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PHD THESIS

Solutions for the Muskat equation with quadratic growth at infinity

Omar Sánchez Antonio

supervised by
Ángel Castro & Daniel Faraco

A mis padres

A Andrea

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Chapter 1

Introducción

1.1 El problema de Muskat

En esta tesis tratamos con el movimiento de un fluido incompresible en un medio poroso. De forma más precisa, consideramos una situación física propuesta por Morris Muskat en [49], debido a sus aplicaciones para la extracción de petróleo. Más allá de estas aplicaciones, la complejidad de este movimiento ha capturado la atención de muchos matemáticos en las últimas décadas. Existencia local y regularidad, formación de singularidades y el comportamiento a gran escala son, entre otras, algunas de las cuestiones que se han estudiado. Nuestro objetivo será mostrar la existencia de un nuevo tipo de soluciones y estudiar su comportamiento.

El problema de Muskat modela la interacción de dos fluidos inmisibles e incompresibles con diferentes densidades los cuales habitan un medio poroso. Los fluidos están separados por una interfaz, la cual divide el plano $\mathbb{R}^2$ en dos dominios $\Omega_+(t)$ y $\Omega_-(t)$. La ecuación que gobierna la dinámica de los fluidos es la ley de Darcy

$$\frac{\mu}{\kappa}\mathbf{v}^{\pm}(\mathbf{x},t) = -\nabla p^{\pm}(\mathbf{x},t) - \rho^{\pm}(\mathbf{x},t)\mathrm{g}\mathbf{e}_2 \quad \text{en} \quad \Omega_{\pm}.$$

Donde, $\mathbf{v}^{\pm}$ es la velocidad del fluido, $\rho^{\pm}$ la densidad y $p^{\pm}$ la presión de cada $\Omega_{\pm}$. La viscosidad μ, la permeabilidad κ y g la gravedad son constantes y asumiremos que son todas igual a 1. La densidad está definida por

$$\rho(\mathbf{x},t) = \begin{cases} \rho^+(\mathbf{x},t), & \mathbf{x} \in \Omega_+(t), \\ \rho^-(\mathbf{x},t), & \mathbf{x} \in \Omega_-(t), \end{cases} \qquad \text{para} \quad \mathbf{x} = (x,y) \in \mathbb{R}^2,$$

y satisface la ecuación de conservación de masa

$$\partial_t \rho(\mathbf{x},t) + \mathbf{v}(\mathbf{x},t) \cdot \nabla \rho(\mathbf{x},t) = 0 \quad \text{en} \quad \mathbb{R}^2,$$

en un sentido débil. En esta tesis, asumiremos que la densidad es una función paso y, entonces $\rho^{\pm}(\mathbf{x},t) = \rho^{\pm}$, con ρ^+ y ρ^- valores constantes. La velocidad está dada por

$$\mathbf{v}(\mathbf{x},t) = \begin{cases} \mathbf{v}^+(\mathbf{x},t), & \mathbf{x} \in \Omega_+(t), \\ \mathbf{v}^-(\mathbf{x},t), & \mathbf{x} \in \Omega_-(t), \end{cases}$$

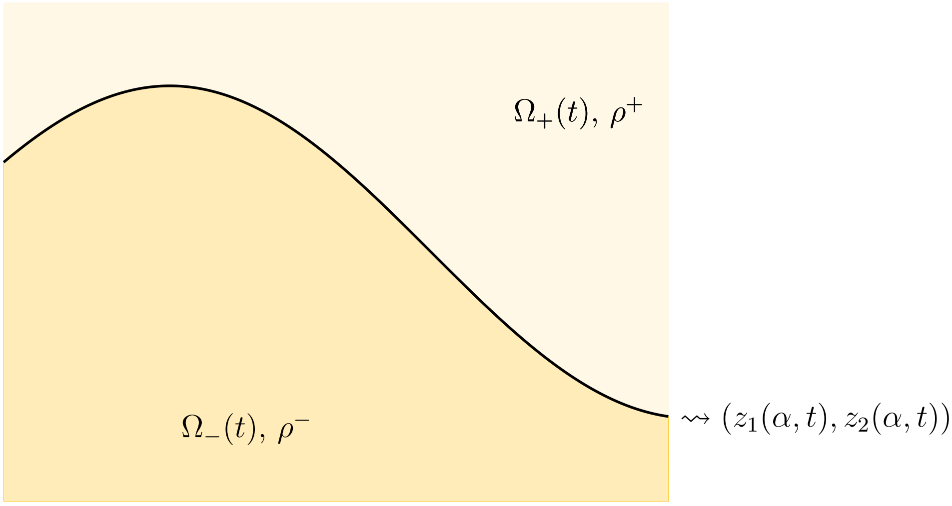
y además asumiremos que los fluidos son incompresibles, es decir

$$\mathrm{div}(\mathbf{v}^{\pm})(\mathbf{x},t) = 0 \quad \text{en} \quad \Omega_{\pm}.$$

La interfaz que comparten los fluidos está parametrizada por una curva arbitraria

$$\partial\Omega_{\pm}(t) = \{\mathbf{z}(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) : \alpha \in \mathbb{R}\},$$

como se ve en la siguiente figura.



Además, la ley de Darcy y la condición de incompresibilidad son suplementadas por las siguientes condiciones de frontera

$$\begin{aligned}(\mathbf{v}^+ - \mathbf{v}^-)\cdot\mathbf{n} = 0 \quad \text{en} \quad \partial\Omega_\pm,\\ p^+ = p^- \quad \text{en} \quad \partial\Omega_\pm,\end{aligned}$$

donde $\mathbf{n}$ denota el vector normal unitario $\partial\Omega_-$, que apunta a Ω_-

$$\mathbf{n} = -\frac{\partial_\alpha^\perp \mathbf{z}(\alpha,t)}{|\partial_\alpha \mathbf{z}(\alpha,t)|^2}.$$

Nótese, que la primera condición de frontera implica que $\operatorname{div}(\mathbf{v}) = 0$ en un sentido débil. La formulación matemática de este problema es la misma que la de dos fluidos incompresibles en una celda de Hele-Shaw, ver [57]. Formalmente, el problema de Muskat se puede reducir al estudio de una ecuación de evolución para la interfaz, la cual está dada por

$$\partial_t \mathbf{z}(\alpha,t) = \frac{\rho^- - \rho^+}{2\pi} PV \int_{\mathbb{R}} \frac{z_1(\alpha,t) - z_1(\alpha-\beta,t)}{|\mathbf{z}(\alpha,t) - \mathbf{z}(\alpha-\beta,t)|^2} (\partial_\alpha \mathbf{z}(\alpha,t) - \partial_\alpha \mathbf{z}(\alpha-\beta,t))\, \mathrm{d}\beta. \tag{1.1.1}$$

Cuando la interfaz está parametrizada por la gráfica de una función $(x, h(x,t))$, la ecuación anterior se reduce a

$$\frac{\mathrm{d}}{\mathrm{d}t} h(x,t) = \frac{\rho^- - \rho^+}{2\pi} PV \int_{\mathbb{R}} \frac{\alpha \cdot (\partial_x h(x,t) - \partial_x h(x-\alpha,t))}{\alpha^2 + (h(x,t) - h(x-\alpha,t))^2}\, \mathrm{d}\alpha. \tag{1.1.2}$$

La estabilidad de la ecuación de Muskat depende fuertemente del signo de la función de Rayleigh-Taylor, la cual está dada por

$$\mathrm{RT}(\mathbf{x},t) = -(\nabla p^-(\mathbf{x},t) - \nabla p^+(\mathbf{x},t))\cdot\mathbf{n}, \quad \mathbf{x}\in\partial\Omega_\pm.$$

En el caso donde parametrizamos la interfaz por $\mathbf{z}(\alpha,t)$ esta función se lee como

$$\mathrm{RT}(\alpha,t) = (\rho^- - \rho^+)\frac{\partial_\alpha z_1(\alpha,t)}{|\partial_\alpha \mathbf{z}(\alpha,t)|^2}.$$

Si parametrizamos la interfaz por el grafo de una función $(x, h(x,t))$ obtenemos

$$\mathrm{RT}(x,t) = (\rho^- - \rho^+)\frac{1}{1 + (\partial_x f(x))^2}.$$

Cuando $\mathrm{RT} > 0$, entonces el fluido más pesado está por debajo, el problema es estable. En este régimen, la existencia local de soluciones es conocida, así como la existencia global para dato inicial pequeño en un espacio de funciones adecuado. Sin embargo, si el fluido más pesado está por encima, la situación es inestable y (1.1.2) está mal planteado. Haremos un resumen de la literatura que trata sobre este problema en la sección 1.3.

1.2 Contenido de la tesis

Esta tesis está dividida en 5 capítulos. En el capítulo 2, estudiaremos la existencia de soluciones de la ecuación de Muskat con crecimiento cuadrático en el infinito. Los primeros estudios que demostraron la existencia de soluciones para el problema de Muskat consideraban el caso de interfaces que eran asintóticamente planas o periódicas (en la variable horizontal). Muy recientemente, varios autores han proporcionado algunos resultados que también prueban la existencia de soluciones con cierto crecimiento en el infinito (ver la sección 1.3 para un resumen). De manera notable, ninguna de estas soluciones han alcanzado un crecimiento cuadrático hasta nuestro resultado.

En el capítulo 2, asumiremos que la interfaz es parametrizada por el grafo de una función h, es decir

$$\partial\Omega_{\pm}(t) = \{(x, h(x,t)) : x \in \mathbb{R}\}$$

y nos enfocaremos en la existencia local de soluciones no triviales de tipo

$$h(x,t) = x^2 + ct + g(x,t),$$

donde g pertenece a un espacio de Sobolev adecuado. En particular g tiende a cero y, de manera notable, las soluciones crecen cuadráticamente en el infinito.

El resultado principal del capítulo 2 es el siguiente y está publicado en [59].

Teorema 1. *Dada $s \geq 3$, $\rho^- > \rho^+$, y $g_0 \in H^s(\mathbb{R})$. Entonces existe un tiempo $T_0 = T(\|g_0\|_{H^s}) > 0$ y una función $g \in L^\infty([0,T_0] : H^s(\mathbb{R})) \cap W^{1,\infty}([0,T_0] : H^{s-1}(\mathbb{R}))$ tal que la función*

$$h(x,t) = x^2 + (\rho^- - \rho^+)t + g(x,t)$$

es solución de la ecuación de Muskat con $h(x,0) = x^2 + g_0(x)$.

Para demostrar el teorema 1, primeramente se encuentra una ecuación de evolución para la función $g(x,t)$ a partir de la ecuación de Muskat. En esta nueva ecuación de evolución, hallamos nuevos núcleos en los cuales se tiene una dependencia explícita en la variable x. La dependencia en x es una de las principales diferencias con la ecuación de Muskat clásica. El primer objetivo será encontrar una estimación *a priori* de la energía de la función $g(x,t)$. Como es usual, la energía se define como la norma en espacios de Sobolev, es decir, $\|g\|_{H^s(\mathbb{R})}$. Dedicamos una sección completa para encontrar la estimación de la energía. El capítulo culmina con el estudio del sistema regularizado de esta ecuación. Las soluciones de la regularización son obtenidas por el Teorema de Picard. La función g en el teorema será obtenida como límite de este sistema regularizado gracias a la estimación de energía.

El capítulo 3 se enfoca en la búsqueda de singularidades tipo *turning* cuando la interfaz es parametrizada por una curva

$$\partial\Omega_{\pm}(t) = \{\mathbf{z}(\alpha,t) : \alpha \in \mathbb{R}\},$$

donde

$$\mathbf{z}(\alpha,t) = \mathbf{d}(\alpha,t) + (\alpha, \alpha^2 + ct).$$

Aquí $\mathbf{d}$ se comporta de manera que la interfaz crece cuadráticamente en el infinito.

Una pregunta natural que surge después de probar la existencia de soluciones con crecimiento cuadrático en el infinito es si estas soluciones persisten para siempre o, por el contrario, si existen soluciones que desarrollan una singularidad. Demostraremos que, de hecho, ocurre el segundo caso mencionado.

En este capítulo el resultado principal es el siguiente.

Teorema 2. *Existen soluciones de la ecuación de Muskat que desarrollan una singularidad de tipo* turning *y que crecen cuadráticamente en el infinito.*

En una singularidad de tipo *turning* la solución empieza en el régimen estable, esto significa que el fluido más denso está por debajo, de modo que puede ser parametrizada como el grafo de un función. Después, en un tiempo finito la solución gira y ya no puede parametrizarse como el grafo de una función, y hay una parte de la interfaz donde el fluido más denso está por encima. En estas singularidades, la solución pasa del régimen estable al régimen inestable Por lo tanto, la función de Rayleigh-Taylor RT pasa de ser positiva $\mathrm{RT} > 0$ a negativa $\mathrm{RT} < 0$.

Para demostrar el teorema 2, primeramente se debe garantizar la existencia local de soluciones para datos iniciales cuya función de Rayleigh-Taylor no es estrictamente positiva. Esto se logra utilizando una versión abstracta del Teorema de Cauchy-Kowaleski, (ver [54] y [53]), por lo que se debe considerar un entorno adecuado para su aplicación. De hecho construimos una escala de Banach $\{X_r\}_{r>0}$, que consiste de funciones analíticas definidas sobre una banda en el plano complejo. Las condiciones que se necesitan verificar en el Teorema de Cauchy-Kowaleski son, la *acotación* y la *condición Lipschitz* del operador (1.4.3) en $\{X_r\}_{r>0}$. La demostración de estas propiedades comprende cuatro secciones de este capítulo. Una vez completada la existencia local en $\{X_r\}_{r>0}$, se demuestra la existencia de soluciones que comienzan en el régimen estable, y en tiempo finito pasan al régimen inestable. El capítulo 4 contiene la demostración de resultados auxiliares usados en los capítulos 2 y 3.

En el capítulo 5, consideramos una modificación de la ecuación de Muskat que tiene en cuenta la tensión superficial. Esta nueva fuerza se introduce a través de una discontinuidad en el salto de la presión a lo largo de la interfaz, proporcional a la curvatura.

En este caso, no tratamos con soluciones que crecen cuadráticamente en el infinito, sino con interfaces asintóticamente planas.

En este capítulo damos una descripción de las soluciones estacionarias para el problema de Muskat con tensión superficial, buscando soluciones 2π-periódicas. Esto puede ser alcanzado reduciendo la ecuación de Muskat (1.2.1) a una ecuación diferencial ordinaria. La ecuación de Muskat con tensión superficial está dada por

$$\begin{aligned}&\partial_t \mathbf{z}(\alpha,t) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(\mathbf{z}(\alpha,t)-\mathbf{z}(\beta,t))^{\perp}}{|\mathbf{z}(\alpha,t)-\mathbf{z}(\beta,t)|^2} \partial_\beta \Big[\gamma\kappa(\mathbf{z}(\beta,t)) + \mathrm{g}\,(\rho^+-\rho^-) z_2(\beta,t)\Big]\,\mathrm{d}\beta,\\ &\mathbf{z}(\alpha,0) = \mathbf{z}^0(\alpha),\end{aligned} \tag{1.2.1}$$

donde $\gamma > 0$ es el parámetro de tensión superficial y κ es la curvatura de $\mathbf{z}$. Para un tratamiento del caso con diferentes viscosidades, ver [47], en este caso hemos considerado viscosidades iguales. Por lo tanto, las soluciones estacionarias de (1.2.1) son soluciones de la ecuación

$$\partial_\alpha \Big[\gamma\kappa(\mathbf{z})(\alpha) + \mathrm{g}(\rho^+-\rho^-) z_2(\alpha)\Big] = 0. \tag{1.2.2}$$

Recíprocamente, una solución de (1.2.2) es una solución estacionaria de (1.2.1) y por lo tanto una solución estacionaria del problema de Muskat. De este modo, una curva $\mathbf{z}\colon \mathbb{R} \to \mathbb{R}^2$ es una solución estacionaria de (1.2.1) si satisface

$$\gamma \frac{z_1' z_2'' - z_1'' z_2'}{(z_1'^2 + z_2'^2)^{3/2}} + \mathrm{g}\,(\rho^+-\rho^-) z_2 = const. \tag{1.2.3}$$

Ehrnstrom, Escher y Matioc en [33] encontraron un valor umbral con la siguiente propiedad: si el coeficiente de tensión superficial se mantiene por debajo de este umbral, existe una solución de (1.2.3), que puede parametrizarse como el grafo de una función 2π-periódica $h(\alpha)$, es decir, $(z_1(\alpha),\, z_2(\alpha)) = (\alpha,\, h(\alpha))$. Además, cuando el coeficiente de tensión superficial se aproxima a este umbral desde abajo, la pendiente máxima de la curva tiende a infinito. Estamos interesados en describir una solución con el siguiente dato inicial

$$\mathbf{z}(0) = (0,0) \quad \text{and} \quad \mathbf{z}'(0) = (-\sigma, 1), \quad \sigma > 0, \tag{1.2.4}$$

donde σ es un parámetro. En lo siguiente $B(x,y)$ representa a la función Beta

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t, \quad \mathrm{Re}(x), \mathrm{Re}(y) > 0.$$

El resultado principal de este capítulo es el siguiente teorema, el cual se encuentra publicado en [58], aquí hemos encontrado un régimen de tensiones superficiales por encima del límite de Ehrnstrom, Escher y Matioc, donde existen soluciones estacionarias, pero que dejan de ser un grafo. Sea

$$\lambda = \mathrm{g}\frac{(\rho^+ - \rho^-)}{\gamma},$$

el valor encontrado por Ehrnstrom, Escher y Matioc está dado por

$$\lambda_* = \frac{1}{2\pi^2}B^2\left(\frac{3}{4}, \frac{1}{2}\right).$$

Teorema 3. *Existe un $\lambda^* > 0$ más pequeño que λ_*, tal que para cada $\lambda \in (\lambda^*, \lambda_*]$ existe un único $\sigma = \sigma(\lambda) \in [0, \infty)$ y una curva $\mathbf{z}$ suave y periódica en la variable horizontal que es solución de la ecuación estacionaria (1.2.2) y que no se auto-interseca.*

Nota 1. *Las soluciones del teorema 3 no pueden ser parametrizadas como el grafo de una función $(\alpha, h(\alpha))$.*

1.3 Resumen de resultados previos

El problema de Muskat ha sido estudiado extensivamente en las últimas décadas. El primer resultado sobre existencia local fue establecido por Yi en [62], usando el método de iteración de Newton. Es importante destacar que en este trabajo no se considera la fuerza de gravedad, pero sí diferentes viscosidades, obteniendo soluciones clásicas del problema.

Ambrose en [6] usando una formulación para el ángulo tangente de la interfaz, demostró existencia local en el espacio $H^s(\mathbb{R})$ con $s \geq 3$, considerando diferentes viscosidades y diferentes densidades.

Por un lado, Caflish, Siegel and Howinson demostraron en [60] el mal planteamiento del problema para el caso inestable con densidades iguales y diferentes viscosidades. Por otro lado, Córdoba y Gancedo en [26] demostraron el mal planteamiento en el caso inestable para diferentes densidades, siguiendo las ideas presentadas [60]. En este mismo artículo, Córdoba y Gancedo demostraron existencia local de soluciones para el caso en 2d y 3d en espacios de Sobolev $H^s(\mathbb{R})$, $s \geq 3$ y $H^s(\mathbb{R}^2)$, $s \geq 4$ respectivamente. Fueron los primeros en introducir la formulación de la ecuación de Muskat (1.1.2).

En [21], Constantin, Gancedo, Shvydkoy y Vicol demostraron existencia local para dato pequeño en $W^{2,p}(\mathbb{R})$ con $p \in (1, \infty]$. Matioc en [46] demostró existencia local en $H^s(\mathbb{R})$ con $s \in (3/2, 2)$ en un escenario funcional analítico cuasilineal. Alazard y Lazar demostraron en [1] existencia local para dato inicial en $\dot{H}^1(\mathbb{R}) \cap \dot{H}^2(\mathbb{R})$ para $s > 3/2$, para obtener este resultado hicieron uso de cálculo paradiferencial, explotanto la parte no lineal de la ecuación de Muskat.

En [18], Cheng, Granero-Belichón and Shkoller demostraron la existencia global para dato pequeño en $H^2(\mathbb{T})$ con diferentes densidades y diferentes viscosidades, los autores utilizaron una formulación Lagrangiana del problema.

Las soluciones de la ecuación de Muskat (1.1.2) satisfacen $L^2(\mathbb{R})$ y $L^\infty(\mathbb{R})$ principios del máximo en tiempo, como demostraron Córdoba y Gancedo en [27] y, Constantin, Córdoba, Gancedo y Strain en [20] respectivamente. Los cuales están dados por

$$\|h(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|h_0\|_{L^\infty(\mathbb{R})},$$
$$\|h(\cdot,t)\|_{L^2(\mathbb{R})} + \frac{\rho^- - \rho^+}{2\pi}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\left(1 + \left(\frac{h(y,s) - h(x,s)}{y - x}\right)^2\right) \mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s = \|h_0\|_{L^2(\mathbb{R})}.$$

Otra propiedad notable de las soluciones, es su invariancia por la escala

$$h_\lambda(x,t) = \lambda^{-1}h(\lambda x, \lambda t),$$

es decir, si h es una solución, entonces h_λ también es solución. Los espacios que son invariantes bajo esta escala son llamados críticos, por ejemplo

$$\dot{H}^{3/2}(\mathbb{R}) \text{ y } \dot{W}^{1,\infty}(\mathbb{R}).$$

En [21], Constantin, Gancedo, Shvydkoy y Vicol, además de demostrar la existencia local, demostraron existencia global, cuando la pendiende de la interfaz h' se mantiene acotada. Posteriormente, en [9] Cameron estableció existencia global en el espacio $C^{1,\epsilon}(\mathbb{R})$ usando un criterio en términos del producto del supremo e ínfimo de la pendiente del dato inicial.

Para datos pequeños, Constantin, Córdoba, Gancedo y Strain en [20] demostraron la existencia global de soluciones en $H^3(\mathbb{R})$ con una derivada pequeña en el álgebra de Wiener $\mathcal{A}(\mathbb{R})$. También establecieron la existencia global de soluciones para dato inicial en $W^{1,\infty}(\mathbb{R})$ con la condición $\|h_0'\|_{L^\infty} < 1$. En un trabajo posterior [19] los mismos autores, junto con Rodríguez-Piazza, extendieron este resultado al caso 3d. En la misma dimensión, Gancedo y Lazar en [40] demostraron existencia global para el espacio crítico $\dot{H}^2(\mathbb{R}^2) \cap \dot{W}^{1,\infty}(\mathbb{R}^2)$.

En [30], Córdoba y Lazar demostraron la existencia global en el espacio $\dot{H}^{3/2}(\mathbb{R}) \cap \dot{H}^{5/2}(\mathbb{R})$ con una condición en $\dot{H}^{3/2}(\mathbb{R})$ haciendo uso de integrales oscilatorias y una nueva formulación de la ecuación de Muskat.

Alazard y Nguyen demostraron en [4], con un enfoque diferente, el mismo resultado de [40] y además la existencia de soluciones para datos iniciales que no son de tipo Lipschitz. En [51] Nguyen estableció la existencia global para dato inicial pequeño en espacios de Besov $\dot{B}^1_{\infty,1}(\mathbb{R}^d)$.

En el caso 3d, Gancedo y Lazar demostraron en [40] la existencia global para el espacio crítico $\dot{H}^2(\mathbb{R}^2) \cap \dot{W}^{1,\infty}(\mathbb{R}^2)$. Alazard y Nguyen probaron en [4], utilizando un enfoque diferente, el mismo resultado que en [40] y establecieron la existencia de soluciones para datos iniciales no-Lipschitz. Nguyen y Pausader demostraron en [52] la existencia local para datos iniciales en el espacio subcrítico $H^s(\mathbb{R}^d)$, donde $s > 1 + d/2$.

En un trabajo posterior [2], Alazard y Nguyen demostraron la existencia local para dato inicial pequeño en el espacio crítico $\dot{W}^{1,\infty}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})$, y la existencia global de soluciones para dato inicial pequeño. En [3] los mismos autores demostraron la existencia local y global para datos iniciales no Lipschitz. Recientemente los mismos autores en [5], demostraron la existencia local para dato pequeño en $H^{3/2}(\mathbb{R})$ y la existencia global en $H^{3/2}(\mathbb{R})$ con una condición pequeña en la norma $\dot{H}^{3/2}(\mathbb{R})$.

En [31] Deng, Lei y Lin construyeron soluciones débiles globales bajo la suposición que la interfaz inicial es decreciente y monótona con un comportamiento asintótico en el infinito, *i.e.* $f_0(x) \to a, x \to \infty$. Cameron en [10] demostró la existencia de soluciones en el caso 3d que son no acotadas y tienen crecimiento sublineal. En [42], García-Juárez, Gómez-Serrano, Nguyen y Pausader demostraron la existencia de soluciones auto-similares. En [41], García-Juárez, Gómez-Serrano, Haziot y Pausader demostraron la existencia local cuando la interfaz inicial tiene múltiples esquinas y crecimiento lineal en el infinito.

Es importante mencionar que ninguno de estos resultados permite crecimiento cuadrático de la interfaz en el infinito.

Córdoba, Córdoba y Gancedo demostraron, en [23], existencia local en $H^k(\mathbb{T})$ con $k \geq 3$, considerando diferentes viscosidades y RT positiva. En un trabajo posterior, los mismos autores en [24], estudiaron el caso 3d para una superficie H^4 considerando diferentes viscosidades. Gancedo, García-Juárez, Patel y Strain demostraron en [37] existencia global en los casos 2d y 3d, también considerando diferentes viscosidades.

En el régimen inestable $\rho^+ > \rho^-$, la ecuación de Muskat está mal planteada, ver por ejemplo [26] y [60], entonces soluciones tipo *mixing* son usadas para describir este escenario. En [11], Castro, Córdoba y Faraco estudiaron este tipo de soluciones haciendo uso de la integración convexa y la teoría de operadores pseudodiferenciales, ver también el trabajo de L. Székelyhidi, [61]. En la misma dirección se puede consultar [16], [55], [8] y [36].

Mengual en [48] estudió el caso inestable con diferentes viscosidades. Recientemente Castro, Faraco y Gebhard en [15], estudiaron la disipación de energía de potencial máxima como un criterio de selección para subsoluciones. Para otros resultados que conciernen a integración convexa aplicado a IPM, ver [25] y [44].

1.3.1 Singularidades

Respecto al estudio de singularidades a tiempo finito, Castro, Córdoba, Fefferman, Gancedo y López-Fernández en [14], demostraron que existe un subconjunto abierto de datos iniciales en H^4 tal que la condición de Rayleigh-Taylor se pierde en tiempo finito. Esto significa que la interfaz inicial es un grafo $\mathrm{RT} > 0$, entonces en un tiempo finito la interfaz deja de ser un grafo, $\mathrm{RT} < 0$. Este fenómeno se conoce como *turning singularity*.

En [12] Castro, Córdoba, Fefferman y Gancedo, demostraron que existen soluciones que pierden la condición de Rayleigh-Taylor y después en un tiempo finito, pierden regularidad. Estas soluciones singulares se han extendido en tiempo como soluciones tipo *mixing*, ver [17].

Córdoba, Gómez-Serrano y Zlatoš demostraron en [28] la existencia de soluciones que inician en el régimen inestable, entonces pasan al régimen estable y finalmente regresan al régimen inestable. Los mismos autores en [29] establecieron la existencia de soluciones que comienzan en el régimen estable, pasan al inestable y finalmente regresan al estable.

Todos estos estudios se han realizado para interfaces asintóticamente planas o periódicas. En esta tesis, nos ocuparemos de soluciones con crecimiento cuadrático en el infinito.

1.3.2 Soluciones estacionarias

La tensión superficial es una fuerza que puede surgir en la interfaz debido a la diferente naturaleza de los fluidos. Normalmente, este efecto se modela a través de la ley de Laplace-Young. Esta ley establece que, en presencia de tensión superficial, la diferencia del límite de la presión sobre la interfaz es proporcional a su curvatura. De hecho,

$$p^+ - p^- = \gamma\,\kappa(z)$$

donde $\gamma > 0$ es un parámetro que mide la intensidad del efecto de la tensión superficial. En este caso, el problema de frontera libre está bien planteado sin importar el signo de la función de Rayleigh-Taylor (ver [32] y [35]). Esto se debe a que la tensión superficial elimina la condición de inestabilidades de Rayleigh-Taylor. Este problema ha sido estudiado extensivamente, ver por ejemplo [43], [34], [56], [7], [39] y [38].

La existencia de soluciones estacionarias para el problema de Muskat con un coeficiente de tensión superficial grande fue estudiada por Ehrnstrom, Escher and Matioc en [33]. En términos generales, demostraron que existe una rama de soluciones, parametrizada por

$$\lambda = g\frac{\rho^+ - \rho^-}{\gamma}$$

de la forma $(\alpha, h_\lambda(\alpha))$. Aquí, $\lambda \in (0, \lambda^*)$, con $\lambda^* > 0$ siendo un número finito y semi-explícito. Acontece que, cuando $\lambda \to \lambda^*$, la pendiente de $h_\lambda(0) \to \infty$. Este hecho hace que Ehrnstrom, Escher y Matioc no continúen las soluciones más allá de λ^*. Realizaremos esta extensión parametrizando la interfaz como una curva

$$\mathbf{z}(\alpha) = (h(\alpha), \alpha).$$

en el intervalo $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Notemos que, dado que $\lambda > 0$, necesitamos que $\rho^+ > \rho^-$. Después de eso, construiremos la solución en el resto de los puntos utilizando las simetrías de la ecuación.

Nótese que, dado que $\lambda > 0$, requerimos que $\rho^+ > \rho^-$.

1.4 El problema de Muskat para la parábola

El primer paso es demostrar que $f(x,t) = x^2 + ct$ es una solución explícita de la ecuación de Muskat. Se tiene el siguiente lema.

Lema 1. *La parábola $f(x,t) = x^2 + ct$ es solución de la ecuación de Muskat con $c = \rho^- - \rho^+ > 0$.*

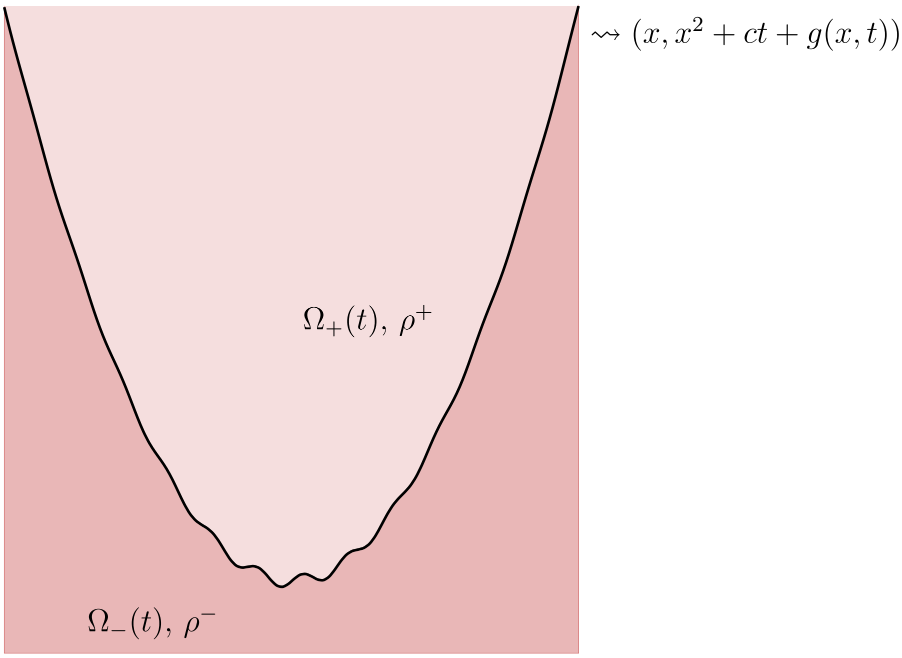
Demostración. Primero calculamos las diferencias

$$\begin{aligned} & f(x) - f(x-\alpha) = \alpha(2x-\alpha), \\ & \partial_x f(x) - \partial_x f(x-\alpha) = 2\alpha, \\ & \partial_t f = c. \end{aligned}$$

Entonces sustituimos en la ecuación de Muskat

$$\begin{aligned} c &= \frac{\rho^- - \rho^+}{2\pi} \int_{\mathbb{R}} \frac{2\alpha^2}{\alpha^2 + \alpha^2(2x-\alpha)^2} \, \mathrm{d}\alpha \\ &= \frac{\rho^- - \rho^+}{\pi} \int_{\mathbb{R}} \frac{1}{1 + (2x-\alpha)^2} \, \mathrm{d}\alpha \\ &= \frac{\rho^- - \rho^+}{\pi} \int_{\mathbb{R}} \frac{1}{1+u^2} \, \mathrm{d}u, \quad u = 2x - \alpha \\ &= \rho^- - \rho^+. \end{aligned}$$

■



La figura anterior ilustra la situación que abordamos. Por renormalización tomaremos $\rho^- - \rho^+ = 2\pi$. Entonces la función $f(x,t) = x^2 + 2\pi t$ es solución explícita de la ecuación de Muskat y representa a una parábola que se mueve a lo largo del eje vertical cuando $t \to +\infty$. Ahora, deduciremos la ecuación de evolución para la función g que se estudia en el capítulo 2.

En lo que sigue, omitiremos la dependencia del tiempo. Primero definimos la diferencia $\delta_\alpha g(x)$ y la pendiente $\Delta_\alpha g(x)$ por

$$\delta_\alpha g(x) := g(x) - g(x-\alpha) \quad \text{y} \quad \Delta_\alpha g(x) := \frac{g(x) - g(x-\alpha)}{\alpha}.$$

Al sustituir la función $h = f + g$ en la ecuación de Muskat (1.1.2), observamos que g satisface

$$\frac{\mathrm{d}}{\mathrm{d}t} g(x) + 2\pi = PV \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha g(x)}{1 + (\Delta_\alpha h(x))^2} \, \mathrm{d}\alpha + PV \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f(x)}{1 + (\Delta_\alpha h(x))^2} \, \mathrm{d}\alpha. \tag{1.4.1}$$

También omitiremos la dependencia en x en la diferencia $\Delta_\alpha g(x)$. Por la definición de f tenemos

$$2\pi = \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2} \, \mathrm{d}\alpha.$$

De este modo añadiendo el término 2π al lado derecho de la ecuación (1.4.1), obtenemos la siguiente expresión

$$\frac{\mathrm{d}}{\mathrm{d}t}g(x) = PV\int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha g}{1+(\Delta_\alpha h)^2}\,\mathrm{d}\alpha + PV\int_{\mathbb{R}} \Delta_\alpha g \frac{(-2)(\Delta_\alpha h + \Delta_\alpha f)}{(1+(\Delta_\alpha h)^2)(1+(\Delta_\alpha f)^2)}\,\mathrm{d}\alpha.$$

Definimos los siguientes núcleos

$$K(x,\alpha) := \frac{1}{1+(\Delta_\alpha h)^2}, \quad G(x,\alpha) := -2\frac{\Delta_\alpha h + \Delta_\alpha f}{(1+(\Delta_\alpha h)^2)(1+(\Delta_\alpha f)^2)}.$$

Entonces la ecuación (1.4.1) es equivalente a la ecuación

$$\frac{\mathrm{d}}{\mathrm{d}t}g(x,t) = PV\int_{\mathbb{R}} \partial_x \Delta_\alpha g(x) K(x,\alpha)\,\mathrm{d}\alpha + PV\int_{\mathbb{R}} \Delta_\alpha g(x) G(x,\alpha)\,\mathrm{d}\alpha. \tag{1.4.2}$$

Enfatizamos que el análisis de la ecuación (1.4.2) para la evolución de $g(x,t)$ presenta diferencias con respecto al análisis de la ecuación (1.1.2) en espacios de Sobolev $H^s(\mathbb{R})$ o $\dot{H}^k(\mathbb{R})$ con $0 \leq k \leq 2$. En efecto, el crecimiento cuadrático en el infinito introduce una degeneración de los núcleos en el infinito que necesitan ser comprendidos. En adición, la dependencia explícita de la variable x conduce al estudio de operadores pseudodiferenciales, lo que es opuesto a los operadores diferenciales que aparecen en el estudio de la ecuación de Muskat clásica. Para aclarar la diferencia entre los núcleos, en la ecuación de Muskat clásica, el núcleo es de la forma

$$K(y, h(x), h(x-y)),$$

mientras que en la ecuación (1.4.2) tenemos dos núcleos de la forma

$$K(x, y, g(x), g(x-y)).$$

De modo que, el objetivo será demostrar la existencia local (1.4.2) para un dato inicial $g(x,0) = g_0(x) \in H^s(\mathbb{R})$.

En la segunda parte de las tesis, para comenzar se tiene el siguiente resultado, que es consecuencia directa del lema anterior.

Corolario 1. *La curva* $\mathbf{p}(\alpha,t) = (\alpha, \alpha^2 + ct)$ *es solución de la ecuación de Muskat con* $c = \rho^- - \rho^+$.

Como en el caso anterior, derivamos una ecuación de evolución para la desviación $\mathbf{d} = \mathbf{z} - \mathbf{p}$, usando la ecuación (1.1.1). Se tiene

$$\begin{aligned}\partial_t \mathbf{d}(\alpha,t) &= PV\int_{\mathbb{R}} \frac{z_1(\alpha,t) - z_1(\alpha-\beta,t)}{|\mathbf{z}(\alpha,t)-\mathbf{z}(\alpha-\beta,t)|^2}(\partial_\alpha \mathbf{z}(\alpha,t) - \partial_\alpha \mathbf{z}(\alpha-\beta,t))\,\mathrm{d}\beta - \partial_t \mathbf{p}(\alpha,t)\\ &:= \mathbf{F}(\mathbf{d})(\alpha,t).\end{aligned}$$

Entonces, $\mathbf{F}(\mathbf{d})(\alpha,t)$ tiene la siguiente expresión

$$\mathbf{F}(\mathbf{d})(\alpha,t) = PV\int_{\mathbb{R}} \frac{z_1(\alpha,t) - z_1(\alpha-\beta,t)}{|\mathbf{z}(\alpha,t)-\mathbf{z}(\alpha-\beta,t)|^2}(\partial_\alpha \mathbf{z}(\alpha,t) - \partial_\alpha \mathbf{z}(\alpha-\beta,t))\,\mathrm{d}\beta - (0, 2\pi). \tag{1.4.3}$$

Dada una desviación inicial $\mathbf{d}^0(\alpha)$, queremos resolver la ecuación anterior (1.4.3) con la condición inicial $\mathbf{d}(\alpha,0) = \mathbf{d}^0(\alpha)$.

Para controlar este tipo de términos, trabajaremos con las transformadas de Hilbert de funciones racionales. Recordemos que la transformada de Hilbert está definida por

$$Hf(x) = \frac{1}{\pi}PV\int_{\mathbb{R}} \frac{f(x-y)}{y}\,\mathrm{d}y$$

y la trasformada de Hilbert truncada está dada por

$$H_{|y|<\epsilon}f(x) = \frac{1}{\pi}PV\int_{|y|<\epsilon} \frac{f(x-y)}{y}\,\mathrm{d}y.$$

Un hecho importante que se usará a lo largo de la tesis es que ambos operadores, la transformada de Hilbert y la trasformada de Hilbert truncada, son acotados de L^2 a L^2.

Chapter 1

Introduction

1.1 The Muskat problem

This thesis deals with the motion of an incompresible and inviscid fluid in a porous medium. More precisely, we will consider a physical situation proposed by Morris Muskat in [49] because of its applications for oil extraction. Beyond these applications, the complexity of this motion has captured the attention of many mathematicians in last decades. Local existence and regularity, singularity formation, and large behaviour are, among others, some questions that have been studied. The main goal for us will be to show the existence of a new type of solutions and to study its behaviour.

The Muskat problem models the interaction of two immiscibles and incomprensible fluids with different densities in a porous medium. The fluids are separated by an interface, which divides the plane $\mathbb{R}^2$ in two domains Ω_+ and Ω_-. The equation governing the dynamic of the fluids is Darcy's law

$$\frac{\mu}{\kappa}\mathbf{v}^{\pm}(\mathbf{x},t) = -\nabla p^{\pm}(\mathbf{x},t) - \rho^{\pm}(\mathbf{x},t)\mathrm{g}\mathbf{e}_2 \quad \text{in} \quad \Omega_{\pm}.$$

Where, $\mathbf{v}^{\pm}$ is the velocity fluid, $\rho^{\pm}$ the density and $p^{\pm}$ the pressure for each $\Omega_{\pm}$. The viscosity μ, the permeability κ and g the gravity are constants and we will assume that they are all equal to 1. The density is defined by

$$\rho(\mathbf{x},t) = \begin{cases} \rho^{+}(\mathbf{x},t), & \mathbf{x} \in \Omega_+(t), \\ \rho^{-}(\mathbf{x},t), & \mathbf{x} \in \Omega_-(t), \end{cases} \quad \text{for} \quad \mathbf{x} = (x,y) \in \mathbb{R}^2,$$

and satisfies the mass conservation equation

$$\partial_t \rho(\mathbf{x},t) + \mathbf{v}(\mathbf{x},t) \cdot \nabla \rho(\mathbf{x},t) = 0 \quad \text{in} \quad \mathbb{R}^2,$$

in a weak sense. In this thesis, we assume that the density is a step function, then $\rho^{\pm}(\mathbf{x},t) = \rho^{\pm}$, where $\rho^{\pm}$ are constants. The velocity is given by

$$v(\mathbf{x},t) = \begin{cases} v^{+}(\mathbf{x},t), & \mathbf{x} \in \Omega_+(t), \\ v^{-}(\mathbf{x},t), & \mathbf{x} \in \Omega_-(t), \end{cases}$$

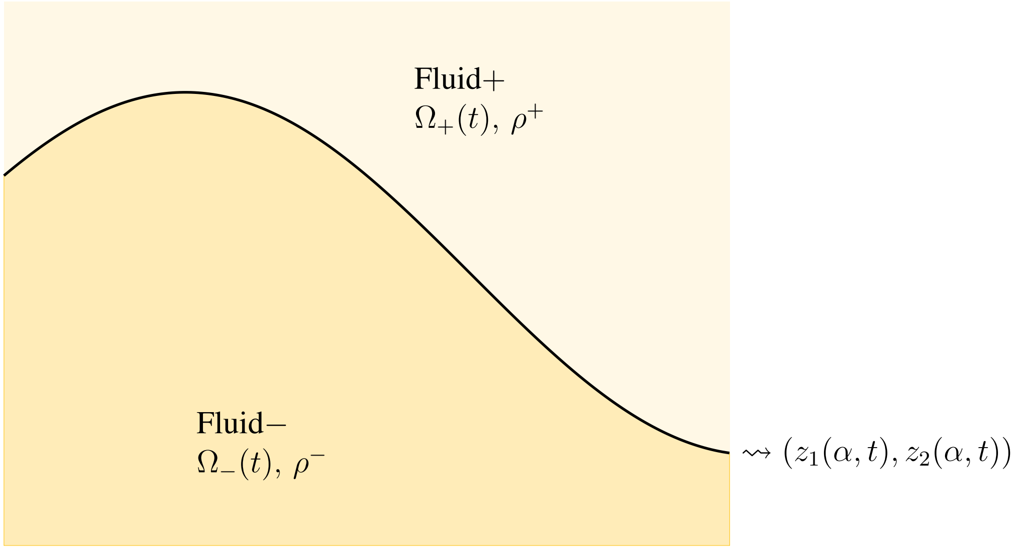
and we will also assume that the fluids are incompressible, that is

$$\mathrm{div}(v^{\pm}) = 0 \quad \text{in} \quad \Omega_{\pm}.$$

The interface shared by the fluids is parameterized by an arbitrary curve

$$\partial\Omega_{\pm}(t) = \{\mathbf{z}(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) : \alpha \in \mathbb{R}\},$$

as we see in the next figure.



Additionally, the Darcy's law and the incompressibility condition are supplemented by the following boundary conditions

$$\begin{aligned}&(\mathbf{v}^+ - \mathbf{v}^-)\cdot\mathbf{n} = 0 \quad \text{in} \quad \partial\Omega_\pm,\\ &p^+ = p^- \quad \text{in} \quad \partial\Omega_\pm,\end{aligned}$$

where $\mathbf{n}$ denotes the unit normal vector to $\partial\Omega_-$, pointing out Ω_-

$$\mathbf{n} = -\frac{\partial_\alpha^\perp \mathbf{z}(\alpha)}{|\partial_\alpha \mathbf{z}(\alpha)|^2}.$$

Notice, the first boundary condition implies that $\mathrm{div}(\mathbf{v}) = 0$ in a weak sense. The mathematical formulation of this problem is the same as that for two incompressible fluids in a Hele-Shaw cell, see [57]. Formally, the Muskat problem can be reduced to the study of an evolution equation for the interface, which is given by

$$\partial_t \mathbf{z}(\alpha,t) = \frac{\rho^- - \rho^+}{2\pi} PV \int_{\mathbb{R}} \frac{z_1(\alpha,t) - z_1(\alpha-\beta,t)}{|\mathbf{z}(\alpha,t) - \mathbf{z}(\alpha-\beta,t)|^2} (\partial_\alpha \mathbf{z}(\alpha,t) - \partial_\alpha \mathbf{z}(\alpha-\beta,t))\,\mathrm{d}\beta. \tag{1.1.1}$$

When the interface is parameterized by the graph of a function $(x,\ h(x))$, the last equation reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} h(x,t) = \frac{\rho^- - \rho^+}{2\pi} PV \int_{\mathbb{R}} \frac{\alpha\cdot(\partial_x h(x,t) - \partial_x h(x-\alpha,t))}{\alpha^2 + (h(x,t) - h(x-\alpha,t))^2}\,\mathrm{d}\alpha. \tag{1.1.2}$$

The stability of the Muskat equation, strongly depends on the sign of the Rayleigh-Taylor function, which is given by

$$\mathrm{RT}(\mathbf{x},t) = -(\nabla p^-(\mathbf{x},t) - \nabla p^+(\mathbf{x},t))\cdot\mathbf{n}, \quad \mathbf{x}\in\partial\Omega_\pm.$$

In the case where we parameterize the interface by $\mathbf{z}(\alpha,t)$ this function reads

$$\mathrm{RT}(\alpha,t) = (\rho^- - \rho^+)\frac{\partial_\alpha z_1(\alpha,t)}{|\partial_\alpha \mathbf{z}(\alpha,t)|^2}.$$

If we parameterize the interface by the graph of a function $(x,\ h(x,t))$ we just have

$$\mathrm{RT}(x,t) = (\rho^- - \rho^+)\frac{1}{1 + (\partial_x f(x))^2}.$$

When $\mathrm{RT} > 0$, thus the heaviest fluid is below, the problem is stable. In this regime, local existence of solutions is very well known, as well as global existence for small initial data in suitable functional spaces. However, if the heaviest fluid is above, the situation is unstable and (1.1.2) is ill-posed. We will review some of the literature dealing with these issues in section 1.3.

1.2 Content of the thesis

This thesis is divided in 5 chapter. In chapter 2, we will study the existence of solutions of the Muskat equation with quadratic growth at the infinity. The first studies showing existence of solutions for the Muskat problem considered the case of either asymptotically flat or periodic (in the horizontal variable) interfaces. Very recently, some results proving also existence of solutions with some growth at the infinity have been provided by several authors (see section 1.3 for a summary). Remarkably, none of these solutions had reached quadratic growth until our result.

In chapter 2, we will assume that the interface is parameterized by the graph of a function h, that is

$$\partial\Omega_{\pm}(t) = \{(x, h(x,t)) : x \in \mathbb{R}\}$$

and we will focus on the local existence of non trivial solutions of the form

$$h(x,t) = x^2 + ct + g(x,t),$$

where g is in a suitable Sobolev space. In particular g tends to zero, and remarkably, the solutions grow quadratically at infinity.

The main result of chapter 2 is the following and is published in [59].

Theorem 1. *Let $s \geq 3$, $\rho^- > \rho^+$, and $g_0 \in H^s(\mathbb{R})$. Then there exists a time $T_0 = T(\|g_0\|_{H^s}) > 0$ and a function $g \in L^\infty([0,T_0] : H^s(\mathbb{R})) \cap W^{1,\infty}([0,T_0] : H^{s-1}(\mathbb{R}))$ such that the function*

$$h(x,t) = x^2 + (\rho^- - \rho^+)t + g(x,t)$$

solves (1.1.2) with $h(x,0) = x^2 + g_0(x)$.

In order to proof theorem 1, firstly we derive an evolution equation for the function $g(x,t)$ from the Muskat equation. In this new evolution equation, we find new kernels that have an explicit dependence on the variable x. The dependence on x is one of the main difference compared to the classical Muskat equation. The first objective is to find an *a priori* estimate of the energy of the function $g(x,t)$. As usual, the energy is defined as the norm in Sobolev spaces, that is, $\|g\|_{H^s(\mathbb{R})}$. We dedicate a complete section to finding the energy estimate. The chapter concludes with the study of a regularization of this equation. Solutions for the regularization are obtained by a Picard Theorem. The function g in the theorem, will be obtained as a limit of these solutions thanks to the energy estimates.

Chapter 3 focuses on the search of *turning* singularities when the interface is parameterized by a curve

$$\partial\Omega_{\pm}(t) = \{\mathbf{z}(\alpha,t) : \alpha \in \mathbb{R}\},$$

where

$$\mathbf{z}(\alpha,t) = \mathbf{d}(\alpha,t) + (\alpha, \alpha^2 + ct).$$

Here $\mathbf{d}$ is such that the interface grows quadratically at the infinity.

A natural question that arises after proving the existence of solutions with quadratic growth at the infinity, is whether these solutions live forever or whether, on the contrary, there exist solutions that develop a singularity. We will prove that, in fact, the second previous case happens.

The main result of this chapter is the following.

Theorem 2. *There are solutions to the Muskat equation that grow quadratically at the infinity and develop a* turning *singularity.*

In a *turning* singularity the solution starts in the stable regime, this means the densest fluid is below, thus the function can be parameterize by the graph of a function. Then, at some finite time, the solution turns, it can not be parameterize as the graph of function, and there is a part of interface where the densest fluid is above. In this singularities the solution passes from the stable to the unstable regime. Therefore, the Rayleigh-Taylor function RT passes from $\mathrm{RT} > 0$ to $\mathrm{RT} < 0$.

In order to prove theorem 2, firstly local existence of solutions must be guaranteed for initial data in the case where the Rayleigh-Taylor function is not strictly positive. This is achieve using an abstract version of Cauchy-Kowaleski's Theorem (see [54] and [53]), then a suitable setting must be considered for its application. Indeed, we construct a scale of Banach spaces $\{X_r\}_{r>0}$, which consists of analytic functions defined over a strip in the complex plane. The conditions of the Cauchy-Kowaleski's Theorem that need to be verified are the *boundedness* and the *Lipschitz condition* of the operator (1.4.4) in $\{X_r\}_{r>0}$. The proof of these properties comprises four sections of this chapter. Once local existence in $\{X_r\}_{r>0}$ is completed, the existence of solutions that start in the stable regime and, in finite time, transition to the unstable regime, is proved. The chapter 4 contains the proofs of auxiliary results used in chapters 2 and 3.

In chapter 5, we consider a modification of the Muskat equation that takes into account surface tension. This new force is introduced through a jump discontinuity of the pressure across the interface, proportional to the curvature.

In this case, we do not deal with solutions which grow quadratically at the infinity, but with asymptotically flat interfaces.

In this chapter we give a description of the stationary solutions for the Muskat problem with surface tension, looking for 2π-periodic solutions. This can be achieved by reducing the Muskat equation (1.2.1) to an ordinary differential equation. The Muskat equation with surface tension is given by

$$\begin{aligned} &\partial_t \mathbf{z}(\alpha,t) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(\mathbf{z}(\alpha,t)-\mathbf{z}(\beta,t))^{\perp}}{|\mathbf{z}(\alpha,t)-\mathbf{z}(\beta,t)|^2} \partial_\beta \Big[\gamma\kappa(\mathbf{z}(\beta,t)) + \mathrm{g}\,(\rho^+-\rho^-) z_2(\beta,t)\Big]\, \mathrm{d}\beta, \\ &\mathbf{z}(\alpha,0) = \mathbf{z}^0(\alpha). \end{aligned} \tag{1.2.1}$$

where $\gamma > 0$ is the surface tension parameter and κ is the curvature of $\mathbf{z}$. For a treatment of the case with different viscosities, see [47], in this case we consider equal viscosities. Therefore, the stationary solutions of (1.2.1) are solutions of the equation

$$\partial_\alpha \Big[\gamma\kappa(\mathbf{z})(\alpha) + \mathrm{g}(\rho^+-\rho^-) z_2(\alpha)\Big] = 0. \tag{1.2.2}$$

Conversely, a solution of (1.2.2) is a stationary solution of (1.2.1) and therefore a stationary solution of the Muskat problem. Thus, a curve $\mathbf{z}\colon \mathbb{R} \to \mathbb{R}^2$ is a stationary solution of (1.2.1) if satisfies

$$\gamma \frac{z_1' z_2'' - z_1'' z_2'}{(z_1'^2 + z_2'^2)^{3/2}} + \mathrm{g}\,(\rho^+-\rho^-) z_2 = const. \tag{1.2.3}$$

Ehrnstrom, Escher and Matioc in [33] found a threshold with the following property: if the surface tension coefficient remains below this threshold, there exists a solution of (1.2.3), that can be parameterized as the graph of a 2π-periodic function $h(\alpha)$, i.e., $(z_1(\alpha),\, z_2(\alpha)) = (\alpha,\, h(\alpha))$. In addition, when the surface tension coefficient approaches to this threshold from below, the maximal slope of the curve tends to infinity. We are interested in describing a solution satisfying the following condition

$$\mathbf{z}(0) = (0,0) \quad \text{and} \quad \mathbf{z}'(0) = (-\sigma, 1), \quad \sigma > 0, \tag{1.2.4}$$

where σ is a parameter. In the following $B(x,y)$ stands for the Beta function,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{d}t, \quad \mathrm{Re}(x), \mathrm{Re}(y) > 0.$$

The main result of this chapter is the following theorem, published in [58]. Here we have found a regime of surface tensions above the limit of Ehrnstrom, Escher, and Matioc, where stationary solutions exist, but which cease to be a graph. Let

$$\lambda = \mathrm{g}\frac{(\rho^+-\rho^-)}{\gamma},$$

the value found by Ehrnstrom, Escher, and Matioc is given by

$$\lambda_* = \frac{1}{2\pi^2} B^2 \left(\frac{3}{4}, \frac{1}{2} \right).$$

Theorem 3. *There exists $\lambda^* > 0$ smaller than λ_*, such that for each $\lambda \in (\lambda^*, \lambda_*]$ there exists a unique $\sigma = \sigma(\lambda) \in [0, \infty)$ and a smooth and periodic in the horizontal variable curve $\mathbf{z}$ which solves the steady-state equation (1.2.2) and that does not self-intersect.*

Remark 1. *The solutions in theorem 3 can not be parameterized as the graph of a function $(\alpha, h(\alpha))$.*

1.3 Summary of previous results

The Muskat problem has been extensively studied in the last decades. The first local existence result was established by Yi in [62], using Newton's iteration method. It is important to note that this work gravity is not considered, but different viscosities are taken into account, obtaining classical solutions to the problem.

Ambrose in [6], using a formulation for the tangent angle, proved local existence in the space $H^s(\mathbb{R})$ where $s \geq 3$, considering different viscosities and different densities.

In one hand, Caflish, Siegel and Howison proved in [60] ill-posedness of the problem for the unstable case with equal densities and different viscosities. On the other hand, Córdoba and Gancedo in [26] proved the ill-posedness in the unstable case for different densities, following the ideas presented in [60]. In this same work Córdoba and Gancedo proved the local existence for the 2d and 3d case in Sovolev spaces $H^s(\mathbb{R})$, $s \geq 3$, and $H^s(\mathbb{R}^2)$, $s \geq 4$ respectively. They were the first who introduced the formulation of the Muskat equation (1.1.2).

In [21], Constantin, Gancedo, Shvydkoy and Vicol proved local existence for initial data in $W^{2,p}(\mathbb{R})$ for $p \in (1, \infty]$. Matioc in [46] proved local existence for initial data $H^s(\mathbb{R})$ with $s \in (3/2, 2)$ in a quasilinear analytic functional setting. Alazard and Lazar established in [1] local existence for initial data in $\dot{H}^1(\mathbb{R}) \cap \dot{H}^s(\mathbb{R})$ with $s > 3/2$, to obtain this result, they make use of paradifferential calculus, exploiting the nonlinear part of the Muskat equation.

In [18] Cheng, Granero-Belichón and Shkoller proved global existence for a small initial data in $H^2(\mathbb{T})$ with different viscosities and densities. The authors make use of a Lagrangian formulation for the problem.

Solutions of the Muskat equation (1.1.2) satisfy a $L^\infty(\mathbb{R})$ and $L^2(\mathbb{R})$ maximum principles, as proven Córdoba and Gancedo in [27] and, Constantin, Córdoba, Gancedo and Strain in [20]. Which are given by

$$\begin{aligned}
&\|h(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|h_0\|_{L^\infty(\mathbb{R})}, \\
&\|h(\cdot, t)\|_{L^2(\mathbb{R})} + \frac{\rho^- - \rho^+}{2\pi} \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \ln \left(1 + \left(\frac{h(y,s) - h(x,s)}{y - x} \right)^2 \right) \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}s = \|h_0\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Another property of solutions, is their scale invariance

$$h_\lambda(x, t) = \lambda^{-1} h(\lambda x, \lambda t),$$

i.e. if h is a solution then h_λ is also a solution. The spaces which are invariant under this scaling are called critical spaces, for example

$$\dot{H}^{3/2}(\mathbb{R}) \text{ and } \dot{W}^{1,\infty}(\mathbb{R}).$$

In [21], Constantin, Gancedo, Shvydkoy and Vicol, besides proving local existence, they proved global existence, when the slope h' remains bounded. Later, in [9], Cameron established global existence in $C^{1,\epsilon}(\mathbb{R})$ using a criteria in terms of the product of the supremum and infimum of the slope of the initial data.

For a small data Constantin, Córdoba, Gancedo and Strain in [20] proved global existence for initial data in $H^3(\mathbb{R})$ with a small derivative in the Wiener algebra $\mathcal{A}(\mathbb{R})$. They also established the existence of global weak solutions for initial data in $W^{1,\infty}(\mathbb{R})$ with the condition $\|h_0'\|_{L^\infty} < 1$. In a subsequent paper [19], the same authors, together with Rodríguez-Piazza, extended these results to the 3d case. In the same dimension Gancedo and Lazar in [40], proved global existence for the critical space $\dot{H}^2(\mathbb{R}^2) \cap \dot{W}^{1,\infty}(\mathbb{R}^2)$.

In [30], Córdoba and Lazar proved global existence for initial data in the space $\dot{H}^{3/2}(\mathbb{R}) \cap \dot{H}^{5/2}(\mathbb{R})$ with a small assumption over $\dot{H}^{3/2}(\mathbb{R})$, by making use of oscillatory integrals and a new formulation of the Muskat equation.

Alazard and Nguyen proved in [4], using a different approach, the same result of [40] and established the existence of solutions for a non-Lipschitz initial data. In [51] Nguyen established the global existence for small initial data in the Besov space $\dot{B}^1_{\infty,1}(\mathbb{R}^d)$.

In the 3d case, Gancedo and Lazar in [40], proved global existence for the critical space $\dot{H}^2(\mathbb{R}^2)\cap\dot{W}^{1,\infty}(\mathbb{R}^2)$. Alazard and Nguyen proved in [4], using a different approach, the same result of [40] and established the existence of solutions for a non-Lipschitz initial data. Nguyen and Pausader proved in [52] the local existence for initial data in the subcritical space $H^s(\mathbb{R}^d)$, where $s > 1 + d/2$.

