \lim_{j\to+\infty} r_j r \|\nabla q\|_\infty |B_r| = 0.$$

Therefore, to obtain (6.38) from (6.39), we are left with the last two terms in (6.39). Recalling the definition of $\widetilde{u}_{r_j}$, we easily see that $\nabla\widetilde{u}_{r_j} = (u_{r_j} - u_0)\nabla\eta + \eta\nabla u_{r_j} + (1-\eta)\nabla u_0$. Thus, expanding the products in the second-to-last term in (6.39), and neglecting the terms that converge to zero because $u_{r_j} \to u_0$ in $L^2(B_r)$ (for that, note that η is smooth, A_{r_j} is uniformly bounded, $u_0 \in H^1(B_r)$ and u_{r_j} have a uniform bound in $H^1(B_r)$), we obtain

$$\begin{aligned}&\lim_{j\to+\infty} \iint_{B_R\setminus B_r} \left(\nabla\widetilde{u}_{r_j} A_{r_j} \nabla\widetilde{u}_{r_j}^{\mathrm{T}} - \nabla u_{r_j} A_{r_j} \nabla u_{r_j}^{\mathrm{T}}\right) dX \\ &\quad= \lim_{j\to+\infty} \iint_{B_R\setminus B_r} \left((\eta^2-1)\nabla u_{r_j} A_{r_j} \nabla u_{r_j}^{\mathrm{T}} + 2\eta(1-\eta)\nabla u_{r_j} A_{r_j} \nabla u_0^{\mathrm{T}} + (1-\eta)^2 \nabla u_0 A_{r_j} \nabla u_0^{\mathrm{T}}\right) dX \\ &\quad= \lim_{j\to+\infty} \iint_{B_R\setminus B_r} (1-\eta^2) \left(\nabla u_0 A_{r_j} \nabla u_0^{\mathrm{T}} - \nabla u_{r_j} A_{r_j} \nabla u_{r_j}^{\mathrm{T}}\right) dX \\ &\quad\leqslant \lim_{j\to+\infty} \iint_{B_R\setminus B_r\cap\{\eta=0\}} \left(\nabla u_0 A_{r_j} \nabla u_0^{\mathrm{T}} - \nabla u_{r_j} A_{r_j} \nabla u_{r_j}^{\mathrm{T}}\right) dX + \lim_{j\to+\infty} \iint_{\{\eta>0\}} \nabla u_0 A_{r_j} \nabla u_0^{\mathrm{T}}\, dX \\ &\quad\leqslant \lambda^{-1} \iint_{\{\eta>0\}} |\nabla u_0|^2\, dX.\end{aligned}$$

In the second equality we have used the weak convergence $\nabla u_{r_j} \rightharpoonup \nabla u_0$ in $L^2(B_r)$; in the next inequality we have used ellipticity of A_{r_j} (HA3) to disregard the term with u_{r_j} in the region where $\eta > 0$; and in the last estimate we have argued as in (6.31), and used the boundedness of A_{r_j} (HA3).

In turn, the last term in (6.39) is easier to deal with:

$$\lim_{j\to+\infty} \iint_{B_R\setminus B_r} q_{r_j}\big(\mathbf{1}_{\Omega_{\tilde{u}_{r_j}}} - \mathbf{1}_{\Omega_{u_{r_j}}}\big)\,dX = \lim_{j\to+\infty} \iint_{B_R\setminus B_r\cap\{\eta=0\}} q_{r_j}\big(\mathbf{1}_{\Omega_{u_0}} - \mathbf{1}_{\Omega_{u_{r_j}}}\big)\,dX + \iint_{\{\eta>0\}} q_{r_j}\big(\mathbf{1}_{\Omega_{\tilde{u}_{r_j}}} - \mathbf{1}_{\Omega_{u_{r_j}}}\big)\,dX \leqslant 2\,\overline{q}\,\big|\{\eta>0\}\big|,$$

simply because the first term in the second expression is non-positive by (6.32).

Step 5: Conclusion of the proof. Plugging these estimates into (6.39) yields

$$\mathrm{LHS} \leqslant \mathrm{RHS} + \lambda^{-1} \iint_{\{\eta>0\}} |\nabla u_0|^2\,dX + 2\,\overline{q}\,\big|\{\eta>0\}\big|.$$

Now, since η is arbitrary, this yields (6.38), hence finishing the proof. □

Let us state a few interesting consequences of the convergence result we have just established:

Corollary 6.27. *Assume* (H*) *and* $0 \in \partial\Omega_u$. *Then, for any* $R > 0$, *if* u_0 *is a blow-up limit as in Lemma 6.26, then:*

1. u_0 *is stationary for* $\mathcal{F}_{\Lambda,0}$, *i.e.* $\delta\mathcal{F}_{\Lambda,0}(u_0)[\xi] = 0$ *for every* $\xi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$,
2. *Furthermore,* u_0 *is a local minimizer of* $\mathcal{F}_{\Lambda,0}$ *in* B_R, *that is, for every* $v \in H^1(B_R)$ *so that* $u_0 - v \in H^1_0(B_R)$ *we have* $\mathcal{F}_{\Lambda,0}(u_0) \leqslant \mathcal{F}_{\Lambda,0}(v)$.

Proof. We show the stationary condition first. Recall that $\mathcal{F}_{\Lambda,0}(u) = \iint_{\mathbb{R}^{n+1}} \nabla u\, A_0\, \nabla u^{\mathrm{T}}\,dX + \Lambda \mathrm{Vol}_{q_0}(\Omega_u)$, where we note that $\mathrm{Vol}_{q_0}(\Omega_u)$ is just $q(0)\,|\Omega_u|$. Using the expressions obtained in Lemma 6.14, noting that A_0 and q_0 are constant functions, we obtain that for $\xi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$,

$$\delta\mathcal{F}_{\Lambda,0}(u)[\xi] = \iint_{\mathbb{R}^{n+1}} \Big(-2\nabla u\, D\xi\, A(0)\, \nabla u^{\mathrm{T}} + \nabla u\, A(0)\, \nabla u^{\mathrm{T}} \operatorname{div}\xi \Big)\,dX + \Lambda q(0) \iint_{\Omega_u} \operatorname{div}\xi\,dX.$$

Now, recall that in Lemma 6.14 we obtained that $\delta\mathcal{F}_\Lambda(u)[\xi] = 0$ for every $\xi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$. Rescaling that, we obtain $\delta\mathcal{F}_{\Lambda,r_j}(u_{r_j})[\xi] = 0$ for every $\xi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ ($\delta\mathcal{F}_{\Lambda,r_j}$ has an explicit expression as in Lemma 6.14, but with A_{r_j}, F_{r_j} and q_{r_j} replacing A, F and q). Now, using all the convergences in Lemmas 6.25 and 6.26, we can take limits in each term of the explicit expression of $\delta\mathcal{F}_{\Lambda,r_j}(u_{r_j})[\xi] = 0$ to obtain $\delta\mathcal{F}_{\Lambda,0}(u_0)[\xi] = 0$, for each $\xi \in C_c^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$.

To prove the local minimality of u_0, we can now use Lemmas 6.25 and 6.26 and simply follow the proof in the second half of [Ve23, Lemma 6.3]: everything now works directly in our setting. □

Our next objective is to show that blow-ups are 1-homogeneous. For this, we will follow the approach in [Ve23, Section 9], which depends on the Weiss monotonicity formula. In this kind of free boundary problems, the Weiss function is typically monotone in r (see e.g. [Ve23, Prop. 9.9]). Nevertheless, in our setting, the presence of $F(X,u)$ and the coefficients $A(X)$ make it more difficult to prove this monotonicity, and in fact we will actually only prove that the derivative of the Weiss function is bounded below (but possibly negative). Fortunately, we will show this suffices to show that blow-up limits are 1-homogeneous.

Lemma 6.28. *Assume* (H*) *and* $0 \in \partial\Omega_u$. *Then any blow-up limit* u_0 *obtained as in Lemma 6.26 is 1-homogeneous.*

Proof. As commented before, we will rely on a Weiss monotonicity formula. However, the fact that our blow-up limits u_0 are minimizers of energies with $A(0)$ instead of the identity, would make us need to adapt the monotonicity formula to these constant matrices. Instead of that, we prefer to change coordinates from the beginning, to find the Laplacian in the limit.

Step 1: A convenient change of variables. The fact that $A(0)$ is symmetric and elliptic (see (HA2) and (HA3)) allows us to find a (unique) symmetric matrix $\sqrt{A(0)}$ which is still elliptic and bounded (as in (HA3)), which constants depending on λ.

With that in mind, define $\widehat{u}_0(X) := u_0(X\sqrt{A(0)})$. Then, Corollary 6.27 implies that $\widehat{u}_0$ is a local minimizer of the energy

$$\widehat{\mathcal{F}}_{\Lambda,0}(v,D) := \iint_D |\nabla v|^2 \, dX + \Lambda q(0)\,|\Omega_v \cap D|\,, \qquad D \subset \mathbb{R}^{n+1}.$$

Indeed, let $R>0$ and $\widehat{v} \in H^1(\mathbb{R}^{n+1})$ with $\widehat{v}-\widehat{u}_0 \in H^1_0(B_R)$. Then, if we define $v(X) := \widehat{v}(X\sqrt{A(0)}^{-1})$ (so that $\widehat{v}(X) = v(X\sqrt{A(0)})$), we have $v-u_0 \in H^1_0(B_{\lambda^{-1/2}R})$ by ellipticity of $A(0)$. Therefore, by Corollary 6.27, $\mathcal{F}_{\Lambda,0}(u_0, B_{\lambda^{-1/2}R}) \leqslant \mathcal{F}_{\Lambda,0}(v, B_{\lambda^{-1/2}R})$, that is,

$$\iint_{\mathbb{R}^{n+1}} \nabla u_0\, A(0)\, \nabla u_0^{\mathrm{T}}\, dX + \Lambda q(0)\left|\Omega_{u_0} \cap B_{\lambda^{1/2}R}\right| \leqslant \iint_{\mathbb{R}^{n+1}} \nabla v\, A(0)\, \nabla v^{\mathrm{T}}\, dX + \Lambda q(0)\left|\Omega_v \cap B_{\lambda^{1/2}R}\right|,$$

and the change of variables $X = y\sqrt{A(0)}$ yields (after canceling out the constant Jacobian)

$$\iint_{\mathbb{R}^{n+1}} \nabla \widehat{u}_0\, \nabla \widehat{u}_0^{\mathrm{T}}\, dy + \Lambda q(0)\left|\Omega_{\widehat{u}_0} \cap B_R\right| \leqslant \iint_{\mathbb{R}^{n+1}} \nabla \widehat{v}\, \nabla \widehat{v}^{\mathrm{T}}\, dX + \Lambda q(0)\,|\Omega_v \cap B_R|\,,$$

i.e. $\widehat{\mathcal{F}}_{\Lambda,0}(\widehat{u}_0, B_R) \leqslant \widehat{\mathcal{F}}_{\Lambda,0}(\widehat{v}, B_R)$.

Furthermore, $\widehat{u}_0$ is a stationary point for $\widehat{\mathcal{F}}_{\Lambda,0}$. Take $\widehat{\xi} \in C^\infty_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$, and define $\xi(X) := \sqrt{A(0)}\,\widehat{\xi}(X\sqrt{A(0)}^{-1}) \in C^\infty_c(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ (so that $\widehat{\xi}(X) = \sqrt{A(0)}^{-1}\xi(X\sqrt{A(0)})$). A change of variables and Corollary 6.27 show that indeed

$$\delta\widehat{\mathcal{F}}_{\Lambda,0}(\widehat{u}_0)[\widehat{\xi}] = \iint_{\mathbb{R}^{n+1}} \Big(-2\nabla\widehat{u}(Y)\, D\widehat{\xi}(Y)\, \nabla\widehat{u}(Y)^{\mathrm{T}}\, dY + \nabla\widehat{u}(Y)\, \nabla\widehat{u}(Y)^{\mathrm{T}} \operatorname{div}\widehat{\xi}(Y)\Big)\, dY$$

$$+\Lambda q(0)\iint\limits_{\mathbb{R}^{n+1}}\mathbf{1}_{\Omega_{\widehat{u}}}(Y)\operatorname{div}\widehat{\xi}(Y)\,dY=\left|\det(A(0))\right|^{-1/2}\delta\mathcal{F}_{\Lambda,0}(u_0)[\xi]=0. \tag{6.40}$$

What we use in the change of variable is that

$$\nabla\widehat{u}(Y)=\nabla u(Y\sqrt{A(0)})\sqrt{A(0)},\qquad \mathbf{1}_{\Omega_{\widehat{u}}}(Y)=\mathbf{1}_{\Omega_u}(Y\sqrt{A(0)}),$$

$$D\widehat{\xi}(Y)=\sqrt{A(0)}^{-1}(D\xi)(Y\sqrt{A(0)})\sqrt{A(0)},\qquad \operatorname{div}\widehat{\xi}(Y)=(\operatorname{div}\xi)(Y\sqrt{A(0)}).$$

To carry out the whole blow-up procedure later, we set

$$\widehat{u}(X):=u(X\sqrt{A(0)}),\qquad \widehat{A}(X):=\sqrt{A(0)}^{-1}A(X\sqrt{A(0)})\sqrt{A(0)}^{-1},$$

$$\widehat{\mathcal{F}}(X,u):=F(X\sqrt{A(0)},u),\qquad \widehat{q}(x):=q(x\sqrt{A(0)}).$$

Then, Lemma 6.14 and a change of coordinates show that $\widehat{u}$ is stationary for

$$\widehat{\mathcal{F}}_\Lambda(v):=\iint\limits_{\mathbb{R}^{n+1}}\nabla v\,\widehat{A}\,\nabla v^{\mathrm{T}}dX-2\iint\limits_{\mathbb{R}^{n+1}}\widehat{F}(X,v(X))\,dX+\Lambda\mathrm{Vol}_{\widehat{q}}(\Omega_v).$$

The change of variables is similar to the one in (6.40), taking also into account that

$$\nabla\widehat{F}(Y,u)=(\nabla F)(Y\sqrt{A(0)},u)\sqrt{A(0)},\quad \nabla\widehat{q}(Y)=(\nabla q)(Y\sqrt{A(0)})\sqrt{A(0)}.$$

Finally, in a similar way, from Corollary 6.20 we infer that $-\operatorname{div}(\widehat{A}\nabla\widehat{u})=\widehat{F}'(X,\widehat{u}(X))$ in $\Omega_{\widehat{u}}$.

Step 2: Equipartition of the energy. Define a modified Weiss function as

$$W_\Lambda(v):=\iint\limits_{B_1}|\nabla v|^2\,dX-\int_{\partial B_1}v^2\,d\mathcal{H}^n+\Lambda q(0)\,|\Omega_v\cap B_1|\,.$$

Let $\widehat{z}_r(X):=|X|\,\widehat{u}_r(X/|X|)$ be the 1-homogeneous extension of $\widehat{u}_r$ from ∂B_1 to B_1. Our goal in this step is to show that there exists $C>0$, uniform on (H*), such that it holds

$$W_\Lambda(\widehat{z}_r)-W_\Lambda(\widehat{u}_r)\geqslant\frac{1}{n}\int_{\partial B_1}\left|X\cdot\nabla\widehat{u}_r-\widehat{u}_r\right|^2\,d\mathcal{H}^n-Cr. \tag{6.41}$$

For this purpose, we follow [Ve23, Lemma 9.8], substantially extending it to introduce $A(X)$, $q(x)$ and $F(X,u)$. First, testing the stationarity of $\widehat{u}$ with respect to $\widehat{\mathcal{F}}_\Lambda$ (see Lemma 6.14) against the same vector field ξ_ε as in [Ve23], and letting $\varepsilon\to0$, we obtain

$$\begin{aligned}0=&-2\iint\limits_{B_r}\nabla\widehat{u}\,\widehat{A}\,\nabla\widehat{u}^{\mathrm{T}}\,dX+(n+1)\iint\limits_{B_r}\nabla\widehat{u}\,\widehat{A}\,\nabla\widehat{u}^{\mathrm{T}}\,dX+\iint\limits_{B_r}\nabla\widehat{u}\,(\nabla\widehat{A}\cdot X)\,\nabla\widehat{u}^{\mathrm{T}}\,dX\\&-2\iint\limits_{B_r}\nabla\widehat{F}(X,\widehat{u})\cdot X\,dX-2(n+1)\iint\limits_{B_r}\widehat{F}(X,\widehat{u})\,dX+\Lambda\iint\limits_{B_r}\mathbf{1}_{\Omega_{\widehat{u}}}\nabla\widehat{q}\cdot X\,dX\end{aligned} \tag{6.42}$$

$$+ (n+1)\,\Lambda \iint_{B_r} \mathbf{1}_{\Omega_{\widehat{u}}}\widehat{q}\,dX + 2r\int_{\partial B_r}(\nabla\widehat{u}\,X^{\mathrm{T}})(\nabla\widehat{u}\,\widehat{A}\,X^{\mathrm{T}})\,d\mathcal{H}^n - r\int_{\partial B_r}\nabla\widehat{u}\,\widehat{A}\,\nabla\widehat{u}^{\mathrm{T}}\,d\mathcal{H}^n$$
$$+ 2r\int_{\partial B_r}\widehat{F}(X,\widehat{u})\,dX - \Lambda\, r\int_{\partial B_r}\mathbf{1}_{\Omega_{\widehat{u}}}\widehat{q}\,d\mathcal{H}^n.$$

Also, the fact that $\operatorname{div}(\widehat{A}\nabla\widehat{u}) = \widehat{F}'(X,\widehat{u})$ in $\Omega_{\widehat{u}}$ (see Step 1) yields

$$\iint_{B_r}\nabla\widehat{u}\,\widehat{A}\,\nabla\widehat{u}^{\mathrm{T}}\,dX = \iint_{B_r}\operatorname{div}(\widehat{u}\,\widehat{A}\,\nabla\widehat{u})\,dX - \iint_{B_r}\widehat{u}\,\operatorname{div}(\widehat{A}\,\nabla\widehat{u})\,dX$$
$$= \int_{\partial B_r}\widehat{u}\,\nabla\widehat{u}\,\widehat{A}\,\frac{X^{\mathrm{T}}}{|X|}\,d\mathcal{H}^n + 2\iint_{B_r}\widehat{u}\,\widehat{F}'(X,\widehat{u})\,dX.$$

Inserting this computation into the first term of (6.42), then rescaling everything to B_1, we obtain

$$\begin{aligned}
0 &= 2\iint_{B_1}\nabla\widehat{u}_r\,\widehat{F}_r'(X,\widehat{u}_r)\,dX + (n+1)\iint_{B_1}\nabla\widehat{u}_r\,\widehat{A}_r\,\nabla\widehat{u}_r^{\mathrm{T}}\,dX + \iint_{B_1}\nabla\widehat{u}_r\,(\nabla\widehat{A}_r\cdot X)\,\nabla\widehat{u}_r^{\mathrm{T}}\,dX\\
&\quad - 2\iint_{B_1}\nabla\widehat{F}_r(X,\widehat{u}_r)\cdot X\,dX - 2(n+1)\iint_{B_1}\widehat{F}_r(X,\widehat{u}_r)\,dX + \Lambda\iint_{B_1}\mathbf{1}_{\Omega_{\widehat{u}_r}}\nabla\widehat{q}_r\cdot X\,dX\\
&\quad + (n+1)\Lambda\iint_{B_1}\mathbf{1}_{\Omega_{\widehat{u}_r}}\widehat{q}_r\,dX - 2\int_{\partial B_1}\widehat{u}_r\,\nabla\widehat{u}_r\,\widehat{A}_r\,X^{\mathrm{T}}\,d\mathcal{H}^n + 2\int_{\partial B_1}(\nabla\widehat{u}_r\,X^{\mathrm{T}})\,(\nabla\widehat{u}_r\,\widehat{A}_r\,X^{\mathrm{T}})\,d\mathcal{H}^n\\
&\quad - \int_{\partial B_1}\nabla\widehat{u}_r\,\widehat{A}_r\,\nabla\widehat{u}_r^{\mathrm{T}}\,d\mathcal{H}^n + 2\int_{\partial B_1}\widehat{F}_r(X,\widehat{u}_r)\,dX - \Lambda\int_{\partial B_1}\mathbf{1}_{\Omega_{\widehat{u}_r}}\widehat{q}_r\,d\mathcal{H}^n.
\end{aligned}\tag{6.43}$$

With this, we find

$$\begin{aligned}
W_\Lambda(\widehat{z}_r) - W_\Lambda(\widehat{u}_r) &= \iint_{B_1}\left|\nabla\widehat{z}_r\right|^2\,dX + \Lambda\,q(0)\left|\Omega_{\widehat{z}_r}\cap B_1\right| - \iint_{B_1}\left|\nabla\widehat{u}_r\right|^2\,dX - \Lambda\,q(0)\left|\Omega_{\widehat{u}_r}\cap B_1\right|\\
&= \frac{1}{n+1}\int_{\partial B_1}\left(\left|\nabla\widehat{u}_r\right|^2 - \left|\nabla\widehat{u}_r\cdot X\right|^2\right)\,d\mathcal{H}^n + \frac{1}{n+1}\int_{\partial B_1}\widehat{u}_r^2\,d\mathcal{H}^n\\
&\quad + \frac{\Lambda}{n+1}\,q(0)\,\mathcal{H}^n(\Omega_{\widehat{u}_r}\cap\partial B_1) - \iint_{B_1}\left|\nabla\widehat{u}_r\right|^2\,dX - \Lambda\,q(0)\left|\Omega_{\widehat{u}_r}\cap B_1\right|\\
&= \frac{1}{n+1}\int_{\partial B_1}\left|\nabla\widehat{u}_r\cdot X - \widehat{u}_r\right|^2\,d\mathcal{H}^n + \frac{1}{n+1}\int_{\partial B_1}\left(\left|\nabla\widehat{u}_r\right|^2 - 2\left|\nabla\widehat{u}_r\cdot X\right|^2\right)\,d\mathcal{H}^n\\
&\quad + \frac{2}{n+1}\int_{\partial B_1}\widehat{u}_r\nabla\widehat{u}_r\cdot X\,d\mathcal{H}^n + \frac{\Lambda}{n+1}\,q(0)\,\mathcal{H}^n(\Omega_{\widehat{u}_r}\cap\partial B_1)\\
&\quad - \iint_{B_1}\left|\nabla\widehat{u}_r\right|^2\,dX - \Lambda\,q(0)\left|\Omega_{\widehat{u}_r}\cap B_1\right|\\
&= \frac{1}{n+1}\int_{\partial B_1}\left|\nabla\widehat{u}_r\cdot X - \widehat{u}_r\right|^2\,d\mathcal{H}^n + \sum_{j=1}^{10}\mathrm{Error}_j.
\end{aligned}\tag{6.44}$$

Here we have used: in the first step, the fact that $\widehat{u}_r \equiv \widehat{z}_r$ on ∂B_1; in the second step, explicit expressions for the terms involving $\widehat{z}_r$ by integrations in polar coordinates (just as in [Ve23, (9.5)]); in the third step, we have simply inserted and removed $\frac{1}{n}\int_{\partial B_1} \left|\nabla \widehat{u}_r \cdot X - \widehat{u}_r\right|^2 d\mathcal{H}^n$. In the last step, we have introduced all the terms in (6.43) (divided by n), and rearranged everything into the following error terms:

$$\begin{aligned}
\text{Error}_1 &:= \frac{1}{n+1}\int_{\partial B_1} \left(\left|\nabla \widehat{u}_r\right|^2 - \nabla \widehat{u}_r\, \widehat{A}_r\, \nabla \widehat{u}_r^{\mathrm{T}}\right) d\mathcal{H}^n,\\
\text{Error}_2 &:= \frac{-2}{n+1}\int_{\partial B_1} \left(\left|\nabla \widehat{u}_r \cdot X\right|^2 - (\nabla \widehat{u}_r\, X^{\mathrm{T}})\,(\nabla \widehat{u}_r\, \widehat{A}_r\, X^{\mathrm{T}})\right) d\mathcal{H}^n,\\
\text{Error}_3 &:= \frac{2}{n+1}\int_{\partial B_1} \left(\widehat{u}_r \nabla \widehat{u}_r \cdot X - \widehat{u}_r\, \nabla \widehat{u}_r\, \widehat{A}_r\, X^{\mathrm{T}}\right) d\mathcal{H}^n,\\
\text{Error}_4 &:= \frac{\Lambda}{n+1}\left(q(0)\,\mathcal{H}^n(\Omega_{\widehat{u}_r} \cap B_1) - \int_{\partial B_1} \mathbf{1}_{\Omega_{\widehat{u}_r}}\, \widehat{q}_r\, d\mathcal{H}^n\right),\\
\text{Error}_5 &:= -\iint_{B_1} \left(\left|\nabla \widehat{u}_r\right|^2 - \nabla \widehat{u}_r\, \widehat{A}_r\, \nabla \widehat{u}_r^{\mathrm{T}}\right) dX, \quad \text{Error}_6 := -\Lambda\left(q(0)\left|\Omega_{\widehat{u}_r} \cap B_1\right| - \iint_{B_1} \mathbf{1}_{\Omega_{\widehat{u}_r}}\, \widehat{q}_r\, dX\right),\\
\text{Error}_7 &:= \frac{1}{n+1}\iint_{B_1} \nabla \widehat{u}_r\, (\nabla \widehat{A}_r \cdot X)\, \nabla \widehat{u}_r^{\mathrm{T}}\, dX, \qquad \text{Error}_8 := -\frac{2}{n+1}\iint_{B_1} \nabla \widehat{F}_r(X, \widehat{u}_r) \cdot X\, dX,\\
\text{Error}_9 &:= -2\iint_{B_1} \widehat{F}_r(X, \widehat{u}_r)\, dX, \qquad \text{Error}_{10} := \frac{\Lambda}{n+1}\iint_{B_1} \mathbf{1}_{\Omega_{\widehat{u}_r}}\, \nabla \widehat{q}_r \cdot X\, dX.
\end{aligned}$$

It is now easy to see that $\left|\text{Error}_j\right| = O(r)$ for every $1 \leqslant j \leqslant 10$, using the estimates in (6.29) and Lemma 6.25 (which clearly also hold for $\widehat{A}, \widehat{F}$ and $\widehat{q}$ because there has only been a linear change of variables by the matrix $\sqrt{A(0)}$, which is elliptic and bounded (HA3)). Moreover, $\widehat{u}_r$ are uniformly bounded in $H^1(B_1)$ and L^∞ by (6.29). More precisely,

$$\begin{aligned}
&|\text{Error}_1|,\ |\text{Error}_2|,\ |\text{Error}_3|,\ |\text{Error}_5| \lesssim \|I - \widehat{A}_r\|_\infty\, \mathfrak{L}^2 = O(r),\\
&|\text{Error}_4|,\ |\text{Error}_6| \lesssim \left\|q(0) - \widehat{q}_r\right\|_\infty = O(r),\\
&|\text{Error}_7| \lesssim r\,\|\nabla \widehat{A}\|_\infty\, \mathfrak{L}^2 = O(r), \qquad |\text{Error}_8| \lesssim r\,\|\nabla \widehat{F}\|_\infty = O(r),\\
&|\text{Error}_9| \lesssim \iint_{B_1} u(rX)\, dX = O(r), \qquad |\text{Error}_{10}| \lesssim r\left\|\nabla \widehat{q}\right\|_\infty = O(r).
\end{aligned}$$

Inserting all these estimates back in (6.44) yields (6.41), as claimed.

Step 3: Blow-up procedure. Take any sequence u_{r_j} (with $r_j \to 0$) converging to a blow-up limit $v := u_0$ as in Lemma 6.26. This convergence takes place in $H^1(B_1)$, where we are assuming that $R > 1$ in Lemma 6.26 for the ease of notation.

It is clear that if we define $\widehat{u}(X) := u(X\sqrt{A(0)})$ and denote its rescalings $\widehat{u}_r$, and if we take $\widehat{v}(X) := u_0(X\sqrt{A(0)}) = \widehat{u}_0(X)$ as in Step 1, the same convergences take place (because we have only made a linear change of variables).

We claim that

$$\ell := \lim_{r \to 0} W_\Lambda(\widehat{u}_r) \in \mathbb{R} \tag{6.45}$$

exists. Indeed, $W_\Lambda(\widehat{u}_r) \geqslant -\int_{\partial B_1} \widehat{u}_r(X)^2\, d\mathcal{H}^n \gtrsim -L^2$ by (6.29). Here, note that [Ve23, Lemmas 9.1 and 9.2] show that $W_\Lambda(\widehat{u}_r)$ is continuous and a.e. differentiable. Moreover, [Ve23, Lemma 9.2] also gives an expression for its derivative, in terms of $\widehat{z}_r$ from Step 2, namely

$$\frac{\partial}{\partial r} W_\Lambda(\widehat{u}_r) = \frac{n+1}{r}\Big(W_\Lambda(\widehat{z}_r) - W_\Lambda(\widehat{u}_r)\Big) + \frac{1}{r}\int_{\partial B_1} \left|X\cdot\nabla\widehat{u}_r - \widehat{u}_r\right|^2 d\mathcal{H}^n, \tag{6.46}$$

so that using (6.41) we obtain

$$\frac{\partial}{\partial r} W_\Lambda(\widehat{u}_r) \geqslant -C.$$

It is elementary that a continuous and a.e. differentiable function which is bounded below in $(0,1)$ and whose derivative is also bounded below in $(0,1)$, like $W_\Lambda(\widehat{u}_r)$, indeed has a finite limit at 0. (Roughly speaking, if the limit did not exist, the function should oscillate "infinitely rapidly", which is prevented the lower bound on the derivative.) Thus we obtain (6.45), as claimed.

From the existence of the limit of $W_\Lambda(\widehat{u}_r)$ at $r = 0$, the 1-homogeneity of blow-up limits is standard (see e.g. [Ve23, Lemma 9.10]). Indeed, we have the rescaling property

$$\lim_{j\to\infty} W_\Lambda(\widehat{u}_{sr_j}) = W_\Lambda(\widehat{v}_s), \qquad \text{for every } s\in(0,1) \tag{6.47}$$

because, as $\widehat{u}_{r_j} \to \widehat{v}$ in $H^1(B_1)$,

$$\iint_{B_1} \left|\nabla\widehat{u}_{sr_j}(X)\right|^2 dX = \iint_{B_1} \left|\nabla\left(\frac{\widehat{u}_{r_j}(sX)}{s}\right)\right|^2 dX \underset{j\to\infty}{\longrightarrow} \iint_{B_1} \left|\nabla\left(\frac{\widehat{v}(sX)}{s}\right)\right|^2 dX = \iint_{B_1} \left|\nabla\widehat{v}_s\right|^2 dX.$$

The terms leading to $\int_{\partial B_1} \widehat{v}_s^2\, d\mathcal{H}^n$ and $\left|\Omega_{\widehat{v}_s} \cap B_1\right|$ can be handled similarly, using the different modes of convergence in Lemma 6.26.