In a posterior work [2], Alazard and Nguyen proved local existence for an initial data in the critical space $\dot{W}^{1,\infty}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})$, and the existence of global solutions for small initial data. In [3] the same authors showed local and global existence for non-Lipschitz initial data. Recently, in [5] they proved local existence for initial data in $H^{3/2}(\mathbb{R})$ and global existence in $H^{3/2}(\mathbb{R})$ with a small condition over $\dot{H}^{3/2}(\mathbb{R})$.

In [31] Deng, Lei and Lin constructed global weak solutions under the assumptions that the initial interface is monotonically decreasing with asymptotic behavior at infinity *i.e.* $f_0(x) \to a, x \to \infty$. Cameron in [10] proved the existence of solutions in the 3d case that are unbounded and has sublinear growth. In [42], García-Juárez, Gómez-Serrano, Nguyen and Pausader proved the existence of self-similar solutions. In [41], García-Juárez, Gómez-Serrano, Haziot and Pausader proved local existence when the initial interface has multiple corners and linear growth at infinity.

It is important to mention that none of these results allow for quadratic growth of the interface at infinity

Córdoba, Córdoba and Gancedo proved, in [23], local existence in $H^k(\mathbb{T})$ with $k \geq 3$, considering different viscosities and positive RT. In a posterior work, the same authors treated in [24] the 3d case for a H^4 surface considering different viscosities. Gancedo, García-Juárez, Patel and Strain in [37] proved global existence for small initial data in both 2d and 3d cases, also considering different viscosities.

In the unstable regime $\rho^+ > \rho^-$, the Muskat equation is ill-posed, see [26] and [60], then mixing solutions are used to describe this scenario. In [11], Castro, Córdoba and Faraco studied this kind of solutions using convex integration and the theory of pseudodifferential operators, ses also the work of L. Székelyhidi, see [61]. In the same direction see [16], [55], [8] and [36].

Mengual in [48] studied the unstable case with different viscosities. Recently Castro, Faraco and Gebhard in [15] studied maximal potential energy dissipation as a selection criterion for subsolutions. For others results concerning convex integration applied to IPM, see [25] and [44].

1.3.1 Singularities

Regarding the study of finite time singularities, Castro, Córdoba, Fefferman, Gancedo and López-Fernández in [14], proved that there is an open subset of initial data in H^4 such that the Rayleigh-Taylor condition breaks down in finite time. This means that the initial interface is a graph $\mathrm{RT} > 0$, then in a finite time the interface is not a graph, $\mathrm{RT} < 0$. This phenomenon is called *turning singularity*.

In [12] Castro, Córdoba, Fefferman and Gancedo, proved that there exist solutions which lose the Rayleigh-Taylor condition and, after that, lose regularity in finite time. These singular solutions have been extended in time as *mixing* solutions in [17].

Córdoba, Gómez-Serrano and Zlatoš proved in [28] the existence of solutions that start in the unstable regime, then become stable and finally return to the unstable regime. The same authors in [29] established the existence of solutions that start in the stable regime, then become unstable and finally return to the stable regime.

All these studies have been realized for either asymptotically flat or periodic interfaces. In this thesis, we will deal with solutions with quadratic growth at the infinity.

1.3.2 Steady-state solutions

Surface tension is a force that can arise on the interface due to the different nature of the fluids. Usually, this effect is modeled through the Laplace-Young law. This law states that, in presence of surface tension, the difference of the limit of pressure over the interface is proportional to its curvature. Indeed,

$$p^+ - p^- = \gamma\,\kappa(z),$$

where $\gamma > 0$ is a parameter which measures the strength of the surface tension effect. In this case, the free boundary evolution problem is well posed no matter the sign of the Rayleigh-Taylor function (see [32] and [35]). This is because the surface tension eliminates the Rayleigh-Taylor instabilities condition. This problem has been studied extensively, let us quote, for example, [43], [34], [56], [7], [39], [38] and [50]

The existence of stationary solutions for the Muskat problem with a large surface tension coefficient were studied by Ehrnstrom, Escher and Matioc in [33]. Roughly speaking, they proved that there a branch of solutions, parameterized by

$$\lambda = g\frac{\rho^+ - \rho^-}{\gamma},$$

of the form $(\alpha, h_\lambda(\alpha))$. Here, $\lambda \in (0, \lambda^*)$, with $\lambda^* > 0$ some finite and semi-explicit number. It happens that, when $\lambda \to \lambda^*$ the slope of $h_\lambda(0) \to \infty$. This fact makes that Ehrnstrom, Escher and Matioc do not continue the solutions further λ^*. We will realize this extension by parameterizing the interface as a curve

$$\mathbf{z}(\alpha) = (h(\alpha),\, \alpha),$$

in the interval $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. After that we will construct the solution in the rest of the points by using the symmetries of the equation.

Notice that since $\lambda > 0$, we required $\rho^+ > \rho^-$.

1.4 The Muskat problem for the parabola

The first step is to prove that $f(x,t) = x^2 + ct$ is an explicit solution of the Muskat equation. We have the following lemma.

Lemma 1. *The parabola $f(x,t) = x^2 + ct$ solves the Muskat equation (1.1.2) with $c = \rho^- - \rho^+ > 0$.*

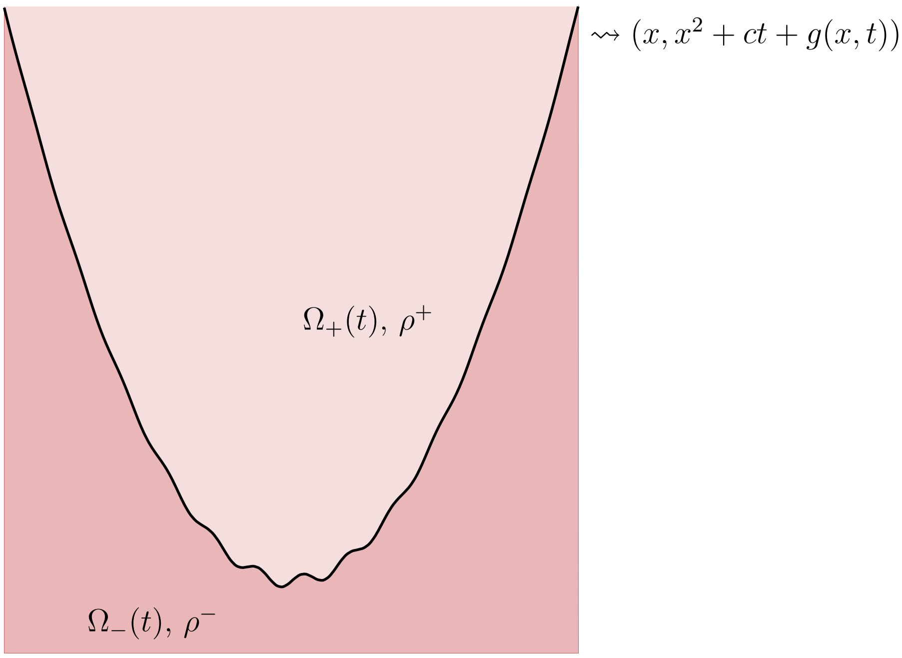
Proof. First we compute the differences

$$\begin{aligned}
&f(x) - f(x-\alpha) = \alpha(2x - \alpha),\\
&\partial_x f(x) - \partial_x f(x-\alpha) = 2\alpha,\\
&\partial_t f = c.
\end{aligned}$$

Then we substitute in the Muskat equation

$$\begin{aligned}
c &= \frac{\rho^- - \rho^+}{2\pi}\int_{\mathbb{R}} \frac{2\alpha^2}{\alpha^2 + \alpha^2(2x-\alpha)^2}\,\mathrm{d}\alpha\\
&= \frac{\rho^- - \rho^+}{\pi}\int_{\mathbb{R}} \frac{1}{1 + (2x-\alpha)^2}\,\mathrm{d}\alpha\\
&= \frac{\rho^- - \rho^+}{\pi}\int_{\mathbb{R}} \frac{1}{1+u^2}\,\mathrm{d}u, \quad u = 2x - \alpha\\
&= \rho^- - \rho^+.
\end{aligned}$$

■



The previous figure ilustrated the situation we are addressing. For renormalization we set $\rho^- - \rho^+ = 2\pi$. The function $f(x,t) = x^2 + 2\pi t$ is an explicit solution of the Muskat equation and represents a parabola moving along the vertical axis as $t \to +\infty$. Now, we will deduce the evolution equation for the function g studied in chapter 2. Henceforth, we will omit the dependence on time. First, we define the difference $\delta_\alpha g(x)$ and the slope $\Delta_\alpha g(x)$ by

$$\delta_\alpha g(x) := g(x) - g(x-\alpha) \quad \text{and} \quad \Delta_\alpha g(x) := \frac{g(x) - g(x-\alpha)}{\alpha}.$$

By substituting in the equation (1.1.2) the function $h := f + g$, we see that g satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} g(x) + 2\pi = PV \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha g(x)}{1 + (\Delta_\alpha h(x))^2}\, \mathrm{d}\alpha + PV \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f(x)}{1 + (\Delta_\alpha h(x))^2}, \mathrm{d}\alpha. \tag{1.4.1}$$

We also omit the dependence on x in the difference $\Delta_\alpha g(z)$. By the definition of f we have

$$2\pi = \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha f}{1 + (\Delta_\alpha f)^2}\, \mathrm{d}\alpha.$$

Thus, adding the term 2π to the right side of (1.4.1), we obtain the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g(x) = PV \int_{\mathbb{R}} \frac{\partial_x \Delta_\alpha g}{1 + (\Delta_\alpha h)^2}\, \mathrm{d}\alpha + PV \int_{\mathbb{R}} \Delta_\alpha g \frac{(-2)(\Delta_\alpha h + \Delta_\alpha f)}{(1 + (\Delta_\alpha h)^2)(1 + (\Delta_\alpha f)^2)}\, \mathrm{d}\alpha.$$

We define the kernels

$$K(x,\alpha) := \frac{1}{1 + (\Delta_\alpha h)^2}, \quad G(x,\alpha) := -2 \frac{\Delta_\alpha h + \Delta_\alpha f}{(1 + (\Delta_\alpha h)^2)(1 + (\Delta_\alpha f)^2)}. \tag{1.4.2}$$

Then (1.4.1) is equivalent to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} g(x,t) = PV \int_{\mathbb{R}} \partial_x \Delta_\alpha g(x) K(x,\alpha)\, \mathrm{d}\alpha + PV \int_{\mathbb{R}} \Delta_\alpha g(x) G(x,\alpha)\, \mathrm{d}\alpha. \tag{1.4.3}$$

Let us emphasize that the analysis of equation (1.4.3) for the evolution of $g(x,t)$ presents severals differences with respect to the analysis of (1.1.2) in $H^s(\mathbb{R})$ or $\dot{H}^k(\mathbb{R})$ spaces, with $0 \le k \le 2$. Indeed, the quadratic growth at infinity introduces a degeneration of the kernels at infinity that need to be understood. In addition, the explicit dependence of x leads to pseudodifferential operators, as opposed to the differential ones which occur in the classical Muskat problem. To clarify the difference between kernels, in the classic Muskat equation, the kernel has the form

$$K(y, h(x), h(x-y)),$$

while in the equation (1.4.3) we have two kernels of the form

$$K(x, y, g(x), g(x-y)).$$

Thus, our task is proving local existence of (1.4.3) with an initial data $g(x,0) = g_0(x) \in H^s(\mathbb{R})$.

In the second part of the thesis, we have the following result, which is a direct consequence of the previous lemma.

Corollary 1. *The curve $\mathbf{p}(\alpha, t) = (\alpha, \alpha^2 + ct)$ solves the Muskat equation (1.1.1) with $c = \rho^- - \rho^+$.*

As in the previous case, we derive an equation for the deviation $\mathbf{d} = \mathbf{z} - \mathbf{p}$, using the Muskat equation (1.1.1). We have

$$\begin{aligned}\partial_t \mathbf{d}(\alpha, t) &= PV \int_{\mathbb{R}} \frac{z_1(\alpha,t) - z_1(\alpha-\beta,t)}{|\mathbf{z}(\alpha,t) - \mathbf{z}(\alpha-\beta,t)|^2} (\partial_\alpha \mathbf{z}(\alpha,t) - \partial_\alpha \mathbf{z}(\alpha-\beta,t))\,\mathrm{d}\beta - \partial_t \mathbf{p}(\alpha,t) \\ &:= \mathbf{F}(\mathbf{d})(\alpha,t).\end{aligned}$$

Then, $\mathbf{F}(\mathbf{d})$ has the following expression

$$\mathbf{F}(\mathbf{d})(\alpha,t) = PV \int_{\mathbb{R}} \frac{z_1(\alpha,t) - z_1(\alpha-\beta,t)}{|\mathbf{z}(\alpha,t) - \mathbf{z}(\alpha-\beta,t)|^2} (\partial_\alpha \mathbf{z}(\alpha,t) - \partial_\alpha \mathbf{z}(\alpha-\beta,t))\,\mathrm{d}\beta - (0, 2\pi). \tag{1.4.4}$$

Given an initial deviation $\mathbf{d}^0(\alpha)$, we want to solve the previous equation (1.4.4) with the initial condition $\mathbf{d}(\alpha, 0) = \mathbf{d}^0(\alpha)$.

To control this type of terms, we deal with the Hilbert transforms of rational functions. Recall the the Hilbert transform is defined by

$$Hf(x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(x-y)}{y}\,\mathrm{d}y$$

and the truncated Hilbert transform is given by

$$H_{|y|<\epsilon} f(x) = \frac{1}{\pi} PV \int_{|y|<\epsilon} \frac{f(x-y)}{y}\,\mathrm{d}y.$$

An important fact that will be used throughout the thesis is that both operators, the Hilbert transform and the truncated Hilbert transform, are bounded from L^2 to L^2.

Chapter 2

Local Existence

Here we prove local existence for the equation (1.4.3). This chapter is organized as follows: The first section 2.1 is dedicated to obtain the appropiate energy estimate for the function g. Section 2.2 is devoted to the study of the regularized system in order to obtain existence of solutions. All the necessary lemmas to prove the energy estimate are presented in Section 4.1, chapter 4. Troughout this chapter, we making use of the following norms

$$\|f\|_{C^k} = \max_{j\geq k} \sup_{x\in\mathbb{R}} \left|\partial_x^k f(x)\right|,$$

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)|.$$

The results in this chapter have been published in [59].

2.1 The energy estimates

In this section we obtain the energy estimate for the function g, we present two main lemmas. Lemma 2 corresponds to the lower order derivative terms, while Lemma 3 deals with the highest derivative terms. Let s be an integer, we consider the energy of the function g as the norm in the Sobolev space $H^s(\mathbb{R})$,

$$E(t) = \frac{1}{2}\|g\|^2_{L^2}(t) + \frac{1}{2}\|\partial_x^s g\|^2_{L^2}(t).$$

In order to prove local existence of solutions in $H^s(\mathbb{R})$ we need an estimate for the evolution in time for the energy $E(t)$. In our case, the estimate will be in polynomial form, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq c\,(E(t) + E(t)^2 + \cdots + E(t)^\ell)$$

for a large integer ℓ. This bound will suffice to prove that the energy of the solution is uniformly bounded in $H^s(\mathbb{R})$ up to some time $T = T(\|g_0\|_{H^s}) > 0$.

2.1.1 Lower derivative

We start by controlling the evolution of the $L^2(\mathbb{R})$ norm of g.

Lemma 2. *Let $g \in H^s(\mathbb{R})$ with $s \geq 3$, then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g\|^2_{L^2}(t) \leq c\left(\|g\|^2_{H^s} + \cdots + \|g\|^5_{H^s}\right). \tag{2.1.1}$$

Proof. Taking the $L^2(\mathbb{R})$ product of g and g_t, given by equation (1.4.3), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g\|_{L^2}^2(t) &= \int_{\mathbb{R}} g(x)\int_{\mathbb{R}} \partial_x\Delta_\alpha g(x)K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x + \int_{\mathbb{R}} g(x)\int_{\mathbb{R}} \Delta_\alpha g(x)G(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x \\ &:= \mathrm{I}+\mathrm{II}.\end{aligned}$$

Bound for I: We use the definition of the slope $\Delta_\alpha g$ to split

$$\begin{aligned}\mathrm{I} &= \int_{\mathbb{R}} g(x)\partial_x g(x)\int_{\mathbb{R}}\frac{1}{\alpha}K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x - \int_{\mathbb{R}} g(x)\int_{\mathbb{R}}\frac{\partial_x g(x-\alpha)}{\alpha}K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x \\ &:= A_1 - A_2.\end{aligned}$$

Using Cauchy-Schwarz inequality and then estimates (4.1.1) from Lemma 18, we find that

$$\begin{aligned}|A_1| &\leq \|g\|_{L^2}\|\partial_x g\|_{L^2}\left\| PV\int_{\mathbb{R}}\frac{1}{\alpha}K(\,\cdot\,,\alpha)\,\mathrm{d}\alpha\right\|_{L^\infty} \\ &\leq c\,(1+\|g\|_{C^2})^3\|g\|_{L^2}\|\partial_x g\|_{L^2}.\end{aligned} \tag{2.1.2}$$

To deal with the term A_2, we split the integral in the *in* and *out* parts. For the *in* part we have the following decomposition

$$\begin{aligned}A_2^{in} &= \int_{\mathbb{R}} g(x)H_{|\alpha|<1}\partial_x g(x)K(x,0)\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} g(x)\int_{|\alpha|<1}\frac{\partial_x g(x-\alpha)}{\alpha}\big[K(x,\alpha)-K(x,0)\big]\,\mathrm{d}\alpha\,\mathrm{d}x,\end{aligned} \tag{2.1.3}$$

where we use the truncated Hilbert transform $H_{|\alpha|<1}\partial_x g$, and add and subtract the kernel at zero

$$K(x,0) = \frac{1}{1+(\partial_x h(x))^2}.$$

Then applying Cauchy-Schwarz inequality, we obtain that

$$\left|\int_{\mathbb{R}} g(x)H_{|\alpha|<1}\partial_x g(x)K(x,0)\,\mathrm{d}x\right| \leq \|g\|_{L^2}\|\partial_x g\|_{L^2}. \tag{2.1.4}$$

We denote by $D(x,\alpha)$ the difference of kernels

$$D(x,\alpha) := K(x,\alpha) - K(x,0). \tag{2.1.5}$$

By direct calculation, together with the Fundamental Theorem of Calculus we deduce an estimate for the difference (2.1.5)

$$|D(x,\alpha)| = \big|K(x,\alpha)-K(x,0)\big| \leq c\,(1+\|\partial_x^2 g\|_{L^\infty})\,|\alpha|.$$

Hence, for the second integral in (2.1.3), we observe that applying Cauchy-Schwarz inequality we derive the following estimate

$$\begin{aligned}&\left|\int_{\mathbb{R}} g(x)\int_{|\alpha|<1}\frac{\partial_x g(x-\alpha)}{\alpha}D(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x\right| \\ &\qquad \leq \int_{\mathbb{R}}|g(x)|\int_{|\alpha|<1}\frac{|\partial_x g(x-\alpha)|}{|\alpha|}|D(x,\alpha)|\,\mathrm{d}\alpha\,\mathrm{d}x \\ &\qquad \leq c\,(1+\|g\|_{C^2})\int_{|\alpha|<1}\int_{\mathbb{R}}|g(x)||\partial_x g(x-\alpha)|\,\mathrm{d}x\,\mathrm{d}\alpha \\ &\qquad \leq c\,(1+\|g\|_{C^2})\|g\|_{L^2}\|\partial_x g\|_{L^2}.\end{aligned} \tag{2.1.6}$$

For the *out* part, we apply Cauchy-Schwarz inequality respect to x

$$|A_2^{out}| = \left| \int_{\mathbb{R}} g(x) \int_{|\alpha|>1} \frac{\partial_x g(x-\alpha)}{\alpha} K(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x \right| \\ \leq \|g\|_{L^2} \left(\int_{\mathbb{R}} \left| \int_{|\alpha|>1} \frac{\partial_x g(x-\alpha)}{\alpha} K(x,\alpha)\, \mathrm{d}\alpha \right|^2 \mathrm{d}x \right)^{1/2}.$$

Now we use Cauchy-Schwarz inequality respect to α

$$|A_2^{out}| \leq \|g\|_{L^2} \left(\int_{\mathbb{R}} \left(\int_{|\alpha|>1} \partial_x g(x-\alpha)^2\, \mathrm{d}\alpha \right) \left(\int_{|\alpha|>1} \frac{1}{\alpha^2} K(x,\alpha)^2\, \mathrm{d}\alpha \right) \mathrm{d}x \right)^{1/2} \\ \leq \|g\|_{L^2} \|\partial_x g\|_{L^2} \left(\int_{|\alpha|>1} \frac{1}{\alpha^2} \int_{\mathbb{R}} K(x,\alpha)^2\, \mathrm{d}x\, \mathrm{d}\alpha \right)^{1/2}.$$

The estimate (4.1.16) in Lemma 23 states that

$$\int_{\mathbb{R}} K(x,\alpha)^2\, \mathrm{d}x \leq c\,(1 + \|\partial_x g\|_{L^\infty}).$$

Therefore putting together the estimates (2.1.2), (2.1.4), (2.1.6) and the inequalities for the *out* part we obtain the following bound

$$|\,\mathrm{I}\,| \leq c\,(1 + \|g\|_{C^2})^3 \|g\|_{L^2} \|\partial_x g\|_{L^2}.$$

Bound for II: For the *in* part, using the Fundamental Theorem of Calculus we have the following formula for the slope

$$\Delta_\alpha g = \int_0^1 \partial_x g(x + (s-1)\alpha)\, \mathrm{d}s, \tag{2.1.7}$$

hence we obtain that

$$\mathrm{II}^{in} = \int_0^1 \int_{\mathbb{R}} g(x) \int_{|\alpha|<1} \partial_x g(x + (s-1)\alpha) G(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x\, \mathrm{d}s.$$

From the definition (1.4.2) we deduce that

$$|G(x,\alpha)| = \left| -2 \frac{\Delta_\alpha h + \Delta_\alpha f}{(1 + (\Delta_\alpha h)^2)(1 + (\Delta_\alpha f)^2)} \right| \leq 2.$$

Now applying the Cauchy-Schwarz inequality yields

$$|\mathrm{II}^{in}| \leq 2\, \|g\|_{L^2} \|\partial_x g\|_{L^2}. \tag{2.1.8}$$

For the *out* part, expanding $\Delta_\alpha g$ we split the integral in two terms

$$\mathrm{II}^{out} = \int_{\mathbb{R}} g(x)^2 \int_{|\alpha|>1} \frac{1}{\alpha} G(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x - \int_{\mathbb{R}} g(x) \int_{|\alpha|>1} \frac{g(x-\alpha)}{\alpha} G(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x \\ := A_3 + A_4.$$

In A_3, we control the inner integral using the estimates (4.1.7) from Lemma 19, hence

$$|A_3| \leq \left| \int_{\mathbb{R}} g(x)^2 \int_{|\alpha|>1} \frac{1}{\alpha} G(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x \right| \\ \leq \|g\|_{L^2}^2 \left\| PV \int_{|\alpha|>1} \frac{1}{\alpha} G(\,\cdot\,,\alpha)\, \mathrm{d}\alpha \right\|_{L^\infty} \leq c\,(1 + \|g\|_{L^\infty})^2 \|g\|_{L^2}^2.$$

Now for A_4, we follow the same technique used in A_2^{out}. First, applying Cauchy-Schwarz inequality first respect to x and then respect to α, we deduce that

$$|A_4| \leq \|g\|_{L^2}^2 \left(\int_{|\alpha|>1} \frac{1}{\alpha^2} \int_{\mathbb{R}} G(x,\alpha)^2 \, \mathrm{d}x \, \mathrm{d}\alpha \right)^{1/2}.$$

The estimate (4.1.17) in Lemma 24, says that

$$\int_{\mathbb{R}} G(x,\alpha)^2 \, \mathrm{d}x \leq c \, (1 + \|\partial_x g\|_{L^\infty})^3.$$

Then the last inequality and (2.1.8) conclude the proof

$$|\,\mathrm{II}\,| \leq c \, (1 + \|g\|_{C^1})^2 \|g\|_{H^1}^2.$$

■

2.1.2 High derivative

Now we move to the second part of the energy, which involves the derivative of order s of g. We will prove the following lemma.

Lemma 3. *Let $g \in H^s(\mathbb{R})$ with $s \geq 3$, then*

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_x^3 g\|_{L^2}^2(t) \leq c \left(\|g\|_{H^s}^2 + \cdots + \|g\|_{H^s}^5 \right). \tag{2.1.9}$$

Proof. We take $s = 3$ and compute $\partial_x^3 g_t$ from the equation (1.4.3). We have two terms

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_x^3 g\|_{L^2}^2 &= \int_{\mathbb{R}} \partial_x^3 g(x) \partial_x^3 \int_{\mathbb{R}} \partial_x \Delta_\alpha g(x) K(x,\alpha) \, \mathrm{d}\alpha \, \mathrm{d}x + \int_{\mathbb{R}} \partial_x^3 g(x) \partial_x^3 \int_{\mathbb{R}} \Delta_\alpha g(x) G(x,\alpha) \, \mathrm{d}\alpha \, \mathrm{d}x \\ &:= \mathrm{III} + \mathrm{IV}. \end{aligned}$$

We use the Leibniz product rule to get the next decomposition

$$\mathrm{III} := J_1 + 3J_2 + 3J_3 + J_4.$$

The goal is obtain a polynomial bound for each J_i. We start by getting a bound for J_1.

Bound for J_1: This term is the most singular because four derivatives acting on g. We expand $\partial_x^4 \Delta_\alpha g$ and add and subtract the kernel at zero $K(x,0)$, we have

$$\begin{aligned} J_1 = &\int_{\mathbb{R}} \partial_x^3 g(x) \partial_x^4 g(x) \int_{\mathbb{R}} \frac{1}{\alpha} K(x,\alpha) \, \mathrm{d}\alpha \, \mathrm{d}x - \int_{\mathbb{R}} K(x,0) \partial_x^3 g(x) \int_{\mathbb{R}} \frac{\partial_x^4 g(x-\alpha)}{\alpha} \, \mathrm{d}\alpha \, \mathrm{d}x \\ &- \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} \frac{\partial_x^4 g(x-\alpha)}{\alpha} \big[K(x,\alpha) - K(x,0) \big] \, \mathrm{d}\alpha \, \mathrm{d}x. \end{aligned}$$

Recall that the kernel at zero is given by

$$K(x,0) = \frac{1}{1 + (\partial_x h(x))^2}.$$

Using

$$\partial_x^3 g(x) \partial_x^4 g(x) = \frac{1}{2} \partial_x [\partial_x^3 g(x)]^2$$

and integration by parts, we obtain that

$$\int_{\mathbb{R}} \partial_x^3 g(x)\partial_x^4 g(x) \int_{\mathbb{R}} \frac{1}{\alpha} K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}} [\partial_x^3 g(x)]^2 \partial_x \int_{\mathbb{R}} \frac{1}{\alpha} K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x.$$

The fact that $H\partial_x = \Lambda$ implies

$$\begin{aligned} J_1 = &-\frac{1}{2}\int_{\mathbb{R}} \left[\partial_x^3 g(x)\right]^2 \int_{\mathbb{R}} \frac{1}{\alpha}\partial_x K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x - \int_{\mathbb{R}} K(x,0)\partial_x^3 g(x)\Lambda\partial_x^3 g(x)\,\mathrm{d}x \\ &- \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} \frac{\partial_x^4 g(x-\alpha)}{\alpha} D(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x, \end{aligned} \tag{2.1.10}$$

where $D(x,\alpha)$ is the difference $K(x,\alpha)-K(x,0)$. Now we use the Córdoba-Córdoba pointwise inequality, see [22], then we obtain that

$$\partial_x^3 g(x)\Lambda\partial_x^3 g(x) \geq \frac{1}{2}\Lambda[\partial_x^3 g(x)]^2.$$

Due to $K(x,0)>0$, we get that

$$\begin{aligned} J_1 \leq &-\frac{1}{2}\int_{\mathbb{R}} \left[\partial_x^3 g(x)\right]^2 \int_{\mathbb{R}} \frac{1}{\alpha}\partial_x K(x,\alpha) d\alpha dx - \frac{1}{2}\int_{\mathbb{R}} \Lambda K(x,0)[\partial_x^3 g(x)]^2\,\mathrm{d}x \\ &- \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} \frac{\partial_x^4 g(x-\alpha)}{\alpha} D(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x. \end{aligned}$$

Using the inequalities (4.1.9) and (4.1.12) in Lemma 20 and Lemma 21, we conclude that the first two terms above are bounded. That is, the $L^\infty(\mathbb{R})$ norms of the inner integral and the operator $\Lambda K(x,0)$ are bounded

$$\left\| PV\int_{\mathbb{R}} \frac{1}{\alpha}\partial_x K(\,\cdot\,,\alpha)\,\mathrm{d}\alpha \right\|_{L^\infty} \leq c\,(1+\|g\|_{C^{2,\delta}})^2$$

and

$$\|\Lambda K(x,0)\|_{L^\infty} \leq c\,(1+\|g\|_{C^{2,\delta}}).$$

In the last two inequalities we take $\delta = 1/2$ because $H^3(\mathbb{R}) \hookrightarrow C^{2,1/2}(\mathbb{R})$. It remains to get the bound for the term with the difference $D(x,\alpha)$ in (2.1.10). First, we note that by the chain rule $\partial_\alpha\partial_x^3 g(x-\alpha) = -\partial_x^4 g(x-\alpha)$. From here, integration by parts yields

$$-\int_{\mathbb{R}} \partial_x^3 g(x)\int_{\mathbb{R}} \frac{\partial_x^4 g(x-\alpha)}{\alpha} D(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x = -\int_{\mathbb{R}} \partial_x^3 g(x)\int_{\mathbb{R}} \partial_x^3 g(x-\alpha)\partial_\alpha\left(\frac{D(x,\alpha)}{\alpha}\right)\mathrm{d}\alpha\,\mathrm{d}x. \tag{2.1.11}$$

Denote

$$\Phi(x,\alpha) := \partial_\alpha[D(x,\alpha)/\alpha].$$

We split the integral (2.1.11) into the *in* and *out* parts. For the *in* part, we observe that

$$\begin{aligned} &\left| \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|<1} \partial_x^3 g(x-\alpha)\Phi(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x \right| \\ &\qquad \leq \int_{\mathbb{R}}\int_{|\alpha|<1} |\partial_x^3 g(x)||\partial_x^3 g(x-\alpha)|\big|\Phi(x,\alpha)\big|\,\mathrm{d}\alpha\,\mathrm{d}x \\ &\qquad \leq \frac{1}{2}\int_{\mathbb{R}}\int_{|\alpha|<1} \Big[|\partial_x^3 g(x)|^2 + |\partial_x^3 g(x-\alpha)|^2\Big]\big|\Phi(x,\alpha)\big|\,\mathrm{d}\alpha\,\mathrm{d}x \\ &\qquad \leq \frac{1}{2}\int_{\mathbb{R}} |\partial_x^3 g(x)|^2 \int_{|\alpha|<1} \big|\Phi(x,\alpha)\big|\,\mathrm{d}\alpha\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}}\int_{|\alpha|<1} |\partial_x^3 g(x-\alpha)|^2\big|\Phi(x,\alpha)\big|\,\mathrm{d}\alpha\,\mathrm{d}x, \end{aligned}$$

where we have used Young's inequality

$$ab \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2.$$

Using estimate (4.1.13) from Lemma 22, we get

$$\left|\Phi(x,\alpha)\chi_{|\alpha|<1}(\alpha)\right| \leq c\,(1+\|g\|_{C^{2,\delta}})^2|\alpha|^{\delta-1}, \tag{2.1.12}$$

which is integrable near to the origin. For the second integral we change variables $\beta = \alpha$ and $y = x - \alpha$ to get

$$\int_{\mathbb{R}}\int_{|\alpha|<1}|\partial_x^3 g(x-\alpha)|^2\left|\Phi(x,\alpha)\right| \mathrm{d}\alpha\,\mathrm{d}x = \int_{\mathbb{R}}|\partial_y^3 g(y)|^2\int_{|\beta|<1}|\Phi(y+\beta,\beta)|\,\mathrm{d}\beta\,\mathrm{d}y$$

and we have the same control (2.1.12) over $|\Phi(y+\beta,\beta)|$. Hence the *in* part is bounded

$$\left|-\int_{\mathbb{R}}\partial_x^3 g(x-\alpha)\int_{|\alpha|<1}\frac{\partial_x^4 g(x-\alpha)}{\alpha}D(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x\right| \leq c\,(1+\|g\|_{C^{2,\delta}})^2\|\partial_x^3 g\|_{L^2}^2. \tag{2.1.13}$$

Now we focus on the *out* part. First, we note that

$$\left|\Phi(x,\alpha)\chi_{|\alpha|>1}(\alpha)\right| \leq \frac{|D(x,\alpha)|}{\alpha^2} + \frac{|\partial_\alpha K(x,\alpha)|}{|\alpha|}.$$

Then we split in two parts. For the first part we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\partial_x^3 g(x)\int_{|\alpha|>1}\partial_x^3 g(x-\alpha)\Phi(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x\right| \\
&\qquad\leq \int_{\mathbb{R}}|\partial_x^3 g(x)|\int_{|\alpha|>1}|\partial_x^3 g(x-\alpha)|\left|\frac{\partial_\alpha K(x,\alpha)}{\alpha}\right|\mathrm{d}\alpha\,\mathrm{d}x \\
&\qquad\quad + \int_{\mathbb{R}}|\partial_x^3 g(x)|\int_{|\alpha|>1}|\partial_x^3 g(x-\alpha)|\left|\frac{D(x,\alpha)}{\alpha^2}\right|\mathrm{d}\alpha\,\mathrm{d}x.
\end{aligned} \tag{2.1.14}$$

The second line of (2.1.14) can be bounded by applying Cauchy-Schwarz inequality, first with respect to x, and then with respect to α. Then we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}}|\partial_x^3 g(x)|\int_{|\alpha|>1}|\partial_x^3 g(x-\alpha)|\left|\frac{\partial_\alpha K(x,\alpha)}{\alpha}\right|\mathrm{d}\alpha\,\mathrm{d}x \\
&\qquad\leq \|\partial_x^3 g\|_{L^2}\left(\int_{\mathbb{R}}\|\partial_x^3 g\|_{L^2}^2\left(\int_{|\alpha|>1}\frac{\partial_\alpha K(x,\alpha)^2}{\alpha^2}\,\mathrm{d}\alpha\right)\mathrm{d}x\right)^{1/2} \\
&\qquad\leq \|\partial_x^3 g\|_{L^2}^2\left(\int_{|\alpha|>1}\frac{1}{\alpha^2}\int_{\mathbb{R}}\partial_\alpha K(x,\alpha)^2\,\mathrm{d}x\,\mathrm{d}\alpha\right)^{1/2} \\
&\qquad\leq c\,\|\partial_x^3 g\|_{L^2}^2(1+\|\partial_x g\|_{L^\infty})^3.
\end{aligned}$$

In the last inequality we applied the estimates (4.1.18) of Lemma 25. For the second term in the right hand side of (2.1.14), we apply Cauchy-Schwarz inequality with respect to x and then use Minkowski's integral inequality. Also, we note that the difference satisfies

$$|D(x,\alpha)| \leq 2.$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}}|\partial_x^3 g(x)|\int_{|\alpha|>1}|\partial_x^3 g(x-\alpha)|\left|\frac{D(x,\alpha)}{\alpha^2}\right|\mathrm{d}\alpha\,\mathrm{d}x \\
&\qquad\leq \|\partial_x^3 g\|_{L^2}\left(\int_{\mathbb{R}}\left|\int_{|\alpha|>1}\frac{\partial_x^3 g(x-\alpha)}{\alpha^2}D(x,\alpha)\,\mathrm{d}\alpha\right|^2\mathrm{d}x\right)^{1/2} \\
&\qquad\leq \|\partial_x^3 g\|_{L^2}\int_{|\alpha|>1}\left(\int_{\mathbb{R}}\left[\frac{\partial_x^3 g(x-\alpha)}{\alpha^2}\right]^2\mathrm{d}x\right)^{1/2}\mathrm{d}\alpha \\
&\qquad\leq c\,\|\partial_x^3 g\|_{L^2}^2.
\end{aligned}$$

Therefore, by joining the estimates for the *out* part and (2.1.13), we deduce that

$$|J_1| \leq c\,(1+\|g\|_{C^{2,1/2}})^3\|g\|_{H^3}^2. \tag{2.1.15}$$

Bound for J_2: The second term J_2 is similar to J_1, expanding $\partial_x^3\Delta g$ we have

$$J_2=\int_{\mathbb{R}}\partial_x^3 g(x)^2\int_{\mathbb{R}}\frac{1}{\alpha}\partial_x K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x-\int_{\mathbb{R}}\partial_x^3 g(x)\int_{\mathbb{R}}\frac{\partial_x^3 g(x-\alpha)}{\alpha}\partial_x K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x.$$

For the last integral in J_2 the change of variable $x-\alpha=y$, leads to

$$\int_{\mathbb{R}}\partial_x^3 g(x)\int_{\mathbb{R}}\frac{\partial_x^3 g(x-\alpha)}{\alpha}\partial_x K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x=\int_{\mathbb{R}}\int_{\mathbb{R}}\partial_x^3 g(x)\partial_y^3 g(y)\frac{\partial_x K(x,x-y)}{x-y}\,\mathrm{d}y\,\mathrm{d}x.$$

Define $\zeta(x,y)$ as the kernel

$$\zeta(x,y):=\frac{\partial_x K(x,x-y)}{x-y}=\frac{-2}{x-y}\frac{\left(\dfrac{h(x)-h(y)}{x-y}\right)\left(\dfrac{\partial_x h(x)-\partial_x h(y)}{x-y}\right)}{\left(1+\left(\dfrac{h(x)-h(y)}{x-y}\right)^2\right)^2}.$$

We observe that $\zeta(x,y)=-\zeta(y,x)$, then change of variables $x=y$ and $y=x$ implies that

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}&\partial_x^3 g(x)\partial_y^3 g(y)\zeta(x,y)\,\mathrm{d}x\,\mathrm{d}y\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}\partial_y^3 g(y)\partial_x^3 g(x)\zeta(y,x)\,\mathrm{d}y\,\mathrm{d}x\\&=-\int_{\mathbb{R}}\int_{\mathbb{R}}\partial_x^3 g(x)\partial_y^3 g(y)\zeta(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Therefore the second integral in J_2 is zero and the first integral has the same bound (2.1.15) of J_1. That is

$$|J_2| \leq c\,(1+\|g\|_{C^{2,1/2}})^3\|g\|_{H^3}^2. \tag{2.1.16}$$

Bound for J_3: We split in the *in* and *out* parts

$$\begin{aligned}J_3&=\int_{\mathbb{R}}\partial_x^3 g(x)\int_{|\alpha|<1}\partial_x^2\Delta_\alpha g\,\partial_x^2 K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x+\int_{\mathbb{R}}\partial_x^3 g(x)\int_{|\alpha|>1}\partial_x^2\Delta_\alpha g\,\partial_x^2 K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x\\&:=J_3^{in}+J_3^{out}.\end{aligned}$$

Using the estimate (4.1.15) we have the following bound

$$|\partial_x^2 K(x,\alpha)|\leq c\left(1+\|\partial_x^2 g\|_{L^\infty}\right)^2+c\,\|g\|_{C^{2,\delta}}\cdot|\alpha|^{\delta-1},$$

where $\delta=1/2$. We use the Fundamental Theorem of Calculus to obtain the following formula

$$\partial_x^2\Delta_\alpha g=\int_0^1\partial_x^3 g(x+(s-1)\alpha)\,\mathrm{d}s. \tag{2.1.17}$$

Then using (2.1.17) and Cauchy-Schwarz inequality with respect to x we obtain that

$$\begin{aligned}|J_3^{in}|&\leq\int_0^1\int_{\mathbb{R}}|\partial_x^3 g(x)|\int_{|\alpha|<1}|\partial_x^3 g(x+(s-1)\alpha)||\partial_x^2 K(x,\alpha)|\,\mathrm{d}\alpha\,\mathrm{d}x\,\mathrm{d}s\\&\leq c\,(1+\|g\|_{C^{2,\delta}})^2\int_0^1\int_{|\alpha|<1}(1+|\alpha|^{\delta-1})\int_{\mathbb{R}}|\partial_x^3 g(x)||\partial_x^3 g(x+(s-1)\alpha)|\,\mathrm{d}x\,\mathrm{d}\alpha\,\mathrm{d}s\\&\leq c\,(1+\|g\|_{C^{2,\delta}})^2\|\partial_x^3 g\|_{L^2}^2.\end{aligned} \tag{2.1.18}$$

where as in the previous term J_2, we take $\delta = 1/2$. Now, for the *out* part expanding $\partial_x^2 \Delta_\alpha g$ we have

$$\begin{aligned} J_3^{out} = \int_{\mathbb{R}} \partial_x^3 g(x) \partial_x^2 g(x) \int_{|\alpha|>1} \frac{1}{\alpha} \partial_x^2 K(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x \\ - \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{\partial_x^2 g(x-\alpha)}{\alpha} \partial_x^2 K(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x. \end{aligned} \tag{2.1.19}$$

For the first integral in the right hand side of (2.1.19) we apply Cauchy-Schwarz inequality and use the estimate (4.1.19) of Lemma 26. We deduce that

$$\begin{aligned} &\left| \int_{\mathbb{R}} \partial_x^3 g(x) \partial_x^2 g(x) \int_{|\alpha|>1} \frac{1}{\alpha} \partial_x^2 K(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x \right| \\ &\qquad \le \|\partial_x^3 g\|_{L^2} \|\partial_x^2 g\|_{L^2} \left\| \int_{|\alpha|>1} \frac{1}{\alpha} \partial_x^2 K(x,\alpha)\, \mathrm{d}\alpha \right\|_{L^\infty} \\ &\qquad \le c\,(1+\|g\|_{C^2})^2 \|\partial_x^3 g\|_{L^2} \|\partial_x^2 g\|_{L^2}. \end{aligned} \tag{2.1.20}$$

For the second integral in the right hand side of (2.1.19), we observe

$$\partial_x^2 K(x,\alpha) = [\partial_x \Delta_\alpha h]^2 B_1(x,\alpha) + \partial_x^2 \Delta_\alpha g B_2(x,\alpha), \tag{2.1.21}$$

where

$$B_1(x,\alpha) := -2K(x,\alpha)^2 + 8(\Delta_\alpha h)^2 K(x,\alpha)^3 \quad \text{and} \quad B_2(x,\alpha) := -2\Delta_\alpha h K(x,\alpha)^2.$$

Then expanding the sum in (2.1.21) we obtain that

$$\int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{\partial_x^2 g(x-\alpha)}{\alpha} \partial_x^2 K(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x := J_{3,1}^{out} + J_{3,2}^{out},$$

where

$$J_{3,1}^{out} = \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{\partial_x^2 g(x-\alpha)}{\alpha} [\partial_x \Delta_\alpha h]^2 B_1(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x$$

and

$$J_{3,2}^{out} = \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{\partial_x^2 g(x-\alpha)}{\alpha} [\partial_x^2 \Delta_\alpha g] B_2(x,\alpha)\, \mathrm{d}\alpha\, \mathrm{d}x.$$

We notice that

$$|B_1(x,\alpha)| \le 10\, K(x,\alpha),$$

which is square integrable with respect to x, by Lemma 23. Using the Fundamental Theorem of Calculus we deduce that

$$|\partial_x \Delta_\alpha h| \le 2\,(1+\|\partial_x^2 g\|_{L^\infty}). \tag{2.1.22}$$

Hence applying Cauchy-Schwarz inequality, first respect to x, then respect to α. We find that

$$\begin{aligned} |J_{3,1}^{out}| &\le \|\partial_x^3 g\|_{L^2} \left(\int_{\mathbb{R}} \left| \int_{|\alpha|>1} \frac{\partial_x^2 g(x-\alpha)}{\alpha} [\partial_x \Delta_\alpha h]^2 B_1(x,\alpha)\, \mathrm{d}\alpha \right|^2 \mathrm{d}x \right)^{1/2} \\ &\le c\,(1+\|\partial_x^2 g\|_{L^\infty})^2 \|\partial_x^3 g\|_{L^2} \|\partial_x^2 g\|_{L^2} \left(\int_{|\alpha|>1} \frac{1}{\alpha^2} \int_{\mathbb{R}} B_1(x,\alpha)^2\, \mathrm{d}x\, \mathrm{d}\alpha \right)^{1/2} \\ &\le c\,(1+\|g\|_{C^2})^3 \|\partial_x^2 g\|_{L^2} \|\partial_x^3 g\|_{L^2}. \end{aligned}$$

The second term $J_{3,2}^{out}$ has a similar bound. In that case we use the following bounds

$$|B_2(x,\alpha)| \le 2K(x,\alpha) \quad \text{and} \quad |\partial_x^2 \Delta_\alpha g| \le 2\|\partial_x^2 g\|_{L^\infty} |\alpha|^{-1}.$$

Therefore by joining the estimates (2.1.18) and (2.1.20) and the inequalities for $J^{out}_{3,1}$ and $J^{out}_{3,2}$ we conclude that

$$|J_3| \leq c\,(1+\|g\|_{C^{2,1/2}})^3\|g\|^2_{H^3}. \tag{2.1.23}$$

Bound for J_4: We notice that

$$\partial_x^3 K(x,\alpha) = \partial_x\Delta_\alpha h B_3(x,\alpha) + \partial_x^2\Delta_\alpha g B_4(x,\alpha) + \partial_x^3\Delta_\alpha g B_5(x,\alpha), \tag{2.1.24}$$

where

$$\begin{aligned} B_3(x,\alpha) &:= \Big[24(\Delta_\alpha h)K(x,\alpha)^3 - 48(\Delta_\alpha h)^3 K(x,\alpha)^4\Big](\partial_x\Delta_\alpha h)^2, \\ B_4(x,\alpha) &:= 3\big[-2K(x,\alpha)^3 + 8(\Delta_\alpha h)^2K(x,\alpha)^4\big](\partial_x\Delta_\alpha h), \\ B_5(x,\alpha) &:= -2\Delta_\alpha h K(x,\alpha)^2. \end{aligned} \tag{2.1.25}$$

Then expanding the sum in (2.1.24) we decompose $J_4 := J_{4,1} + J_{4,2} + J_{4,3}$ with

$$\begin{aligned} J_{4,1} &= \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} (\partial_x\Delta_\alpha g)\partial_x\Delta_\alpha h B_3(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x, \\ J_{4,2} &= \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} (\partial_x\Delta_\alpha g)\partial_x^2\Delta_\alpha g B_4(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x, \\ J_{4,3} &= \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} (\partial_x\Delta_\alpha g)\partial_x^3\Delta_\alpha g B_5(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x. \end{aligned}$$

Using the Fundamental Theorem of Calculus we have the following formula

$$\partial_x\Delta_\alpha g = \int_0^1 \partial_x^2 g(x+(s-1)\alpha)\,\mathrm{d}s. \tag{2.1.26}$$

Notice

$$|B_3(x,\alpha)| \leq c\,(1+\|\partial_x^2 g\|_{L^\infty})^2, \tag{2.1.27}$$

then the estimate (2.1.22) together with the Cauchy-Schwarz inequality yields to the following bound

$$\begin{aligned} |J^{in}_{4,1}| &\leq c\int_0^1\int_{|\alpha|<1}\int_{\mathbb{R}} |\partial_x^3 g(x)||\partial_x^2 g(x+(s-1)\alpha)||\partial_x\Delta_\alpha h||B_3(x,\alpha)|\,\mathrm{d}x\,\mathrm{d}\alpha\,\mathrm{d}s \\ &\leq c\,(1+\|\partial_x^2 g\|_{L^\infty})^3\|\partial_x^3 g\|_{L^2}\|\partial_x^2 g\|_{L^2}. \end{aligned} \tag{2.1.28}$$

For the *out* part, we expand $\partial_x\Delta_\alpha g$ and take $J^{out}_{4,1} := L_1 + L_2$, where

$$\begin{aligned} L_1 &= \int_{\mathbb{R}} \partial_x^3 g(x)\partial_x g(x)\int_{|\alpha|>1}\frac{1}{\alpha}\partial_x\Delta_\alpha h B_3(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x, \\ L_2 &= -\int_{\mathbb{R}} \partial_x^3 g(x)\int_{|\alpha|>1}\frac{\partial_x g(x-\alpha)}{\alpha}\partial_x\Delta_\alpha h B_3(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x. \end{aligned}$$

Now we expand the sum $\partial_x\Delta_\alpha h = 2 + \partial_x\Delta_\alpha g$ and decompose further $L_1 := S_1 + S_2$ for

$$\begin{aligned} S_1 &= 2\int_{\mathbb{R}} \partial_x^3 g(x)\partial_x g(x)\left\{\int_{|\alpha|>1}\frac{1}{\alpha}B_3(x,\alpha)\,\mathrm{d}\alpha\right\}\mathrm{d}x = 2\int_{\mathbb{R}}\partial_x^3 g(x)\partial_x g(x)\eta(x)\,\mathrm{d}x, \\ S_2 &= \int_{\mathbb{R}} \partial_x^3 g(x)\partial_x g(x)\int_{|\alpha|>1}\frac{\partial_x\Delta_\alpha g}{\alpha}B_3(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x. \end{aligned}$$

In order to get a bound of S_1 we need an estimate for $\eta(x)$. First we observe from (2.1.25) that

$$B_3(x,\alpha) = \gamma(x,\alpha)\Big[4 + 4\partial_x\Delta_\alpha g + (\partial_x\Delta_\alpha g)^2\Big],$$

where

$$\gamma(x,\alpha) := 24(\Delta_\alpha h)K(x,\alpha)^3 - 48(\Delta_\alpha h)^3 K(x,\alpha)^4. \tag{2.1.29}$$

We expand $B_3(x,\alpha)$ and decompose $\eta(x) := 4\eta_1(x) + \eta_2(x)$ for

$$\begin{aligned} \eta_1(x) &= PV\int_{|\alpha|>1} \frac{1}{\alpha}\gamma(x,\alpha)\,\mathrm{d}\alpha, \\ \eta_2(x) &= PV\int_{|\alpha|>1} \frac{1}{\alpha}\gamma(x,\alpha)(4\partial_x\Delta_\alpha g + (\partial_x\Delta_\alpha g)^2)\,\mathrm{d}\alpha. \end{aligned} \tag{2.1.30}$$

We derive the bound for η_2 from the estimate $|\gamma(x,\alpha)| < c$ and the following inequality

$$|4\partial_x\Delta_\alpha g + (\partial_x\Delta_\alpha g)^2| \le 8\frac{\|\partial_x g\|_{L^\infty}}{|\alpha|} + 4\frac{\|\partial_x g\|_{L^\infty}^2}{|\alpha|^2}.$$

Hence

$$|\eta_2(x)| \le c\,(\|\partial_x g\|_{L^\infty} + \|\partial_x g\|_{L^\infty}^2).$$

While for η_1, the estimate (4.1.21) in Lemma 28 states that

$$|\eta_1(x)| \le c\,(1 + \|g\|_{L^\infty})^3.$$

By joining the inequalities for η_1 and η_2 we obtain the next estimate

$$\|\eta\|_{L^\infty} \le c\,(1 + \|g\|_{C^1})^3.$$

Thus, applying the Cauchy-Schwarz inequality we complete the estimate for S_1. We have that

$$|S_1| \le 4\int_{\mathbb{R}} |\partial_x^3 g(x)||\partial_x g(x)||\eta(x)|\,\mathrm{d}x \le c\,(1 + \|g\|_{C^1})^3\|\partial_x g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

The inner integral in S_2 is easily bounded by using the estimate (2.1.27), we conclude that

$$\begin{aligned} \left|\int_{|\alpha|>1} \frac{\partial_x\Delta_\alpha g}{\alpha} B_3(x,\alpha)\,\mathrm{d}\alpha\right| &\le c\,(1 + \|\partial_x^2 g\|_{L^\infty})^3 \int_{|\alpha|>1} |\alpha|^{-2}\,\mathrm{d}\alpha \\ &\le c\,(1 + \|\partial_x^2 g\|_{L^\infty})^3. \end{aligned}$$

Then, similarly to S_1, we apply the Cauchy-Schwarz inequality and use the previous bound to obtain that

$$|S_2| \le c\,(1 + \|g\|_{C^2})^3\|\partial_x g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

The last inequality completes the estimate for L_1. Now we move to L_2, analogously we take $L_2 := S_3 + S_4$, where

$$\begin{aligned} S_3 &= \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{2\partial_x g(x-\alpha)}{\alpha} B_3(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x, \\ S_4 &= \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{\partial_x g(x-\alpha)}{\alpha} \partial_x\Delta_\alpha g B_3(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x. \end{aligned}$$

Notice that $|\gamma(x,\alpha)| < c\,K(x,\alpha)$. Using the bound (2.1.27) we derive the following estimate

$$|B_3(x,\alpha)| \le 4cK(x,\alpha) + 4c\frac{\|\partial_x g\|_{L^\infty}}{|\alpha|} + c\frac{\|\partial_x g\|_{L^\infty}^2}{|\alpha|^2}.$$

The last bound together with the Cauchy-Schwarz inequality with respect to x and Minkowski's integral inequality leads to

$$|S_3| \leq c\,\|\partial_x^3 g\|_{L^2}\|\partial_x g\|_{L^2}\left(\int_{|\alpha|>1}\frac{1}{\alpha^2}\int_{\mathbb{R}} K(x,\alpha)^2\,\mathrm{d}x\,\mathrm{d}\alpha\right)^{1/2}$$
$$+\,c\,\|\partial_x g\|_{L^\infty}\|\partial_x^3 g\|_{L^2}\|\partial_x g\|_{L^2}\int_{|\alpha|>1}\frac{1}{|\alpha|^2}\,\mathrm{d}\alpha + c\,\|\partial_x g\|_{L^\infty}^2\|\partial_x^3 g\|_{L^2}\|\partial_x g\|_{L^2}\int_{|\alpha|>1}\frac{1}{|\alpha|^3}\,\mathrm{d}\alpha.$$

Then the estimate (4.1.16) in Lemma 23 implies that

$$|S_3| \leq c\,(1+\|g\|_{C^1})^2\|\partial_x g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

For S_4, we use the bound (2.1.27) to obtain that

$$|\partial_x\Delta_\alpha g B_3(x,\alpha)| \leq c\,(1+\|\partial_x^2 g\|_{L^\infty})^2\|\partial_x g\|_{L^2}|\alpha|^{-1}.$$

Now, we apply the Cauchy-Schwarz and Minkoswki's integral inequalities. Then we conclude the following bound

$$|S_4| \leq c\,(1+\|g\|_{C^2})^3\|\partial_x g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

The last inequality completes the estimate for the *out* part L_2^{out}. Hence estimate (2.1.28) and bounds for L_1 and L_2 implies that

$$|J_{4,1}| \leq c\,(1+\|g\|_{C^2})^3\|g\|_{H^3}^2. \tag{2.1.31}$$

For $J_{4,2}^{in}$ using the formula (2.1.17) we find that

$$J_{4,2}^{in} = \int_0^1\int_{\mathbb{R}}\partial_x^3 g(x)\int_{|\alpha|<1}\partial_x^3 g(x+(s-1)\alpha)\partial_x\Delta_\alpha g B_4(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x\,\mathrm{d}s.$$

From the identities (2.1.25) and the estimate (2.1.22) we deduce that

$$|B_4(x,\alpha)| \leq c\,(1+\|\partial_x^2 g\|_{L^\infty}). \tag{2.1.32}$$

Using the estimates (2.1.32), (2.1.26), (2.1.17) together with the Cauchy-Schwarz inequality, we infer

$$|J_{4,2}^{in}| \leq c\,(1+\|\partial_x^2 g\|_{L^\infty})\|\partial_x^2 g\|_{L^\infty}\|\partial_x^3 g\|_{L^2}^2. \tag{2.1.33}$$

For the *out* part, expanding $\partial_x^2\Delta_\alpha g$ we split $J_{4,2}^{out} := L_3 + L_4$ for

$$L_3 = \int_{\mathbb{R}}\partial_x^3 g(x)\partial_x^2 g(x)\int_{|\alpha|>1}\frac{1}{\alpha}\partial_x\Delta_\alpha g B_4(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x,$$

$$L_4 = -\int_{\mathbb{R}}\partial_x^3 g(x)\int_{|\alpha|>1}\frac{\partial_x^2 g(x-\alpha)}{\alpha}\partial_x\Delta_\alpha g B_4(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x.$$

Recall that $|\partial_x\Delta_\alpha g| \leq 2\|\partial_x g\|_{L^\infty}|\alpha|^{-1}$. We use the estimate (2.1.32) and analogously to S_2 we obtain that

$$|L_3| \leq c\,(1+\|\partial_x^2 g\|_{L^\infty})\|\partial_x g\|_{L^\infty}\|\partial_x^2 g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

The estimate for L_4 is easy, because is similar to S_4, then we have the following bound

$$|L_4| \leq c\,(1+\|\partial_x^2 g\|_{L^\infty})\|\partial_x g\|_{L^\infty}\|\partial_x^2 g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

Using the estimates for L_3, L_4 and the bound (2.1.33) we obtain that

$$|J_{4,2}| \leq c\,(1+\|g\|_{C^2})^2\|g\|_{H^3}^2. \tag{2.1.34}$$

Finally for $J_{4,3}$, expanding $\partial_x^3\Delta_\alpha g$, we split $J_{4,3} := L_5 + L_6$ for

$$L_5 = \int_{\mathbb{R}} [\partial_x^3 g(x)]^2 \int_{\mathbb{R}} \frac{1}{\alpha}[\partial_x\Delta_\alpha g] B_5(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x,$$
$$L_6 = \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} \frac{\partial_x^3 g(x-\alpha)}{\alpha}\partial_x\Delta_\alpha g B_5(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x.$$

Using the indentities (2.1.25) we have the next bound

$$|\partial_x\Delta_\alpha g B_5(x,\alpha)| \leq 2\|\partial_x^2 g\|_{L^\infty}.$$

Using the last bound and estimates (4.1.9) together with the Cauchy-Schwarz and Minkowski's integral inequalities, we deduce

$$|L_5| \leq c\,(1+\|g\|_{C^{2,1/2}})^2\|\partial_x^3 g\|_{L^2}^2. \tag{2.1.35}$$

For $L_6 := L_6^{in} + L_6^{out}$, the *out* part is easy controlled by using Cauchy-Schwarz inequality and Minkowski's integral inequality

$$\begin{aligned}|L_6^{out}| &\leq 2\|\partial_x g\|_{L^\infty}\int_{\mathbb{R}}|\partial_x^3 g(x)|\int_{|\alpha|>1}\frac{|\partial_x^3 g(x-\alpha)|}{|\alpha|^2}\,\mathrm{d}\alpha\,\mathrm{d}x\\ &\leq c\,\|\partial_x g\|_{L^\infty}\|\partial_x^3 g\|_{L^2}^2.\end{aligned} \tag{2.1.36}$$

For the *in* part, we add and subtract $\partial_x^2 g(x)$ and $B_5(x,0)$ in order to get $L_6^{in} := N_1 + N_2 + N_3$ for

$$N_1 = \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|<1} \frac{\partial_x^3 g(x-\alpha)}{\alpha}\big(\partial_x\Delta_\alpha g - \partial_x^2 g(x)\big)B_5(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x,$$
$$N_2 = \int_{\mathbb{R}} \partial_x^3 g(x)\partial_x^2 g(x) \int_{|\alpha|<1} \frac{\partial_x^3 g(x-\alpha)}{\alpha}\big(B_5(x,\alpha) - B_5(x,0)\big)\,\mathrm{d}\alpha\,\mathrm{d}x,$$
$$N_3 = \int_{\mathbb{R}} \partial_x^3 g(x)\partial_x^2 g(x) B_5(x,0) H_{|\alpha|<1}\partial_x^3 g(x)\,\mathrm{d}x.$$