Then, (6.45) and (6.47) yield

$$\frac{\partial}{\partial s} W_\Lambda(\widehat{v}_s) = 0, \qquad s\in(0,1).$$

Inserting this back in our formula (6.46) for the derivative of W_Λ, and noting that, as discussed in Step 1, $\widehat{v}$ is a minimizer of $\widehat{\mathcal{F}}_{\Lambda,0}$ (which coincides with W_Λ up to the term supported on ∂B_1), we infer that

$$\int_{\partial B_1} \left|X\cdot\nabla\widehat{v}_s - \widehat{v}_s\right|^2 d\mathcal{H}^n = 0, \qquad s\in(0,1).$$

It is well known that this implies the 1-homogeneity of $\widehat{v}$ (see e.g. [Ve23, Lemma 9.3]), which clearly implies the 1-homogeneity of $v := u_0$. □

6.9.2 Overdetermined boundary condition

To establish the Neumann boundary condition on $\partial\Omega_u$, we need to work a bit more with blow-ups. First, we prove an easy auxiliary lemma, which again shows F essentially disappears after the blow-up.

Lemma 6.29 (PDE for blow-up limits)**.** *Assume* (H*). *Let u_0 be a blow-up limit for u at $0\in\partial\Omega_u$. Then Ω_{u_0} is open and $-\operatorname{div}(A_0\nabla u_0) = 0$ in Ω_{u_0}.*

Proof. Since u_{r_j} are Lipschitz with constant L (Proposition 6.19), by Arzelà-Ascoli and the uniform convergence in Lemma 6.26, u_0 is Lipschitz, too. Concretely, continuity implies that Ω_{u_0} is open. In turn, the PDE follows as in Corollary 6.20 from the fact that u_0 is a local minimizer of $\mathcal{F}_{\Lambda,0}$ in B_1 (see Corollary 6.27). □

Next, we prove a couple of auxiliary lemmas that will help us classify the blow-up limits.

Lemma 6.30 (Non-degeneracy of blow-up limits)**.** *Assume* (H*). *Let u_0 be a blow-up limit for u at $0 \in \partial\Omega_u$. Then $u_0 \not\equiv 0$.*

Proof. Since $0 \in \partial\Omega_u$, the non-degeneracy property (Proposition 6.22) implies, for r small enough, say $0 < r < r_0$, that $\fint_{\partial B_r} u \, d\mathcal{H}^n \geqslant \kappa_0 r$. It is easy to see, using a change of variables, that this implies (if r_j is small enough) that $\fint_{\partial B_r} u_{r_j} \, d\mathcal{H}^n \geqslant \kappa_0 r$ for $r \in (0,1)$. (Note that the constant does not change after the rescaling). Using also (6.29), we can estimate

$$\mathfrak{L}\,|\Omega_{u_{r_j}} \cap B_1| \geqslant \iint_{B_1} u_{r_j}\, dX = \int_0^1 \int_{\partial B_r} u_{r_j}\, d\mathcal{H}^n\, dr \gtrsim \int_0^1 \kappa_0 r^{n+1}\, dr \gtrsim \kappa_0,$$

so $|\Omega_{u_{r_j}} \cap B_1| \gtrsim 1$ uniformly on j. By the convergence of indicator functions in Lemma 6.26, we obtain $\left|\Omega_{u_0} \cap B_1\right| \gtrsim 1$. The result follows. □

Lemma 6.31 (Upper densities)**.** *Assume* (H*). *Then* $\lim_{r\to 0} \dfrac{\left|\Omega_u \cap B_r(x_0)\right|}{|B_r|} < 1$ *for any* $x_0 \in \partial\Omega_u$.

Proof. Using the function h from Lemma 6.17, which is L-superharmonic (because $F' \geqslant 0$, see (HF3)), we can compute, for small r,

$$h(X) \gtrsim \fint_{\partial B_r(x_0)} h\, d\mathcal{H}^n \geqslant \kappa_0 r, \qquad X \in B_{r/2}(x_0).$$

Here we have used Lemma 6.6 (applied to the function $-h(X) + \|u\|_\infty$, which is non-negative and L-subharmonic) and Proposition 6.22. This estimate and (6.24) allow us to follow the proof of [Ve23, Lemma 5.1]. Indeed, since by (6.29) it holds $u \leqslant L\varepsilon r$ in $B_{\varepsilon r}(x_0)$, we can compute (using ε small enough so that $\varepsilon L \ll \kappa_0$)

$$\begin{aligned} |B_{\varepsilon r}| &\lesssim \frac{1}{r} \iint_{B_{\varepsilon r}(x_0)} (h-u)\, dX \leqslant \frac{1}{r} \iint_{B_r(x_0)} (h-u)\, dX \\ &\lesssim |B_r|^{1/2} \left(\iint_{B_r(x_0)} \left|\nabla(h-u)\right|^2 dX \right)^{1/2} \lesssim |B_r|^{1/2} \left(\lambda^{-1}\overline{q}\,\Lambda \left|\{u=0\} \cap B_r(x_0)\right| \right)^{1/2}, \end{aligned}$$

where we have used Poincaré's inequality and Lemma 6.17. The result then follows. □

We are ready to finish, obtaining the overdetermined condition as a consequence of a classification of the possible blow-up limits. In general, it will only hold in the viscosity sense:

Definition 6.32 (Viscosity sense)**.** Given $B : \mathbb{R}^{n+1} \to \mathbb{R}$, we say that $\nabla u(x)\, A(x)\, \nabla u(x)^{\mathrm{T}} \leqslant B(x)$ in the viscosity sense at $x_0 \in \partial\Omega_u$ if whenever $\varphi \in C^\infty(\mathbb{R}^{n+1})$ touches u from below at x_0 (i.e. $\varphi(x_0) = u(x_0)$ and $\varphi \leqslant u$ in a neighborhood of x_0), we have $\nabla\varphi(x_0)\, A(x_0)\, \nabla\varphi(x_0)^{\mathrm{T}} \leqslant B(x_0)$. In turn, $\nabla u(x)\, A(x)\, \nabla u(x)^{\mathrm{T}} \geqslant B(x)$ is defined similarly. Thus, one says $\nabla u(x)\, A(x)\, \nabla u(x)^{\mathrm{T}} = B(x)$ in the *viscosity sense* at a point $x_0 \in \partial\Omega_u$ when both conditions hold.

Proposition 6.33 (Overdetermined boundary condition)**.** *Assume* (H*)*. Then*

$$\nabla u(x)\, A(x)\, \nabla u(x)^{\mathrm{T}} = \Lambda\, q(x)$$

in the viscosity sense for all $x \in \partial\Omega_u$.

Proof. With the auxiliary results we have established, the proof parallels [Ve23, Prop. 9.18]. Let us see the differences. To classify blow-up limits in our context, it is more convenient to run the proof in different coordinates, so that in the limit we encounter the Laplacian (instead of the constant coefficient PDE in Lemma 6.29).

Step 1: Upper bound. Assume that $0 \in \partial\Omega_u$ (as discussed along this section, we can do this without loss of generality). As in the proof of Lemma 6.28, define

$$\widehat{u}(X) := u(X\sqrt{A(0)}), \qquad \widehat{u}_0(X) := u_0(X\sqrt{A(0)}),$$

so that the rescalings $\widehat{u}_{r_j}$ converge to $\widehat{u}_0$. It is easy to see that $-\Delta\widehat{u}_0 = 0$ in $\Omega_{\widehat{u}_0}$ (changing variables in Lemma 6.29, or directly because $\widehat{u}_0$ is a local minimizer of $\widehat{\mathcal{F}}_{\Lambda,0}$, see Step 1 of Lemma 6.28).

Let $\varphi \in C^\infty(\mathbb{R}^{n+1})$ touch u from below at 0. Then, $\widehat{\varphi}(X) := \varphi(X\sqrt{A(0)})$ touches $\widehat{u}$ from below at 0. If we rescale $\widehat{\varphi}_{r_j}(X) := r_j^{-1}\widehat{\varphi}(r_j X)$, it is clear that $\widehat{\varphi}_{r_j} \longrightarrow \widehat{\varphi}_0$ uniformly in B_1, as $j \to \infty$, perhaps up to a subsequence, because $\widehat{\varphi}$ is smooth. The smoothness of $\widehat{\varphi}$ implies that $\widehat{\varphi}_0(X) = \nabla\widehat{\varphi}(0) \cdot X$. Without loss of generality, using a rotation we may write $\widehat{\varphi}_0(X) = B\, X_{n+1}$ for $B := \big|\nabla\widehat{\varphi}(0)\big|$. Moreover, we can assume that $B > 0$, for otherwise $\big|\nabla\varphi(0)\big| \leqslant \sqrt{\Lambda}$ is trivially satisfied. Since $\widehat{u}_0 \geqslant \widehat{\varphi}_0$ because $\widehat{\varphi}$ touches $\widehat{u}$ from below, this implies that $\widehat{u}_0 > 0$ in $\{X_{n+1} > 0\}$.

Thus $\widehat{u}_0$ is a 1-homogeneous (see Lemma 6.28) harmonic function in $\{X_{n+1} > 0\} \subset \{\widehat{u}_0 > 0\}$. Therefore, by [Ve23, Lemma 9.16],

$$\text{either} \quad \widehat{u}_0(X) = \alpha X_{n+1}^+, \quad \text{or} \quad \widehat{u}_0(X) = \alpha X_{n+1}^+ + \beta X_{n+1}^-, \qquad \text{for some } \alpha, \beta > 0.$$

But the latter possibility contradicts Lemma 6.31, so $\widehat{u}_0(X) = \alpha X_{n+1}^+$ for some $\alpha > 0$. Hence testing the stationarity of $\widehat{u}_0$ for $\widehat{F}_{\Lambda,0}$ (see Step 1 in Lemma 6.28) against $\xi(X) := (0, \dots, 0, \xi_{n+1}(X))$, we find:

$$0 = \delta\widehat{\mathcal{F}}_{\Lambda,0}(\widehat{u}_0)[\xi] = \iint_{\mathbb{R}^{n+1}_+} \left(-2\alpha^2 \partial_{x_{n+1}}\xi_{n+1} + \alpha^2 \operatorname{div}\xi + \Lambda q(0) \operatorname{div}\xi\right) dX$$

$$= \iint_{\mathbb{R}^{n+1}_+} (-\alpha^2 + \Lambda q(0)) \partial_{x_{n+1}}\xi_{n+1}\, dX.$$

Since ξ is arbitrary, this yields $\alpha = \sqrt{\Lambda q(0)}$.

Thus we have shown that

$$\widehat{u}_0(X) = \sqrt{\Lambda q(0)}\left(\frac{\nabla\widehat{\varphi}(0)}{\left|\nabla\widehat{\varphi}(0)\right|}\cdot X\right)_+,$$

which implies that

$$u_0(X) = \frac{\sqrt{\Lambda q(0)}}{(\nabla\varphi(0)\,A(0)\,\nabla\varphi(0)^{\mathrm{T}})^{1/2}}\,(\nabla\varphi(0)\cdot X)_+.$$

Note that since $u_0 \geqslant \varphi_0$ and $u_0(0) = 0 = \varphi_0(0)$, we have

$$\left|\nabla\varphi(0)\right| = \left|\nabla\varphi_0(0)\right| \leqslant \left|\nabla u_0(0)\right| = \sqrt{\Lambda q(0)}\frac{\left|\nabla\varphi(0)\right|}{(\nabla\varphi(0)\,A(0)\,\nabla\varphi(0)^{\mathrm{T}})^{1/2}},$$

which finally gives $\nabla\varphi(0)\,A(0)\,\nabla\varphi(0)^{\mathrm{T}} \leqslant \Lambda q(0)$.

Step 2: Lower bound. Similarly, let $\varphi \in C^\infty(\mathbb{R}^{n+1})$ touch u from above at $0 \in \partial\Omega_u$, and change coordinates again so that $\widehat{\varphi}$ touches $\widehat{u}$ from above at 0. Doing again the blow-up procedure, we obtain blow-up limits $\widehat{u}_0$ and $\widehat{\varphi}_0$. By Lemma 6.30, $\widehat{u}_0 \not\equiv 0$, whence $B := \left|\nabla\widehat{\varphi}(0)\right| > 0$ because $\widehat{\varphi}$ touches $\widehat{u}$ from above. Therefore, since $\widehat{u}_0 \leqslant \widehat{\varphi}_0$, it holds $\{\widehat{u}_0 > 0\} \subset \{\widehat{\varphi}_0 > 0\} \subset \{X \in \mathbb{R}^{n+1} : \nabla\varphi(0)\cdot X > 0\}$, where we note that the last set is simply a (possibly rotated) half-space. Thus, since $\widehat{u}_0$ is 1-homogeneous and harmonic, [Ve23, Lemma 9.15] implies that $\widehat{u}_0(X) = \alpha\left(\frac{\nabla\widehat{\varphi}(0)}{\left|\nabla\widehat{\varphi}(0)\right|}\cdot X\right)_+$ for some $\alpha > 0$. Arguing as in Step 1, we obtain $\alpha = \sqrt{\Lambda q(0)}$, and later $\nabla\varphi(0)\,A(0)\,\nabla\varphi(0)^{\mathrm{T}} \geqslant \Lambda\, q(0)$. □

6.9.3 Decomposition and properties of the free boundary

As is customary in elliptic free boundary problems, the boundary $\partial\Omega_u$ will consist of a part which is regular, and a singular part that is small in some sense (in our setting, in the sense of Hausdorff dimension).

To make this precise, we will only need minor modifications to the classical approach, as presented in [Ve23, Sections 6–10]. This is because the improvement of flatness for rather general PDE (the essential step when showing regularity of the free boundary) was already developed by De Silva, Ferrari and Salsa in [DFS], extending the strategy of De Silva in [De11].

We start by the definition of the regular and singular parts of the boundary. We have to make a modification: as already seen in Proposition 6.33, we expect the blow-ups limits after the change of variables to have the shape $\widehat{u}_0(X) = (X\cdot\widehat{\nu})_+$ for some unit vector $\widehat{\nu} \in \partial B_1$. Therefore, before the change of variables, these blow-up limits should be $u_0(X) = \widehat{u}_0(X\sqrt{A(0)}^{-1}) = (X\cdot\sqrt{A(0)}^{-1}\widehat{\nu})_+ =: (X\cdot\nu)_+$, where now $|\nu| \in [\lambda^{1/2}, \lambda^{-1/2}]$ by (HA3). Since the distortion is always uniformly controlled (depending on the constant λ), the subsequent arguments will work without major modifications.

Definition 6.34 (Regular and singular parts of the boundary)**.** We define the *regular part* of the boundary, $\mathrm{Reg}(\partial\Omega_u)$, as the set of points $x_0 \in \partial\Omega_u$ such that there exists some blow-up limit u_0 at x_0 taking the form

$$u_0(X) = h_{\nu,x_0}(X) := \sqrt{\Lambda q(x_0)}\,((X - x_0)\cdot\nu)_+$$

for some vector $\nu \in \mathbb{R}^{n+1}$ with $|\nu| \in [\lambda^{1/2}, \lambda^{-1/2}]$. The *singular part* of the boundary is $\mathrm{Sing}(\partial\Omega_u) := \partial\Omega_u\backslash\mathrm{Reg}(\partial\Omega_u)$.

The smoothness of the regular part is proven using robust techniques (see e.g. [Ve23, Corollary 8.2]) that need essentially no modifications in our setting:

Proposition 6.35 (Regularity of $\mathrm{Reg}(\partial\Omega_u)$)**.** *Assume (H*). Then, there exists $r_0 > 0$ so that, for every $x_0 \in \mathrm{Reg}(\partial\Omega_u)$, $\partial\Omega_u \cap B_{r_0}(x_0)$ is a $C^{1,\alpha}$ regular manifold, for some $\alpha > 0$. Furthermore, if F, A and q are smooth, so is $\partial\Omega_u$ in a (possibly smaller) ball around x_0.*

Proof. Up to a translation and dilation (with constants depending on Λ and $\underline{q}, \overline{q}$), we may assume $x_0 = 0$ and $u_0(X) = (X \cdot \nu)_+$ for some $\nu \in B_{\lambda^{-1/2}} \setminus B_{\lambda^{1/2}}$. Then, given $\delta > 0$, by Lemma 6.26, it holds $\|u_{r_j} - u_0\|_{L^\infty(B_1)} < \delta$ for j large enough, from where it easily follows that $\{u_{r_j} = 0\} \cap B_1 \subset \{X \cdot \nu \leqslant \delta\} \cap B_1$.

To obtain, $\{X \cdot \nu \leqslant -\delta\} \cap B_1 \subset \{u_{r_j} = 0\} \cap B_1$, we first use [Ve23, Lemma 6.5] to obtain that $\overline{\Omega_{u_{r_j}}} \to \overline{\Omega_{u_0}} = \{X \cdot \nu \geqslant 0\}$ locally Hausdorff in B_2. This can be applied here because of the convergence in Lemma 6.26, the Lipschitz continuity in Proposition 6.19, and the non-degeneracy in Proposition 6.22 and Lemma 6.30. Then, pick $X \in \{X \cdot \nu \leqslant -\delta\} \cap B_1$, which implies that $\mathrm{dist}(X, \overline{\Omega_{u_0}}) \geqslant \delta|\nu| \geqslant \delta\lambda^{1/2}$. Combining this with the fact that, for j large enough, $\mathrm{dist}(\overline{\Omega_{u_{r_j}}}, \overline{\Omega_{u_0}}) \leqslant \delta\lambda^{1/2}/2$ by the Hausdorff convergence, we trivially have $\mathrm{dist}(X, \overline{\Omega_{u_{r_j}}}) \geqslant \delta\lambda^{1/2}/2$, whence $u_{r_j}(X) = 0$.

Overall, we have obtained that, for any $\delta > 0$, if we consider $j \gg 1$,

$$\{X \cdot \nu \leqslant -\delta\} \cap B_1 \ \subset\ \{u_{r_j} = 0\} \cap B_1 \ \subset\ \{X \cdot \nu \leqslant \delta\} \cap B_1.$$

A simple rescaling yields

$$\{X \cdot \nu \leqslant -\delta r_j\} \cap B_{r_j} \ \subset\ \{u = 0\} \cap B_{r_j} \ \subset\ \{X \cdot \nu \leqslant \delta r_j\} \cap B_{r_j}.$$

Hence, denoting $\widehat{\nu} := \nu/|\nu| \in \partial B_1$, and recalling that $|\nu| \in [\lambda^{1/2}, \lambda^{-1/2}]$, this yields

$$\{X \cdot \widehat{\nu} \leqslant -\delta\lambda^{-1/2} r_j\} \cap B_{r_j} \ \subset\ \{u = 0\} \cap B_{r_j} \ \subset\ \{X \cdot \widehat{\nu} \leqslant \delta\lambda^{-1/2} r_j\} \cap B_{r_j}.$$

Since $Lu = F'(X, u(X)) \in L^\infty(\Omega_u \cap B_1)$ (because it is controlled by M_1) and we have also shown the overdetermined boundary condition $|\nabla u \sqrt{A}|^2 = \Lambda q$ in a viscosity sense in Lemma 6.33, applying [DFS, Th. 1.4], we obtain the desired $C^{1,\alpha}$ regularity in $B_{r_j/2}$ for $j \gg 1$.

More precisely, one should divide by $\Lambda q(x)$ to obtain the equations

$$\begin{cases} -\operatorname{div}\left(\dfrac{A(X)}{\Lambda q(X)} \nabla u(X)\right) = \dfrac{F'(X, u(X))}{\Lambda q(X)} - \dfrac{\nabla u(X) A(X) \nabla q(X)^{\mathrm{T}}}{\Lambda q(X)^2} & \text{in } \Omega_u \cap B_1, \\ \nabla u(x) \dfrac{A(x)}{\Lambda q(x)} \nabla u(x)^{\mathrm{T}} = 1 & \text{on } \partial\Omega_u \cap B_1, \end{cases}$$

where the right-hand-side of the first equation is uniformly bounded by (HF3), (6.15), (Hq1), (Hq2), (HA3) and Proposition 6.19; and where the coefficients $A(X)/(\Lambda q(X))$ are Lipschitz, symmetric, elliptic and bounded because of (HA1), (HA2), (HA3), (Hq1) and (Hq2). [DFS, Theorem 1.4] applies directly to this problem.

Lastly, let us derive higher regularity of $\mathrm{Reg}(\partial\Omega_u)$ whenever F, A and q are smooth. We would like to simply invoke the classical Kinderlehrer–Nirenberg self-improvement result [KN77, Theorem 2], but

that requires $u \in C^2(\overline{\Omega_u} \cap B_r)$ for some small r, and we have not yet established that. Therefore, we prefer to follow another approach.

Start by using elliptic regularity to derive that $u \in C^{1,\alpha}(\overline{\Omega_u} \cap B_{r_0/2})$ (see e.g. [GT77, Corollary 8.36]). This allows us to follow the first part of the proof of [LZ23, Theorem 1.31] to show that $u \in C^{2,\tilde{\alpha}}(\overline{\Omega_u} \cap B_r)$ and $\partial\Omega_u \cap B_r \in C^{2,\tilde{\alpha}}$ for some $\tilde{\alpha} > 0$, if $r > 0$ is small enough. Indeed, adapting the proof of [LZ23, Theorem 1.31] to our setting is possible because it is essentially a repeated application of [LZ23, Theorem 1.24], which is very versatile and adapts to our PDE, as explained in [LZ23, Remark 1.30]. Note also that the oblique derivative condition over the boundary in [LZ23, Theorem 1.24] is just our overdetermined boundary condition.

Now that $\partial\Omega_u \cap B_r$ is $C^{2,\tilde{\alpha}}$, classical elliptic regularity yields $u \in C^{2,\tilde{\alpha}}(\overline{\Omega_u} \cap B_{r/2})$ (see e.g. [GT77, Theorem 6.19]). And then we can invoke [KN77, Theorem 2] to infer that $\partial\Omega_u$ is actually smooth in a neighborhood of the origin, hence finishing the proof. (Alternatively, one could have continued adapting the proof in [LZ23, Theorem 1.31] by iterating [LZ23, Theorem 1.29], which is again possible in our setting thanks to [LZ23, Remark 1.30]). □

We are now ready to derive the classical bounds for the Hausdorff dimension of the singular set of the free boundary:

Proposition 6.36 (Dimension of the singular set)**.** *Assume* (H*). *Then there exists* $n^* \in \{5, 6, 7\}$ *such that*

- *if* $n + 1 < n^*$, *then* $\mathrm{Sing}(\partial\Omega_u)$ *is empty,*
- *if* $n + 1 = n^*$, *then* $\mathrm{Sing}(\partial\Omega_u)$ *is a locally finite set,*
- *if* $n + 1 > n^*$, *then* $\dim_{\mathcal{H}}(\mathrm{Sing}(\partial\Omega_u)) \leqslant (n + 1) - n^*$.

Proof. The result will follows from the abstract result in [Ve23, Prop. 10.13] after we verify that the hypotheses of that result hold in our setting.

The non-degeneracy was shown in Proposition 6.22. Then, we proved the convergence of blow-ups in Lemma 6.26. Moreover, the homogeneity and minimality of blow-up limits (with respect to $\mathcal{F}_{\Lambda,0}$, exactly the same energy functional as in [Ve23] because F terms disappeared in the limit) was obtained in Lemmas 6.26 and 6.28. Lastly, we have verified the uniform ε-regularity property in the proof of Proposition 6.35.

The definition of the uniform ε-regularity property must be modified slightly with respect to that in [Ve23]: the shape of our blow-up limits is given by $h_{\nu,x_0}(X) = \sqrt{\Lambda q(x_0)}((X-x_0)\cdot\nu)_+$ for $\nu \in B_{\lambda^{1/2},\lambda^{-1/2}}$; whereas in [Ve23], the vector should lie exactly on ∂B_1, and $q(x_0)$ is not present. In any case, since these modifications are uniformly controlled (with constants depending on λ, q and $\overline{q}$), the program in [Ve23, Prop. 10.13] carries over to the present case almost verbatim. Indeed, our new definition of uniform ε-regularity is sufficient to run the proof of [Ve23, Lemma 10.7], which is the main ingredient that needs to be adapted in [Ve23, Prop. 10.13] to work in our setting.

Also, the fact that [Ve23, Prop. 10.13] is only stated for bounded sets is not a problem, since we could just truncate to B_j and then consider the union with $j \in \mathbb{N}$. Recall that we showed that Ω_u is bounded in Proposition 6.24. □

6.10 Proof of Theorem 6.2, the main result

We are now ready to prove our main result, Theorem 6.2. Fix $m > 0$ and admissible $F(X,u), A(X)$ and $q(x)$, satisfying the periodicity hypothesis (HPer) (see Definition 6.3).

Let us fix some $R \gg 1$ to be determined later. Now take a cutoff $\varphi_R \in C_c^\infty(\mathbb{R}^{n+1})$ satisfying

$$0 \leqslant \varphi_R \leqslant 1 \text{ in } \mathbb{R}^{n+1}, \quad \varphi_R \equiv 1 \text{ in } B_R, \quad \varphi_R \equiv 0 \text{ in } \mathbb{R}^{n+1}\backslash B_{2R}.$$

We then define the smoothly truncated energy

$$\mathcal{F}_{0,R}(v,D) := \iint_D |\nabla v|^2\,dX - 2\iint_D F_R(X,v)\,dX, \quad D \subset \mathbb{R}^{n+1},$$

with $F_R(X,v) := F(X,v)\,\varphi_R(X)$. As usual, we will omit D when $D = \mathbb{R}^{n+1}$. Since φ_R only depends on X, is bounded between 0 and 1, and is smooth, it is clear that F_R is admissible[3], and in fact fulfills (HF4) and (HF5) with the same constants as F. In what follows, we shall make sure that no constant depends on R, but only on the constants appearing in the admissibility of F (equivalently, F_R), A and q, and of course m and n.

Step 1. Existence of minimizers of $F_{0,R}$ in $\mathbb{K}_{\leqslant m}$. First let us note that Faber-Krahn's inequality provides a Poincaré inequality in $\mathbb{K}_{\leqslant m}$. Indeed, given any $u \in \mathbb{K}_{\leqslant m}$, the symmetrization u^* of u satisfies $u^* \in H_0^1(B^{\tilde{m}})$ (see (6.28)), so that we obtain

$$\iint_{\mathbb{R}^{n+1}} u^2\,dX = \iint_{B^{\tilde{m}}} (u^*)^2\,dX \leqslant \lambda_1(B^{\tilde{m}})^{-1} \iint_{B^{\tilde{m}}} \left|\nabla u^*\right|^2\,dX \leqslant \lambda_1(B^{\tilde{m}})^{-1} \iint_{\mathbb{R}^{n+1}} |\nabla u|^2\,dX. \tag{6.48}$$

Having obtained this Poincaré inequality, the existence of minimizers of $\mathcal{F}_{0,R}$ in $\mathbb{K}_{\leqslant m}$ follows from standard arguments. First, the fact that $u \in \mathbb{K}_{\leqslant m}$ (and (6.28)), (HF2), (HF4), (HA3), and (6.48) imply

$$\begin{aligned}\mathcal{F}_{0,R}(u) &\geqslant \lambda \iint_{\mathbb{R}^{n+1}} |\nabla u|^2\,dX - 2N\left|\{u>0\}\right| - 2b\iint_{\mathbb{R}^{n+1}} u^2\,dX \\ &\geqslant \left(\lambda - \frac{2b}{\lambda_1(B^m)}\right)\iint_{\mathbb{R}^{n+1}} |\nabla u|^2\,dX - 2N\frac{m}{\underline{q}},\end{aligned} \tag{6.49}$$

so that the smallness of b in (HF4) gives $\mathcal{F}_{0,R}(u) \geqslant -2Nm/\underline{q}$ for every $u \in \mathbb{K}_{\leqslant m}$. That is, $\mathcal{F}_{0,R}$ is bounded below uniformly, with constants independent of R.

This allows us to take a minimizing sequence $u_j \in \mathbb{K}_{\leqslant m}$ (i.e. $\mathcal{F}_{0,R}(u_j) \longrightarrow \inf_{\mathbb{K}_{\leqslant m}} \mathcal{F}_{0,R}$ as $j \to +\infty$). Concretely, testing with the constant zero function, we obtain that $\mathcal{F}_{0,R}(u_j) \leqslant 1$ for every j (up to taking a subsequence). This and (6.49) ensure that as long as $b < \lambda\lambda_1(B^m)/2$ we have the uniform bound

$$\iint_{\mathbb{R}^{n+1}} \left|\nabla u_j\right|^2\,dX \leqslant \left(1 + 2N\frac{m}{\underline{q}}\right)\left(\lambda - \frac{2b}{\lambda_1(B^m)}\right)^{-1}. \tag{6.50}$$

[3]To verify (HF6) with F_R, note the following: since (HF6) holds for F, by density, there exists $u_0 \in C_c^\infty(\mathbb{R}^{n+1})$ with $\mathcal{F}_0(u_0) < 0$. Then, taking R large enough so that $\operatorname{supp} u_0 \subset B_{R/10}$, (HF6) is satisfied with F_R.

Thus, by the Poincaré inequality in (6.48), u_j are also bounded in $L^2(\mathbb{R}^{n+1})$, with uniform constants. Therefore, by the Banach–Alaoglu compactness theorem, there exists $u_0 \in H^1(\mathbb{R}^{n+1})$ such that $u_j \rightharpoonup u_0$ in $H^1(\mathbb{R}^{n+1})$ (up to a subsequence). Therefore, by the Rellich–Kondrachov theorem in the bounded set B_{2R}, we get (up to a subsequence) $u_j \to u_0$ in $L^2(B_{2R})$, so (up to a further subsequence) we can also get pointwise a.e. convergence $u_j \to u_0$ in B_{2R}. This convergence, along with the smoothness of F_R (recall (HF1)) and the fact that $\operatorname{supp} F_R \subset B_{2R}$, lower semi-continuity of weak convergence (recall (HA1)) and the Dominated Convergence Theorem, imply

$$\begin{aligned}\mathcal{F}_{0,R}(u_0) &= \iint_{\mathbb{R}^{n+1}} |\nabla u_0|^2 \, dX - 2 \iint_{B_{2R}} F_R(X, u_0)\, dX \\ &\leqslant \liminf_{j\to+\infty} \iint_{\mathbb{R}^{n+1}} \left|\nabla u_j\right|^2 dX - \lim_{j\to+\infty} 2 \iint_{B_{2R}} F_R(X, u_j)\, dX = \liminf_{j\to+\infty} \mathcal{F}_{0,R}(u_j), \quad (6.51)\end{aligned}$$

whence u_0 minimizes $\mathcal{F}_{0,R}$. Moreover, by Mazur's theorem on weak convergences and convexity, $u_0 \geqslant 0$ in $\mathbb{R}^{n+1}$.

Lastly, to see that $\operatorname{Vol}_q(\Omega_{u_0}) \leqslant m$, it suffices (by the Monotone Convergence Theorem) to check that $\operatorname{Vol}_q(\Omega_{u_0} \cap B_r) \leqslant m$ for every $r > 0$. Given one such $r > 0$, using standard compactness results as above we obtain that $u_j \to u_0$ a.e. in B_r (up to a subsequence). This clearly implies that $\mathbf{1}_{\{u_0>0\}}(X) \leqslant \liminf_{j\to+\infty} \mathbf{1}_{\{u_j>0\}}(X)$ a.e. in B_r. Integrating, this yields $\operatorname{Vol}_q(\Omega_{u_0} \cap B_r) \leqslant m$ because $u_j \in \mathbb{K}_{\leqslant m}$. This finishes the proof that $u_0 \in \mathbb{K}_{\leqslant m}$, so u_0 is a minimizer of $\mathcal{F}_{0,R}$ in $\mathbb{K}_{\leqslant m}$.

Step 2. Properties of the minimizer of $\mathcal{F}_{0,R}$. The fact that u_0 has the properties listed in the conclusions of Theorem 6.2 is a consequence of all the previous sections, because (H*) is because (using u_0, F_R and $\mathcal{F}_{0,R}$ in place of u, F and $\mathcal{F}_0$) with constants that do not depend on R, as discussed right before Step 1.

Indeed, in Lemma 6.13 we proved that $u_0 \in \mathbb{K}_{=m}$, in Proposition 6.19 we showed that u_0 is Lipschitz and $\{u_0 > 0\}$ is open, and in Proposition 6.24 we obtained that $\{u_0 > 0\}$ is also bounded. We also proved the equations (6.6) in Corollary 6.20 (in the first equation we have F_R on the right-hand-side instead of F) and Proposition 6.33. And lastly, the properties of the regular and singular parts of the boundary follow from Propositions 6.35 and 6.36.

To finish, we need to show that u_0 solves the equation (6.6) with F in the right-hand-side instead of F_R, and also that the support of u_0 is uniformly bounded. We shall next show that this follows using the periodicity assumption (HPer).

Step 3. Boundedness of supports, and transference to the original F. Recall that we showed in Proposition 6.24 that Ω_{u_0} has a finite number $N' \leqslant N$ of enlarged connected components (see Definition 6.23), and each of these enlarged connected components has diameter controlled by D. The important point here is that N and D only depend on m, n and the admissibility constants of F_R, A and q. And since the admissibility constants of F_R and F are the same, N and D are independent of R.

Now label all the enlarged connected components of Ω_{u_0} as $\operatorname{ECC}(V_1), \ldots, \operatorname{ECC}(V_{N'})$. Using the x-periodicity of F, A and q, we will translate them towards the origin, without increasing the energy or changing the q-volume of the support. Writing T for the length of the period in (HPer), we may assume that $D \geqslant 10T$. Concretely, given $W_j := \operatorname{ECC}(V_j)$, consider a translation of W_j by a whole number of periods in each direction (say by the vector Tv_j with $v_j \in \mathbb{Z}^{n+1}$) so that $\widetilde{W}_j := W_j - Tv_j \subset$

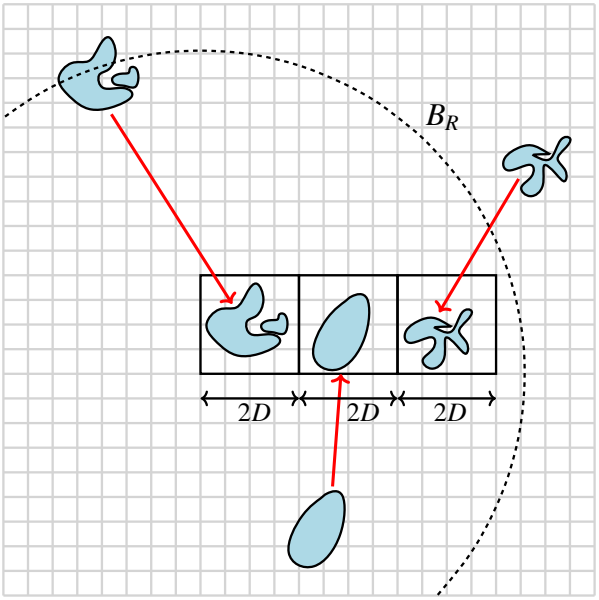


Figure 6.3: Example of how one can move connected components of Ω_u to fixed cubes of length K in a region around the origin, under the periodicity assumption (HPer).

$(2Dj, 0, \dots, 0) + [0, 2D]^{n+1}$ (we can do this because $D \geqslant 10T$, see Figure 6.3). This way, it is clear that $\widetilde{W}_j \subset [0, 2DN']^{n+1} \subset B_R$ if we choose R large enough.

After translating, we have a function $\widetilde{u}_0 \in H^1(\mathbb{R}^{n+1})$ which satisfies $\widetilde{u}_0(X) := u_0(X + Tv_j)$ on each set $\widetilde{W}_j$. The point is that $\mathcal{F}_{0,R}(\widetilde{u}_0, \widetilde{W}_j) \leqslant \mathcal{F}_{0,R}(u_0, W_j)$. To see this, note that, of course, the Dirichlet energies are the same because A is periodic; for the other terms, using the periodicity of F,

$$\iint_{\widetilde{W}_j} F_R(X, \widetilde{u}_0(X))\, dX = \iint_{W_j} F_R(X - Tv_j, u_0(X))\, dX = \iint_{W_j} F(X - Tv_j, u_0(X))\, \varphi_R(X - Tv_j)\, dX$$
$$= \iint_{W_j} F(X, u_0(X))\, \varphi_R(X - Tv_j)\, dX \geqslant \iint_{W_j} F(X, u_0(X))\, \varphi_R(X)\, dX.$$

In the last step we have used the fact that $F \geqslant 0$ (see (HF2) and (HF3)) and $\varphi_R(X - Tv_j) = 1 \geqslant \varphi_R(X)$ for $X \in W_j$ (indeed, in such case, $X - Tv_j \in \widetilde{W}_j \subset B_R$). The analysis of the q-volume is analogous, because q is periodic.

Therefore, after translating (one by one) all the N' enlarged connected components of Ω_{u_0}, we infer that $\Omega_{\widetilde{u}_0} \subset B_R$ if R is large enough. Moreover, recalling that u_0 is a minimizer of $\mathcal{F}_{0,R}$, so is $\widetilde{u}_0$. With large R, we have that $F_R \equiv F$ in $\Omega_{\widetilde{u}_0}$, whence $\widetilde{u}_0$ actually solves the PDE in (6.6) with F (because it solves it with F_R, as we discussed in Step 2). Therefore, $u := \widetilde{u}_0$ has all the properties listed in Theorem 6.2, and the theorem follows.

6.11 Overdetermined problems on manifolds

We shall next sketch why the approach developed in this chapter works equally well for semilinear equations on compact manifolds, with only minor modifications.

Throughout this section, let M be a compact connected manifold of class C^∞, endowed with a smooth Riemannian metric g. We consider the following analog of (6.2):

$$\begin{cases} -\Delta_g v = f(X, v) & \text{in } \Omega \subset M, \\ v = 0 \quad \text{and} \quad |\nabla_g u|_g^2 = c\, q(x) & \text{on } \partial\Omega. \end{cases} \tag{6.52}$$

Here Δ_g and ∇_g respectively denote the Laplace–Beltrami operator and the covariant derivative on the Riemannian manifold (M, g). To control the size of Ω, we will impose the condition

$$\mathrm{Vol}_{q,g}(\Omega) := \iint_\Omega q\, d\mu_g = m\,, \tag{6.53}$$

where $d\mu_g$ denotes the Riemannian measure.

Our main result for this overdetermined problem is the following. Here the definition of "admissible" is essentially as in the case of overdetermined problems on $\mathbb{R}^{n+1}$, with some slight modifications in the constants involved that we will address later.

Theorem 6.37. *Let M be a smooth, compact and connected Riemannian manifold of dimension $n+1 \leqslant 4$. For any constant $0 < m < \mathrm{Vol}_{q,g}(M)$ and for any pair of smooth admissible functions $q : M \to (0, \infty)$ and $f : M \times [0, \infty) \to \mathbb{R}$, there exists an open set $\Omega \subset M$ with smooth boundary satisfying the q-volume constraint (6.53) and a constant $c > 0$ for which the overdetermined boundary value problem (6.52) admits a positive solution $v \in C^\infty(\overline{\Omega})$.*

As before, this result follows from analogous result about minimizers of the functional

$$\mathcal{F}_0(u) := \iint_M |\nabla_g u|_g^2\, d\mu_g - 2 \iint_M F(X, u)\, d\mu_g$$

on the space

$$\mathbb{K}_{\leqslant m} := \big\{ u \in H^1(M) :\ u \geqslant 0,\ \ \mathrm{Vol}_{q,q}(\{u > 0\}) \leqslant m \big\}.$$

We still use the notation $F(X, v) := \int_0^t f(X, t)\, dt$. Specifically, we prove the following result:

Theorem 6.38. *Let M be a smooth, compact and connected Riemannian manifold of dimension $n+1$. Let $0 < m < \mathrm{Vol}_{q,g}(M)$ and assume that $F(X, u)$ and $q(x)$ are admissible. Then there exists a Lipschitz continuous minimizer u of $\mathcal{F}_0$ in $\mathbb{K}_{\leqslant m}$, which actually belongs to $\mathbb{K}_{=m}$. Moreover, $\{u > 0\}$ is open and*

$$\begin{cases} -\Delta_g u = f(X, u) & \text{in } \{u > 0\}, \\ |\nabla_g u|_g^2 = c\, q & \text{on } \partial\{u > 0\} \text{ in the viscosity sense} \end{cases}$$

for some constant $c > 0$. Furthermore, the free boundary can be decomposed as a disjoint union

$$\partial\{u > 0\} = \mathrm{Reg}(\{u > 0\}) \cup \mathrm{Sing}(\{u > 0\}),$$

where:

1. $\mathrm{Reg}(\{u > 0\})$ *is a* $C^{1,\alpha}$ *submanifold of dimension* n *for some* $\alpha > 0$*, open within* $\partial\{u > 0\}$*, and* $|\nabla_g u|_g^2 = c\,q$ *holds pointwise on this set. If* f *and* q *are smooth,* $\mathrm{Reg}(\{u > 0\})$ *is smooth as well.*
2. $\mathrm{Sing}(\{u > 0\})$ *is a closed set within* $\partial\{u > 0\}$ *of Hausdorff dimension at most* $(n+1) - 5$*. Moreover* $\mathrm{Sing}(\{u > 0\})$ *is empty if* $n + 1 \leqslant 4$*, and consists at most of a countable set of points if* $n + 1 = 5$*.*

Let us now sketch the few modifications that are needed to establish this result.

First, note that the results in Sections 6.4, 6.5, 6.7 and 6.9, except for Proposition 6.9, are all of local nature. As the manifold is compact, we can cover it with finitely many charts of (for instance) local normal coordinates. In each chart, which covers a small enough geodesic ball, Equation (6.52) reads as

$$-\sum_{i,j} \partial_{x_i}\big(\sqrt{|g|}g^{ij}\partial_{x_j}u\big) = \sqrt{|g|}\,f(X,v)\,,$$

and the Riemannian measure is $d\mu_g = \sqrt{|g|}\,dX$. Here g^{ij} is the inverse matrix of the Riemannian metric, and $|g|$ denotes the determinant of the metric, in these coordinates. So locally, this equation fits within the framework of (6.2) with the admissible matrix $A^{ij} := \sqrt{|g|}g^{ij}$. Therefore, all the results of local nature still hold in manifolds, on small enough geodesic balls.

It is also not hard to check that, although the arguments in Section 6.8 are not of local nature, they also hold on a compact manifold. Similarly, the proofs in Section 6.6 carry over verbatim to compact manifolds, essentially because they only use general ideas of the calculus of variations. The only point which we want to comment on is the assumption that the exterior of B is connected in Lemma 6.15, as can be seen in [Ve23, Prop. 11.10]: it is not hard to see that manifolds do not introduce any complications here because, just as in the case of $\mathbb{R}^{n+1}$, the complement of a small enough ball in a manifold is always connected (it is diffeomorphic to M minus a point).

Regarding Proposition 6.9, one should simply work in small balls. We can actually provide an alternative proof that shows boundedness in any small ball, which by compactness of M gives the result we want. Let us sketch the argument, again considering balls B that are smaller than the injectivity radius. Then, we can work in local coordinates, considering an admissible coefficient matrix $A(X)$. Since we know $u \in H^1(M)$, necessarily $u \in L^2(B)$. As in (6.12), we infer $f(X) := F'(X,u(X)) \in L^2(B)$. If we define $w_{-\Delta}(X) := \iint_B \Gamma_{-\Delta}(X,y)f(y)\,dy$, we obtain (if $n+1 \geqslant 3$; otherwise we can simply follow Case 2 in Proposition 6.9) $w_{-\Delta} \in L^{2^*}(B)$ by [GT77, Lemma 7.12]. By the comparability of fundamental solutions discussed in Proposition 6.9, it also holds $w(X) := \iint_B \Gamma_L(X,y)f(y)\,dy \in L^{2^*}(B)$. And then, using Lemma 6.6 as in Proposition 6.9 (which is local, so it works here too) we infer $u \in L^{2^*}(B)$. Repeating this self-improvement process we will eventually reach sufficiently large exponents to use [GT77, Lemma 7.12] to obtain $w_{-\Delta} \in L^\infty(B)$, which implies $u \in L^\infty(B)$ as before. This completes the proof because, as mentioned above, M is covered by finitely many balls of this kind.

Lastly, to adapt Section 6.10, we first note that we do not need to introduce the truncation function φ_R to ensure that no mass is lost in the limit in (6.51): this follows because, as M is compact, one can simply use Rellich–Kondrachov on the whole manifold. Therefore, we may work directly with F the whole time, and Step 3 is not needed.

The only change is in the derivation of (6.48). Instead of using symmetrization, on the compact Riemannian manifold, we need to use the existence of sets which minimize the first eigenvalue λ_1 among those which are quasi-open and have prescribed volume m, which is ensured by [LS20, Th. 1.1] (their

proof works without any modification with our modified volume $\mathrm{Vol}_{q,g}$ under the hypotheses (Hq1) and (Hq2)). To put it differently, for each $m \in (0, \mathrm{Vol}_{q,g}(M))$ there exists some $\tilde{\lambda}(m) > 0$ such that

$$\iint_D |\nabla_g w|_g^2 \, d\mu_g \geqslant \tilde{\lambda}(m) \iint_D w^2 \, d\mu_g$$

for any quasi-open set D with $\mathrm{Vol}_{q,g}(D) = m$ and any $w \in H_0^1(D)$.

With this in hand, since for any $u \in H^1$ and we have that $\{u > 0\}$ is quasi-open (more precisely, there is a representative in the equivalence class of u in H^1 for which this holds, by [HKM06, Th. 4.4]), we may apply [LS20, Th. 1.1] to derive (6.48). In the subsequent arguments, the only change we need on the admissibility assumptions is that the constant b in (HF4) must be smaller than $\tilde{\lambda}(m)/2$ times the ellipticity constant of the coefficient matrix induced by the metric g in the normal coordinates, as explained above. The other arguments carry over directly, completing the proof of Theorem 6.38.

6.12 Non-periodic admissible nonlinearities do not have minimizers with uniformly bounded supports

In this last section, we show that some key estimates can fail when the admissibility and periodicity conditions do not hold.

Concerning admissibility, the following elementary calculation shows that quadratic growth of F on u, as in (HF4), is indeed the threshold to be able to find minimizers of $\mathcal{F}_0$. A minor variation of this argument applies to the case $F(X,u) = cu^p$ for any $p > 2$ and any $c > 0$. So from this point of view, (HF4) is sharp.

Proposition 6.39. *Let $F(X,u) := bu^2$ with some constant $b > \lambda_1(B^m)/2$. Suppose that $A := I$ and $q := 1$. Then*

$$\inf_{\varphi \in \mathbb{K}_{\leqslant m}} \mathcal{F}_0(\varphi) = -\infty.$$

Proof. Let $\varphi_1 \in H_0^1(B^m) \subset H^1(\mathbb{R}^{n+1})$ the first Dirichlet eigenfunction of the Laplacian in the ball B^m. Then, it is easy to compute that $\mathcal{F}_0(\varphi_1) = (\lambda_1(B^m) - 2b) \iint_{B^m} \varphi_1^2 \, dX < 0$. Therefore, for any $\tau > 0$,

$$\mathcal{F}_0(\tau\varphi_1) = \tau^2 \mathcal{F}_0(\varphi_1),$$

which tends to $-\infty$ as $\tau \to +\infty$. □

The following proposition, which is less elementary, shows that if F is not periodic, there is no hope for minimizers to have uniformly bounded supports:

Proposition 6.40. *There are admissible A, F, q, and $m > 0$, for which $\mathrm{diam}(\Omega_u)$ is arbitrarily large. More specifically, $\mathrm{diam}(\Omega_u)$ cannot be controlled uniformly with respect to m, n and the admissibility constants of A, F, q.*

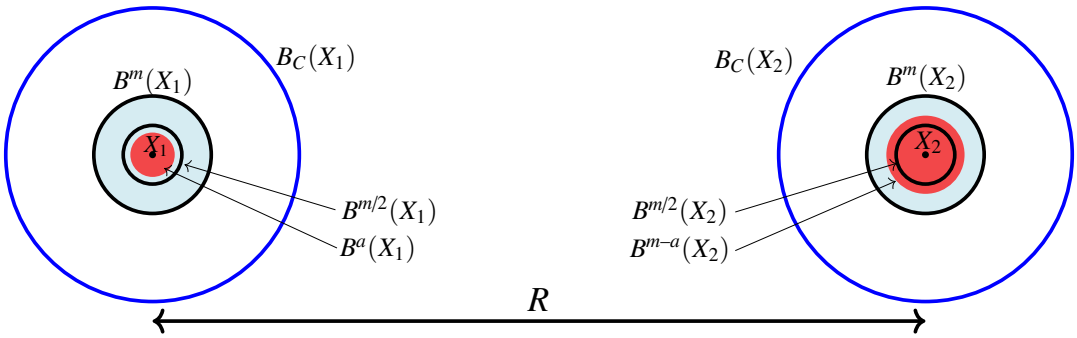


Figure 6.4: We can force Ω_u to accumulate in two regions which are far apart.

Proof. Let us set $A := I$ and $q := 1$, and assume F is only non-zero in two balls $B^m(X_1), B^m(X_2)$, which are separated by a distance $R \gg 1$ still unspecified. Concretely, set $F(X,u) := \varphi(X)\,u$ for

$$\varphi(X) := \begin{cases} \phi(X - X_1) & \text{for } X \in B^m(X_1), \\ \phi(X - X_2) & \text{for } X \in B^m(X_2), \\ 0 & \text{elsewhere,} \end{cases} \qquad \text{where} \quad \begin{cases} \phi \in C^\infty_c(B^m), \text{ radially decreasing,} \\ \phi \equiv 1 \text{ in } B^{m/2}, \\ 0 < \phi < 1 \text{ in } B^m \backslash B^{m/2}. \end{cases}$$

The reader should actually think that ϕ is very close to 0 in $B^m \backslash B^{m/2}$.

Step 1: Ω_u consists of one or two separated balls. Let us consider the associated minimizer u of $\mathcal{F}_0$. Given any connected component V of Ω_u intersecting $B^m(X_1)$, if we define $\mathcal{J}_V := \big\{\mathbf{j} \in \mathbb{Z}^{n+1} : V \cap (\mathbf{j} + [0,1)^{n+1}) \neq \varnothing\big\}$, following the proof of Proposition 6.24 we obtain $|\mathcal{J}_V| \leqslant C$ for some uniform C. Therefore, the connectedness of V implies that $\operatorname{diam}(V) \leqslant C$, i.e. $V \subset B_C(X_1)$.

Of course, the same happens with any other connected component of Ω_u intersecting $B^m(X_1)$, and with those intersecting $B^m(X_2)$. It is also easy to check that every connected component of Ω_u must intersect $B^m(X_1)$ or $B^m(X_2)$, because otherwise it would only increase $\mathcal{F}_0(u)$ (since F is supported in $B^m(X_1) \cup B^m(X_2)$), and u is a minimizer of $\mathcal{F}_0$. Therefore, we have shown that $\Omega_u \subset B_C(X_1) \cup B_C(X_2)$. Thus, if we choose $R \gg C$, these two balls are disjoint and the support of u is separated clearly in two parts. By an easy symmetrization argument inside each of these two balls (recall that ϕ is radially decreasing), we deduce that (see Figure 6.4)

$$\Omega_u = B^a(X_1) \cup B^{m-a}(X_2) \quad \text{for some } 0 \leqslant a \leqslant m.$$

Our goal is to show that it must be $0 < a < m$, so that $\operatorname{diam}(\Omega_u) \geqslant R$, whence the arbitrariness of R would give the result.

Step 2: Energy if there is only one ball. For the sake of contradiction, let us suppose that $a = 0$. For simplicity of the notation along this step, set $X_2 = 0$. By Corollary 6.20, $-\Delta u = \phi$ in B^m. And the maximum principle informs us that $u \leqslant w$, where $-\Delta w = 1$ in B^m and $\{w > 0\} = B^m$. Therefore, testing the equation of u against u itself, we obtain

$$\mathcal{F}_0(u) = -\iint_{B^m} \phi u \, dX \geqslant -\iint_{B^m} \phi w \, dX = -\iint_{B^{m/2}} w \, dX - \iint_{B^m \backslash B^{m/2}} \phi w \, dX.$$

Since the last term can be made arbitrarily small using appropriate choices of ϕ, let us focus on the first one. In fact, it is easy to check that $w(X) = -\frac{|X|^2}{2(n+1)} + \frac{\rho^2}{2(n+1)}$, where $\rho > 0$ is defined so that $\left|B_\rho\right| = |B^m|$. Similarly, find $r > 0$ so that $|B_r| = \left|B^{m/2}\right|$. Then, by an easy integration,

$$\iint\limits_{B^{m/2}} w\, dX = -\frac{\mathcal{H}^n(\partial B_1)}{2(n+1)}\frac{r^{n+3}}{n+3} + \frac{\rho}{2(n+1)}\frac{m}{2}.$$

Step 3: A better candidate. On the other hand, if we concentrate the mass around both X_1 and X_2, we can obtain smaller energies. Indeed, take $v \in H^1(\mathbb{R}^{n+1})$ satisfying $\Delta v = -1$ in $B^{m/2}(X_1) \cup B^{m/2}(X_2)$, with $\{v > 0\} = B^{m/2}(X_1) \cup B^{m/2}(X_2)$. Then, by a similar calculation as above,

$$\mathcal{F}_0(v) = 2\left(-\frac{\mathcal{H}^n(\partial B_1)}{2(n+1)}\frac{r^{n+3}}{n+3} + \frac{r}{2(n+1)}\frac{m}{2}\right).$$

Noting that $\rho = 2^{1/(n+1)}r$ and that $m \approx r^{n+1}$, it is easy to check that if r is large enough (equivalently, if m is large enough) it holds

$$\begin{aligned}\mathcal{F}_0(v) = -\frac{\mathcal{H}^n(\partial B_1)}{2(n+1)(n+3)}2r^{n+3} + \frac{m}{4(n+1)}2r \\ < -\frac{\mathcal{H}^n(\partial B_1)}{2(n+1)(n+3)}r^{n+3} + \frac{m}{4(n+1)}2^{1/(n+1)}r = \iint\limits_{B^{m/2}} w\, dX.\end{aligned}$$

Since these estimates are independent on the choice of ϕ and we can make this choice so that $\mathcal{F}_0(u)$ is as close to $\iint_{B^{m/2}} w\, dX$ as we wish, we obtain the desired contradiction because for these ϕ, we have $\mathcal{F}_0(v) < \mathcal{F}_0(u)$. □

CHAPTER 7

The Kato square root problem for elliptic operators in non-divergence form

So far, we have studied very fine properties of solutions of PDE and boundaries of the domains. In this last chapter, we will consider a much more *foundational* problem: for equations in non-divergence form, namely when $Lu = \sum a_{ij}\partial_{x_i}\partial_{x_j}u$ (instead of the elliptic operators in divergence form in Definition 2.27), the theory is not very developed if the coefficients are rough (if they are regular, the non-divergence and the divergence formulations are essentially interchangeable). Therefore, we will focus in a problem that has a simple formulation, yet having very relevant and deep consequences: the Kato square root problem.

Indeed, a milestone in the theory of elliptic PDE with rough coefficients in divergence form (which, as just said, is much more developed than the non-divergence one), was the solution of the Kato square root conjecture by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian in [AHLMT02].

Theorem 7.1. *Let $L = -\operatorname{div} A\nabla$ be a second order elliptic operator in divergence form, where $A \in L^\infty(\mathbb{R}^n)$. Then the domain of $\sqrt{L}$ coincides with $H^1(\mathbb{R}^n)$ and*

$$\|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)},$$

with constants uniform on dimension, and ellipticity and boundedness of A.

As explained in the introduction, not only the result that they showed was important, but also all the techniques involved and developed in the proof, which mix with mastery the areas of Functional Analysis, PDE and, above all, Harmonic Analysis.

In this last chapter, we show some partial progress to solve the Kato square root problem for rough coefficients in non-divergence form. For this, we follow the well-established strategy by [AHLMT02], but several improvements are necessary. First, in this non-divergence context, it is more natural to work in weighted spaces which are possibly not Hilbert: we are not able to address this difficulty yet without further assumptions. Second, and probably most importantly, the lack of symmetry: the adjoint of a non-divergence form operator is *not* another non-divergence form operator (as happens for divergence form operators), but a double-divergence form operator. These are much more poorly behaved, which

makes the Kato problem for the adjoint operator (needed for the lower bound in the Kato estimate) much more challenging.

For the moment, we are able to solve the problem under the additional assumption that the coefficients have sufficiently small BMO norm, which makes the situation easier because allows us to work in Hilbert spaces. Furthermore, the small BMO condition is fairly mild.

This chapter is based on the paper [EHH24], done in collaboration with Luis Escauriaza and Steve Hofmann.

We suggest the reader to have Section 7.4 handy while reading the chapter, for it has the key ideas and provides a simplified and understandable sketch of the proofs that will be carried out in all detail along the rest of the chapter.

7.1 Further historical context . . . 272
7.2 Preliminaries . . . 273
7.3 Main result . . . 276
7.4 Main ideas and outline of the chapter . . . 278
7.5 Auxiliary results . . . 279
7.5.1 Notation . . . 279
7.5.2 Gaussian bounds for kernels of semigroups . . . 279
7.5.3 Weighted Littlewood-Paley theory . . . 280
7.5.4 Uniform bounds for some operators . . . 281
7.5.5 Off-diagonal estimates . . . 282
7.5.6 Estimates for differences and gradients . . . 284
7.6 The Kato problem for L . . . 286
7.7 The adjoint Kato problem, for $\widetilde{L}$ . . . 289
7.7.1 Reduction to a quadratic estimate . . . 289
7.7.2 Reduction to a Carleson measure estimate . . . 291
7.7.3 The $T(b)$ argument . . . 299

7.1 Further historical context

In [AHLMT02], the authors resolved in the affirmative the long-standing square root problem of Kato for divergence form complex-elliptic operators in $\mathbb{R}^n$. This was the culmination of a series of previous results: [CMM82] (treating the 1-d case); [CDM83] and [FJK85, FJK84] (treating small perturbations of the constant coefficient case); [HM02] (the 2-d case); [AHLT01] (small perturbations of the real-symmetric case, sometimes referred to as the restricted Kato problem); and [HLM02] (the case that the heat kernel satisfies a Gaussian upper bound and Nash-type Hölder continuity estimates). The solution of the Kato problem, and the circle of ideas involved in its proof, led to subsequent breakthroughs in the theory of elliptic boundary value problems, see, e.g., [AAAHK11], [Ho06] [HKMP15a, HKMP15b], [AKM06], [AA11], [AAH08], [AAM10a, AAM10b], [CR15], [HLM19], [EH18], [HLMP22, HLMP22], [BHLMP20]. See also the significant related ground-breaking work in the parabolic setting: [Ny17], [AEN18, AEN20].

In this chapter, we initiate the study of the square root problem in the non-divergence setting in dimensions greater than 1, with the eventual goal of developing applications to the theory of boundary value problems, as has been done in the aforementioned divergence form case. Previously, the non-divergence problem had been treated only in the 1-dimensional setting, in [KM85]; in fact, in that paper the authors treat the more general class of operators of the form $L = -aDbD$, where D denotes the ordinary differentiation operator on the line, and a, b are arbitrary bounded accretive complex-valued functions on the line.

At present, we are able to treat only the case of real coefficients. On the other hand, we point out that in the divergence form case, there is no known proof for real, non-symmetric coefficients that is fundamentally easier than the proof in the general case, owing to the non-selfadjointness of non-symmetric divergence form operators. Moreover, it is the real, non-symmetric case that underlies the breakthrough in the study of the Dirichlet problem obtained in [HKMP15a]. We observe that in the non-divergence setting, we may assume without loss of generality that the coefficient matrix is symmetric, but in contrast to the divergence form case, operators of non-divergence type are inherently non-selfadjoint, even with symmetric coefficients.

A fundamental difficulty that one encounters in the non-divergence setting, is that the Kato problem for non-divergence elliptic operators seems to be most naturally formulated in a weighted L^2 space, and in general, the weight need not belong to the Muckenhoupt A_2 class. Another difficulty inherent to the non-divergence setting is the lack of uniqueness (see the work of Nadirashvili [Na97]). On the other hand, working with real coefficients allows us to make use of the pioneering work of Krylov and Safonov [KS81], as well as the important ideas of Baumann [Ba84] and Escauriaza [Es00]. We shall return to these matters in the sequel. First, let us set notation and definitions.