For N_1, we use

$$|\partial_x\Delta_\alpha g - \partial_x^2 g(x)| \leq |g|_{C^{2,\delta}}|\alpha|^\delta,$$

for $\delta = 1/2$. Due to $|B_5(x,\alpha)| \leq c$, and applying Cauchy-Schwarz inequality with respect to x followed by the Minkoswki integral inequality, we obtain

$$\begin{aligned}|N_1| &\leq c\,\|g\|_{C^{2,\delta}}\|\partial_x^3 g\|_{L^2}\left(\int_{\mathbb{R}}\left|\int_{|\alpha|<1}|\partial_x^3 g(x-\alpha)||\alpha|^{\delta-1}\,\mathrm{d}\alpha\right|^2\mathrm{d}x\right)^{1/2}\\ &\leq c\,\|g\|_{C^{2,\delta}}\|\partial_x^3 g\|_{L^2}\int_{|\alpha|<1}|\alpha|^{\delta-1}\left(\int_{\mathbb{R}}|\partial_x^3 g(x-\alpha)|^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}\alpha \leq c\,\|g\|_{C^{2,\delta}}\|\partial_x^3 g\|_{L^2}^2.\end{aligned}$$

We estimate the second term N_2 using the next inequality

$$|B_5(x,\alpha) - B_5(x,0)| \leq c\,|\Delta_\alpha h - \partial_x h(x)| \leq c\,(1+\|\partial_x^2 g\|_{L^\infty}).$$

Then, by applying the Cauchy-Schwarz and Minkoswki integral we obtain that

$$|N_2| \leq c\,(1+\|g\|_{C^2})^2\|\partial_x^3 g\|_{L^2}^2.$$

Finally using $|B_5(x,0)| \leq c$ and the fact that the truncated Hilbert transform is bounded operator in $L^2(\mathbb{R})$, we obtain that

$$|N_3| \leq c\,\|\partial_x^2 g\|_{L^\infty}\|\partial_x^3 g\|_{L^2}^2.$$

The estimates for N_1, N_2, N_3, the bound (2.1.36) and the estimate (2.1.35) allow us conclude that

$$|J_{4,3}| \leq c\,(1+\|g\|_{C^{2,1/2}})^3\|g\|_{H^3}^2.$$

By joining the estimates (2.1.31), (2.1.34) and the last one, we complete the estimate for J_4. We obtain that

$$|J_4| \leq c\,(1+\|g\|_{C^{2,1/2}})^3\|g\|_{H^3}^2. \tag{2.1.37}$$

We conclude from inequalities (2.1.15), (2.1.16), (2.1.23) and (2.1.37) that

$$|\,\mathrm{III}\,| \leq c\,(1+\|g\|_{C^{2,1/2}})^3\|g\|_{H^3}^2. \tag{2.1.38}$$

Bound for IV: Notice

$$\mathrm{IV} = \int_{\mathbb{R}} \partial_x^3 g(x)\partial_x^3 \int_{\mathbb{R}} \Delta_\alpha g\, G(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x = 2\int_{\mathbb{R}} \partial_x^3 g(x)\int_{\mathbb{R}} \partial_x^3 K(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x.$$

Using (2.1.24) we decompose $\mathrm{IV} := J_5 + J_6 + J_7$ for

$$\begin{aligned}
J_5 &= 2\int_{\mathbb{R}} \partial_x^3 g(x)\int_{\mathbb{R}} \partial_x\Delta_\alpha h B_3(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x,\\
J_6 &= 2\int_{\mathbb{R}} \partial_x^3 g(x)\int_{\mathbb{R}} \partial_x^2\Delta_\alpha g B_4(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x,\\
J_7 &= 2\int_{\mathbb{R}} \partial_x^3 g(x)\int_{\mathbb{R}} \partial_x^3\Delta_\alpha g B_5(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x.
\end{aligned}$$

From the identities (2.1.25) we see that $B_5(x,\alpha) = -2\Delta_\alpha h K(x,\alpha)^2$. Then we estimate J_7 is the same way as J_2. Thus

$$|J_7| \leq c\,(1+\|g\|_{C^{2,1/2}})^2\|g\|_{H^3}^2. \tag{2.1.39}$$

Using the formula (2.1.17) and the inequality (2.1.32) together with the Cauchy-Schwarz inequality we find that

$$\begin{aligned}
|J_6^{in}| &\leq \int_0^1\int_{|\alpha|<1}\int_{\mathbb{R}} |\partial_x^3 g(x)||\partial_x^3 g(x+(s-1)\alpha)||B_4(x,\alpha)|\,\mathrm{d}\alpha\,\mathrm{d}x\,\mathrm{d}s\\
&\leq c\,(1+\|\partial_x^2 g\|_{L^\infty})\|\partial_x^3 g\|_{L^2}^2.
\end{aligned}$$

For the *out* part expanding $\partial_x^2\Delta_\alpha g$ we decompose $J_6^{out} := L_5 + L_6$ for

$$\begin{aligned}
L_5 &= \int_{\mathbb{R}} \partial_x^3 g(x)\partial_x^2 g(x)\left\{\int_{|\alpha|>1}\frac{1}{\alpha}B_4(x,\alpha)\,\mathrm{d}\alpha\right\}\mathrm{d}x,\\
L_6 &= -\int_{\mathbb{R}} \partial_x^3 g(x)\int_{|\alpha|>1}\frac{\partial_x^2 g(x-\alpha)}{\alpha}B_4(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x.
\end{aligned}$$

We denote the inner integral by

$$\nu(x) := PV\int_{|\alpha|>1}\frac{1}{\alpha}B_4(x,\alpha)\,\mathrm{d}\alpha.$$

Now in order to get a bound for ν, we proceed in similar way to η in (2.1.30). Using estimates (4.1.25) in Lemma 29 we obtain that

$$\|\nu\|_{L^\infty} \leq c\,(1+\|g\|_{C^1})^2.$$

Then Cauchy-Schwarz inequality yields to

$$|L_5| \leq c\,(1+\|g\|_{C^1})^3\|\partial_x^2 g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

From identities (2.1.25) we have the next bound

$$|B_4(x,\alpha)| \le c\,(1+\|\partial_x^2 g\|_{L^\infty})K(x,\alpha).$$

For L_6, we apply the Cauchy-Schwarz inequality, first with respect to x and then respect to α, also we use Lemma 25, then we deduce that

$$\begin{aligned}|L_6| &\le c\,(1+\|\partial_x g\|_{L^\infty})\|\partial_x^3 g\|_{L^2}\|\partial_x^2 g\|_{L^2}\left(\int_{|\alpha|>1}\frac{1}{|\alpha|^2}\int_{\mathbb{R}} K(x,\alpha)^2\,\mathrm{d}x\,\mathrm{d}\alpha\right)^{1/2}\\ &\le c\,(1+\|\partial_x g\|_{L^\infty})^2\|\partial_x^3 g\|_{L^2}\|\partial_x^2 g\|_{L^2}.\end{aligned}$$

The bounds for L_5 and L_6 complete the estimate for the *out* part. We conclude

$$|J_6| \le c\,(1+\|g\|_{C^2})^3\|g\|_{L^2}\|\partial_x^3 g\|_{L^2}. \tag{2.1.40}$$

We now estimate J_5, first we note

$$\partial_x\Delta_\alpha h\, B_3(x,\alpha) = \gamma(x,\alpha)\big[8+12\partial_x\Delta_\alpha g+6(\partial_x\Delta_\alpha g)^2+(\partial_x\Delta_\alpha g)^3\big],$$

where $\gamma(x,\alpha)$ is given by (2.1.29). Then we decompose $J_5 := J_{5,1}+J_{5,2}+J_{5,3}+J_{5,4}$ for

$$\begin{aligned}J_{5,1} &= 8\int_{\mathbb{R}}\partial_x^3 g(x)\int_{\mathbb{R}}\gamma(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x,\\ J_{5,2} &= 12\int_{\mathbb{R}}\partial_x^3 g(x)\int_{\mathbb{R}}\gamma(x,\alpha)\partial_x\Delta_\alpha g\,\mathrm{d}\alpha\,\mathrm{d}x,\\ J_{5,3} &= 6\int_{\mathbb{R}}\partial_x^3 g(x)\int_{\mathbb{R}}\gamma(x,\alpha)(\partial_x\Delta_\alpha g)^2 g\,\mathrm{d}\alpha\,\mathrm{d}x,\\ J_{5,4} &= \int_{\mathbb{R}}\partial_x^3 g(x)\int_{\mathbb{R}}\gamma(x,\alpha)(\partial_x\Delta_\alpha g)^3\,\mathrm{d}\alpha\,\mathrm{d}x.\end{aligned}$$

The bound $|\gamma(x,\alpha)|<c$, the formula (2.1.26) and the Cauchy-Schwarz inequality imply that

$$|J_{5,2}^{in}|+|J_{5,3}^{in}|+|J_{5,4}^{in}| \le c\,(1+\|\partial_x^2 g\|_{L^\infty})^2\|\partial_x^2 g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

The *out* part $J_{5,3}^{out}$ is easily bounded. By expanding $\partial_x\Delta_\alpha g$ we have $J_{5,3}^{out} := S_5+S_6$ for

$$\begin{aligned}S_5 &= 6\int_{\mathbb{R}}\partial_x^3 g(x)\partial_x g(x)\int_{|\alpha|>1}\frac{1}{\alpha}\gamma(x,\alpha)(\partial_x\Delta_\alpha g)\,\mathrm{d}\alpha\,\mathrm{d}x,\\ S_6 &= -6\int_{\mathbb{R}}\partial_x^3 g(x)\int_{|\alpha|>1}\frac{\partial_x g(x-\alpha)}{\alpha}\gamma(x,\alpha)(\partial_x\Delta_\alpha g)\,\mathrm{d}\alpha\,\mathrm{d}x.\end{aligned}$$

The bound $|\partial_x\Delta_\alpha g| \le 2\|\partial_x g\|_{L^\infty}|\alpha|^{-1}$ and the Cauchy-Schwarz inequality yields

$$|S_5| \le c\,\|\partial_x g\|_{L^\infty}\|\partial_x g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

While for S_6 we use Minkowski's integral inequality and we obtain that

$$|S_6| \le c\,\|\partial_x g\|_{L^\infty}\|\partial_x g\|_{L^2}\|\partial_x^3 g\|_{L^2},$$

and this completes the estimate for $J_{5,3}^{out}$. For $J_{5,4}^{out}$ we proceed similarly to $J_{5,3}^{out}$, then we conclude that

$$|J_{5,4}^{out}| \le c\,\|\partial_x g\|_{L^\infty}^2\|\partial_x g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

For $J_{5,2}^{out}$, expanding $\partial_x \Delta_\alpha g$ we get

$$J_{5,2}^{out} = 12 \int_{\mathbb{R}} \partial_x^3 g(x) \partial_x g(x) \eta_1(x) \, \mathrm{d}x - 12 \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{\partial_x g(x-\alpha)}{\alpha} \gamma(x,\alpha) \, \mathrm{d}\alpha \, \mathrm{d}x,$$

where $\eta_1(x)$ is given as in (2.1.30). We use $|\gamma(x,\alpha)| \le c\, K(x,\alpha)$ and the estimate (4.1.21) for $\|\eta_1\|_{L^\infty}$ followed by applying the Cauchy-Schwarz inequality we obtain that

$$|J_{5,2}^{out}| \le c\,(1 + \|g\|_{L^\infty})^3 \|\partial_x g\|_{L^2} \|\partial_x^3 g\|_{L^2}.$$

Finally, from the definition (2.1.29) we split $\gamma(x,\alpha)$ and decompose $J_{5,1} := S_7 + S_8$ for

$$\begin{aligned}
S_7 &= 24 \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} \Delta_\alpha h K(x,\alpha)^3 \, \mathrm{d}\alpha \, \mathrm{d}x, \\
S_8 &= -48 \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} (\Delta_\alpha h)^3 K(x,\alpha)^4 \, \mathrm{d}\alpha \, \mathrm{d}x.
\end{aligned}$$

We define

$$\gamma_f(x,\alpha) := \frac{\Delta_\alpha f}{(1 + (\Delta_\alpha f)^2)^3}$$

and observe

$$\int_{\mathbb{R}} \gamma_f(x,\alpha) d\alpha = 0.$$

Expanding $\Delta_\alpha h$ and adding γ_f, we decompose

$$\begin{aligned}
S_7 &= 24 \int_{\mathbb{R}} \partial_x^3 g(x) \big(\Delta_\alpha f K(x,\alpha)^3 - \gamma_f(x,\alpha)\big) \, \mathrm{d}\alpha \, \mathrm{d}x + 24 \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} \Delta_\alpha g K(x,\alpha)^3 \, \mathrm{d}\alpha \, \mathrm{d}x \\
&:= S_{7,1} + S_{7,2}.
\end{aligned}$$

Using the Fundamental Theorem of Calculus and the Cauchy-Schwarz inequality we obtain that

$$\begin{aligned}
|S_{7,2}^{in}| &\le c \int_0^1 \int_{\mathbb{R}} \int_{|\alpha|<1} |\partial_x^3 g(x)| |\partial_x g(x + (s-1)\alpha)| K(x,\alpha)^3 \, \mathrm{d}\alpha \, \mathrm{d}x \, \mathrm{d}s \\
&\le c\, \|\partial_x g\|_{L^2} \|\partial_x^3 g\|_{L^2}.
\end{aligned}$$

Anagolously to J_6^{out}, we expand $\Delta_\alpha g$ and decompose

$$S_{7,2}^{out} = \int_{\mathbb{R}} \partial_x^3 g(x) g(x) \int_{|\alpha|>1} \frac{1}{\alpha} K(x,\alpha)^3 \, \mathrm{d}\alpha \, \mathrm{d}x - \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{g(x-\alpha)}{\alpha} K(x,\alpha)^3 \, \mathrm{d}\alpha \, \mathrm{d}x.$$

Applying the Cauchy-Schwarz inequality and using the estimates (4.1.16) and (4.1.20) from Lemma 23 and Lemma 27 we deduce that

$$|S_{7,2}^{out}| \le c\,(1 + \|g\|_{L^\infty}) \|g\|_{L^2} \|\partial_x^3 g\|_{L^2}.$$

For the term $S_{7,1}$, we observe

$$\Delta_\alpha f K(x,\alpha)^3 - \gamma_f(x,\alpha) = \Delta_\alpha g\, \Gamma(x,\alpha),$$

where

$$\Gamma(x,\alpha) := -\Delta_\alpha f(\Delta_\alpha f + \Delta_\alpha h) \Bigg[\frac{K(x,\alpha)^3}{1 + (\Delta_\alpha f)^2} + \frac{K(x,\alpha)^2}{(1 + (\Delta_\alpha f)^2)^2} + \frac{K(x,\alpha)}{(1 + (\Delta_\alpha f)^2)^3} \Bigg]. \tag{2.1.41}$$

Notice $|\Gamma(x,\alpha)| \le c$, we obtain a bound for the *in* part

$$|S_{7,1}^{in}| \le c\, \|\partial_x g\|_{L^2} \|\partial_x^3 g\|_{L^2}.$$

Now expanding $\Delta_\alpha g$ we have

$$S_{7,1}^{out} = \int_{\mathbb{R}} \partial_x^3 g(x) g(x) \int_{|\alpha|>1} \frac{1}{\alpha}\Gamma(x,\alpha)\, d\alpha\, dx - \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{g(x-\alpha)}{\alpha}\Gamma(x,\alpha)\, dx\, d\alpha. \tag{2.1.42}$$

Using the estimate (4.1.26) from Lemma 30 and the Cauchy-Schwarz inequality we find that

$$\left| \int_{\mathbb{R}} \partial_x^3 g(x) g(x) \int_{|\alpha|>1} \frac{1}{\alpha}\Gamma(x,\alpha)\, d\alpha\, dx \right| \leq c\,(1+\|g\|_{L^\infty})\|g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

For the second term in (2.1.42) we use the next bound

$$|\Gamma(x,\alpha)| \leq c\, K(x,\alpha) + 2\|g\|_{L^\infty}|\alpha|^{-1}.$$

Then we apply the Cauchy-Schwarz and Minkowski's integral inequalities to obtain that

$$\left| \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{g(x-\alpha)}{\alpha}\Gamma(x,\alpha)\, dx\, d\alpha \right| \leq c\,(1+\|g\|_{L^\infty})\|g\|_{L^2}\|\partial_x^3 g\|_{L^2},$$

and this completes the estimate for $S_{7,1}^{out}$. Therefore

$$|S_7| \leq c\,(1+\|g\|_{L^\infty})\|g\|_{L^2}\|\partial_x^3 g\|_{L^2}. \tag{2.1.43}$$

Finally, for S_8 we expand $(\Delta_\alpha h)^3$ and decompose $S_8 := S_{8,1} + S_{8,2} + S_{8,3} + S_{8,4}$ for

$$\begin{aligned}
S_{8,1} &= \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} (\Delta_\alpha f)^3 K(x,\alpha)^4\, d\alpha\, dx,\\
S_{8,2} &= 3\int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} (\Delta_\alpha f)^2 \Delta_\alpha g K(x,\alpha)^4\, d\alpha\, dx,\\
S_{8,3} &= 3\int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} \Delta_\alpha f (\Delta_\alpha g)^2 K(x,\alpha)^4\, d\alpha\, dx,\\
S_{8,4} &= \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} (\Delta_\alpha g)^3 K(x,\alpha)^4\, d\alpha\, dx.
\end{aligned}$$

Repeating the same argument as in S_7, we find that

$$|S_{8,2} + S_{8,3} + S_{8,4}| \leq c\,(1+\|g\|_{C^1})^2\|g\|_{H^3}^2.$$

For $S_{8,1}$ we consider the function

$$\theta_f(x,\alpha) := \frac{(\Delta_\alpha f)^3}{(1+(\Delta_\alpha f)^2)^4},$$

we observe $\int_{\mathbb{R}} \theta_f\, d\alpha = 0$. Using the Fundamental Theorem of Calculus and adding θ_f we decompose $S_{8,1}$ in the next way

$$\begin{aligned}
S_{8,1} &= \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} \Big[(\Delta_\alpha f)^3 K(x,\alpha)^4 - \theta_f(x,\alpha)\Big]\, d\alpha\, dx\\
&= \int_{\mathbb{R}} \partial_x^3 g(x) \int_{\mathbb{R}} \Delta_\alpha g\, \Theta(x,\alpha)\, d\alpha\, dx\\
&= \int_0^1 \int_{|\alpha|<1} \int_{\mathbb{R}} \partial_x^3 g(x) \partial_x g(x+(s-1)\alpha)\Theta(x,\alpha)\, dx\, d\alpha\, ds\\
&+ \int_{\mathbb{R}} \partial_x^3 g(x) g(x) \int_{|\alpha|>1} \frac{1}{\alpha}\Theta(x,\alpha) d\alpha dx - \int_{\mathbb{R}} \partial_x^3 g(x) \int_{|\alpha|>1} \frac{g(x-\alpha)}{\alpha}\Theta(x,\alpha)\, d\alpha\, dx,
\end{aligned}$$

where

$$\Theta(x,\alpha) := -(\Delta_\alpha f)^3(\Delta_\alpha f+\Delta_\alpha h)\Bigg[\frac{K(x,\alpha)^4}{1+(\Delta_\alpha f)^2}+\frac{K(x,\alpha)^3}{(1+(\Delta_\alpha f)^2)^2} \\ +\frac{K(x,\alpha)^2}{(1+(\Delta_\alpha f)^2)^3}+\frac{K(x,\alpha)}{(1+(\Delta_\alpha f)^2)^4}\Bigg]. \tag{2.1.44}$$

We use a similar bound as in (4.1.23) to obtain that $|\Theta(x,\alpha)| < c\,(1+\|\partial_x g\|_{L^\infty})^2$. Then using Cauchy-Schwarz inequality we find a bound for the *in* part

$$|S^{in}_{8,1}| \leq c\,(1+\|\partial_x g\|_{L^\infty})^2\|\partial_x g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

For the last two terms we use the estimate (4.1.28) in Lemma 31, to control the inside integral, and the estimate

$$|\Theta(x,\alpha)| \leq c\,(1+\|\partial_x g\|_{L^\infty})^2 K(x,\alpha).$$

Then Cauchy-Schwarz inequality implies

$$\left|\int_{\mathbb{R}}\partial_x^2 g(x)g(x)\int_{|\alpha|>1}\frac{1}{\alpha}\Theta(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x\right| \leq c\,(1+\|g\|_{C^1})^2\|g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

Now we apply the Cauchy-Schwarz inequality first with respect to x and then respect to α

$$\left|\int_{\mathbb{R}}\partial_x^2 g(x)\int_{|\alpha|>1}\frac{g(x-\alpha)}{\alpha}\Theta(x,\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x\right| \\ \leq c\,(1+\|g\|_{C^1})^2\|g\|_{L^2}\|\partial_x^3 g\|_{L^2}\left(\int_{|\alpha|>1}\frac{1}{|\alpha|^2}\int_{\mathbb{R}}K(x,\alpha)^2\,\mathrm{d}x\,\mathrm{d}\alpha\right)^{1/2}.$$

The estimate (4.1.16) in Lemma 23 leads to

$$|S^{out}_{8,1}| \leq c\,(1+\|g\|_{C^1})^2\|g\|_{L^2}\|\partial_x^3 g\|_{L^2}.$$

By bringing together the inequalities for $S_{8,1}, S_{8,2}, S_{8,3}, S_{8,4}$ and the bound (2.1.43) we complete the estimate for $J_{5,1}$, and we obtain that

$$|J_{5,1}| \leq c\,(1+\|g\|_{C^1})^2\|g\|^2_{H^3}.$$

The previous estimates for $J_{5,2}, J_{5,3}, J_{5,4}$, and the last one, lead us to conclude that

$$|J_5| \leq c\,(1+\|g\|_{C^2})^3\|g\|^2_{H^3}. \tag{2.1.45}$$

Using the estimates (2.1.39), (2.1.40) and (2.1.45) we deduce

$$|\,\mathrm{IV}\,| \leq c\,(1+\|g\|_{C^{2,1/2}})^3\|g\|^2_{H^3}.$$

Finally, using the estimate (2.1.38) we obtain

$$|\,\mathrm{III}\,|+|\,\mathrm{IV}\,| \leq c\,(1+\|g\|_{C^{2,1/2}})^3\|g\|^2_{H^3}.$$

The Sobolev embedding $H^3(\mathbb{R}) \hookrightarrow C^{2,1/2}(\mathbb{R})$ completes the proof of the lemma. ■

From the inequalities (2.1.1) and (2.1.9) in Lemma 2 and Lemma 3 we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(1+\|g(t)\|_{H^3}) \leq c\left(1+\|g(t)\|_{H^3}\right)^4.$$

We integrate in time to obtain that

$$\|g(t)\|_{H^3} \leq \frac{\|g_0\|_{H^3}}{\left(1-c[\phi(0)]^3 t\right)^{1/3}},$$

where $\phi(0) = 1+\|g_0\|_{H^3}$. Then the solution belongs to $H^3(\mathbb{R})$ up to a time

$$t < \frac{\phi(0)^{-3}}{c} = T^\star.$$

2.2 Regularized system

In this section we regularize the equation (1.4.3), via mollifiers. We consider a function $\chi \in C_c^\infty(\mathbb{R})$ that satisfies

$$\int_{\mathbb{R}} \chi(x)\,\mathrm{d}x = 1, \quad \chi(|x|) = \chi(x) \quad \text{and} \quad \chi \geq 0.$$

For every $\epsilon > 0$ we define $\chi_\epsilon(x) = \epsilon^{-1}\chi(\epsilon^{-1}x)$. We denote the convolution by

$$\chi_\epsilon g(x) := (\chi_\epsilon * g)(x) = \int_{\mathbb{R}} \chi_\epsilon(x-y)g(y)\,\mathrm{d}y.$$

Throughout the section we use the next properties of mollifiers

$$\begin{aligned}
&\|\chi_\epsilon \partial_x^k g\|_{L^\infty}, \|\chi_\epsilon \partial_x^k g\|_{L^2} \leq c(\epsilon)\|g\|_{L^2},\\
&\partial_x^s \chi_\epsilon g = \chi_\epsilon \partial_x^s g,\\
&\|\chi_\epsilon g - g\|_{H^{s-1}} \leq c\,\epsilon \|g\|_{H^s}.
\end{aligned} \tag{2.2.1}$$

Now we define the regularized system as follows

$$\begin{aligned}
&M^\epsilon(g^\epsilon) := \chi_\epsilon \int_{\mathbb{R}} \partial_x \Delta_\alpha(\chi_\epsilon g^\epsilon)(x)\, K^\epsilon(x,\alpha)\,\mathrm{d}\alpha + \chi_\epsilon \int_{\mathbb{R}} \Delta_\alpha(\chi_\epsilon g^\epsilon)(x)\, G^\epsilon(x,\alpha)\,\mathrm{d}\alpha,\\
&g^\epsilon(x,0) = g_0(x),
\end{aligned} \tag{2.2.2}$$

where the regularized kernels are defined by

$$\begin{aligned}
K^\epsilon(x,\alpha) &:= \frac{1}{1 + (\Delta_\alpha(\chi_\epsilon g^\epsilon) + \Delta_\alpha f)^2},\\
G^\epsilon(x,\alpha) &:= -\frac{2\Delta_\alpha f + \Delta_\alpha(\chi_\epsilon g^\epsilon)}{(1 + (\Delta_\alpha(\chi_\epsilon g^\epsilon) + \Delta_\alpha f)^2)(1 + (\Delta_\alpha f)^2)}.
\end{aligned} \tag{2.2.3}$$

In the next lemma we apply the Picard theorem, see [45], to the regularized system (2.2.2), where we consider the open set $\mathcal{O} \subset H^s(\mathbb{R})$ defined by $\mathcal{O} = \{g \in H^s(\mathbb{R}) : \|g\|_{H^s} < c\}$ for $s \geq 3$.

Lemma 4. *Let $\epsilon > 0$, then there exists a time $T_\epsilon > 0$ and a solution $g^\epsilon(x,t) \in C^1([0,T_\epsilon] : \mathcal{O})$ to the regularized system (2.2.2) such that $g^\epsilon(x,0) = g_0(x)$ for $s \geq 3$.*

Proof. Take $g_1, g_2 \in \mathcal{O} \subset H^s(\mathbb{R})$. We define the auxiliary operator

$$\mathfrak{M}^\epsilon(g)(x) := \int_{\mathbb{R}} \partial_x \Delta_\alpha(\chi_\epsilon g^\epsilon)(x) K^\epsilon(x,\alpha)\,\mathrm{d}\alpha + \int_{\mathbb{R}} \Delta_\alpha(\chi_\epsilon g^\epsilon)(x) G^\epsilon(x,\alpha)\,\mathrm{d}\alpha.$$

We observe that $M^\epsilon = \chi_\epsilon * \mathfrak{M}^\epsilon$. By applying the triangle inequality we have

$$\|\mathfrak{M}^\epsilon(g_1) - \mathfrak{M}^\epsilon(g_2)\|_{L^2} \leq \|R_1\|_{L^2} + \|R_2\|_{L^2},$$

for

$$\begin{aligned}
R_1(x) &:= \int_{\mathbb{R}} \partial_x \Delta_\alpha(\chi_\epsilon g_1) K_1^\epsilon(x,\alpha)\,\mathrm{d}\alpha - \int_{\mathbb{R}} \partial_x \Delta_\alpha(\chi_\epsilon g_2) K_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha,\\
R_2(x) &:= \int_{\mathbb{R}} \Delta_\alpha(\chi_\epsilon g_1) G_1^\epsilon(x,\alpha)\,\mathrm{d}\alpha - \int_{\mathbb{R}} \Delta_\alpha(\chi_\epsilon g_2) G_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha,
\end{aligned}$$

where $K_i^\epsilon(x,\alpha)$ and $G_i^\epsilon(x,\alpha)$ are the respective kernels for the functions g_1 and g_2. For R_1, we note that by adding and subtracting $\partial_x\Delta_\alpha(\chi_\epsilon g_2)K_1^\epsilon(x,\alpha)$, we find that

$$R_1(x) = \int_{\mathbb{R}} \Big[\partial_x\Delta_\alpha(\chi_\epsilon g_1) - \partial_x\Delta_\alpha(\chi_\epsilon g_2)\Big] K_1^\epsilon(x,\alpha)\,\mathrm{d}\alpha \\ - \int_{\mathbb{R}} \partial_x\Delta_\alpha(\chi_\epsilon g_2)\Big[K_2^\epsilon(x,\alpha) - K_1^\epsilon(x,\alpha)\Big]\,\mathrm{d}\alpha.$$

We have the following identities

$$\begin{aligned}
&\partial_x\Delta_\alpha(\chi_\epsilon g_1) - \partial_x\Delta_\alpha(\chi_\epsilon g_2) = \frac{1}{\alpha}\chi_\epsilon(\partial_x g_1(x) - \partial_x g_2(x)) - \frac{1}{\alpha}\chi_\epsilon(\partial_x g_1(x-\alpha) - \partial_x g_2(x-\alpha)),\\
&\partial_x\Delta_\alpha(\chi_\epsilon g_2)\big[K_2^\epsilon(x,\alpha) - K_1^\epsilon(x,\alpha)\big] = \Big[\Delta_\alpha(\chi_\epsilon g_1) - \Delta_\alpha(\chi_\epsilon g_2)\Big]B_\epsilon(x,\alpha),\\
&B_\epsilon(x,\alpha) = \partial_x\Delta_\alpha(\chi_\epsilon g_2)(2\Delta_\alpha f + \chi_\epsilon\Delta_\alpha g_1 + \chi_\epsilon\Delta_\alpha g_2)K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha).
\end{aligned} \tag{2.2.4}$$

Using the formulas (2.2.4), we obtain the next decomposition

$$\begin{aligned}
R_1(x) &= \chi_\epsilon[\partial_x g_1(x) - \partial_x g_2(x)]\int_{\mathbb{R}} \frac{1}{\alpha}K_1^\epsilon(x,\alpha)\,\mathrm{d}\alpha + \chi_\epsilon[g_1(x) - g_2(x)]\int_{\mathbb{R}}\frac{1}{\alpha}B_\epsilon(x,\alpha)\,\mathrm{d}\alpha\\
&+ \int_{\mathbb{R}} \frac{\chi_\epsilon[\partial_x g_1(x-\alpha) - \partial_x g_2(x-\alpha)]}{\alpha}K_1^\epsilon(x,\alpha)\,\mathrm{d}\alpha\\
&+ \int_{\mathbb{R}} \frac{\chi_\epsilon[g_1(x-\alpha) - g_2(x-\alpha)]}{\alpha}B_\epsilon(x,\alpha)\,\mathrm{d}\alpha\\
&:= T_1(x) + T_2(x) + T_3(x) + T_4(x).
\end{aligned}$$

We use the estimate (4.1.1) in Lemma 18 to get a bound for T_1. Now, we use the properties (2.2.1) to obtain

$$\|T_1\|_{L^2} \le c(\|g_1\|_{L^2},\epsilon)\,\|g_1 - g_2\|_{L^2}.$$

For T_2 we decompose the integral in the next way

$$PV\int_{\mathbb{R}}\frac{1}{\alpha}B_\epsilon(x,\alpha)\,\mathrm{d}\alpha := Q_1(x) + Q_2(x) + Q_3(x),$$

for

$$\begin{aligned}
Q_1(x) &= 2\int_{\mathbb{R}}\frac{1}{\alpha}\Delta_\alpha f\cdot\partial_x\Delta_\alpha(\chi_\epsilon g_2)\cdot K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha,\\
Q_2(x) &= \int_{\mathbb{R}}\frac{1}{\alpha}\Delta_\alpha(\chi_\epsilon g_1)\cdot\partial_x\Delta_\alpha(\chi_\epsilon g_2)\cdot K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha,\\
Q_3(x) &= \int_{\mathbb{R}}\frac{1}{\alpha}\Delta_\alpha(\chi_\epsilon g_2)\cdot\partial_x\Delta_\alpha(\chi_\epsilon g_2)\cdot K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha.
\end{aligned}$$

Using the next estimates

$$\begin{aligned}
&|\Delta_\alpha(\chi_\epsilon g_i)| \le \|\chi_\epsilon g_i\|_{L^\infty}|\alpha|^{-1} \quad \text{for} \quad i = 1,2,\\
&|\partial_x\Delta_\alpha(\chi_\epsilon g_2)| \le \|\chi_\epsilon\partial_x g_2\|_{L^\infty}|\alpha|^{-1}
\end{aligned}$$

and the next bound

$$|\Delta_\alpha f\, K_1^\epsilon(x,\alpha)| \le |(\Delta_\alpha f + \Delta_\alpha(\chi_\epsilon g_1))K_1^\epsilon(x,\alpha)| + |\Delta_\alpha(\chi_\epsilon g_1)\cdot K_1^\epsilon(x,\alpha)| \le c\,(1 + \|\chi_\epsilon\partial_x g_1\|_{L^\infty})$$

we find that

$$\left|Q_1(x)^{out}+Q_2(x)^{out}+Q_3(x)^{out}\right|\le c\,\|\chi_\epsilon\partial_x g_2\|_{L^\infty}(1+\|\chi_\epsilon g_1\|_{L^\infty}+\|\chi_\epsilon g_2\|_{L^\infty}+\|\partial_x\chi_\epsilon g_1\|_{L^\infty}).$$

To estimate $Q_2(x)^{in}$, we add and subtract $\chi_\epsilon\partial_x g_1(x)$, $\chi_\epsilon\partial_x^2 g_2(x)$, $K_1^\epsilon(x,0)$ and $K_2^\epsilon(x,0)$. We obtain

$$\begin{aligned}
Q_2(x)^{in}=&\int_{|\alpha|<1}\frac{1}{\alpha}\big[\Delta_\alpha\chi_\epsilon g_1-\chi_\epsilon\partial_x g_1(x)\big]\partial_x\Delta_\alpha(\chi_\epsilon g_2)K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha\\
&+\chi_\epsilon\partial_x g_1(x)\int_{|\alpha|<1}\frac{1}{\alpha}\big[\partial_x\Delta_\alpha(\chi_\epsilon g_2)-\chi_\epsilon\partial_x^2 g_2(x)\big]K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha\\
&+\chi_\epsilon\partial_x g_1(x)\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\big[K_1^\epsilon(x,\alpha)-K_1^\epsilon(x,0)\big]K_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha\\
&+\chi_\epsilon\partial_x g_1(x)\chi_\epsilon\partial_x^2 g_2(x)K_1^\epsilon(x,0)\int_{|\alpha|<1}\frac{1}{\alpha}\big[K_2^\epsilon(x,\alpha)-K_2^\epsilon(x,0)\big]\,\mathrm{d}\alpha,
\end{aligned}$$

where the regularized kernels at zero are

$$\begin{aligned}
K_1^\epsilon(x,0)&=\frac{1}{1+(\partial_x f(x)+\chi_\epsilon\partial_x g_1(x))^2},\\
K_2^\epsilon(x,0)&=\frac{1}{1+(\partial_x f(x)+\chi_\epsilon\partial_x g_2(x))^2}.
\end{aligned}$$

In a similar way to (4.1.4) and (4.1.5) we have the following inequalities

$$\begin{aligned}
&|\Delta_\alpha\chi_\epsilon g_1-\chi_\epsilon\partial_x g_1(x)|\le c\,\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty}|\alpha|,\\
&|\partial_x\Delta_\alpha\chi_\epsilon g_2-\chi_\epsilon\partial_x^2 g_2(x)|\le c\,\|(\partial_x\chi_\epsilon)\partial_x^2 g_2\|_{L^\infty}|\alpha|,\\
&|K_1^\epsilon(x,\alpha)-K_1^\epsilon(x,0)|\le c\,(1+\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty})\,|\alpha|,\\
&|K_2^\epsilon(x,\alpha)-K_2^\epsilon(x,0)|\le c\,(1+\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty})\,|\alpha|.
\end{aligned}\tag{2.2.5}$$

Hence, we deduce the following

$$\begin{aligned}
\left|Q_2(x)^{in}\right|\le c\Big(&\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty}\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty}+\|\chi_\epsilon\partial_x g_1\|_{L^\infty}\|(\partial_x\chi_\epsilon)\partial_x^2 g_2\|_{L^\infty}\\
&+\|\chi_\epsilon\partial_x g_1\|_{L^\infty}\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty}(1+\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty}+\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty})\Big).
\end{aligned}$$

Similarly to the last term, we derive that

$$\begin{aligned}
\left|Q_3(x)^{in}\right|\le c\Big(&\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty}\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty}+\|\chi_\epsilon\partial_x g_2\|_{L^\infty}\|(\partial_x\chi_\epsilon)\partial_x^2 g_2\|_{L^\infty}\\
&+\|\chi_\epsilon\partial_x g_2\|_{L^\infty}\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty}(1+\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty}+\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty})\Big).
\end{aligned}$$

We recall the definition of the auxiliary function

$$F(x)=\frac{1}{1+x^2}$$

then we decompose $Q_1(x)^{in}$ by adding and subtracting $\chi_\epsilon\partial_x^2 g_2(x)$ and $F(\Delta_\alpha f)$. We take $Q_1(x)^{in}:=\mathfrak{I}_1(x)+\mathfrak{I}_2(x)+\mathfrak{I}_3(x)+\mathfrak{I}_4(x)$ for

$$\begin{aligned}
\mathfrak{I}_1(x)&=\int_{|\alpha|<1}\frac{1}{\alpha}[\partial_x\Delta_\alpha\chi_\epsilon g_2-\chi_\epsilon\partial_x^2 g_2(x)]\Delta_\alpha f K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha,\\
\mathfrak{I}_2(x)&=\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha f\big[K_2^\epsilon(x,\alpha)-F(\Delta_\alpha f)\big]K_1^\epsilon(x,\alpha)\,\mathrm{d}\alpha,\\
\mathfrak{I}_3(x)&=\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)\big[K_1^\epsilon(x,\alpha)-F(\Delta_\alpha f)\big]\,\mathrm{d}\alpha,\\
\mathfrak{I}_4(x)&=\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)^2\,\mathrm{d}\alpha.
\end{aligned}\tag{2.2.6}$$

A direct computation yields to

$$K_1^\epsilon(x,\alpha)-F(\Delta_\alpha f)=-\Delta_\alpha\chi_\epsilon g_1(2\Delta_\alpha f+\Delta_\alpha\chi_\epsilon g_1)K_1^\epsilon(x,\alpha)F(\Delta_\alpha f).$$

Now, we decompose $\mathfrak{I}_3(x)$ by adding and subtracting $\chi_\epsilon\partial_x g_1(x)$ and $K_1^\epsilon(x,0)$, then we obtain that

$$\begin{aligned}
\mathfrak{I}_3(x)=&-2\int_{|\alpha|<1}\frac{1}{\alpha}(\Delta_\alpha f)^2\big[\Delta_\alpha\chi_\epsilon g_1-\chi_\epsilon\partial_x g_1(x)\big]K_1^\epsilon(x,\alpha)F(\Delta_\alpha f)^2\,\mathrm{d}\alpha\\
&-2\chi_\epsilon\partial_x^2 g_2(x)\chi_\epsilon\partial_x g_1(x)\int_{|\alpha|<1}\frac{1}{\alpha}(\Delta_\alpha f)^2\big[K_1^\epsilon(x,\alpha)-K_1^\epsilon(x,0)\big]F(\Delta_\alpha f)^2\,\mathrm{d}\alpha\\
&-2\chi_\epsilon\partial_x^2 g_2(x)\chi_\epsilon\partial_x g_1(x)K_1^\epsilon(x,0)\int_{|\alpha|<1}\frac{1}{\alpha}(\Delta_\alpha f)^2F(\Delta_\alpha f)^2\,\mathrm{d}\alpha\\
&-\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha f\big[\Delta_\alpha\chi_\epsilon g_1-\chi_\epsilon\partial_x g_1(x)\big]\Delta_\alpha\chi_\epsilon g_1K_1^\epsilon(x,\alpha)F(\Delta_\alpha f)^2\,\mathrm{d}\alpha\\
&-\chi_\epsilon\partial_x^2 g_2(x)\chi_\epsilon\partial_x g_1(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha f\big[\Delta_\alpha\chi_\epsilon g_1-\chi_\epsilon\partial_x g_1(x)\big]K_1^\epsilon(x,\alpha)F(\Delta_\alpha f)^2\,\mathrm{d}\alpha\\
&-\chi_\epsilon\partial_x^2 g_2(x)(\chi_\epsilon\partial_x g_1(x))^2\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha f\big[K_1^\epsilon(x,\alpha)-K_1^\epsilon(x,0)\big]F(\Delta_\alpha f)^2\,\mathrm{d}\alpha\\
&-\chi_\epsilon\partial_x^2 g_2(x)(\chi_\epsilon\partial_x g_1(x))^2K_1^\epsilon(x,0)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha fF(\Delta_\alpha f)^2\,\mathrm{d}\alpha.
\end{aligned}$$

Hence, using the estimates (2.2.5) we find that

$$\big|\mathfrak{I}_3(x)\big|\le c\Big(\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty}+(1+\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty})(\|\chi_\epsilon\partial_x g_1\|_{L^\infty}+\|\chi_\epsilon\partial_x g_1\|_{L^\infty}^2)\Big). \tag{2.2.7}$$

For the second term in (2.2.6) we add and subtract $F(\Delta_\alpha f)$, and hence

$$\begin{aligned}
\mathfrak{I}_2(x)=&\,\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha f\big[K_2^\epsilon(x,\alpha)-F(\Delta_\alpha f)\big]\big[K_1^\epsilon(x,\alpha)-F(\Delta_\alpha f)\big]\,\mathrm{d}\alpha\\
&+\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha fF(\Delta_\alpha f)[K_2^\epsilon(x,\alpha)-F(\Delta_\alpha f)]\,\mathrm{d}\alpha\\
&:=\mathfrak{I}_{2,1}(x)+\mathfrak{I}_{2,2}(x).
\end{aligned}\tag{2.2.8}$$

The term $\mathfrak{I}_{2,2}(x)$ in the last decomposition (2.2.8) can be bounded in similar way to (2.2.7). While for the first one, we observe that

$$\begin{aligned}
\mathfrak{I}_{2,1}(x)=&\,\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha fK_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha\\
&-\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha fK_1^\epsilon(x,\alpha)F(\Delta_\alpha f)\,\mathrm{d}\alpha\\
&-\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha fK_2^\epsilon(x,\alpha)F(\Delta_\alpha f)\,\mathrm{d}\alpha\\
&+\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha fF(\Delta_\alpha f)^2\,\mathrm{d}\alpha:=\mathfrak{N}_1(x)+\mathfrak{N}_2(x)+\mathfrak{N}_3(x)+\mathfrak{N}_4(x).
\end{aligned}\tag{2.2.9}$$

The term $\mathfrak{N}_4(x)$ is bounded by lemma (17). For $\mathfrak{N}_2(x)$ we decompose by adding and subtracting $K_1^\epsilon(x,0)$ then we have

$$\begin{aligned}
\mathfrak{N}_2(x)=&-\chi_\epsilon\partial_x^2 g_2(x)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha f\big[K_1^\epsilon(x,\alpha)-K_1^\epsilon(x,0)\big]F(\Delta_\alpha f)\,\mathrm{d}\alpha\\
&-\chi_\epsilon\partial_x^2 g_2(x)K_1^\epsilon(x,0)\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha fF(\Delta_\alpha f)\,\mathrm{d}\alpha.
\end{aligned}$$

Using the estimates (2.2.5) we find that

$$|\mathfrak{N}_2(x)| \leq c\,\|\chi_\epsilon \partial_x^2 g_2\|_{L^\infty}(1 + \|\chi_\epsilon \partial_x^2 g_1\|_{L^\infty}).$$

Similarly we get

$$|\mathfrak{N}_3(x)| \leq c\,\|\chi_\epsilon \partial_x^2 g_2\|_{L^\infty}(1 + \|\chi_\epsilon \partial_x^2 g_2\|_{L^\infty}).$$

For the remaining term in (2.2.9) we add and subtract $F(\Delta_\alpha f), K_1^\epsilon(x,0), K_2^\epsilon(x,0)$ and $\chi_\epsilon \partial_x g_1(x)$. We find that

$$\begin{aligned}
\mathfrak{N}_1(x) = &-2\chi_\epsilon \partial_x^2 g_2(x) \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f\big[\Delta_\alpha \chi_\epsilon g_1 - \chi_\epsilon \partial_x g_1(x)\big] K_1^\epsilon(x,\alpha) K_2^\epsilon(x,\alpha) F(\Delta_\alpha f)\,\mathrm{d}\alpha \\
&- 2\chi_\epsilon \partial_x^2 g_2(x) \chi_\epsilon \partial_x g_1(x) \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f\big[K_1^\epsilon(x,\alpha) - K_1^\epsilon(x,0)\big] K_2^\epsilon(x,\alpha) F(\Delta_\alpha f)\,\mathrm{d}\alpha \\
&- 2\chi_\epsilon \partial_x^2 g_2(x) \chi_\epsilon \partial_x g_1(x) K_1^\epsilon(x,0) \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f\big[K_2^\epsilon(x,\alpha) - K_2^\epsilon(x,0)\big] F(\Delta_\alpha f)\,\mathrm{d}\alpha \\
&- 2\chi_\epsilon \partial_x^2 g_2(x) \chi_\epsilon \partial_x g_1(x) K_1^\epsilon(x,0) K_2^\epsilon(x,0) \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)\,\mathrm{d}\alpha \\
&- \chi_\epsilon \partial_x^2 g_2(x) \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f\big[\Delta_\alpha \chi_\epsilon g_1 - \chi_\epsilon \partial_x g_1(x)\big] \Delta_\alpha \chi_\epsilon g_1 K_1^\epsilon(x,\alpha) K_2^\epsilon(x,\alpha) F(\Delta_\alpha f)\,\mathrm{d}\alpha \\
&- \chi_\epsilon \partial_x^2 g_2(x) \chi_\epsilon \partial_x g_1(x) \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f[\Delta_\alpha \chi_\epsilon g_1 - \chi_\epsilon \partial_x g_1(x)\big] K_1^\epsilon(x,\alpha) K_2^\epsilon(x,\alpha) F(\Delta_\alpha f)\,\mathrm{d}\alpha \\
&- \chi_\epsilon \partial_x^2 g_2(x) (\chi_\epsilon \partial_x g_1(x))^2 \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f\big[K_1^\epsilon(x,\alpha) - K_1^\epsilon(x,0)\big] K_2^\epsilon(x,\alpha) F(\Delta_\alpha f)\,\mathrm{d}\alpha \\
&- \chi_\epsilon \partial_x^2 g_2(x) (\chi_\epsilon \partial_x g_1(x))^2 K_1^\epsilon(x,0) \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f\big[K_2^\epsilon(x,\alpha) - K_2^\epsilon(x,0)\big] F(\Delta_\alpha f)\,\mathrm{d}\alpha \\
&- \chi_\epsilon \partial_x^2 g_2(x) (\chi_\epsilon \partial_x g_1(x))^2 K_1^\epsilon(x,0) K_2^\epsilon(x,0) \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)\,\mathrm{d}\alpha \\
&+ \chi_\epsilon \partial_x^2 g_2(x) \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)\big[K_1^\epsilon(x,\alpha) - K_1^\epsilon(x,0)\big]\,\mathrm{d}\alpha \\
&+ \chi_\epsilon \partial_x^2 g_2(x) K_1^\epsilon(x,0) \int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)\,\mathrm{d}\alpha.
\end{aligned}$$

Using the bounds (2.2.5) we deduce the next estimate

$$\begin{aligned}
\big|\mathfrak{N}_1(x)\big| \leq c\Big\{&1 + \|\chi_\epsilon \partial_x^2 g_1\|_{L^\infty} + \|\chi_\epsilon \partial_x g_2\|_{L^\infty}\|\chi_\epsilon \partial_x^2 g_1\|_{L^\infty} \\
&+ (\|\chi_\epsilon \partial_x^2 g_1\|_{L^\infty} + \|\chi_\epsilon \partial_x g_1\|_{L^\infty}^2)(1 + \|\chi_\epsilon \partial_x^2 g_2\|_{L^\infty} + \|\chi_\epsilon \partial_x^2 g_1\|_{L^\infty})\Big\}\|\chi_\epsilon \partial_x^2 g_2\|_{L^\infty}.
\end{aligned}$$

The last inequality completes the estimate for the *in* part $Q_1(x)^{in}$. Now, we use the properties of mollifiers (2.2.1) and we conclude that

$$\big|Q_1(x)^{in}\big| \leq c(\epsilon)\,(1 + \|g_1\|_{L^2})^3(1 + \|g_2\|_{L^2})^3\|g_2\|_{L^2}.$$

Therefore

$$\|T_2\|_{L^2} \leq c\,(\|g_1\|_{L^2}, \|g_2\|_{L^2}, \epsilon)\,\|g_1 - g_2\|_{L^2}.$$

Now we move to T_3, for the *out* part using the Cauchy-Schwarz inequality with respect to α, we find the following bound

$$\|T_3^{out}\|_{L^2} \leq \|\chi_\epsilon(\partial_x g_1 - \partial_x g_2)\|_{L^2}\left(\int_{|\alpha|>1} \frac{1}{\alpha^2}\int_{\mathbb{R}} K_1^\epsilon(x,\alpha)^2\,\mathrm{d}x\,\mathrm{d}\alpha\right)^{1/2}$$

which is enough to control the *out* part. For the *in* part we add and subtract the term $K_1^\epsilon(x,0)$. This leads to the next decomposition

$$\begin{aligned}\int_{|\alpha|<1} &\frac{\chi_\epsilon(\partial_x g_1(x-\alpha)-\partial_x g_2(x-\alpha))}{\alpha}K_1^\epsilon(x,\alpha)\,\mathrm{d}\alpha\\ &= \int_{|\alpha|<1}\chi_\epsilon(\partial_x g_1(x-\alpha)-\partial_x g_2(x-\alpha))\frac{1}{\alpha}\Big[K_1^\epsilon(x,\alpha)-K_1^\epsilon(x,0)\Big]\,\mathrm{d}\alpha\\ &+K_1^\epsilon(x,0)\int_{|\alpha|<1}\frac{\chi_\epsilon(\partial_x g_1(x-\alpha)-\partial_x g_2(x-\alpha))}{\alpha}\,\mathrm{d}\alpha.\end{aligned}$$

From the above, a truncated Hilbert transform arises. Applying the Minkowski's integral inequality and using the estimates (2.2.5) we obtain that

$$\begin{aligned}\|T_3^{in}\|_{L^2} &\le \left\|\int_{|\alpha|<1}\chi_\epsilon(\partial_x g_1(x-\alpha)-\partial_x g_2(x-\alpha))\frac{K_1^\epsilon(x,\alpha)-K_1^\epsilon(x,0)}{\alpha}\,\mathrm{d}\alpha\right\|_{L^2}\\ &+\left\|K_1^\epsilon(x,0)H_{|\alpha|<1}\chi_\epsilon(\partial_x g_1-\partial_x g_2)(x)\right\|_{L^2}\\ &\le c\int_{|\alpha|<1}(1+\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty})\left(\int_{\mathbb{R}}\chi_\epsilon\big[\partial_x g_1(x-\alpha)-\partial_x g_2(x-\alpha)\big]^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}\alpha\\ &+\|K_1^\epsilon(x,0)\|_{L^\infty}\|\chi_\epsilon(\partial_x g_1-\partial_x g_2)\|_{L^2}\\ &\le c\,(\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty},\epsilon)\|\chi_\epsilon(\partial_x g_1-\partial_x g_2)\|_{L^2}.\end{aligned}$$

We use the properties of mollifiers (2.2.1) to conclude that

$$\|T_3\|_{L^2}\le c\,(\|g_1\|_{L^2},\epsilon)\|g_1-g_2\|_{L^2}.$$

For T_4 we expand the sum in $B_\epsilon(x,\alpha)$, see the definitions (2.2.4), and we repeat the argument used in T_3. We have the next decomposition

$$\begin{aligned}B_\epsilon(x,\alpha) = 2\Delta_\alpha f K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\partial_x\Delta_\alpha(\chi_\epsilon g_2)+\Delta_\alpha(\chi_\epsilon g_1)K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\partial_x\Delta_\alpha(\chi_\epsilon g_2)\\ +\Delta_\alpha(\chi_\epsilon g_2)K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\partial_x\Delta_\alpha(\chi_\epsilon g_2).\end{aligned}$$

For the second term in $B_\epsilon(x,\alpha)$ we add and subtract the terms $\chi_\epsilon\partial_x g_1(x)$, $\chi_\epsilon\partial_x^2 g_2(x)$, $K_1^\epsilon(x,0)$ and $K_2^\epsilon(x,0)$ in order to obtain

$$\begin{aligned}\partial_x\Delta_\alpha(\chi_\epsilon g_2)\Delta_\alpha(\chi_\epsilon g_1)K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha) &= \big[\partial_x\Delta_\alpha\chi_\epsilon g_2-\chi_\epsilon\partial_x^2 g_2(x)\big]\Delta_\alpha g_1K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\\ &+\chi_\epsilon\partial_x^2 g_2(x)\big[\Delta_\alpha\chi_\epsilon g_1-\chi_\epsilon\partial_x g_1(x)\big]K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\\ &+\chi_\epsilon\partial_x^2 g_2(x)\chi_\epsilon\partial_x g_1(x)\big[K_1^\epsilon(x,\alpha)-K_1^\epsilon(x,0)\big]K_2^\epsilon(x,\alpha)\\ &+\chi_\epsilon\partial_x^2 g_2(x)\chi_\epsilon\partial_x g_1(x)K_1^\epsilon(x,0)\big[K_2^\epsilon(x,\alpha)-K_2^\epsilon(x,0)\big]\\ &+\chi_\epsilon\partial_x^2 g_2(x)\chi_\epsilon\partial_x g_1(x)K_1^\epsilon(x,0)K_2^\epsilon(x,0).\end{aligned}$$

Now, we use the last decomposition and the estimates (2.2.5) together with the Minkowski's integral inequality to obtain that

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}\left|\int_{|\alpha|<1}\frac{\chi_\epsilon(g_1(x-\alpha)-g_2(x-\alpha))}{\alpha}\partial_x\Delta_\alpha(\chi_\epsilon g_2)\Delta_\alpha(\chi_\epsilon g_1)K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)\,\mathrm{d}\alpha\right|^2\mathrm{d}x\right)^{1/2}\\
&\le\int_{|\alpha|<1}\|(\partial_x\chi_\epsilon)\partial_x^2 g_2\|_{L^\infty}\|\chi_\epsilon\partial_x g_1\|_{L^\infty}\left(\int_{\mathbb{R}}\chi_\epsilon(g_1(x-\alpha)-g_2(x-\alpha))^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}\alpha\\
&\le\int_{|\alpha|<1}\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty}\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty}\left(\int_{\mathbb{R}}\chi_\epsilon(g_1(x-\alpha)-g_2(x-\alpha))^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}\alpha\\
&+\int_{|\alpha|<1}\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty}\|\chi_\epsilon\partial_x g_1\|_{L^\infty}(1+\|\chi_\epsilon\partial_x^2 g_1\|_{L^\infty})\left(\int_{\mathbb{R}}\chi_\epsilon(g_1(x-\alpha)-g_2(x-\alpha))^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}\alpha\\
&+\int_{|\alpha|<1}\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty}\|\chi_\epsilon\partial_x g_1\|_{L^\infty}(1+\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty})\left(\int_{\mathbb{R}}\chi_\epsilon(g_1(x-\alpha)-g_2(x-\alpha))^2\,\mathrm{d}x\right)^{1/2}\mathrm{d}\alpha\\
&+\|\chi_\epsilon\partial_x^2 g_2\|_{L^\infty}\|\chi_\epsilon\partial_x g_1\|_{L^\infty}\|H_{|\alpha|<1}\chi_\epsilon(g_1-g_2)\|_{L^2}\\
&\le c\,(\|g_1\|_{L^2},\|g_2\|_{L^2},\epsilon)\|g_1-g_2\|_{L^2}.
\end{aligned}
$$

Analogously, we obtain a similar bound for the third term in $B_\epsilon(x,\alpha)$. For the first term in $B_\epsilon(x,\alpha)$ we decompose

$$2\Delta_\alpha f K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)=2(\Delta_\alpha f+\chi_\epsilon g_1)K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha)-2\chi_\epsilon\Delta_\alpha g_1 K_1^\epsilon(x,\alpha)K_2^\epsilon(x,\alpha),$$

and repeat the previous argument. Thus, we conclude that

$$\|T_4\|_{L^2}\le c\,(\|g_1\|_{L^2},\|g_2\|_{L^2},\epsilon)\|g_1-g_2\|_{L^2}.$$

By joining the estimates for T_1, T_2, T_3 and T_4 we obtain the bound for R_1. For R_2 we observe from the definitions (2.2.3) and (2.2.4) the following

$$R_2(x)=2\int_{\mathbb{R}}\left[K_1^\epsilon(x,\alpha)-K_2^\epsilon(x,\alpha)\right]\mathrm{d}\alpha=2\int_{\mathbb{R}}\left[\Delta_\alpha(\chi_\epsilon g_1)-\Delta_\alpha(\chi_\epsilon g_2)\right]B_\epsilon(x,\alpha)\,\mathrm{d}\alpha.$$

Thus, similarly to R_1 we obtain the next estimate

$$\|R_2\|_{L^2}\le c(\|g_1\|_{L^2},\|g_2\|_{L^2},\epsilon)\|g_1-g_2\|_{L^2}.$$

Therefore using the properties of mollifiers (2.2.1) together with the bounds for R_1 and R_2, we deduce that

$$\|M^\epsilon(g_1)-M^\epsilon(g_2)\|_{H^s}\le c\,\epsilon^{-s}\|\mathfrak{M}^\epsilon(g_1)-\mathfrak{M}^\epsilon(g_2)\|_{L^2}\le c(\|g_1\|_{L^2},\|g_2\|_{L^2},\epsilon)\|g_1-g_2\|_{L^2}.$$

Finally, we conclude

$$\|M^\epsilon(g_1)-M^\epsilon(g_2)\|_{H^s}\le c(\|g_1\|_{L^2},\|g_2\|_{L^2},\epsilon)\|g_1-g_2\|_{H^s}.$$

Thus the operator M^ϵ is locally Lipschitz on the open set $\mathcal{O}$. The Picard theorem implies that there exists an unique solution $g^\epsilon\in C^1([0,T_\epsilon]:\mathcal{O})$ of (2.2.2) which completes the proof. ■

Due to the properties of mollifiers (2.2.1) we use the energy estimate obtained in section 2.1 and the time of existence $T_\epsilon>0$ can be changed for a time that depends only on the initial data $g_0\in H^s(\mathbb{R})$. That is

$$\|g^\epsilon(t)\|_{H^3}\le\frac{\|g_0\|_{H^3}}{\left(1-c[\phi(0)]^3 t\right)^{1/3}},\tag{2.2.10}$$

and it follows that $g^\epsilon(\,\cdot\,,t)\in H^3(\mathbb{R})$ when $t<T^\star$. The next step is to prove that the regularized system forms a Cauchy sequence with respect to the norm $L^2(\mathbb{R})$ which is the next lemma where we choose $T_0<T^*$.

Lemma 5. *The sequence of regularized solutions forms a Cauchy sequence in $C([0,T_0]:L^2(\mathbb{R}))$ and we have the estimate*

$$\|g^\epsilon - g^{\epsilon'}\|_{L^2}(t) \le c\,(T_0)\,(\epsilon+\epsilon'),$$

for $\epsilon \neq \epsilon'$ and therefore there exists a limit function $g \in C([0,T_0):L^2(\mathbb{R}))$ such that $g^\epsilon \to g$.