7.2 Preliminaries

As opposed to all the previous chapters, in this chapter there are no boundary value problems, or even boundaries; so our dimension of interest will be the ambient dimension. For this reason, we will work in $\mathbb{R}^n$ and, similarly, we will use lower-case letters to denote points at all times. We hope this does not confuse the reader.

Definition 7.2 (Operators in non-divergence form)**.** We will say that the operator L is a *second-order elliptic operator in non-divergence form* on $\mathbb{R}^n$ if

$$Lu = -\sum_{i,j=1}^{n} a_{ij}\partial_{x_i}\partial_{x_j}u,$$

where $A = (a_{ij}(\cdot))$ is a real and measurable coefficient matrix which (without loss of generality) we can take to be symmetric, and for which we also assume, for some $\lambda > 0$,

$$A(x)\xi\cdot\xi \geqslant \lambda|\xi|^2 \quad \text{and} \quad \left|A(x)\xi\cdot\zeta\right| \leqslant \lambda^{-1}|\xi||\zeta| \qquad \text{for all } \xi,\zeta\in\mathbb{R}^n \text{ and a.e. } x\in\mathbb{R}^n, \tag{7.1}$$

Definition 7.3 (Adjoint operator)**.** Given L a second-order operator in non-divergence form, we have also its adjoint operator

$$L^*u = -\sum_{i,j=1}^{n} \partial_{x_i}\partial_{x_j}(a_{ij}u).$$

Following [Es00], we say that the function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is a solution of the adjoint equation $L^*u = 0$ if for every $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} u(x) L\varphi(x) dx = 0.$$

The reader is also advised to review the theory of Muckenhoupt weights, and actually all Section 2.2.1, since it will be ubiquitous along the chapter. With this in mind, we recall a fundamentally important property of equations in non-divergence form.

Lemma 7.4 ([Es00, Theorem 1.1]). *Let L be a second-order elliptic operator in non-divergence form, and L^* its adjoint. Then there exists a non-negative solution W of the adjoint equation $L^*W = 0$ in $\mathbb{R}^n$, satisfying $W(B_1(0)) = |B_1(0)|$, which we call the* `global non-negative adjoint solution`*. Furthermore, W satisfies a Reverse Hölder property with exponent $\frac{n}{n-1}$ (see Theorem 2.4), so $W \in A_\infty$ (c.f. Definition 2.3). Moreover, the $RH_{n/(n-1)}$ constants depend only on dimension and ellipticity.*

If the coefficients of L are smooth, or even belong to VMO, then W (with the stated normalization) is unique. In general, it need not be unique. On the other hand, for any given L, any choice of such a W will enjoy the same quantitative estimates, with uniform control of all relevant constants. In the case that W is not unique, we may therefore simply fix an arbitrary choice of W.

We recall the well known fact that A_∞ weights are doubling. In our case, this means that there exists a constant $C_D = C_D([W]_{A_\infty}) \geqslant 1$ such that $W(2B) \leqslant C_D W(B)$ for every ball B.

From now on we will work most of the time in the weighted Hilbert space

$$L^2_W := L^2(\mathbb{R}^n, W(x)dx).$$

In the non-divergence setting, this space is more natural in many ways than unweighted L^2. In particular, the following identity holds, as may be seen formally by using $L^*W = 0$ and integrating by parts:

$$\int_{\mathbb{R}^n} uLuWdx = \int_{\mathbb{R}^n} A\nabla u \cdot \nabla u W dx. \tag{7.2}$$

In fact, one may readily deduce that (7.2) holds when the coefficients are smooth, and more generally, it also holds at least when the coefficients have sufficiently small BMO norm (depending only on dimension and ellipticity), for all $u \in \mathcal{D}(L)$ (the domain of L), defined by

$$\mathcal{D}(L) := \left\{ u \in L^2_W : Lu \in L^2_W \right\}.$$

Indeed, if the coefficients have sufficiently small BMO norm, then $W \in A_2$ (see [Es94, Theorem 1.2], and its proof[1]). In turn, using this fact, one may prove[2] the regularity estimate

$$\|\nabla^2 u\|_{L^2_W} \lesssim \|f\|_{L^2_W} \tag{7.3}$$

for solutions of the Poisson problem $Lu = f \in L^2_W$, and hence, that

$$\mathcal{D}(L) = H^2_W(\mathbb{R}^n) := \left\{ u \in L^2_W : \nabla u, \nabla^2 u \in L^2_W \right\}. \tag{7.4}$$

The identity (7.2) then follows readily for all $u \in \mathcal{D}(L)$.

[1] In fact, [Es94] precedes [Es00] chronologically; the result in [Es94] treats explicitly local versions of W, but the same arguments may be applied to the global W constructed in [Es00].

[2] We caution the reader that the proof of (7.3) is a somewhat non-trivial matter, and is based on the boundedness of the commutator $[T, b]$, where T is a singular integral and $b \in$ BMO [CRW76], along with a suitable expansion in terms of spherical harmonics; see [CFL91, CFL93] for related results in the unweighted case.

Remark 7.5. We observe that $H^2_W(\mathbb{R}^n)$ is dense in L^2_W when $W \in A_2$ (indeed, even $\mathscr{C}^\infty_c$ is dense in that case), and therefore L is densely defined when the coefficients have sufficiently small BMO norm.

Definition 7.6 ($\widetilde{L}$, the normalized adjoint)**.** We shall also consider the *normalized adjoint* of L, which we denote by $\widetilde{L}$, and which we define to be the adjoint of L with respect to the space L^2_W. Thus, $\widetilde{L}$ is given, at least for smooth coefficients, by the formula

$$\widetilde{L}u := -\frac{1}{W}\sum_{i,j=1}^{n} \partial_{x_i}\partial_{x_j}(a_{ij}uW) = \frac{1}{W}L^*(uW). \tag{7.5}$$

If the coefficients are merely measurable, then we interpret $\widetilde{L}$ in the weak sense: we say that $u \in L^2_W$ belongs to $\mathcal{D}(\widetilde{L})$, the domain of $\widetilde{L}$, provided that there is a function $g \in L^2_W$ such that for every $v \in \mathcal{D}(L)$,

$$\int_{\mathbb{R}^n} Lv\, u\, W dx = \int_{\mathbb{R}^n} vgW dx\,,$$

and in this case we set $\widetilde{L}u = g$.

Just as for (7.2), integrating by parts and using $L^*W = 0$, we obtain (at least in the case of smooth coefficients, and for $u \in H^2_W$)

$$\int_{\mathbb{R}^n} u\widetilde{L}uW dx = \int_{\mathbb{R}^n} A\nabla u \cdot \nabla u W dx. \tag{7.6}$$

We shall henceforth make the *qualitative* assumption that the coefficients a_{ij} are smooth, with qualitative L^∞ bounds on ∇a_{ij} and $\nabla^2 a_{ij}$. Thus, (7.6) will be valid in the sequel, for $u \in H^2_W$. On the other hand, we emphasize that our *quantitative* bounds will never depend on smoothness, nor on estimates for the derivatives of a_{ij}.

As a consequence of (7.2), (7.6), (7.3), and (7.4)[3], we have the following result, which establishes the functional analytical framework needed to justify all the forthcoming arguments. At this point, the reader may want to review Section 2.6 or have [Ha06, EHMT24] handy.

Lemma 7.7. *The operators L and $\widetilde{L}$ defined above, viewed as unbounded operators on L^2_W, are each closed, sectorial and m-accretive. Hence, each has an m-accretive square root (see [Ka66, Theorem 3.35, p. 281], or [Ha06, Sections 3 and 7]). Moreover, L generates a heat semigroup $z \mapsto e^{-zL}$, which is well-defined and analytic in a sector containing the positive real axis (hence, the analogous statement is also true for $\widetilde{L}$).*

Proof. Let us proceed in a series of steps.

Step 1: L and $\widetilde{L}$ are closed operators. For $\widetilde{L}$, this follows immediately from the fact that L is densely defined (see [Ka66, p. 168]). Thus we consider L. Suppose that $\{u_j\}_j \subset \mathcal{D}(L)$, that $u_j \to u$ in L^2_W, and that

$$f_j := Lu_j \to f \text{ in } L^2_W\,.$$

We need to verify that $u \in \mathcal{D}(L)$, and that $Lu = f$, i.e., that the graph $\{(u, Lu) : u \in \mathcal{D}(L)\}$ is a closed set in $L^2_W \times L^2_W$. Applying (7.2) and (7.3) to $u_j - u_{j'}$, we see that $\{\nabla u_j\}_j$ and $\{\nabla^2 u_j\}_j$ are each convergent in L^2_W, thus, $\{u_j\}_j$ is convergent in H^2_W, and since $u_j \to u$ in L^2_W, we see that $u \in H^2_W = \mathcal{D}(L)$, and that

[3] (7.4) is used to show that (7.2) holds for all $u \in \mathcal{D}(L)$.

$u_j \to u$ in H^2_W. In particular, $\partial_{x_i}\partial_{x_j}u_j \to \partial_{x_i}\partial_{x_j}u$ in L^2_W for each $i, j = 1, 2, ..., n$, hence $Lu_j \to Lu$, so that $Lu = f$, as desired.

Step 2: L and $\widetilde{L}$ are sectorial. It follows readily from (7.2) (respectively, (7.6)) that the numerical ranges

$$\Theta := \left\{\langle Lu, u\rangle \in \mathbb{C} : \|u\|_{L^2_W} = 1\right\}, \quad \widetilde{\Theta} := \left\{\langle \widetilde{L}u, u\rangle \in \mathbb{C} : \|u\|_{L^2_W} = 1\right\}$$

are each contained in a sector $S_\omega := \{z \in \mathbb{C} : |\arg z| \leqslant \omega\} \cup \{0\}$, with $0 < \omega < \pi/2$, depending only on ellipticity. We omit the standard argument.

Step 3: L and $\widetilde{L}$ are m-accretive. By [Ka66, Problem 3.31, p 279], it suffices to show that L is m-accretive. To this end, set

$$\Delta := \mathbb{C}\backslash S_\omega\,,$$

and for $\zeta \in \Delta$, set $\delta = \delta(\zeta) := \operatorname{dist}(\zeta, S_\omega)$. By symmetry, we also have $\delta = \operatorname{dist}(\overline{\zeta}, S_\omega)$. We now claim that

$$\|(L-\zeta)u\|_{L^2_W} \geqslant \delta\|u\|_{L^2_W}\,, \quad u \in \mathcal{D}(L)\,,\ \zeta \in \Delta\,. \tag{7.7}$$

Indeed, to verify the claim, we may assume without loss of generality that $\|u\|_{L^2_W} = 1$, in which case

$$\|(L-\zeta)u\|_{L^2_W} \geqslant |\langle (L-\zeta)u, u\rangle| = |\langle Lu, u\rangle - \zeta| \geqslant \delta\,,$$

since $\langle Lu, u\rangle \in \Theta \subset S_\omega$.

Fix $\zeta \in \Delta$. Then $L - \zeta$ is 1-1 on $\mathcal{D}(L)$, and has closed range (since L is a closed operator). Similarly, $\widetilde{L} - \overline{\zeta}$ is 1-1 on $\mathcal{D}(\widetilde{L})$. Since L is densely defined,

$$\mathcal{N}(\widetilde{L} - \overline{\zeta}) = \mathcal{R}(L-\zeta)^\perp\,,$$

i.e., the null space of $\widetilde{L} - \overline{\zeta}$ is the orthogonal complement of the range of $L - \zeta$. Thus, $L - \zeta$ has dense range, since $\widetilde{L} - \overline{\zeta}$ is 1-1. Hence, $L - \zeta$ is invertible as a mapping from $\mathcal{D}(L)$ onto L^2_W. Combined with the estimate (7.7), this shows that L is m-accretive (see [Ka66, p. 279]).

Step 4: The heat semi-group. Given the preceeding properties of L, we have existence, uniqueness, and analyticity of a contraction semigroup e^{-zL}, for z in the open sector $S^0_\alpha := \{z \in \mathbb{C} : |\arg z| < \alpha\}$, provided $0 < \alpha < \pi/2 - \omega$. See, e.g., pp. 480-493, especially Theorem 1.24, p. 492, in [Ka66]. □

7.3 Main result

Our main result is the following:

Theorem 7.8. *Let L be a second-order elliptic operator in non-divergence form with smooth real coefficients satisfying (7.1), and let W be the associated global non-negative adjoint solution provided by Lemma 7.4. If $W \in A_2$ (see Definition 2.3), then we have*

$$\|\sqrt{L}f\|_{L^2_W} \approx \|\nabla f\|_{L^2_W} \approx \|\sqrt{\widetilde{L}}f\|_{L^2_W},$$

where the implicit constants depend only on n, λ and $[W]_{A_2}$.

The main goal of this chapter is to prove this Theorem. Hence, from here on we will always impose the extra assumption that $W \in A_2$, along with the qualitative assumption that the coefficients are smooth. The result will follow at once from Theorems 7.28 and 7.34.

Some additional remarks are in order.

Remark 7.9. As mentioned above, in general W belongs to the class A_∞, and thus $W \in A_p$ for some p depending on dimension and ellipticity, but p may be strictly greater than 2, and in fact in the general case we have no precise upper bound on p. Thus, our result is only a partial one, and does not address the fundamental challenge of treating the non-A_2 case. On the other hand, as noted above, if the coefficients have sufficiently small BMO norm, then $W \in A_2$, and thus our result does apply in that setting.

Remark 7.10. In the case that the coefficient matrix has sufficiently small BMO norm, then as also noted above, we may identify the domain $\mathcal{D}(L)$ as the weighted Sobolev space $H^2_W(\mathbb{R}^n)$ (see (7.4)). Hence, by combining several known (or at least implicit) results, we may identify the domain of $\sqrt{L}$ as the Sobolev space $H^1_W(\mathbb{R}^n) := \{u \in L^2_W(\mathbb{R}^n) : \nabla u \in L^2_W(\mathbb{R}^n)\}$; this corresponds to the estimate $\|\sqrt{L} f\|_{L^2_W} \lesssim \|\nabla f\|_{L^2_W}$. Indeed, in [DY02] it is shown that the operator L has a bounded holomorphic functional calculus in (unweighted) L^2 (even in L^p), provided that the BMO norm of the coefficients is sufficiently small. Under the same smallness assumption, the arguments of [DY02] may be extended to the weighted case considered here, to deduce that L has a bounded holomorphic functional calculus in L^2_W. Combining the results of [Ya84] and [Mc86], we find that $\mathcal{D}(\sqrt{L})$ is the complex interpolation space mid-way between L^2_W and $\mathcal{D}(L) = H^2_W$, i.e., $\mathcal{D}(\sqrt{L}) = H^1_W$.

The analogous strategy fails for $\widetilde{L}$, as we have no idea how to identify $\mathcal{D}(\widetilde{L})$ (similarly, the square root problem in the divergence form case entailed the same difficulty).

Remark 7.11. The assumption of smoothness of the coefficients is purely qualitative, and our quantitative estimates will not depend on smoothness, but only on the stated parameters n, λ and $[W]_{A_2}$. However, it is not clear at present how to make sense of the identity (7.6) for non-smooth coefficients, and as a consequence, in the absence of smoothness, we do not know how to prove certain estimates which rely on (7.6), such as Lemma 7.20 (viii) (in the case of measurable coefficients, we know how to give only a formal proof of the latter, assuming a priori finiteness of $\|\nabla e^{-t^2\widetilde{L}} f\|_{L^2_W}$).

On the other hand, as noted above, identity (7.2) holds without smoothness, in the case that the coefficients have sufficiently small BMO norm. Under the latter scenario, we require identity (7.6) and its consequences (and thus, the qualitative, a priori assumption of smoothness of the coefficients) in two places: 1) in the proof of Theorem 7.34 (the square root problem for $\widetilde{L}$), where estimate (7.6) is heavily used, and 2) in the proof of the m-accretivity of L given above, where we used (7.6) to establish density of the range of $L - \zeta$. Otherwise, (7.6) is not used in the proof of Theorem 7.28 (the square root problem of L).

Although our operators L and $\widetilde{L}$ are not self-adjoint, there is a nice identity relating these two non-divergence operators with another one which is in divergence form, but degenerate elliptic[4]. Indeed, if we let $\widetilde{\operatorname{div}}$ denote the normalized divergence, defined for an $\mathbb{R}^n$-valued function $\mathbf{v}$ by $\widetilde{\operatorname{div}}\,\mathbf{v} := \frac{1}{W}\operatorname{div}(W\mathbf{v})$, then $\widetilde{\operatorname{div}}$ is precisely the adjoint operator to $-\nabla$ inside L^2_W, and we also have, using $L^* W = 0$,

$$Lu + \widetilde{L}u = -2\widetilde{\operatorname{div}}(A\nabla u). \tag{7.8}$$

[4]Thus, our results here are somewhat related to those of Cruz-Uribe and Rios [CR15], and Auscher, Rosén and Rule [ARR15].

In the case of non-smooth coefficients, we interpret the latter identity in the weak sense described above: for $u, \varphi \in H^2_W$,

$$\int_{\mathbb{R}^n} \big(\varphi Lu \, + \, uL\varphi\big) W dx = 2 \int_{\mathbb{R}^n} A\nabla u \cdot \nabla \varphi \, W dx \, .$$

The identity (7.8) will be of great use to us in the sequel.

7.4 Main ideas and outline of the chapter

First, in Section 7.5, we give some definitions and estimates for some of the operators that which will appear repeatedly across the chapter. Concretely, it is crucial that we know how to derive boundedness and off-diagonal bounds for the gradient of the semigroup, because these will be key for (7.37) and (7.38), which are fundamental for the $T(b)$ argument. All the estimates are weighted and hence more technical to be shown, but they are very close in spirit to the original proofs in [AHLMT02]. Concretely, the estimates in the last subsection show a clever self-improvement: if we use the fact that our operators annihilate constants to introduce averages at suitably chosen scales, Poincaré's inequality gives us several estimates that will prove to be useful later.

In Section 7.6, we prove $\| \sqrt{L} f \|_{L^2_W} \lesssim \|\nabla f\|_{L^2_W}$, which turns out to be a relatively easy consequence of Littlewood-Paley theory. Indeed, the form of L (it is in non-divergence form, so it annihilates not only constants but also first degree monomials) allows us to give a fairly short and self-contained proof by almost-orthogonality arguments.

Section 7.7, where we prove $\| \sqrt{\widetilde{L}} f \|_{L^2_W} \lesssim \|\nabla f\|_{L^2_W}$, is the core of the paper. In the first subsection, after representing the square root via an integral against semigroups (using Functional Calculus), we are able to reduce matters (just by duality) to a square function estimate, similarly as what we did in Section 7.6

In the next subsection, a careful first order approximation of the operator allows us to reduce matters to a simpler square function estimate (similarly in spirit to a $T1$ theorem), which after suitable smoothing using approximations of the identity P_t turns out to be equivalent to a Carleson measure estimate. For these reductions, heavily relying on almost-orthogonality arguments, we combine estimates on the global behavior of our semigroups (via off-diagonal estimates) jointly with fine pointwise estimates, and also the crucial relation (7.8) (which helps us moving derivatives to and from the arguments, whenever it is needed).

In the last subsection of Section 7.7, we prove the Carleson measure estimate by testing against suitable functions which oscillate in the appropriate dyadic region (as seen in Lemma 7.52): it is crucial that we have reduced to a dyadic Carleson measure estimate because this allows us to consider only one scale at a time (so the construction of the test functions is much easier), and then invoke the powerful tool of extrapolation of Carleson measures (although this will be somehow hidden because we will just follow previous papers for this part of the argument). Since these functions are constructed using the semigroup of our operator, and we have collected a number of estimate for this in the previous sections, the proof of the Carleson-like estimate for the constructed functions follows rapidly.

7.5 Auxiliary results

7.5.1 Notation

We suggest the reader to review the general notation in Section 2.1. Apart from that, we will use some more specific notation in this chapter.

- We recall the reader that, unlike in previous chapters, it makes more sense (since there are no boundaries of domains involved) to work in $\mathbb{R}^n$ as ambient space, and always use lower-case letters for points in $\mathbb{R}^n$.

- In the proofs of the results from now on, we will omit dependencies of constants on n, λ, $[W]_{A_\infty}$ and $[W]_{A_2}$ – treating them as harmless constants – although we will make these dependencies explicit in the statements.

- As explained before, we set $L^2_W := L^2(\mathbb{R}^n, W(x)dx)$, and we define the weighted Sobolev space $H^2_W := \{u \in L^2_W : \nabla u \in L^2_W, \nabla^2 u \in L^2_W\}$. We will also write $L^2_W(E) := L^2(E, W(x)dx)$ for any subset $E \subset \mathbb{R}^n$.

- We will denote vector-valued functions with boldface letters, e.g., $\mathbf{v} := (v_1, ..., v_n)$.

- We will frequently use cubes in our proofs: every time we cover $\mathbb{R}^n$ (or some portion of it) by cubes, we mean we are using a covering by cubes of the dyadic grid $\{2^j\mathbf{k} + [0, 2^j)^n : j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n\}$. Anytime we use the letter Q, we will be referring to a dyadic cube. For such a cube Q, we let $\ell(Q)$ denote its sidelength.

- If two operators are written one after another, we mean that they are composed. For instance, given two operators U and V, we mean $UV(f) := U(V(f))$. In contrast, the dot $\cdot$ between operators will mean pointwise multiplication of operators, i.e. $(U \cdot V)(f) = U(f)V(f)$.

- We let $\mathcal{M}$ and $\mathcal{M}_W$ denote, respectively, the classical Hardy-Littlewood maximal operator (recall Section 2.2.1), and the Hardy-Littlewood maximal function with respect to the measure $W(x)dx$, that is,

$$\mathcal{M}f(x) := \sup_{B \ni x} \fint_B |f(y)|\, dy = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy\,,$$

and

$$\mathcal{M}_W f(x) := \sup_{B \ni x} \fint_B |f(y)|\, W(y)dy = \sup_{B \ni x} \frac{1}{W(B)} \int_B |f(y)|\, W(y)dy.$$

Since W is doubling (because $W \in A_\infty$), $\mathcal{M}_W$ is bounded on L^p_W for every $1 < p < \infty$. We will use this fact in the sequel.

7.5.2 Gaussian bounds for kernels of semigroups

From now on, we will use many times the parabolic semigroup (with elliptic homogeneity) e^{-t^2L}, whose kernel is the fundamental solution $\Gamma_{t^2}(x, y)$; i.e. we have $e^{-t^2L}f(x) = \int_{\mathbb{R}^n} \Gamma_{t^2}(x, y)f(y)dy$ for sufficiently regular f. The fundamental solution satisfies the following Gaussian estimate:

Lemma 7.12 ([Es00, Theorem 1.2]). *The kernel $\Gamma_{t^2}(\cdot,\cdot)$ of e^{-t^2L} satisfies the Gaussian bounds*

$$\Gamma_{t^2}(x,y) \lesssim \min\left\{\frac{1}{W(B_t(x))}, \frac{1}{W(B_t(y))}\right\} e^{-c\frac{|x-y|^2}{t^2}} W(y), \tag{7.9}$$

and

$$\max\left\{\frac{1}{W(B_t(x))}, \frac{1}{W(B_t(y))}\right\} e^{-\frac{|x-y|^2}{ct^2}} W(y) \lesssim \Gamma_{t^2}(x,y), \tag{7.10}$$

where the implicit constants and c depend on n and λ.

Remark 7.13. The results stated above as Lemma 7.4 and Lemma 7.12, are stated in [Es00] explicitly for smooth coefficients, but as the author points out, "the usual compactness arguments", and the uniformity of the estimates depending only on n and λ, allow one to deduce the existence of (non-unique) W and Γ verifying the same bounds, in the general case of bounded measurable coefficients.

Remark 7.14. The doubling property of W, combined with the exponential decay factor, allow us to interchange "min" and "max" in (7.9) and (7.10), modulo an adjustment of the constants.

Remark 7.15. The (absolute values of) the kernels of the operators $t^2Le^{-t^2L}$ and $t^4L^2e^{-t^2L}$ also satisfy the upper bound (7.9), by analyticity of the semigroup $z \mapsto e^{-zL}$ in a sector.

7.5.3 Weighted Littlewood-Paley theory

We already gave a careful presentation of Littlewood-Paley theory in Section 2.2.3. In this context, we will need weighted analogues of those results. We expose them here without proof because they are standard analogues of the ones presented in Section 2.2.3.

Lemma 7.16. *Let $W \in A_2$, and let $K_t f := k_t * f$, with k defined on $\mathbb{R}^n$ satisfying $|k(x)| \leqslant (1+|x|)^{-n-1}$, where $k_t(x) := t^{-n}k(x/t)$. Then*

$$\|\sup_{t>0}|K_t f|\|_{L^2_W} \lesssim \|\mathcal{M}f\|_{L^2_W} \lesssim \|f\|_{L^2_W},$$

where $\mathcal{M}$ is the classical Hardy-Littlewood maximal operator, and the implicit constant depends on n and $[W]_{A_2}$.

Lemma 7.17. *Let $W \in A_2$, and let $Q_s f := \psi_s * f$, where $\psi \in \mathcal{S}$ and satisfies $\int_{\mathbb{R}^n}\psi = 0$. Then*

$$\int_0^\infty \|Q_s f\|^2_{L^2_W}\,\frac{ds}{s} \lesssim \|f\|^2_{L^2_W},$$

where the implicit constant depends on n, ψ, and $[W]_{A_2}$. Moreover, if in addition ψ is radial and non-trivial, then using a slight abuse of notation and then normalizing, we may assume that $\int_0^\infty \hat{\psi}(s)^2\,\frac{ds}{s} = 1$, in which case we have the Calderón reproducing formula

$$\int_0^\infty Q_s^2 f\,\frac{ds}{s} = f \in L^2_W.$$

Remark 7.18. Regarding the last pair of lemmata:

- We will often denote by Q_t the operators satisfying the hypotheses in Lemma 7.17, while we use P_t for "nice" approximate identities (i.e. $P_t f := \varphi_t * f$, with $\varphi \in \mathcal{S}$ radial, and $\int\varphi = 1$).

- It is easy to check if P_t is a nice approximate identity, then $Q_t := t\partial_{x_i} P_t$ satisfies the hypotheses of the first part of Lemma 7.17 (where ∂_{x_i} denotes the partial derivative in any direction x_i).
- We will frequently further assume that the kernel k in Lemma 7.16 (in particular, φ and ψ as above) satisfies $\operatorname{supp} k \subset B_1(0)$; in this case, we shall refer to K_t (in particular, P_t or Q_t) as having a "compactly supported kernel".
- We will use repeatedly the fact that P_t and Q_s commute with derivatives, for they are convolution operators.

The following is an easy consequence of Lemma 7.17, by standard "almost-orthogonality" arguments, as in Section 2.2.3.

Lemma 7.19. *Let $\{Q_s\}_{s>0}$ be a family operators satisfying the conditions in Lemma 7.17, and R_t be a family of operators, bounded on $L^2_W(\mathbb{R}^n)$ for each fixed $t > 0$, and satisfying, for some $\alpha > 0$, the almost-orthogonality condition*

$$\|R_t Q_s\|_{L^2_W \to L^2_W} \leqslant C_1 \min\left\{\frac{t}{s}, \frac{s}{t}\right\}^{\alpha},$$

where C_1 is a uniform constant which does not depend on t, s. Then R_t satisfies the square function estimate

$$\int_0^\infty \|R_t f\|^2_{L^2_W} \frac{dt}{t} \lesssim \|f\|^2_{L^2_W},$$

where the implicit constant depends on n, C_1, α and $[W]_{A_2}$.

7.5.4 Uniform bounds for some operators

Let us show that some operators related with the semigroups are uniformly bounded, which we will use throughout the text.

Lemma 7.20. *The following operators are $L^2_W \to L^2_W$ bounded, uniformly on t, with norm depending on n, λ and the doubling constant for W:*

(i) $e^{-t^2 L}$,	*(ii)* $t^2 L e^{-t^2 L}$,	*(iii)* $t^4 L^2 e^{-t^2 L}$,	*(iv)* $t\nabla e^{-t^2 L}$,
(v) $e^{-t^2 \widetilde{L}}$,	*(vi)* $t^2 \widetilde{L} e^{-t^2 \widetilde{L}}$,	*(vii)* $t^4 \widetilde{L}^2 e^{-t^2 \widetilde{L}}$,	*(viii)* $t\nabla e^{-t^2 \widetilde{L}}$,
(ix) $te^{-t^2 L}\operatorname{div}$,	*(x)* $te^{-t^2 \widetilde{L}}\operatorname{div}$,	*(xi)* $t^3 L e^{-t^2 L}\operatorname{div}$,	*(xii)* $t^3 \widetilde{L} e^{-t^2 \widetilde{L}}\operatorname{div}$.

Remark 7.21. Some of the operators are in fact defined for functions $\mathbf{f} \in (L^2_W)^n$. In this case, we obviously mean that each of their components are bounded.

Proof. (i) - (iii). Recall that $\mathcal{M}_W$ denotes the Hardy-Littlewood maximal operator with respect to the weighted measure $W(x)dx$. If we let T_t denote the operator under consideration in (i), (ii) or (iii), it suffices to observe that in each case we have the pointwise bound

$$\sup_t |T_t f| \lesssim \mathcal{M}_W f,$$

using only the Gaussian bounds in Lemma 7.12 (and Remark 7.15 in the case of (ii) and (iii)), and the doubling property of W. We omit the routine details.

To prove (iv), we set $u := e^{-t^2L}f$. Note that by the identity (7.2), we can "interpolate" the estimates in (i) and (ii) using Cauchy-Schwarz as follows:

$$\int_{\mathbb{R}^n} |t\nabla u|^2\, W dx \lesssim \int_{\mathbb{R}^n} t^2 A\nabla u\cdot\nabla u W dx = \int_{\mathbb{R}^n} t^2 Luu W dx$$
$$\leqslant \left(\int_{\mathbb{R}^n} \left|t^2 Lu\right|^2 W dx\right)^{1/2} \left(\int_{\mathbb{R}^n} |u|^2\, W dx\right)^{1/2} \lesssim \int_{\mathbb{R}^n} |f|^2\, W dx,$$

which shows (iv).

In turn, (v)-(vii) follow by duality from (i)-(iii), while (viii) follows "interpolating" (v) and (vi) in the same way that we did it for (iv), this time using (7.6) instead of (7.2).

Then, (ix) and (x) follow by duality from (viii), and (iv), respectively.

Lastly, (xi) (resp. (xii)) follows by first dualizing, and then "interpolating" (vi) and (vii) using (7.6) (resp. (ii) and (iii) using (7.2)). □

7.5.5 Off-diagonal estimates

Definition 7.22. We say that the operators S_t **satisfy off-diagonal estimates** (aka Gaffney estimates), if there exist $c, C > 0$ (independent of t) such that it holds for every $t > 0$, and every pair of measurable sets E and F,

$$\int_F \left|S_t f(x)\right|^2 W(x)dx \leqslant Ce^{-c\frac{\operatorname{dist}(E,F)^2}{t^2}} \int_E \left|f(x)\right|^2 W(x)dx \qquad \text{if } \operatorname{supp} f \subset E\,.$$

Lemma 7.23. *The following operators satisfy off-diagonal estimates, with constants depending only on n, λ and the doubling constant of W:*

(i) e^{-t^2L},	*(ii)* $t^2Le^{-t^2L}$,	*(iii)* $t^4L^2e^{-t^2L}$,	*(iv)* $t\nabla e^{-t^2L}$,
(v) $e^{-t^2\widetilde{L}}$,	*(vi)* $t^2\widetilde{L}e^{-t^2\widetilde{L}}$,	*(vii)* $t^4\widetilde{L}^2e^{-t^2\widetilde{L}}$,	*(viii)* $te^{-t^2\widetilde{L}}\widetilde{\operatorname{div}}$,
(ix) $t\nabla e^{-t^2\widetilde{L}}$,	*(x)* $te^{-t^2L}\operatorname{div}$.		

Proof. Fix sets $E, F \subset \mathbb{R}^n$. We may assume that $d := \operatorname{dist}(E,F) > 5t$, as otherwise we may invoke Lemma 7.20.

The bound for (i) is a straightforward consequence of the pointwise bounds in Lemma 7.12. We omit the routine details. The off-diagonal estimates for (ii)-(iii) follow in the same way as for (i), using Remark 7.15.

Let us now treat (iv). We argue as in the proof of Caccioppoli's inequality. Let $u := e^{-t^2L}f$, with f supported in E. Choose $\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$, where $0 \leqslant \psi \leqslant 1$ satisfies $\psi \equiv 1$ on F, $\operatorname{dist}(\operatorname{supp}\psi, E) \geqslant d/2$ (denoting $d := \operatorname{dist}(E,F)$), and $\|\psi\|_\infty + d\,\|\nabla\psi\|_\infty + d^2\left\|\nabla^2\psi\right\|_\infty \leqslant C$. For future reference, we note that

$$|\nabla\psi|^2 + |\nabla^2\psi| \lesssim d^{-2}\,\mathbf{1}_{\operatorname{supp}\psi}\,. \tag{7.11}$$

Clearly, we have

$$\int_F |t\nabla u|^2\, W dx \leqslant t^2 \int_{\mathbb{R}^n} |\nabla u|^2\,\psi^2 W dx. \tag{7.12}$$

Compute, using (7.2) and the symmetry of A:

$$\int_{\mathbb{R}^n} A\nabla(u\psi)\cdot\nabla(u\psi)Wdx = \int_{\mathbb{R}^n} L(u\psi)u\psi W = -\sum_{i,j=1}^n \int_{\mathbb{R}^n} a_{ij}\partial_{x_i}\partial_{x_j}(u\psi)u\psi Wdx$$
$$= \sum_{i,j=1}^n \left(-\int_{\mathbb{R}^n} a_{ij}\partial_{x_i}\partial_{x_j}uu\psi^2 Wdx - 2\int_{\mathbb{R}^n} a_{ij}\partial_{x_i}u\partial_{x_j}\psi u\psi Wdx - \int_{\mathbb{R}^n} a_{ij}\partial_{x_i}\partial_{x_j}\psi u^2\psi Wdx \right)$$
$$= \int_{\mathbb{R}^n} Luu\psi^2 Wdx - 2\int_{\mathbb{R}^n} A\nabla u\cdot\nabla\psi u\psi Wdx + \int_{\mathbb{R}^n} L\psi u^2\psi Wdx. \quad (7.13)$$

Also,

$$\int_{\mathbb{R}^n} A\nabla(u\psi)\cdot\nabla(u\psi)Wdx \geqslant \lambda\int_{\mathbb{R}^n} \left|\nabla(u\psi)\right|^2 Wdx$$
$$= \lambda\left(\int_{\mathbb{R}^n} |\nabla u|^2\psi^2 Wdx + 2\int_{\mathbb{R}^n} \nabla u\cdot\nabla\psi u\psi Wdx + \int_{\mathbb{R}^n} |\nabla\psi|^2 u^2 Wdx\right) \quad (7.14)$$
$$\geqslant \lambda\int_{\mathbb{R}^n} |\nabla u|^2\psi^2 Wdx + 2\lambda\int_{\mathbb{R}^n} \nabla u\cdot\nabla\psi u\psi Wdx.$$

Combining (7.13) and (7.14), we may dominate the right hand side of (7.12) by

$$t^2\int_{\mathbb{R}^n} |\nabla u|^2\psi^2 Wdx \lesssim t^2\int_{\mathbb{R}^n} |\nabla u|\,|\nabla\psi|\,|u\psi|\,Wdx + t^2\int_{\mathbb{R}^n} \left|Luu\psi^2\right|W + t^2\int_{\mathbb{R}^n} \left|L\psi u^2\psi\right|W$$
$$=: I + II + III\,. \quad (7.15)$$

For $\varepsilon > 0$ to be chosen momentarily, we have

$$I \lesssim \varepsilon t^2\int_{\mathbb{R}^n} |\nabla u|^2\psi^2 Wdx + \varepsilon^{-1}\left(\frac{t}{d}\right)^2\int_{\operatorname{supp}\psi} u^2 Wdx\,,$$

where we have used "Cauchy's inequality with ε" in term I, and then (7.11). Choosing ε small enough, we may hide the first term, and use the off-diagonal bounds for (i) to obtain the desired bound for the second term, since $t \leqslant d$. We estimate term II using the Cauchy-Schwarz inequality, along with the off-diagonal bounds for (i) and (ii).

The operators (v)-(viii) are dual to the first four, and (x) is dual to (ix). It therefore remains to treat (ix). To this end, we now set $u := e^{-t^2\widetilde{L}}f$, with f supported in E, so with ψ as above, again using (7.12), we have

$$\int_F |t\nabla u|^2 Wdx \leqslant \int_{\mathbb{R}^n} |t\nabla u|^2\psi^2 Wdx \lesssim t^2\int_{\mathbb{R}^n} a_{ij}\partial_{x_j}u\partial_{x_i}u\,\psi^2 Wdx$$
$$= t^2\int_{\mathbb{R}^n} \partial_{x_j}u\,\partial_{x_i}(a_{ij}Wu)\,\psi^2 dx - t^2\int_{\mathbb{R}^n} u\partial_{x_j}u\,\partial_{x_i}(a_{ij}W)\,\psi^2 dx =: \widetilde{I} + \widetilde{II}\,,$$

where summation over i, j is understood. We first note that

$$\widetilde{I} = -t^2\int_{\mathbb{R}^n} u\,\partial_{x_j}\partial_{x_i}(a_{ij}Wu)\,\psi^2 dx - 2t^2\int_{\mathbb{R}^n} u\,\partial_{x_i}(a_{ij}Wu)\,\psi\partial_{x_j}\psi\,dx$$

$$= t^2 \int_{\operatorname{supp}\psi} u\widetilde{L}u\,\psi^2 W dx \;-\; 2t^2 \int_{\mathbb{R}^n} u\,\partial_{x_i}(a_{ij}Wu)\,\psi\partial_{x_j}\psi\,dx \;=:\; \widetilde{I}_1 + \widetilde{I}_2\,.$$

We obtain the desired estimate for $\widetilde{I}_1$ by the off-diagonal bounds for (v) and (vi) like for term II in (7.15). Also, integrating by parts,

$$\widetilde{I}_2 \;=\; 2t^2 \int_{\mathbb{R}^n} u\partial_{x_i}u\,a_{ij}\,\psi\partial_{x_j}\psi\,W dx \;+\; 2t^2 \int_{\mathbb{R}^n} u^2\,a_{ij}(\partial_{x_i}\psi\partial_{x_j}\psi + \psi\partial_{x_i}\partial_{x_j}\psi)\,W dx \;=:\; \widetilde{I}_{2,1} + \widetilde{I}_{2,2}\,.$$

We will cancel term $\widetilde{I}_{2,1}$ momentarily, but it may also be handled directly, exactly like term I in (7.15), using Cauchy's inequality with ε, hiding the small term, and bounding the other term using the off-diagonal bounds for (v). We estimate term $\widetilde{I}_{2,2}$ like terms I and III in (7.15), using (7.11), the fact that $t \leqslant d$, and the off-diagonal bounds for (v).

Since W is an adjoint solution,

$$\begin{aligned}\widetilde{II} \;&=\; t^2 \int u^2\,\psi\partial_{x_j}\psi\,\partial_{x_i}(a_{ij}W)dx \\ &=\; -2t^2 \int u\partial_{x_i}u\,\psi\partial_{x_j}\psi\,a_{ij}W dx \;-\; t^2 \int u^2\,(\partial_{x_i}\psi\partial_{x_j}\psi + \psi\partial_{x_i}\partial_{x_j}\psi)\,a_{ij}W dx \;=:\; \widetilde{II}_1 + \widetilde{II}_2\,.\end{aligned}$$

Observe that $\widetilde{II}_1 \equiv -\widetilde{I}_{2,1}$, and $\widetilde{II}_2 \equiv -\frac{1}{2}\widetilde{I}_{2,2}$. □

Let us finish with two elementary results that we prefer to state here in an abstract form to use them repeatedly later.

Lemma 7.24. *Let K_t be a convolution type operator as in Lemma 7.16, with a* `compactly supported kernel`*. If $h = h(x,t)$ is a function such that $f \mapsto h(\cdot,t)K_t f$ is bounded on L^2_W, uniformly in t (i.e. $\big\|h(\cdot,t)K_t\big\|_{L^2_W \to L^2_W} \lesssim 1$ uniformly in t), then $h(\cdot,t)K_t$ satisfies off-diagonal estimates.*

Lemma 7.25. *Let $\{U_t\}_{t>0}$ and $\{U'_t\}_{t>0}$ be two families of operators, each satisfying off-diagonal estimates, then the composition $U_t U'_t$ also satisfies off-diagonal estimates for each t.*

7.5.6 Estimates for differences and gradients

Lemma 7.26. *For $f \in \operatorname{Lip}$ and $t \leqslant \ell(Q)$, we have*

$$\int_Q \left|e^{-t^2L}f(x) - f(x)\right|^2 W(x)dx \lesssim t^2\,\|\nabla f\|^2_\infty\,W(Q)$$

and

$$\int_Q \left|e^{-t^2\widetilde{L}}f(x) - f(x)\right|^2 W(x)dx \lesssim t^2\,\|\nabla f\|^2_\infty\,W(Q),$$

where the implicit constants depend on n, λ and $[W]_{A_2}$.

Proof. Let us show the first estimate. Cover Q by non-overlapping cubes Q_k with sidelength $t/2 < \ell(Q_k) \leqslant t$. Note that $e^{-t^2L}\mathbf{1} = \mathbf{1}$, since $L\mathbf{1} = 0$. Letting $[f]_E$ denote the average of f over the set E,

$$\left\|e^{-t^2L}f - f\right\|^2_{L^2_W(Q)} = \sum_k \left\|e^{-t^2L}(f - [f]_{2Q_k}) - (f - [f]_{2Q_k})\right\|^2_{L^2_W(Q_k)} =: \sum_k A_k^2 \tag{7.16}$$

We then have

$$A_k \leqslant \left\| e^{-t^2L}\left((f-[f]_{2Q_k})\mathbf{1}_{2Q_k}\right)\right\|_{L^2_W(Q_k)} + \left\|f-[f]_{2Q_k}\right\|_{L^2_W(Q_k)} + \sum_{j=1}^{\infty}\left\| e^{-t^2L}\left((f-[f]_{2Q_k})\mathbf{1}_{2^{j+1}Q_k\setminus 2^jQ_k}\right)\right\|_{L^2_W(Q_k)} =: I^{(k)} + \sum_{j=1}^{\infty} II_j^{(k)}. \quad (7.17)$$

Using the boundedness of e^{-t^2L} from Lemma 7.20 and the weighted version of Poincaré's inequality [HKM06, 15.26], we deduce

$$I^{(k)} \lesssim \left\|f-[f]_{2Q_k}\right\|_{L^2_W(2Q_k)} \lesssim \ell(Q_k)\,\|\nabla f\|_{L^2_W(2Q_k)} \leqslant t\,\|\nabla f\|_{L^2_W(2Q_k)} \lesssim t\,\|\nabla f\|_{\infty}\,\sqrt{W(Q_k)}.$$

For convenience of notation in the rest of this argument, we replace the constant c by $4c$ in the off-diagonal estimates for e^{-t^2L} from Lemma 7.23. Thus,

$$\begin{aligned} II_j^{(k)} &\lesssim e^{-c4^{j+1}\frac{\ell(Q_k)^2}{t^2}}\left\|f-[f]_{2Q_k}\right\|_{L^2_W(2^{j+1}Q_k)} \\ &\leqslant e^{-c4^j}\left\|f-[f]_{2^{j+1}Q_k}\right\|_{L^2_W(2^{j+1}Q_k)} + \sum_{i=1}^{j} e^{-c4^j}\left\|[f]_{2^{i+1}Q_k}-[f]_{2^iQ_k}\right\|_{L^2_W(2^{j+1}Q_k)} \\ &\lesssim e^{-c4^j}\left\|f-[f]_{2^{j+1}Q_k}\right\|_{L^2_W(2^{j+1}Q_k)} + \sum_{i=1}^{j} e^{-c4^j}C_D^{(j-i)/2}\left\|f-[f]_{2^{i+1}Q_k}\right\|_{L^2_W(2^{i+1}Q_k)}. \end{aligned}$$

Hence, summing on j and interchanging the order of summation we obtain

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j} e^{-c4^j}C_D^{(j-i)/2}\left\|f-[f]_{2^{i+1}Q_k}\right\|_{L^2_W(2^{i+1}Q_k)} \lesssim \sum_{i=1}^{\infty} e^{-\frac{c}{2}4^i}\left\|f-[f]_{2^{i+1}Q_k}\right\|_{L^2_W(2^{i+1}Q_k)},$$

and therefore we have, by Poincaré's inequality,

$$\begin{aligned} \sum_{j=1}^{\infty} II_j^{(k)} &\lesssim \sum_{j=1}^{\infty} e^{-\frac{c}{2}4^j}\left\|f-[f]_{2^{j+1}Q_k}\right\|_{L^2_W(2^{j+1}Q_k)} \lesssim \sum_{j=1}^{\infty} e^{-\frac{c}{2}4^j}2^j\ell(Q_k)\,\|\nabla f\|_{L^2_W(2^{j+1}Q_k)} \\ &\lesssim t\sum_{j=1}^{\infty} e^{-\frac{c}{4}4^j}\,\|\nabla f\|_{\infty}\,\sqrt{W(2^{j+1}Q_k)} \lesssim t\,\|\nabla f\|_{\infty}\sum_{j=1}^{\infty} e^{-\frac{c}{5}4^j}\,\sqrt{W(Q_k)} \lesssim t\,\|\nabla f\|_{\infty}\,\sqrt{W(Q_k)}. \end{aligned}$$

Thus, going back to A_k (see (7.16)-(7.17)) we obtain

$$A_k \lesssim t\,\|\nabla f\|_{\infty}\,\sqrt{W(Q_k)},$$

and therefore, since the cubes Q_k are non-overlapping,

$$\left\|e^{-t^2L}f-f\right\|^2_{L^2_W(Q)} \lesssim t^2\,\|\nabla f\|^2_{\infty}\sum_k W(Q_k) = t^2\,\|\nabla f\|^2_{\infty}\,W(Q).$$

The corresponding estimate for $e^{-t^2\widetilde{L}}$ holds with exactly the same proof, for the operator is also bounded and satisfies off-diagonal estimates (see Lemmas 7.20 and 7.23), and $e^{-t^2\widetilde{L}}\mathbf{1} = \mathbf{1}$ since $\widetilde{L}\mathbf{1} = \frac{1}{W}L^*W = 0$. □

Lemma 7.27. *For $f \in \mathrm{Lip}$ and $t \leqslant \ell(Q)$, we have*

$$\int_Q \left|\nabla e^{-t^2 L} f(x)\right|^2 W(x)dx \lesssim \|\nabla f\|_\infty^2 \, W(Q)$$

and

$$\int_Q \left|\nabla e^{-t^2 \widetilde{L}} f(x)\right|^2 W(x)dx \lesssim \|\nabla f\|_\infty^2 \, W(Q),$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

Proof. The reader can check that the proof is the same as for Lemma 7.26, this time using the operators $t\nabla e^{-t^2 L}$ and $t\nabla e^{-t^2 \widetilde{L}}$. Indeed, the proof of the preceeding Lemma used only the boundedness of the operator (Lemma 7.20), the off-diagonal estimates (Lemma 7.23), and the conservation property which allows us to subtract constants. In this case,

$$\left\|\nabla e^{-t^2 L} f\right\|_{L^2_W(Q)}^2 = \sum_k \left\|\nabla e^{-t^2 L}(f - [f]_{2Q_k})\right\|_{L^2_W(Q_k)}^2 ,$$

and similarly for $\widetilde{L}$, because $e^{-t^2 L}\mathbf{1} = \mathbf{1} = e^{-t^2 \widetilde{L}}\mathbf{1}$, as before. After this, the proof is the same as that of Lemma 7.26. □

7.6 The Kato problem for L

In this section our goal is to prove the following result:

Theorem 7.28. *It holds*

$$\left\| \sqrt{L} f\right\|_{L^2_W} \lesssim \|\nabla f\|_{L^2_W} ,$$

where the hidden constant depends on n, λ and $[W]_{A_2}$.

As noted above (see Remark 7.10), in the case that the coefficient matrix has small enough BMO norm, we could deduce Theorem 7.28 as an easy consequence of certain known results. Instead, following the easier part of our proof of Theorem 7.34 below, we shall give a self-contained, direct argument, which does not rely on an explicit assumption of smallness in BMO, but only on the validity of (7.2), and the assumption that $W \in A_2$.

To prove the theorem, let us use the representation of the square root operator via the Functional Calculus formula

$$\sqrt{L} f = a \int_0^\infty t^3 L^2 e^{-2t^2 L} f \frac{dt}{t},$$

where $a = (\int_0^\infty t^3 e^{-2t^2} \frac{dt}{t})^{-1} = \sqrt{\frac{128}{\pi}}$ is just a normalizing constant, to estimate, using duality and later Cauchy-Schwarz,

$$\left|\langle \sqrt{L} f, g\rangle_{L^2_W}\right|^2 \leqslant a^2 \left(\int_0^\infty \left\|tLe^{-t^2 L} f(x)\right\|_{L^2_W}^2 \frac{dt}{t}\right) \left(\int_0^\infty \left\|t^2 \widetilde{L} e^{-t^2 \widetilde{L}} g(x)\right\|_{L^2_W}^2 \frac{dt}{t}\right).$$

With this decomposition, we will finish the proof of Theorem 7.28 by duality once we prove the following Lemma 7.29 and Lemma 7.31.

The desired bound for the second factor is the following.

Lemma 7.29. *It holds*

$$\int_0^\infty \int_{\mathbb{R}^n} \left| t^2 \widetilde{L} e^{-t^2 \widetilde{L}} g(x) \right|^2 W(x) dx \frac{dt}{t} \lesssim \|g\|_{L^2_W}^2 , \tag{7.18}$$

where the implicit constant depends only on n, λ and the doubling constant for W.

We could obtain the conclusion of the lemma by invoking the abstract McIntosh and Yagi theorem [Ya84], [Mc86], but instead, we will give a self-contained and more elementary proof, using quasi-orthogonality arguments.

Proof. We abbreviate $V_t := t^2 L e^{-t^2 L}$. Its adjoint within L^2_W is $\widetilde{V}_t := t^2 \widetilde{L} e^{-t^2 \widetilde{L}}$. To make the argument rigorous, given a small positive ε, we set $V_t \equiv 0 \equiv \widetilde{V}_t$ whenever $t \leqslant \varepsilon$ or $t \geqslant 1/\varepsilon$, and we obtain quantitative bounds that are uniform in ε. We compute, using duality, Fubini and Cauchy-Schwarz,

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^n} |\widetilde{V}_t g(x)|^2 W(x) dx \frac{dt}{t} &= \int_0^\infty \int_{\mathbb{R}^n} \widetilde{V}_t g(x) \widetilde{V}_t g(x) W(x) dx \frac{dt}{t} \\
&= \int_0^\infty \int_{\mathbb{R}^n} g(x) V_t \widetilde{V}_t g(x) W(x) dx \frac{dt}{t} = \int_{\mathbb{R}^n} \int_0^\infty V_t \widetilde{V}_t g(x) \frac{dt}{t} g(x) W(x) dx \\
&\leqslant \|g\|_{L^2_W} \left\| \int_0^\infty V_t \widetilde{V}_t g \frac{dt}{t} \right\|_{L^2_W} .
\end{aligned} \tag{7.19}$$

To deal with the second term, let us first establish a useful fact:

Claim 7.30. *It holds, for any $t, s > 0$,*

$$\|\widetilde{V}_t V_s\|_{L^2_W \to L^2_W} \lesssim \min\left\{ \frac{s}{t}, \frac{t}{s} \right\} .$$

Proof of the claim. We may assume that $\varepsilon < s, t < 1/\varepsilon$. If $s \leqslant t$, we can compute using (7.8) and the uniform bounds from Lemma 7.20,

$$\begin{aligned}
\|\widetilde{V}_t V_s\| = \|t^2 s^2 \widetilde{L} e^{-t^2 \widetilde{L}} L e^{-s^2 L}\| &\lesssim \|t^2 s^2 \widetilde{L} e^{-t^2 \widetilde{L}} \widetilde{L} e^{-s^2 L}\| + \|t^2 s^2 \widetilde{L} e^{-t^2 \widetilde{L}} \operatorname{div} A \nabla e^{-s^2 L}\| \\
&\leqslant \frac{s^2}{t^2} \|t^4 \widetilde{L}^2 e^{-t^2 \widetilde{L}}\| \|e^{-s^2 L}\| + \frac{s}{t} \|t^3 \widetilde{L} e^{-t^2 \widetilde{L}} \operatorname{div} \|A\| \| s \nabla e^{-s^2 L}\| \lesssim \frac{s^2}{t^2} + \frac{s}{t} \lesssim \frac{s}{t}.
\end{aligned}$$

In a similar fashion we can compute, when $s > t$, using again (7.8) and Lemma 7.20,

$$\begin{aligned}
\|\widetilde{V}_t V_s\| = \|t^2 s^2 e^{-t^2 \widetilde{L}} \widetilde{L} L e^{-s^2 L}\| &\lesssim \|t^2 s^2 e^{-t^2 \widetilde{L}} L L e^{-s^2 L}\| + \|t^2 s^2 e^{-t^2 \widetilde{L}} \operatorname{div} A \nabla L e^{-s^2 L}\| \\
&\leqslant \frac{t^2}{s^2} \|e^{-t^2 \widetilde{L}}\| \|s^4 L^2 e^{-s^2 L}\| + \frac{t}{s} \|t e^{-t^2 \widetilde{L}} \operatorname{div} \| \|A\| \| s^3 \nabla L e^{-s^2 L}\| \lesssim \frac{t^2}{s^2} + \frac{t}{s} \lesssim \frac{t}{s}.
\end{aligned}$$

□

With this almost-orthogonality result, we can estimate the last term in (7.19) as follows:

$$\left\| \int_0^\infty V_t \widetilde{V}_t g \frac{dt}{t} \right\|_{L^2_W}^2 = \int_{\mathbb{R}^n} \left(\int_0^\infty V_t \widetilde{V}_t g(x) \frac{dt}{t} \right) \left(\int_0^\infty V_s \widetilde{V}_s g(x) \frac{ds}{s} \right) W(x) dx$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} V_t \widetilde{V}_t g(x) V_s \widetilde{V}_s g(x) W(x) dx \frac{dt}{t} \frac{ds}{s} \\
&= \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \widetilde{V}_t g(x) \widetilde{V}_t V_s \widetilde{V}_s g(x) W(x) dx \frac{dt}{t} \frac{ds}{s} \\
&\leqslant \int_0^\infty \int_0^\infty \left(\int_{\mathbb{R}^n} \left| \widetilde{V}_t g \right|^2 W dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \left| \widetilde{V}_t V_s \widetilde{V}_s g \right|^2 W dx \right)^{1/2} \frac{dt}{t} \frac{ds}{s} \\
&\leqslant \int_0^\infty \int_0^\infty \min\left\{ \frac{s}{t}, \frac{t}{s} \right\} \left(\int_{\mathbb{R}^n} \left| \widetilde{V}_t g \right|^2 W dx \right)^{1/2} \left(\int_{\mathbb{R}^n} \left| \widetilde{V}_s g \right|^2 W dx \right)^{1/2} \frac{dt}{t} \frac{ds}{s} \\
&\lesssim \int_0^\infty \int_0^\infty \min\left\{ \frac{s}{t}, \frac{t}{s} \right\} \int_{\mathbb{R}^n} \left| \widetilde{V}_t g(x) \right|^2 W(x) dx \frac{dt}{t} \frac{ds}{s} \\
&\lesssim \int_0^\infty \int_{\mathbb{R}^n} \left| \widetilde{V}_t g(x) \right|^2 W(x) dx \frac{dt}{t},
\end{aligned}
$$

Plugging this estimate into (7.19), we have

$$
\int_0^\infty \int_{\mathbb{R}^n} \left| \widetilde{V}_t g(x) \right|^2 W(x) dx \frac{dt}{t} \lesssim \|g\|_{L^2_W} \left(\int_0^\infty \int_{\mathbb{R}^n} \left| \widetilde{V}_t g(x) \right|^2 W(x) dx \frac{dt}{t} \right)^{1/2},
$$

from which the result readily follows (recall that we have effectively truncated so $\varepsilon < t < 1/\varepsilon$, hence the integrals are finite). □

Let us turn our attention to the other square function estimate.

Lemma 7.31. *It holds*

$$
\int_0^\infty \int_{\mathbb{R}^n} \left| tLe^{-t^2 L} f(x) \right|^2 W(x) dx \frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_W},
$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

We will devote the rest of the section to the proof of Lemma 7.31. As above, set $V_t := t^2 L e^{-t^2 L}$ and decompose, with the help of an approximate identity P_t,

$$
tLe^{-t^2 L} = t^{-1} V_t (I - P_t) + t^{-1} V_t P_t =: R_t + T_t. \tag{7.20}
$$

The proof of Lemma 7.31, and hence of Theorem 7.28, will come immediately from the next two lemmas.

Lemma 7.32. *With the notations of (7.20), we have*

$$
\int_0^\infty \|R_t f\|^2_{L^2_W} \frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_W},
$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

Proof. Choose the approximate identity $P_t := e^{-t^2(-\Delta)}$. With it, we can compute, using the Fundamental Theorem of Calculus,

$$
\begin{aligned}
R_t f &= t^{-1} V_t (I - P_t) f = V_t \left(\frac{1}{t} (I - P_t) f \right) = V_t \left(-\frac{1}{t} \int_0^t \frac{\partial}{\partial s} P_s f ds \right) \\
&= -2 V_t \left(\frac{1}{t} \int_0^t s P_s \Delta f ds \right) = -2 V_t \left(\frac{1}{t} \int_0^t s P_s \operatorname{div} \nabla f ds \right) =: 2 V_t \left(\frac{1}{t} \int_0^t \vec{Q}_s \nabla f ds \right).
\end{aligned}
$$

Now, using the boundedness on L^2_W of $V_t = t^2 L e^{-t^2 L}$ (see Lemma 7.20), Hardy's inequality and the fact that $\vec{Q}_s$ satisfies the square function estimate of Lemma 7.17 (see Remark 7.18), we obtain the desired estimate:

$$\int_0^\infty \|R_t f\|^2_{L^2_W} \frac{dt}{t} = \int_0^\infty \int_{\mathbb{R}^n} \left| 2V_t \left(\frac{1}{t} \int_0^t \vec{Q}_s \nabla f ds \right)(x) \right|^2 W(x) dx \frac{dt}{t}$$
$$\lesssim \int_0^\infty \int_{\mathbb{R}^n} \left| \frac{1}{t} \int_0^t \vec{Q}_s \nabla f(x) ds \right|^2 W(x) dx \frac{dt}{t} \lesssim \int_0^\infty \int_{\mathbb{R}^n} \left| \vec{Q}_t \nabla f(x) \right|^2 W(x) dx \frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_W}.$$

□

Lemma 7.33. *With the notations of* (7.20)*, we have*

$$\int_0^\infty \|T_t f\|^2_{L^2_W} \frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_W},$$

where the hidden constant depends on n, λ and $[W]_{A_2}$.

Proof. Simply compute, using the boundedness of $e^{-t^2 L}$ from Lemma 7.20, and the square function bounds of Lemma 7.17 (see Remark 7.18),

$$\int_0^\infty \|T_t f\|^2_{L^2_W} \frac{dt}{t} \lesssim \int_0^\infty \|tLP_t f\|^2_{L^2_W} \frac{dt}{t} \lesssim \sum_{i,j=1}^n \int_0^\infty \left\| t\partial_{x_i}\partial_{x_j} P_t f \right\|^2_{L^2_W} \frac{dt}{t}$$
$$= \sum_{i,j=1}^n \int_0^\infty \left\| t\partial_{x_i} P_t \partial_{x_j} f \right\|^2_{L^2_W} \frac{dt}{t} =: \sum_{i,j=1}^n \int_0^\infty \left\| Q_t^{(i)} \partial_{x_j} f \right\|^2_{L^2_W} \frac{dt}{t}$$
$$\lesssim \sum_{j=1}^n \left\| \partial_{x_j} f \right\|^2_{L^2_W} = \|\nabla f\|^2_{L^2_W}.$$

□

7.7 The Kato problem for $\widetilde{L}$

In this section our goal is to prove the following result, which is really the main result in this chapter:

Theorem 7.34. *It holds*

$$\| \sqrt{\widetilde{L}} f \|_{L^2_W} \lesssim \|\nabla f\|_{L^2_W},$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

7.7.1 Reduction to a quadratic estimate

To prove Theorem 7.34, let us again use the representation of the square root operator via the formula

$$\sqrt{\widetilde{L}} f = a \int_0^\infty t^3 \widetilde{L}^2 e^{-2t^2 \widetilde{L}} f \frac{dt}{t},$$

so that

$$\left|\langle \sqrt{\widetilde{L}} f, g\rangle_{L^2_W}\right|^2 \leqslant a^2 \left(\int_0^\infty \left\|t\widetilde{L}e^{-t^2\widetilde{L}} f(x)\right\|^2_{L^2_W} \frac{dt}{t}\right)\left(\int_0^\infty \left\|t^2 L e^{-t^2 L} g(x)\right\|^2_{L^2_W} \frac{dt}{t}\right).$$

Theorem 7.34 then follows immediately from Lemma 7.35 and Theorem 7.36 below.