Proof. Taking the $L^2(\mathbb{R})$ product, we add and subtract $M^{\epsilon'}(g^\epsilon)$, then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|g^\epsilon - g^{\epsilon'}\|_{L^2}^2 &= \int_{\mathbb{R}}(g^\epsilon - g^{\epsilon'})(M^\epsilon(g^\epsilon) - M^{\epsilon'}(g^{\epsilon'}))dx\\ &= \int_{\mathbb{R}}(g^\epsilon - g^{\epsilon'})(M^\epsilon(g^\epsilon) - M^{\epsilon'}(g^{\epsilon}))dx + \int_{\mathbb{R}}(g^\epsilon - g^{\epsilon'})(M^{\epsilon'}(g^\epsilon) - M^{\epsilon'}(g^{\epsilon'})dx\\ &= \int_{\mathbb{R}}(g^\epsilon - g^{\epsilon'})(M^\epsilon(g^\epsilon) - M^{\epsilon'}(g^{\epsilon}))dx + \int_{\mathbb{R}}\chi_{\epsilon'}(g^\epsilon - g^{\epsilon'})(\mathfrak{M}^{\epsilon'}(g^\epsilon) - \mathfrak{M}^{\epsilon'}(g^{\epsilon'}))dx.\end{aligned}$$

To deal with the integral above, we rewrite

$$\int_{\mathbb{R}}\chi_{\epsilon'}(g^\epsilon - g^{\epsilon'})(\mathfrak{M}^{\epsilon'}(g^\epsilon) - \mathfrak{M}^{\epsilon'}(g^{\epsilon'}))dx = \int_{\mathbb{R}}(f^\epsilon - f^{\epsilon'})(M(f^\epsilon) - M(f^{\epsilon'}))dx,$$

where $f^\epsilon = \chi_{\epsilon'}g^\epsilon, f^\epsilon = \chi_{\epsilon'}g^{\epsilon'}$ and M denotes the operator defined by equation (1.4.3). We write $M = M_K + M_G$ where K and G denote the respective kernels in (1.4.3). Then

$$\int_{\mathbb{R}}(f^\epsilon - f^{\epsilon'})(M_K(f^\epsilon) - M_K(f^{\epsilon'}))dx = A+B+C,$$

with

$$\begin{aligned}A &= \int_{\mathbb{R}}(f^\epsilon - f^{\epsilon'})(x)\partial_x(f^\epsilon - f^{\epsilon'})(x)\int_{\mathbb{R}}\frac{1}{\alpha}\frac{1}{1+(\Delta_\alpha f + \Delta_\alpha f^\epsilon)^2}d\alpha\\ B &= -\int_{\mathbb{R}}(f^\epsilon - f^{\epsilon'})(x)\int_{\mathbb{R}}\frac{\partial_x(f^\epsilon - f^{\epsilon'})(x-\alpha)}{\alpha}\frac{1}{1+(\Delta_\alpha f + \Delta_\alpha f^\epsilon)^2}d\alpha dx\\ C &= \int_{\mathbb{R}}(f^\epsilon - f^{\epsilon'})(x)\int_{\mathbb{R}}\partial_x\Delta f^{\epsilon'}\left[\frac{1}{1+(\Delta_\alpha f + \Delta_\alpha f^\epsilon)^2} - \frac{1}{1+(\Delta_\alpha f + \Delta_\alpha f^{\epsilon'})^2}\right]d\alpha dx.\end{aligned}$$

By using integrations by parts we obtain that

$$A = -\frac{1}{2}\int_{\mathbb{R}}(f^\epsilon - f^{\epsilon'})^2(x)\partial_x\int_{\mathbb{R}}\frac{1}{\alpha}\frac{1}{1+(\Delta_\alpha f + \Delta_\alpha f^\epsilon)^2}d\alpha.$$

Thus, from the energy estimate, we infer

$$A \le c(\|f^\epsilon\|_{H^3})\|f^\epsilon - f^{\epsilon'}\|_{L^2}^2 \le c(T_0)\|g^\epsilon - g^{\epsilon'}\|_{L^2}^2.$$

To deal with B, we integrate by parts, with respect to α, hence

$$\begin{aligned}B = &-\int_{\mathbb{R}}(f^\epsilon - f^{\epsilon'})(x)\int_{\mathbb{R}}\frac{1}{\alpha}\frac{\partial_\alpha(f^\epsilon - f^{\epsilon'})(x) - \partial_\alpha(f^\epsilon - f^{\epsilon'})(x-\alpha)}{1+(\Delta_\alpha f + \Delta_\alpha f^\epsilon)^2}d\alpha dx\\ &\int_{\mathbb{R}}(f^\epsilon - f^{\epsilon'})(x)\int_{\mathbb{R}}\Big[(f^\epsilon - f^{\epsilon'})(x) - (f^\epsilon - f^{\epsilon'})(x-\alpha)\Big]\partial_\alpha\left[\frac{1}{\alpha}\frac{1}{1+(\Delta_\alpha f + \Delta_\alpha f^\epsilon)^2}\right]d\alpha dx.\end{aligned}$$

Then we repeat the argument used in (2.1.11) to obtain that

$$B \le c(\|f^\epsilon\|_{H^3})\|f^\epsilon - f^{\epsilon'}\|_{L^2}^2 \le c(T_0)\|g^\epsilon - g^{\epsilon'}\|_{L^2}^2.$$

By computing the difference in C, we can repeat the previous arguments to obtain that

$$C \leq c(\|f^{\epsilon}\|_{H^3}, \|f^{\epsilon'}\|_{H^3})\|f^{\epsilon} - f^{\epsilon'}\|_{L^2}^2 \leq c(T_0)\|g^{\epsilon} - g^{\epsilon'}\|_{L^2}^2,$$

and we can use a similar technique to deal with the kernel G. We deduce by using the energy estimate for the L^2 norm, that

$$\left| \int_{\mathbb{R}} (f^{\epsilon} - f^{\epsilon'})(M(f^{\epsilon}) - M(f^{\epsilon'}))dx \right| \leq c(T_0)\|f^{\epsilon} - f^{\epsilon'}\|_{L^2}^2,$$

where $c(T_0)$ is independend of ϵ.

To deal with the integral above, we rewrite

$$\int_{\mathbb{R}} \chi_{\epsilon'}(g^{\epsilon} - g^{\epsilon'})(\mathfrak{M}^{\epsilon'}(g^{\epsilon}) - \mathfrak{M}^{\epsilon'}(g^{\epsilon'}))\, \mathrm{d}x = \int_{\mathbb{R}} (f^{\epsilon} - f^{\epsilon'})(M(f^{\epsilon}) - M(f^{\epsilon'}))\, \mathrm{d}x,$$

where $f^{\epsilon} = \chi_{\epsilon'} g^{\epsilon}$, $f^{\epsilon} = \chi_{\epsilon'} g^{\epsilon'}$ and M denotes the operator defined by equation (1.4.3). Using a similar analysis to that one for than the one for the energy estimates, we obtain that

$$\int_{\mathbb{R}} (f^{\epsilon} - f^{\epsilon'})(M(f^{\epsilon}) - M(f^{\epsilon'}))\, \mathrm{d}x \leq c(T_0)\|f^{\epsilon} - f^{\epsilon'}\|_{L^2}^2,$$

where $c(T_0)$ is independend of ϵ. Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g^{\epsilon} - g^{\epsilon'}\|_{L^2}^2 \leq \|g^{\epsilon} - g^{\epsilon'}\|_{L^2}\|M^{\epsilon}(g^{\epsilon}) - M^{\epsilon'}(g^{\epsilon})\|_{L^2} + c(T_0)\|g^{\epsilon} - g^{\epsilon'}\|_{L^2}^2.$$

Now, for the first term in the last inequality we add and subtract $\mathfrak{M}^{\epsilon}(g^{\epsilon})$ and $\mathfrak{M}^{\epsilon'}(g^{\epsilon})$, then we get

$$\begin{aligned}\|M^{\epsilon}(g^{\epsilon}) - M^{\epsilon'}(g^{\epsilon})\|_{L^2} \leq{}& \|\chi_{\epsilon}\mathfrak{M}^{\epsilon}(g^{\epsilon}) - \mathfrak{M}^{\epsilon}(g^{\epsilon})\|_{L^2} + \|\chi_{\epsilon'}\mathfrak{M}^{\epsilon'}(g^{\epsilon}) - \mathfrak{M}^{\epsilon'}(g^{\epsilon})\|_{L^2} \\ &+ \|\mathfrak{M}^{\epsilon}(g^{\epsilon}) - \mathfrak{M}^{\epsilon'}(g^{\epsilon})\|_{L^2}.\end{aligned}$$

Using the properties of mollifiers (2.2.1) we deduce that

$$\|M^{\epsilon}(g^{\epsilon}) - M^{\epsilon'}(g^{\epsilon})\|_{L^2} \leq c\,\epsilon\|\mathfrak{M}^{\epsilon}(g^{\epsilon})\|_{H^1} + c\,\epsilon'\|\mathfrak{M}^{\epsilon'}(g^{\epsilon})\|_{H^1} + \|\mathfrak{M}^{\epsilon}(g^{\epsilon}) - \mathfrak{M}^{\epsilon'}(g^{\epsilon})\|_{L^2}. \tag{2.2.11}$$

The bound for the last term in (2.2.11) is obtained by applying the Lemma 4 with $g_1 = \chi_{\epsilon} g^{\epsilon}$ and $g_2 = \chi_{\epsilon'} g^{\epsilon}$, that is

$$\|\mathfrak{M}^{\epsilon}(g^{\epsilon}) - \mathfrak{M}^{\epsilon'}(g^{\epsilon})\|_{L^2} \leq c(T_0)\|\chi_{\epsilon} g^{\epsilon} - \chi_{\epsilon'} g^{\epsilon}\|_{L^2}.$$

Hence by adding and subtracting g^{ϵ} and using the properties of mollifiers (2.2.1) we find that

$$\begin{aligned}\|\chi_{\epsilon} g^{\epsilon} - \chi_{\epsilon'} g^{\epsilon}\|_{L^2} &= \|\chi_{\epsilon} g^{\epsilon} - g^{\epsilon} + g^{\epsilon} - \chi_{\epsilon'} g^{\epsilon}\|_{L^2} \\ &\leq \|\chi_{\epsilon} g^{\epsilon} - g^{\epsilon}\|_{L^2} + \|\chi_{\epsilon'} g^{\epsilon} - g^{\epsilon}\|_{L^2} \\ &\leq c\,\epsilon\|g^{\epsilon}\|_{H^1} + c\,\epsilon'\|g^{\epsilon}\|_{H^1}.\end{aligned}$$

Because the solutions g^{ϵ} are uniformly bounded by relation (2.2.10), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g^{\epsilon} - g^{\epsilon'}\|_{L^2}^2 \leq c\,(T_0)\|g^{\epsilon} - g^{\epsilon'}\|_{L^2}^2 + c\,(T_0)(\epsilon + \epsilon')\|g^{\epsilon} - g^{\epsilon'}\|_{L^2}.$$

Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g^{\epsilon} - g^{\epsilon'}\|_{L^2} \leq c\,(T_0)\big[\epsilon + \epsilon' + \|g^{\epsilon} - g^{\epsilon'}\|_{L^2}\big].$$

Finally, we integrate with respect to t to conclude that

$$\|g^{\epsilon} - g^{\epsilon'}\|_{L^2}(t) \leq c\,(T_0)(\epsilon + \epsilon'),$$

and this completes the proof. ∎

Now we prove the main result

Proof of the main Theorem 1. By Lemma 4, there exists solutions $\{g^\epsilon\}$ of the regularized problem and from the energy estimate they are uniformly bounded in $H^3(\mathbb{R})$. These solutions can be continued for all time, see theorem 3.3 in [45]. By Lemma 5 the solutions $\{g^\epsilon\}$ forms a Cauchy sequence in $C([0,T_0]:L^2(\mathbb{R}))$ hence $\{g^\epsilon\}$ converges to a function $g \in C([0,T_0]:L^2(\mathbb{R}))$. Now we use Sobolev interpolation, for any $0 < s < 3$, there exists a constant $c_s > 0$ such that

$$\|f\|_{H^s} \leq c_s \|f\|_{L^2}^{1-s/3}\|f\|_{H^3}^{s/3} \quad \text{for all} \quad f \in H^3(\mathbb{R}).$$

We apply the previous inequality to the difference $g^\epsilon - g^{\epsilon'}$ to derive the following

$$\begin{aligned}
\|g^\epsilon - g^{\epsilon'}\|_{H^s} &\leq c_s \|g^\epsilon - g^{\epsilon'}\|_{L^2}^{1-s/3}\|g^\epsilon - g^{\epsilon'}\|_{H^3}^{s/3} \\
&\leq c(s,T_0)(\epsilon+\epsilon')^{1-s/3}\|g^\epsilon - g^{\epsilon'}\|_{H^3}^{s/3} \\
&\leq c(s,T_0)(\epsilon+\epsilon')^{1-s/3}.
\end{aligned}$$

Then $\{g^\epsilon\}$ forms a Cauchy sequence in $H^s(\mathbb{R})$, and this implies strong convergence in the space $C([0,T_0]:H^s(\mathbb{R}))$ for $s<3$ and the limit function g satisfies the equation (1.4.3).

For the rest of the proof we follow several steps.

Step 1: Fix $t \in [0,T_0]$, we use the energy estimate to obtain that $\{g^\epsilon(\cdot,t)\}$ is a sequence uniformly bounded in $H^3(\mathbb{R})$. The Banach-Alaoglu theorem implies that there exists a subsequence $\{g^\epsilon(\cdot,t)\}$ that converges weakly to some function $\tilde{g}(\cdot,t) \in H^3(\mathbb{R})$.

Step 2: The weak limit and the strong limit are equal pointwise in time, that is, $\tilde{g}(\cdot,t) = g(\cdot,t)$, where g is the function of the strong convergence in $H^s(\mathbb{R})$ for all $t \in [0,T_0]$. We take $\varphi \in H^{-s}(\mathbb{R})$ and for $g \in H^s(\mathbb{R})$ we denote $\langle g,\varphi\rangle_s$ as the dual pairing of $H^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$ through the $L^2(\mathbb{R})$ product. Using the weak convergence

$$\langle g^\epsilon(\cdot,t),\varphi\rangle_3 \to \langle \tilde{g}(\cdot,t),\varphi\rangle_3, \quad \text{as} \quad \epsilon \to 0 \quad \text{for all} \quad \varphi \in H^{-3}(\mathbb{R}),$$

and the inclusion $L^2(\mathbb{R}) \subset H^{-3}(\mathbb{R})$, we see that

$$\int_{\mathbb{R}} \big[g^\epsilon(x,t) - \tilde{g}(x,t)\big]\varphi(x)\,\mathrm{d}x \to 0, \quad \text{as} \quad \epsilon \to 0 \quad \text{for all} \quad \varphi \in L^2(\mathbb{R}).$$

The strong convergence in $H^s(\mathbb{R})$ implies weak convergence in $H^s(\mathbb{R})$, thus for the same function $\varphi \in L^2(\mathbb{R})$ we have

$$\langle g(\cdot,t)^\epsilon - g(\cdot,t),\varphi\rangle_s \to 0, \quad \text{as} \quad \epsilon \to 0.$$

Therefore if $\tilde{g}(\cdot,t) \neq g(\cdot,t)$ we get

$$\langle g(\cdot,t) - \tilde{g}(\cdot,t),\varphi\rangle_0 = \langle g(\cdot,t) - g^\epsilon(\cdot,t),\varphi\rangle_0 + \langle g^\epsilon(\cdot,t) - \tilde{g}(\cdot,t),\varphi\rangle_0 \to 0$$

and we have a contradition, therefore the weak limit $\tilde{g}(\cdot,t)$ is equal pointwise in time to the strong limit $g(\cdot,t)$. Hence $g(\cdot,t) \in H^3(\mathbb{R})$ for every $t \in [0,T_0]$.

Step 3: The limit function $g \in C_{\mathrm{w}}([0,T_0]:H^3(\mathbb{R}))$. Using that $H^{-s}(\mathbb{R})$ is dense in $H^{-3}(\mathbb{R})$ for $s<3$, we take $\varphi \in H^{-3}(\mathbb{R})$ and $\epsilon > 0$, then there exists $\varphi' \in H^{-s}(\mathbb{R})$ such that

$$\|\varphi - \varphi'\|_{H^{-3}} < \epsilon.$$

The uniform bound for g^ϵ together with the triangle inequality and the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
|\langle g^\epsilon(\cdot,t) - g(\cdot,t),\varphi\rangle_3| &\leq |\langle g^\epsilon(\cdot,t) - g(\cdot,t),\varphi-\varphi'\rangle_3| + |\langle (g^\epsilon - g)(\cdot,t),\varphi'\rangle_3| \\
&\leq 2c(T_0)\|\varphi-\varphi'\|_{H^{-3}} + \|\varphi'\|_{H^{-s}}\|g^\epsilon(t) - g(t)\|_{H^s}.
\end{aligned}$$

Using the strong convergence in $H^s(\mathbb{R})$ we have

$$|\langle g^\epsilon(\cdot,t) - g(\cdot,t), \varphi\rangle_3| \leq \epsilon\, c\,(T_0).$$

The last inequality implies that

$$\langle g^\epsilon(\cdot,t), \varphi\rangle_3 \to \langle g(\cdot,t), \varphi\rangle_3$$

as $\epsilon \to 0$ uniformly, therefore the limit $\langle g(\cdot,t), \phi\rangle_3$ is a continuous function in time over $[0, T_0]$, and the arbitrary choice of $\varphi \in H^{-3}(\mathbb{R})$ implies that $g \in C_{\mathrm{w}}([0,T_0] : H^3(\mathbb{R}))$. ■

Remark 2. *The limit solution belongs to $H^3(\mathbb{R})$ for every $t \in [0, T_0]$ and we have*

$$g \in L^\infty([0,T_0] : H^3(\mathbb{R})).$$

We observe that this argument is not sufficient to prove the continuity in time of the limit solution, due to the loss of parabolicity in the equation.

Chapter 3

Turning Singularities

This chapter focuses on the search of *turning singularities* for solutions with quadratic growth at the infinity. As commented in the introduction, this kind of singularities have been already studied in [14, 12, 17, 28, 29] for either asymptotically flat or periodic interfaces.

Recall that a *turning singularities* means that the initial interface can be parameterized by the graph of a function but, at some finite time, the solution *turns* and loses this property. Thus, the solution initially satisfies $\mathrm{RT} > 0$ and then, in a finite time the interface, satisfies $\mathrm{RT} < 0$. Here RT denotes the Rayleigh-Taylor condition. Notice that, in this turning singularities, the solution passes from the stable to the unstable regime. In order to do that we do not parameterize the interface by

$$\{\mathbf{x} \in \mathbb{R}^2 \,:\, \mathbf{x} = (x,\, f(x,t))\}$$

but by

$$\{\mathbf{x} \in \mathbb{R}^2 \,:\, \mathbf{x} = \mathbf{z}(\alpha,t) = (z_1(\alpha,t),\, z_2(\alpha,t))\}$$

for some parameter $\alpha \in \mathbb{R}$.

Now, in order to prove the existence of this singularities we will consider an initial data that both, has vertical slope at some point (in the rest of the point is locally the graph of function) and yields a velocity which makes the interface to turn (passing to the unstable regime). Since, for such an initial interface, the Rayleigh-Taylor function is zero (at the point with vertical slope) we can not apply the results in the chapter 2. Instead of that, we will consider analytic initial data. For this kind of data we will be able to show existence of solutions even if the Rayleigh-Taylor function is negative.

The chapter starts with the analytic setting in order to obtain local existence for analytic curves. The main tool to prove this result will be the Cauchy-Kowaleski's Theorem.

3.1 Analytic Setting

Given $r > 0$, we take the complex strip $S_r \subset \mathbb{C}$, which is defined by

$$S_r = \left\{\, \alpha + i\xi : \alpha \in \mathbb{R}, \quad |\xi| < r \,\right\}.$$

Now, we consider analytic functions f in S_r such that the restriction to ∂S_r is square integrable, *i.e.* functions $f \in L^2(\partial S_r)$ such that the norm

$$\|f\|^2_{L^2(\partial S_r)} := \sum_{\pm} \int_{\mathbb{R}} |f(\alpha \pm ir)|^2_* \,\mathrm{d}\alpha$$

is finite. Here $|\cdot|_*$ denotes the complex modulus. Let $\mathbf{d}(\alpha,\cdot) \in \mathbb{R}^2$ be a function where each component d_i is analytic on the complex strip for $\alpha \in S_r$. For an integer $k \geq 0$, we define the space

$$X_{r,k} = \left\{ \begin{array}{c} \mathbf{d}(\alpha,\cdot) : \mathbb{R} \to \mathbb{R}^2 : \ \mathbf{d} = (d_1, d_2), \ d_i \text{ is real in } \mathbb{R}, \text{and can be extended analytically to } S_r, \\ \text{and } d_i, \ \partial_\alpha^k d_i \in L^2(\partial S_r) \end{array} \right\}$$

endowed with the norm

$$\|\mathbf{d}\|^2_{X_{r,k}} = \|\mathbf{d}\|^2_{L^2(\partial S_r)} + \|\partial_\alpha^k \mathbf{d}\|^2_{L^2(\partial S_r)} \tag{3.1.1}$$

which in the Fourier side is equivalent to

$$\|\mathbf{d}\|^2_{X_{r,k}} = 2\sum_n (1 + |2\pi n|^{2k}) |\mathcal{F}(\mathbf{d})(n)|_*^2 \cosh(4\pi n r).$$

We notice that $\{X_{r,k}\}_{r>0}$ form a Banach scale, which means that $X_{r,k} \subset X_{r',k}$ for $r' < r$. This inclusion arises from the following inequality

$$\|\mathbf{d}\|_{X_{r',k}} \leq \|\mathbf{d}\|_{X_{r,k}}. \tag{3.1.2}$$

3.1.1 The Muskat equation

The case when the interface is a general curve, the Muskat problem can be reduced to a contour equation for the curve $\mathbf{z} = (z_1, z_2)$, which is given by the following equation

$$\begin{aligned} \partial_t \mathbf{z}(\alpha,t) &= \frac{\rho^- - \rho^+}{2\pi} PV \int_{\mathbb{R}} \frac{z_1(\alpha,t) - z_1(\alpha-\beta,t)}{|\mathbf{z}(\alpha,t) - \mathbf{z}(\alpha-\beta,t)|^2} (\partial_\alpha \mathbf{z}(\alpha,t) - \partial_\alpha \mathbf{z}(\alpha-\beta,t))\,\mathrm{d}\beta \\ &:= \mathbf{M}(\mathbf{z})(\alpha,t) \end{aligned} \tag{3.1.3}$$

where $|\mathbf{z}|$ is the module defined by

$$|\mathbf{z}|^2 := (z_1)^2 + (z_2)^2.$$

We notice that these types of curves are not necessarily the graph of a function. This means that the Rayleigh-Taylor condition can be positive or negative. Now, considering a curve

$$\mathbf{z}(\alpha,t) := \mathbf{d}(\alpha,t) + \mathbf{p}(\alpha,t) \quad \text{where} \quad \mathbf{p}(\alpha,t) := (\alpha, \alpha^2 + 2\pi t). \tag{3.1.4}$$

We know, from Corollary 1, that a parabola is a particular solution of the Muskat equation (3.1.3). Now, we derive an equation for the *deviation* $\mathbf{d} = \mathbf{z} - \mathbf{p}$, using the Muskat equation (3.1.3). That is

$$\begin{aligned} \partial_t \mathbf{d}(\alpha,t) &= \mathbf{M}(\mathbf{z})(\alpha,t) - \partial_t \mathbf{p}(\alpha,t) \\ &:= \mathbf{F}(\mathbf{d})(\alpha,t). \end{aligned}$$

We renormalize, by setting $\rho^- - \rho^+ = 2\pi$. Then, $\mathbf{F}(\mathbf{d})$ is expressed by

$$\mathbf{F}(\mathbf{d})(\alpha,t) = PV \int_{\mathbb{R}} \frac{z_1(\alpha,t) - z_1(\alpha-\beta,t)}{|\mathbf{z}(\alpha,t) - \mathbf{z}(\alpha-\beta,t)|^2} (\partial_\alpha \mathbf{z}(\alpha,t) - \partial_\alpha \mathbf{z}(\alpha-\beta,t))\,\mathrm{d}\beta - (0, 2\pi). \tag{3.1.5}$$

In order to simplify the notation, we omit the time dependence t. We denote the differences $\Delta z_i(\alpha,\beta)$ by

$$\begin{aligned} \Delta z_1(\alpha,\beta) &:= z_1(\alpha) - z_1(\alpha-\beta), \\ \Delta z_2(\alpha,\beta) &:= z_2(\alpha) - z_2(\alpha-\beta), \end{aligned} \tag{3.1.6}$$

and the modulus $Q(\alpha,\beta)$ by

$$Q(\alpha,\beta) := (\Delta z_1(\gamma,\beta))^2 + (\Delta z_2(\gamma,\beta))^2 = |\mathbf{z}(\alpha) - \mathbf{z}(\alpha-\beta)|^2. \tag{3.1.7}$$

Using the notation (3.1.6) and (3.1.7), we rewrite the operator $\mathbf{F}$ defined in (3.1.5) as follows

$$\mathbf{F}(\mathbf{d})(\alpha) = PV \int_{\mathbb{R}} \frac{\Delta z_1(\alpha,\beta)}{Q(\alpha,\beta)} \big(\partial_\alpha \Delta d_1(\alpha,\beta), \partial_\alpha \Delta z_2(\alpha,\beta)\big)\, \mathrm{d}\beta - (0, 2\pi),$$

where we used the following property

$$\partial_\alpha \Delta z_1(\gamma,\beta) = \partial_\alpha \Delta d_1(\gamma,\beta).$$

The operator $\mathbf{F}$ which is defined over a deviation $\mathbf{d}(\alpha)$, it can be seen as two operators $\mathbf{F} = (F_1, F_2)$, which are given by

$$F_1(\mathbf{d})(\alpha) = PV \int_{\mathbb{R}} \frac{\Delta z_1(\alpha,\beta)}{Q(\alpha,\beta)} \partial_\alpha \Delta d_1(\alpha,\beta)\, \mathrm{d}\beta,$$
$$F_2(\mathbf{d})(\alpha) = PV \int_{\mathbb{R}} \frac{\Delta z_1(\alpha,\beta)}{Q(\alpha,\beta)} \partial_\alpha \Delta z_2(\alpha,\beta)\, \mathrm{d}\beta - 2\pi.$$

3.1.2 The arc-chord condition

A curve $\mathbf{z}$ fulfills the arc-chord condition if there exists a constant $R > 0$ such that for all $\alpha, \beta \in \mathbb{R}$, it is satisfied that

$$|\beta| \leq R\,|\mathbf{z}(\alpha) - \mathbf{z}(\alpha - \beta)|.$$

We then define the function

$$G(\mathbf{z})(\alpha,\beta) := \frac{\beta^2}{|\mathbf{z}(\alpha) - \mathbf{z}(\alpha-\beta)|^2},$$

and we notice that $\mathbf{z}(\alpha)$ satisfies the arc-chord condition if $G(\mathbf{z})$ is always bounded. In this work, we assume that the curve $\mathbf{z} = \mathbf{d} + \mathbf{p}$ satisfies the arc-chord condition. Finally we define

$$\|G(\mathbf{z})\|_{L^\infty(\mathbb{R})} = \sup_{\alpha,\beta\in\mathbb{R}} |G(\mathbf{z})(\alpha,\beta)|.$$

3.1.3 The complex extensions

We will need to extend the operator $\mathbf{F}$ to the complex strip S_r, by allowing the coordinates z_i to be defined over S_r, whose restriction to the real line is a real function. Then, the extended operator is defined by

$$\mathbf{F}(\mathbf{d})(\alpha + i\xi) = PV \int_{\mathbb{R}} \frac{\Delta z_1(\alpha+i\xi,\beta)}{Q(\alpha+i\xi,\beta)} \big(\partial_\alpha \Delta d_1(\alpha+i\xi,\beta), \partial_\alpha \Delta z_2(\alpha+i\xi,\beta)\big)\, \mathrm{d}\beta - (0, 2\pi), \tag{3.1.8}$$

where $|\xi| < r$. The modulus extended is given by

$$Q(\alpha+i\xi,\beta) = |\mathbf{z}(\alpha+i\xi) - \mathbf{z}(\alpha+i\xi-\beta)|^2 = |\Delta z_1(\alpha+i\xi,\beta)|^2 + |\Delta z_2(\alpha+i\xi,\beta)|^2.$$

Furthermore, we extend the arc-chord condition over S_r, as follows

$$\|G(\mathbf{z})\|_{L^\infty(S_r)} = \sup_{\alpha+i\xi\in S_r, \beta\in\mathbb{R}} |G(\mathbf{z})(\alpha+i\xi,\beta)|_*.$$

where recall $|\cdot|_*$ is the complex modulus.

3.2 The Cauchy-Kowaleski's Theorem

In order to prove the local existence result for analytic initial data, we will use the Cauchy-Kowaleski Theorem, see for example [54] and [53]. We fix $r_0, R_0 > 0$ and consider r and R such that $0 < r < r_0$ and $0 < R_0 < R$. We define the open set

$$O_R = \left\{ \mathbf{d} \in X_{r,k} : \|\mathbf{d}\|_{X_{r,k}} < R \ \text{ and } \ \|G(\mathbf{z})\|_{L^\infty(S_r)} < R \right\}. \tag{3.2.1}$$

Notice that O_R is an open set in $X_{r,k}$, due to the arc-chord condition is an open condition in this space. Thus the operator $\mathbf{F}$ maps from $O_R \subset X_{r,k}$ to $X_{r',k}$ for $r' < r$, that is

$$\mathbf{F} \colon O_R \to X_{r',k}.$$

In our case we will take $k = 3$, we estate the following lemma.

Lemma 6. *Consider $0 \leq r' < r$ and the open set O_R defined in (3.2.1). The operator $\mathbf{F} \colon O_R \to X_{r',3}$ is a continuous mapping. In addition, there exists a constant $c_R > 0$ which only depends on $R > 0$, such that*

1. *The operator is bounded*
$$\|\mathbf{F}(\mathbf{d})\|_{X_{r',3}} \leq \frac{c_R}{r - r'} \|\mathbf{d}\|_{X_{r,3}}, \tag{3.2.2}$$
2. *and is Lipschitz*
$$\|\mathbf{F}(\mathbf{d}^1) - \mathbf{F}(\mathbf{d}^2)\|_{X_{r',3}} \leq \frac{c_R}{r - r'} \|\mathbf{d}^1 - \mathbf{d}^2\|_{X_{r,3}}, \tag{3.2.3}$$

for $\mathbf{d}, \mathbf{d}^1, \mathbf{d}^2 \in O_R$ deviations.

The consequence of the previous lemma and the abstract Cauchy-Kowaleski's theorem in [54], is the local existence theorem for analytic curves.

Theorem 4. *Let $\mathbf{d}^0 \in X_{r_0,3}$ an initial analytic deviation such that $\mathbf{z}^0(\alpha) = \mathbf{d}^0(\alpha) + (\alpha, \alpha^2)$ satisfies the arc-chord conditon $\|G(\mathbf{z}^0)\|_{L^\infty(S_{r_0})} < R_0$. Then there exists an analytic solution for the Muskat equation (3.1.3) of the form*

$$\mathbf{z}(\alpha, t) = \mathbf{d}(\alpha, t) + (\alpha, \alpha^2 + 2\pi t),$$

in $C([-T, T] : X_{r,3})$ for a small $T > 0$.

The first part of Lemma 6 concerns the boundedness of the operator $\mathbf{F}$. In the following section we will prove the property (3.2.2). We achieve this by proving that this property holds true for each coordinate of the operator $\mathbf{F} = (F_1, F_2)$. We start with the term of lower order in the norm, see the definition (3.1.1). In order to simplify the notation we will take $\gamma' := \alpha \pm ir' \in S_{r'}$. Additionally, observe that the complex extension (3.1.8) for coordinate F_1, are given by

$$\begin{aligned} F_1(\mathbf{d})(\gamma') &= PV \int_{\mathbb{R}} \frac{\Delta z_1(\gamma', \beta)}{Q(\gamma', \beta)} \partial_\alpha \Delta d_1(\gamma', \beta) \, \mathrm{d}\beta, \\ F_2(\mathbf{d})(\gamma') &= PV \int_{\mathbb{R}} \frac{\Delta z_1(\gamma', \beta)}{Q(\gamma', \beta)} \partial_\alpha \Delta z_2(\gamma', \beta) \, \mathrm{d}\beta - 2\pi. \end{aligned} \tag{3.2.4}$$

Here, we recall that the restrictions to the real line are real.

3.3 Boundedness of $\mathbf{F}$

The goal of this section is to establish the boundeness in $L^2(\partial S_{r'})$ of the lower derivative of $\mathbf{F}$. We will prove this property as a consequence of four lemmas.

3.3.1 Boundedness of F_1

We denote by $c_R > 0$ a constant that depends only on R. To simplify the notation, this constant may change from one line to another. We omit the principal value notation PV in some integrals, but all of them should be understood in that sense. The first lemma deals with the first coordinate and states the following.

Lemma 7. *Given a deviation* $\mathbf{d} \in O_R$, *the following estimate holds*

$$\|F_1(\mathbf{d})\|_{L^2(\partial S_{r'})} \leq c_R \|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}. \tag{B1}$$

Proof. In order to obtain the $L^2(\partial S_{r'})$ bound, we decompose the integral $F_1(\mathbf{d})$ by expanding the difference

$$\partial_\alpha \Delta d_1(\gamma', \beta) = \partial_\alpha d_1(\gamma') - \partial_\alpha d_1(\gamma' - \beta).$$

Hence, we write

$$\begin{aligned} F_1(\mathbf{d})(\gamma') =& \partial_\alpha d_1(\gamma') \int_{\mathbb{R}} \frac{\Delta z_1(\gamma', \beta)}{Q(\gamma', \beta)} \, \mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha d_1(\gamma' - \beta) \frac{\Delta z_1(\gamma', \beta)}{Q(\gamma', \beta)} \, \mathrm{d}\beta. \\ :=& I_1(\gamma') - I_2(\gamma'). \end{aligned} \tag{3.3.1}$$

To estimate I_1 in (3.3.1), we require a bound in $L^\infty(\partial S_{r'})$ of the following integral

$$PV \int_{\mathbb{R}} \frac{\Delta z_1(\gamma', \beta)}{Q(\gamma', \beta)} \, \mathrm{d}\beta. \tag{3.3.2}$$

The estimate of (3.3.2) is important, because as we will observe in the subsequents terms, a similar technique will be employed. First, we multiply and divide the integrand by β^2, and then we split the integral in the *in* and *out* parts. For the *in* part, adding and subtracting $\partial_\alpha z_1(\gamma')$ and $1/|\partial_\alpha \mathbf{z}(\gamma')|^2$ we obtain the following decomposition

$$\begin{aligned} \int_{|\beta|<1} \frac{\Delta z_1(\gamma', \beta)}{\beta} \frac{\beta^2}{Q(\gamma', \beta)} \frac{\mathrm{d}\beta}{\beta} =& \int_{|\beta|<1} \left(\frac{\Delta z_1(\gamma', \beta)}{\beta} - \partial_\alpha z_1(\gamma') \right) \frac{\beta^2}{Q(\gamma', \beta)} \frac{\mathrm{d}\beta}{\beta} \\ &+ \partial_\alpha z_1(\gamma') \int_{|\beta|<1} \left(\frac{\beta^2}{Q(\gamma', \beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \right) \frac{\mathrm{d}\beta}{\beta} \end{aligned} \tag{3.3.3}$$

where

$$|\partial_\alpha \mathbf{z}(\gamma')|^2 = (\partial_\alpha z_1(\gamma'))^2 + (\partial_\alpha z_2(\gamma'))^2.$$

First, we will establish the control in $L^\infty(\partial S_{r'})$ over the first term in (3.3.3), by finding a bound for the integrand. We use the Fundamental Theorem of Calculus to obtain the following formula

$$\frac{\Delta z_1(\gamma', \beta)}{\beta} - \partial_\alpha z_1(\gamma') = \int_0^1 \big(\partial_\alpha z_1(\gamma' + (s-1)\beta) - \partial_\alpha z_1(\gamma') \big) \, \mathrm{d}s.$$

Notice, using definition (3.1.4) we have that $\partial_\alpha z_1(\gamma') = \partial_\alpha d_1(\gamma') + 1$. Hence, we rewrite the previous formula as follows

$$\frac{\Delta z_1(\gamma', \beta)}{\beta} - \partial_\alpha z_1(\gamma') = \int_0^1 \big(\partial_\alpha d_1(\gamma' + (s-1)\beta) - \partial_\alpha d_1(\gamma') \big) \, \mathrm{d}s.$$

Then, we deduce the following inequality

$$\left| \frac{\Delta z_1(\gamma', \beta)}{\beta} - \partial_\alpha z_1(\gamma') \right|_* \leq \|\partial_\alpha^2 d_1\|_{L^\infty(\partial S_{r'})} |\beta|. \tag{A1}$$

We assume that $\mathbf{d} \in O_R$, which implies that the curve $\mathbf{z} = \mathbf{d} + \mathbf{p}$ satisfies the arc-chord condition. Hence

$$\left| \frac{\beta^2}{Q(\gamma', \beta)} \right|_* \leq c_R. \tag{A2}$$

Thus, using the inequalities (A1) and (A2) we obtain the estimate for the first integral in (3.3.3), given by

$$\left| \int_{|\beta|<1} \left(\frac{\Delta z_1(\gamma',\beta)}{\beta} - \partial_\alpha z_1(\gamma') \right) \frac{\beta^2}{Q(\gamma',\beta)} \frac{\mathrm{d}\beta}{\beta} \right|_* \leq c_R.$$

The bound for the second integral in (3.3.3) follows from the estimation in the difference on the kernels, as Lemma 32 states. We deduce

$$\left| \partial_\alpha z_1(\gamma') \int_{|\beta|<1} \left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \right) \frac{\mathrm{d}\beta}{\beta} \right|_* \leq c_R.$$

Therefore

$$\left| PV \int_{|\beta|<1} \frac{\Delta z_1(\gamma',\beta)}{Q(\gamma',\beta)} \,\mathrm{d}\beta \right|_* < c_R. \tag{3.3.4}$$

To deal with the *out* part in (3.3.2), we use the definition (3.1.4) to expand

$$\Delta z_1(\gamma',\beta) = \Delta d_1(\gamma',\beta) + \Delta p_1(\gamma',\beta).$$

We obtain the next decomposition

$$\int_{|\beta|>1} \frac{\Delta z_1(\gamma',\beta)}{Q(\gamma',\beta)} d\beta = \int_{|\beta|>1} \frac{\Delta d_1(\gamma',\beta)}{Q(\gamma',\beta)} \,\mathrm{d}\beta + \int_{|\beta|>1} \frac{\Delta p_1(\gamma',\beta)}{Q(\gamma',\beta)} \,\mathrm{d}\beta. \tag{3.3.5}$$

In the first integral of (3.3.5), we use the inequality

$$\|\Delta d_1\|_{L^\infty(\partial S_{r'})} \leq 2\|d_1\|_{L^\infty(\partial S_{r'})} < c_R. \tag{A3}$$

We make use of the arc-chord condition (A2) and (A3). Thus

$$\left| \int_{|\beta|>1} \frac{\Delta d_1(\gamma',\beta)}{\beta^2} \frac{\beta^2}{Q(\gamma',\beta)} \,\mathrm{d}\beta \right|_* \leq 2\|d_1\|_{L^\infty(S_r)} R \int_{|\beta|>1} |\beta|^{-2} \,\mathrm{d}\beta < c_R. \tag{3.3.6}$$

For the second integral in (3.3.5), by the definition (3.1.4) we have that $\Delta p_1(\gamma',\beta) = \beta$. Therefore, by adding and subtracting $\beta^2/Q^{\mathbf{p}}(\gamma',\beta)$, we find the next decomposition

$$\int_{|\beta|>1} \frac{1}{\beta} \frac{\beta^2}{Q(\gamma',\beta)} \,\mathrm{d}\beta = \int_{|\beta|>1} \left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{\beta^2}{Q^{\mathbf{p}}(\gamma',\beta)} \right) \frac{\mathrm{d}\beta}{\beta} + \int_{|\beta|>1} \frac{\beta^2}{Q^{\mathbf{p}}(\gamma',\beta)} \frac{\mathrm{d}\beta}{\beta}, \tag{3.3.7}$$

where

$$Q^{\mathbf{p}}(\gamma',\beta) := |\mathbf{p}(\gamma') - \mathbf{p}(\gamma'-\beta)|^2. \tag{3.3.8}$$

The Lemma 33 states that

$$\left| \frac{\beta^2}{Q(\gamma',\beta)} - \frac{\beta^2}{Q^{\mathbf{p}}(\gamma',\beta)} \right|_* \leq c_R|\beta|^{-1},$$

then we obtain integrability for the first integral in the righ-hand side of (3.3.7). Regarding the last integral of (3.3.7), we use Lemma 34, therefore we conclude

$$\left| \int_{|\beta|>1} \frac{1}{\beta} \frac{\beta^2}{Q(\gamma',\beta)} \,\mathrm{d}\beta \right|_* \leq c_R.$$

Combining the previous inequality with (3.3.6), we deduce

$$\left| PV \int_{|\beta|>1} \frac{\Delta z_1(\gamma',\beta)}{Q(\gamma',\beta)} \,\mathrm{d}\beta \right|_* \leq c_R.$$

By joining the last bound for the *out* part, and the inequality (3.3.4) for the *in* part, we find that

$$\left\| PV\int_{\mathbb{R}} \frac{\Delta z_1(\gamma',\beta)}{Q(\gamma',\beta)}\,\mathrm{d}\beta \right\|_{L^\infty(\partial S_{r'})} < c_R.$$

The last $L^\infty(\partial S_{r'})$, bound leads to an estimate for I_1, see (3.3.1). By taking the $L^2(\partial S_{r'})$ norm of I_1, we find the following bound

$$\begin{aligned}\|I_1\|_{L^2(\partial S_{r'})} &= \left\|\partial_\alpha d_1(\gamma')\int_{\mathbb{R}} \frac{\Delta z_1(\gamma',\beta)}{Q(\gamma',\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_{r'})}\\ &\le \|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}\left\| PV\int_{\mathbb{R}} \frac{\Delta z_1(\gamma',\beta)}{Q(\gamma',\beta)}\,\mathrm{d}\beta \right\|_{L^\infty(\partial S_{r'})} \le c_R\|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}.\end{aligned}$$

We get

$$\|I_1\|_{L^2(\partial S_{r'})} \le c_R\|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}. \tag{3.3.9}$$

Now we move to the second integral in (3.3.1), denoted as I_2. Similar to the previous term, we require an $L^2(\partial S_{r'})$ bound. By Splitting in the *in* and *out* parts, then adding and subtracting $\partial_\alpha z_1(\gamma')$ and $1/|\partial_\alpha \mathbf{z}(\gamma')|^2$, we obtain the following decomposition

$$\begin{aligned}I_2^{in}(\gamma') = \int_{|\beta|<1} &\frac{\partial_\alpha d_1(\gamma'-\beta)}{\beta}\frac{\Delta z_1(\gamma',\beta)}{\beta}\frac{\beta^2}{Q(\gamma',\beta)}\,\mathrm{d}\beta\\ &= \int_{|\beta|<1} \frac{\partial_\alpha d_1(\gamma'-\beta)}{\beta}\left(\frac{\Delta z_1(\gamma',\beta)}{\beta} - \partial_\alpha z_1(\gamma')\right)\frac{\beta^2}{Q(\gamma',\beta)}\,\mathrm{d}\beta\\ &+ \partial_\alpha z_1(\gamma')\int_{|\beta|<1} \frac{\partial_\alpha d_1(\gamma'-\beta)}{\beta}\left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right)\mathrm{d}\beta\\ &+ \frac{\partial_\alpha z_1(\gamma')}{|\partial_\alpha \mathbf{z}(\gamma')|^2}H_{|\beta|<1}\partial_\alpha d_1(\gamma').\end{aligned} \tag{3.3.10}$$

Now, we observe that the last term is a truncated Hilbert transform of $\partial_\alpha d_1$. We deduce

$$\left\|\frac{\partial_\alpha z_1(\gamma')}{|\partial_\alpha \mathbf{z}(\gamma')|^2}H_{|\beta|<1}\partial_\alpha d_1(\gamma')\right\|_{L^2(\partial S_{r'})} \le c_R\|H_{|\beta|<1}\partial_\alpha d_1\|_{L^2(\partial S_{r'})}.$$

To estimate fhe first integral in (3.3.10) we consider the arc-chord condition (A2) and estimate (A1). Then, we compute the $L^2(\partial S_{r'})$ norm and apply the Minkoski's integral inequality. We obtain that

$$\begin{aligned}\Bigg\|\int_{|\beta|<1} &\frac{\partial_\alpha d_1(\gamma'-\beta)}{\beta}\left(\frac{\Delta z_1(\gamma',\beta)}{\beta} - \partial_\alpha z_1(\gamma')\right)\frac{\beta^2}{Q(\gamma',\beta)}\,\mathrm{d}\beta\Bigg\|_{L^2(\partial S_{r'})}\\ &\le c_R\left\|\int_{|\beta|<1}\partial_\alpha d_1(\gamma'-\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_{r'})}\\ &\le c_R\int_{|\beta|<1}\|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}\,\mathrm{d}\beta\\ &\le c_R\|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}.\end{aligned}$$

For the second integral of (3.3.10), we consider the estimate from Lemma 32. Together with the Minkowski's integral inequality, we deduce

$$\left\|\partial_\alpha z_1(\gamma')\int_{|\beta|<1} \frac{\partial_\alpha d_1(\gamma'-\beta)}{\beta}\left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right)\mathrm{d}\beta\right\|_{L^2(\partial S_{r'})} \le c_R\|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}.$$

Combining the last two inequalities, we deduce

$$\|I_2^{in}\|_{L^2(\partial S_{r'})} \leq c_R \|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}. \tag{3.3.11}$$

Now we treat with the *out* part. We expand

$$\Delta z_1(\gamma', \beta) = \Delta d_1(\gamma', \beta) + \beta$$

and we obtain the following decomposition

$$I_2^{out}(\gamma') = \int_{|\beta|>1} \partial_\alpha d_1(\gamma' - \beta) \frac{\Delta d_1(\gamma', \beta)}{Q(\gamma', \beta)} \, \mathrm{d}\beta + \int_{|\beta|>1} \partial_\alpha d_1(\gamma' - \beta) \frac{\beta}{Q(\gamma', \beta)} \, \mathrm{d}\beta.$$

To estimate the first integral in the last expression, we use the bound (A3) and the arc-chord condition (A2). Then we obtain

$$\left| \int_{|\beta|>1} \frac{\partial_\alpha d_1(\gamma' - \beta)}{\beta^2} \frac{\Delta d_1(\gamma', \beta)\beta^2}{Q(\gamma', \beta)} \, \mathrm{d}\beta \right|_* \leq c_R \int_{|\beta|>1} \frac{|\partial_\alpha d_1(\gamma' - \beta)|_*}{|\beta|^2} \, \mathrm{d}\beta.$$

Thus, by taking the $L^2(\partial S_{r'})$ norm and making use of the Minkowski's integral inequality, we derive

$$\begin{aligned} \left(\int_{\mathbb{R}} \left| \int_{|\beta|>1} \partial_\alpha d_1(\gamma' - \beta) \frac{\Delta d_1(\gamma', \beta)}{Q(\gamma', \beta)} \, \mathrm{d}\beta \right|_*^2 \mathrm{d}\alpha \right)^{1/2} &\leq \int_{|\beta|>1} |\beta|^{-2} \left(\int_{\mathbb{R}} |\partial_\alpha d_1(\gamma' - \beta)|_*^2 \, \mathrm{d}\alpha \right)^{1/2} \mathrm{d}\beta \\ &\leq c_R \|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}. \end{aligned} \tag{3.3.12}$$

Regarding the second term of I_2^{out}, we add and subtract $\beta^2 / Q^{\mathrm{P}}(\gamma', \beta)$, the term $Q^{\mathrm{P}}(\gamma', \beta)$ is defined in equation (3.3.8). We have the following decomposition

$$\begin{aligned} \int_{|\beta|>1} \frac{\partial_\alpha d_1(\gamma' - \beta)}{\beta} \frac{\beta^2}{Q(\gamma', \beta)} \, \mathrm{d}\beta = &\int_{|\beta|>1} \frac{\partial_\alpha d_1(\gamma' - \beta)}{\beta} \left(\frac{\beta^2}{Q(\gamma', \beta)} - \frac{\beta^2}{Q^{\mathrm{P}}(\gamma', \beta)} \right) \mathrm{d}\beta \\ &+ \int_{|\beta|>1} \frac{\partial_\alpha d_1(\gamma' - \beta)}{\beta} \frac{\beta^2}{Q^{\mathrm{P}}(\gamma', \beta)} \, \mathrm{d}\beta. \end{aligned} \tag{3.3.13}$$

For the first integral in the decomposition (3.3.13), we take the $L^2(\partial S_{r'})$ norm. We use the estimate from Lemma 33, and then we apply the Minkowski's integral inequality. We deduce

$$\begin{aligned} &\left(\int_{\mathbb{R}} \left| \int_{|\beta|>1} \frac{\partial_\alpha d_1(\gamma' - \beta)}{\beta} \left(\frac{\beta^2}{Q(\gamma', \beta)} - \frac{\beta^2}{Q^{\mathrm{P}}(\gamma', \beta)} \right) \mathrm{d}\beta \right|_*^2 \mathrm{d}\alpha \right)^{1/2} \\ &\qquad \leq c_R \left(\int_{\mathbb{R}} \left(\int_{|\beta|>1} |\partial_\alpha d_1(\gamma' - \beta)|_* |\beta|^{-2} \, \mathrm{d}\beta \right)^2 \mathrm{d}\alpha \right)^{1/2} \\ &\qquad \leq c_R \, \|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}. \end{aligned} \tag{3.3.14}$$

The second integral of (3.3.13) is bounded by applying the Cauchy-Schwarz inequality with respect to β and the integrability of $Q^{\mathrm{P}}(\gamma', \beta)$. This yields

$$\begin{aligned} &\left(\int_{\mathbb{R}} \left| \int_{|\beta|>1} \frac{\partial_\alpha d_1(\gamma' - \beta)}{\beta} \frac{\beta^2}{Q^{\mathrm{P}}(\gamma', \beta)} \, \mathrm{d}\beta \right|_*^2 \mathrm{d}\alpha \right)^{1/2} \\ &\qquad \leq \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\partial_\alpha d_1(\gamma' - \beta)|_*^2 \, \mathrm{d}\beta \right] \left[\int_{|\beta|>1} \frac{1}{\beta^2} \frac{\beta^4}{Q^{\mathrm{P}}(\gamma', \beta)^2} \, \mathrm{d}\beta \right] \mathrm{d}\alpha \right)^{1/2} \\ &\qquad \leq \left(\int_{\mathbb{R}} |\partial_\alpha d_1(\gamma')|_*^2 \, \mathrm{d}\alpha \right)^{1/2} \left(\int_{|\beta|>1} |\beta|^{-2} \int_{\mathbb{R}} \frac{\beta^4}{Q^{\mathrm{P}}(\gamma', \beta)^2} \, \mathrm{d}\alpha \, \mathrm{d}\beta \right)^{1/2} \\ &\qquad \leq c_R \, \|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}. \end{aligned} \tag{3.3.15}$$

Notice that, in the third line of (3.3.15) we use the Lemma 38, which estates

$$\left| \int_{\mathbb{R}} \frac{\beta^4}{Q^{\mathrm{P}}(\gamma',\beta)^2}\,\mathrm{d}\alpha \right|_* \leq c_R.$$

Therefore, from (3.3.12), (3.3.14) and (3.3.15), we derive the following estimate

$$\|I_2^{out}\|_{L^2(\partial S_{r'})} \leq c_R \|\partial_\alpha d_1\|_{L^2(\partial S_{r'})}.$$

The last inequality together with the estimates for I_2^{in}, bound (3.3.11), and the estimate for I_1, bound (3.3.9), allow us to deduce an $L^2(\partial S_{r'})$ bound for the first coordinate of the operator which completes the proof. ■

3.3.2 Boundedness of F_2

Lemma 8. *Given a deviation* $\mathbf{d} \in O_R$, *the following estimate holds*

$$\|F_2(\mathbf{d})\|_{L^2(\partial S_{r'})} \leq c_R \Big[\|\partial_\alpha d_2\|_{L^2(\partial S_{r'})} + \|\mathbf{d}\|_{L^2(\partial S_{r'})} \Big]. \tag{B2}$$

Proof. By definition (3.1.4), we expand

$$\partial_\alpha \Delta z_2(\gamma',\beta) = \partial_\alpha \Delta d_2(\gamma',\beta) + 2\beta.$$

and we obtain the following decomposition

$$F_2(\mathbf{d})(\gamma') = PV \int_{\mathbb{R}} \frac{\Delta z_1(\gamma',\beta)}{Q(\gamma',\beta)} \partial_\alpha \Delta d_2(\gamma',\beta)\,\mathrm{d}\beta + PV \int_{\mathbb{R}} \frac{\Delta z_1(\gamma',\beta) 2\beta}{Q(\gamma',\beta)}\,\mathrm{d}\beta - 2\pi. \tag{3.3.16}$$

To estimate the first integral in (3.3.16), we notice that has the same structure $F_1(\mathbf{d})$, see the definition (3.2.4). We deduce the following $L^2(\partial S_{r'})$ bound

$$\left\| PV \int_{\mathbb{R}} \frac{\Delta z_1(\gamma',\beta)}{Q(\gamma',\beta)} \partial_\alpha \Delta d_2(\gamma',\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_{r'})} \leq c_R \|\partial_\alpha d_2\|_{L^2(\partial S_{r'})}. \tag{3.3.17}$$

To deal with the remaining terms in (3.3.16), we expand

$$\Delta z_1(\gamma',\beta) = \Delta d_1(\gamma',\beta) + \beta.$$

We obtain the following decomposition

$$PV \int_{\mathbb{R}} \frac{\Delta z_1(\gamma',\beta) 2\beta}{Q(\gamma',\beta)}\,\mathrm{d}\beta - 2\pi = \int_{\mathbb{R}} \frac{\Delta d_1(\gamma',\beta) 2\beta}{Q(\gamma',\beta)}\,\mathrm{d}\beta + \int_{\mathbb{R}} \frac{2\beta^2}{Q(\gamma',\beta)}\,\mathrm{d}\beta - 2\pi. \tag{3.3.18}$$

We notice, that the first integral in (3.3.18) is similar to $F_1(\mathbf{d})$, the first coordinate. Instead of having $\Delta z_1(\gamma',\beta)$ and $\partial_\alpha \Delta d_1(\gamma',\beta)$, we have $\Delta d_1(\gamma',\beta)$ and 2β. Thus, the $L^2(\partial S_{r'})$ is obtained in similar way and we deduce the following

$$\left\| PV \int_{\mathbb{R}} \frac{\Delta d_1(\gamma',\beta) 2\beta}{Q(\gamma',\beta)}\,\mathrm{d}\beta \right\|_{L^2(\partial S_{r'})} \leq c_R \|d_1\|_{L^2(\partial S_{r'})}. \tag{3.3.19}$$

For the final terms in (3.3.18), we notice that

$$\int_{\mathbb{R}} \frac{2\beta^2}{Q^{\mathrm{P}}(\gamma',\beta)}\,\mathrm{d}\beta = 2\pi.$$

By using the previous identity, we rewrite the difference in the following way

$$\begin{aligned}\int_{\mathbb{R}} \frac{2\beta^2}{Q(\gamma',\beta)}\,\mathrm{d}\beta - 2\pi = 2\int_{\mathbb{R}}\left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{\beta^2}{Q^{\mathrm{P}}(\gamma',\beta)}\right)\mathrm{d}\beta \\ = \int_{\mathbb{R}} \Delta d_1(\gamma',\beta)\mathbf{K}_1(\gamma',\beta)\,\mathrm{d}\beta + \int_{\mathbb{R}} \Delta d_2(\gamma',\beta)\mathbf{K}_2(\gamma',\beta)\,\mathrm{d}\beta \\ := I_{5,1}(\gamma') + I_{5,2}(\gamma'),\end{aligned}$$

where the kernels are given by

$$\begin{aligned}\mathbf{K}_1(\gamma',\beta) &= -2\beta^2\frac{(2\beta+\Delta d_1(\gamma',\beta))}{Q(\gamma',\beta)Q^{\mathrm{P}}(\gamma',\beta)},\\ \mathbf{K}_2(\gamma',\beta) &= -2\beta^2\frac{(2\beta(2\gamma'-\beta)+\Delta d_2(\gamma',\beta))}{Q(\gamma',\beta)Q^{\mathrm{P}}(\gamma',\beta)}.\end{aligned} \tag{3.3.20}$$

By expanding the numerator in $\mathbf{K}_1(\gamma',\beta)$, we rewrite $I_{5,1}$, as follows

$$\begin{aligned}I_{5,1}(\gamma') = \int_{\mathbb{R}} \Delta d_1(\gamma',\beta)\Big[-\frac{2\beta^3}{Q(\gamma',\beta)Q^{\mathrm{P}}(\gamma',\beta)}\Big]\,\mathrm{d}\beta \\ + \int_{\mathbb{R}} \Delta d_1(\gamma',\beta)\Big[-\frac{2\beta^2\Delta d_1(\gamma',\beta)}{Q(\gamma',\beta)Q^{\mathrm{P}}(\gamma',\beta)}\Big]\,\mathrm{d}\beta := I_{5,1,1}(\gamma') + I_{5,1,2}(\gamma').\end{aligned} \tag{3.3.21}$$

To estimate $I_{5,1,1}$ we expand

$$\Delta d_1(\gamma',\beta) = d_1(\gamma') - d_1(\gamma'-\beta).$$

We find the next decomposition

$$-\frac{1}{2}I_{5,1,1}(\gamma') := d_1(\gamma')\int_{\mathbb{R}} \frac{\beta^3}{Q(\gamma',\beta)Q^{\mathrm{P}}(\gamma',\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}} d_1(\gamma'-\beta)\frac{\beta^3}{Q(\gamma',\beta)Q^{\mathrm{P}}(\gamma',\beta)}\,\mathrm{d}\beta. \tag{3.3.22}$$

For the first integral in (3.3.22), to obtain an $L^2(\partial S_{r'})$ bound, we require an estimate in the $L^\infty(\partial S_{r'})$ norm of the following integral

$$PV\int_{\mathbb{R}} \frac{\beta^3}{Q(\gamma',\beta)Q^{\mathrm{P}}(\gamma',\beta)}\,\mathrm{d}\beta.$$

We follow the technique used for (3.3.2). We decompose in the *in* and *out* parts. For the *in* part, first we multiply and divide by β, then we add and subtract $1/|\partial_\alpha \mathbf{z}(\gamma)|^2$. We obtain

$$\begin{aligned}\int_{|\beta|<1} \frac{1}{\beta}\frac{\beta^4}{Q(\gamma',\beta)Q^{\mathrm{P}}(\gamma',\beta)}\,\mathrm{d}\beta = \int_{|\beta|<1}\frac{1}{\beta}\left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right)\frac{\beta^2}{Q^{\mathrm{P}}(\gamma',\beta)}\,\mathrm{d}\beta \\ + \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\int_{|\beta|<1}\frac{1}{\beta}\frac{\beta^2}{Q^{\mathrm{P}}(\gamma',\beta)}\,\mathrm{d}\beta.\end{aligned} \tag{3.3.23}$$

Considering the bound from Lemma 32, we have

$$\left|\frac{\beta^2}{Q(\gamma',\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right|_* \leq c_R|\beta|.$$

Additionally, from the definition (3.1.4), it follows that

$$\left|\frac{\beta^2}{Q^{\mathrm{P}}(\gamma',\beta)}\right|_* \leq c_R.$$

Using the last two bounds, we control the first integral in the right-hand side of (3.3.23). Regarding the last integral in (3.3.23), we use the Lemma 34 and the arc-chord condition (A2) to obtain an $L^\infty(\partial S_{r'})$ bound. Thus, we deduce

$$\left| \int_{|\beta|<1} \frac{1}{\beta} \frac{\beta^4}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta \right|_* \leq c_R. \tag{3.3.24}$$

While for the *out* part, by adding and subtracting $\beta^2/Q^{\mathbf{P}}(\gamma',\beta)$, we obtain the following

$$\int_{|\beta|>1} \frac{1}{\beta} \frac{\beta^4}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta = \int_{|\beta|>1} \frac{1}{\beta} \left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{\beta^2}{Q^{\mathbf{P}}(\gamma',\beta)} \right) \frac{\beta^2}{Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta$$
$$+ \int_{|\beta|>1} \frac{1}{\beta} \frac{\beta^4}{Q^{\mathbf{P}}(\gamma',\beta)^2} \,\mathrm{d}\beta.$$

The first integral in the previous equation can be bounded by Lemma 33. Additionally, Lemma 36 states that the last integral is bounded. We deduce

$$\left| \int_{|\beta|>1} \frac{1}{\beta} \frac{\beta^4}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta \right|_* \leq c_R. \tag{3.3.25}$$

Joining the inequalities (3.3.24) and (3.3.25), and taking the $L^2(\partial S_{r'})$ norm, we get

$$\left\| d_1(\gamma) \int_{\mathbb{R}} \frac{\beta^3}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta \right\|_{L^2(\partial S_{r'})} \leq c_R \|d_1\|_{L^2(\partial S_{r'})}. \tag{3.3.26}$$

The bound for the second integral in (3.3.22) is obtained in a similar way to I_2, see equation (3.3.1). For the *in* part by using a similar decomposition as in (3.3.10). We have

$$\int_{|\beta|<1} \frac{d_1(\gamma'-\beta)}{\beta} \frac{\beta^4}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta$$
$$= \int_{|\beta|<1} \frac{d_1(\gamma'-\beta)}{\beta} \left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \right) \frac{\beta^2}{Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta$$
$$+ \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \int_{|\beta|<1} \frac{d_1(\gamma'-\beta)}{\beta} \frac{\beta^2}{Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta.$$

While the decomposition for the *out* part, in a similar way to (3.3.13), is given by

$$\int_{|\beta|>1} \frac{d_1(\gamma'-\beta)}{\beta} \frac{\beta^4}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta$$
$$= \int_{|\beta|<1} \frac{d_1(\gamma'-\beta)}{\beta} \left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{\beta^4}{Q^{\mathbf{P}}(\gamma',\beta)} \right) \frac{\beta^2}{Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta$$
$$+ \int_{|\beta|>1} \frac{d_1(\gamma'-\beta)}{\beta} \frac{\beta^4}{Q^{\mathbf{P}}(\gamma',\beta)^2} \,\mathrm{d}\beta.$$

By using the same arguments as in I_2^{in} and I_2^{out}, see (3.3.12) and (3.3.15), we obtain an $L^2(\partial S_{r'})$ bound for the second part of (3.3.22). We have that

$$\left\| \int_{\mathbb{R}} d_1(\gamma'-\beta) \frac{\beta^3}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)} \,\mathrm{d}\beta \right\|_{L^2(\partial S_{r'})} \leq c_R \|d_1\|_{L^2(\partial S_{r'})}. \tag{3.3.27}$$

Thus, the inequalities (3.3.26) and (3.3.27) allow us to conclude the following bound

$$\|I_{5,1,1}\|_{L^2(\partial S_{r'})} \leq c_R \|d_1\|_{L^2(\partial S_{r'})}. \tag{3.3.28}$$

The second integral in (3.3.21), denoted as $I_{1,5,2}$ has the following decomposition

$$-\frac{1}{2}I_{5,1,2}(\gamma') := d_1(\gamma')\int_{\mathbb{R}} \frac{\beta^2\Delta d_1(\gamma',\beta)}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}} d_1(\gamma'-\beta)\frac{\beta^2\Delta d_1(\gamma',\beta)}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\,\mathrm{d}\beta.$$

Once again, splitting in the *in* and *out* parts. For the *in* part, we add and subtract $\partial_\alpha d_1(\gamma')\beta$, then we obtain the next decomposition

$$\begin{aligned}\int_{|\beta|<1} \frac{\beta^2\Delta d_1(\gamma',\beta)}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\,\mathrm{d}\beta = &\int_{|\beta|<1}\left(\frac{\Delta d_1(\gamma',\beta)}{\beta} - \partial_\alpha d_1(\gamma')\right)\left[\frac{\beta^3}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\right]\mathrm{d}\beta\\ &- \partial_\alpha d_1(\gamma')\int_{|\beta|<1}\frac{\beta^3}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\,\mathrm{d}\beta.\end{aligned}$$

We can observe that, the last integral has the same kernel as that found in $I_{5,1,1}$. Thus, the bound follows from inequalities (3.3.24) and (3.3.27). We have

$$\left\| d_1(\gamma')\int_{|\beta|<1} \frac{\beta^2\Delta d_1(\gamma',\beta)}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_{r'})} \leq c_R\|d_1\|_{L^2(\partial S_{r'})}. \tag{3.3.29}$$

In a similar way, we infer

$$\left\| \int_{|\beta|<1} d_1(\gamma'-\beta)\left[\frac{\beta^2\Delta d_1(\gamma',\beta)}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\right]\mathrm{d}\beta\right\|_{L^2(\partial S_{r'})} \leq c_R\|d_1\|_{L^2(\partial S_{r'})}. \tag{3.3.30}$$

To estimate the *out* part of $I_{5,2,2}$, see equation (3.3.21), from the inequality (A3) and the arc-chord condition (A2), we deduce the next inequality

$$\left|\frac{\beta^2\Delta d_1(\gamma',\beta)}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\right|_* \leq c_R|\beta|^{-2}.$$

Thus, applying the Cauchy-Schwarz together with the Minkowski's integral inequalities, we infer

$$\left\| \int_{|\beta|>1} d_1(\gamma'-\beta)\left[\frac{\beta^2\Delta d_1(\gamma',\beta)}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\right]\mathrm{d}\beta\right\|_{L^2(\partial S_{r'})} \leq c_R\|d_1\|_{L^2(\partial S_{r'})}. \tag{3.3.31}$$

Therefore, from (3.3.29), (3.3.30) and (3.3.31), we get

$$\|I_{5,1,2}\|_{L^2(\partial S_{r'})} \leq c_R\|d_1\|_{L^2(\partial S_{r'})} \tag{3.3.32}$$

and this complete the estimate of $I_{5,1}$. Now we move on to the estimate $I_{5,2}$. We have a similar situation. By expanding $\mathbf{K}_2(\gamma',\beta)$, defined in (3.3.20), we have the next decomposition

$$\begin{aligned}I_{5,2}(\gamma') = &\int_{\mathbb{R}} \Delta d_2(\gamma',\beta)\left[-\frac{2\beta^2\Delta d_2(\gamma',\beta)}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\right]\mathrm{d}\beta\\ &+ \int_{\mathbb{R}} \Delta d_2(\gamma',\beta)\left[-\frac{4\beta^3(2\gamma'-\beta)}{Q(\gamma',\beta)Q^{\mathbf{P}}(\gamma',\beta)}\right]\mathrm{d}\beta := I_{5,2,1}(\gamma') + I_{5,2,2}(\gamma').\end{aligned}$$

The integral $I_{5,2,1}$ is similar to $I_{5,1,2}$, instead of having $\Delta d_1(\gamma',\beta)$ we have $\Delta d_2(\gamma',\beta)$. Thus we deduce the following inequality

$$\|I_{5,2,1}\|_{L^2(\partial S_{r'})} \leq c_R\|d_2\|_{L^2(\partial S_{r'})}. \tag{3.3.33}$$

For the second term $I_{5,2,2}$, we expand

$$\Delta d_2(\gamma',\beta) = d_2(\gamma') - d_2(\gamma'-\beta).$$

We have that

$$-\frac{1}{4}I_{5,2,2}(\gamma') = d_2(\gamma')\int_{\mathbb{R}} \frac{\beta^3(2\gamma'-\beta)}{Q(\gamma',\beta)Q^{\mathbf{p}}(\gamma',\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}} d_2(\gamma'-\beta)\frac{\beta^3(2\gamma'-\beta)}{Q(\gamma',\beta)Q^{\mathbf{p}}(\gamma',\beta)}\,\mathrm{d}\beta.$$

In order to obtain the $L^2(\partial S_{r'})$, we require an $L^\infty(\partial S_{r'})$ bound for the following integral

$$PV\int_{\mathbb{R}} \frac{\beta^3(2\gamma'-\beta)}{Q(\gamma',\beta)Q^{\mathbf{p}}(\gamma',\beta)}\,\mathrm{d}\beta.$$

We proceed as in the integral (3.3.2). For the *in* part, first we multiply and divide by β, then add and subtract $1/|\partial_\alpha \mathbf{z}(\gamma')|^2$, we have the following decomposition

$$\begin{aligned}\int_{|\beta|<1} \frac{\beta^2}{Q(\gamma',\beta)}\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)}\frac{\mathrm{d}\beta}{\beta} &= \int_{|\beta|<1}\left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right)\frac{\beta^2(2\gamma'-\beta)}{Q(\gamma',\beta)}\frac{\mathrm{d}\beta}{\beta}\\ &+ \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\int_{|\beta|<1}\frac{1}{\beta}\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)}\,\mathrm{d}\beta.\end{aligned}$$

For the right-hand side above, by using inequalities (4.2.1) and (A9) in the first integral and Lemma 35 in the second integral, we infer

$$\left|\int_{|\beta|<1} \frac{\beta^2}{Q(\gamma',\beta)}\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)}\frac{\mathrm{d}\beta}{\beta}\right|_* \leq c_R. \tag{3.3.34}$$

Regarding the *out* part we have the next decomposition

$$\begin{aligned}\int_{|\beta|>1} \frac{\beta^2}{Q(\gamma',\beta)}\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)}\frac{\mathrm{d}\beta}{\beta} &= \int_{|\beta|>1}\left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{\beta^2}{Q^{\mathbf{p}}(\gamma',\beta)}\right)\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)}\frac{\mathrm{d}\beta}{\beta}\\ &+ \int_{|\beta|>1}\frac{1}{\beta}\frac{\beta^4(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

By using Lemma 33 and Lemma 37 and (A9), we infer

$$\left|\int_{|\beta|>1} \frac{\beta^2}{Q(\gamma',\beta)}\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)}\frac{\mathrm{d}\beta}{\beta}\right|_* \leq c_R. \tag{3.3.35}$$

Thus, taking the $L^2(\partial S_{r'})$ norm and considering, the bounds (3.3.34) and (3.3.35), we deduce

$$\left\| d_2(\gamma')\int_{\mathbb{R}} \frac{\beta^4(2\gamma'-\beta)}{Q(\gamma',\beta)Q^{\mathbf{p}}(\gamma',\beta)}\frac{\mathrm{d}\beta}{\beta}\right\|_{L^2(\partial S_{r'})} \leq c_R\|d_2\|_{L^2(\partial S_{r'})}. \tag{3.3.36}$$

Now, we move to the second integral in $I_{5,2,1}$. We follow the same technique as in (3.3.10). For the *in* part we have the next decomposition

$$\begin{aligned}&\int_{|\beta|<1} d_2(\gamma'-\beta)\frac{\beta^3(2\gamma'-\beta)}{Q(\gamma',\beta)Q^{\mathbf{p}}(\gamma',\beta)}\,\mathrm{d}\beta\\ &\qquad= \int_{|\beta|<1}\frac{d_2(\gamma'-\beta)}{\beta}\left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right)\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)}\,\mathrm{d}\beta\\ &\qquad+ \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\int_{|\beta|<1}\frac{d_2(\gamma'-\beta)}{\beta}\left(\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)} - \frac{2\gamma'}{|\partial_\alpha \mathbf{p}(\gamma')|^2}\right)\mathrm{d}\beta\\ &\qquad+ \frac{2\gamma'}{|\partial_\alpha \mathbf{z}(\gamma')|^2|\partial_\alpha \mathbf{p}(\gamma')|^2}H_{|\beta|<1}d_2(\gamma').\end{aligned}$$

We notice from Corollary 2 and estimate (A8) that

$$\left|\frac{2\gamma'}{|\partial_\alpha \mathbf{z}(\gamma')|^2 |\partial_\alpha \mathbf{p}(\gamma')|^2}\right|_* \leq c_R \quad \text{and} \quad \left|\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)} - \frac{2\gamma'}{|\partial_\alpha \mathbf{p}(\gamma')|^2}\right|_* \leq c_R|\beta|.$$

Hence, using the Lemma 32 and applying the Minkowski's integral inequality, we deduce the following

$$\left\| \int_{|\beta|<1} d_2(\gamma'-\beta) \frac{\beta^3(2\gamma'-\beta)}{Q(\gamma',\beta)Q^{\mathbf{p}}(\gamma',\beta)} \,\mathrm{d}\beta \right\|_{L^2(\partial S_{r'})} \leq c_R \|d_2\|_{L^2(\partial S_{r'})}.$$

The bound for the *out* part can be obtained in a similar way to (3.3.13). We have the following decomposition

$$\begin{aligned} \int_{|\beta|>1} d_2(\gamma'-\beta) &\frac{\beta^3(2\gamma'-\beta)}{Q(\gamma',\beta)Q^{\mathbf{p}}(\gamma',\beta)} \,\mathrm{d}\beta \\ &= \int_{|\beta|<1} \frac{d_2(\gamma'-\beta)}{\beta} \left(\frac{\beta^2}{Q(\gamma',\beta)} - \frac{\beta^2}{Q^{\mathbf{p}}(\gamma',\beta)} \right) \frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)} \,\mathrm{d}\beta \\ &+ \int_{|\beta|<1} \frac{d_2(\gamma'-\beta)}{\beta} \frac{\beta^4(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)^2} \,\mathrm{d}\beta. \end{aligned} \tag{3.3.37}$$

For the first integral in the right-hand side of (3.3.37) we use the Lemma 33, and regarding the last integral, we follow the same technique as in (3.3.15). We get

$$\left\| \int_{|\beta|>1} d_2(\gamma'-\beta) \frac{\beta^3(2\gamma'-\beta)}{Q(\gamma',\beta)Q^{\mathbf{p}}(\gamma',\beta)} \,\mathrm{d}\beta \right\|_{L^2(\partial S_{r'})} \leq c_R \|d_2\|_{L^2(\partial S_{r'})}. \tag{3.3.38}$$

From inequalities (3.3.36) and (3.3.38), we infer

$$\|I_{5,2,2}\|_{L^2(\partial S_{r'})} \leq c_R \|d_2\|_{L^2(\partial S_{r'})}. \tag{3.3.39}$$

Combining the inequalities (3.3.17), (3.3.19), (3.3.28), (3.3.32), (3.3.33) and (3.3.39) finishes the proof. ■

The conclusion of this section follows from the inequalities (B1) and (B2) together with the property (3.1.2). We infer

$$\|\mathbf{F}(\mathbf{d})\|_{L^2(\partial S_{r'})} \leq \frac{c_R}{r-r'} \|\mathbf{d}\|_{L^2(\partial S_r)}. \tag{B3}$$

3.4 Boundedness of $\partial_\alpha^3 \mathbf{F}$

The next step in the proof concerns to the high order derivative. We will prove the following $L^2(\partial S_{r'})$ estimate

$$\|\partial_\alpha^3 \mathbf{F}(\mathbf{d})\|_{L^2(\partial S_{r'})} \leq \frac{c_R}{r-r'} \|\mathbf{d}\|_{X_{r,3}}.$$

A preliminary step is to use the property of the Banach scale, given by

$$\|\partial_\alpha^3 \mathbf{F}(\mathbf{d})\|_{L^2(\partial S_{r'})} \leq \frac{c_R}{r-r'} \|\partial_\alpha^2 \mathbf{F}(\mathbf{d})\|_{L^2(\partial S_r)}. \tag{3.4.1}$$

Thus, the proof of the boundedness will be complete if we prove that

$$\|\partial_\alpha^2 \mathbf{F}(\mathbf{d})\|_{L^2(\partial S_r)} \leq c_R \|\mathbf{d}\|_{X_{r,3}}.$$

In a similar way to the lower order order term, we begin with the first coordinate of the operator.

3.4.1 Boundedness of $\partial_\alpha^2 F_1$

As in the previous section, we obtain this inequality as a consequence of several lemmas. The first lemma estates the following

Lemma 9. *Given a deviation* $\mathbf{d} \in O_R$*, the following estimate holds*

$$\|\partial_\alpha^2 F_1(\mathbf{d})\|_{L^2(\partial S_r)} \leq c_R\Big[\|\mathbf{d}\|_{L^2(\partial S_r)} + \|\partial_\alpha \mathbf{d}\|_{L^2(\partial S_r)} + \|\partial_\alpha^2 \mathbf{d}\|_{L^2(\partial S_r)} + \|\partial_\alpha^3 d_1\|_{L^2(\partial S_r)}\Big]. \tag{B4}$$

Proof. We denote $\gamma = \alpha \pm ir$. We compute the second order derivative, and decompose in the following way

$$\partial_\alpha^2 F_1(\mathbf{d})(\gamma) = (\mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3 + \mathrm{J}_4 + \mathrm{J}_5 + \mathrm{J}_6)(\gamma)$$

for

$$\begin{aligned}
\mathrm{J}_1(\gamma) &:= \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)} \partial_\alpha^3 \Delta d_1(\gamma,\beta)\,\mathrm{d}\beta,\\
\mathrm{J}_2(\gamma) &:= 3\int_{\mathbb{R}} \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)} \partial_\alpha^2 \Delta d_1(\gamma,\beta)\,\mathrm{d}\beta,\\
\mathrm{J}_3(\gamma) &:= -2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha^2 \Delta d_1(\gamma,\beta)\partial_\alpha Q(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\
\mathrm{J}_4(\gamma) &:= -2\int_{\mathbb{R}} \frac{\partial_\alpha \Delta d_1(\gamma,\beta)^2 \partial_\alpha Q(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\
\mathrm{J}_5(\gamma) &:= -\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)\partial_\alpha^2 Q(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\
\mathrm{J}_6(\gamma) &:= 2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)\partial_\alpha Q(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta.
\end{aligned} \tag{3.4.2}$$

We will estimate term by term, as each term has a different kernel.

Bound for J_1: For the first integral J_1, by expanding

$$\partial_\alpha^3 \Delta d_1(\gamma,\beta) = \partial_\alpha^3 d_1(\gamma) - \partial_\alpha^3 d_1(\gamma-\beta),$$

we get

$$\mathrm{J}_1 = \partial_\alpha^3 d_1(\gamma)\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha^3 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta. \tag{3.4.3}$$

Notice, in both integrals we have the same kernel that we found in the equation (3.3.1). Hence, we argue in a similar way to I_1 and I_2 in Lemma 7. We deduce the next $L^2(\partial S_r)$ estimate

$$\|\mathrm{J}_1\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha^3 d_1\|_{L^2(\partial S_r)}. \tag{B5}$$

Bound for J_2: For the second term J_2, by expanding the second order derivative

$$\partial_\alpha^2 \Delta d_1(\gamma,\beta) = \partial_\alpha^2 d_1(\gamma) - \partial_\alpha^2 d_1(\gamma-\beta)$$

we have

$$\frac{1}{3}\mathrm{J}_2(\gamma) = \partial_\alpha^2 d_1(\gamma)\int_{\mathbb{R}} \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta. \tag{3.4.4}$$

In order to obtain an $L^2(\partial S_r)$ bound for J_2, we can observe that we need a bound for the following integral

$$PV\int_{\mathbb{R}} \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta. \tag{3.4.5}$$

The strategy is similar to the one used for (3.3.2) in I_1. For the *in* part, we multiply and divide by β^2, then adding and subtracting $\partial_\alpha^2 d_1(\gamma)$ and $1/|\partial_\alpha \mathbf{z}(\gamma')|^2$. We obtain

$$\int_{|\beta|<1} \frac{\partial_\alpha^2 \Delta d_1(\gamma,\beta)}{\beta^2}\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta = \int_{|\beta|<1}\frac{1}{\beta}\left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right)\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta$$
$$+ \partial_\alpha^2 d_1(\gamma)\int_{|\beta|<1}\frac{1}{\beta}\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma)|^2}\right)\mathrm{d}\beta.$$

The Fundamental Theorem of Calculus give us the next formula

$$\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma) = \int_0^1 \left(\partial_\alpha^2 d_1(\gamma+(s-1)\beta) - \partial_\alpha^2 d_1(\gamma)\right)\mathrm{d}s.$$

And we deduce the bound

$$\left|\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right|_* \leq \|d_1\|_{C^{2,1/2}(\partial S_r)}. \tag{A4}$$

We use the inequalities (A4) the arc-chord condition (A2) and by Lemma 32, we can deduce the next bound for the *in* part

$$\left|\int_{|\beta|<1} \frac{\partial_\alpha^2 \Delta d_1(\gamma,\beta)}{\beta^2}\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right|_* \leq c_R. \tag{3.4.6}$$

Regarding the *out* part, we use arc chord condition (A2), and the following bound

$$\|\partial_\alpha \Delta d_1\|_{L^\infty(\partial S_r)} \leq 2\|\partial_\alpha d_1\|_{L^\infty(\partial S_r)} < c_R \tag{A5}$$

and we deduce

$$\left|\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta^2}\frac{\beta^2}{Q(\gamma,\beta)}\right|_* \leq c_R|\beta|^{-2}. \tag{3.4.7}$$

Thus, the next bound follows easily

$$\left|\int_{|\beta|>1} \frac{\partial_\alpha^2 \Delta d_1(\gamma,\beta)}{\beta^2}\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right|_* \leq c_R.$$

We combine, the previous estimate and (3.4.6). Hence, we deduce the $L^\infty(\partial S_r)$ estimate of the integral (3.4.5), given by

$$\left\|PV\int_{\mathbb{R}} \frac{\partial_\alpha^2 \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_r)} \leq c_R.$$

Now, by taking the $L^2(\partial S_r)$ norm of the first integral in (3.4.4), we have

$$\begin{aligned}\left\|\partial_\alpha^2 d_1(\gamma)\int_{\mathbb{R}} \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} &\leq \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}\left\|\int_{\mathbb{R}} \frac{\partial_\alpha^2 \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_r)} \\ &\leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.\end{aligned} \tag{3.4.8}$$

The second integral in (3.4.4), is bounded in similar way to I_2 in Lemma 7, see the decomposition (3.3.10). For the *in* part, we consider the inequality (A4). Thus

$$\left\|\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.9}$$

Regarding the *out* part, we use the inequality (3.4.7) in Lemma 7 and apply the Minkowski's integral inequality, then we get

$$\begin{aligned}\left\| \int_{|\beta|>1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} &\le c_R\left\| \int_{|\beta|>1} |\partial_\alpha^2 d_1(\gamma-\beta)|_*^2|\beta|^{-2}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\\ &\le c_R\int_{|\beta|>1}\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}|\beta|^{-2}\,\mathrm{d}\beta\\ &\le c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}\end{aligned} \tag{3.4.10}$$

Using the inequalities (3.4.9) and (3.4.10), we obtain the $L^2(\partial S_r)$ bound for the second term of (3.4.4). Combining with the inequality (3.4.8) we complete the $L^2(\partial S_r)$ bound of $\mathrm{J}_2(\gamma)$, given by

$$\|\mathrm{J}_2\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{B6}$$

Bound for J_3: To estimate this term, we compute the derivative $\partial_\alpha Q(\gamma,\beta)$ which is given by

$$\begin{aligned}\partial_\alpha Q(\gamma,\beta) = 2\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta) + 2\Delta d_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta) + 4\Delta d_2(\alpha,\beta)\beta\\ + 2\beta(2\gamma-\beta)\partial_\alpha\Delta d_2(\gamma,\beta) + 4\beta^2(2\gamma-\beta).\end{aligned} \tag{3.4.11}$$

We expand the difference

$$\partial_\alpha^2\Delta d_1(\gamma,\beta) = \partial_\alpha^2 d_1(\gamma) - \partial_\alpha^2 d_1(\gamma-\beta)$$

and we decompose J_3 as follows

$$-\frac{1}{2}\mathrm{J}_3(\gamma) = \partial_\alpha^2 d_1(\gamma)\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta. \tag{3.4.12}$$

In order to obtain the $L^2(\partial S_r)$ bound for the first integral in (3.4.12), we need an $L^\infty(\partial S_r)$ estimate of the following integral

$$PV\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta = \sum_{i=1}^{5}K_i(\gamma), \tag{3.4.13}$$

where

$$\begin{aligned}K_1(\gamma) &:= 2\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\ K_2(\gamma) &:= 2\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\Delta d_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\ K_3(\gamma) &:= 4\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\Delta d_2(\gamma,\beta)\beta}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\ K_4(\gamma) &:= 2\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)\beta(2\gamma-\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\ K_5(\gamma) &:= 4\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned} \tag{3.4.14}$$

The $L^\infty(\partial S_r)$ estimate for each K_i, will be obtained following the same ideas of Lemma 7. We denote by K_1^{in} the *in* part. We multiply and divide by β^4, then, adding and subtracting $\partial_\alpha z_1(\gamma), \partial_\alpha^2 d_1(\gamma)$ and $1/|\partial_\alpha\mathbf{z}(\gamma)|^2$, we obtain the following decomposition

$$\begin{aligned}\frac{1}{2}K_1^{in}(\gamma) &= \int_{|\beta|<1}\left(\frac{\Delta z_1(\gamma,\beta)}{\beta} - \partial_\alpha z_1(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta}\\ &+ \partial_\alpha z_1(\gamma)\int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta}\\ &+ \partial_\alpha z_1(\gamma)\partial_\alpha^2 d_1(\gamma)\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned} \tag{3.4.15}$$

The last integral decomposes as follows

$$\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta=\int_{|\beta|<1}\left(\frac{\beta^2}{Q(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^2}{Q(\gamma,\beta)}\frac{\mathrm{d}\beta}{\beta}+\frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta. \tag{3.4.16}$$

We notice that the last integral is bounded by (3.3.4) in Lemma 7. Now, we employ the Fundamental Theorem of Calculus to obtain the next inequality

$$\left|\frac{\Delta z_1(\gamma,\beta)}{\beta}\right|_*\leq\int_0^1\left|\partial_\alpha z_1(\gamma+(s-1)\beta)\right|_*\mathrm{d}s\leq c_R. \tag{A6}$$

Using the inequalities (A1), (A6), (A4) and Lemma 32, together with the arc-chord condition (A2), we deduce a bound for the remaining terms. We infer

$$|K_1^{in}(\gamma)|_*\leq c_R. \tag{3.4.17}$$

Then, using the estimate (A6), inequality (A5) and the arc-chord condition (A2), we deduce

$$\left|\frac{\Delta z_1(\alpha,\gamma)^2\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)}\right|_*\leq c_R|\beta|^{-2} \tag{3.4.18}$$

thus

$$\frac{1}{2}|K_1^{out}(\gamma)|_*\leq\int_{|\beta|>1}\left|\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\right|_*\mathrm{d}\beta\leq c_R\int_{|\beta|>1}|\beta|^{-2}\,\mathrm{d}\beta.$$

We combine the previous estimate with the inequality (3.4.17) to obtain that

$$\|K_1\|_{L^\infty(\partial S_r)}\leq c_R \tag{3.4.19}$$

To estimate K_2, we notice that we have kernel similar to the previous term K_1. In this case, we add and subtract $\partial_\alpha d_2(\gamma)$ and $\partial_\alpha^2 d_2(\gamma)$, to obtain the following decomposition

$$\begin{aligned}\frac{1}{2}K_2^{in}(\gamma)=&\int_{|\beta|<1}\left(\frac{\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha d_2(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta}\\&+\partial_\alpha d_2(\gamma)\int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_2(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta}\\&+\partial_\alpha d_2(\gamma)\partial_\alpha^2 d_2(\gamma)\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

The bound for the last integral in the above decomposition can be deduced from (3.4.16) in (3.4.15). Then, by using the inequalities (A1), (A4) and (A2). We obtain

$$|K_2^{in}(\gamma)|_*\leq c_R. \tag{3.4.20}$$

The *out* part, will once again be bounded by using the inequalities (A6), (A2) and the next inequality

$$\|\Delta d_2\|_{L^\infty(\partial S_r)},\|\partial_\alpha\Delta d_2\|_{L^\infty(\partial S_r)}<c_R. \tag{A7}$$

From the previous estimate and (A2) and (A7), we infer the following bound

$$\left|\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\Delta d_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta^3}\frac{\Delta z_1(\gamma,\beta)}{\beta}\right|_*\leq c_R|\beta|^{-3}. \tag{3.4.21}$$

We derive the estimate

$$\frac{1}{2}|K_2^{out}(\gamma)|_* \leq \int_{|\beta|>1}\left|\frac{\Delta z_1(\gamma,\beta)\Delta d_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\right|_* \leq c_R\int_{|\beta|>1}|\beta|^{-3}\,\mathrm{d}\beta.$$

We combine the last estimate with (3.4.20) to obtain

$$\|K_2\|_{L^\infty(\partial S_r)} \leq c_R. \tag{3.4.22}$$

The next term K_3 has a kernel similar to K_2. In K_3^{in}, we add and subtract $\partial_\alpha z_1(\gamma)$, $\partial_\alpha d_2(\gamma)$ and $1/|\partial_\alpha\mathbf{z}(\gamma)|^2$, then we obtain the following decomposition

$$\begin{aligned}\frac{1}{4}K_3^{in}(\gamma) &= \int_{|\beta|<1}\left(\frac{\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha d_2(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta}\\ &\quad+\partial_\alpha d_2(\gamma)\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

Then analogously to K_2^{in}, we use the estimates (A1), (A6) and the arc-chord condition (A2). We deduce

$$|K_3^{in}(\gamma)|_* \leq c_R. \tag{3.4.23}$$

Regarding the *out* part, we use (A6), (A7) and the arc-chord condition (A2), to obtain a bound for the kernel

$$\left|\frac{\Delta z_1(\gamma,\beta)\Delta d_2(\gamma,\beta)}{\beta^3}\frac{\beta^4}{Q(\gamma,\beta)^2}\right|_* \leq c_R|\beta|^{-2}. \tag{3.4.24}$$

Then we deduce the next inequality

$$\frac{1}{4}|K_3^{out}(\gamma)|_* \leq \int_{|\beta|>1}\left|\frac{\Delta z_1(\gamma,\beta)\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\right|_* \leq c_R\int_{|\beta|>1}|\beta|^{-3}\,\mathrm{d}\beta.$$

Together with the inequality (3.4.23) and the above estimate, we obtain that

$$\|K_3\|_{L^\infty(\partial S_r)} \leq c_R. \tag{3.4.25}$$

To estimate K_4, for the *in* part, we add and subtract $\partial_\alpha z_1(\gamma)$,$\partial_\alpha^2 d_2(\gamma)$, $1/|\partial_\alpha\mathbf{z}(\gamma)|^2$ and $2\gamma/|\partial_\alpha\mathbf{z}(\gamma)|2$, hence we have the next decomposition

$$\begin{aligned}\frac{1}{2}K_4^{in}(\gamma) &= \int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_2(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\beta(2\gamma-\beta)}{\beta}\frac{\mathrm{d}\beta}{\beta}\\ &\quad+\partial_\alpha^2 d_2(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2}{Q(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^2}{Q(\gamma,\beta)}\frac{\beta(2\gamma-\beta)}{\beta}\frac{\mathrm{d}\beta}{\beta}\\ &\quad+\frac{\partial_\alpha^2 d_2(\gamma)}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)(2\gamma-\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta,\end{aligned}$$

where, the last integral decomposes

$$\begin{aligned}\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)(2\gamma-\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta &= \int_{|\beta|<1}\left(\frac{\Delta z_1(\gamma,\beta)}{\beta}-\partial_\alpha z_1(\gamma)\right)\frac{\beta^2}{Q(\gamma,\beta)}\frac{\beta(2\gamma-\beta)}{\beta}\frac{\mathrm{d}\beta}{\beta}\\ &\quad+\partial_\alpha z_1(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)}-\frac{2\gamma}{|\partial_\alpha\mathbf{z}(\gamma)^2|}\right)\frac{\mathrm{d}\beta}{\beta}.\end{aligned}$$

We use the inequalities (A1), (A4), (A9) and Corollary 2 to conclude that

$$|K_4^{in}(\gamma)|_* \leq c_R. \tag{3.4.26}$$

For the *out* part, by using the definition (3.1.4), we expand

$$\Delta z_1(\gamma,\beta) = \Delta d_1(\gamma,\beta) + \beta,$$

hence

$$\frac{1}{4}K_4^{out}(\gamma) = \int_{|\beta|>1} \frac{\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)\beta(2\gamma-\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta + \int_{|\beta|>1} \frac{\beta\partial_\alpha\Delta d_2(\gamma,\beta)\beta(2\gamma-\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

Considering the inequalities (A3), (A7) and (A2), we deduce that

$$|K_4^{out}(\gamma)|_* \le c_R \int_{|\beta|>1} |\beta|^{-3}\,\mathrm{d}\beta + c_R\int_{|\beta|>1}|\beta|^{-2}\,\mathrm{d}\beta < c_R.$$

Hence, the previous inequality and estimate (3.4.26) yields to

$$\|K_4\|_{L^\infty(\partial S_r)} \le c_R. \tag{3.4.27}$$

The term K_5, shares a kernel similar to K_4, in this case we have β explicitly, instead of $\partial_\alpha\Delta d_2(\gamma,\beta)$. For the *in* part, we have the following decomposition

$$\begin{aligned}\frac{1}{4}K_5^{in}(\gamma) = &\int_{|\beta|<1}\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^2}{Q(\gamma,\beta)}\frac{(2\gamma-\beta)}{\beta}\,\mathrm{d}\beta\\ &+\frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)(2\gamma-\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.\end{aligned}\tag{3.4.28}$$

Using the inequalities (A6), (A9) and Lemma 32 we deduce

$$|K_5^{in}(\gamma)|_* \le c_R. \tag{3.4.29}$$

Regarding the *out* part we expand $\Delta z_1(\gamma,\beta)$. Thus

$$\frac{1}{4}K_5^{out}(\gamma) = \int_{|\beta|>1}\frac{\Delta d_1(\gamma,\beta)}{\beta}\frac{\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)}\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta + \int_{|\beta|>1}\frac{1}{\beta}\frac{\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)}\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta.$$

Using the inequalities (A3), (A9) and (A2), we derive

$$\left|\int_{|\beta|>1}\frac{\Delta d_1(\gamma,\beta)}{\beta}\frac{\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)}\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right|_* \le c_R. \tag{3.4.30}$$

For the last integral in $K_5^{out}(\gamma)$, adding and subtracting $\beta^2/Q^{\mathbf{P}}(\gamma,\beta)$ we obtain the next decomposition

$$\begin{aligned}\int_{|\beta|>1}\frac{1}{\beta}\frac{\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)}\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta = &\int_{|\beta|>1}\frac{1}{\beta}\frac{\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)}\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{\beta^2}{Q^{\mathbf{P}}(\gamma,\beta)}\right)\mathrm{d}\beta\\ &+\int_{|\beta|>1}\frac{1}{\beta}\frac{\beta^2(2\gamma-\beta)}{Q^{\mathbf{P}}(\gamma,\beta)}\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{\beta^2}{Q^{\mathbf{P}}(\gamma,\beta)}\right)\mathrm{d}\beta\\ &+\int_{|\beta|>1}\frac{1}{\beta}\frac{\beta^4(2\gamma-\beta)}{Q^{\mathbf{P}}(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}\tag{3.4.31}$$

The first terms in the right-hand side of (3.4.31) are bounded by using the estimate from Lemma 33. The last integral is bounded by considering Lemma 37, that is

$$\left|\int_{|\beta|>1}\frac{1}{\beta}\frac{\beta^4(2\gamma-\beta)}{Q^{\mathbf{P}}(\gamma,\beta)^2}\,\mathrm{d}\beta\right|_* < c_R.$$

We deduce

$$|K_5^{out}(\gamma)|_* \leq c_R. \tag{3.4.32}$$

Then, using the bound (3.4.29) and (3.4.32), we obtain

$$\|K_5\|_{L^\infty(\partial S_r)} \leq c_R \tag{3.4.33}$$

By combining the estimates (3.4.19), (3.4.22), (3.4.25), (3.4.33) and (3.4.27), we obtain the following estimate

$$\left\| PV \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^\infty(\partial S_r)} \leq c_R.$$

Now, by taking the $L^2(\partial S_r)$ norm, we get

$$\begin{aligned}
&\left\| \partial_\alpha^2 d_1(\gamma) \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \\
&\qquad\qquad \leq \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)} \left\| \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^\infty(\partial S_r)} \\
&\qquad\qquad \leq c_R \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.
\end{aligned} \tag{3.4.34}$$

The last bound correspond to the first integral in (3.4.12). To deal with the second integral in (3.4.12), by using the expression for $\partial_\alpha Q(\gamma,\beta)$, we have

$$PV \int_{\mathbb{R}} \partial_\alpha^2 d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta = \sum_{i=1}^{5} K_i^*(\gamma) \tag{3.4.35}$$

for

$$\begin{aligned}
K_1^*(\gamma) &:= 2\int_{\mathbb{R}} \partial_\alpha^2 d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta, \\
K_2^*(\gamma) &:= 2\int_{\mathbb{R}} \partial_\alpha^2 d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)\Delta d_2(\gamma,\beta)\partial_\alpha \Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta, \\
K_3^*(\gamma) &:= 4\int_{\mathbb{R}} \partial_\alpha^2 d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)\Delta d_2(\gamma,\beta)\beta}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta, \\
K_4^*(\gamma) &:= 2\int_{\mathbb{R}} \partial_\alpha^2 d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_2(\gamma,\beta)\beta(2\gamma-\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta, \\
K_5^*(\gamma) &:= 4\int_{\mathbb{R}} \partial_\alpha^2 d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.
\end{aligned} \tag{3.4.36}$$

Recall that we need an $L^2(\partial S_r)$ estimate for each K_i^*. For the first term in (3.4.36). We split in the *in* and *out* parts. To estimate the *in* part we use a similar decomposition as in K_1, see the decomposition (3.4.15). Thus

$$\begin{aligned}
\frac{1}{2}K_1^{*,in}(\gamma) &= \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\Delta z_1(\gamma,\beta)}{\beta} - \partial_\alpha z_1(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta} \\
&\quad + \partial_\alpha z_1(\gamma)\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta} \\
&\quad + \partial_\alpha z_1(\gamma)\partial_\alpha^2 d_1(\gamma)\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.
\end{aligned}$$

The last integral decomposes

$$\begin{aligned}\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \\ &= \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\beta^2}{Q(\gamma,\beta)}-\frac{1}{|\partial_\alpha \mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^2}{Q(\gamma,\beta)}\frac{\mathrm{d}\beta}{\beta} \\ &+ \frac{1}{|\partial_\alpha \mathbf{z}(\gamma)|^2}\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.\end{aligned} \tag{3.4.37}$$

Note that the last integral above, can be bounded following the estimates for I_2^{in} in Lemma 7. Hence

$$\left\| \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.$$

For the remaining terms, we combine the last inequality together with the estimates (A6), (A1), (A4), (A2) together with Lemma 32 and making use the Minkowski's integral inequality, we deduce

$$\begin{aligned}\|K_1^{*,in}\|_{L^2(\partial S_r)} &\le c_R \int_{|\beta|<1}\left(\int_{\mathbb{R}} |\partial_\alpha^2 d_1(\gamma-\beta)|_*^2\,\mathrm{d}\alpha\right)^{1/2}\mathrm{d}\beta + c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)} \\ &\le c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.\end{aligned} \tag{3.4.38}$$

Regarding the *out* part, we use the inequality (3.4.18) to infer a similar bound as in K_1^{out}. Then, applying the Minkowski's integral inequality, we get

$$\begin{aligned}\frac{1}{2}\|K_1^{*,out}\|_{L^2(\partial S_r)} &\le \left\| \int_{|\beta|>1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \\ &\le c_R\int_{|\beta|>1} \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}|\beta|^{-2}\,\mathrm{d}\beta \\ &\le c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.\end{aligned} \tag{3.4.39}$$

From the estimates (3.4.38) and (3.4.39), we derive

$$\|K_1^*\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.40}$$

The next terms K_2^* and K_3^* share a kernel similar to K_1^*. Hence we replicate the procedure used for K_1^*. We have

$$\begin{aligned}\frac{1}{2}K_2^{in,*}(\gamma) &= \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha d_2(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\partial_\alpha \Delta d_2(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta} \\ &+ \partial_\alpha d_2(\gamma)\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\partial_\alpha \Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_2(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta} \\ &+ \partial_\alpha d_2(\gamma)\partial_\alpha^2 d_2(\gamma)\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned} \tag{3.4.41}$$

The last integral is bounded as in the previous term $K_1^{in,*}$. Then, by considering the inequalities (A1), (A4), the arc-chord condition (A2) and Lemma 32, we obtain the following $L^2(\partial S_r)$ bound

$$\|K_2^{*,in}\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.42}$$

To estimate the *out* part $K_2^{*,out}$, we use inequality (3.4.21) and applying the Minkowski's integral inequality, we infer the following bound

$$\|K_2^{*,out}\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.43}$$

The inequalities (3.4.42) and (3.4.43) allow us to conclude that

$$\|K_2^*\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.44}$$

In the next term the decomposition is shorter, due to the explicit β in the kernel, following the decomposition for K_3^{in}

$$\frac{1}{4}K_3^{in,*}(\gamma) = \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\Delta d_2(\gamma,\beta)}{\beta} - \partial_\alpha d_2(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta}$$
$$+ \partial_\alpha d_2(\gamma)\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

Then using the Minkowski's integral inequality, we get

$$\|K_3^{*,in}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.45}$$

Once again, we use the Minkowski's integral inequality and estimate (3.4.24) to deduce the following

$$\|K_3^{*,out}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.46}$$

Therefore, using (3.4.46) and (3.4.45), we arrive to

$$\|K_3^*\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.47}$$

To estimate K_4^*, we have the next decomposition

$$\begin{aligned}
&\frac{1}{2}K_4^{in,*}(\gamma)\\
&= \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_2(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\beta(2\gamma-\beta)}{\beta}\frac{\mathrm{d}\beta}{\beta}\\
&+ \partial_\alpha^2 d_2(\gamma)\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^2}{Q(\gamma,\beta)}\frac{\beta(2\gamma-\beta)}{\beta}\frac{\mathrm{d}\beta}{\beta}\\
&+ \frac{\partial_\alpha^2 d_2(\gamma)}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)(2\gamma-\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.
\end{aligned} \tag{3.4.48}$$

Where, the last integral decomposes as follows

$$\begin{aligned}
&\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)(2\gamma-\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\\
&\qquad= \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\Delta z_1(\gamma,\beta)}{\beta} - \partial_\alpha z_1(\gamma)\right)\frac{\beta^2}{Q(\gamma,\beta)}\frac{\beta(2\gamma-\beta)}{\beta}\frac{\mathrm{d}\beta}{\beta}\\
&\qquad+ \partial_\alpha z_1(\gamma)\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)} - \frac{2\gamma}{|\partial_\alpha\mathbf{z}(\gamma)^2|}\right)\frac{\mathrm{d}\beta}{\beta}\\
&\qquad+ \frac{2\gamma\partial_\alpha z_1(\gamma)}{|\partial_\alpha\mathbf{z}(\gamma)|^2}H_{|\beta|<1}\partial_\alpha^2 d_1(\gamma).
\end{aligned}$$

Regarding the last term, we use the inequality (A8) given by

$$\left|\frac{2\gamma}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right|_* \leq c_R.$$

and Corollary 2. Thus by taking the $L^2(\partial S_r)$ norm and making use of the Minkowski's integral inequality, we derive

$$\left\| \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)(2\gamma-\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.$$

The remaining integrals in decomposition (3.4.48) can be bounded in an analogous way to $K_1^{*,in}$, by using the estimates (A1), (A4), (A9), (A2), and Lemma 32. Thus from the Minkowski's integral inequality we get

$$\|K_4^{*,in}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.49}$$

For the *out* part, we expand $\Delta z_1(\gamma,\beta)$. Then we have the following decomposition

$$\begin{aligned}\frac{1}{2}K_4^{*,out}(\gamma) &= \int_{|\beta|>1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)\beta(2\gamma-\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &+\int_{|\beta|>1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\beta\partial_\alpha\Delta d_2(\gamma,\beta)\beta(2\gamma-\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

Once again we consider, the inequalities (A3), (A7) and (A2) and (A9) to obtain that a bound for both kernels

$$\left|\frac{\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta^3}\frac{\beta^4(2\gamma-\beta)}{Q(\gamma,\beta)^2}\right|_* \leq c_R|\beta|^{-3}$$

and

$$\left|\frac{\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta^2}\frac{\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)}\frac{\beta^2}{Q(\gamma,\beta)}\right|_* \leq c_R|\beta|^{-2}.$$

Thus, from the Minkowski's integral inequality we derive

$$\|K_4^{*,out}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.50}$$

By joining both estimates (3.4.49) and (3.4.50) we deduce that

$$\|K_4^{*}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.51}$$

To estimate K_5^*, we follow the estimate for K_4^*, instead of having $\Delta d_2(\gamma,\beta)$, we have β. We decompose

$$\begin{aligned}\frac{1}{4}K_5^{in,*}(\gamma) &= \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\left(\frac{\beta^2}{Q(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\beta^2}{Q(\gamma,\beta)}\frac{(2\gamma-\beta)}{\beta}\,\mathrm{d}\beta\\ &+\frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)(2\gamma-\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.\end{aligned}$$

Thus, we obtain the following $L^2(\partial S_r)$ bound

$$\|K_5^{*,in}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.52}$$

For the *out* part, we expand $\Delta z_1(\gamma,\beta)$, then we have the next decomposition

$$\begin{aligned}\frac{1}{4}K_5^{*,out}(\gamma) &= \int_{|\beta|>1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta d_1(\gamma,\beta)}{\beta}\frac{\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)}\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta\\ &+\int_{|\beta|>1} \frac{\partial_\alpha^2 d_1(\gamma-\beta)}{\beta}\frac{\beta^2(2\gamma-\beta)}{Q(\gamma,\beta)}\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta.\end{aligned}$$

We follow a similar a decomposition as in (3.4.31) and use the estimates (3.3.15), then we get

$$\|K_5^{*,out}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.53}$$

Thus, estimates (3.4.52) and (3.4.53) yields to

$$\|K_5^*\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.54}$$

Then from the inequalities (3.4.40), (3.4.44), (3.4.47), (3.4.51) and (3.4.54), we infer

$$\begin{aligned}\left\| \int_{\mathbb{R}} \partial_\alpha^2 d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} &\leq \sum_{i=1}^{5} \|K_i^*\|_{L^2(\partial S_r)} \\ &\leq c_R \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.\end{aligned} \tag{3.4.55}$$

Hence, we use the previous inequality (3.4.55) and the bound (3.4.34), see the decomposition (3.4.12), to obtain that

$$\|\mathrm{J}_3\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{B7}$$

Bound for J_4: This following term, has a kernel similar to J_3. Expanding $\partial_\alpha \Delta d_1(\gamma,\beta)$, we get

$$\begin{aligned}-\frac{1}{2}\mathrm{J}_4(\gamma) = \partial_\alpha d_1(\gamma) \int_{\mathbb{R}} \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta \\ - \int_{\mathbb{R}} \partial_\alpha d_1(\gamma-\beta) \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned} \tag{3.4.56}$$

We will now estimate the integral

$$PV \int_{\mathbb{R}} \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta$$

in $L^\infty(\partial S_r)$. We use the expasion (3.4.11) for $\partial_\alpha Q(\gamma,\beta)$ as in J_3. Then the $L^\infty(\partial S_r)$ estimate follows from the estimates for K_i. We modify the estimates by changing $\Delta z_1(\gamma,\beta)$ by $\partial_\alpha \Delta d_1(\gamma,\beta)$. We deduce

$$\left\| PV \int_{\mathbb{R}} \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^\infty(\partial S_r)} \leq c_R.$$

Under the same argument of (3.4.55), for the second integral of (3.4.56) we infer

$$\left\| PV \int_{\mathbb{R}} \partial_\alpha d_1(\gamma-\beta) \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

Then, joining the two estimates above, we obtain that

$$\|\mathrm{J}_4\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}. \tag{B8}$$

Bound for J_5: To estimate the next term J_5, we deal with $\partial_\alpha^2 Q(\gamma,\beta)$ which is given by

$$\begin{aligned}\partial_\alpha^2 Q(\gamma,\beta) = 2\partial_\alpha \Delta d_1(\gamma,\beta)^2 + 2\Delta z_1(\gamma,\beta)\partial_\alpha^2 \Delta d_1(\gamma,\beta) + 2\Delta d_2(\gamma,\beta)\partial_\alpha^2 \Delta d_2(\gamma,\beta) \\ + 2\partial_\alpha \Delta d_2(\gamma,\beta)^2 + 8\beta \partial_\alpha \Delta d_2(\gamma,\beta) + 2\beta(2\gamma-\beta)\partial_\alpha^2 \Delta d_2(\gamma,\beta) + 8\beta^2.\end{aligned} \tag{3.4.57}$$

Where we have used the fact that

$$\Delta z_2 = \Delta d_2 + \Delta p_2 \quad \text{where} \quad \Delta p_2 = \beta(2\gamma-\beta), \quad \partial_\alpha \Delta p_2 = 2\beta.$$

By substituting the derivative, we decompose

$$\mathrm{J}_5(\gamma) = (J_{5,1} + J_{5,2} + J_{5,3} + J_{5,4} + J_{5,5} + J_{5,6} + J_{5,7})(\gamma)$$

for

$$
\begin{aligned}
J_{5,1}(\gamma) &:= -2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^3}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\
J_{5,2}(\gamma) &:= -2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha \Delta d_1(\gamma,\beta)\partial_\alpha^2 \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\
J_{5,3}(\gamma) &:= -2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)\partial_\alpha^2 \Delta d_2(\gamma,\beta)\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\
J_{5,4}(\gamma) &:= -2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)\partial_\alpha \Delta d_2(\gamma,\beta)^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\
J_{5,5}(\gamma) &:= -8\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)\beta\partial_\alpha \Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\
J_{5,6}(\gamma) &:= -8\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta,\\
J_{5,7}(\gamma) &:= -2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)\beta(2\gamma-\beta)\partial_\alpha^2 \Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.
\end{aligned} \tag{3.4.58}
$$

Bound for $J_{5,1}$: To estimate this term, we expand

$$\partial_\alpha \Delta d_1(\gamma,\beta) = \partial_\alpha d_1(\gamma) - \partial_\alpha d_1(\gamma-\beta)$$

to obtain the following

$$-\frac{1}{2}J_{5,1}(\gamma) = \partial_\alpha d_1(\gamma)\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta. \tag{3.4.59}$$

In order to obtain the $L^2(\partial S_r)$ estimate, first we will prove that the integral

$$PV\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta$$

is bounded in $L^\infty(\partial S_r)$. For the *in* part, we multiply and divide by β^4. Then by adding and subtracting $\partial_\alpha z_1(\gamma)$, $\partial_\alpha^2 d_1(\gamma)$ and $1/|\partial_\alpha \mathbf{z}(\gamma)|^4$. We have that

$$
\begin{aligned}
&\int_{|\beta|<1} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\
&\quad= \int_{|\beta|<1}\left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{\beta}\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta}\frac{\beta^4}{Q(\gamma,\beta)^2}\frac{\mathrm{d}\beta}{\beta}\\
&\quad+ \partial_\alpha^2 d_1(\gamma)\int_{|\beta|<1} \frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta
\end{aligned} \tag{3.4.60}
$$

and

$$
\begin{aligned}
&\int_{|\beta|<1} \frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\
&\quad= \int_{|\beta|<1}\left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\
&\quad+ \partial_\alpha^2 d_1(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\\
&\quad+ \frac{\partial_\alpha^2 d_1(\gamma)}{|\partial_\alpha \mathbf{z}(\gamma)|^2}\int_{|\beta|<1} \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.
\end{aligned} \tag{3.4.61}
$$

We make use of the estimate (3.3.4) in Lemma 7 for the last integral above. Next, we use the inequalities (A4), (A6), the arc-chord condition (A2), and Lemma 32, to obtain that

$$\left| \int_{|\beta|<1} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2} \mathrm{d}\beta \right|_* \leq c_R.$$

Regarding the *out* part, by using (A6), (A5) and (A2), we infer

$$\left| \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2} \right|_* \leq c_R |\beta|^{-3}. \tag{3.4.62}$$

and hence

$$\left| \int_{|\beta|>1} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2} \,\mathrm{d}\beta \right|_* \leq c_R \int_{|\beta|>1} |\beta|^{-3} \,\mathrm{d}\beta < c_R.$$

Therefore, by taking the $L^2(\partial S_r)$ bound, we get

$$\begin{aligned} &\left\| \partial_\alpha d_1(\gamma) \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2} \,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \\ &\qquad \leq \|\partial_\alpha d_1\|_{L^2(\partial S_r)} \left\| PV \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2} \,\mathrm{d}\beta \right\|_{L^\infty(\partial S_r)} \\ &\qquad \leq c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}. \end{aligned} \tag{3.4.63}$$

For the second integral in (3.4.59), we have a similar decomposition

$$\begin{aligned} &\int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2} \,\mathrm{d}\beta \\ &\quad = \int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta) \left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma) \right) \frac{\Delta z_1(\gamma,\beta)}{\beta} \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} \frac{\beta^4}{Q(\gamma,\beta)^2} \frac{\mathrm{d}\beta}{\beta} \\ &\quad + \partial_\alpha^2 d_1(\gamma) \int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta) \frac{\beta \Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \,\mathrm{d}\beta \end{aligned} \tag{3.4.64}$$

and

$$\begin{aligned} &\int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta) \frac{\beta \Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \,\mathrm{d}\beta \\ &\qquad = \int_{|\beta|<1} \left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma) \right) \frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2} \,\mathrm{d}\beta \\ &\qquad + \partial_\alpha^2 d_1(\gamma) \int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta) \left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma)|^2} \right) \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)} \,\mathrm{d}\beta \\ &\qquad + \frac{\partial_\alpha^2 d_1(\gamma)}{|\partial_\alpha \mathbf{z}(\gamma)|^2} \int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)} \,\mathrm{d}\beta. \end{aligned} \tag{3.4.65}$$

We use the Minkowski's integral inequality and, once again, we use inequalities (A6), (A5), (A2) to obtain that

$$\left\| \int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2} \,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

To estimate the *out* part, we use the bound (3.4.62) and apply the Minkowski's integral inequality. We deduce

$$\begin{aligned} \left\| \int_{|\beta|>1} \partial_\alpha d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2} \,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} &\leq \int_{|\beta|>1} |\beta|^{-3} \|\partial_\alpha d_1\|_{L^2(\partial S_r)} \,\mathrm{d}\beta \\ &\leq c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}. \end{aligned}$$

Plugging the last two $L^2(\partial S_r)$ inequalities, we derive the following

$$\Big\| \int_{\mathbb{R}} \partial_\alpha d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \Big\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}. \tag{3.4.66}$$

We combine the inequalities (3.4.66) and (3.4.63), and we obtain

$$\|J_{5,1}\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}. \tag{3.4.67}$$

Bound for $J_{5,2}$: We expand $\partial_\alpha^2 \Delta d_1(\gamma,\beta)$ to obtain that

$$-\frac{1}{2} J_{5,2} = \partial_\alpha^2 d_1(\gamma) \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha^2 d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

To estimate the *in* part, we have

$$\begin{aligned}\int_{|\beta|<1} \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta &= \int_{|\beta|<1} \left(\frac{\Delta z_1(\gamma,\beta)}{\beta} - \partial_\alpha z_1(\gamma) \right) \frac{\beta \Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \\ &+ \partial_\alpha z_1(\gamma) \int_{|\beta|<1} \frac{\beta \Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

We use (A1), (A6), (A5), (A2) to control the first integral above. For the second integral, we follow the decomposition (3.4.61). Then, we infer

$$\Big| \int_{|\beta|<1} \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\mathrm{d}\beta \Big|_* \le c_R.$$

Regarding the *out* part, by using (A6), (A5) and (A2), we deduce

$$\Big| \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \Big|_* \le c_R |\beta|^{-2} \tag{3.4.68}$$

thus

$$\Big| \int_{|\beta|>1} \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\mathrm{d}\beta \Big|_* \le c_R.$$

and hence

$$\begin{aligned}&\Big\| \partial_\alpha^2 d_1(\gamma) \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \Big\|_{L^2(\partial S_r)} \\ &\qquad \le \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)} \Big\| \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \Big\|_{L^\infty(\partial S_r)} \\ &\qquad \le c_R \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.\end{aligned} \tag{3.4.69}$$

To deal with the second part of $J_{5,2}$, we decompose the *in* as follows

$$\begin{aligned}&\int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta) \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \\ &\qquad = \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta) \left(\frac{\Delta z_1(\gamma,\beta)}{\beta} - \partial_\alpha z_1(\gamma) \right) \frac{\beta \Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \\ &\qquad + \partial_\alpha z_1(\gamma) \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta) \frac{\beta \Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

Taking the $L^2(\partial S_r)$ and Uusing the Minkowski's integral inequality and estimates (A6), (A5), (A2), we deduce

$$\Big\| \int_{|\beta|<1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\Big\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.$$

To estimate the *out* part, we use the bound (3.4.68) and apply the Minkowski's integral inequality. We deduce

$$\begin{aligned}\Big\| \int_{|\beta|>1} \partial_\alpha^2 d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\Big\|_{L^2(\partial S_r)} &\leq \int_{|\beta|>1}|\beta|^{-2}\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}\,\mathrm{d}\beta\\ &\leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.\end{aligned}$$

We combine the previous estimates and (3.4.69), then we infer

$$\|J_{5,2}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{3.4.70}$$

Bound for $J_{5,3}$: To estimate $J_{5,3}$, we expand $\partial_\alpha^2\Delta d_2(\gamma,\beta)$, we have

$$\begin{aligned}-\frac{1}{2}J_{5,3}(\gamma) = \partial_\alpha^2 d_2(\gamma)\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta&\\ -\int_{\mathbb{R}}\partial_\alpha^2 d_2(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta&\end{aligned}$$

We replace $\Delta d_2(\gamma,\beta)$ by $\Delta z_1(\gamma,\beta)$ in the previous estimates for $J_{5,2}$, we obtain

$$\|J_{5,3}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_2\|_{L^2(\partial S_r)}. \tag{3.4.71}$$

Bound for $J_{5,4}$: To deal with this term, we expand $\partial_\alpha\Delta d_2(\gamma,\beta)$, we have

$$\begin{aligned}-\frac{1}{2}J_{5,4}(\gamma) = \partial_\alpha d_2(\gamma)\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta&\\ -\int_{\mathbb{R}}\partial_\alpha^2 d_2(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.&\end{aligned}$$

Now, we replace $\Delta d_2(\gamma,\beta)$ by $\partial_\alpha\Delta d_2(\gamma,\beta)$ in the estimation of $J_{5,3}$. We deduce

$$\|J_{5,4}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}. \tag{3.4.72}$$

Bound for $J_{5,5}$: To estimate $J_{5,5}$, we expand

$$\partial_\alpha\Delta d_2(\gamma,\beta) = \partial_\alpha d_2(\gamma) - \partial_\alpha d_2(\gamma-\beta).$$