We estimate the second square function via the following lemma.

Lemma 7.35. *It holds*

$$\int_0^\infty \int_{\mathbb{R}^n} \left|t^2 L e^{-t^2 L} g(x)\right|^2 W(x)dx\frac{dt}{t} \lesssim \|g\|^2_{L^2_W},$$

where the implicit constant depends on n, λ and $[W]_{A_\infty}$.

Proof. The lemma can be proved either by invoking the McIntosh and Yagi theorem [Ya84], [Mc86], or via a self-contained elementary proof using quasi-orthogonality. For the latter path, the proof follows that of Lemma 7.29 *mutatis mutandis*, simply reversing the roles of V_t and $\widetilde{V}_t$. We omit the details. □

Let us now turn the attention to the other square function estimate, which is in fact the core of this chapter.

Theorem 7.36. *It holds*

$$\int_0^\infty \int_{\mathbb{R}^n} \left|t\widetilde{L}e^{-t^2\widetilde{L}} f(x)\right|^2 W(x)dx\frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_W},$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

The rest of this section is devoted to the proof of Theorem 7.36. We start by splitting

$$t\widetilde{L}e^{-t^2\widetilde{L}} = t\widetilde{L}e^{-t^2\widetilde{L}}(I - P_t^2) + t\widetilde{L}e^{-t^2\widetilde{L}}P_t^2 =: \widetilde{R}_t + \widetilde{T}_t, \tag{7.21}$$

where P_t is a nice approximate identity with a smooth compactly supported convolution kernel $\varphi_t(x) = t^{-n}\varphi(x/t)$, which we take to be even. For future reference, let us record the following well-known observation:

$$t\partial_t\big(\widehat{\varphi}(t\xi)\big)^2 = 2(\nabla\widehat{\varphi})(t\xi)\cdot t\xi\widehat{\varphi}(t\xi) = c\,\widehat{(x\varphi(x))}(t\xi)\cdot\widehat{(\nabla\varphi)}(t\xi) =: c\,\widehat{\psi^{(1)}}(t\xi)\widehat{\psi^{(2)}}(t\xi),$$

where c is a harmless constant, and $\psi^{(1)}(x) := x\varphi(x)$ and $\psi^{(2)} := \nabla\varphi$ are both $\mathscr{C}^\infty_c$ functions with mean value zero (here we are using that φ is even, in the case of $\psi^{(1)}$). Hence,

$$t\partial_t P_t^2 = c\,Q_t^{(1)}Q_t^{(2)},$$

where $Q_t^{(k)}$ is the convolution kernel with kernel $\psi_t^{(k)}(x) := t^{-n}\psi^{(k)}(x/t)$, $k = 1, 2$, and therefore each of $Q_t^{(1)}$, $Q_t^{(2)}$ satisfies the square function bound of Lemma 7.17 (and each is bounded on L^2_W, uniformly in t).

Lemma 7.37. *With the notations of (7.21), we have*

$$\int_0^\infty \|\widetilde{R}_t f\|^2_{L^2_W}\frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_W}, \tag{7.22}$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

Proof. Using the preceding observations, we may follow the proof of Lemma 7.32, invoking Lemma 7.20 to obtain that $t^2\widetilde{L}e^{-t^2\widetilde{L}}$ is L^2_W bounded, to obtain (7.22). □

Applying now (7.8) to $u = P_t^2 f$ we obtain

$$\widetilde{T}_t f = te^{-t^2\widetilde{L}}\widetilde{L}P_t^2 f = -te^{-t^2\widetilde{L}}LP_t^2 f - 2te^{-t^2\widetilde{L}}\widetilde{\operatorname{div}}(A\nabla(P_t^2 f)). \tag{7.23}$$

Lemma 7.38. *We have*

$$\int_0^\infty \left\| te^{-t^2\widetilde{L}}LP_t^2 f\right\|^2_{L^2_W}\frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_W},$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

Proof. The proof is the same as that in Lemma 7.33 once we use that $e^{-t^2\widetilde{L}} : L^2_W \to L^2_W$ is uniformly bounded by Lemma 7.20, and that P_t are uniformly bounded on L^2_W by Lemma 7.16. □

Therefore, to finish the proof of Theorem 7.36 (and hence of Theorem 7.34), it remains to show

$$\int_0^\infty \left\| te^{-t^2\widetilde{L}}\widetilde{\operatorname{div}}(A\nabla(P_t^2 f))\right\|^2_{L^2_W}\frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_W}. \tag{7.24}$$

7.7.2 Reduction to a Carleson measure estimate

For $\mathbf{g} = (g_1, g_2, ..., g_n)$, write

$$\theta_t\mathbf{g} := te^{-t^2\widetilde{L}}\widetilde{\operatorname{div}}(A\mathbf{g})\left(= te^{-t^2\widetilde{L}}\frac{1}{W}\operatorname{div}(WA\mathbf{g})\right). \tag{7.25}$$

With this notation, the remaining estimate (7.24) becomes

$$\int_0^\infty \left\|\theta_t\nabla(P_t^2 f)\right\|^2_{L^2_W}\frac{dt}{t} \lesssim \|\nabla f\|^2_{L^2_W}. \tag{7.26}$$

Let us also define the operator

$$\widetilde{\theta}_t\mathbf{g} := te^{-t^2\widetilde{L}}\left(\widetilde{\operatorname{div}}(A\mathbf{g}) - \frac{1}{2}\sum_{i,j=1}^n a_{ij}\partial_{x_i}\mathbf{g}_j\right),$$

so that, taking $\mathbf{g} = \nabla u$, and using (7.8),

$$\widetilde{\theta}_t\nabla u = -\frac{1}{2}te^{-t^2\widetilde{L}}\widetilde{L}u. \tag{7.27}$$

It will be convenient to use both operators at different stages of the proof. Note that trivially, $\widetilde{\theta}_t\mathbf{e} = \theta_t\mathbf{e}$, for any constant vector $\mathbf{e}$. In particular, if $\mathbf{1}$ denotes the $n \times n$ identity matrix, then

$$\widetilde{\theta}_t\mathbf{1} = \theta_t\mathbf{1},$$

where we naturally define $\widetilde{\theta}_t\mathbf{1} = \theta_t\mathbf{1}$ as a vector-valued function whose k^{th} entry is $\widetilde{\theta}_t\mathbf{e}^k = \theta_t\mathbf{e}^k$, with $\mathbf{e}^k$ equal to the standard unit basis vector in the x_k direction.

To prove (7.24), as in the divergence form case treated in [AHLMT02], we begin with a "$T1$" reduction.

Lemma 7.39. *We have*

$$\int_0^\infty \left\| \theta_t P_t^2 \mathbf{g} - (\theta_t \mathbf{1}) \cdot (P_t^2 \mathbf{g}) \right\|_{L^2_W}^2 \frac{dt}{t} \lesssim \|\mathbf{g}\|_{L^2_W}^2 ,$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

Proof. Write $U_t \mathbf{g} := \theta_t P_t^2 \mathbf{g} - (\theta_t \mathbf{1}) \cdot (P_t^2 \mathbf{g})$. By Lemma 7.19, it suffices to show that

$$\|U_t\|_{L^2_W \to L^2_W} \lesssim 1$$

uniformly on t, and for some $\alpha > 0$, and for any nice operator Q_s as in Lemma 7.17, with a compactly supported kernel,

$$\|U_t Q_s\|_{L^2_W \to L^2_W} \lesssim \min\left\{ \frac{s}{t}, \frac{t}{s} \right\}^\alpha .$$

These two estimates, and hence the conclusion of Lemma 7.39, will follow at once from the next claims and Lemma 7.16.

Claim 7.40. *We have, uniformly on t,*

$$\|\theta_t P_t\|_{L^2_W \to L^2_W} \lesssim 1,$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

Proof of the claim. Just compute, with the aid of Lemmas 7.16 and 7.20,

$$\|\theta_t P_t\|_{L^2_W \to L^2_W} \leqslant \|t e^{-t^2 \tilde{L}} \widetilde{\operatorname{div}}\|_{L^2_W \to L^2_W} \|A\|_\infty \|P_t\|_{L^2_W \to L^2_W} \lesssim 1.$$

□

Claim 7.41. *We have, uniformly on t,*

$$\left\| (\theta_t \mathbf{1}) \cdot P_t \right\|_{L^2_W \to L^2_W} \lesssim 1,$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

Proof of the claim. This proof will follow that of [CR15, (4.10)]. Let us cover $\mathbb{R}^n$ by cubes Q_k satisfying $t/2 < \ell(Q_k) \leqslant t$. In this way, we obtain

$$\int_{\mathbb{R}^n} \left| (\theta_t \mathbf{1})(x) \cdot P_t \mathbf{g}(x) \right|^2 W(x) dx = \sum_k \int_{Q_k} \left| (\theta_t \mathbf{1})(x) \cdot P_t \mathbf{g}(x) \right|^2 W(x) dx.$$

We first establish an L^∞ bound for $P_t \mathbf{g}(x)$, in the cube Q_k. Note that for $x \in Q_k$ we have that $P_t \mathbf{g}(x) = P_t(\mathbf{g} \mathbf{1}_{3Q_k})(x)$ because $t \leqslant 2\ell(Q_k)$ (and $\operatorname{supp} \varphi \subset B(0,1)$). Hence,

$$\begin{aligned}
\left| P_t(\mathbf{g} \mathbf{1}_{3Q_k})(x) \right|^2 &\leqslant \left(\|\varphi_t\|_\infty \int_{3Q_k} \left| \mathbf{g}(y) \right| dy \right)^2 \leqslant \left(t^{-n} \int_{3Q_k} \left| \mathbf{g}(y) \right| dy \right)^2 \\
&\approx \left(\fint_{3Q_k} \left| \mathbf{g}(y) \right| dy \right)^2 \leqslant \frac{1}{|3Q_k|^2} \int_{3Q_k} \left| \mathbf{g}(y) \right|^2 W(y) dy \int_{3Q_k} W^{-1}(y) dy \\
&= \frac{W(3Q_k) W^{-1}(3Q_k)}{|3Q_k|^2} \fint_{3Q_k} \left| \mathbf{g}(y) \right|^2 W(y) dy \lesssim \fint_{3Q_k} \left| \mathbf{g}(y) \right|^2 W(y) dy
\end{aligned}$$

where we used the definition of an A_2 weight in the last step (see Definition 2.3).

We next claim that

$$\frac{1}{W(Q_k)}\int_{Q_k}\left|(\theta_t\mathbf{b})(x)\right|^2 W(x)dx \lesssim \|\mathbf{b}\|_\infty^2\,. \tag{7.28}$$

Taking this claim for granted momentarily, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\left|(\theta_t\mathbf{1})(x)\cdot P_t\mathbf{g}(x)\right|^2 W(x)dx &\lesssim \sum_k \int_{Q_k}\left|(\theta_t\mathbf{1})(x)\right|^2 \fint_{3Q_k}\left|\mathbf{g}(y)\right|^2 W(y)dy\, W(x)dx\\ &\lesssim \sum_k \frac{1}{W(Q_k)}\int_{Q_k}\left|(\theta_t\mathbf{1})(x)\right|^2 W(x)dx\int_{3Q_k}\left|\mathbf{g}(y)\right|^2 W(y)dy\\ &\lesssim \sum_k\int_{3Q_k}\left|\mathbf{g}(y)\right|^2 W(y)dy \lesssim \int_{\mathbb{R}^n}\left|\mathbf{g}(y)\right|^2 W(y)dy,\end{aligned}$$

where in the last two steps we used first (7.28), and then the bounded overlap property of the cubes $3Q_k$.

It remains to verify (7.28). We dualize: choose $\mathbf{h}=(h_1,h_2,...,h_n)\in L^2_W$, with $\operatorname{supp}\mathbf{h}\subset Q_k$, and write

$$\begin{aligned}\int_{\mathbb{R}^n}\theta_t\mathbf{b}(x)\cdot\mathbf{h}(x)W(x)dx &= \int_{\mathbb{R}^n} te^{-t^2\widetilde{L}}\widetilde{\operatorname{div}}A\mathbf{b}(x)\cdot\mathbf{h}(x)W(x)dx\\ &= \int_{\mathbb{R}^n} A\mathbf{b}(x)\cdot t\nabla e^{-t^2L}\mathbf{h}(x)W(x)dx \lesssim \|\mathbf{b}\|_\infty\int_{\mathbb{R}^n}\left|t\nabla e^{-t^2L}\mathbf{h}(x)\right|W(x)dx\\ &\leqslant \|\mathbf{b}\|_\infty\left(\int_{2Q_k}\left|t\nabla e^{-t^2L}\mathbf{h}(x)\right|W(x)dx+\sum_{j=2}^\infty\int_{2^jQ_k\setminus 2^{j-1}Q_k}\left|t\nabla e^{-t^2L}\mathbf{h}(x)\right|W(x)dx\right)\\ &=: \|\mathbf{b}\|_\infty\left(I^{(k)}+II^{(k)}\right).\end{aligned}\tag{7.29}$$

For the first term we may simply compute, using Jensen's inequality and the boundedness of $t\nabla e^{-t^2L}$ from Lemma 7.20,

$$I^{(k)}\lesssim\left(W(Q_k)\int_{2Q_k}\left|t\nabla e^{-t^2L}\mathbf{h}(x)\right|^2W(x)dx\right)^{1/2}\lesssim\sqrt{W(Q_k)}\,\|\mathbf{h}\|_{L^2_W}\,.$$

For the second term we use Jensen again, and later the off-diagonal estimates from Lemma 7.23 (taking advantage of $\ell(Q_k)\approx t$) to obtain

$$\begin{aligned}II^{(k)} &\lesssim \sum_{j=2}^\infty\left(W(2^jQ_k)\int_{2^jQ_k\setminus2^{j-1}Q_k}\left|t\nabla e^{-t^2L}\mathbf{h}(x)\right|^2W(x)dx\right)^{1/2}\\ &\lesssim\sum_{j=2}^\infty\left(C_D^jW(Q_k)e^{-c4^j}\int_{Q_k}\left|\mathbf{h}(x)\right|^2W(x)dx\right)^{1/2}\lesssim\sqrt{W(Q_k)}\,\|\mathbf{h}\|_{L^2_W}\,.\end{aligned}$$

With the estimates for $I^{(k)}$ and $II^{(k)}$, we can substitute back in (7.29) and obtain

$$\int_{\mathbb{R}^n}\theta_t\mathbf{b}(x)\cdot\mathbf{h}(x)W(x)dx\lesssim\sqrt{W(Q_k)}\,\|\mathbf{b}\|_\infty\,\|\mathbf{h}\|_{L^2_W}\,,$$

which after squaring gives (7.28) by duality, as desired. This completes the proof of Claim 7.41. □

Claim 7.42. *Suppose $s \leqslant t$. Then*

$$\|U_t Q_s\|_{L^2_W \to L^2_W} \lesssim \frac{s}{t}\,.$$

Proof of the claim. Note that we have the pointwise estimate

$$|P_t Q_s f| \lesssim \frac{s}{t}\,\mathcal{M}^2 f\,, \tag{7.30}$$

where $\mathcal{M}^2 := \mathcal{M} \circ \mathcal{M}$ is the iterated Hardy-Littlewood maximal operator (with respect to Lebesgue measure). One may verify (7.30) by a standard argument using the size estimates and compact support of the kernels of P_t and Q_s, along with the smoothness of the former, and the cancellation property of the latter. We omit the well-known details. Since $\mathcal{M}$ is bounded on L^2_W (recall that $W \in A_2$), we find, with the aid of Claims 7.40 and 7.41, and (7.30):

$$\begin{aligned}\|U_t Q_s\|_{L^2_W \to L^2_W} &\leqslant \|\theta_t P_t\|_{L^2_W \to L^2_W}\, \|P_t Q_s\|_{L^2_W \to L^2_W} + \left\|(\theta_t \mathbf{1}) \cdot P_t\right\|_{L^2_W \to L^2_W}\, \|P_t Q_s\|_{L^2_W \to L^2_W} \\ &\lesssim \|P_t Q_s\|_{L^2_W \to L^2_W} \lesssim \frac{s}{t}\,.\end{aligned}$$

□

Claim 7.43. *We have, uniformly on t,*

$$\|U_t \mathbf{g}\|_{L^2_W} \lesssim t\, \|\nabla \mathbf{g}\|_{L^2_W}\,.$$

Proof. The proof is inspired by [AAAHK11, Lemma 3.5], and in fact is similar in spirit to that of Lemma 7.26, relying strongly in a decomposition in subcubes of the right size to use Poincaré's inequality, and some boundedness and off-diagonal estimates. Nevertheless, let us show it in detail, because some parts of it will be reused later. Cover $\mathbb{R}^n$ by a grid of non-overlapping dyadic cubes Q_k with sidelength $t/2 < \ell(Q_k) \leqslant t$. Using the easy fact that $U_t \mathbf{1} = 0$ we compute

$$\begin{aligned}\|U_t \mathbf{g}\|^2_{L^2_W} &= \sum_k \left\|U_t\left(\mathbf{g} - [\mathbf{g}]_{2Q_k}\right)\right\|^2_{L^2_W(Q_k)} \\ &\lesssim \sum_k \left\|\theta_t P_t^2\left(\mathbf{g} - [\mathbf{g}]_{2Q_k}\right)\right\|^2_{L^2_W(Q_k)} + \sum_k \left\|(\theta_t \mathbf{1}) \cdot P_t^2\left(\mathbf{g} - [\mathbf{g}]_{2Q_k}\right)\right\|^2_{L^2_W(Q_k)} =: A + B. \end{aligned} \tag{7.31}$$

Let us first deal with A, denoting $S_t := \theta_t P_t^2$ because we intend to reuse some computations later on. For each term in the series, simply using linearity and the triangle inequality

$$\begin{aligned}\left\|S_t\left(\mathbf{g} - [\mathbf{g}]_{2Q_k}\right)\right\|_{L^2_W(Q_k)} &\leqslant \left\|S_t\left(\left(\mathbf{g} - [\mathbf{g}]_{2Q_k}\right)\mathbf{1}_{2Q_k}\right)\right\|_{L^2_W(Q_k)} \\ &\quad + \sum_{j=1}^{\infty} \left\|S_t\left(\left(\mathbf{g} - [\mathbf{g}]_{2Q_k}\right)\mathbf{1}_{2^{j+1}Q_k \setminus 2^j Q_k}\right)\right\|_{L^2_W(Q_k)} =: I^{(k)} + \sum_{j=1}^{\infty} II_j^{(k)}. \end{aligned} \tag{7.32}$$

Using the boundedness of S_t (in this case, this follows from Lemmas 7.16 and 7.20) and Poincaré's inequality, we deduce

$$I^{(k)} \lesssim \left\|\mathbf{g} - [\mathbf{g}]_{2Q_k}\right\|_{L^2_W(2Q_k)} \lesssim \ell(Q_k)\, \|\nabla \mathbf{g}\|_{L^2_W(2Q_k)} \leqslant t\, \|\nabla \mathbf{g}\|_{L^2_W(2Q_k)}\,. \tag{7.33}$$

And for the other terms, we can use the off-diagonal estimates for S_t (in this case, this follows from Lemmas 7.23 and 7.25), and taking advantage of $\ell(Q_k) \approx t$ and Poincaré, we obtain, similarly to the situation in Lemma 7.26,

$$\begin{aligned}\sum_{j=1}^{\infty} II_j^{(k)} &\lesssim e^{-c4^j} \left\|\mathbf{g} - [\mathbf{g}]_{2Q_k}\right\|_{L^2_W(2^{j+1}Q_k)} \\ &\lesssim \sum_{j=1}^{\infty} e^{-c4^j} \left\|\mathbf{g} - [\mathbf{g}]_{2^{j+1}Q_k}\right\|_{L^2_W(2^{j+1}Q_k)} + \sum_{j=1}^{\infty}\sum_{i=1}^{j} e^{-c4^j} C_D^{(j-i)/2} \left\|\mathbf{g} - [\mathbf{g}]_{2^{i+1}Q_k}\right\|_{L^2_W(2^{i+1}Q_k)} \\ &\lesssim \sum_{j=1}^{\infty} e^{-\frac{c}{2}4^j} 2^j \ell(Q_k) \|\nabla \mathbf{g}\|_{L^2_W(2^{j+1}Q_k)} \lesssim t \sum_{j=1}^{\infty} e^{-\frac{c}{4}4^j} \|\nabla \mathbf{g}\|_{L^2_W(2^{j+1}Q_k)}.\end{aligned}$$

Thus, going back to (7.32) we obtain

$$\left\|S_t(\mathbf{g} - [\mathbf{g}]_{2Q_k})\right\|_{L^2_W(Q_k)} \lesssim t \|\nabla \mathbf{g}\|_{L^2_W(2Q_k)} + t \sum_{j=1}^{\infty} e^{-\frac{c}{4}4^j} \|\nabla \mathbf{g}\|_{L^2_W(2^{j+1}Q_k)},$$

and hence

$$A \lesssim t^2 \sum_k \|\nabla \mathbf{g}\|^2_{L^2_W(2Q_k)} + t^2 \sum_k \left(\sum_{j=1}^{\infty} e^{-\frac{c}{4}4^j} \|\nabla \mathbf{g}\|_{L^2_W(2^{j+1}Q_k)} \right)^2 =: t^2(A_1 + A_2).$$

By bounded overlap of the cubes $2Q_k$ we easily get

$$A_1 \lesssim \|\nabla \mathbf{g}\|^2_{L^2_W}.$$

For the other term, we note that $|x-y| \lesssim 2^j \ell(Q_k) \approx 2^j t$, whenever $x \in Q_k$, and $y \in 2^{j+1}Q_k$. We further note that $W(Q_k) \approx W(B_t(x))$, for $x \in Q_k$, and that for all $x \in \mathbb{R}^n$,

$$e^{-\frac{c}{4}4^j} \int_{|x-y| \lesssim 2^j t} W(B_t(x))^{-1} W(x)\, dx \lesssim e^{-\frac{c}{8}4^j},$$

by the doubling property of W. We now use these observations, along with Cauchy-Schwarz, Fubini's theorem, and the fact that the cubes Q_k are non-overlapping, to obtain

$$\begin{aligned}A_2 &\leqslant \sum_k \left(\sum_{j=1}^{\infty} e^{-\frac{c}{4}4^j} \right) \left(\sum_{j=1}^{\infty} e^{-\frac{c}{4}4^j} \|\nabla \mathbf{g}\|^2_{L^2_W(2^{j+1}Q_k)} \right) = \sum_k \sum_{j=1}^{\infty} e^{-\frac{c}{4}4^j} \|\nabla \mathbf{g}\|^2_{L^2_W(2^{j+1}Q_k)} \\ &\lesssim \sum_k \sum_{j=1}^{\infty} e^{-\frac{c}{4}4^j} \int_{Q_k} W(B_t(x))^{-1} W(x) \int_{|x-y| \lesssim 2^j t} |\nabla \mathbf{g}(y)|^2 W(y)\, dy dx \\ &= \sum_{j=1}^{\infty} e^{-\frac{c}{4}4^j} \int_{\mathbb{R}^n} W(B_t(x))^{-1} W(x) \int_{|x-y| \lesssim 2^j t} |\nabla \mathbf{g}(y)|^2 W(y)\, dy dx \\ &\lesssim \sum_{j=1}^{\infty} e^{-\frac{c}{8}4^j} \int_{\mathbb{R}^n} |\nabla \mathbf{g}(y)|^2 W(y)\, dy \lesssim \|\nabla \mathbf{g}\|^2_{L^2_W}.\end{aligned}$$

Consequently, we have shown that

$$A \lesssim t^2 \|\nabla \mathbf{g}\|^2_{L^2_W} .$$

We can apply a similar, but simpler argument to handle term B in (7.31). We now set $S_t := (\theta_t \mathbf{1}) \cdot P_t^2$, and note that S_t is uniformly bounded on L^2_W, by Claim 7.41 and Lemma 7.16. Moreover, the kernel of P_t^2 is compactly supported in the ball of radius $2t$, so the same is true for S_t. Hence, for the current version of S_t, we obtain a simplified variant of (7.32), in which only the term $I^{(k)}$ appears, enjoying the same bound as in (7.33). Thus,

$$B \lesssim t^2 \|\nabla \mathbf{g}\|^2_{L^2_W} .$$

The proof of Claim 7.43 is now complete. □

Claim 7.44. *For $t \leqslant s$, we have*

$$\|U_t Q_s\|_{L^2_W \to L^2_W} \lesssim \frac{t}{s}.$$

Proof of the claim. Using Claim 7.43 and Lemma 7.16, we have

$$\|U_t Q_s \mathbf{g}\|_{L^2_W} \lesssim t \|\nabla Q_s \mathbf{g}\|_{L^2_W} = \frac{t}{s} \|s \nabla Q_s \mathbf{g}\|_{L^2_W} \lesssim \frac{t}{s} \|\mathbf{g}\|_{L^2_W} ,$$

as desired. □

As noted above, the preceding claims conclude the proof of Lemma 7.39. □

We are now ready to reduce matters to a Carleson measure estimate. Recall that to prove Theorem 7.36 (and hence Theorem 7.34), it suffices to verify estimate (7.26) (equivalently, (7.24)).

Lemma 7.45. *Theorem 7.36 (and hence Theorem 7.34) follows from the Carleson measure estimate*

$$\sup_Q \frac{1}{W(Q)} \int_0^{\ell(Q)} \int_Q \left|\widetilde{\theta}_t \mathbf{1}(x)\right|^2 W(x) \frac{dxdt}{t} < \infty. \tag{7.34}$$

Proof. Recalling that $\widetilde{\theta}_t \mathbf{1} = \theta_t \mathbf{1}$, we see that by Lemma 7.39 and a weighted version of Carleson's embedding inequality (see [CR15, Lemma 2.2]), the estimate (7.34) implies (7.26). □

Our goal then, is to prove (7.34). To this end, let us first establish a few more estimates to be used in the sequel. We define the dyadic averaging operator by

$$A_t \mathbf{f}(x) := \fint_{Q_{x,t}} \mathbf{f}(y) dy,$$

where $Q_{x,t}$ is the half-open dyadic cube containing x for which $t/2 < \ell(Q_{x,t}) \leqslant t$.

Lemma 7.46. *We have*

$$\int_0^\infty \left\|(\theta_t \mathbf{1}) \cdot (P_t^2 - A_t)\mathbf{g}\right\|^2_{L^2_W} \frac{dt}{t} \lesssim \|\mathbf{g}\|^2_{L^2_W} ,$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

Proof. The proof of this estimate will be very similar to that of Lemma 7.39. We set

$$\widetilde{U}_t := (\theta_t \mathbf{1}) \cdot (P_t^2 - A_t)\,,$$

and note that it is enough to show that $\widetilde{U}_t$ satisfies the hypotheses of the weighted Littlewood-Paley almost-orthogonality result Lemma 7.19. The uniform boundedness of $\widetilde{U}_t$ arises immediately from that of $(\theta_t \mathbf{1}) \cdot P_t^2$ (see Claim 7.41 and Lemma 7.16), along with the following result:

Claim 7.47. *We have, uniformly on t,*

$$\big\|(\theta_t \mathbf{1}) \cdot A_t\big\|_{L^2_W \to L^2_W} \lesssim 1.$$

Proof of the claim. The proof is the same as that of Claim 7.41, which treated $(\theta_t \mathbf{1}) \cdot P_t$. Indeed, the only properties of P_t that were used in that argument were the size and support condition of its kernel. The kernel of A_t enjoys similar properties, in fact

$$\big|A_t \mathbf{f}(x)\big|^2 \leqslant \left(\fint_{Q_k} \big|\mathbf{f}(y)\big|\, dy \right)^2 \lesssim \fint_{Q_k} \big|\mathbf{f}(y)\big|^2\, W(y) dy,$$

hence, the same proof may be repeated. □

To prove the quasi-orthogonality with the Q_s operators, the next result will be useful.

Claim 7.48. *We have, uniformly on t,*

$$\|\widetilde{U}_t \mathbf{g}\|_{L^2_W} \lesssim t\, \|\nabla \mathbf{g}\|_{L^2_W}\,.$$

Proof. The proof is similar to that of Claim 7.43, but simpler: now one has to deal only with terms like "B" associated to $S_t = (\theta_t \mathbf{1}) \cdot A_t$ in (7.31), so that there is no "tail" as in (7.32), but rather only a local term analogous to $I^{(k)}$. We omit the routine details. □

The following two claims finish the proof of Lemma 7.46, and are analogous to those in the proof of Lemma 7.39.