Thus,

$$-\frac{1}{8}J_{5,5}(\gamma) = \partial_\alpha d_2(\gamma)\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\beta}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha d_2(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\beta}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

The $L^2(\partial S_r)$ bound can be deduced from the estimates for $J_{5,5}$, see third line of (3.4.60) decomposition. We infer

$$\|J_{5,5}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}. \tag{3.4.73}$$

Bound for $J_{5,6}$: We now expand

$$\partial_\alpha\Delta d_1(\gamma,\beta) = \partial_\alpha d_1(\gamma) - \partial_\alpha d_1(\gamma-\beta),$$

we get

$$-\frac{1}{8}J_{5,6}(\gamma) := \partial_\alpha d_1(\gamma)\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

Using the decomposition (3.4.16) for K_1 and decomposition (3.4.37) for K_1^*, in the estimation of J_3. We deduce the following

$$\|J_{5,6}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha d_1\|_{L^2(\partial S_r)}. \tag{3.4.74}$$

Bound for $J_{5,7}$: We expand $\partial_\alpha^2\Delta d_1(\gamma,\beta)$, we get

$$\begin{aligned}-\frac{1}{2}J_{5,7}(\gamma) = \partial_\alpha^2 d_2(\gamma)\int_{\mathbb{R}}&\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\beta(2\gamma-\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &- \int_{\mathbb{R}}\partial_\alpha^2 d_2(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\beta(2\gamma-\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

To estimate $J_{5,7}$, arguing as in estimates of K_4 and K_4^* in the estimation of J_3, we derive the following

$$\|J_{5,7}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_2\|_{L^2(\partial S_r)}. \tag{3.4.75}$$

Combining the estimates (3.4.67), (3.4.70), (3.4.71), (3.4.72), (3.4.73), (3.4.74) and (3.4.75), we can derive the $L^2(\partial S_r)$ bound for J_5, given by

$$\|\mathrm{J}_5\|_{L^2(\partial S_r)} \leq c_R\Big(\|\partial_\alpha\mathbf{d}\|_{L^2(\partial S_r)} + \|\partial_\alpha^2\mathbf{d}\|_{L^2(\partial S_r)}\Big), \quad \text{where} \quad \mathbf{d} = (d_1, d_2). \tag{B9}$$

Bound for J_6: For the last term, we compute $\partial_\alpha Q(\gamma,\beta)^2$, given by

$$\begin{aligned}&\partial_\alpha Q(\gamma,\beta)^2\\ &= \Big\{2\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta) + 2\Big(\Delta d_2(\gamma,\beta)+\Delta p_2(\gamma,\beta)\Big)\Big(\partial_\alpha\Delta d_2(\gamma,\beta)+\partial_\alpha\Delta p_2(\gamma,\beta)\Big)\Big\}^2.\end{aligned} \tag{3.4.76}$$

We substitute the expression (3.4.76) in J_6. We obtain

$$\begin{aligned}\frac{1}{2}\mathrm{J}_6(\gamma) &= PV\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^3}\partial_\alpha Q(\gamma,\beta)^2\,\mathrm{d}\beta\\ &= J_{6,1}(\gamma)+\cdots+J_{6,14}(\gamma)\end{aligned}$$

for

$$
\begin{aligned}
J_{6,1}(\gamma) &:= 8\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^3}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,2}(\gamma) &:= 4\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\Delta d_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,3}(\gamma) &:= 4\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\Delta d_2(\gamma,\beta)\partial_\alpha\Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,4}(\gamma) &:= 4\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\Delta p_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,5}(\gamma) &:= 4\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\Delta p_2(\gamma,\beta)\partial_\alpha\Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,6}(\gamma) &:= 2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)^2\partial_\alpha\Delta d_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,7}(\gamma) &:= 4\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)^2\partial_\alpha\Delta d_2(\gamma,\beta)\partial_\alpha\Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,8}(\gamma) &:= 2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)^2\partial_\alpha\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,9}(\gamma) &:= 2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)^2\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,10}(\gamma) &:= 8\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)\Delta p_2(\gamma,\beta)\partial_\alpha\Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,11}(\gamma) &:= 4\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)\Delta p_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,12}(\gamma) &:= 4\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)\Delta p_2(\gamma,\beta)\partial_\alpha\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,13}(\gamma) &:= 4\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)\Delta p_2(\gamma,\beta)^2\partial_\alpha\Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
J_{6,14}(\gamma) &:= 2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta p_2(\gamma,\beta)^2\partial_\alpha\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta.
\end{aligned}
\tag{3.4.77}
$$

Bound for $J_{6,1}$: To estimate thist term, we expand

$$\partial_\alpha\Delta d_1(\gamma,\beta) = \partial_\alpha d_1(\gamma) - \partial_\alpha d_1(\gamma-\beta)$$

we have

$$
\begin{aligned}
\frac{1}{8}J_{6,1}(\gamma) = \partial_\alpha d_1(\gamma)\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta&\\
-\int_{\mathbb{R}} \partial_\alpha d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta.&
\end{aligned}
\tag{3.4.78}
$$

We decompose the *in* part as follows

$$
\begin{aligned}
&\int_{|\beta|<1} \frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\
&\qquad = \int_{|\beta|<1}\left(\frac{\Delta z_1(\gamma,\beta)}{\beta} - \partial_\alpha z_1(\gamma)\right)\frac{\beta\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta + \partial_\alpha z_1(\gamma)\mathcal{W}_1(\gamma)
\end{aligned}
$$

where

$$\mathcal{W}_1(\gamma) := \int_{|\beta|<1} \frac{\beta\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta. \tag{3.4.79}$$

We further decompose

$$\begin{aligned}\mathcal{W}_1(\gamma) &= \int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right)\frac{\beta^2\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\ &\quad + \partial_\alpha^2 d_1(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &\quad + \frac{\partial_\alpha^2 d_1(\gamma)}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\frac{K_1^{in}(\gamma)}{2}.\end{aligned}$$

Using the estimates for K_1 in the estimation of J_3 together with inequalities (A4), (A6), (A5), (4.2.1) and the arc-chord condition (A2), we infer

$$|\mathcal{W}_1(\gamma)|_* \le c_R.$$

Then, using the last bound and estimates (A1),(A6), (A5), (A2), we deduce the following

$$\left|\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\right|_* \le c_R.$$

Regarding the *out* part, from inequalities (A6), (A5), (A2), we infer the following

$$\left|\frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\right|_* \le c_R|\beta|^{-3}. \tag{3.4.80}$$

Hence

$$\left|\int_{|\beta|>1}\frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\right|_* \le \int_{|\beta|<1}\left|\frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\right|_*\mathrm{d}\beta \le c_R.$$

By taking the $L^2(\partial S_r)$ we derive the following

$$\begin{aligned}&\left\|\partial_\alpha d_1(\gamma)\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\\ &\qquad \le \|\partial_\alpha d_1\|_{L^2(\partial S_r)}\left\|\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_r)}\\ &\qquad \le c_R\|\partial_\alpha d_1\|_{L^2(\partial S_r)}.\end{aligned} \tag{3.4.81}$$

To estimate the second integral in $J_{6,1}$ equation (3.4.78), we consider a similar decomposition. We have

$$\begin{aligned}&\int_{|\beta|<1}\partial_\alpha d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^3\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\ &\quad = \int_{|\beta|<1}\partial_\alpha d_1(\gamma-\beta)\left(\frac{\Delta z_1(\gamma,\beta)}{\beta} - \partial_\alpha z_1(\gamma)\right)\frac{\beta\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\ &\quad + \partial_\alpha z_1(\gamma)\mathcal{W}_1^*[\partial_\alpha d_1](\gamma).\end{aligned}$$

Where $\mathcal{W}_1^*[f]$, is given by

$$\mathcal{W}_1^*[f](\gamma) := \int_{|\beta|<1} f(\gamma-\beta)\frac{\beta\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta. \tag{3.4.82}$$

We further decompose

$$\mathcal{W}_1^*[\partial_\alpha d_1](\gamma) = \int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta)\left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right)\frac{\beta^2 \Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta$$
$$+ \partial_\alpha^2 d_1(\gamma)\int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta)\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta$$
$$+ \frac{\partial_\alpha^2 d_1(\gamma)}{|\partial_\alpha \mathbf{z}(\gamma)|^2}\frac{\mathcal{K}_1^{*,in}[\partial_\alpha d_1](\gamma)}{2}.$$

Where $\mathcal{K}_1^*[f]$ is given by

$$\mathcal{K}_1^*[f](\gamma) := \int_{|\beta|<1} f(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta. \tag{3.4.83}$$

In view of the bound (3.4.40) for $K_1^{*,in}$, the inequalities (A4), (A6), (A5), (A2) and (4.2.1) together with the Minkowski's integral inequality, we obtain

$$\|\mathcal{W}_1^*[\partial_\alpha d_1]\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

Taking the $L^2(\partial S_r)$ norm, a making use of the Minkowski's integral inequality and considering the control of $\mathcal{W}_1^*[\partial_\alpha d_1]$, estimates (A1), we infer

$$\left\| \int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^3 \partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}. \tag{3.4.84}$$

To estimate the *out* part, we consider the inequality (3.4.80) and employ Minkowski's integral inequality

$$\left\| \int_{|\beta|>1} \partial_\alpha d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^3 \partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

Using the last bound, together with (3.4.84) and (3.4.81), we deduce

$$\|J_{6,1}\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,2}$: We expand

$$\partial_\alpha \Delta d_2(\gamma,\beta) = \partial_\alpha d_2(\gamma) - \partial_\alpha d_2(\gamma-\beta)$$

we get

$$\frac{1}{4}J_{6,2}(\gamma) = \partial_\alpha d_2(\gamma)\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)^2 \Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta$$
$$- \int_{\mathbb{R}} \partial_\alpha d_2(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)^2 \Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta.$$

We decompose the *in* parts as follows,

$$\int_{|\beta|<1} \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)^2 \Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta$$
$$= \int_{|\beta|<1} \left(\frac{\Delta d_2(\gamma,\beta)}{\beta} - \partial_\alpha d_2(\gamma)\right)\frac{\beta \Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta$$
$$+ \partial_\alpha d_2(\gamma)\mathcal{W}_1(\gamma)$$

and

$$\begin{aligned}\int_{|\beta|<1} \partial_\alpha d_2(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta \\ = \int_{|\beta|} \partial_\alpha d_2(\gamma-\beta)\left(\frac{\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha d_2(\gamma)\right)\frac{\beta\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta \\ + \partial_\alpha d_2(\gamma)\mathcal{W}_1^*[\partial_\alpha d_2](\gamma),\end{aligned} \tag{3.4.85}$$

where $\mathcal{W}_1$ is defined in (3.4.79) and $\mathcal{W}_1^*[\,\cdot\,]$ is defined in (3.4.82). Using the bound for $\mathcal{W}_1$ derived in the estimation of $J_{6,1}$ together with estimates (A1), (A6), (A5), (A2), we infer

$$\left|\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\right|_* \le c_R.$$

Hence

$$\left\|\partial_\alpha d_2(\gamma)\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

To estimate the second integral of $J_{6,2}$, first we estimate $\mathcal{W}_1^*[\partial_\alpha d_2]$. Using the bound already derived for $\mathcal{W}_1^*[\partial_\alpha d_1]$ in the estimation of $J_{6,1}$, we deduce

$$\|\mathcal{W}_1^*[\partial_\alpha d_2]\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Now, taking the $L^2(\partial S_r)$ in (3.4.85), and making use of the Minkowski's integral inequality, we derive

$$\left\|\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Regarding the *out* part, we consider the following bound

$$\left|\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\right|_* \le c_R|\beta|^{-4}. \tag{3.4.86}$$

Taking the $L^2(\partial S_r)$ and making use of the Minkowski's integral inequality, we infer

$$\|J_{6,2}\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}$$

Bound for $J_{6,3}$: By expanding $\Delta d_2(\gamma,\beta)$ we obtain that

$$\frac{1}{8}J_{6,3}(\gamma) = d_2(\gamma)\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\beta}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta - \int_{\mathbb{R}} d_2(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\beta}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta.$$

In this case $\partial_\alpha\Delta p_2(\gamma,\beta) = 2\beta$. We can rewrite the *in* part as

$$\frac{1}{8}J_{6,3}^{in}(\gamma) = d_2(\gamma)\mathcal{W}_1(\gamma) - \mathcal{W}_1^*[d_2](\gamma).$$

The estimate can be deduce from the estimates for $J_{6,1}$ and the bound

$$\left|\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2\beta}{Q(\gamma,\beta)^3}\right|_* \le c_R|\beta|^{-2}.$$

Hence

$$\|J_{6,3}\|_{L^2(\partial S_r)} \le c_R\|d_2\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,4}$: To estimate this term, we expand

$$\partial_\alpha \Delta d_2(\gamma,\beta) = \partial_\alpha d_2(\gamma) - \partial_\alpha d_2(\gamma,\beta).$$

Then

$$\frac{1}{4} J_{6,4}(\gamma) = \partial_\alpha d_2(\gamma) \int_{\mathbb{R}} \mathbf{w}_1(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha d_2(\gamma-\beta) \mathbf{w}_1(\gamma,\beta)\,\mathrm{d}\beta, \tag{3.4.87}$$

where we denote

$$\mathbf{w}_1(\gamma,\beta) = \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)^2 \Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)^3}.$$

Recall, from definition (3.1.4) that $\Delta p_2(\gamma,\beta) = \beta(2\gamma-\beta)$. For the *in* part, we have

$$\begin{aligned}\int_{|\beta|<1} \mathbf{w}_1(\gamma,\beta)\,\mathrm{d}\beta = &\int_{|\beta|<1} \frac{1}{\beta}\left(\frac{\beta \Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)} - \frac{2\gamma}{|\partial_\alpha z_1(\gamma)|^2}\right) \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta \\ &+ \frac{2\gamma}{|\partial_\alpha \mathbf{z}(\gamma)|^2} \mathcal{W}_2(\gamma)\end{aligned}$$

where $\mathcal{W}_2$ is given by

$$\mathcal{W}_2(\gamma) := \int_{|\beta|<1} \frac{1}{\beta} \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta. \tag{3.4.88}$$

Then we further decompose

$$\begin{aligned}\mathcal{W}_2(\gamma) := &\int_{|\beta|<1} \left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right) \frac{\Delta z_1(\gamma,\beta)^2 \partial_\alpha \Delta d_1(\gamma,\beta) 2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \\ &+ \partial_\alpha^2 d_1(\gamma) \frac{K_1^{in}(\gamma)}{2}.\end{aligned}$$

We use the bound (3.4.40) for K_1^{in} in the estimation of J_3 and inequalities (A4), (A6), (A5), (A2). We deduce

$$\left|\mathcal{W}_2(\gamma)\right|_* < c_R.$$

From Corollary 2 and estimate (A8), we derive the following

$$\left|\int_{|\beta|<1} \mathbf{w}_1(\gamma,\beta)\,\mathrm{d}\beta\right|_* \le c_R. \tag{3.4.89}$$

Regarding the *out* part, we use inequality (A9) given by

$$\left|\frac{\Delta p_2(\gamma,\beta)}{\beta} \frac{\beta^2}{Q(\gamma,\beta)}\right|_* \le c_R, \tag{3.4.90}$$

then, using the inequalities (A6), (A5), (3.4.90) and (A2), we infer the following

$$\left|\mathbf{w}_1(\gamma,\beta)\right|_* \le c_R |\beta|^{-3}. \tag{3.4.91}$$

Thus

$$\left|\int_{|\beta|>1} \mathbf{w}_1(\gamma,\beta)\,\mathrm{d}\beta\right|_* \le \int_{|\beta|>1} \left|\mathbf{w}_1(\gamma,\beta)\right|_* \mathrm{d}\beta \le c_R.$$

Now, by taking the $L^2(\partial S_r)$ norm and combining the previous estimates with (3.4.89), we infer

$$\left\|\partial_\alpha d_2(\gamma) \int_{\mathbb{R}} \mathbf{w}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha d_2\|_{L^2(\partial S_r)} \tag{3.4.92}$$

To estimate the second integral in (3.4.87), we decompose the *in* part as follows

$$\begin{aligned}\int_{|\beta|<1} \partial_\alpha d_2(\gamma-\beta)\mathbf{w}_1(\gamma,\beta)\,\mathrm{d}\beta \\ &= \int_{|\beta|<1} \frac{\partial_\alpha d_2(\gamma-\beta)}{\beta}\left(\frac{\beta\Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)} - \frac{2\gamma}{|\partial_\alpha z_1(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta \\ &+ \frac{2\gamma}{|\partial_\alpha \mathbf{z}(\gamma)|^2}\mathcal{W}_2^*[\partial_\alpha d_2](\gamma).\end{aligned}$$

Where $\mathcal{W}_2^*[f]$ is given by

$$\mathcal{W}_2^*[f](\gamma) := \int_{|\beta|<1} \frac{f(\gamma-\beta)}{\beta}\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

We decompose

$$\begin{aligned}\mathcal{W}_2^*[\partial_\alpha d_2](\gamma) = \int_{|\beta|<1} \partial_\alpha d_2(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \\ + \frac{\partial_\alpha^2 d_1(\gamma)\mathcal{K}_1^*[\partial_\alpha d_2](\gamma)}{2}.\end{aligned}$$

In view of the bound (3.4.40) for $K_1^{*,in}$, the inequalities (A4), (A5), (A2) the Minkowski's integral inequality, we deduce

$$\|\mathcal{W}_2^*[\partial_\alpha d_2]\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Thus the $L^2(\partial S_r)$ bound follows from the Minkowski's integral inequality and (A1), (A6), (A5), (A8) the arc-chord condition (A2) and Corollary 2. We derive

$$\left\|\int_{|\beta|<1} \partial_\alpha d_2(\gamma-\beta)\mathbf{w}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}. \tag{3.4.93}$$

For the *out* part, we consider the bound (3.4.91) and apply the Minkowski's integral inequality. We deduce that

$$\left\|\int_{|\beta|>1} \partial_\alpha d_2(\gamma-\beta)\mathbf{w}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

The last estimate together with (3.4.93) and (3.4.92) allow us conclude

$$\|J_{6,4}\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,5}$: To handle this term, we expand $\partial_\alpha\Delta d_1(\gamma,\beta)$, we obtain

$$\frac{1}{8}J_{6,5}(\gamma) = \partial_\alpha d_1(\gamma)\int_{\mathbb{R}} \mathbf{w}_2(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha d_1(\gamma-\beta)\mathbf{w}_2(\gamma,\beta)\,\mathrm{d}\beta \tag{3.4.94}$$

where

$$\mathbf{w}_2(\gamma,\beta) := \frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)\Delta p_2(\gamma,\beta)\beta}{Q(\gamma,\beta)^3}.$$

We decompose the *in* part

$$\begin{aligned}\int_{|\beta|<1} \mathbf{w}_2(\gamma,\beta)\,\mathrm{d}\beta \\ &= \int_{|\beta|<1}\left(\frac{\beta\Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)} - \frac{2\gamma}{|\partial_\alpha \mathbf{z}(\gamma)|^4}\right)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \\ &+ \frac{\gamma}{|\partial_\alpha \mathbf{z}(\gamma)|^2}K_1^{in}(\gamma).\end{aligned}$$

Once again, from inequalities (A6), (A5), (A2), (A8) and Corollary 2, together with the estimate (3.4.40) for K_1^*, we deduce

$$\left| \int_{|\beta|<1} \mathbf{w}_2(\gamma,\beta)\,\mathrm{d}\beta \right|_* \leq c_R.$$

For the *out* part, we consider the following

$$\left|\mathbf{w}_2(\gamma,\beta)\right|_* \leq c_R|\beta|^{-2}. \tag{3.4.95}$$

Thus

$$\left| \int_{|\beta|>1} \mathbf{w}_2(\gamma,\beta)\,\mathrm{d}\beta \right|_* \leq \int_{|\beta|>1} \left|\mathbf{w}_2(\gamma,\beta)\right|_* \mathrm{d}\beta \leq c_R.$$

Taking the $L^2(\partial S_r)$ norm we get

$$\left\| \partial_\alpha d_1(\gamma) \int_{\mathbb{R}} \mathbf{w}_2(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

To estimate the second integral in (3.4.94), we have

$$\begin{aligned}
&\int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta)\mathbf{w}_2(\gamma,\beta)\,\mathrm{d}\beta \\
&\qquad = \int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta)\left(\frac{\beta\Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)} - \frac{2\gamma}{|\partial_\alpha \mathbf{z}(\gamma)|^4}\right)\frac{\Delta z_1(\gamma,\beta)^2\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta \\
&\qquad + \frac{\gamma}{|\partial_\alpha \mathbf{z}(\gamma)|^2}\mathcal{K}_1^*[\partial_\alpha d_1](\gamma).
\end{aligned}$$

Once again, the $L^2(\partial S_r)$ estimate follows from inequalities (A6), (A5), (A2), (32) and estimate for (3.4.83), together with the Minkowski's integral inequality. We deduce

$$\left\| \int_{|\beta|<1} \partial_\alpha d_1(\gamma-\beta)\mathbf{w}_2(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

Regarding the *out* part, we use the inequality (3.4.95). We can infer that

$$\|J_{6,5}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,6}$: We expand

$$\partial_\alpha \Delta d_2(\gamma,\beta) = \partial_\alpha d_2(\gamma) - \partial_\alpha d_2(\gamma-\beta),$$

then

$$\frac{1}{2}J_{6,6}(\gamma) = \partial_\alpha d_2(\gamma)\int_{\mathbb{R}} \mathbf{w}_3(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha d_2(\gamma-\beta)\mathbf{w}_3(\gamma,\beta)\,\mathrm{d}\beta. \tag{3.4.96}$$

where we denote

$$\mathbf{w}_3(\gamma,\beta) := \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)^2\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}.$$

We decompose the *in* parts as follows

$$\begin{aligned}
&\int_{|\beta|<1} \mathbf{w}_3(\gamma,\beta)\,\mathrm{d}\beta \\
&\qquad = \int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_2(\gamma)\right)\frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta \\
&\qquad + \partial_\alpha^2 d_2(\gamma)\mathcal{W}_3(\gamma),
\end{aligned}$$

in the last integral, we define

$$\mathcal{W}_3(\gamma) := \int_{|\beta|<1} \frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta. \tag{3.4.97}$$

We decompose further

$$\begin{aligned}\mathcal{W}_3(\gamma) = &\int_{|\beta|<1}\left(\frac{\Delta d_2(\gamma,\beta)}{\beta} - \partial_\alpha d_2(\gamma)\right)\frac{\beta^2\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\ &+ \partial_\alpha d_2(\gamma)\mathcal{W}_4(\gamma),\end{aligned}$$

where

$$\mathcal{W}_4(\gamma) := \int_{|\beta|<1} \frac{\beta^2\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta. \tag{3.4.98}$$

and

$$\begin{aligned}\mathcal{W}_4(\gamma) = &\int_{|\beta|<1}\left(\frac{\Delta d_2(\gamma,\beta)}{\beta} - \partial_\alpha d_2(\gamma)\right)\frac{\beta^3\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\ &+ \partial_\alpha d_2(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &+ \frac{\partial_\alpha d_2(\gamma)}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right)\frac{\beta^2\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &+ \frac{\partial_\alpha d_2(\gamma)\partial_\alpha^2 d_1(\gamma)}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\int_{|\beta|<1}\frac{\beta^2\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

Firstly, we estimate $\mathcal{W}_4$, we use equation (3.4.16) from the estimation of J_3, to control the last integral. From inequalities (A1), (A6), (A5), (A2) and Lemma 32, we infer the following

$$|\mathcal{W}_4(\gamma)|_* \le c_R.$$

Secondly, the estimate for $\mathcal{W}_3$, follows from the previous estimate along with the same inequalities previously used. We infer

$$|\mathcal{W}_3(\gamma)|_* \le c_R.$$

Hence, we deduce

$$\left|\int_{|\beta|<1}\mathbf{w}_3(\gamma,\beta)\,\mathrm{d}\beta\right|_* \le c_R. \tag{3.4.99}$$

The *out* part, is bounded by considering the inequality

$$\big|\mathbf{w}_3(\gamma,\beta)\big|_* \le c_R|\beta|^{-5} \tag{3.4.100}$$

Thus

$$\left|\int_{|\beta|>1}\mathbf{w}_3(\gamma,\beta)\,\mathrm{d}\beta\right|_* \le c_R\int_{|\beta|>1}|\beta|^{-5}\,\mathrm{d}\beta \le c_R.$$

Taking the $L^2(\partial S_r)$ norm, and considering the previous inequality and estimate (3.4.99), we derive the following

$$\left\|\partial_\alpha d_2(\gamma)\int_{|\beta|>1}\mathbf{w}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

For the second integral in (3.4.96), we use a similar decomposition, then we have

$$\begin{aligned}&\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_3(\gamma,\beta)\,\mathrm{d}\beta\\&=\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_2(\gamma)\right)\frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\&+\partial_\alpha^2 d_2(\gamma)\mathcal{W}_3^*[\partial_\alpha d_2](\gamma).\end{aligned}$$

Where we define

$$\mathcal{W}_3^*[f](\gamma):=\int_{|\beta|<1}f(\gamma-\beta)\frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta. \tag{3.4.101}$$

We further decompose

$$\begin{aligned}&\mathcal{W}_3^*(\gamma)[\partial_\alpha d_2](\gamma)\\&=\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\left(\frac{\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha d_2(\gamma)\right)\frac{\beta^2\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\&+\partial_\alpha d_2(\gamma)\mathcal{W}_4^*[\partial_\alpha d_2](\gamma),\end{aligned}$$

for

$$\mathcal{W}_4^*[f](\gamma):=\int_{|\beta|<1}f(\gamma-\beta)\frac{\beta^2\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta. \tag{3.4.102}$$

and

$$\begin{aligned}\mathcal{W}_4^*[\partial_\alpha d_2](\gamma)&=\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\left(\frac{\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha d_2(\gamma)\right)\frac{\beta^3\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\&+\partial_\alpha d_2(\gamma)\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\left(\frac{\beta^2}{Q(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\&+\frac{\partial_\alpha d_2(\gamma)}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1(\gamma)\right)\frac{\beta^2\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\&+\frac{\partial_\alpha d_2(\gamma)\partial_\alpha^2 d_1(\gamma)}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\frac{\beta^2\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

We make use of the Minkowsk's integral inequality, to obtain

$$\left\|\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Regarding the *out* part, from the inequality (3.4.100) together with the Minkowski's inequality, we infer

$$\left\|\int_{|\beta|>1}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

and we deduce that

$$\|J_{6,6}\|_{L^2(\partial S_r)}\le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,7}$: We move on to estimate $J_{6,7}$. By expanding $\partial_\alpha\Delta d_2(\gamma,\beta)$, and using the definitions (3.4.97) and (3.4.101), we rewrite the *in* part as follows

$$\frac{1}{8}J_{6,7}^{in}(\gamma)=\partial_\alpha d_2(\gamma)\mathcal{W}_3(\gamma)-\mathcal{W}_3^*[\partial_\alpha d_2](\gamma).$$

The bound in $L^2(\partial S_r)$ follows from the previous estimate of $J_{6,6}$. Hence, we deduce the following

$$\|J_{6,7}\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,8}$: Using the definitions (3.4.98) and (3.4.102), we have

$$\frac{1}{8} J_{6,8}(\gamma) = d_2(\gamma)\mathcal{W}_4(\gamma) - \mathcal{W}_4^*[d_2](\gamma).$$

Hence

$$\|J_{6,8}\|_{L^2(\partial S_r)} \leq c_R \|d_2\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,9}$: To estimate this term, we expand $\partial_\alpha \Delta d_2(\gamma,\beta)$, we have

$$\frac{1}{2} J_{6,9}(\gamma) = \partial_\alpha d_2(\gamma) \int_{\mathbb{R}} \mathbf{w}_4(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha d_2(\gamma-\beta)\mathbf{w}_4(\gamma,\beta)\,\mathrm{d}\beta, \tag{3.4.103}$$

where we denote

$$\mathbf{w}_4(\gamma,\beta) := \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)\partial_\alpha \Delta d_2(\gamma,\beta)\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}.$$

Then, we decompose the *in* part as follows

$$\begin{aligned}\int_{|\beta|<1} \mathbf{w}_4(\gamma,\beta)\,\mathrm{d}\beta = &\int_{|\beta|<1} \left(\frac{\beta^2 \Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^2} - \frac{4\gamma^2}{|\partial_\alpha \mathbf{z}(\gamma)|^4}\right) \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)\partial_\alpha \Delta d_2(\gamma,\beta)}{\beta^2 Q(\gamma,\beta)}\,\mathrm{d}\beta \\ &+ \frac{4\gamma^2}{|\partial_\alpha \mathbf{z}(\gamma)|^4}\mathcal{W}_5(\gamma).\end{aligned} \tag{3.4.104}$$

For

$$\mathcal{W}_5(\gamma) := \int_{|\beta|<1} \frac{1}{\beta^2} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)\partial_\alpha \Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.$$

We further decompose

$$\begin{aligned}\mathcal{W}_5(\gamma) = &\int_{|\beta|<1} \left(\frac{\partial_\alpha \Delta d_2(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_2(\gamma)\right) \frac{\Delta z_1(\gamma,\beta)\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta \\ &+ \partial_\alpha^2 d_2(\gamma) \int_{|\beta|<1} \left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1(\gamma)\right) \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta \\ &\partial_\alpha^2 d_1(\gamma)\partial_\alpha^2 d_2(\gamma) \int_{|\beta|<1} \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.\end{aligned}$$

We use the estimate (3.3.4) in Lemma 7, then we obtain the estimates for the last integral. From estimates (A4), (A5) and (A2), we infer

$$\left|\mathcal{W}_5(\gamma)\right|_* \leq c_R.$$

Hence, using a Corollary 3 we control the first integral in the right-hand side of (3.4.104). We infer

$$\left|\int_{|\beta|<1} \mathbf{w}_4(\gamma,\beta)\,\mathrm{d}\beta\right|_* \leq c_R.$$

Regarding the *out* part, we use estimates (A6), (A5), (3.4.90) and (A2). We have

$$\left|\mathbf{w}_4(\gamma,\beta)\right|_* \leq c_R |\beta|^{-2}. \tag{3.4.105}$$

In view of the above inequality, we get

$$\left|\int_{|\beta|>1} \mathbf{w}_4(\gamma,\beta)\,\mathrm{d}\beta\right|_* \leq \int_{|\beta|>1} \left|\mathbf{w}_4(\gamma,\beta)\right|_* \mathrm{d}\beta \leq c_R.$$

Taking the $L^2(\partial S_r)$ norm in the first term in (3.4.103), we infer the following

$$\left\|\partial_\alpha d_2(\gamma)\int_{\mathbb{R}}\mathbf{w}_4(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

The second integral in (3.4.103), can be bounded by making a similar decomposition

$$\begin{aligned}&\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_4(\gamma,\beta)\,\mathrm{d}\beta\\&\qquad=\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\left(\frac{\beta^2\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^2}-\frac{4\gamma^2}{|\partial_\alpha\mathbf{z}(\gamma)|^4}\right)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta^2Q(\gamma,\beta)}\,\mathrm{d}\beta\\&\qquad+\frac{4\gamma^2}{|\partial_\alpha\mathbf{z}(\gamma)|^4}\mathcal{W}_5^*[\partial_\alpha d_2](\gamma)\end{aligned}$$

where we denote

$$\mathcal{W}_5^*[f](\gamma):=\int_{|\beta|<1}f(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta^2Q(\gamma,\beta)}\,\mathrm{d}\beta.$$

We further decompose

$$\begin{aligned}\mathcal{W}_5^*[\partial_\alpha d_2](\gamma)&=\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_2(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta Q(\gamma,\beta)}\,\mathrm{d}\beta\\&+\partial_\alpha^2 d_2(\gamma)\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1(\gamma)\right)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\\&+\partial_\alpha^2 d_2(\gamma)\partial_\alpha^2 d_1(\gamma)\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.\end{aligned}$$

To deal with the last integral, we use estimates for I_2^{in} in Lemma 7. Then taking the $L^2(\partial S_r)$ norm and using the same estimates used for $\mathcal{W}_5$, as well as the Minkowski's integral inequality, we deduce

$$\left\|\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_4(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

The out part is bounded by considering (3.4.105). Thus, we infer

$$\|J_{6,9}\|_{L^2(\partial S_r)}\le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,10}$: By expanding $\Delta d_2(\gamma,\beta)$ we have

$$\frac{1}{16}J_{6,10}(\gamma)=d_2(\gamma)\int_{\mathbb{R}}\mathbf{w}_5(\gamma,\beta)\,\mathrm{d}\beta-\int_{\mathbb{R}}d_2(\gamma-\beta)\mathbf{w}_5(\gamma,\beta)\,\mathrm{d}\beta,$$

where we denote

$$\mathbf{w}_5(\gamma,\beta)=\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)\Delta p_2(\gamma,\beta)\beta}{Q(\gamma,\beta)^3}.$$

We decompose the *in* part

$$\begin{aligned}&\int_{|\beta|<1}\mathbf{w}_5(\gamma,\beta)\,\mathrm{d}\beta\\&\qquad=\int_{|\beta|<1}\left(\frac{\beta\Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)}-\frac{2\gamma}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\&\qquad+\frac{2\gamma}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\mathcal{W}_6(\gamma),\end{aligned}$$

for
$$\mathcal{W}_6(\gamma) = \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

Now, we decompose $\mathcal{W}_6$, as follows
$$\mathcal{W}_6(\gamma) = \int_{|\beta|<1}\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta^2 Q(\gamma,\beta)}\,\mathrm{d}\beta + \mathcal{W}_5(\gamma).$$

From the estimation for $\mathcal{W}_5$ and estimates (A6), (A5) (A2) and Lemma 32, we deduce the following
$$|\mathcal{W}_6|_* \le c_R$$

and from Corollary 2, we infer
$$\left|\int_{|\beta|<1}\mathbf{w}_5(\gamma,\beta)\,\mathrm{d}\beta\right|_* \le c_R.$$

Regarding the *out* part, we use estimates (A6), (A5) and (3.4.90). We have
$$\left|\mathbf{w}_5(\gamma,\beta)\right|_* \le c_R|\beta|^{-3}.$$

Thus by taking the $L^2(\partial S_r)$ norm, together with the estimate of $\mathcal{W}_6$, we can deduce the following
$$\left\|d_2(\gamma)\int_{\mathbb{R}}\mathbf{w}_5(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Next, using the decomposition
$$\begin{aligned}&\int_{|\beta|<1} d_2(\gamma-\beta)\mathbf{w}_5(\gamma,\beta)\,\mathrm{d}\beta\\ &\quad= \int_{|\beta|<1} d_2(\gamma-\beta)\left(\frac{\beta\Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)} - \frac{2\gamma}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &\quad+\frac{2\gamma}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\mathcal{W}_6^*[d_2](\gamma),\end{aligned}$$

where
$$\mathcal{W}_6^*[f](\gamma) := \int_{\mathbb{R}} f(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

and
$$\mathcal{W}_6^*[f](\gamma) = \int_{|\beta|<1}\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha\mathbf{z}(\gamma)|^2}\right)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{\beta^2 Q(\gamma,\beta)}\,\mathrm{d}\beta + \mathcal{W}_5^*[f](\gamma).$$

Using estimates for $\mathcal{W}_5^*[\,\cdot\,]$ together with Lemma 32 and Corollary 2, we deduce
$$\left\|\int_{|\beta|<1} d_2(\gamma-\beta)\mathbf{w}_5(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R\|d_2\|_{L^2(\partial S_r)}.$$

We can infer
$$\|J_{6,10}\|_{L^2(\partial S_r)} \le c_R\|d_2\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,11}$: We expand $\partial_\alpha\Delta d_2(\gamma,\beta)$,
$$\frac{1}{4}J_{6,11}(\gamma) = \partial_\alpha d_2(\gamma)\int_{\mathbb{R}}\mathbf{w}_6(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_6(\gamma,\beta)\,\mathrm{d}\beta, \tag{3.4.106}$$

where

$$\mathbf{w}_6(\gamma,\beta) = \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta d_2(\gamma,\beta)\Delta p_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^3}.$$

We decompose the *in* part as follows

$$\begin{aligned}&\int_{|\beta|<1}\mathbf{w}_6(\gamma,\beta)\,\mathrm{d}\beta\\ &\quad=\int_{|\beta|<1}\left(\frac{\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha d_2(\gamma)\right)\frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta p_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &\quad+\partial_\alpha d_2(\gamma)\int_{|\beta|<1}\mathbf{w}_6(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned}$$

Regarding the *out* part, we use estimates (A6), (A5), (3.4.90) and (A2). We deduce the following

$$\left|\mathbf{w}_6(\gamma,\beta)\right|_* \le c_R|\beta|^{-3}.$$

This yields

$$\left|\int_{\mathbb{R}}\mathbf{w}_6(\gamma,\beta)\,\mathrm{d}\beta\right|_* \le \int_{|\beta|<1}\left|\mathbf{w}_6(\gamma,\beta)\right|_*\mathrm{d}\beta \le c_R.$$

Thus by taking the $L^2(\partial S_r)$ norm, together with the estimate of previous estimate, we can deduce the following

$$\left\|\partial_\alpha d_2(\gamma)\int_{\mathbb{R}}\mathbf{w}_6(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

To deal with the second integral in (3.4.106), we decompose

$$\begin{aligned}&\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_6(\gamma,\beta)\,\mathrm{d}\beta\\ &=\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\left(\frac{\Delta d_2(\gamma,\beta)}{\beta}-\partial_\alpha d_2(\gamma)\right)\frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta p_2(\gamma,\beta)\partial_\alpha\Delta d_2(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &+\partial_\alpha d_2(\gamma)\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_5(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned}$$

Taking the $L^2(\partial S_r)$ norm, and making use the Minkowski's integral inequality, together with the estimates for $J_{6,10}$, we infer

$$\left\|\int_{\mathbb{R}}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_6(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Then

$$\|J_{6,11}\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,12}$: We expand $\Delta d_2(\gamma,\beta)$, then we get

$$\frac{1}{16}J_{6,12}(\gamma) = d_2(\gamma)\int_{\mathbb{R}}\mathbf{w}_7(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}} d_2(\gamma-\beta)\mathbf{w}_7(\gamma,\beta)\,\mathrm{d}\beta. \tag{3.4.107}$$

Where we denote

$$\mathbf{w}_7(\gamma,\beta) = \frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)\Delta p_2(\gamma,\beta)\beta^2}{Q(\gamma,\beta)^3}$$

and

$$\int_{|\beta|<1} \mathbf{w}_7(\gamma,\beta)\,\mathrm{d}\beta$$
$$= \int_{|\beta|<1} \left(\frac{\beta \Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)} - \frac{2\gamma}{|\partial_\alpha \mathbf{z}(\gamma)|^2} \right) \frac{\beta \Delta z_1(\gamma,\beta) \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta$$
$$+ \frac{2\gamma}{|\partial_\alpha \mathbf{z}(\gamma)|^2} \int_{|\beta|<1} \left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)^2}{\beta} - \partial_\alpha^2 d_1(\gamma) \right) \frac{\beta^2 \Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta$$
$$+ \frac{2\gamma \partial_\alpha^2 d_1(\gamma)}{|\partial_\alpha \mathbf{z}(\gamma)|^2} \int_{|\beta|<1} \frac{\beta^2 \Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

By using the decomposition (3.4.16) from the estimation of J_3, we obtain the estimates for the last integral. Then, by using inequalities (A1), (A5), (A4), (A2), (3.4.90) and Corollary 2, we derive

$$\left| \int_{|\beta|<1} \mathbf{w}_7(\gamma,\beta)\,\mathrm{d}\beta \right|_* \le c_R.$$

Regarding the *out* part we consider

$$\left| \mathbf{w}_7(\gamma,\beta) \right|_* \le c_R |\beta|^{-2}. \tag{3.4.108}$$

Thus

$$\left| \int_{|\beta|<1} \mathbf{w}_7(\gamma,\beta)\,\mathrm{d}\beta \right|_* \le \int_{|\beta|>1} \left| \mathbf{w}_7(\gamma,\beta) \right|_* \mathrm{d}\beta \le c_R.$$

To estimate the second integral in (3.4.107), we have

$$\int_{|\beta|<1} d_2(\gamma-\beta) \mathbf{w}_7(\gamma,\beta)\,\mathrm{d}\beta$$
$$= \int_{|\beta|<1} d_2(\gamma-\beta) \left(\frac{\beta \Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)} - \frac{2\gamma}{|\partial_\alpha \mathbf{z}(\gamma)|^2} \right) \frac{\beta \Delta z_1(\gamma,\beta) \partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta$$
$$+ \frac{2\gamma}{|\partial_\alpha \mathbf{z}(\gamma)|^2} \int_{|\beta|<1} d_2(\gamma-\beta) \left(\frac{\partial_\alpha \Delta d_1(\gamma,\beta)^2}{\beta} - \partial_\alpha^2 d_1(\gamma) \right) \frac{\beta^2 \Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta$$
$$+ \frac{2\gamma \partial_\alpha^2 d_1(\gamma)}{|\partial_\alpha \mathbf{z}(\gamma)|^2} \int_{|\beta|<1} d_2(\gamma-\beta) \frac{\beta^2 \Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

To handle with the last integral, we apply the bound for K_1^* from the estimation of J_3. Then, by taking the $L^2(\partial S_r)$ norm and using the same estimates used for $\mathcal{W}_6$, and the Minkowski's integral inequality, we infer

$$\left\| \int_{|\beta|<1} d_2(\gamma-\beta) \mathbf{w}_7(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \le c_R \|d_2\|_{L^2(\partial S_r)}.$$

The out part is bounded by considering (3.4.108). Thus

$$\|J_{6,12}\|_{L^2(\partial S_r)} \le c_R \|d_2\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,13}$: To estimate $J_{6,13}$, we decompose

$$\frac{1}{8} J_{6,13}(\gamma) = \partial_\alpha d_2(\gamma) \int_{\mathbb{R}} \mathbf{w}_8(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha d_2(\gamma-\beta) \mathbf{w}_8(\gamma,\beta)\,\mathrm{d}\beta \tag{3.4.109}$$

where

$$\mathbf{w}_8(\gamma,\beta) = \frac{\Delta z_1(\gamma,\beta) \partial_\alpha \Delta d_1(\gamma,\beta) \Delta p_2(\gamma,\beta)^2 \partial_\alpha \Delta p_2(\gamma,\beta)}{Q(\gamma,\beta)^3}.$$

We decompose the *in* part as follows

$$\begin{aligned}\int_{|\beta|<1}\mathbf{w}_8(\gamma,\beta)\,\mathrm{d}\beta &= \int_{|\beta|<1}\left(\frac{\beta^2\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^2}-\frac{4\gamma^2}{|\partial_\alpha\mathbf{z}(\gamma)|^4}\right)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta Q(\gamma,\beta)}\,\mathrm{d}\beta\\ &+\frac{4\gamma^2}{|\partial_\alpha\mathbf{z}(\gamma)|^4}\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta Q(\gamma,\beta)}\,\mathrm{d}\beta.\end{aligned}\tag{3.4.110}$$

Using the decomposition (3.4.61) we control the last integral. We consider (A6), (A5), (A2) and then using Corollary 3 we control the first integral in the right-hand side of (3.4.110). We obtain

$$\left|\int_{|\beta|<1}\mathbf{w}_8(\gamma,\beta)\,\mathrm{d}\beta\right|\leq c_R.$$

Regarding the *out* part we consider

$$\left|\mathbf{w}_8(\gamma,\beta)\right|_*\leq c_R|\beta|^{-4}.\tag{3.4.111}$$

Thus

$$\left|\int_{|\beta|>1}\mathbf{w}_8(\gamma,\beta)\,\mathrm{d}\beta\right|_*\leq\int_{|\beta|>1}\left|\mathbf{w}_8(\gamma,\beta)\right|_*\mathrm{d}\beta\leq c_R.$$

Regarding the second part of (3.4.109), for the *in* part we have the following decomposition

$$\begin{aligned}\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_8(\gamma,\beta)\,\mathrm{d}\beta &= \int_{|\beta|<1}\left(\frac{\beta^2\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^2}-\frac{4\gamma^2}{|\partial_\alpha\mathbf{z}(\gamma)|^4}\right)\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta Q(\gamma,\beta)}\,\mathrm{d}\beta\\ &+\frac{4\gamma^2}{|\partial_\alpha\mathbf{z}(\gamma)|^4}\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\frac{\beta\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{\beta Q(\gamma,\beta)}\,\mathrm{d}\beta.\end{aligned}$$

Then, using decomposition (3.4.65) we control the last integral and we infer the following

$$\left\|\int_{|\beta|<1}\partial_\alpha d_2(\gamma-\beta)\mathbf{w}_8(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\leq c_R\|\partial_\alpha d_2\|_{L^2(\partial S_r)}.$$

Finally the *out* part is bounded by considering the inequality (3.4.111), we deduce

$$\|J_{6,13}\|_{L^2(\partial S_r)}\leq c_R\|d_2\|_{L^2(\partial S_r)}.$$

Bound for $J_{6,14}$: Finally, to deal with $J_{6,14}$ we define

$$\mathbf{w}_9(\gamma,\beta)=\frac{\Delta z_1(\gamma,\beta)\Delta p_2(\gamma,\beta)^2\partial_\alpha\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}$$

then we get

$$\frac{1}{8}J_{6,14}(\gamma)=\partial_\alpha d_1(\gamma)\int_{\mathbb{R}}\mathbf{w}_9(\gamma,\beta)\,\mathrm{d}\beta-\int_{\mathbb{R}}\partial_\alpha d_1(\gamma-\beta)\mathbf{w}_9(\gamma,\beta)\,\mathrm{d}\beta.$$

Once again, we decompose the integral in the *in* and *out* part, thus

$$\begin{aligned}\int_{|\beta|<1}\mathbf{w}_9(\gamma,\beta)\,\mathrm{d}\beta &= \int_{|\beta|<1}\left(\frac{\beta^2\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^2}-\frac{4\gamma^2}{|\partial_\alpha\mathbf{z}(\gamma)|^4}\right)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\\ &+\frac{4\gamma^2}{|\partial_\alpha\mathbf{z}(\gamma)|^4}\int_{|\beta|<1}\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.\end{aligned}$$

We use the estimate (3.3.4) in Lemma 7 to control the last integral above. Then, by using estimates (A1), (A5), (A2), (3.4.90) and Corollary 3, we can conclude that

$$\left|\int_{|\beta|<1}\mathbf{w}_9(\gamma,\beta)\,\mathrm{d}\beta\right|_* \leq c_R.$$

For the *out* part, we decompose

$$\begin{aligned}&\int_{|\beta|>1}\mathbf{w}_9(\gamma,\beta)\,\mathrm{d}\beta\\&=\int_{|\beta|>1}\frac{\Delta d_1(\gamma,\beta)}{\beta^2}\frac{\beta^4\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\&+\int_{|\beta|>1}\frac{1}{\beta}\left(\frac{\beta^2}{Q(\gamma,\beta)}-\frac{\beta^2}{Q^{\mathbf{P}}(\gamma,\beta)}\right)\frac{\Delta p_2(\gamma,\beta)^2\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\&+\int_{|\beta|>1}\frac{1}{\beta}\frac{\Delta p_2(\gamma,\beta)^2}{Q^{\mathbf{P}}(\gamma,\beta)}\left(\frac{\beta^4}{Q(\gamma,\beta)^2}-\frac{\beta^4}{Q^{\mathbf{P}}(\gamma,\beta)^2}\right)\mathrm{d}\beta+\int_{|\beta|>1}\frac{1}{\beta}\frac{\beta^4\Delta p_2(\gamma,\beta)^2}{Q^{\mathbf{P}}(\gamma,\beta)^3}\,\mathrm{d}\beta.\end{aligned}$$

From estimates (A6), (A5), (A9) and (A2) together with Lemma 33, we deduce

$$\left|\int_{|\beta|>1}\mathbf{w}_9(\gamma,\beta)\,\mathrm{d}\beta\right|_* \leq c_R.$$

To estimate the second integral, we decompose

$$\begin{aligned}\int_{|\beta|<1}\partial_\alpha d_1(\gamma-\beta)\mathbf{w}_9(\gamma,\beta)\,\mathrm{d}\beta&=\int_{|\beta|<1}\partial_\alpha d_1(\gamma-\beta)\left(\frac{\beta^2\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^2}-\frac{4\gamma^2}{|\partial_\alpha\mathbf{z}(\gamma)|^4}\right)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\\&+\frac{4\gamma^2}{|\partial_\alpha\mathbf{z}(\gamma)|^4}\int_{|\beta|<1}\partial_\alpha d_1(\gamma-\beta)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.\end{aligned}$$

Then, the Minkowski's integral inequality yields to

$$\left\|\int_{|\beta|<1}\partial_\alpha d_1(\gamma-\beta)\mathbf{w}_9(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\leq c_R\|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

Regarding the *out* part, we have

$$\begin{aligned}\int_{|\beta|>1}\partial_\alpha d_1(\gamma-\beta)\mathbf{w}_9(\gamma,\beta)\,\mathrm{d}\beta&=\int_{|\beta|>1}\frac{\Delta d_1(\gamma,\beta)}{\beta^2}\frac{\beta^4\Delta p_2(\gamma,\beta)^2}{Q(\gamma,\beta)^3}\,\mathrm{d}\beta\\&+\int_{|\beta|>1}\frac{\partial_\alpha d_1(\gamma-\beta)}{\beta}\left(\frac{\beta^2}{Q(\gamma,\beta)}-\frac{\beta^2}{Q^{\mathbf{P}}(\gamma,\beta)}\right)\frac{\Delta p_2(\gamma,\beta)^2\beta^2}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta\\&+\int_{|\beta|>1}\frac{\partial_\alpha d_1(\gamma-\beta)}{\beta}\frac{\Delta p_2(\gamma,\beta)^2}{Q^{\mathbf{P}}(\gamma,\beta)}\left(\frac{\beta^4}{Q(\gamma,\beta)^2}-\frac{\beta^4}{Q^{\mathbf{P}}(\gamma,\beta)^2}\right)\mathrm{d}\beta\\&+\int_{|\beta|>1}\frac{\partial_\alpha d_1(\gamma-\beta)}{\beta}\frac{\beta^4\Delta p_2(\gamma,\beta)^2}{Q^{\mathbf{P}}(\gamma,\beta)^3}\,\mathrm{d}\beta.\end{aligned}$$

As before, we take the $L^2(\partial S_r)$ norm and apply the Minkowski's integral inequality in the first two terms, while for the last one, we proceed as in (3.3.15). Thus

$$\left\|\int_{|\beta|>1}\partial_\alpha d_1(\gamma-\beta)\mathbf{w}_9(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\leq c_R\|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

Hence, we conclude that

$$\|J_{6,14}\|_{L^2(\partial S_r)}\leq c_R\|\partial_\alpha d_1\|_{L^2(\partial S_r)}.$$

By combining all the $L^2(\partial S_r)$ estimates for $J_{6,i}$ we obtain that

$$\|\mathrm{J}_6\|_{L^2(\partial S_r)}\leq c_R\|\mathbf{d}\|_{L^2(\partial S_r)}+c_R\|\partial_\alpha\mathbf{d}\|_{L^2(\partial S_r)},\quad\text{where}\quad \mathbf{d}=(d_1,d_2). \tag{B10}$$

Then, by collecting estimates (B5), (B6), (B7), (B8), (B9) and (B10), we complete the proof. ■

3.4.2 Boundedness of $\partial_\alpha^2 F_2$

Lemma 10. *Given a deviation $\mathbf{d}\in O_R$, the following estimate holds*

$$\|\partial_\alpha^2 F_2(\mathbf{d})\|_{L^2(\partial S_r)} \leq c_R\left[\|\mathbf{d}\|_{L^2(\partial S_r)} + \|\partial_\alpha \mathbf{d}\|_{L^2(\partial S_r)} + \|\partial_\alpha^2 \mathbf{d}\|_{L^2(\partial S_r)} + \|\partial_\alpha^3 d_2\|_{L^2(\partial S_r)}\right]. \tag{B11}$$

Proof. Now we deal with the second order derivative of F_2. This derivative is given by

$$\partial_\alpha^2 F_2(\mathbf{d})(\gamma) = \partial_\alpha^2\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\partial_\alpha\Delta d_2(\gamma,\beta)\,\mathrm{d}\beta + 2\partial_\alpha^2\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\beta}{Q(\gamma,\beta)}\,\mathrm{d}\beta. \tag{3.4.112}$$

As in the lower derivative, the first integral in (3.4.112) has a similar structure to the first coordinate of the operator given in (3.2.4). Thus, we can deduce from the estimate for $\partial_\alpha^2 F_1(\mathbf{d})$ the following

$$\begin{aligned}&\left\|\partial_\alpha^2\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\\ &\qquad\leq c_R\Big[\|\mathbf{d}\|_{L^2(\partial S_r)} + \|\partial_\alpha \mathbf{d}\|_{L^2(\partial S_r)} + \|\partial_\alpha^2 \mathbf{d}\|_{L^2(\partial S_r)} + \|\partial_\alpha^3 d_2\|_{L^2(\partial S_r)}\Big].\end{aligned}$$

Now, our task will be to obtain an estimate for the second integral in (3.4.112). We decompose

$$\partial_\alpha^2\int_{\mathbb{R}}\frac{\Delta z_1(\gamma,\beta)\beta}{Q(\gamma,\beta)}\,\mathrm{d}\beta = (\mathrm{J}_7+\mathrm{J}_8+\mathrm{J}_9+\mathrm{J}_{10})(\gamma)$$

for

$$\begin{aligned}\mathrm{J}_7(\gamma) &:= \int_{\mathbb{R}}\beta\frac{\partial_\alpha^2\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta,\\ \mathrm{J}_8(\gamma) &:= -2\int_{\mathbb{R}}\beta\frac{\partial_\alpha\Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta,\\ \mathrm{J}_9(\gamma) &:= -\int_{\mathbb{R}}\beta\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2}\partial_\alpha^2 Q(\gamma,\beta)\,\mathrm{d}\beta,\\ \mathrm{J}_{10}(\gamma) &:= 2\int_{\mathbb{R}}\beta\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^3}\partial_\alpha Q(\gamma,\beta)^2\,\mathrm{d}\beta.\end{aligned} \tag{3.4.113}$$

For the first term J_7, we expand

$$\partial_\alpha^2\Delta d_1(\gamma,\beta) = \partial_\alpha^2 d_1(\gamma) - \partial_\alpha^2 d_1(\gamma-\beta).$$

Then

$$\mathrm{J}_7(\gamma) = \partial_\alpha^2 d_1(\gamma)\int_{\mathbb{R}}\frac{\beta}{Q(\gamma,\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha^2 d_1(\gamma-\beta)\frac{\beta}{Q(\gamma,\beta)}\,\mathrm{d}\beta. \tag{3.4.114}$$

In order to obtain an $L^2(\partial S_r)$ estimate, first we need an $L^\infty(\partial S_r)$ of the first integral. This estimate can be deduced from Lemma 7. We derive

$$\left|\int_{\mathbb{R}}\frac{\beta}{Q(\gamma,\beta)}\,\mathrm{d}\beta\right|_* \leq c_R.$$

To deal with second part of (3.4.114) we have

$$\begin{aligned}\int_{|\beta|<1}\frac{\partial_\alpha^2 d_1(\gamma-\beta)}{\beta}\frac{\beta^2}{Q(\gamma,\beta)}\,\mathrm{d}\beta &= \int_{|\beta|<1}\frac{\partial_\alpha^2 d_1(\gamma-\beta)}{\beta}\left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma)|^2}\right)\mathrm{d}\beta\\ &\quad + \frac{1}{|\partial_\alpha \mathbf{z}(\gamma)|^2}H_{|\beta|<1}\partial_\alpha^2 d_1(\gamma).\end{aligned}$$

Then by taking the $L^2(\partial S_r)$ norm and making use of the Minkowski's integral inequality, we get

$$\left\| \int_{|\beta|<1} \frac{\partial_\alpha^2 d_1(\gamma-\beta)}{\beta} \frac{\beta^2}{Q(\gamma,\beta)} \,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.$$

Regarding the *out*, we have the following decomposition, then

$$\begin{aligned}\int_{|\beta|>1} \frac{\partial_\alpha^2 d_1(\gamma-\beta)}{\beta} \frac{\beta^2}{Q(\gamma,\beta)} \,\mathrm{d}\beta &= \int_{|\beta|<1} \frac{\partial_\alpha^2 d_1(\gamma-\beta)}{\beta} \left(\frac{\beta^2}{Q(\gamma,\beta)} - \frac{\beta^2}{Q^{\mathbf{P}}(\gamma,\beta)} \right) \mathrm{d}\beta \\ &\quad + \int_{|\beta|>1} \frac{\partial_\alpha^2 d_1(\gamma-\beta)}{\beta} \frac{\beta^2}{Q^{\mathbf{P}}(\gamma,\beta)} \,\mathrm{d}\beta.\end{aligned}$$

We deduce that

$$\left\| \int_{|\beta|<1} \frac{\partial_\alpha^2 d_1(\gamma-\beta)}{\beta} \frac{\beta^2}{Q(\gamma,\beta)} \,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}.$$

Hence

$$\|\mathrm{J}_7\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha^2 d_1\|_{L^2(\partial S_r)}. \tag{B12}$$

To estimate J_8, we expand $\partial_\alpha \Delta d_1(\gamma,\beta)$, thus

$$-\frac{1}{2}\mathrm{J}_8(\gamma) = \partial_\alpha d_1(\gamma) \int_{\mathbb{R}} \frac{\beta}{Q(\gamma,\beta)} \partial_\alpha Q(\gamma,\beta) \,\mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha d_1(\gamma-\beta) \frac{\beta}{Q(\gamma,\beta)} \partial_\alpha Q(\gamma,\beta) \,\mathrm{d}\beta. \tag{3.4.115}$$

The kernel in (3.4.115) has the same structure as in (3.4.12), which is the decomposition for J_3. Therefore by replacing $\Delta z_1(\gamma,\beta)$ with β in the estimation for J_3, we can deduce

$$\|\mathrm{J}_8\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha d_1\|_{L^2(\partial S_r)}. \tag{B13}$$

For the next term J_9, we note that J_5 has a similar kernel, then we can deduce

$$\|\mathrm{J}_9\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha \mathbf{d}\|_{L^2(\partial S_r)} + c_R \|\partial_\alpha^2 \mathbf{d}\|_{L^2(\partial S_r)}. \tag{B14}$$

To estimate the final term J_{10}, we use the same technique used in J_6. We obtain the following bound

$$\|\mathrm{J}_{10}\|_{L^2(\partial S_r)} \le c_R \|\mathbf{d}\|_{L^2(\partial S_r)} + c_R \|\partial_\alpha \mathbf{d}\|_{L^2(\partial S_r)}. \tag{B15}$$

By combining the $L^2(\partial S_r)$ estimates (B12), (B13), (B14) and (B15) we deduce the following bound

$$\left\| \partial_\alpha^2 \int_{\mathbb{R}} \frac{\beta \Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)} \,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \le c_R \|\partial_\alpha \mathbf{d}\|_{L^2(\partial S_r)} + c_R \|\partial_\alpha^2 \mathbf{d}\|_{L^2(\partial S_r)}.$$

The last inequality together with the bound for the first integral of (3.4.112) concludes the proof. ■

From the previous two lemmas, estimates (B4) and (B11), we infer

$$\begin{aligned}\|\partial_\alpha^2 \mathbf{F}(\mathbf{d})\|_{L^2(\partial S_r)} &\le c_R \left[\|\partial_\alpha^2 F_1(\mathbf{d})\|_{L^2(\partial S_r)} + \|\partial_\alpha^2 F_2(\mathbf{d})\|_{L^2(\partial S_r)} \right] \\ &\le c_R \|\mathbf{d}\|_{X_{r,3}},\end{aligned} \tag{3.4.116}$$

which is the desired estimate of the section. Thus using the property (3.4.1) we conclude

$$\|\partial_\alpha^3 \mathbf{F}(\mathbf{d})\|_{L^2(\partial S_{r'})} \le \frac{c_R}{r-r'} \|\mathbf{d}\|_{X_{r,3}}.$$

Now, we combine the last inequality with the estimate for the lower order term (B3). We deduce

$$\|\mathbf{F}(\mathbf{d})\|_{X_{r',3}} \le \frac{c_R}{r-r'} \|\mathbf{d}\|_{X_{r,3}}$$

which is the boundedness condition (3.2.2) with $k=3$ of the main Lemma 6.

3.5 Lipschitz Condition of $\mathbf{F}$

The second part of Lemma 6 follows similar steps from the first part, now for the difference $\mathbf{d}^1 - \mathbf{d}^2$, where $\mathbf{d}^1, \mathbf{d}^2 \in O_R$ are two deviations. We will prove this property as a consequence of four lemmas. Throughout this section, we denote, for $i = 1, 2$

$$\begin{aligned}
&Q_i(\gamma', \beta) = |\mathbf{z}^i(\gamma') - \mathbf{z}^i(\gamma' - \beta)|^2, \\
&\mathbf{z}^i(\gamma', \beta) = \mathbf{d}^i(\gamma', \beta) + \mathbf{p}^i(\gamma', \beta), \\
&\Delta z_i^1(\gamma', \beta) - \Delta z_i^2(\gamma', \beta) = \Delta(\mathbf{z}^1 - \mathbf{z}^2)_i(\gamma', \beta), \\
&\Delta d_i^1(\gamma', \beta) - \Delta d_i^2(\gamma', \beta) = \Delta(\mathbf{d}^1 - \mathbf{d}^2)_i(\gamma', \beta).
\end{aligned}$$

We omit the principal value notation PV in some integrals, but all of them should be understood in that sense.

3.5.1 Lipschitz Condition of F_1

We start by establishing an $L^2(\partial S_{r'})$ bound for the difference of the lower order term. We have the following lemma.

Lemma 11. *Given two deviations* $\mathbf{d}^1, \mathbf{d}^2 \in O_R$, *the following inequality holds*

$$\|F_1(\mathbf{d}^1) - F_1(\mathbf{d}^2)\|_{L^2(\partial S_{r'})} \leq c_R\Big[\|\mathbf{d}^1 - \mathbf{d}^2\|_{L^2(\partial S_{r'})} + \|\partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)_1)\|_{L^2(\partial S_{r'})}\Big]. \tag{L1}$$

Proof. We decompose the difference by adding and subtracting mixed terms. We obtain

$$\begin{aligned}
&F_1(\mathbf{d}^1)(\gamma') - F_1(\mathbf{d}^2)(\gamma') \\
&\qquad = \int_{\mathbb{R}} \frac{\Delta z_1^1(\gamma, \beta)}{Q_1(\gamma, \beta)} \partial_\alpha \Delta d_1^1(\gamma, \beta)\, \mathrm{d}\beta - \int_{\mathbb{R}} \frac{\Delta z_1^2(\gamma, \beta)}{Q_2(\gamma, \beta)} \partial_\alpha \Delta d_1^2(\gamma, \beta)\, \mathrm{d}\beta \\
&\qquad = D_1(\gamma') + D_2(\gamma') + D_3(\gamma')
\end{aligned} \tag{3.5.1}$$

for

$$\begin{aligned}
D_1(\gamma') &:= \int_{\mathbb{R}} \partial_\alpha \Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma, \beta) \frac{\Delta z_1^1(\gamma', \beta)}{Q_1(\gamma', \beta)}\, \mathrm{d}\beta, \\
D_2(\gamma') &:= \int_{\mathbb{R}} \Delta(\mathbf{z}^1 - \mathbf{z}^2)_1(\gamma', \beta) \frac{\partial_\alpha \Delta d_1^2(\gamma', \beta)}{Q_1(\gamma', \beta)}\, \mathrm{d}\beta, \\
D_3(\gamma') &:= \int_{\mathbb{R}} \partial_\alpha \Delta d_1^2(\gamma', \beta) \Delta z_1^2(\gamma', \beta) \left[\frac{1}{Q_1(\gamma', \beta)} - \frac{1}{Q_2(\gamma', \beta)}\right] \mathrm{d}\beta.
\end{aligned}$$

Bound for D_1: To estimate $D_1(\gamma')$, we expand $\partial_\alpha \Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma', \beta)$, we get

$$D_1(\gamma') = \partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma') \int_{\mathbb{R}} \frac{\Delta z_1^1(\gamma', \beta)}{Q_1(\gamma', \beta)}\, \mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma' - \beta) \frac{\Delta z_1^1(\gamma', \beta)}{Q_1(\gamma', \beta)}\, \mathrm{d}\beta. \tag{3.5.2}$$

We observe in (3.5.2), that the integrals has the same kernel as in J_1, see equation (3.4.3). Thus we can deduce the following $L^2(\partial S_{r'})$ bound

$$\|D_1\|_{L^2(\partial S_{r'})} \leq c_R \|\partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_{r'})}. \tag{D1}$$

Bound for D_2: We notice that

$$\Delta(\mathbf{z}^1 - \mathbf{z}^2)_1(\gamma', \beta) = \Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma, \beta).$$

Then we have the same situation as in D_1, expanding $\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma',\beta)$, we get

$$D_2(\gamma')=(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma')\int_{\mathbb{R}}\frac{\partial_\alpha\Delta d_1^2(\gamma',\beta)}{Q_1(\gamma',\beta)}\,\mathrm{d}\beta-\int_{\mathbb{R}}(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma'-\beta)\frac{\partial_\alpha\Delta d_1^2(\gamma',\beta)}{Q_1(\gamma',\beta)}\,\mathrm{d}\beta.$$

Notice that, the kernels is similar to the kernel in (3.4.4). Therefore, we can deduce the following bound

$$\|D_2\|_{L^2(\partial S_{r'})}\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_{r'})}. \tag{D2}$$

Bound for D_3: In D_3 we compute the difference to obtain an explicit $\Delta(\mathbf{d}^1-\mathbf{d}^2)_i(\gamma,\beta)$ term. We have

$$\begin{aligned}\frac{1}{Q_1(\gamma',\beta)}-\frac{1}{Q_2(\gamma',\beta)}&=\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma',\beta)\frac{\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\\&\quad+\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma',\beta)\frac{\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}.\end{aligned}\tag{3.5.3}$$

Then we obtain the next decomposition

$$D_3(\gamma')=D_{3,1}(\gamma')+D_{3,2}(\gamma')$$

for

$$\begin{aligned}D_{3,1}(\gamma')&:=\int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma',\beta)\mathbf{S}_1(\gamma',\beta)\,\mathrm{d}\beta,\\D_{3,2}(\gamma')&:=\int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma',\beta)\mathbf{S}_2(\gamma',\beta)\,\mathrm{d}\beta,\end{aligned}$$

where $\mathbf{S}_i(\gamma',\beta)$ are the following kernels

$$\mathbf{S}_i(\gamma',\beta):=\frac{\partial_\alpha\Delta d_1^2(\gamma',\beta)\Delta z_1^2(\gamma',\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_i(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)},\quad i=1,2.$$

To estimate $D_{3,1}(\gamma)$ we expand $\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)$. We get

$$D_{3,1}(\gamma)=(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma')\int_{\mathbb{R}}\mathbf{S}_1(\gamma',\beta)\,\mathrm{d}\beta-\int_{\mathbb{R}}(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma'-\beta)\mathbf{S}_1(\gamma',\beta)\,\mathrm{d}\beta. \tag{3.5.4}$$

We require an $L^\infty(\partial S_{r'})$ bound for the first integral in (3.5.4). For the *in* part we have

$$\begin{aligned}\int_{|\beta|<1}\mathbf{S}_1(\gamma',\beta)\,\mathrm{d}\beta&=\int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1^2(\gamma',\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma')\right)\frac{\beta\Delta z_1^2(\gamma',\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta\\&\quad+\partial_\alpha^2 d_1^2(\gamma')\int_{|\beta|<1}\frac{\beta\Delta z_1^2(\gamma',\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta,\end{aligned}$$

where the last integral decomposes as follows

$$\begin{aligned}&\int_{|\beta|<1}\frac{\beta\Delta z_1^2(\gamma',\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta\\&\qquad=\int_{|\beta|<1}\left(\frac{\Delta z_1^2(\gamma',\beta)}{\beta}-\partial_\alpha z_1^2(\gamma')\right)\frac{\beta^2\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta.\\&\qquad\quad+\partial_\alpha z_1^2(\gamma')\int_{|\beta|<1}\frac{\beta^2\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta.\end{aligned}\tag{3.5.5}$$

Moreover, we decompose the last integral above as follows

$$\begin{aligned}\int_{|\beta|<1} &\frac{\beta^2\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta \\ &= \int_{|\beta|<1}\left(\frac{\Delta(\mathbf{z}^2+\mathbf{z}^1)_1(\gamma',\beta)}{\beta}-\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma')\right)\frac{\beta^3}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta \\ &\quad+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma')\int_{|\beta|<1}\frac{\beta^3}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta.\end{aligned} \tag{3.5.6}$$

And for the last integral above, we have

$$\begin{aligned}\int_{|\beta|<1}\frac{\beta^3}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta &= \int_{|\beta|<1}\left(\frac{\beta^2}{Q_1(\gamma',\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^1(\gamma')|^2}\right)\frac{\beta}{Q_2(\gamma',\beta)}\,\mathrm{d}\beta \\ &\quad+\frac{1}{|\partial_\alpha\mathbf{z}^1(\gamma')|^2}\int_{|\beta|<1}\frac{1}{\beta}\left(\frac{\beta^2}{Q_1(\gamma',\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^2(\gamma')|^2}\right)\mathrm{d}\beta.\end{aligned}$$

From estimates (A1), (A4), (A5), (A2), and Lemma 32, we can deduce the following

$$\left|\int_{|\beta|<1}\mathbf{S}_1(\gamma',\beta)\,\mathrm{d}\beta\right|_*\le c_R. \tag{3.5.7}$$

Regarding the out part, we use (A6), (A5), (A2) to obtain the following bound

$$\left|\mathbf{S}_1(\gamma',\beta)\right|_*\le c_R|\beta|^{-2}. \tag{3.5.8}$$

Thus

$$\left|\int_{|\beta|>1}\mathbf{S}_1(\gamma',\beta)\,\mathrm{d}\beta\right|_*\le\int_{|\beta|>1}\left|\mathbf{S}_1(\gamma',\beta)\right|_*\mathrm{d}\beta\le c_R\int_{|\beta|>1}|\beta|^{-2}\,\mathrm{d}\beta. \tag{3.5.9}$$

By joining the estimates (3.5.7) and (3.5.9), we get

$$\left|\int_{\mathbb{R}}\mathbf{S}_1(\gamma',\beta)\,\mathrm{d}\beta\right|_*\le c_R.$$

Now, by taking the $L^2(\partial S_{r'})$ norm, we get obtain that

$$\begin{aligned}&\left\|(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma')\int_{\mathbb{R}}\mathbf{S}_1(\gamma',\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_{r'})} \\ &\quad\le\left\|\int_{\mathbb{R}}\mathbf{S}_1(\gamma',\beta)\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_{r'})}\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_{r'})} \\ &\quad\le\left\{\left\|\int_{\mathbb{R}}\mathbf{S}_{1,1}(\gamma',\beta)\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_{r'})}+\left\|\int_{\mathbb{R}}\mathbf{S}_{1,2}(\gamma',\beta)\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_{r'})}\right\}\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_{r'})} \\ &\quad\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_{r'})}.\end{aligned} \tag{3.5.10}$$

In the second integral of (3.5.4), we use a similar decomposition and we apply the Minkowski's integral inequality. For the *in* part, we have

$$\begin{aligned}\int_{|\beta|<1}&(\mathbf{d}^2-\mathbf{d}^1)(\gamma'-\beta)_1\mathbf{S}_1(\gamma',\beta)\,\mathrm{d}\beta \\ &=\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)(\gamma'-\beta)_1\mathbf{S}_1\left(\frac{\partial_\alpha\Delta d_1^2(\gamma',\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma')\right)\frac{\beta\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta \\ &+\partial_\alpha^2 d_1^2(\gamma')\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)(\gamma'-\beta)_1\frac{\beta\Delta z_1^2(\gamma',\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}\,\mathrm{d}\beta.\end{aligned}$$

Where the last integral decomposes as follows

$$\begin{aligned}
&\int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta) \frac{\beta \Delta z_1^2(\gamma', \beta) \Delta(\mathbf{z}^1 + \mathbf{z}^2)_1(\gamma', \beta)}{Q_1(\gamma', \beta) Q_2(\gamma', \beta)} \,\mathrm{d}\beta \\
&\qquad = \int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta) \left(\frac{\Delta z_1^2(\gamma', \beta)}{\beta} - \partial_\alpha z_1^2(\gamma') \right) \frac{\beta^2 \Delta(\mathbf{z}^1 + \mathbf{z}^2)_1(\gamma', \beta)}{Q_1(\gamma', \beta) Q_2(\gamma', \beta)} \,\mathrm{d}\beta. \\
&\qquad + \partial_\alpha z_1^2(\gamma') \int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta) \frac{\beta^2 \Delta(\mathbf{z}^1 + \mathbf{z}^2)_1(\gamma', \beta)}{Q_1(\gamma', \beta) Q_2(\gamma', \beta)} \,\mathrm{d}\beta.
\end{aligned} \tag{3.5.11}$$

Again, we decompose the last integral above as follows

$$\begin{aligned}
&\int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta) \frac{\beta^2 \Delta(\mathbf{z}^1 + \mathbf{z}^2)_1(\gamma', \beta)}{Q_1(\gamma', \beta) Q_2(\gamma', \beta)} \,\mathrm{d}\beta \\
&= \int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta) \left(\frac{\Delta(\mathbf{z}^2 + \mathbf{z}^1)_1(\gamma', \beta)}{\beta} - \partial_\alpha(\mathbf{z}^1 + \mathbf{z}^2)_1(\gamma') \right) \frac{\beta^3}{Q_1(\gamma', \beta) Q_2(\gamma', \beta)} \,\mathrm{d}\beta. \\
&+ \partial_\alpha(\mathbf{z}^1 + \mathbf{z}^2)_1(\gamma') \int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta) \frac{\beta^3}{Q_1(\gamma', \beta) Q_2(\gamma', \beta)} \,\mathrm{d}\beta,
\end{aligned} \tag{3.5.12}$$

and for the last integral we have

$$\begin{aligned}
&\int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta) \frac{\beta^3}{Q_1(\gamma', \beta) Q_2(\gamma', \beta)} \,\mathrm{d}\beta \\
&\qquad = \int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta) \left(\frac{\beta^2}{Q_1(\gamma', \beta)} - \frac{1}{|\partial_\alpha \mathbf{z}^1(\gamma')|^2} \right) \frac{\beta}{Q_2(\gamma', \beta)} \,\mathrm{d}\beta \\
&\qquad + \frac{1}{|\partial_\alpha \mathbf{z}^1(\gamma')|^2} \int_{|\beta|<1} \frac{(\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta)}{\beta} \left(\frac{\beta^2}{Q_2(\gamma', \beta)} - \frac{1}{|\partial_\alpha \mathbf{z}^2(\gamma', \beta)|^2} \right) \mathrm{d}\beta \\
&\qquad + \frac{H_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta)}{|\partial_\alpha \mathbf{z}^1(\gamma')|^2 |\partial_\alpha \mathbf{z}^2(\gamma')|^2}.
\end{aligned}$$

From (A1), (A4), (A5), (A2), and Lemma 32, together with the Minkowski's integral inequality, we obtain the following estimate

$$\left\| \int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta) \mathbf{S}_1(\gamma', \beta) \,\mathrm{d}\beta \right\|_{L^2(\partial S_{r'})} \le c_R \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_{r'})}. \tag{3.5.13}$$

Regarding the *out* part, we use the bound (3.5.8) and the Minkowski's integral inequality. Then we can deduce that

$$\begin{aligned}
\left\| \int_{|\beta|>1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma' - \beta) \mathbf{S}_1(\gamma', \beta) \,\mathrm{d}\beta \right\|_{L^2(\partial S_{r'})} &\le c_R \int_{|\beta|>1} |\beta|^{-2} \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_{r'})} \,\mathrm{d}\beta \\
&\le c_R \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_{r'})} \,\mathrm{d}\beta.
\end{aligned} \tag{3.5.14}$$

From estimates (3.5.10), (3.5.13) and (3.5.14), we conclude that

$$\|D_{3,1}\|_{L^2(\partial S_{r'})} \le c_R \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_{r'})}. \tag{3.5.15}$$

To estimate $D_{3,2}$, we expand

$$\Delta(\mathbf{z}^1 + \mathbf{z}^2)_2(\gamma', \beta) = \Delta(\mathbf{d}^1 + \mathbf{d}^2)_2(\gamma, \beta) + 2\beta(2\gamma' - \beta).$$

Then we rewrite

$$D_{3,2}(\gamma') := \int_{\mathbb{R}} \Delta(\mathbf{d}^2 - \mathbf{d}^1)_2(\gamma',\beta)\mathbf{S}_{2,1}(\gamma',\beta)\,\mathrm{d}\beta + \int_{\mathbb{R}} \Delta(\mathbf{d}^2 - \mathbf{d}^1)_2(\gamma',\beta)\mathbf{S}_{2,2}(\gamma',\beta)\,\mathrm{d}\beta \tag{3.5.16}$$

for

$$\mathbf{S}_{2,1}(\gamma',\beta) := \frac{\partial_\alpha \Delta d_1^2(\gamma',\beta)\Delta z_1^2(\gamma',\beta)\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)},$$
$$\mathbf{S}_{2,2}(\gamma',\beta) := 2\frac{\partial_\alpha \Delta d_1^2(\gamma',\beta)\Delta z_1^2(\gamma',\beta)\beta(2\gamma'-\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}.$$

The estimate for the first integral in $D_{3,2}$ follows the same lines as the estimate of $D_{3,1}$, by replacing $\Delta(\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma',\beta)$ and $\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma',\beta)$ by $\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma',\beta)$ and $\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma',\beta)$. We infer

$$\left\| \int_{\mathbb{R}} \Delta(\mathbf{d}^2 - \mathbf{d}^1)_2(\gamma',\beta)\mathbf{S}_{2,1}(\gamma',\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_{r'})} \le c_R \|(\mathbf{d}^2-\mathbf{d}^1)_2\|_{L^2(\partial S_{r'})}.$$

Next, we deal with the second integral. We expand $\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)$ and obtain the following

$$\begin{aligned}\int_{\mathbb{R}} &\Delta(\mathbf{d}^2 - \mathbf{d}^1)_2(\gamma',\beta)\mathbf{S}_{2,2}(\gamma',\beta)\,\mathrm{d}\beta \\ &= (\mathbf{d}^2-\mathbf{d}^1)_2(\gamma')\int_{\mathbb{R}} \mathbf{S}_{2,2}(\gamma',\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}} (\mathbf{d}^2-\mathbf{d}^1)_2(\gamma'-\beta)\mathbf{S}_{2,2}(\gamma',\beta)\,\mathrm{d}\beta.\end{aligned} \tag{3.5.17}$$

Now, we will estimate the first integral of the right-hand side above. For the *in* part we have

$$\begin{aligned}\frac{1}{2}\int_{|\beta|<1} \mathbf{S}_{2,2}(\gamma',\beta)\,\mathrm{d}\beta = &\int_{|\beta|<1} \left(\frac{\beta^2(2\gamma'-\beta)}{Q_1(\gamma',\beta)} - \frac{2\gamma'}{|\partial_\alpha \mathbf{z}^1(\gamma')|^2}\right)\frac{\Delta z_1^2(\gamma',\beta)\partial_\alpha\Delta d_1^2(\gamma',\beta)}{\beta Q_2(\gamma',\beta)}\,\mathrm{d}\beta \\ &+ \frac{2\gamma'}{|\partial_\alpha \mathbf{z}^1(\gamma')|^2}\int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1^2(\gamma',\beta)}{\beta} - \partial_\alpha^2 d_1^2(\gamma')\right)\frac{\Delta z_1^2(\gamma',\beta)}{Q_2(\gamma',\beta)}\,\mathrm{d}\beta \\ &+ \frac{2\gamma'\partial_\alpha^2 d_1^2(\gamma')}{|\partial_\alpha \mathbf{z}^1(\gamma')|^2}\int_{|\beta|<1}\frac{\Delta z_1^2(\gamma',\beta)}{Q_2(\gamma',\beta)}\,\mathrm{d}\beta.\end{aligned}$$

We observe that the last integral is bounded in the same manner as I_1 in Lemma 7. From estimates (A6), (A4), (A5), (A2), and Corollary 2, we deduce

$$\left| \int_{|\beta|<1} \mathbf{S}_{2,2}(\gamma',\beta)\,\mathrm{d}\beta \right|_* \le c_R.$$

Regarding the *out* part, by using estimates (A6), (A5), and (A2), we obtain the following bound

$$\left|\mathbf{S}_{2,2}(\gamma',\beta)\right|_* \le c_R|\beta|^{-2}. \tag{3.5.18}$$

Hence

$$\left| \int_{|\beta|>1} \mathbf{S}_{2,2}(\gamma',\beta)\,\mathrm{d}\beta \right|_* \le \int_{|\beta|>1} \left|\mathbf{S}_{2,2}(\gamma',\beta)\right|_*\,\mathrm{d}\beta \le c_R.$$

Now, taking the $L^2(\partial S_{r'})$ norm and considering the two previous estimates together with the inequality (3.5.1), we derive

$$\left\|(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma')\int_{\mathbb{R}} \mathbf{S}_{2,2}(\gamma',\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_{r'})} \le c_R\|(\mathbf{d}^2-\mathbf{d}^1)_2\|_{L^2(\partial S_{r'})}.$$

In order to obtain an estimate for the second part in (3.5.17), we decompose the *in* part as follows

$$\begin{aligned}
&\frac{1}{2}\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma'-\beta)\mathbf{S}_{2,2}(\gamma',\beta)\,\mathrm{d}\beta\\
&\qquad=\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma'-\beta)\left(\frac{\beta^2(2\gamma'-\beta)}{Q_1(\gamma',\beta)}-\frac{2\gamma'}{|\partial_\alpha\mathbf{z}^1(\gamma')|^2}\right)\frac{\Delta z_1^2(\gamma',\beta)\partial_\alpha\Delta d_1^2(\gamma',\beta)}{\beta Q_2(\gamma',\beta)}\,\mathrm{d}\beta\\
&\qquad+\frac{2\gamma'}{|\partial_\alpha\mathbf{z}^1(\gamma')|^2}\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma'-\beta)\left(\frac{\partial_\alpha\Delta d_1^2(\gamma',\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma')\right)\frac{\Delta z_1^2(\gamma',\beta)}{Q_2(\gamma',\beta)}\,\mathrm{d}\beta\\
&\qquad+\frac{2\gamma'\partial_\alpha^2 d_1^2(\gamma')}{|\partial_\alpha\mathbf{z}^1(\gamma')|^2}\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma'-\beta)\frac{\Delta z_1^2(\gamma',\beta)}{Q_2(\gamma',\beta)}\,\mathrm{d}\beta.
\end{aligned}$$

We notice that, using the estimates of I_2 in Lemma 7, we control the last integral above. For the remaining terms, we take the $L^2(\partial S_{r'})$ norm and consider estimates (A6), (A4), (A5), (A2) together with Corollary 2. Then making use of the Minkowski's integral inequality, we infer

$$\left\|\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma'-\beta)\mathbf{S}_{2,2}(\gamma',\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_{r'})}\le c_R\|(\mathbf{d}^2-\mathbf{d}^1)_2\|_{L^2(\partial S_{r'})}.$$

Finally by using the estimate (3.5.18), we control the *out* part, and hence

$$\|D_{3,2}\|_{L^2(\partial S_{r'})}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_{r'})}.$$

We combine the above inequality with inequality (3.5.15), then we deduce

$$\|D_3\|_{L^2(\partial S_{r'})}\le c_R\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_{r'})}. \tag{D3}$$

Combining the estimates (D1), (D2) and (D3) we obtain the desired estimate (L1). ■

3.5.2 Lipschitz Condition of F_2

Lemma 12. *Given two deviations* $\mathbf{d}^1,\mathbf{d}^2\in O_R$, *the following inequality holds*

$$\|F_2(\mathbf{d}^1)-F_2(\mathbf{d}^2)\|_{L^2(\partial S_{r'})}\le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_{r'})}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_{r'})}\Big]. \tag{L2}$$

Proof. For the second coordinate F_2, we have the following decomposition

$$F_2(\mathbf{d}^1)(\gamma')-F_2(\mathbf{d}^2)(\gamma')=\mathcal{D}_1(\gamma)+\mathcal{D}_2(\gamma)+\mathcal{D}_3(\gamma)+\mathcal{D}_4(\gamma),$$

for

$$\begin{aligned}
\mathcal{D}_1(\gamma)&:=\int_{\mathbb{R}}\partial_\alpha\Delta d_2^1(\gamma',\beta)\frac{\Delta z_1^1(\gamma',\beta)}{Q_1(\gamma',\beta)}\,\mathrm{d}\beta-\int_{\mathbb{R}}\partial_\alpha\Delta d_2^2(\gamma',\beta)\frac{\Delta z_1^2(\gamma',\beta)}{Q_2(\gamma',\beta)}\,\mathrm{d}\beta,\\
\mathcal{D}_2(\gamma)&:=2\int_{\mathbb{R}}\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma',\beta)\beta}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta,\\
\mathcal{D}_3(\gamma)&:=2\int_{\mathbb{R}}\Delta d_1^2(\gamma',\beta)\beta\left[\frac{1}{Q_1(\gamma',\beta)}-\frac{1}{Q_2(\gamma',\beta)}\right]\mathrm{d}\beta,\\
\mathcal{D}_4(\gamma)&:=\int_{\mathbb{R}}\beta^2\left[\frac{1}{Q_1(\gamma',\beta)}-\frac{1}{Q_2(\gamma',\beta)}\right]\mathrm{d}\beta.
\end{aligned}$$

Bound for $\mathcal{D}_1$: To estimate $\mathcal{D}_1(\gamma)$, we follow the same lines as the estimate of the first coordinate F_1 by replacing $\partial_\alpha \Delta d_1^i(\gamma',\beta)$ by $\partial_\alpha \Delta d_2^i(\gamma',\beta)$ in the equation (3.5.1). Then we can deduce

$$\|\mathcal{D}_1\|_{L^2(\partial S_{r'})} \leq c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_{r'})} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_{r'})}\Big]. \tag{D4}$$

Bound for $\mathcal{D}_2$: By expanding $\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$, we get

$$\frac{1}{2}\mathcal{D}_2(\gamma) = (\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{\mathbb{R}}\frac{\beta}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}}(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\frac{\beta}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta.$$

We notice that we have a similar kernels as in equation (3.3.18), thus the $L^2(\partial S_{r'})$ bound is automatic

$$\|\mathcal{D}_2\|_{L^2(\partial S_{r'})} \leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_{r'})}. \tag{D5}$$

Bound for $\mathcal{D}_3$: To estimate $\mathcal{D}_3(\gamma')$, we use the difference (3.5.3), to obtain the next decomposition

$$\mathcal{D}_3(\gamma') = \mathcal{D}_{3,1}(\gamma') + \mathcal{D}_{3,2}(\gamma')$$

for

$$\mathcal{D}_{3,1}(\gamma') := \int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma',\beta)\mathcal{S}_1(\gamma',\beta)\,\mathrm{d}\beta,$$
$$\mathcal{D}_{3,2}(\gamma') := \int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma',\beta)\mathcal{S}_2(\gamma',\beta)\,\mathrm{d}\beta,$$

where the kernels are given by

$$\mathcal{S}_i(\gamma',\beta) := 2\frac{\Delta d_1^2(\gamma',\beta)\beta\Delta(\mathbf{z}^1+\mathbf{z}^2)_i(\gamma',\beta)}{Q_1(\gamma',\beta)Q_2(\gamma',\beta)}, \quad i=1,2.$$

We follow the estimates of $D_{3,1}$ and $D_{3,2}$ by replacing $\Delta z_1^2(\gamma',\beta)$ and $\partial_\alpha\Delta d_1^2(\gamma',\beta)$ by $\Delta d_1^2(\gamma',\beta)$ and 2β to obtain an estimate for $\mathcal{D}_{3,1}$ and $\mathcal{D}_{3,2}$. Therefore

$$\|\mathcal{D}_3\|_{L^2(\partial S_{r'})} \leq c_R\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_{r'})}. \tag{D6}$$

Bound for $\mathcal{D}_4$: Finally, the estimate for $\mathcal{D}_4$, follows from the estimate from the previous $\mathcal{D}_3$, by replacing $2\Delta d_1^2(\gamma,\beta)$ by β. Then we deduce

$$\|\mathcal{D}_4\|_{L^2(\partial S_{r'})} \leq c_R\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_{r'})}. \tag{D7}$$

By joining the previous estimates (D4), (D5), (D6) and (D7), we obtain the desired inequality (L2). ■

By combining the inequalities (L1) and (L2), we infer

$$\|\mathbf{F}(\mathbf{d}^1)-\mathbf{F}(\mathbf{d}^2)\|_{L^2(\partial S_{r'})} \leq c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]. \tag{L3}$$

which is the Lipschitz condition for the lower order term.

3.6 Lipschitz Condition of $\partial_\alpha^3\mathbf{F}$

Now we move to the high order derivative. First we will use the Banach scale property, given by

$$\|\partial_\alpha^3\mathbf{F}(\mathbf{d}^1)-\partial_\alpha^3\mathbf{F}(\mathbf{d}^2)\|_{L^2(\partial S_{r'})} \leq \frac{c_R}{r-r'}\|\partial_\alpha^2\mathbf{F}(\mathbf{d}^1)-\partial_\alpha^2\mathbf{F}(\mathbf{d}^2)\|_{L^2(\partial S_r)}.$$

Thus, the proof will be complete if we prove the following inequality

$$\|\partial_\alpha^2\mathbf{F}(\mathbf{d}^1)-\partial_\alpha^2\mathbf{F}(\mathbf{d}^2)\|_{L^2(\partial S_r)} \leq c_R\|\mathbf{d}^1-\mathbf{d}^2\|_{X_{r,3}}. \tag{L4}$$

3.6.1 Lipschitz Condition of $\partial_\alpha^2 F_1$

For the second order derivative, we will use the decomposition (3.4.2), then we write

$$\partial_\alpha^2 F_1(\mathbf{d})(\gamma) = \mathrm{J}_1(\mathbf{d})(\gamma) + \mathrm{E}_1(\mathbf{d})(\gamma), \tag{3.6.1}$$

where $\mathrm{J}_1(\mathbf{d})$ is given by

$$\mathrm{J}_1(\mathbf{d})(\gamma) = PV\int_{\mathbb{R}} \partial_\alpha^3 \Delta d_1(\gamma,\beta)\frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta.$$

Lemma 13. *Given two deviations* $\mathbf{d}^1,\mathbf{d}^2\in O_R$*, the following inequality holds*

$$\begin{aligned}&\|\mathrm{J}_1(\mathbf{d}^1)-\mathrm{J}_1(\mathbf{d}^2)\|_{L^2(\partial S_r)}\\ &\qquad\le c_R\Big[\|\mathbf{d}^2-\mathbf{d}^1\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}+\|\partial_\alpha^3(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].\end{aligned} \tag{L5}$$

Proof. Now, for two deviations $\mathbf{d}^1,\mathbf{d}^2\in O_R$, the difference of the singular terms can be written as

$$\mathrm{J}_1(\mathbf{d}^1)(\gamma)-\mathrm{J}_1(\mathbf{d}^2)(\gamma) = (D_4+D_5+D_6)(\gamma),$$

for

$$\begin{aligned}D_4(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^3\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\frac{\Delta z_1^1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta,\\ D_5(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\frac{\partial_\alpha^3\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta,\\ D_6(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^3\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\left[\frac{1}{Q_1(\gamma,\beta)}-\frac{1}{Q_2(\gamma,\beta)}\right]\mathrm{d}\beta.\end{aligned}$$

To estimate D_4 we expand $\partial_\alpha^3\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$, then we get

$$D_4(\gamma) = \partial_\alpha^3(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{\mathbb{R}}\frac{\Delta z_1^1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha^3(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\frac{\Delta z_1^1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta. \tag{3.6.2}$$

We observe that the last decomposition is similar to (3.5.2), thus we can deduce the following

$$\|D_4\|_{L^2(\partial S_r)}\le c_R\|\partial_\alpha^3(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \tag{D8}$$

For the term next term D_5, we expand

$$\partial_\alpha^3\Delta d_1^2(\gamma,\beta) = \partial_\alpha^3 d_1(\gamma)-\partial_\alpha^3 d_1(\gamma-\beta).$$

Then we get

$$D_5(\gamma) = \partial_\alpha^3 d_1^2(\gamma)\int_{\mathbb{R}}\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha^3 d_1^2(\gamma-\beta)\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta. \tag{3.6.3}$$

Considering

$$\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta) = \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta).$$

Thus, in order to obtain the $L^2(\partial S_r)$ bound we estimate the $L^\infty(\partial S_r)$ norm of the first integral in (3.6.3). We decompose the *in* part in the following way

$$\begin{aligned}\int_{|\beta|<1}\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta &= \int_{|\beta|<1}\left(\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right)\frac{\beta^2}{Q_1(\gamma,\beta)}\frac{\mathrm{d}\beta}{\beta}\\ &\quad+\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2}{Q_1(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\right)\frac{\mathrm{d}\beta}{\beta}.\end{aligned}$$

We use the Fundamental Theorem of Calculus to obtain the next formula

$$\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)=\int_0^1\Big(\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma+(s-1)\beta)-\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\Big)\,\mathrm{d}s.$$

Thus, it follows

$$\left|\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right|_*\leq c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}|\beta|. \tag{3.6.4}$$

From (3.6.4) the arc-chord condition (A2) and estimates Lemma 32, we can deduce the following bound

$$\left|\int_{|\beta|<1}\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right|_*\leq c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^1(\partial S_r)}. \tag{3.6.5}$$

The *out* part is easily bounded by considering (A3), we get

$$\left|\int_{|\beta|>1}\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right|_*\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}. \tag{3.6.6}$$

Now, by taking the $L^2(\partial S_r)$ norm, and using the bounds (3.6.5) and (3.6.6), we arrive to the next estimate

$$\left\|\partial_\alpha^3 d_1^2(\gamma)\int_{\mathbb{R}}\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\leq c_R\|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}\|(\mathbf{d}^2-\mathbf{d}^1)_1\|_{C^2(\partial S_r)}. \tag{3.6.7}$$

To estimate the second integral in (3.6.3), we decompose the *in* part in the following way

$$\begin{aligned}
&\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\\
&\qquad=\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\left(\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right)\frac{\beta^2}{Q_1(\gamma,\beta)}\frac{\mathrm{d}\beta}{\beta}\\
&\qquad+\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\left(\frac{\beta^2}{Q_1(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\right)\frac{\mathrm{d}\beta}{\beta}\\
&\qquad+\frac{\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}H_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma).
\end{aligned}$$

Now, we will take the $L^2(\partial S_r)$ norm. For the first term in (3.6.2) we make use of the Minkowski's integral inequality, the arc-chord condition (A2) and (3.6.4). For the remaining terms, we make use of the Cauchy-Schwarz inequality and Lemma 32, then we get

$$\begin{aligned}
&\left\|\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\\
&\qquad\leq\int_{|\beta|<1}\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}\|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}\,\mathrm{d}\beta\\
&\qquad+c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}\\
&\qquad+c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\|H_{|\beta|<1}\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}.
\end{aligned}$$

Regarding the *out* part, we consider the following inequality

$$\left|\partial_\alpha^3 d_1^2(\gamma-\beta)\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\right|_*\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}|\partial_\alpha^3 d_1^2(\gamma-\beta)|_*|\beta|^{-2}.$$

Thus, by taking the $L^2(\partial S_r)$ norm and using the Minkowski's integral inequality we get

$$\left\| \int_{|\beta|>1} \partial_\alpha^3 d_1^2(\gamma-\beta)\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}\|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}. \tag{3.6.8}$$

From (3.6.7) and (3.6.8), we can deduce the next estimate

$$\|D_5\|_{L^2(\partial S_r)} \le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)} + c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \tag{D9}$$

For D_6, we use the equation (3.5.3), then we decompose

$$D_6(\gamma) = D_{6,1}(\gamma) + D_{6,2}(\gamma)$$

for

$$\begin{aligned} D_{6,1}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^3\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta,\\ D_{6,2}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^3\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2\mathbf{S}_4(\gamma,\beta)\,\mathrm{d}\beta, \end{aligned}$$

where the kernels are given by

$$\begin{aligned} \mathbf{S}_3(\gamma,\beta) &:= \frac{\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)},\\ \mathbf{S}_4(\gamma,\beta) &:= \frac{\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}. \end{aligned}$$

To estimate $D_{6,1}$, we expand $\partial_\alpha^3\Delta d_1^2(\gamma,\beta)$, we get

$$D_{6,1}(\gamma) := \partial_\alpha^3 d_1^2(\gamma)\int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha^3 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta. \tag{3.6.9}$$

We decompose the *in* part as follows

$$\begin{aligned} &\int_{|\beta|<1}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\\ &\quad= \int_{|\beta|<1}\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)}{\beta} - \partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\right)\frac{\beta^4\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)\beta^3}\,\mathrm{d}\beta\\ &\quad+\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{|\beta|<1}\beta\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta. \end{aligned} \tag{3.6.10}$$

In one hand, the last integral above correspond to (3.5.5), and therefore is bounded. On the other hand, from (A6) and the arc-chord condition (A2), we infer

$$\left|\frac{\beta^4\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)\beta^3}\right|_* \le c_R|\beta|^{-1}. \tag{3.6.11}$$

By taking the $L^2(\partial S_r)$ norm and making use of the Minkowski's integral inequality and (3.6.4), we get

$$\begin{aligned} &\left\|\partial_\alpha^3 d_1^2(\gamma)\int_{|\beta|<1}\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)}{\beta} - \partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\right)\frac{\beta^4\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)(\gamma,\beta)_1}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)\beta^3}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\\ &\qquad\le c_R\int_{|\beta|<1}\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}\|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}\,\mathrm{d}\beta\\ &\qquad\le c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}\|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}. \end{aligned}$$

Regarding the last integral in (3.6.10), by taking the $L^2(\partial S_r)$ norm, and making use of the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}\left\| \partial_\alpha^3 d_1^2(\gamma)\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{|\beta|<1}\beta\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}&\\ \leq \|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\left\|\int_{|\beta|<1}\beta\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_r)}&.\end{aligned}$$

By using the inequalities (A1), (A5), (A2) we get

$$\left|\int_{|\beta|<1}\beta\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\right|_* \leq c_R.$$

Thus, we conclude that

$$\begin{aligned}\left\|\partial_\alpha^3 d_1^2(\gamma)\int_{|\beta|<1}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}&\\ \leq c_R\Big[\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big]&.\end{aligned} \tag{3.6.12}$$

For the *out* part we consider the following inequality, we get

$$\left|\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_3(\gamma,\beta)\right|_* \leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}|\beta|^{-2}.$$

Thus, by taking the $L^2(\partial S_r)$ norm, we get

$$\left\|\partial_\alpha^3 d_1^2(\gamma)\int_{|\beta|>1}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}. \tag{3.6.13}$$

The inequalities (3.6.12) and (3.6.13) yields to

$$\begin{aligned}\left\|\partial_\alpha^3 d_1^2(\gamma)\int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}&\\ \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big]&.\end{aligned}$$

To estimate the second integral in (3.6.9), we decompose the *in* part as follows

$$\begin{aligned}&\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\\ &=\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\right)\frac{\beta^4\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)\beta^3}\,\mathrm{d}\beta\\ &+\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\beta\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned} \tag{3.6.14}$$

Thus by taking the $L^2(\partial S_r)$ norm and applying the Minkowski's integral inequality we get

$$\begin{aligned}\left\|\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\right)\frac{\beta^4\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)\beta^3}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}&\\ \leq c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}\|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}&.\end{aligned}$$

To deal with the second integral in the right-hand side of (3.6.14), we use the estimates from the decomposition (3.5.11). We deduce

$$
\begin{aligned}
&\left\|\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\beta\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\\
&\qquad\qquad\leq\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1)\|_{L^2(\partial S_r)}\left\|\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\beta\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\\
&\qquad\qquad\leq c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1)\|_{L^2(\partial S_r)}.
\end{aligned}
$$

Regarding the *out* part, we use the inequality (3.6.11), then we obtain

$$
\left\|\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\leq c_R\|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}.
$$

Thus, we can conclude that

$$
\|D_{6,1}\|_{L^2(\partial S_r)}\leq c_R\Big[\|(\mathbf{d}^2-\mathbf{d}^1)_1\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].
$$

The $L^2(\partial S_r)$ control of $D_{6,2}$ is similar to the control of $D_{3,2}$, see equation (3.5.16). By expanding $\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)$ in the kernel $\mathbf{S}_4(\gamma,\beta)$, de obtain

$$
\begin{aligned}
D_{6,2,1}(\gamma)&=\int_{\mathbb{R}}\partial_\alpha^3\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2\mathbf{S}_{4,1}(\gamma,\beta)\,\mathrm{d}\beta,\\
D_{6,2,2}(\gamma)&=\int_{\mathbb{R}}\partial_\alpha^3\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2\mathbf{S}_{4,2}(\gamma,\beta)\,\mathrm{d}\beta.
\end{aligned}
$$

Where

$$
\begin{aligned}
\mathbf{S}_{4,1}(\gamma,\beta)&:=\frac{\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)},\\
\mathbf{S}_{4,2}(\gamma,\beta)&:=2\frac{\Delta z_1^2(\gamma,\beta)\beta(2\gamma-\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}.
\end{aligned}
$$

The estimate for $D_{6,2,1}$ can be deduced from the estimates for $D_{6,1}$. Then

$$
\|D_{6,2,1}\|_{L^2(\partial S_r)}\leq c_R\Big[\|(\mathbf{d}^2-\mathbf{d}^1)_2\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}\Big].
$$

Next, we estimate $D_{6,2,2}$, we expand $\partial_\alpha^3 d_1^2(\gamma,\beta)$ then

$$
D_{6,2,2}(\gamma)=\partial_\alpha^3 d_1^2(\gamma)\int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_2\mathbf{S}_{4,2}(\gamma,\beta)\,\mathrm{d}\beta-\int_{\mathbb{R}}\partial_\alpha^3 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2\mathbf{S}_{4,2}(\gamma,\beta)\,\mathrm{d}\beta. \tag{3.6.15}
$$

We will estimate the first integral in the right-hand side of (3.6.15). For the *in* part, we decompose

$$
\begin{aligned}
&\frac{1}{2}\int_{|\beta|<1}\Delta(\mathbf{d}^2-\mathbf{d}^1)_2\mathbf{S}_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\\
&\qquad=\int_{|\beta|<1}\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma)\right)\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\
&\qquad+\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma)\int_{|\beta|<1}\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\,\mathrm{d}\beta,
\end{aligned}
$$

and the last integral above decomposes

$$\int_{|\beta|<1}\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\,\mathrm{d}\beta=\int_{|\beta|<1}\left(\frac{\beta^2(2\gamma-\beta)}{Q_1(\gamma,\beta)}-\frac{2\gamma}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)}{\beta Q_2(\gamma,\beta)}\,\mathrm{d}\beta$$
$$+\frac{2\gamma}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1^2(\gamma,\beta)}{Q_2(\gamma,\beta)}\,\mathrm{d}\beta.$$

From Corollary 2 and estimates (A6), (A2) we control the first term in the right-hand side above. While for the last integral we use (3.3.4) and (A8). We deduce

$$\left|\int_{|\beta|<1}\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\,\mathrm{d}\beta\right|_*\le c_R$$

and hence

$$\left\|\partial_\alpha^3 d_1^2(\gamma)\int_{|\beta|<1}\Delta(\mathbf{d}^2-\mathbf{d}^1)_2\mathbf{S}_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}$$
$$\le c_R\|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha^2(\mathbf{d}^2-\mathbf{d}^1)_2\|_{L^\infty(\partial S_r)}$$
$$+\|\partial_\alpha^3 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_2\|_{L^2(\partial S_r)}\left|\int_{|\beta|<1}\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\,\mathrm{d}\beta\right|_*$$
$$\le c_R\Big[\|(\mathbf{d}^2-\mathbf{d}^1)_2\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}\Big].$$

To deal with the *out* part, we have the following bound

$$\big|\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{4,2}(\gamma,\beta)\big|_*\le c_R\|(\mathbf{d}^2-\mathbf{d}^1)_2\|_{L^\infty(\partial S_r)}|\beta|^{-2}.$$

We infer

$$\left\|\partial_\alpha^3 d_1^2(\gamma)\int_{|\beta|>1}\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le_R\|(\mathbf{d}^2-\mathbf{d}^1)_2\|_{L^\infty(\partial S_r)}.$$

To estimate the second part in (3.6.15), we decompose in similar way to the previous term. We have

$$\frac{1}{2}\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{4,2}(\gamma,\beta)\,\mathrm{d}\beta$$
$$=\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma)\right)\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\,\mathrm{d}\beta$$
$$+\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma)\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\,\mathrm{d}\beta,$$

and the last integral decomposes

$$\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\,\mathrm{d}\beta$$
$$=\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\left(\frac{\beta^2(2\gamma-\beta)}{Q_1(\gamma,\beta)}-\frac{2\gamma}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)}{\beta Q_2(\gamma,\beta)}\,\mathrm{d}\beta$$
$$+\frac{2\gamma}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\int_{|\beta|<1}\partial_\alpha^3 d_1^2(\gamma-\beta)\frac{\Delta z_1^2(\gamma,\beta)}{Q_2(\gamma,\beta)}\,\mathrm{d}\beta.$$

Thus the Minkowski's integral inequality yields

$$\|D_{6,2}\|_{L^2(\partial S_r)}\le c_R\Big[\|(\mathbf{d}^2-\mathbf{d}^1)_2\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}\Big].$$

And hence

$$\|D_6\|_{L^2(\partial S_r)} \leq c_R\Big[\|\mathbf{d}^2-\mathbf{d}^1\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]. \tag{D10}$$

By combining the inequalities (D8), (D9) and (D10), we derive the desired estimate (L5). ■

Now, we move to the second part of (3.6.1). We have the following lemma.

Lemma 14. *Given two deviations* $\mathbf{d}^1, \mathbf{d}^2 \in O_R$, *the following inequality holds*

$$\begin{aligned}\|\mathrm{E}_1(\mathbf{d}^1)-\mathrm{E}_1(\mathbf{d}^2)\|_{L^2(\partial S_r)} \leq c_R\Big[&\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)} + \|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}\\ &+ \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big].\end{aligned} \tag{L6}$$

Proof. The remainder term $\mathrm{E}_1(\mathbf{d})$ contains the $\mathrm{J}_i(\mathbf{d})$'s terms as in decomposition (3.4.2). We write

$$\mathrm{E}_1(\mathbf{d})(\gamma) = \sum_{i=2}^{6}\mathrm{J}_i(\mathbf{d})(\gamma). \tag{3.6.16}$$

Bound for J_2: To estimate the difference with J_2, adding and subtracting mixed terms we have the following decomposition

$$\mathrm{J}_2(\mathbf{d}^1)(\gamma) - \mathrm{J}_2(\mathbf{d}^2)(\gamma) = (D_7+D_8+D_9)(\gamma),$$

for

$$\begin{aligned}D_7(\gamma) &:= 3\int_{\mathbb{R}} \partial_\alpha^2\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\frac{\partial_\alpha\Delta d_1^1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta,\\ D_8(\gamma) &:= 3\int_{\mathbb{R}} \partial_\alpha^2\Delta d_1^2(\gamma,\beta)\frac{\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta,\\ D_9(\gamma) &:= 3\int_{\mathbb{R}} \partial_\alpha^2\Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\left[\frac{1}{Q_1(\gamma,\beta)}-\frac{1}{Q_2(\gamma,\beta)}\right]\mathrm{d}\beta.\end{aligned} \tag{3.6.17}$$

For the first term, we expand $\partial_\alpha^2\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$, then we get

$$\frac{1}{3}D_7(\gamma) = \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{\mathbb{R}}\frac{\partial_\alpha\Delta d_1^1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\frac{\partial_\alpha\Delta d_1^1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta.$$

We notice that D_7 is similar to J_2, see equation (3.4.4) in the estimation for F_1. Then we follow the estimates (3.4.5) to (3.4.10). We deduce

$$\|D_7\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \tag{D11}$$

For the second term D_8, by expanding $\partial_\alpha^2\Delta d_1^2(\gamma,\beta) = \partial_\alpha^2 d_1^2(\gamma) - \partial_\alpha^2 d_1^2(\gamma-\beta)$, we get

$$\frac{1}{3}D_8(\gamma) = \partial_\alpha^2 d_1^2(\gamma)\int_{\mathbb{R}}\frac{\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta. \tag{3.6.18}$$

In order to obtain an $L^2(\partial S_r)$ estimate, we require a bound in $L^\infty(\partial S_r)$, for the first integral. We split the integral in the *in* and *out* parts. We find that

$$\begin{aligned}\int_{|\beta|<1}\frac{\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta = &\int_{|\beta|>1}\left(\frac{\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta} - \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right)\frac{\beta^2}{Q_1(\gamma,\beta)}\frac{\mathrm{d}\beta}{\beta}\\ &+ \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\frac{1}{\beta}\left(\frac{\beta^2}{Q_1(\gamma,\beta)} - \frac{1}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\right)\mathrm{d}\beta.\end{aligned}$$

By using the fundamental Theorem of Calculus we obtain the next estimate

$$\begin{aligned}
&\left|\frac{\partial_\alpha \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta} - \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right| \\
&\qquad\qquad \leq \int_0^1 \left|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma+(s-1)\beta) - \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right|_* \mathrm{d}s \\
&\qquad\qquad \leq \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^\delta(\partial S_r)}|\beta|^\delta, \quad \text{for} \quad \delta\in(0,1).
\end{aligned} \tag{3.6.19}$$

By taking the $L^2(\partial S_r)$ norm, using inequalities (3.6.19), (A2) and Lemma 32, we get

$$\begin{aligned}
&\left\|\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|>1}\frac{\partial_\alpha \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \\
&\qquad \leq c_R\|\partial_\alpha^2 d_1^2(\gamma)\|_{L^2(\partial S_r)}\Big[\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^\delta(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].
\end{aligned}$$

Regarding, the *out* part is easily bounded by considering (A5) and (A2), therefore we get

$$\left|\int_{|\beta|>1}\frac{\partial_\alpha \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right|_* \leq c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}.$$

Now, by taking the $L^2(\partial S_r)$ norm, we get

$$\left\|\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|>1}\frac{\partial_\alpha \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}.$$

Thus,

$$\begin{aligned}
&\left\|\partial_\alpha^2 d_1^2(\gamma)\int_{\mathbb{R}}\frac{\partial_\alpha \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \\
&\qquad\qquad \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^{2,\delta}(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].
\end{aligned}$$

For the second integral of (3.6.18), we decompose in an analogous way

$$\begin{aligned}
&\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\partial_\alpha \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta \\
&\qquad = \int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\left(\frac{\partial_\alpha \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta} - \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right)\frac{\beta^2}{Q_1(\gamma,\beta)}\frac{\mathrm{d}\beta}{\beta} \\
&\qquad + \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\frac{\partial_\alpha^2 d_1^2(\gamma-\beta)}{\beta}\left(\frac{\beta^2}{Q_1(\gamma,\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}^1(\gamma)|^2}\right)\mathrm{d}\beta \\
&\qquad + \frac{\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)}{|\partial_\alpha \mathbf{z}^1(\gamma)|^2}H_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma).
\end{aligned}$$

Thus, by taking the $L^2(\partial S_r)$ norm and using the arc-chord condition (A2), Lemma 32 together with the Minkowski's integral inequality along with the Cauchy-Schwarz inequality, we infer

$$\begin{aligned}
&\left\|\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\partial_\alpha \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \\
&\qquad\qquad \leq c_R\Big[\|(\mathbf{d}^2-\mathbf{d}^1)_1\|_{C^{2,\delta}(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].
\end{aligned}$$

For the *out* part, once again, we use the Minkoswki's integral inequality, then we get

$$\left\|\int_{|\beta|>1}\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\partial_\alpha \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}.$$

Thus, we deduce

$$\left\| \int_{\mathbb{R}} \partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^{2,\delta}(\partial S_r)}+\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].$$

and we can conclude

$$\|D_8\|_{L^2(\partial S_r)} \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^{2,\delta}(\partial S_r)}+\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big]. \tag{D12}$$

For the final term in (3.6.17) we compute the difference using (3.5.3), we decompose

$$\frac{1}{3}D_9(\gamma) = D_{9,1}(\gamma)+D_{9,2}(\gamma)$$

for

$$\begin{aligned} D_{9,1}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^2\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta,\\ D_{9,2}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^2\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_6(\gamma,\beta)\,\mathrm{d}\beta, \end{aligned} \tag{3.6.20}$$

where

$$\begin{aligned} \mathbf{S}_5(\gamma,\beta) &:= \frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)},\\ \mathbf{S}_6(\gamma,\beta) &:= \frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}. \end{aligned}$$

The next step is obtain the $L^2(\partial S_r)$ bound, we expand

$$\partial_\alpha^2\Delta d_1^2(\gamma,\beta) = \partial_\alpha^2 d_1^2(\gamma) - \partial_\alpha^2 d_1^2(\gamma-\beta),$$

then we get

$$\begin{aligned} D_{9,1}(\gamma) = \partial_\alpha^2 d_1^2(\gamma)&\int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\\ &-\int_{\mathbb{R}} \partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta. \end{aligned} \tag{3.6.21}$$

In order to obtain the $L^2(\partial S_r)$ estimate, we control the $L^\infty(\partial S_r)$ of the first integral. We split in the *in* and *out* part, for the *in* part we see

$$\begin{aligned} \int_{|\beta|<1} &\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\\ &= \int_{|\beta|<1}\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\beta)}{\beta} - \partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\right)\beta\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\\ &\quad+\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{|\beta|<1}\beta\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta. \end{aligned}$$

To deal with the last integral above, we use the estimates for $\mathbf{S}_1(\gamma,\beta)$, see decomposition (3.5.6). By replacing $\Delta z_1^2(\gamma,\beta)$ by β. Then we can deduce the following

$$\left|\int_{|\beta|<1}\beta\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\right|_* \leq c_R.$$

Now, by taking the $L^2(\partial S_r)$ norm, we obtain that

$$\left\|\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\Big[\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].$$

For the *out* part, we consider the following

$$\big|\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\big|_* \leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}|\beta|^{-2}. \tag{3.6.22}$$

Therefore, by taking the $L^2(\partial S_r)$, we find that

$$\left\|\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|>1}\Delta(\mathbf{d}^2-\mathbf{d}^1)(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)}.$$

Thus we can deduce that

$$\left\|\partial_\alpha^2 d_1^2(\gamma)\int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big]. \tag{3.6.23}$$

To estimate the second integral of (3.6.21) we use a similar decomposition

$$\begin{aligned}\int_{|\beta|<1}&\partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_3(\gamma,\beta)\,\mathrm{d}\beta\\ &=\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\beta)}{\beta}-\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\right)\beta\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\\ &+\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\beta\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned}$$

Note that the last integral in the previous term, can be bounded as in (3.5.12). Now, by taking the $L^2(\partial S_r)$ norm and making use of the Minkowski's integral inequality, the Cauchy-Schwarz inequality, estimates (3.6.4), we infer

$$\begin{aligned}\left\|\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}&\\ \leq c_R\|\partial_\alpha^2(\mathbf{d}^2-\mathbf{d}^1)_1\|_{L^\infty(\partial S_r)}\|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}&\\ +\|\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1\|_{L^2(\partial S_r)}\left\|\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\beta\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}&.\end{aligned}$$

Making use of equation (3.5.12), we can deduce a bound for the last $L^2(\partial S_r)$ norm

$$\left\|\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\beta\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}<c_R.$$

For the *out* part, we consider the bound (3.6.22), then by taking the $L^2(\partial S_r)$ norm and making use of the Minkowski's integral inequality, we get

$$\left\|\int_{|\beta|>1}\partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\leq c_R\|(\mathbf{d}^2-\mathbf{d}^1)\|_{L^\infty(\partial S_r)}.$$

Thus,

$$\begin{aligned} &\left\| \int_{\mathbb{R}} \partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_5(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \\ &\qquad\qquad \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1\|_{L^2(\partial S_r)}\Big]. \end{aligned} \tag{3.6.24}$$

From inequalities (3.6.23) and (3.6.24), we obtain that

$$\|D_{9,1}\|_{L^2(\partial S_r)} \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1\|_{L^2(\partial S_r)}\Big].$$

Now, the estimate for the second term $D_{9,2}$ can be deduced from the estimation for $D_{6,2}$. We obtain

$$\|D_{9,2}\|_{L^2(\partial S_r)} \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_2\|_{L^2(\partial S_r)}\Big].$$

Hence

$$\|D_9\|_{L^2(\partial S_r)} \leq c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)\|_{L^2(\partial S_r)}\Big]. \tag{D13}$$

By joining the inequalities (D11), (D12) and (D13), we can complete the estimate for the difference with J_2, which is given by

$$\|\mathrm{J}_2(\mathbf{d}^1)-\mathrm{J}_2(\mathbf{d}^2)\|_{L^2(\partial S_r)} \leq c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)\|_{L^2(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^2-\mathbf{d}^1)_1\|_{L^2(\partial S_r)}\Big]. \tag{L7}$$

Bound for J_3: Now, we move to the difference with J_3, from equation (3.4.2) this term is given by

$$\mathrm{J}_3(\mathbf{d}) := -2\int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\partial_\alpha^2\Delta d_1(\gamma,\beta)\partial_\alpha Q(\gamma,\beta)}{Q(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

We compute the difference $\mathrm{J}_3(\mathbf{d}^1)-\mathrm{J}_3(\mathbf{d}^2)$. We add and subtract mixed terms and decompose as follows

$$-\frac{1}{2}\Big(J_3(\mathbf{d}^1)(\gamma) - J_3(\mathbf{d}^2)(\gamma)\Big) = (D_{10}+D_{11}+D_{12}+D_{13})(\gamma),$$

for

$$\begin{aligned} D_{10}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^2\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\left[\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right]\mathrm{d}\beta, \\ D_{11}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\left[\frac{\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right]\mathrm{d}\beta, \\ D_{12}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha(Q_1-Q_2)(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\left[\frac{1}{Q_1(\gamma,\beta)^2}\right]\mathrm{d}\beta, \\ D_{13}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha Q_2(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\left[\frac{1}{Q_1(\gamma,\beta)^2} - \frac{1}{Q_2(\gamma,\beta)^2}\right]\mathrm{d}\beta. \end{aligned} \tag{3.6.25}$$

To estimate D_{10}, we expand $\partial_\alpha^2\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$, then we get

$$\begin{aligned} D_{10}(\gamma) = \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)(\gamma)&\int_{\mathbb{R}} \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \\ &\qquad - \int_{\mathbb{R}} \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)(\gamma-\beta)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta. \end{aligned}$$

We notice that D_{10} has the same kernel as J_3, in (3.4.12). Then, the $L^2(\partial S_r)$ estimate can be deduced from equations (3.4.11) to (3.4.55). We obtain that

$$\|D_{10}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \tag{D14}$$

We move on to estimate D_{11}. By expanding $\partial_\alpha^2 \Delta d_1^2(\gamma,\beta)$, we have that

$$\begin{aligned} D_{11}(\gamma) = \partial_\alpha^2 d_1^2(\gamma) \int_{\mathbb{R}} \frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \\ - \int_{\mathbb{R}} \partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta. \end{aligned} \tag{3.6.26}$$