Claim 7.49. *We have, uniformly for $t \leqslant s$,*

$$\|\widetilde{U}_t Q_s\|_{L^2_W \to L^2_W} \lesssim \frac{t}{s}.$$

Proof. In view of Claim 7.48, repeating the proof of Claim 7.44, we simply write

$$\|\widetilde{U}_t Q_s\|_{L^2_W \to L^2_W} \lesssim t\, \|\nabla Q_s\|_{L^2_W \to L^2_W} \lesssim \frac{t}{s}\,.$$

□

Claim 7.50. *We have, uniformly in $s \leqslant t$, and for some fixed $\alpha > 0$,*

$$\|\widetilde{U}_t Q_s\|_{L^2_W \to L^2_W} \lesssim \left(\frac{s}{t}\right)^{\alpha}.$$

Proof. On the one hand, as in Claim 7.42, and using the boundedness of Claim 7.41,

$$\left\|(\theta_t \mathbf{1}) \cdot P_t^2 Q_s\right\|_{L^2_W \to L^2_W} \lesssim \|P_t Q_s\|_{L^2_W \to L^2_W} \lesssim \frac{s}{t}.$$

On the other hand, by [AHLMT02, proof of Lemma 4.7], we have the unweighted quasi-orthogonality estimate

$$\|A_t Q_s\|_{L^2 \to L^2} \lesssim \left(\frac{s}{t}\right)^{\alpha},$$

for some exponent $\alpha > 0$, uniformly for $s \leqslant t$. Consequently, we may use the technique of Duoandikoetxea and Rubio de Francia [DR86], in which one first self-improves the weight W, and then uses Stein-Weiss interpolation with change of measure [SW58], to deduce the weighted quasi-orthogonality estimate

$$\|A_t Q_s\|_{L^2_W \to L^2_W} \lesssim \left(\frac{s}{t}\right)^{\beta},$$

for some positive $\beta < \alpha$ (see Lemma 2.5 in [CR15] for more details). Hence, by Claim 7.47,

$$\left\|(\theta_t \mathbf{1}) \cdot A_t Q_s\right\|_{L^2_W \to L^2_W} = \left\|(\theta_t \mathbf{1}) \cdot A_t^2 Q_s\right\|_{L^2_W \to L^2_W} \lesssim \|A_t Q_s\|_{L^2_W \to L^2_W} \lesssim \left(\frac{s}{t}\right)^{\beta}.$$

□

Collecting all the above claims, the proof of Lemma 7.46 is completed. □

Corollary 7.51. *We have the square function bound*

$$\int_0^\infty \left\|\widetilde{\theta}_t \nabla f - (\widetilde{\theta}_t \mathbf{1}) \cdot A_t \nabla f\right\|_{L^2_W}^2 \frac{dt}{t} \lesssim \|\nabla f\|_{L^2_W}^2,$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

Proof. With Lemma 7.46 in hand, since $\widetilde{\theta}_t \mathbf{1} = \theta_t \mathbf{1}$, it is enough to prove the following:

$$\int_0^\infty \left\|\widetilde{\theta}_t \nabla f - (\theta_t \mathbf{1}) \cdot P_t^2 \nabla f\right\|_{L^2_W}^2 \frac{dt}{t} \lesssim \|\nabla f\|_{L^2_W}^2.$$

To this end, we write

$$\widetilde{\theta}_t \nabla f - (\theta_t \mathbf{1}) \cdot P_t^2 \nabla f = \widetilde{\theta}_t \nabla (I - P_t^2) f + \left[\widetilde{\theta}_t \nabla P_t^2 f - (\theta_t \mathbf{1}) \cdot P_t^2 \nabla f\right] =: Y_t f + Z_t f.$$

By (7.27),

$$-2 Y_t = t \widetilde{L} e^{-t^2 \widetilde{L}} (I - P_t^2) =: \widetilde{R}_t,$$

where $\widetilde{R}_t$ is precisely the same operator defined in (7.21), enjoying the square function bound established in Lemma 7.37. In addition, again using (7.27),

$$-2\, \widetilde{\theta}_t \nabla P_t^2 = t \widetilde{L} e^{-t^2 \widetilde{L}} P_t^2 =: \widetilde{T}_t,$$

where $\widetilde{T}_t$ is precisely the same operator defined in (7.21). We now repeat the splitting of $\widetilde{T}_t$, exactly as in (7.23):

$$\widetilde{T}_t f = -t e^{-t^2 \widetilde{L}} L P_t^2 f - 2t e^{-t^2 \widetilde{L}} \widetilde{\operatorname{div}}(A \nabla (P_t^2 f)).$$

Note that the the second term equals $-2\theta_t \nabla P_t^2 f$ (see (7.25)). Combining these observations, we see that

$$Z_t f \;=\; -\frac{1}{2}\widetilde{T}_t f - (\theta_t \mathbf{1})\cdot P_t^2 \nabla f \;=\; \frac{1}{2} t e^{-t^2\widetilde{L}} L P_t^2 f \;+\; \Big[\theta_t P_t^2 \nabla f - (\theta_t \mathbf{1})\cdot P_t^2 \nabla f\Big] =: E_t f + U_t f\,,$$

where the term $E_t f$ (which is actually the error $(\widetilde{\theta}_t - \theta_t)\nabla P_t^2 f$), satisfies the desired square function bound, by Lemma 7.38. The last term also enjoys the desired square function bound, by Lemma 7.39. This concludes the proof of the Corollary. □

7.7.3 The $T(b)$ argument

Recall that our goal is to prove the Carleson measure estimate (7.34). We now turn to this task, which will finish the proof of Theorem 7.34 (and therefore also the proof of Theorem 7.8). Our arguments here will be an adaptation of the proof of the Kato conjecture in the divergence form setting, see [AHLMT02], and in particular, the extension of that proof to the degenerate elliptic case in [CR15].

We note that by the doubling property of W, we may assume that the supremum in (7.34) is taken over dyadic cubes Q. Given any such cube Q, a sufficiently small number $\varepsilon \in (0,1)$ to be chosen, and $v \in \mathbb{R}^n$ with $|v| = 1$, we define

$$f^{\varepsilon}_{Q,v} := e^{-(\varepsilon \ell(Q))^2 \widetilde{L}}(\mathbf{\Phi}_Q \chi_Q \cdot v),$$

where $\mathbf{\Phi}_Q(x) = x - x_Q$, x_Q denotes the center of Q, and $\chi_Q \in \mathscr{C}_0^\infty$ is a cut-off function such that $\chi_Q \equiv 1$ in $2Q$, $\operatorname{supp}\chi_Q \subset 4Q$ and $\|\chi_Q\|_\infty + \ell(Q)\|\nabla\chi_Q\|_\infty + \ell(Q)^2\|\nabla^2\chi_Q\|_\infty \lesssim 1$. Clearly,

$$\big\|\nabla(\mathbf{\Phi}_Q \chi_Q \cdot v)\big\|_\infty \lesssim 1\,, \tag{7.35}$$

and also

$$\int_{\mathbb{R}^n} |\mathbf{\Phi}_Q \chi_Q \cdot v|^2 W(x)dx \;\lesssim\; \ell(Q)^2 W(Q)\,. \tag{7.36}$$

The following estimates hold for $f^{\varepsilon}_{Q,v}$, with constants that are uniform on Q, v and ε:

$$\int_{5Q} \big|f^{\varepsilon}_{Q,v} - \mathbf{\Phi}_Q \chi_Q \cdot v\big|^2 W dx \lesssim \varepsilon^2 \ell(Q)^2 W(Q), \tag{7.37}$$

$$\int_{5Q} \big|\nabla f^{\varepsilon}_{Q,v}\big|^2 W dx \;+\; \int_{5Q} \Big|\nabla\Big(f^{\varepsilon}_{Q,v} - \mathbf{\Phi}_Q \chi_Q \cdot v\Big)\Big|^2 W dx \;\lesssim\; W(Q), \tag{7.38}$$

These estimates follow at once from (7.35), Lemmas 7.26 and 7.27 (with $t = \varepsilon\ell(Q)$), and the doubling property of W.

The proof of (7.34) (and hence of Theorem 7.34 by Corollary 7.45) follows from the next two lemmas.

Lemma 7.52. *There exists $0 < \varepsilon = \varepsilon(\lambda, n, [W]_{A_2}) \ll 1$ and a finite set V of unit vectors in $\mathbb{R}^n$, whose cardinality depends only on ε and n, such that*

$$\sup_Q \frac{1}{W(Q)} \int_0^{\ell(Q)} \int_Q \big|(\widetilde{\theta}_t \mathbf{1})(x)\big|^2 W(x) \frac{dxdt}{t}$$
$$\lesssim \sum_{v\in V} \sup_Q \frac{1}{W(Q)} \int_0^{\ell(Q)} \int_Q \Big|(\widetilde{\theta}_t \mathbf{1})(x)\cdot\Big(A_t \nabla f^{\varepsilon}_{Q,v}\Big)(x)\Big|^2 W(x)\frac{dxdt}{t},$$

where the implicit constant depends on n, λ and $[W]_{A_2}$.

Proof. The reader may check that the proof of [CR15, Lemma 5.1] (which in turn is an adaptation to the weighted case of [AHLMT02, Lemma 5.4]) works perfectly well in our situation: as long as $W \in A_2$ and $f^{\varepsilon}_{Q,v}$ satisfies the estimates (7.37) and (7.38), the proof in [CR15] goes through[5]. □

With Lemma 7.52 in hand, estimate (7.34) will follow immediately from the next lemma.

Lemma 7.53. *For every cube Q and unit vector v, we have*

$$\int_0^{\ell(Q)} \int_Q \left|(\widetilde{\theta}_t \mathbf{1})(x) \cdot \left(A_t \nabla f^{\varepsilon}_{Q,v}\right)(x)\right|^2 W(x) \frac{dxdt}{t} \lesssim W(Q),$$

where the implicit constant depends on $n, \lambda, [W]_{A_2}$ and ε, but is uniform on Q and v.

Proof. Fix Q and v, and abbreviate $f := f^{\varepsilon}_{Q,v}$. By Corollary 7.51, we have

$$\begin{aligned}\int_0^{\ell(Q)} \int_Q \left|(\widetilde{\theta}_t \mathbf{1})(x) \cdot (A_t \nabla f)(x)\right|^2 W(x) \frac{dxdt}{t} \\ \lesssim \|\nabla f\|^2_{L^2_W} + \int_0^{\ell(Q)} \int_Q \left|(\widetilde{\theta}_t \nabla f)(x)\right|^2 W(x) \frac{dxdt}{t} =: I + II \lesssim W(Q) + II,\end{aligned}$$

where in the last step, we have used (7.38) to obtain the desired bound for term I.

Term II can be treated as follows, using (7.27), Lemma 7.20, and the definition of $f = f^{\varepsilon}_{Q,v}$,

$$\begin{aligned}II &\approx \int_0^{\ell(Q)} \int_Q \left|te^{-t^2\widetilde{L}} \widetilde{L} f(x)\right|^2 W(x) \frac{dxdt}{t} \\ &\lesssim \int_0^{\ell(Q)} tdt \int_{\mathbb{R}^n} \left|\widetilde{L} e^{-(\varepsilon \ell(Q))^2 \widetilde{L}} (\mathbf{\Phi}_Q \chi_Q \cdot v)(x)\right|^2 W(x) dx \\ &\approx \ell(Q)^2 \int_{\mathbb{R}^n} \left|\widetilde{L} e^{-(\varepsilon \ell(Q))^2 \widetilde{L}} (\mathbf{\Phi}_Q \chi_Q \cdot v)(x)\right|^2 W(x) dx \\ &\lesssim \varepsilon^{-4} \ell(Q)^{-2} \int_{\mathbb{R}^n} |\mathbf{\Phi}_Q \chi_Q \cdot v|^2 W(x) dx \lesssim \varepsilon^{-4} W(Q),\end{aligned}$$

where in the last two steps we have first used Lemma 7.20 (vi) with $t = \varepsilon \ell(Q)$, and then (7.36). Since ε has been fixed depending only on allowable parameters, the dependence on ε is harmless.

Collecting all the preceeding estimates, we have finished the proof. □

[5] To clarify a possible point of confusion, we mention that in [AHLMT02] and [CR15], the unit vectors were taken in $\mathbb{C}^n$, because in the divergence form setting of those papers, one treats the case of complex coefficients; at present, our results in the non-divergence form case treat only the case of real coefficients, so we need only consider real unit vectors.

Bibliography

[AB98] A. Aftalion, and J. Busca, *Radial symmetry of overdetermined boundary value problems in exterior domains*, Arch. Rat. Mech. Anal. **143** (1998), 195–206. Cited in: 223

[ABR99] A. Aftalion, J. Busca, and W. Reichel, *Approximate radial symmetry for overdetermined boundary value problems*, Adv. Diff. Eqs. **4** (1999), 907–932. Cited in: 223

[ADN59] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I*, Comm. Pure Appl. Math. **12** (1959), 623–727. Cited in: 54

[ADN64] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II*, Comm. Pure Appl. Math. **17** (1964), 35–92. Cited in: 54

[AAC86] N. Aguilera, H.W. Alt, and L.A. Caffarelli, *An optimization problem with volume constraint*, SIAM J. Control Optim. **24** (1986), 191–198. Cited in: 227

[Ai02] H. Aikawa, *Hölder continuity of the Dirichlet solution for a general domain*, Bull. London Math. Soc **34** (2002), 691–702. Cited in: 5, 6, 51, 52, 56, 58, 74, 75, 78, 81, 83

[AG89] G. Alessandrini, and N. Garofalo, *Symmetry for degenerate parabolic equations*, Arch. Rat. Mech. Anal. **108** (1989), 161–174. Cited in: 223

[AAAHK11] M.A. Alfonseca, P. Auscher, A. Axelsson, S. Hofmann, and S. Kim, *Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients*, Adv. Math. **226** (2011), 4533–4606. Cited in: 53, 272, 294

[AC81] H.W. Alt, and L.A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 105–144. Cited in: 11, 221, 222, 224, 226, 227, 236, 237, 241, 242, 245, 246, 247

[An86] A. Ancona, *On strong barriers and an inequality of Hardy for domains in $\mathbb{R}^n$*, J. London Math. Soc. **34** (1986), 274–290. Cited in: 85

[Ar67] D.G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Am. Math. Soc. **73** (1967), 890–896. Cited in: 46

[AA11] P. Auscher, and A. Axelsson, *Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I*, Invent. Math. **184** (2011), 47–115. Cited in: 12, 272

[AAH08] P. Auscher, A. Axelsson, and S. Hofmann, *Functional calculus of Dirac operators and complex perturbations of Neumann and regularity problems*, J. Funct. Anal. **255** (2008), 374–448. Cited in: 272

[AAM10a] P. Auscher, A. Axelsson, and McIntosh, *Solvability of elliptic systems with square integrable boundary data*, Ark. Mat. **48** (2010), 253–287. Cited in: 12, 272

[AAM10b] P. Auscher, A. Axelsson, and A. McIntosh, *On a quadratic estimate related to the Kato conjecture and boundary value problems*, Contemporary Mathematics, AMS, **505** (2010), 105–129. Cited in: 12, 272

[AEN18] P. Auscher, M. Egert, and K. Nyström, *The Dirichlet problem for second order parabolic operators in divergence form*, J. Ec. Polytech. Math. **5** (2018), 407–441. Cited in: 272

[AEN20] P. Auscher, M. Egert, and K. Nyström, *L^2 well-posedness of boundary value problems and the Kato square root problem for parabolic systems with measurable coefficients*, J. Eur. Math. Soc. **22** (2020), 2943–3058. Cited in: 272

[AHLMT02] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian, *The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$*, Ann. of Math. **156** (2002), 633–654. Cited in: 12, 50, 271, 272, 278, 291, 298, 299, 300

[AHLT01] P. Auscher, S. Hofmann, J.L. Lewis, and P. Tchamitchian, *Extrapolation of Carleson measures and the analyticity of Kato's square root operators*, Acta Math. **187** (2001), 161–190. Cited in: 12, 272

[ARR15] P. Auscher, A. Rosén, and D. Rule, *Boundary value problems for degenerate elliptic equations and systems*, Ann. Sci. Éc. Norm. Supér. **48** (2015), 951–1000. Cited in: 53, 277

[AT98] *Square root problem for divergence operators and related topics*, Astérisque **249**, Soc. Math. France, Paris, 1998. Cited in: 12

[AKM06] A. Axelsson, S. Keith, and A. McIntosh, *Quadratic estimates and functional calculi of perturbed Dirac operators*, Invent. Math **163** (2006), 455–497. Cited in: 12, 272

[Az21] J. Azzam, *Semi-uniform domains and the A_∞ property for harmonic measure*, Int. Math. Res. Not. **9** (2021), 6717–6771. Cited in: 106

[AGMT23] J. Azzam, J. Garnett, M. Mourgoglou, and X. Tolsa, *Uniform rectifiability, elliptic measure, square functions, and ε-approximability via an ACF monotonicity formula*, Int. Math. Res. Not. **13** (2023), 10837–10941. Cited in: 7, 106, 108, 136

[AHMMT20] J. Azzam, S. Hofmann, J.M. Martell, M. Mourgoglou, and X. Tolsa, *Harmonic measure and quantitative connectivity: geometric characterization of the L^p-solvability of the Dirichlet problem*, Invent. Math. **222** (2020), 881–993. Cited in: 6, 7, 8, 53, 55, 97, 103, 104, 106, 148

[AHMM+16] J. Azzam, S. Hofmann, J.M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa, and A. Volberg, *Rectifiability of harmonic measure*. Geom. Funct. Anal. **26** (2016), 703–728. Cited in: 7, 105

[AHMNT17] J. Azzam, S. Hofmann, J.M. Martell, K. Nyström, and T. Toro, *A new characterization of chord-arc domains*, J. Eur. Math. Soc. **19** (2017), 967–981. Cited in: 7, 53, 55, 105, 156

[Ba11] M. Badger, *Harmonic Polynomials and Free Boundary Regularity for Harmonic Measure from Two Sides*, ProQuest LLC, Ann Arbor, MI, 2011, Thesis (Ph.D.)–University of Washington. Cited in: 76

[Ba84] P. Bauman, *Positive solutions of elliptic equations in nondivergence form and their adjoints*, Ark. Mat. **22** (1984), 153–173. Cited in: 273

[BS88] C. Bennett, and R. Sharpley, *Interpolation of operators*, Volume 129 of Pure and Applied Mathematics, Academic Press Inc., Boston, MA, 1988. Cited in: 60

[BJ90] C.J. Bishop, and P.W. Jones, *Harmonic measure and arclength*, Ann. of Math. **132** (1990), 511–547. Cited in: 105

[BH15] I. Blank, and Z. Hao, *The mean value theorem and basic properties of the obstacle problem for divergence form elliptic operators*, Commun. Anal. Geom. **23** (2015), 129–158. Cited in: 236

[BFHH25] S. Bortz, S. Ferris, P. Hidalgo-Palencia, and S. Hofmann, *A variable coefficient free boundary problem for L^p solvability of parabolic Dirichlet problem in graph domains*, arxiv preprint 2503.00873. Cited in: 3, 9, 158

[BHHLN22a] S. Bortz, J. Hoffman, S. Hofmann, J.L. Luna-Garcia, and K. Nyström, *Coronizations and big pieces in metric spaces*, Ann. Inst. Fourier (Grenoble) **72** (2022), 2037–2078. Cited in: 162

[BHHLN22b] S. Bortz, J. Hoffman, S. Hofmann, J.L. Luna-Garcia, and K. Nyström, *On big pieces approximations of parabolic hypersurfaces*, Ann. Fenn. Math. **47** (2022), 533–571. Cited in: 161

[BHHLN23a] S. Bortz, J. Hoffman, S. Hofmann, J.L. Luna-Garcia, and K. Nyström, *Carleson measure estimates for caloric functions and parabolic uniformly rectifiable sets*, Anal. PDE **16** (2023), 1061–1088. Cited in: 170, 195, 217

[BHHLN23b] S. Bortz, J. Hoffman, S. Hofmann, J.L. Luna-Garcia, and K. Nyström, *Corona decompositions for parabolic uniformly rectifiable sets*, J. Geom. Anal. **33** (2023), Paper No. 96, 67pp. Cited in: 161, 166, 170, 171

[BHL20] S. Bortz, S. Hofmann, and J.L. Luna Garcia, *T1 and Tb theorems and applications*, IAS/Park City Mathematical Series, Harmonic Analysis and Applications **27** (2020), 155–198. Cited in: 23

[BHLMP20] S. Bortz, S. Hofmann, J.L. Luna Garcia, S. Mayboroda, and B. Poggi, *Critical Perturbations for Second Order Elliptic Operators. Part I: Square function bounds for layer potentials*, Anal. PDE **15** (2022), 1215–1286. Cited in: 272

[BHMN25] S. Bortz, S. Hofmann, J.M. Martell, and K. Nyström, *Solvability of the L^p Dirichlet problem for the heat equation is equivalent to parabolic uniform rectifiability in the case of a parabolic Lipschitz graph*, Invent. Math. **239** (2025), 165–217. Cited in: 9, 10, 157, 158, 159, 161, 163, 164, 170, 171, 175, 177, 178, 180, 195, 198, 201, 202, 204, 207, 208, 209, 211

[BHMN25b] S. Bortz, S. Hofmann, J.M. Martell, and K. Nyström, *Solvability of the L^p Dirichlet problem for the heat equation implies parabolic uniform rectifiability*, in preparation. Cited in: 213

[Bo87] J. Bourgain, *On the Hausdorff dimension of harmonic measure in higher dimensions*, Invent. Math. **87** (1987), 477–483. Cited in: 34, 35, 36

[BNST08] B. Brandolini, C. Nitsch, P. Salani, and C. Trombetti, *Serrin-type overdetermined problems: an alternative proof*, Arch. Rat. Mech. Anal. **190** (2008), 267–280. Cited in: 223

[Ca98] L.A. Caffarelli, *The Obstacle Problem*. The Fermi Lectures, Accademia Nazionale Dei Lincei Scuola Normale Superiore, 1998. Cited in: 236

[CC95] L. Caffarelli, and X. Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, vol. 43, American Mathematical Society, Providence, RI, 1995. Cited in: 11

[CFMS81] L.A. Caffarelli, E.B. Fabes, S. Mortola, and S. Salsa, *Boundary behavior of non-negative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J. **30** (1981), 621–640. Cited in: 7, 36, 232

[CFK81] L.A. Caffarelli, E.B. Fabes and C.E. Kenig, *Completely singular elliptic-harmonic measures*, Indiana Univ. Math. J. **30** (1981), 917–924. Cited in: 7, 104, 232

[Ca78] A.P. Calderón, *Commutators, singular integrals on Lipschitz curves and applications*, Proc. I.C.M. Helsinski (1978), 85–96. Cited in: 6

[CDMT22] M. Cao, Ó. Domínguez, J.M. Martell, and P. Tradacete, *On the A_∞ condition for elliptic operators in 1-sided NTA domains satisfying the capacity density condition*, Forum Math. Sigma **10** (2022), Paper No. e59. Cited in: 119

[CHM24] M. Cao, P. Hidalgo-Palencia, and J.M. Martell, *Carleson measure estimates, corona decompositions, and perturbation of elliptic operators without connectivity*, Math. Ann. **390** (2024), 95–156. Cited in: 3, 7, 104

[CHMPZ25] M. Cao, P. Hidalgo-Palencia, J.M. Martell, C. Prisuelos-Arribas, and Z. Zhao, *Elliptic operators in rough sets, and the Dirichlet problem with boundary data in Hölder spaces*, J. Funct. Anal. **288** (2025), 68pp. Cited in: 2, 3, 5, 7, 39, 52

[CMM22] M. Cao, J.J. Marín, and J.M. Martell, *Extrapolation on function and modular spaces, and applications*, Adv. Math. **406** (2022), 108520. Cited in: 54

[CHM19] J. Cavero, S. Hofmann, and J.M. Martell, *Perturbations of elliptic operators in* 1*-sided chord-arc domains. Part I: Small and large perturbation for symmetric operators*, Trans. Amer. Math. Soc. **371** (2019), 2797–2835. Cited in: 105

[CHMT20] J. Cavero, S. Hofmann, J.M. Martell, and T. Toro, *Perturbations of elliptic operators in 1-sided chord-arc domains. Part II: Non-symmetric operators and Carleson measure estimates*, Trans. Amer. Math. Soc. **373** (2020), 7901–7935. Cited in: 55, 88, 105, 107, 109, 131, 143, 152, 156

[CFL91] F. Chiarenza, M. Frasca, and P. Longo, *Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40** (1991), 149–168. Cited in: 274

[CFL93] F. Chiarenza, M. Frasca, and P. Longo, *$W^{2,p}$-solvability of the Dirichlet problem for non-divergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336** (1993), 841–853. Cited in: 274

[Ch90] M. Christ, *A Tb theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math. **60/61** (1990), 601–628. Cited in: 26, 116

[CS09] A. Cianchi, and P. Salani, *Overdetermined anisotropic elliptic problems*, Math. Ann. **345** (2009), 859–881. Cited in: 223

[CDM83] R. Coifman, D. Deng, and Y. Meyer, *Domaine de la racine carrée de certains operateurs differentiels accrétifs*, Ann. Inst. Fourier **33** (1983), 123–134. Cited in: 12, 272

[CMM82] R. Coifman, A. McIntosh, and Y. Meyer. *L'intégrale de Cauchy définit un opérateur borné sur $L^2(\mathbb{R})$ pour les courbes lipschitziennes*, Ann. of Math. **116** (1982), 361–387. Cited in: 7, 12, 272

[CMS85] R. Coifman, Y. Meyer, and E.M. Stein, *Some new function spaces and their applications to Harmonic analysis*, J. Funct. Anal. **62** (1985), 304–335. Cited in: 21

[CRW76] R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635. Cited in: 274

[CR15] D. Cruz-Uribe, and C. Ríos, *The Kato problem for operators with weighted ellipticity*, Trans. Amer. Math. Soc. **367** (2015), 4727–4756. Cited in: 272, 277, 292, 296, 298, 299, 300

[Da77] B.E.J. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal. **65** (1977), 275–288. Cited in: 6, 7, 9, 36, 53, 103, 157, 160

[DK87] B.E.J. Dahlberg, and C.E. Kenig, *Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains*, Ann. of Math. **125** (1987), 437–465. Cited in: 54, 173

[DPR99] L. Damascelli, F. Pacella, and M. Ramaswamy, *Symmetry of ground states of p-Laplace equations via the moving plane method*, Arch. Rat. Mech. Anal. **148** (1999), 291–308. Cited in: 223

[DFM19] G. David, J. Feneuil, and S. Mayboroda, *Dahlberg's theorem in higher co-dimension*, J. Funct. Anal. **276** (2019), 2731–2820. Cited in: 54

[DJ90] G. David and D. Jerison, *Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals*, Indiana Univ. Math. J. **39** (1990), 831–845. Cited in: 53, 105

[DJ84] G. David and J.L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. **120** (1984), 371–397. Cited in: 12, 161

[DM22] G. David and S. Mayboroda, *Approximation of Green functions and domains with uniformly rectifiable boundaries of all dimensions*, Adv. Math. **410** (2022), 108717. Cited in: 105, 110

[DM23] G. David and S. Mayboroda, *Harmonic measure is absolutely continuous with respect to the Hausdorff measure on all low-dimensional uniformly rectifiable sets*, Int. Math. Res. Not. **11** (2023), 9319–9426. Cited in: 54

[DS91] G. David and S. Semmes, *Singular integrals and rectifiable sets in $\mathbb{R}^n$. Au-delà des graphes lipschitziens*, Astérisque **193** (1991). Cited in: 7, 26, 116, 170, 171, 195

[DS93] G. David and S. Semmes, *Analysis of and on uniformly rectifiable sets*, Mathematical Monographs and Surveys **38**, AMS 1993. Cited in: 7, 25, 26, 112, 114, 116, 148, 162, 170

[De11] D. De Silva, *Free boundary regularity for a problem with right hand side*, Interfaces Free Bound. **13** (2011), 223–238. Cited in: 222, 226, 259

[DFS] D. De Silva, F. Ferrari, and S. Salsa, *Regularity of the free boundary for two-phase problems governed by divergence form equations and applications*, Nonlinear Anal. **138** (2016), 3–30. Cited in: 259, 260

[DPW15] M. Del Pino, F. Pacard, and J. Wei, *Serrin's overdetermined problem and constant mean curvature surfaces*, Duke Math. J. **164** (2015), 2643–2722. Cited in: 223

[DS15] E. Delay, and P. Sicbaldi, *Extremal domains for the first eigenvalue in a general compact Riemannian manifold*, Discrete Contin. Dyn. Syst. **35** (2015), 5799–5825. Cited in: 222, 223

[DHP23] M. Dindoš, S. Hofmann, and J. Pipher, *Regularity and Neumann problems for operators with real coefficients satisfying Carleson conditions*, J. Funct. Anal **285** (2023), article 10024. Cited in: 8

[DPP17] M. Dindoš, J. Pipher, and S. Petermichl, *BMO solvability and the A_∞ condition for second order parabolic operators*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), 1155–1180. Cited in: 160, 161

[DEP19] M. Domínguez-Vázquez, A. Enciso, and D. Peralta-Salas, *Solutions to the overdetermined boundary problem for semilinear equations with position-dependent nonlinearities*, Adv. Math. **351** (2019), 718–760. Cited in: 222, 223

[DEP21] M. Domínguez-Vázquez, A. Enciso, and D. Peralta-Salas, *Piecewise smooth stationary Euler flows with compact support via overdetermined boundary problems*, Arch. Ration. Mech. Anal. **239** (2021), 1327–1347. Cited in: 223

[DEP23] M. Domínguez-Vázquez, A. Enciso, and D. Peralta-Salas, *Overdetermined boundary problems with non constant Dirichlet and Neumann data*, Anal. PDE **16** (2023), 1989–2003. Cited in: 222, 223

[Du01] J. Duoandikoetxea, *Fourier Analysis*, American Mathematical Soc. **29**, 2001. Cited in: 18, 19, 20, 22, 23

[DR86] J. Duoandikoetxea, J.L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541–561. Cited in: 298

[DY02] X.T. Duong and L. Yan, *Bounded holomorphic functional calculus for non-divergence form differential operators*, Differ. Integral Equ. **15** (2002), 709–730. Cited in: 277

[EHMT24] M. Egert, R. Haller, S. Monniaux, and P. Tolksdorf, *Harmonic analysis techniques for elliptic operators*, Lecture notes, 27th International Internet Seminar, available online at https://www.mathematik.tu-darmstadt.de/media/analysis/lehrmaterial_anapde/hallerd/ISem_complete_lecture_notes.pdf, 2024. Cited in: 47, 48, 49, 50, 275

[EFR24] A. Enciso, A. Fernández, and D. Ruiz, *Smooth nonradial stationary Euler flows on the plane with compact support*, arxiv preprint 2406.04414. Cited in: 223

[EFRS23] A. Enciso, A. Fernández, D. Ruiz, and P. Sicbaldi, *A Schiffer-type problem for annuli with applications to stationary planar Euler flows*, Duke Math. J., in press. Cited in: 223

[EHR24] A. Enciso, P. Hidalgo-Palencia, and X. Ros-Oton, *Positive solutions to general semilinear overdetermined boundary problems*, arxiv preprint 2412.11902. Cited in: 3, 10, 222

[En17] M. Engelstein, *A free boundary problem for the parabolic Poisson kernel*, Adv. Math. **314** (2017), 835–947. Cited in: 170

[Es94] L. Escauriaza, *Weak-type (1,1) inequalities and regularity properties of adjoint and normalized adjoint solutions to linear nondivergence form operators with VMO coefficients*, Duke Math J. **74** (1994), 177–201. Cited in: 274

[Es00] L. Escauriaza, *Bounds for the fundamental solution of elliptic and parabolic equations*, Commun. Partial Diff. Equ. **25** (2000), 821–845. Cited in: 273, 274, 280

[EHH24] L. Escauriaza, P. Hidalgo-Palencia, and S. Hofmann, *On the Kato problem for elliptic operators in non-divergence form*, Vietnam J. Math **52** (2024), 1067–1096. Cited in: 3, 12, 272

[EH18] L. Escauriaza, and S. Hofmann, *Kato square root problem with unbounded leading coefficients*, Proc. Amer. Math. Soc. **146** (2018), 5295–5310. Cited in: 272

[Ev98] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, 1998. Cited in: 6, 28, 29, 243

[EG82] L.C. Evans, and R.F. Gariepy, *Wiener's criterion for the heat equation*, Arch. Rational Mech. Anal. **78** (1982), 293–314. Cited in: 8

[EG92] L.C. Evans, and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, 1992. Cited in: 30, 166

[FGL89] E.B. Fabes, N. Garofalo, and E. Lanconelli, *Wiener's criterion for divergence form parabolic operators with C^1-Dini continuous coefficients*, Duke Math. J. **59** (1989), 191–232. Cited in: 8

[FGS84] E.B. Fabes, N. Garofalo, and Sandro Salsa, *Comparison theorems for temperatures in non-cylindrical domains*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **77** (1984), 1–12. Cited in: 172, 174

[FGS86] E.B. Fabes, N. Garofalo, and Sandro Salsa, *A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations*, Illinois J. Math. **30** (1986), 536–565. Cited in: 44, 45, 172, 174, 181

[FJK84] E.B. Fabes, D. Jerison, and C. Kenig, *Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure*, Ann. of Math. **119** (1984), 121–141. Cited in: 272

[FJK85] E.B. Fabes, D. Jerison, and C. Kenig, *Multilinear square functions and partial differential equations*, Amer. J. of Math. **107** (1985), 1325–1367. Cited in: 12, 272

[FJN76] E.B. Fabes, R.L. Johnson, and U. Neri, *Spaces of harmonic functions representable by Poisson integrals of functions in* BMO *and* $L_{p,\lambda}$, Indiana Univ. Math. J. **25** (1976), 159–170. Cited in: 54

[FN80] E.B. Fabes and U. Neri, *Dirichlet problem in Lipschitz domains with* BMO *data*, Proc. Amer. Math. Soc. **78** (1980), 33–39. Cited in: 54

[FS97] E.B. Fabes, and M.V. Safonov, *Behavior near the boundary of positive solutions of second order parabolic equations*, Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996) **3** (1997), 871–882. Cited in: 172, 174

[FSY99] E.B. Fabes, M.V. Safonov, and Y. Yuan, *Behavior near the boundary of positive solutions of second order parabolic equations II*, Trans. Amer. Math. Soc. **351** (1999), 4947–4961. Cited in: 172, 174

[FV10] A. Farina, and E. Valdinoci, *Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems*, Arch. Rat. Mech. Anal. **195** (2010), 1025–1058. Cited in: 223

[FV13] A. Farina, and E. Valdinoci, *On partially and globally overdetermined problems of elliptic type*, Amer. J. Math. **135** (2013), 1699–1726. Cited in: 223

[FMW17] M.M. Fall, I. A. Minlend, and T. Weth, *Unbounded periodic solutions to Serrin's overdetermined boundary value problem*, Arch. Ration. Mech. Anal. **233** (2017), 737–759. Cited in: 223

[FMW18] M.M. Fall, I. A. Minlend, and T. Weth, *Serrin's overdetermined problem on the sphere*, Calc. Var. Partial Differ. Equ. **57** (2018), 24 pp. Cited in: 223

[FMW23] M. M. Fall, I. A. Minlend, and T. Weth, *The Schiffer problem on the cylinder and on the 2-sphere*, J. Eur. Math. Soc., in press. Cited in: 223

[Fe69] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. Cited in: 25

[Fe71] C. Fefferman, *The multiplier problem for the ball*, Ann. of Math. **94** (1971), 330–336. Cited in: 22

[FKP91] R. Fefferman, C.E. Kenig, and J. Pipher, *The theory of weights and the Dirichlet problem for elliptic equations*, Ann. of Math. **134** (1991), 65–124. Cited in: 7, 10, 53, 104, 105, 107

[FL23] J. Feneuil, and L. Li, *A Green function characterization of uniformly rectifiable sets of any codimension*, Adv. Math. **430** (2023), 45 pp. Cited in: 160, 162

[FR22] X. Fernández-Real, and X. Ros-Oton, *Regularity theory for elliptic PDE*, Zurich Lectures in Advanced Mathematics, EMS books, 2022. Cited in: 33, 34, 169

[FZ24] A. Figalli, and Y.RY. Zhang, *Serrin's overdetermined problem in rough domains*, arxiv preprint 2407.02293. Cited in: 10, 221, 223

[FG08] I. Fragalà, and F. Gazzola, *Partially overdetermined elliptic boundary value problems*, J. Differ. Equ. **245** (2008), 1299–1322. Cited in: 223

[FGK06] I. Fragalà, F. Gazzola, and B. Kawohl, *Overdetermined problems with possibly degenerate ellipticity, a geometric approach*, Math. Z. **254** (2006), 117–132. Cited in: 223

[GMT18] J. Garnett, M. Mourgoglou, and X. Tolsa, *Uniform rectifiability from Carleson measure estimates and ε-approximability of bounded harmonic functions*, Duke Math. J. **167** (2018), 1473–1524. Cited in: 7, 106, 108, 110, 121, 136, 153

[GL88] N. Garofalo, and E. Lanconelli, *Wiener's criterion for parabolic equations with variable coefficients and its consequences*, Trans. Amer. Math. Soc. **308** (1988), 811–836. Cited in: 8

[GL89] N. Garofalo, and J.L. Lewis, *A symmetry result related to some overdetermined boundary value problems*, Amer. J. Math. **111** (1989), 9–33. Cited in: 223

[GS99] N. Garofalo, and E. Sartori, *Symmetry in exterior boundary value problems for quasilinear elliptic equations via blow-up and a priori estimates*, Adv. Diff. Eq. **4** (1999), 137–161. Cited in: 223

[Ge20] A. Genschaw, *Absolute continuity of parabolic measure and the initial-Dirichlet problem*, PhD dissertation available at ProQuest LLC, Ann Arbor, MI, 2019. Cited in: 46

[GH20] A. Genschaw, and S. Hofmann, *A weak reverse Hölder inequality for caloric measure*, J. Geom. Anal. **30** (2020), 1530–1564. Cited in: 43, 44, 46, 160, 161, 173, 174, 175, 176

[GH21] A. Genschaw, and S. Hofmann, *BMO solvability and absolute continuity of caloric measure*, Potential Anal. **54** (2021), 451–469. Cited in: 161

[GM12] M. Giaquinta and L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*. Second edition. Lecture Notes. Scuola Normale Superiore di Pisa (New Series), 2012. Cited in: 148

[GT77] D. Gilbarg, and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin-New York, 1977. Cited in: 2, 6, 24, 28, 29, 31, 32, 33, 34, 37, 231, 234, 235, 242, 243, 261, 266

[Gr08a] L. Grafakos, *Classical Fourier Analysis*, volume 249 of Graduate Texts in Mathematics, Springer, New York, 2nd edition, 2008. Cited in: 18, 21, 22, 25

[Gr08b] L. Grafakos. *Modern Fourier Analysis*, volume 250 of Graduate Texts in Mathematics, Springer, New York, 2nd edition, 2008. Cited in: 18, 20, 23

[GW82] M. Grüter, and K.O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982), 303–342. Cited in: 32, 35

[GS96] B. Gustafsson, and H. Shahgholian, *Existence and geometric properties of solutions of a free boundary problem in potential theory*, J. Reine Angew. Math. **473** (1996), 137–179. Cited in: 222, 226

[Ha06] M. Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, vol. 169, Birkhäuser Verlag, Basel, 2006. Cited in: 47, 48, 49, 50, 275

[HL11] Q. Han, and F. Lin, *Elliptic partial differential equations*, American Mathematical Soc. 1, 2011. Cited in: 30, 31, 33, 84

[HHST+22] S.V. Haziot, V.M. Hur, W.A. Strauss, J.F. Toland, E Wahlén, S. Walsh, M.H. Wheeler, *Traveling water waves — the ebb and flow of two centuries*, Quart. Appl. Math. **80** (2022) 317–401. Cited in: 223

[He01] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer-Verlag, New York, 2001. Cited in: 75

[HKM06] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Dover Publications, Inc., Mineola, NY, 2006, Unabridged republication of the 1993 original. Cited in: 30, 37, 38, 40, 43, 55, 77, 82, 84, 86, 267, 285

[He69] L.L. Helms, *Introduction to potential theory*, Pure and Applied Mathematics 22, Wiley-Interscience, New York-London-Sidney, 1969. Cited in: 53

[Hi21] P. Hidalgo-Palencia, *The elliptic measure in rough domains*, Master thesis available at the webpage of the author, 2021. Cited in: 38

[Ho97] S. Hofmann, *Parabolic singular integrals of Calderón-type, rough operators, and caloric layer potentials*, Duke Math. J. **90** (1997), 209–259. Cited in: 161

[Ho06] S. Hofmann, *Local Tb Theorems and applications in PDE*, Proceedings of the ICM Madrid, Vol. **II**, pp. 1375–1392, European Math. Soc., 2006. Cited in: 272

[HL96] S. Hofmann, and J.L. Lewis, *L^2 solvability and representation by caloric layer potentials in time- varying domains*, Ann. of Math. **144** (1996), 349–420. Cited in: 157, 161, 169, 211

[HL01] S. Hofmann, and J.L. Lewis, *The Dirichlet problem for parabolic operators with singular drift terms*, Mem. Amer. Math. Soc. **151** (2001), 113 pp. Cited in: 160, 161

[HL05] S. Hofmann, and J.L. Lewis, *The L^p Neumann problem for the heat equation in non-cylindrical domains*, J. Funct. Anal. **220** (2005), 1–54. Cited in: 161

[HLN03] S. Hofmann, J.L. Lewis, and K. Nyström, *Existence of big pieces of graphs for parabolic problems*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 355–384. Cited in: 170

[HLN04] S. Hofmann, J.L. Lewis, and K. Nyström, *Caloric measure in parabolic flat domains*, Duke Math. J. **122** (2004), 281–346. Cited in: 170

[HKMP15a] S. Hofmann, C. Kenig, S. Mayboroda, and J. Pipher, *Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators*, J. Amer. Math. Soc. **28** (2015), 483–529. Cited in: 7, 53, 88, 272, 273

[HKMP15b] S. Hofmann, C. Kenig, S. Mayboroda, and J. Pipher, *The Regularity problem for second order elliptic operators with complex-valued bounded measurable coefficients*, Math. Ann. **361** (2015), 863–907. Cited in: 272

[HK07] S. Hofmann, and S. Kim, *The Green function estimates for strongly elliptic systems of second order*, Manuscripta Math. **124** (2007), 139–172. Cited in: 32, 35

[HLM02] S. Hofmann, M. Lacey, and A. McIntosh. *The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds*, Ann. of Math. **156** (2002), 623–631. Cited in: 12, 272

[HL18] S. Hofmann, and P. Le, BMO *solvability and absolute continuity of harmonic measure*, J. Geom. Anal. **28** (2018), 3278–3299. Cited in: 53, 104, 106

[HLMN17] S. Hofmann, P. Le, J.M. Martell, and K. Nyström, *The weak-A_∞ property of harmonic and p-harmonic measures implies uniform rectifiability*, Anal. PDE **10** (2017), 513–558. Cited in: 7, 55, 106, 108, 111, 125, 144, 145, 147, 148, 149, 150, 162

[HLM19] S. Hofmann, P. Le, and A.J. Morris, *Carleson measure estimates and the Dirichlet problem for degenerate elliptic equations*, Anal. PDE **12** (2019), 2095–2146. Cited in: 53, 88, 272

[HL96] S. Hofmann, and J.L. Lewis, L^2 *solvability and representation by caloric layer potentials in time-varying domains*, Ann. of Math. **144** (1996), 349–420. Cited in: 9

[HLMP22] S. Hofmann, L. Li, S. Mayboroda, and J. Pipher, L^p *theory for the square roots and square functions of elliptic operators having a BMO anti-symmetric part*, Math. Z. **301** (2022), 935–976. Cited in: 272

[HLMP22] S. Hofmann, L. Li, S. Mayboroda, and J. Pipher, *The Dirichlet problem for elliptic operators having a BMO anti-symmetric part*, Math. Ann. **382** (2022), 103–168. Cited in: 272

[HM14] S. Hofmann, and J.M. Martell, *Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in* L^p, Ann. Sci. Éc. Norm. Supér. **47** (2014), 577–654. Cited in: 7, 53, 55, 105, 117, 118, 153

[HM15] S. Hofmann and J.M. Martell, *Uniform Rectifiability and harmonic measure IV: Ahlfors regularity plus Poisson kernels in* L^p *implies uniform rectifiability*, arxiv preprint 1505.06499. Cited in: 7, 106

[HMM16] S. Hofmann, J.M. Martell, and S. Mayboroda, *Uniform rectifiability, Carleson measure estimates, and approximation of harmonic functions*, Duke Math. J. **165** (2016), 2331–2389. Cited in: 7, 106, 108, 118, 144

[HMM24] S. Hofmann, J.M. Martell, and S. Mayboroda, *Transference of scale-invariant estimates from Lipschitz to Non-tangentially accessible to Uniformly rectifiable domains*, Anal. PDE. **17** (2024), 3251–3334. Cited in: 144

[HMMTZ21] S. Hofmann, J.M. Martell, S. Mayboroda, T. Toro, and Z. Zhao, *Uniform rectifiability and elliptic operators satisfying a Carleson measure condition*, Geom. Funct. Anal. **31** (2021), 325–401. Cited in: 7, 55, 88, 105, 144, 149, 156, 162

[HMT] S. Hofmann, J.M. Martell, and T. Toro, *Elliptic operators on non-smooth domains*, Book in preparation. Cited in: 38

[HMT17] S. Hofmann, J.M. Martell, and T. Toro, A_∞ *implies NTA for a class of variable coefficient elliptic operators*, J. Differ. Equ. **263** (2017), 6147–6188. Cited in: 7, 105, 108, 119, 148, 149, 161, 163, 189, 190

[HMU14] S. Hofmann, J.M. Martell, and I. Uriarte-Tuero, *Uniform rectifiability and harmonic measure, II: Poisson kernels in* L^p *imply uniform rectifiability*, Duke Math. J. **163** (2014), 1601–1654. Cited in: 7, 53, 55, 105

[HMaM15] S. Hofmann, S. Mayboroda, and M. Mourgoglou, *Layer potentials and boundary value problems for elliptic equations with complex* L^∞ *coefficients satisfying the small Carleson measure norm condition*, Adv. Math. **270** (2015), 480–564. Cited in: 7, 53

[HM02] S. Hofmann and A. McIntosh, *The solution of the Kato problem in two dimensions*, Proceedings of the Conference on Harmonic Analysis and PDE held in El Escorial, Spain in July 2000; Publ. Mat. vol. extra (2002), 143–160. Cited in: 272

[HMMM17] S. Hofmann, D. Mitrea, M. Mitrea, and A.J. Morris, *L^p-square function estimates on spaces of homogeneous type and on uniformly rectifiable sets*, Mem. Amer. Math. Soc. **245** (2017), 108 pp. Cited in: 116

[HMiM15] S. Hofmann, M. Mitrea, and A.J. Morris, *The method of layer potentials in L^p and endpoint spaces for elliptic operators with L^∞ coefficients*, Proc. Lond. Math. Soc. **111** (2015), 681–716. Cited in: 53

[JK82] D.S. Jerison, and C. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. Math. **46** (1982), 80–147. Cited in: 7, 36, 38, 53

[Jo65] F. John, *Quasi-isometric mappings*, Seminari 1962/63 Anal. Alg. Geom. e Topol., Vol. 2, Ist. Naz. Alta Mat, pages 462–473, Ed. Cremonese, Rome, 1965. Cited in: 201

[Ka98] B. Kawohl, *Symmetry or not?*, Math. Intelligencer **20** (1998), 16–22. Cited in: 223

[Ka66] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966. Cited in: 275, 276

[KW80] R. Kaufman, and J.M. Wu, *Singularity of parabolic measures*, Compos. Math. **40** (1980), 243–250. Cited in: 9, 157, 161, 169

[Ke94] C. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, Regional Conference Series in Mathematics 83, American Mathematical Society, 1994. Cited in: 36, 38, 78

[KKPT16] C. Kenig, B. Kirchheim, J. Pipher, and T. Toro, *Square functions and the A_∞ property of elliptic measures*, J. Geom. Anal. **26** (2016), 2383–2410. Cited in: 160

[KKPT00] C. Kenig, H. Koch, J. Pipher, and T. Toro, *A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations*, Adv. Math. **153** (2000), 231–298. Cited in: 53, 160

[KM85] C. Kenig and Y. Meyer, *The Cauchy integral on Lipschitz curves and the square root of second order accretive operators are the same*, Recent Progress in Fourier Analysis, I. Peral, ed., North Holland Math Studies **111** (1985), 123–145. Cited in: 273

[KP01] C. Kenig and J. Pipher, *The Dirichlet problem for elliptic equations with drift terms*, Publ. Mat. **45** (2001), 199–217. Cited in: 7, 105, 158, 160

[KT97] C. Kenig, and T. Toro, *Harmonic measure on locally flat domains*, Duke Math J. **87** (1997), 501–551. Cited in: 170

[KT99] C. Kenig, and T. Toro, *Free boundary regularity for harmonic measure and Poisson kernels*, Ann. of Math. **150** (1999), 369–454. Cited in: 170

[KT03] C. Kenig, and T. Toro, *Poisson kernel characterizations of Reifenberg flat chord arc domains*, Ann. Sci. École Norm. Sup. **36** (2003), 323–401. Cited in: 170

[KN77] D. Kinderlehrer, and L. Nirenberg, *Regularity in free boundary problems*, Ann. Sc. Norm. Sup. Pisa **4** (1977), 373–391. Cited in: 260, 261

[KS81] N. Krylov, and M. Safonov, *A certain property of solutions of parabolic equations with measurable coefficients*, Math. USSR Izv. **16** (1981), 151–164. Cited in: 273

[KP98] S. Kumaresan, and J. Prajapat, *Serrin's result for hyperbolic space and sphere*, Duke Math. J. **91** (1998), 17–28. Cited in: 223

[LS20] J. Lamboley, and P. Sicbaldi, *Existence and regularity of Faber-Krahn minimizers in a Riemannian manifold*, J. Math. Pures Appl. **141** (2020), 137–183. Cited in: 266, 267

[La73] E. Lanconelli, *Sul problema di Dirichlet per l'equazione del calore*, Ann. Math. Pura Appl. **97** (1973), 83–114. Cited in: 8

[Le96] C. Lederman, *A free boundary problem with a volume penalization*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **23** (1996), 249–300. Cited in: 222, 226

[LM95] J.L. Lewis, and M.A. Murray, *The method of layer potentials for the heat equation in time-varying domains*, Mem. Amer. Math. Soc. **114** (1995), 157 pp. Cited in: 9, 157, 161, 172, 174, 176

[LN07] J.L. Lewis, and K. Nyström, *On a parabolic symmetry problem*, Rev. Mat. Iberoam. **23** (2007), 513–536. Cited in: 163, 185, 187

[LS88] J.L. Lewis, and J. Silver, *Parabolic measure and the Dirichlet problem for the heat equation in two dimensions*, Indiana Univ. Math. J. **37** (1988), 801–839. Cited in: 161, 169

[LZ23] Y. Lian, and K. Zhang, *Boundary pointwise regularity and applications to the regularity of free boundaries*, Calc. Var. **62** (2023), 32 pp. Cited in: 261

[Li96] G.M. Lieberman, *Second order parabolic differential equations*, World scientific, 1996. Cited in: 43, 44, 180, 183, 184

[LSW63] W. Littman, G. Stampacchia, and H.F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm.-Sci. **17** (1963), 43–77. Cited in: 4, 6, 8, 32, 37, 51, 57, 74, 86, 234, 236

[MS79] R. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. Math. **33** (1979), 257–270. Cited in: 101

[MP20a] R. Magnanini, and G. Poggesi, *Nearly optimal stability for Serrin's problem and the soap bubble theorem*, Calc. Var. Partial Differ. Equ. **59** (2020), 23 pp. Cited in: 223

[MP20b] R. Magnanini, and G. Poggesi, *Serrin's problem and Alexandrov's soap bubble theorem: enhanced stability via integral identities*, Indiana Univ. Math. J. **69** (2020), 1181–1205. Cited in: 223

[MMM20] J.J. Marín, J.M. Martell, and M. Mitrea, *The generalized Hölder and Morrey-Campanato Dirichlet problems for elliptic systems in the upper-half space*, Potential Anal. **53** (2020), 947–976. Cited in: 54, 62

[MMMM22] J.J. Marín, J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea, *Singular integral operators, quantitative flatness, and boundary problems*, Progress in Mathematics, vol. 344, Birkhäuser/Springer, Cham, 2022. Cited in: 54

[MMMM16] J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea, *The Dirichlet problem for elliptic systems with data in Köthe function spaces*, Rev. Mat. Iberoam. **32** (2016), 913–970. Cited in: 54

[MMMM19] J.M. Martell, D. Mitrea, I. Mitrea, and M. Mitrea, *The* BMO*-Dirichlet problem for elliptic systems in the upper half-space and quantitative characterization of* VMO, Anal. PDE. **12** (2019), 605–720. Cited in: 54

[Ma95] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995. Cited in: 25

[MZ19] S. Mayboroda, and Z. Zhao, *Square function estimates, the* BMO *Dirichlet problem, and absolute continuity of harmonic measure on lower-dimensional sets*, Anal. PDE **12** (2019), 1843–1890. Cited in: 54

[Ma14] V.G. Maz'ya, *Notes on Hölder regularity of a boundary point with respect to an elliptic operator of second order*, J. Math. Sci. **196** (2014), 572–577. Cited in: 86

[Mc85] A. McIntosh, *Square roots of elliptic operators*, J. Funct. Anal. **61** (1985), 307–327. Cited in: 12

[Mc86] A. McIntosh, *Operators which have an H^∞ functional calculus*, Miniconference on Operator Theory and Partial Differential Equations, 1986; Proceedings of the Centre for Mathematical Analysis, ANU, Canberra, 1986, 210–231. Cited in: 277, 287, 290

[MPT13] E. Milakis, J. Pipher, and T. Toro, *Harmonic analysis on chord arc domains*, J. Geom. Anal. **23** (2013), 2091–2157. Cited in: 107

[MM80] L. Modica, and S. Mortola, *Construction of a singular elliptic-harmonic measure*, Manuscripta Math. **33** (1980), 81–98. Cited in: 7, 104, 232

[Mo64] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134. Cited in: 44

[MP21] M. Mourgoglou, and C. Puliatti, *Blow-ups of caloric measure in time varying domains and applications to two-phase problems*, J. Math. Pures Appl. **152** (2021), 1–68. Cited in: 9, 43, 44, 46

[MT20] M. Mourgoglou, and X. Tolsa, *Harmonic measure and Riesz transform in uniform and general domains*, J. Reine Angew. Math. **758** (2020), 183–221. Cited in: 106

[MT24a] M. Mourgoglou and X. Tolsa, *The regularity problem for the Laplace equation in rough domains*, Duke Math. J. **173** (2024), 1731–1837. Cited in: 8

[MT24b] M. Mourgoglou and X. Tolsa, *Solvability of the Neumann problem for elliptic equations in chord-arc domains with very big pieces of good superdomains*, arxiv preprint 2407.20385. Cited in: 8

[Mu72] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226. Cited in: 160

[Mu85] M.A.M. Murray, *Commutators with fractional differentiation and BMO Sobolev spaces*, Indiana Univ. Math. J. **34** (1985), 205–215. Cited in: 161

[Na97] N.S. Nadirashvili, *Non-uniqueness in the martingale problem and the Dirichlet problem for uniformly elliptic operators*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **24** (1997), 537–550. Cited in: 273

[Na58] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954. Cited in: 44, 46

[NTV14] F. Nazarov, X. Tolsa, and A. Volberg, *On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: The case of codimension 1*, Acta Math. **213** (2014), 237–321. Cited in: 7, 108

[Ny97] K. Nyström, *The Dirichlet problem for second order parabolic operators*, Indiana Univ. Math. J. **46** (1997), 183–245. Cited in: 162, 172, 173, 174, 176, 218

[Ny17] K. Nyström, *L^2 Solvability of boundary value problems for divergence form parabolic equations with complex coefficients*, J. Differ. Equ. **262** (2017), 2808–2939. Cited in: 272

[NS17] K. Nyström, and M. Strömqvist, *On the parabolic Lipschitz approximation of parabolic uniform rectifiable sets*, Rev. Mat. Iberoam. **33** (2017), 1397–1422. Cited in: 161

[PS09] F. Pacard, and P. Sicbaldi, *Extremal domains for the first eigenvalue of the Laplace-Beltrami operator*, Ann. Inst. Fourier **59** (2009), 515–542. Cited in: 11, 222, 223

[Pe35] I. Petrowsky, *Zur ersten Randwertaufgabe der Wärmeleitungsgleichung*, Compositio Math. **1** (1935), 383–419. Cited in: 8

[Pr98] J. Prajapat, *Serrin's result for domains with a corner or cusp*, Duke Math. J. **91** (1998), 29–31. Cited in: 223

[PT23] M. Prats, and X. Tolsa, *Notes on harmonic measure*, to appear, lecture notes available online at https://mat.uab.cat/~xtolsa/mesuraharmonica.pdf, 2023. Cited in: 30, 34, 36, 37, 38

[QX19] Z. Qian, and G. Xi, *Parabolic equations with singular divergence-free drift vector fields*, J. Lond. Math. Soc. **100** (2019), 17–40. Cited in: 46

[Ra77] J.W.S. Rayleigh, *The Theory of Sound*, Macmillan, London, 1877. Cited in: 223

[Ri03] J. Rivera-Noriega, *Absolute continuity of parabolic measure and area integral estimates in non-cylindrical domains*, Indiana Univ. Math. J. **52** (2023), 477–525. Cited in: 160

[RR16] F. Riesz, and M. Riesz, *Über die randwerte einer analtischen Funktion*, Compte Rendues du Quatrième Congrès des Mathématiciens Scandinaves, Stockholm 1916, Almqvists and Wilksels, Uppsala, 1920. Cited in: 105

[RRS20] A. Ros, D. Ruiz, and P. Sicbaldi, *Solutions to overdetermined elliptic problems in nontrivial exterior domains*, J. Eur. Math. Soc. **22** (2020), 253–281. Cited in: 223

[Ru22] D. Ruiz, *Nonsymmetric sign-changing solutions to overdetermined elliptic problems in bounded domains*, J. Eur. Math. Soc., in press. Cited in: 223

[Ru23] D. Ruiz, *Symmetry results for compactly supported steady solutions of the 2D Euler equations*, Arch. Ration. Mech. Anal. **247** (2023), 25 pp. Cited in: 223

[RSW22] D. Ruiz, P. Sicbaldi, and J. Wu, *Overdetermined elliptic problems in onduloid-type domains with general nonlinearities*, J. Funct. Anal. **283** (2022), 26 pp. Cited in: 223

[Sa18] M. Safonov, *On the boundary estimates for second-order elliptic equations*, Complex Var. Elliptic Equ. **63** (2018), 1123–1141. Cited in: 232

[Se71] J. Serrin, *A symmetry problem in potential theory*, Arch. Rat. Mech. Anal. **43** (1971), 304–318. Cited in: 10, 221, 223, 226

[St70] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, Princeton University Press, Princeton, N.J., 1970. Cited in: 18, 22

[St93] E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, Princeton University Press, Princeton, N.J., 1993. Cited in: 18, 21

[SW58] E.M. Stein, and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. **87** (1958), 159–172. Cited in: 298

[St80] R.S. Strichartz, *Bounded mean oscillation and Sobolev spaces*, Indiana Univ. Math. J. **29** (1980), 539–558. Cited in: 169

[St79] J.O. Strömberg, *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana Univ. Math. J. **28** (1979), 511–544. Cited in: 201

[Sh03] Z. Shen, *Boundary value problems in Morrey spaces for elliptic systems on Lipschitz domains*, Amer. J. Math. **125** (2003), 1079–1115. Cited in: 54

[Si87] D. Siegel, *The Dirichlet problem in a half-space and a new Phragmén-Lindelöf principle*, Maximum principles and eigenvalue problems in partial differential equations (Knoxville, TN, 1987), 208–217; Pitman Res. Notes Math. Ser. 175, Longman Sci. Tech., Harlow, 1988. Cited in: 53

[ST96] D. Siegel, and E.O. Talvila, *Uniqueness for the n-dimensional half space Dirichlet problem*, Pacific J. Math. **175** (1996), 571–587. Cited in: 53

[Si02] B. Sirakov, *Overdetermined Elliptic Problems in Physics*, Berestycki, H., Pomeau, Y. (eds), Nonlinear PDE's in Condensed Matter and Reactive Flows. NATO Science Series, vol 569. Springer, Dordrecht, 2002. Cited in: 223

[To96] J.F. Toland, *Stokes waves*, Topol. Methods Nonlinear Anal. **7** (1996) 1–48. Cited in: 223

[Tr14] M. Traizet, *Classification of the solutions to an overdetermined elliptic problem in the plane*, Geom. Func. Anal. **24** (2014), 690–720. Cited in: 223

[Ve23] B. Velichkov, *Regularity of the One-Phase Free Boundaries*, Lect. Notes Unione Mat. Ital., Springer, Cham, 2023. Cited in: 11, 222, 225, 226, 227, 228, 233, 236, 237, 239, 240, 241, 242, 243, 244, 247, 251, 252, 253, 255, 256, 257, 258, 259, 260, 261, 266

[Wa12] N.A. Watson, *Introduction to heat potential theory*, Math. Surveys Monogr. **182**, American Mathematical Society, Providence, RI, 2012. Cited in: 43, 44

[Wi24] N. Wiener, *The Dirichlet problem*, J. Math. and Phys. **3** (1924), 127–146. Cited in: 4

[Ya84] A. Yagi, *Coincidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'opérateurs*, C. R. Acad. Sc. Paris **299** (1984), 173–176. Cited in: 277, 287, 290

[Yo91] H. Yoshida, *Harmonic majorization of a subharmonic function on a cone or on a cylinder*, Pacific J. Math. **148** (1991), 369–395. Cited in: 53

[YM98] H. Yoshida, and I. Miyamoto, *Solutions of the Dirichlet problem on a cone with continuous data*, J. Math. Soc. Japan **50** (1998), 71–93. Cited in: 53

[Zh18] Z. Zhao, BMO *solvability and* A_∞ *condition of the elliptic measures in uniform domains*, J. Geom. Anal. **28** (2018), 866–908. Cited in: 38, 54