Once again, we split in the *in* and *out* part. For the *in* part we have the following decomposition

$$\begin{aligned} \int_{|\beta|<1} \frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \\ = \int_{|\beta|<1}\left(\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta} - \partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right)\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \\ + \partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta. \end{aligned}$$

Using the inequalities (A6), (A5), (3.4.90) and (A2) , we infer

$$\left|\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right|_* \leq c_R|\beta|^{-1}. \tag{3.6.27}$$

Then, by taking the $L^2(\partial S_r)$ norm and using the previous estimate, we deduce the following

$$\begin{aligned} \left\|\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \\ \leq c_R\|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)} \\ + \|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\left\|\int_{|\beta|<1}\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_r)}. \end{aligned} \tag{3.6.28}$$

To complete the estimate, we require to bound the $L^\infty(\partial S_r)$ norm of the following integral

$$\int_{|\beta|<1}\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

We can replace $\Delta z_1(\gamma,\beta)$ with β in the expression (3.4.13). Then, by repeating the argument used in the estimation for J_3, we obtain

$$\left|\int_{|\beta|<1}\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\right|_* \leq c_R.$$

Regarding the *out* part, by considering the inequalities (A1), (A5) and (A2) we have the following bound

$$\left|\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right|_* \leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}|\beta|^{-2}. \tag{3.6.29}$$

Thus the Minkowski's integral inequality yields to

$$\left\|\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|>1}\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}. \tag{3.6.30}$$

From inequalities (3.6.28) and (3.6.30), we have that

$$\begin{aligned} \left\|\partial_\alpha^2 d_1^2(\gamma)\int_{\mathbb{R}}\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \\ \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big]. \end{aligned} \tag{3.6.31}$$

To deal with the second part of (3.6.26), we split in the *in* and *out* parts. For the *in* part we have the following decomposition

$$\begin{aligned}\int_{|\beta|<1} &\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &=\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\left(\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right)\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &+\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

By taking the $L^2(\partial S_r)$ norm and making use of the Minkowski's integral inequality, inequalities (3.6.4) and (3.6.27), we derive the following

$$\begin{aligned}\Bigg\|\int_{|\beta|<1}&\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Bigg\|_{L^2(\partial S_r)}\\ &\le c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}\\ &+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Bigg\|\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Bigg\|_{L^\infty(\partial S_r)}.\end{aligned}\tag{3.6.32}$$

The las term in (3.6.32), can be bounded following estimates for (3.4.35). Thus

$$\Bigg\|\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Bigg\|_{L^\infty(\partial S_r)}\le c_R.$$

While for the *out* part we use the bound (3.6.29). Then, by taking the $L^2(\partial S_r)$ norm and by using the Minkowski's integral inequality, we get

$$\Bigg\|\int_{|\beta|>1}\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Bigg\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}.\tag{3.6.33}$$

Combining the inequalities (3.6.31), (3.6.32) and (3.6.33), we infer

$$\|D_{11}\|_{L^2(\partial S_r)}\le c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].\tag{D15}$$

To estimate D_{12}, we compute

$$\begin{aligned}\partial_\alpha(Q_1-Q_2)(\gamma,\beta)&=\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)\\ &+\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)\\ &+\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)\\ &+\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta).\end{aligned}\tag{3.6.34}$$

By substituting the last expression in D_{11}, we decompose it as follows

$$D_{12}(\gamma)=(D_{12,1}+D_{12,2}+D_{12,3}+D_{12,4})(\gamma),$$

for

$$\begin{aligned}D_{12,1}(\gamma)&:=\int_{\mathbb{R}}\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta,\\ D_{12,2}(\gamma)&:=\int_{\mathbb{R}}\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\mathbf{N}_2(\gamma,\beta)\,\mathrm{d}\beta,\\ D_{12,3}(\gamma)&:=\int_{\mathbb{R}}\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\mathbf{N}_3(\gamma,\beta)\,\mathrm{d}\beta,\\ D_{12,4}(\gamma)&:=\int_{\mathbb{R}}\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\mathbf{N}_4(\gamma,\beta)\,\mathrm{d}\beta,\end{aligned}$$

where $\mathbf{N_i}(\gamma,\beta)$ are the kernels given by

$$\mathbf{N}_1(\gamma,\beta) = \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha \Delta(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2},$$
$$\mathbf{N}_2(\gamma,\beta) = \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha \Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2},$$
$$\mathbf{N}_3(\gamma,\beta) = \frac{\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2},$$
$$\mathbf{N}_4(\gamma,\beta) = \frac{\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}.$$

For the first term we expand $\partial_\alpha^2 \Delta d_1^2(\gamma,\beta)$, then we get

$$\begin{aligned} D_{12,1}(\gamma) = \partial_\alpha^2 d_1^2(\gamma)\int_{\mathbb{R}} \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta \\ - \int_{\mathbb{R}} \partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta. \end{aligned} \tag{3.6.35}$$

Once again, we split in the *in* and *out* parts. For the *in* part, we have the following decomposition

$$\begin{aligned} &\int_{|\beta|<1} \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta \\ &\quad= \int_{|\beta|<1}\left(\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta} - \partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right)\beta\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta \\ &\quad+ \partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\beta\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta. \end{aligned}$$

Then, by taking the $L^2(\partial S_r)$ norm and considering the inequality (3.6.4), we get

$$\begin{aligned} &\left\|\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \\ &\quad\le c_R\|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)} \\ &\quad+ \|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\left\|\int_{|\beta|<1}\beta\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_r)}. \end{aligned} \tag{3.6.36}$$

To deal with the last term in (3.6.36), we decompose the *in* part as follows

$$\begin{aligned} \int_{|\beta|<1}\beta\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta &= \int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta(\mathbf{d}^1+\mathbf{d}^2)(\gamma,\beta)}{\beta} - \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)\right)\frac{\beta^2\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \\ &+ \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\frac{\beta^2\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta. \end{aligned}$$

To estimate the last integral above, we follow the decomposition (3.4.16) used in the estimation for J_3. From inequalities (A1), (A5), the arc-chord condition (A2) and Lemma 32, we infer

$$\left\|\int_{|\beta|<1}\beta\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_r)} \le c_R.$$

Regarding the *out* part, by considering (A3), (A5), (A2), we obtain that

$$\left|\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\right|_* \le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}|\beta|^{-3}. \tag{3.6.37}$$

Then, by taking the $L^2(\partial S_r)$ norm we get

$$\left\| \partial_\alpha^2 d_1^2(\gamma) \int_{|\beta|>1} \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \leq c_R \|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}. \tag{3.6.38}$$

By joining inequalities (3.6.36) and (3.6.38), we can deduce that

$$\begin{aligned} &\left\|\partial_\alpha^2 d_1^2(\gamma) \int_{\mathbb{R}} \Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \\ &\qquad\qquad \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big]. \end{aligned} \tag{3.6.39}$$

To deal with the second integral in (3.6.35), we decompose the *in* part as follows

$$\begin{aligned} &\int_{|\beta|<1} \partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta \\ &\qquad = \int_{|\beta|<1} \partial_\alpha^2 d_1^2(\gamma-\beta)\left(\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta} - \partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right)\beta\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta \\ &\qquad + \partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1} \partial_\alpha^2 d_1^2(\gamma-\beta)\beta\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta \end{aligned}$$

and the last integral decomposes as follows

$$\begin{aligned} &\int_{|\beta|<1} \partial_\alpha^2 d_1^2(\gamma-\beta)\beta\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta \\ &\qquad = \int_{|\beta|<1} \partial_\alpha^2 d_1^2(\gamma-\beta)\left(\frac{\partial_\alpha(\mathbf{d}^1+\mathbf{d}^2)(\gamma,\beta)}{\beta} - \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)\right)\frac{\beta^2\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \\ &\qquad + \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1} \partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\beta^2\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta. \end{aligned}$$

Note that the last integral can be bounded following the decomposition (3.4.37). Then, by taking the $L^2(\partial S_r)$ norm, considering the bound (3.6.4) and applying the Minkowski's integral inequality, we derive

$$\begin{aligned} &\left\|\int_{|\beta|<1} \partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_R)} \\ &\qquad \leq c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)} \\ &\qquad + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\left\|\int_{|\beta|<1} \partial_\alpha^2 d_1^2(\gamma-\beta)\beta\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}. \end{aligned} \tag{3.6.40}$$

Then, from inequalities (A1), (A5), the arc-chord condition (A2) and Lemma 32 we infer

$$\left\| \int_{|\beta|<1} \partial_\alpha^2 d_1^2(\gamma-\beta)\beta\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}.$$

Regarding the *out* part, we use the inequality (3.6.37), thus we can conclude from inequality (3.6.40) that

$$\begin{aligned} &\left\|\int_{\mathbb{R}} \partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\mathbf{N}_1(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_R)} \\ &\qquad\qquad \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big]. \end{aligned}$$

By joining the inequalities (3.6.39) and (3.6.1), we deduce

$$\|D_{12,1}\|_{L^2(\partial S_r)} \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].$$

To estimate $D_{12,2}$, we follow the estimates (3.6.36) to (3.6.1), by replacing $\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$ and $\mathbf{N}_1(\gamma,\beta)$ by $\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)$ and $\mathbf{N}_2(\gamma,\beta)$. We deduce

$$\|D_{12,2}\|_{L^2(\partial S_r)} \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}\Big].$$

To estimate $D_{12,3}$, we replace $\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$ and $\mathbf{N}_1(\gamma,\beta)$ by $\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$ and $\mathbf{N}_3(\gamma,\beta)$, in estimates (3.6.36) to (3.6.1). We infer

$$\|D_{12,3}\|_{L^2(\partial S_r)} \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^{2,\delta}(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].$$

Finally, to estimate $D_{12,4}$, we expand

$$\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta) = \Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta) + 2\beta(2\gamma-\beta).$$

Then we decompose in the following way

$$\begin{aligned} D_{12,4}(\gamma) = &\int_{\mathbb{R}} \partial_\alpha^2\Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\mathbf{N}_{4,1}(\gamma,\beta)\,\mathrm{d}\beta \\ &\qquad + \int_{\mathbb{R}} \partial_\alpha^2\Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\mathbf{N}_{4,2}(\gamma,\beta)\,\mathrm{d}\beta. \end{aligned}$$

Where the kernels are given by

$$\begin{aligned} \mathbf{N}_{4,1}(\gamma,\beta) &= \frac{\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}, \\ \mathbf{N}_{4,2}(\gamma,\beta) &= 2\frac{\Delta z_1^2(\gamma,\beta)\beta(2\gamma-\beta)}{Q_1(\gamma,\beta)^2}. \end{aligned}$$

Thus, the $L^2(\partial S_r)$ bound for the first integral is automatic, and is given by

$$\begin{aligned} &\left\|\int_{\mathbb{R}} \partial_\alpha^2\Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\mathbf{N}_{4,1}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \\ &\qquad\qquad \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{C^{2,\delta}(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}\Big]. \end{aligned}$$

To estimate the integral with the kernel $\mathbf{N}_{4,2}(\gamma,\beta)$, we notice that this kernel is similar to the kernel K_4 which appears in (3.4.14) and the kernel K_4^* in (3.4.36). We deduce

$$\begin{aligned} &\left\|\int_{\mathbb{R}} \partial_\alpha^2\Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\mathbf{N}_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \\ &\qquad\qquad \leq c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{C^{2,\delta}(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}\Big]. \end{aligned}$$

The last bound completes the $L^2(\partial S_r)$ estimate for D_{12}. Then we arrive to the following inequality

$$\|D_{12}\|_{L^2(\partial S_r)} \leq c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]. \tag{D16}$$

Finally, for D_{13}, we compute the difference as in (3.5.3), to obtain the next decomposition

$$D_{13}(\gamma) = (D_{13,1}+D_{13,2}+D_{13,3}+D_{13,4})(\gamma)$$

for

$$\begin{aligned}
D_{13,1}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta,\\
D_{13,2}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\mathbf{S}_8(\gamma,\beta)\,\mathrm{d}\beta,\\
D_{13,3}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\mathbf{S}_9(\gamma,\beta)\,\mathrm{d}\beta,\\
D_{13,4}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\mathbf{S}_{10}(\gamma,\beta)\,\mathrm{d}\beta,
\end{aligned}$$

where the kernels are given by

$$\begin{aligned}
\mathbf{S}_7(\gamma,\beta) &= \frac{\partial_\alpha Q_2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)},\\
\mathbf{S}_8(\gamma,\beta) &= \frac{\partial_\alpha Q_2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)^2},\\
\mathbf{S}_9(\gamma,\beta) &= \frac{\partial_\alpha Q_2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)},\\
\mathbf{S}_{10}(\gamma,\beta) &= \frac{\partial_\alpha Q_2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)^2}.
\end{aligned}$$

To estimate $D_{13,1}$, we expand $\partial_\alpha^2 d_1^2(\gamma,\beta) = \partial_\alpha^2 d_1^2(\gamma) - \partial_\alpha^2 d_1^2(\gamma-\beta)$. We obtain

$$\begin{aligned}
D_{13,1}(\gamma) = \partial_\alpha^2 d_1^2(\gamma)\int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta&\\
-\int_{\mathbb{R}} \partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_7(\gamma,\beta)d\beta.&
\end{aligned}\tag{3.6.41}$$

In order to obtain a $L^2(\partial S_r)$ bound for $D_{13,1}$, we estimate the first integral in $L^\infty(\partial S_r)$. For the *in* we have the following decomposition

$$\begin{aligned}
&\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1} \Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\\
&\qquad= \partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)}{\beta} - \partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\right)\beta\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\\
&\qquad+ \partial_\alpha^2 d_1^2(\gamma)\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{|\beta|<1}\beta\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta.
\end{aligned}\tag{3.6.42}$$

From (A6) and the arc-chord condition (A2), we have

$$\left|\beta\mathbf{S}_7(\gamma,\beta)\right|_* \le c_R|\beta|^{-1}\tag{3.6.43}$$

Thus, by taking the $L^2(\partial S_r)$ norm in (3.6.42), using the estimate (3.6.4) and the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
&\left\|\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\\
&\qquad\le c_R\|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}\\
&\qquad+ c_R\|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\left\|\int_{|\beta|<1}\beta\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_r)}.
\end{aligned}\tag{3.6.44}$$

To deal with the last term in (3.6.44), we decompose in the following way

$$\begin{aligned}\int_{|\beta|<1}\beta\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta &= \int_{|\beta|<1}\left(\frac{\Delta(\mathbf{z}^1+\mathbf{z}^2)(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\right)\frac{\beta^2\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\ &+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2}{Q_2(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &+\frac{\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\partial_\alpha Q_2(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned}\tag{3.6.45}$$

We notice that the last integral above is similar to (3.4.13). Then the bound follows from the estimates for K_i and (A6), the arc-chord condition (A2) and Lemma 32. Thus, we obtain that

$$\left|\int_{|\beta|<1}\beta\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\right|_*\leq c_R.$$

Regarding the *out* part, we have the following inequality

$$\left|\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_7(\gamma,\beta)\right|_*\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}|\beta|^{-2}.\tag{3.6.46}$$

Then, by taking the $L^2(\partial S_r)$ norm and using the Minkowski's integral inequality, we get

$$\left\|\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|>1}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}.\tag{3.6.47}$$

Now, we deal with the second integral in (3.6.41). For the *in* part we have the following decomposition

$$\begin{aligned}&\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\\ &\qquad=\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\right)\beta\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\\ &\qquad+\partial_\alpha^2 d_1^2(\gamma)\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\beta\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned}$$

By taking the $L^2(\partial S_r)$ norm and using inequalities (3.6.43), (3.6.4), together with the Minkowski's and the Cauchy-Schwarz inequalities, we infer

$$\begin{aligned}&\left\|\int_{\mathbb{R}}\partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\\ &\leq c_R\|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}\\ &+c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\left\|\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\beta\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}.\end{aligned}\tag{3.6.48}$$

The last term in (3.6.48), decomposes further as

$$\begin{aligned}&\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\beta\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta\\ &=\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\left(\frac{\Delta(\mathbf{z}^1+\mathbf{z}^2)(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\right)\frac{\beta^2\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\ &+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\left(\frac{\beta^2}{Q_2(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &+\frac{\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\int_{|\beta|<1}\partial_\alpha^2 d_1^2(\gamma-\beta)\frac{\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\partial_\alpha Q_2(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned}\tag{3.6.49}$$

We notice that the last integral above is similar to (3.4.35). Then the bound follows from the estimates for K_i^*. We obtain that

$$\left\| \int_{|\beta|<1} \partial_\alpha^2 d_1^2(\gamma-\beta)\beta \mathbf{S}_7(\gamma,\beta)\, \mathrm{d}\beta \right\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha^2 d_1^2\|_{L^2(\partial S_r)}.$$

For the *out* part, we use the bound (3.6.46), then we can conclude

$$\begin{aligned} &\left\| \int_{\mathbb{R}} \partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_7(\gamma,\beta)\, \mathrm{d}\beta \right\|_{L^2(\partial S_r)} \\ &\qquad\qquad \leq c_R \Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)} \Big]. \end{aligned} \tag{3.6.50}$$

Thus, by joining inequalities (3.6.44), (3.6.47) and (3.6.50) we can deduce the following bound

$$\|D_{13,1}\|_{L^2(\partial S_r)} \leq c_R \Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)} \Big].$$

Now, the bound for $D_{13,2}$ is automatic

$$\|D_{13,2}\|_{L^2(\partial S_r)} \leq c_R \Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)} \Big].$$

To estimate $D_{13,3}$ we expand

$$\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta) = \Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta) + 2\beta(2\gamma-\beta),$$

then we decompose in the following way

$$\begin{aligned} D_{13,3}(\gamma) = &\int_{\mathbb{R}} \partial_\alpha^2 \Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{9,1}(\gamma,\beta)\, \mathrm{d}\beta \\ &\qquad + \int_{\mathbb{R}} \partial_\alpha^2 \Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{9,2}(\gamma,\beta)\, \mathrm{d}\beta. \end{aligned}$$

Where the kernels are given by

$$\begin{aligned} \mathbf{S}_{9,1}(\gamma,\beta) &= \frac{\partial_\alpha Q_2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2 Q_2(\gamma,\beta)}, \\ \mathbf{S}_{9,2}(\gamma,\beta) &= 2\frac{\partial_\alpha Q_2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\beta(2\gamma-\beta)}{Q_1(\gamma,\beta)^2 Q_2(\gamma,\beta)}. \end{aligned}$$

Hence, the bound for the first integral, can be deduced from the previous bound for $D_{13,1}$. By replacing $\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)$ and $\mathbf{S}_7$ by $\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)$ and $\mathbf{S}_{9,1}(\gamma,\beta)$. Then we get

$$\begin{aligned} &\left\| \int_{\mathbb{R}} \partial_\alpha^2 \Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{9,1}(\gamma,\beta)\, \mathrm{d}\beta \right\|_{L^2(\partial S_r)} \\ &\qquad\qquad \leq c_R \Big[\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)} \Big]. \end{aligned}$$

To estimate the second integral, we decompose $\partial_\alpha^2 \Delta d_1^2(\gamma,\beta)$, then we get

$$\begin{aligned} &\int_{\mathbb{R}} \partial_\alpha^2 \Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{9,2}(\gamma,\beta)\, \mathrm{d}\beta \\ &= \partial_\alpha^2 d_1^2(\gamma) \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{9,2}(\gamma,\beta)\, \mathrm{d}\beta - \int_{\mathbb{R}} \partial_\alpha^2 d_1^2(\gamma-\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{9,2}(\gamma,\beta)\, \mathrm{d}\beta \end{aligned}$$

We decompose

$$\begin{aligned}\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{9,2}(\gamma,\beta)\,\mathrm{d}\beta \\ = \partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\left(\frac{\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma)\right)\beta\mathbf{S}_{9,2}(\gamma,\beta)\,\mathrm{d}\beta \\ +\partial_\alpha^2 d_1^2(\gamma)\partial_\alpha(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma)\int_{|\beta|<1}\beta\mathbf{S}_{9,2}(\gamma,\beta)\,\mathrm{d}\beta\end{aligned}$$

and

$$\begin{aligned}\int_{|\beta|<1}\beta\mathbf{S}_{9,2}(\gamma,\beta)d\beta = \int_{|\beta|<1}\left(\frac{\beta^2(2\gamma-\beta)}{Q_2(\gamma,\beta)}-\frac{2\gamma}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \\ +\frac{2\gamma}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\partial_\alpha Q_2(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned}$$

Notice, we will deal with the same kernel, then by taking the $L^2(\partial S_r)$, we can deduce the following

$$\begin{aligned}\left\|\int_{\mathbb{R}}\partial_\alpha^2\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{9,2}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \\ \le c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}\Big].\end{aligned}$$

We derive

$$\|D_{13}\|_{L^2(\partial S_r)}\le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]. \tag{D17}$$

By joining the bounds (D14), (D15), (D16) and (D17), we complete the estimate for the difference

$$\|\mathrm{J}_3(\mathbf{d}^1)-\mathrm{J}_3(\mathbf{d}^2)\|_{L^2(\partial S_r)}\le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}+\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]. \tag{L8}$$

Bound for J_4: Now we move to the difference with J_4, adding and subtracting mixed terms, we obtain the following decomposition

$$-\frac{1}{2}\Big(\mathrm{J}_4(\mathbf{d}^1)(\gamma)-\mathrm{J}_4(\mathbf{d}^2)(\gamma)\Big)=(D_{14}+D_{15}+D_{16}+D_{17})(\gamma),$$

for

$$\begin{aligned}D_{14}(\gamma)&:=\int_{\mathbb{R}}\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\left[\frac{\partial_\alpha\Delta d_1^1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right]\mathrm{d}\beta,\\ D_{15}(\gamma)&:=\int_{\mathbb{R}}\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\left[\frac{\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right]\mathrm{d}\beta,\\ D_{16}(\gamma)&:=\int_{\mathbb{R}}\partial_\alpha(Q_1-Q_2)(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)^2\left[\frac{2}{Q_1(\gamma,\beta)^2}\right]\mathrm{d}\beta,\\ D_{17}(\gamma)&:=\int_{\mathbb{R}}\partial_\alpha Q_2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)^2\left[\frac{1}{Q_1(\gamma,\beta)^2}-\frac{1}{Q_2(\gamma,\beta)^2}\right]\mathrm{d}\beta.\end{aligned}$$

We notice that D_{14} and D_{15} share a similar kernel as in D_{10} and D_{11}. Hence the following $L^2(\partial S_r)$ bounds are automatic

$$\|D_{14}\|_{L^2(\partial S_r)}+\|D_{15}\|_{L^2(\partial S_r)}\le c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^{2,\delta}(\partial S_r)}+\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big]. \tag{D18}$$

To estimate D_{16} and D_{17}, we observe that have kernels similar to D_{12} and D_{13}. We deduce the following bounds

$$\begin{aligned}&\|D_{16}\|_{L^2(\partial S_r)}+\|D_{17}\|_{L^2(\partial S_r)}\\&\qquad\leq c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}+\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big].\end{aligned}\tag{D19}$$

From (D18) and (D19) we conclude

$$\|\mathrm{J}_4(\mathbf{d}^1)-\mathrm{J}_4(\mathbf{d}^2)\|_{L^2(\partial S_r)}\leq c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}+\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big].\tag{L9}$$

Bound for J_5: In the next difference we have to deal with $\mathrm{J}_5(\mathbf{d})$. Then we compute the difference, adding and subtracting mixed terms. We find the next decomposition

$$-\big(J_5(\mathbf{d}^1)(\gamma)-J_5(\mathbf{d}^2)(\gamma)\big)=(D_{18}+D_{19}+D_{20}+D_{21})(\gamma),$$

for

$$\begin{aligned}D_{18}(\gamma)&:=\int_{\mathbb{R}}\frac{\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\Delta z_1^1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\partial_\alpha^2Q_1(\gamma,\beta)\,\mathrm{d}\beta,\\D_{19}(\gamma)&:=\int_{\mathbb{R}}\frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\partial_\alpha^2Q_1(\gamma,\beta)\,\mathrm{d}\beta,\\D_{20}(\gamma)&:=\int_{\mathbb{R}}\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\partial_\alpha^2(Q_1-Q_2)(\gamma,\beta)\,\mathrm{d}\beta,\\D_{21}(\gamma)&:=\int_{\mathbb{R}}\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha^2Q_2(\gamma,\beta)\left[\frac{1}{Q_1(\gamma,\beta)^2}-\frac{1}{Q_2(\gamma,\beta)^2}\right]\mathrm{d}\beta.\end{aligned}$$

To estimate D_{18}, we expand $\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$ then we have

$$\begin{aligned}D_{18}(\gamma):=\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)&\int_{\mathbb{R}}\frac{\Delta z_1^1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\partial_\alpha^2Q_1(\gamma,\beta)\,\mathrm{d}\beta\\&-\int_{\mathbb{R}}\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\frac{\Delta z_1^1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\partial_\alpha^2Q_1(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned}\tag{3.6.51}$$

Then, by using the expression (3.4.57), we expand $\partial_\alpha^2Q_1(\gamma,\beta)$ and from the decomposition (3.4.58), we can deduce the following bound

$$\|D_{18}\|_{L^2(\partial S_r)}\leq c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}.\tag{D20}$$

The same argument can be used for D_{19}, we derive

$$\|D_{19}\|_{L^2(\partial S_r)}\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}.\tag{D21}$$

Now, we move to D_{20}. In this case we compute the second order derivative of the difference $\partial_\alpha^2(Q_1-Q_2)(\gamma,\beta)$, which is given by

$$\begin{aligned}\partial_\alpha^2Q_1(\gamma,\beta)-\partial_\alpha^2Q_2(\gamma,\beta)&=\partial_\alpha^2\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)\\&+2\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma,\beta)\\&+\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha^2\Delta(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma,\beta)\\&+\partial_\alpha^2\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)\\&+2\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)\\&+\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha^2\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta).\end{aligned}$$

Then, we decompose

$$D_{20}(\gamma) = D_{20,1}(\gamma) + D_{20,2}(\gamma) + D_{20,3}(\gamma) + D_{20,4}(\gamma) + D_{20,5}(\gamma) + D_{20,6}(\gamma),$$

for

$$\begin{aligned}
D_{20,1}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma,\beta)\mathbf{M}_1(\gamma,\beta)\,\mathrm{d}\beta,\\
D_{20,2}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha\Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma,\beta)\mathbf{M}_2(\gamma,\beta)\,\mathrm{d}\beta,\\
D_{20,3}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^2\Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma,\beta)\mathbf{M}_3(\gamma,\beta)\,\mathrm{d}\beta,\\
D_{20,4}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{d}^1 - \mathbf{d}^2)_2(\gamma,\beta)\mathbf{M}_4(\gamma,\beta)\,\mathrm{d}\beta,\\
D_{20,5}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha\Delta(\mathbf{d}^1 - \mathbf{d}^2)_2(\gamma,\beta)\mathbf{M}_5(\gamma,\beta)\,\mathrm{d}\beta,\\
D_{20,6}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^2\Delta(\mathbf{d}^1 - \mathbf{d}^2)_2(\gamma,\beta)\mathbf{M}_6(\gamma,\beta)\,\mathrm{d}\beta,
\end{aligned}$$

where the kernels are given by

$$\begin{aligned}
\mathbf{M}_1(\gamma,\beta) &= \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha^2\Delta(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2},\\
\mathbf{M}_2(\gamma,\beta) &= 2\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2},\\
\mathbf{M}_3(\gamma,\beta) &= \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2},\\
\mathbf{M}_4(\gamma,\beta) &= \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha^2\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2},\\
\mathbf{M}_5(\gamma,\beta) &= 2\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2},\\
\mathbf{M}_6(\gamma,\beta) &= \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}.
\end{aligned}$$

To estimate $D_{20,1}$ we expand the second order derivative $\partial_\alpha^2\Delta(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma,\beta)$, then we have

$$\begin{aligned}
D_{20,1}(\gamma) = \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)&\int_{\mathbb{R}} \frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\
&+\int_{\mathbb{R}} \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma-\beta)\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.
\end{aligned} \tag{3.6.52}$$

In order to obtain the $L^2(\partial S_r)$ estimate, we decompose in the *in* and *out* part as follows. For the *in* part, we have the following decomposition

$$\begin{aligned}
&\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1} \frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\
&= \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\left(\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta} - \partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\right)\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\
&+ \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.
\end{aligned}$$

Thus by taking the $L^2(\partial S_r)$ norm, considering the estimates from (3.4.61), inequality (3.6.4) together with the bound

$$\left|\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right|_* \le c_R|\beta| \tag{3.6.53}$$

we get

$$\begin{aligned}\Big\|\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)&\int_{|\beta|<1}\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Big\|_{L^2(\partial S_r)}\\ &\le c_R\|\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}\\ &+\|\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}\Big\|\int_{|\beta|<1}\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Big\|_{L^\infty(\partial S_r)}.\end{aligned} \tag{3.6.54}$$

We notice that the last term is bounded in a similar way to K_3^{in}, as seen in inequality (3.4.23). We have

$$\Big\|\int_{|\beta|<1}\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Big\|_{L^\infty(\partial S_r)}\le c_R.$$

For the *out* part, from inequalities (A6), (A5) and (A2), we have that

$$\left|\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right|_*\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}|\beta|^{-3}. \tag{3.6.55}$$

Then, by taking the $L^2(\partial S_r)$ norm, we have that

$$\begin{aligned}\Big\|\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)\int_{|\beta|>1}\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Big\|_{L^2(\partial S_r)}&\\ \le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}.&\end{aligned} \tag{3.6.56}$$

From inequalities (3.6.54) and (3.6.56) we can deduce

$$\begin{aligned}\Big\|\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)\int_{\mathbb{R}}\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Big\|_{L^2(\partial S_r)}&\\ \le c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].&\end{aligned} \tag{3.6.57}$$

The second part of (3.6.52) can be bounded in a similar way. We decompose the *in* part, as follows

$$\begin{aligned}&\int_{|\beta|<1}\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma-\beta)\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &=\int_{|\beta|<1}\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma-\beta)\Big(\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\Big)\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &+\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{|\beta|<1}\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma-\beta)\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

To deal with the last estimate above we use the decomposition (3.4.65). Thus by taking the $L^2(\partial S_r)$ and making use of the Minkowski's integral inequality, the Cauchy-Schwarz inequality, together with the inequalities (3.6.4) and (3.6.53), we infer

$$\begin{aligned}\Big\|\int_{|\beta|<1}\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma-\beta)\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Big\|_{L^2(\partial S_r)}&\\ \le c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}&\\ +c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\|\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1\|_{L^2(\partial S_r)}.&\end{aligned} \tag{3.6.58}$$

Regarding the *out* part we use the inequality (3.6.55), and then apply the Minkowski integral inequality. We have that

$$\begin{aligned}\Big\|\int_{|\beta|>1}\partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma-\beta)\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\Big\|_{L^2(\partial S_r)} \\ \le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^\infty(\partial S_r)}.\end{aligned}\tag{3.6.59}$$

From inequalities (3.6.57), (3.6.58) and (3.6.59) we can conclude that

$$\|D_{20,1}\|_{L^2(\partial S_r)}\le c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].$$

To estimate $D_{20,2}$ we use the same argument as in $D_{20,1}$. By replacing $\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$ and $\partial_\alpha^2\Delta(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma,\beta)$ by $\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$ and $\partial_\alpha\Delta(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma,\beta)$, we can conclude that

$$\|D_{20,2}\|_{L^2(\partial S_r)}\le c_r\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{C^{2,\delta}(\partial S_r)}+\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big]$$

For the next term $D_{20,3}$, we expand $\partial_\alpha^2\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$, then we get the following decomposition

$$D_{20,3}(\gamma)=\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{\mathbb{R}}\mathbf{M}_3(\gamma,\beta)\,\mathrm{d}\beta-\int_{\mathbb{R}}\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\mathbf{M}_3(\gamma,\beta)\,\mathrm{d}\beta.\tag{3.6.60}$$

Now, we split in the *in* and *out* part. The *in* part decomposes as follows

$$\begin{aligned}&\int_{|\beta|<1}\mathbf{M}_3(\gamma,\beta)d\beta\\ &\quad=\int_{|\beta|<1}\Big(\frac{\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\Big)\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &\quad+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\int_{|\beta|<1}\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

To estimate the last integral we follow the decomposition (3.4.61). Together with inequalities (3.6.4) and (3.6.53), we infer

$$\Big|\int_{|\beta|<1}\mathbf{M}_3(\gamma,\beta)\,\mathrm{d}\beta\Big|_*\le c_R.$$

For the *out* part by inequalities (A6), (A5) and (A2), it follows that

$$|\mathbf{M}_3(\gamma,\beta)|_*\le c_R|\beta|^{-2}.\tag{3.6.61}$$

Thus we can conclude that

$$\Big|\int_{\mathbb{R}}\mathbf{M}_3(\gamma,\beta)\,\mathrm{d}\beta\Big|_*\le c_R.$$

Then by taking the $L^2(\partial S_r)$ norm we derive

$$\Big\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{\mathbb{R}}\mathbf{M}_3(\gamma,\beta)\,\mathrm{d}\beta\Big\|_{L^2(\partial S_R)}\le c_R\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}.\tag{3.6.62}$$

The second part of (3.6.60) can be bounded in a similar way, we decompose the *in* part as follows

$$\begin{aligned}&\int_{|\beta|<1}\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\mathbf{M}_3(\gamma,\beta)\,\mathrm{d}\beta\\ &=\int_{|\beta|<1}\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\Big(\frac{\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\Big)\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\int_{|\beta|<1}\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

Thus, by taking the $L^2(\partial S_r)$ norm, using the inequalities (3.6.4) and (3.6.53), and applying the Minkowski's integral inequality, we infer

$$\begin{aligned}
&\left\| \int_{|\beta|<1} \partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\mathbf{M}_3(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \\
&\quad \le c_R \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)} \\
&\quad + c_R\|\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1\|_{L^\infty(\partial S_r)} \left\| \int_{|\beta|<1} \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)}.
\end{aligned} \tag{3.6.63}$$

The last term in (3.6.63), can be bounded by making use the inequality for (3.4.65). Thus we can deduce the following

$$\left\| \int_{|\beta|<1} \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\frac{\beta\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

The *out* part is bounded by considering the inequality (3.6.61). From inequalities (3.6.62) and (3.6.63) it follows that

$$\|D_{20,3}\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

To estimate $D_{20,4}$, we proceed as in $D_{20,1}$, by expanding $\partial_\alpha\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta)$ we get

$$\begin{aligned}
D_{20,4}(\gamma) = \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma)\int_{\mathbb{R}} \frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \\
+\int_{\mathbb{R}} \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma-\beta)\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.
\end{aligned}$$

Thus, following the estimate for (3.6.52), we can deduce

$$\|D_{20,4}\|_{L^2(\partial S_r)} \le c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}\Big]$$

The estimation for $D_{20,5}$, once again, follows from the estimation for $D_{20,1}$. Then we have that

$$\|D_{20,5}\|_{L^2(\partial S_r)} \le c_R\Big[\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{C^{2,\delta}(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}\Big].$$

Finally, to obtain a bound for $D_{20,6}$ we expand inside the kernel $\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)$ then we get

$$\begin{aligned}
D_{20,6,1}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^2\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\mathbf{M}_{6,1}(\gamma,\beta)\,\mathrm{d}\beta, \\
D_{20,6,2}(\gamma) &:= \int_{\mathbb{R}} \partial_\alpha^2\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\mathbf{M}_{6,2}(\gamma,\beta)\,\mathrm{d}\beta.
\end{aligned}$$

Thus, the bound for $D_{20,6,1}(\gamma)$ follows from the estimate for $D_{20,3}(\gamma)$ and we have

$$\|D_{20,6,1}\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

For $D_{20,6,2}(\gamma)$ we expand $\partial_\alpha^2\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)$. Then we get

$$D_{20,6,2}(\gamma) := \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma)\int_{\mathbb{R}}\mathbf{M}_{6,2}(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma-\beta)\mathbf{M}_{6,2}(\gamma,\beta)\,\mathrm{d}\beta.$$

We notice that in the first integral above is similar to K_4. From estimates (3.4.26), we can derive the following

$$\left| \int_{\mathbb{R}} \mathbf{M}_{6,2}(\gamma,\beta)\,\mathrm{d}\beta \right|_* \le c_R.$$

For the second integral, we notice that is similar to $K_4^*(\gamma)$. By using the estimates (3.4.51), we deduce that

$$\left\| \int_{\mathbb{R}} \partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma-\beta)\mathbf{M}_{6,2}(\gamma,\beta)\,\mathrm{d}\beta \right\|_{L^2(\partial S_r)} \le c_R\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}$$

By joining the estimates for $D_{20,i}$, we can deduce that

$$\|D_{20}\|_{L^2(\partial S_r)} \le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]. \tag{D22}$$

In the next term D_{21} we use the difference given in (3.5.3), to

$$D_{21}(\gamma) = (D_{21,1}+D_{21,2}+D_{21,3}+D_{21,4})(\gamma)$$

for

$$\begin{aligned} D_{21,1}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_{11}(\gamma,\beta)\,\mathrm{d}\beta,\\ D_{21,2}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_{12}(\gamma,\beta)\,\mathrm{d}\beta,\\ D_{21,3}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{13}(\gamma,\beta)\,\mathrm{d}\beta,\\ D_{21,4}(\gamma) &:= \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{14}(\gamma,\beta)\,\mathrm{d}\beta, \end{aligned}$$

where the kernels $\mathbf{S}_i(\gamma,\beta)$ are given by

$$\begin{aligned} \mathbf{S}_{11}(\gamma,\beta) &:= \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)},\\ \mathbf{S}_{12}(\gamma,\beta) &:= \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)^2},\\ \mathbf{S}_{13}(\gamma,\beta) &:= \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)},\\ \mathbf{S}_{14}(\gamma,\beta) &:= \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)^2}. \end{aligned}$$

To estimate $D_{21,1}$, we expand $\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)$, then we get

$$D_{21,1}(\gamma) = (\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{\mathbb{R}}\mathbf{S}_{11}(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}}(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma-\beta)\mathbf{S}_{11}(\gamma,\beta)\,\mathrm{d}\beta. \tag{3.6.64}$$

In order to obtain an $L^2(\partial S_r)$ bound we require an $L^\infty(\partial S_r)$. for the first integral. We decompose in the *in* and *out* parts. Thus

$$\begin{aligned} &\int_{|\beta|<1}\mathbf{S}_{11}(\gamma,\beta)\,\mathrm{d}\beta\\ &\quad= \int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta\Delta z_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha^2Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\ &\quad+\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\beta\mathbf{S}_7(\gamma,\beta)\,\mathrm{d}\beta. \end{aligned} \tag{3.6.65}$$

We notice that the last integral in (3.6.65), can be bounded following the decomposition (3.6.45). By using inequalities (A6), (A5) and the expression (3.4.57), we deduce that

$$\left|\frac{\partial_\alpha^2 Q_2(\gamma,\beta)}{\beta^2}\right|_* \le c_R. \tag{3.6.66}$$

From the previoues bound (3.6.66) and the inequalities (A4), (A5), the arc-chord condition (A2) and Lemma 32, we can derive that

$$\left| \int_{|\beta|<1} \mathbf{S}_{11}(\gamma,\beta) d\beta \right|_* \leq c_R.$$

For the *out* part, from inequalities (3.6.66), (A6), (A5) and (A2), we obtain the following bound

$$\left| \mathbf{S}_{11}(\gamma,\beta) \right|_* \leq c_R |\beta|^{-2}. \tag{3.6.67}$$

Thus, we get

$$\left| \int_{|\beta|>1} \mathbf{S}_{11}(\gamma,\beta)\, \mathrm{d}\beta \right|_* \leq c_R.$$

Therefore by taking the $L^2(\partial S_r)$ norm, we can deduce the following

$$\begin{aligned} \left\| (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma) \int_{\mathbb{R}} \mathbf{S}_{11}(\gamma,\beta)\, \mathrm{d}\beta \right\|_{L^2(\partial S_r)} &\leq \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)} \left\| \int_{\mathbb{R}} \mathbf{S}_{11}(\gamma,\beta)\, \mathrm{d}\beta \right\|_{L^\infty(\partial S_R)} \\ &\leq c_R \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \end{aligned} \tag{3.6.68}$$

Regarding to the second integral in (3.6.64), for the *in* part, we have the following decomposition

$$\begin{aligned} &\int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma - \beta) \mathbf{S}_{11}(\gamma,\beta)\, \mathrm{d}\beta \\ &\qquad = \int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma - \beta) \left(\frac{\partial_\alpha \Delta d_1^2(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1^2(\gamma) \right) \beta \mathbf{S}_7(\gamma,\beta)\, \mathrm{d}\beta \\ &\qquad + \partial_\alpha^2 d_1^2(\gamma) \int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma - \beta) \beta \mathbf{S}_7(\gamma,\beta)\, \mathrm{d}\beta. \end{aligned}$$

The control in $L^2(\partial S_r)$, follows the same lines as in (3.6.49). Thus by taking the $L^2(\partial S_r)$ norm and making use the Minkowski's integral inequality we get that

$$\left\| \int_{|\beta|<1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma - \beta) \mathbf{S}_{11}(\gamma,\beta)\, \mathrm{d}\beta \right\|_{L^2(\partial S_r)} \leq c_R \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \tag{3.6.69}$$

Regarding the *out* part, we use the inequality (3.6.67) and apply the Minkowski's integral inequality. Then we have that

$$\left\| \int_{|\beta|>1} (\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma - \beta) \mathbf{S}_{11}(\gamma,\beta)\, \mathrm{d}\beta \right\|_{L^2(\partial S_r)} \leq c_R \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \tag{3.6.70}$$

From the inequalities (3.6.68), (3.6.69) and (3.6.70), we conclude that

$$\|D_{21,1}\|_{L^2(\partial S_r)} \leq c_R \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

For the second term $D_{21,2}$, we can argue in a similar way as in the previous term. Then we deduce that

$$\|D_{21,2}\|_{L^2(\partial S_r)} \leq c_R \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

To estimate $D_{21,3}$, we expand inside the kernel the next term

$$\Delta(\mathbf{z}^1 + \mathbf{z}^2)_2(\gamma,\beta) = \Delta(\mathbf{d}^1 + \mathbf{d}^2)_2(\gamma,\beta) + 2\beta(\gamma - \beta).$$

Then we decompose in the following way

$$D_{21,3}(\gamma) := D_{21,3,1}(\gamma) + D_{21,3,2}(\gamma),$$

for

$$D_{21,3,1}(\gamma) := \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{13,1}(\gamma,\beta)\,\mathrm{d}\beta,$$
$$D_{21,3,2}(\gamma) := \int_{\mathbb{R}} \Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{13,2}(\gamma,\beta)\,\mathrm{d}\beta,$$

where the kernels are given by

$$\mathbf{S}_{13,1}(\gamma,\beta) = \frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)},$$
$$\mathbf{S}_{13,2}(\gamma,\beta) = 2\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)\beta(2\gamma-\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)}.$$

The $L^2(\partial S_r)$ bound for $D_{21,3,1}$, follows from the estimate for $D_{21,1}$. We have that

$$\|D_{21,3,1}\|_{L^2(\partial S_r)} \le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

To deal with $D_{21,3,2}(\gamma)$, we expand $\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)$. We have that

$$D_{21,3,2}(\gamma) = (\mathbf{d}^2-\mathbf{d}^1)_2(\gamma)\int_{\mathbb{R}} \mathbf{S}_{13,2}(\gamma,\beta)\,\mathrm{d}\beta - \int_{\mathbb{R}} (\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\mathbf{S}_{13,2}(\gamma,\beta)\,\mathrm{d}\beta. \tag{3.6.71}$$

For the *in* part we have the following decomposition

$$\begin{aligned}
&\frac{1}{2}\int_{|\beta|<1} \mathbf{S}_{13,2}(\gamma,\beta)d\beta \\
&\quad= \int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta} - \partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)}\,\mathrm{d}\beta \\
&\quad+ \partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2(2\gamma-\beta)}{Q_2(\gamma,\beta)} - \frac{2\gamma}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\right)\frac{(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \\
&\quad+ \frac{\partial_\alpha^2 d_1^2(\gamma)2\gamma}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.
\end{aligned} \tag{3.6.72}$$

From the inequalities (3.6.66), (A4), (A5), (A2) and Corollary 2, we obtain that

$$\left|\int_{|\beta|<1} \mathbf{S}_{13,2}(\gamma,\beta)\,\mathrm{d}\beta\right|_* \le c_R.$$

For the *out* part, using the same bounds as in (3.6.67), we obtain the following

$$\left|\,\mathbf{S}_{13,2}(\gamma,\beta)\,\right|_* \le c_R|\beta|^{-2}, \tag{3.6.73}$$

this inequality yields to

$$\left|\int_{|\beta|>1} \mathbf{S}_{13,2}(\gamma,\beta)\,\mathrm{d}\beta\right|_* \le c_R.$$

Thus, by taking the $L^2(\partial S_r)$ norm, we obtain that

$$\begin{aligned}
\left\|(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma)\int_{\mathbb{R}} \mathbf{S}_{13,2}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} &\le \|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\left\|\int_{\mathbb{R}} \mathbf{S}_{13,2}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^\infty(\partial S_R)} \\
&\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.
\end{aligned}$$

Finally, for the second term in (3.6.71), we decompose the *in* part, as follows

$$\begin{aligned}
&\frac{1}{2}\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\mathbf{S}_{13,2}(\gamma,\beta)\,\mathrm{d}\beta\\
&\quad=\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\
&\quad+\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\left(\frac{\beta^2}{Q_2(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha^2 Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\
&\quad+\frac{\partial_\alpha^2 d_1^2(\gamma)}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma)\partial_\alpha^2 Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.
\end{aligned}$$

Then, by taking the $L^2(\partial S_r)$, and using the same estimates as in (3.6.72), together with the Minkowski's integral inequality, we get

$$\left\|\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\mathbf{S}_{11}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

Regarding to the *out* part we use the inequality (3.6.73). We deduce that

$$\|D_{21,3,2}\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

Hence

$$\|D_{21,3}\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

To deal with $D_{21,4}$, we can argue as in the previous term, this yields to

$$\|D_{21,4}\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

By combining the previous estimates for $D_{21,i}$, we get

$$\|D_{21}\|_{L^2(\partial S_r)}\le c_R\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}. \tag{D23}$$

The last inequality together with (D20), (D21) and (D22) allows us to derive the complete estimate for the difference

$$\begin{aligned}
&\|\mathrm{J}_5(\mathbf{d}^1)-\mathrm{J}_5(\mathbf{d}^2)\|_{L^2(\partial S_r)}\\
&\le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)}+\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}+\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big].
\end{aligned}\tag{L10}$$

Bound for J_6: For the last term J_6, as in the previous terms, we add and subtract mixed terms, to obtain the following decomposition

$$\frac{1}{2}(J_6(\mathbf{d}^1)(\gamma)-J_6(\mathbf{d}^2)(\gamma))=(D_{22}+D_{23}+D_{24}+D_{25})(\gamma),$$

for

$$\begin{aligned}
D_{22}(\gamma)&:=\int_{\mathbb{R}}\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\frac{\Delta z_1^1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
D_{23}(\gamma)&:=\int_{\mathbb{R}}\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
D_{24}(\gamma)&:=\int_{\mathbb{R}}\left[\partial_\alpha Q_1(\gamma,\beta)^2-\partial_\alpha Q_2(\gamma,\beta)^2\right]\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta,\\
D_{25}(\gamma)&:=\int_{\mathbb{R}}\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2\left[\frac{1}{Q_1(\gamma,\beta)^3}-\frac{1}{Q_2(\gamma,\beta)^3}\right]\mathrm{d}\beta.
\end{aligned}\tag{3.6.74}$$

For the first term D_{22}, we expand $\partial_\alpha \Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma, \beta)$, then we get

$$D_{22}(\gamma) = \partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma) \int_{\mathbb{R}} \frac{\Delta z_1^1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta \\ - \int_{\mathbb{R}} \partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma - \beta)\frac{\Delta z_1^1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta,$$

we observe that D_{21} has the same kernel as we found in J_6, see equation (3.4.2). Therefore, the $L^2(\partial S_r)$ bound is automatic. Thus,

$$\|D_{22}\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \tag{D24}$$

To estimate D_{23}, we proceed in a similar way. By expanding $\Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma, \beta)$, we have that

$$D_{23}(\gamma) = (\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma) \int_{\mathbb{R}} \frac{\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta \\ - \int_{\mathbb{R}} (\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma - \beta)\frac{\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta,$$

Then we have that

$$\|D_{23}\|_{L^2(\partial S_r)} \leq c_R \|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \tag{D25}$$

In the next term D_{24}, we use equation (3.6.34) to decompose

$$\partial_\alpha Q_1(\gamma,\beta)^2 - \partial_\alpha Q_2(\gamma,\beta)^2 = (\partial_\alpha Q_1(\gamma,\beta) - \partial_\alpha Q_2(\gamma,\beta))(\partial_\alpha Q_1(\gamma,\beta) + \partial_\alpha Q_2(\gamma,\beta))$$

then substituting the previous term in D_{24}, we get

$$D_{24}(\gamma) = D_{24,1}(\gamma) + D_{24,2}(\gamma),$$

for

$$D_{24,1}(\gamma); = \int_{\mathbb{R}} \partial_\alpha(Q_1 - Q_2)(\gamma,\beta)\frac{\partial_\alpha \Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta,$$
$$D_{24,2}(\gamma) := \int_{\mathbb{R}} \partial_\alpha(Q_1 - Q_2)(\gamma,\beta)\frac{\partial_\alpha \Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta.$$

Then by using the expression (3.6.34) we decompose further,

$$D_{24,1}(\gamma) = D_{24,1,1}(\gamma) + D_{24,1,2}(\gamma) + D_{24,1,3}(\gamma) + D_{24,1,4}(\gamma)$$

for

$$D_{24,1,1}(\gamma) = \int_{\mathbb{R}} \Delta(\mathbf{z}^1 - \mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\,\mathrm{d}\beta,$$
$$D_{24,1,2}(\gamma) = \int_{\mathbb{R}} \partial_\alpha \Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_2^*(\gamma,\beta)\,\mathrm{d}\beta,$$
$$D_{24,1,3}(\gamma) = \int_{\mathbb{R}} \Delta(\mathbf{z}^1 - \mathbf{z}^2)_2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_3^*(\gamma,\beta)\,\mathrm{d}\beta,$$
$$D_{24,1,4}(\gamma) = \int_{\mathbb{R}} \partial_\alpha \Delta(\mathbf{d}^1 - \mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_4^*(\gamma,\beta)\,\mathrm{d}\beta.$$

where each kernel is given by

$$\mathbf{N}_1^*(\gamma,\beta) = \frac{\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^3},$$
$$\mathbf{N}_2^*(\gamma,\beta) = \frac{\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^3},$$
$$\mathbf{N}_3^*(\gamma,\beta) = \frac{\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^3},$$
$$\mathbf{N}_4^*(\gamma,\beta) = \frac{\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^3}.$$

For the first term $D_{24,1,2}$, we expand $\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)$. We obtain

$$\begin{aligned} D_{24,1,2}(\gamma) = (\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)&\int_{\mathbb{R}}\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\,\mathrm{d}\beta \\ &-\int_{\mathbb{R}}(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\,\mathrm{d}\beta. \end{aligned} \tag{3.6.75}$$

In order to obtain the $L^2(\partial S_r)$ bound, we first find the $L^\infty(\partial S_r)$ bound of the first integral. We decompose the *in* part as follows

$$\begin{aligned} &\int_{|\beta|<1}\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\,\mathrm{d}\beta \\ &= \int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\right)\frac{\beta\partial_\alpha Q_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta \\ &+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta^2\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta \\ &+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2}{Q_1(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta \\ &+\frac{\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\partial_\alpha^2 d_1^2(\gamma)}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta. \end{aligned} \tag{3.6.76}$$

Combining the estimates (A6), (A5) and the expression (3.4.57) we deduce that

$$\left|\frac{\partial_\alpha Q_1(\gamma,\beta)}{\beta^2}\right|_* \leq c_R. \tag{3.6.77}$$

Thus, from inequalities (3.6.77), (A4), (A5), (A2) and Lemma 32 we control the first three terms in (3.6.76). To estimate the last term in (3.6.76), we use the bound for J_3, see estimates for (3.4.14). We obtain that

$$\left|\int_{|\beta|<1}\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\,\mathrm{d}\beta\right|_* \leq c_R.$$

For the *out* part, by combining (3.6.77) (A6), (A5) and (A2) we get

$$\left|\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\right|_* \leq c_R|\beta|^{-2}. \tag{3.6.78}$$

Thus

$$\left|\int_{|\beta|>1}\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\,\mathrm{d}\beta\right|_* \leq c_R.$$

By taking the $L^2(\partial S_r)$ norm, we get

$$\left\|(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma)\int_{\mathbb{R}}\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \tag{3.6.79}$$

The estimate of the second part in (3.6.75) can be deduce in the same manner. We decompose the *in* part as follows

$$\begin{aligned}
&\int_{|\beta|<1}(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\,\mathrm{d}\beta\\
&=\int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\right)\frac{\beta\partial_\alpha Q_1(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta\\
&+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\int_{|\beta|<1}(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta^2\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta\\
&+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\left(\frac{\beta^2}{Q_1(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\
&+\frac{\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\partial_\alpha^2 d_1^2(\gamma)}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\int_{|\beta|<1}(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.
\end{aligned}$$

Now, by taking the $L^2(\partial S_r)$ and making use of the Minkowski's integral inequality, we deduce the following

$$\left\|\int_{|\beta|<1}(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

Regarding the *out* part, we use the inequality (3.6.78) and the Minkowski's integral inequality. We infer

$$\left\|\int_{\mathbb{R}}(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma-\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_1^*(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

We combine (3.6.79) with the previous estimate, we deduce

$$\|D_{24,1,1}\|_{L^2(\partial S_r)}\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

The bounds for $D_{24,1,2}$ and $D_{24,1,3}$ can be deduce from the previous estimate, then we have that

$$\|D_{24,1,2}\|_{L^2(\partial S_r)}\leq c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}$$

and

$$\|D_{24,1,3}\|_{L^2(\partial S_r)}\leq c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

To deal with $D_{21,1,4}$ we expand

$$\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)=\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta)+2\beta(2\gamma-\beta).$$

Then we decompose

$$D_{24,1,4}(\gamma)=D_{24,1,4,1}(\gamma)+D_{24,1,4,2}(\gamma)$$

for

$$\begin{aligned}
D_{24,1,4,1}(\gamma)&=\int_{\mathbb{R}}\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_{4,1}^*(\gamma,\beta)\,\mathrm{d}\beta,\\
D_{24,1,4,2}(\gamma)&=\int_{\mathbb{R}}\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}_{4,2}^*(\gamma,\beta)\,\mathrm{d}\beta.
\end{aligned}$$

Where the kernels are given by

$$\mathbf{N}^*_{4,1}(\gamma,\beta) := \frac{\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^3},$$
$$\mathbf{N}^*_{4,2}(\gamma,\beta) := 2\frac{\beta(2\gamma-\beta)\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^3}.$$

To estimate $D_{24,1,4,1}$, we argue as in the estimation for $D_{21,1,1}$. Hence we obtain

$$\|D_{24,1,4,1}\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

For the next term $D_{24,1,4,2}$ we expand $\partial_\alpha\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)$, then we have

$$\begin{aligned}D_{24,1,4,2}(\gamma) = \partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma)&\int_{\mathbb{R}}\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}^*_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\\ &-\int_{\mathbb{R}}\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma-\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}^*_{4,2}(\gamma,\beta)\,\mathrm{d}\beta.\end{aligned} \tag{3.6.80}$$

We will estimate the $L^\infty(\partial S_r)$ norm of the first integral in the previous term. We decompose the *in* part as follows

$$\begin{aligned}&\frac{1}{2}\int_{|\beta|<1}\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}^*_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\\ &\quad=\int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta\\ &\quad+\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2(2\gamma-\beta)}{Q_1(\gamma,\beta)}-\frac{2\gamma}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &\quad+\frac{2\gamma\partial_\alpha^2 d_1^2(\gamma)}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned} \tag{3.6.81}$$

From inequalities (3.6.77), (A4), (A5), (A2) and Corollary 2, we derive the following

$$\left|\int_{|\beta|<1}\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}^*_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\right|_* \leq c_R$$

Regarding the *out* part, by combining the inequalities (3.6.77), (A6), (A5) and (A2), we get

$$\left|\int_{|\beta|>1}\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}^*_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\right|_* \leq c_R\int_{|\beta|>1}|\beta|^{-2}d\beta < c_R.$$

By taking the $L^2(\partial S_r)$ norm and making use of the above estimates, we get

$$\left\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma)\int_{\mathbb{R}}\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}^*_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

To estimate the second integral in (3.6.80), we decompose the *in* part as follows

$$\begin{aligned}&\frac{1}{2}\int_{|\beta|<1}\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma-\beta)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}^*_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\\ &=\int_{|\beta|<1}\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta\\ &+\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma-\beta)\left(\frac{\beta^2(2\gamma-\beta)}{Q_1(\gamma,\beta)}-\frac{2\gamma}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta\\ &+\frac{2\gamma\partial_\alpha^2 d_1^2(\gamma)}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\int_{|\beta|<1}\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma-\beta)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.\end{aligned}$$

Then, by taking the $L^2(\partial S_r)$ norm and making use of the Minkowski's integral inequality we get

$$\Big\|\int_{|\beta|<1}\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma)\partial_\alpha Q_1(\gamma,\beta)\mathbf{N}^*_{4,2}(\gamma,\beta)\,\mathrm{d}\beta\Big\|_{L^2(\partial S_r)}\le c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)},$$

where we have used the estimates (3.6.77), (A4), (A5), (A2) and Corollary 2. This yields

$$\|D_{24,1,4,2}\|_{L^2(\partial S_r)}\le c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

Therefore

$$\|D_{24,1,4}\|_{L^2(\partial S_r)}\le c_R\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

Combining the estimates $D_{24,1,i}$, we obtain

$$\|D_{24,1}\|_{L^2(\partial S_r)}\le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big].$$

The estimate of the second $D_{24,2}$ can be deduce in the same manner as in $D_{24,1}$. We infer the following

$$\|D_{24,2}\|_{L^2(\partial S_r)}\le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]$$

and then

$$\|D_{24}\|_{L^2(\partial S_r)}\le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big].\tag{D26}$$

It remains to prove the estimate for D_{25}. By computing the difference of cubes, we obtain that

$$D_{25}(\gamma)=D_{25,1}(\gamma)+D_{25,2}(\gamma)+D_{25,3}(\gamma),$$

for

$$D_{25,1}(\gamma):=\int_{\mathbb{R}}\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^2}\left[\frac{1}{Q_1(\gamma,\beta)}-\frac{1}{Q_2(\gamma,\beta)}\right]\mathrm{d}\beta,$$
$$D_{25,2}(\gamma):=\int_{\mathbb{R}}\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\left[\frac{1}{Q_1(\gamma,\beta)}-\frac{1}{Q_2(\gamma,\beta)}\right]\mathrm{d}\beta,$$
$$D_{25,3}(\gamma):=\int_{\mathbb{R}}\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_2(\gamma,\beta)^2}\left[\frac{1}{Q_1(\gamma,\beta)}-\frac{1}{Q_2(\gamma,\beta)}\right]\mathrm{d}\beta.$$

To estimate $D_{25,1}$ we use (3.5.3) to decompose further. We have

$$D_{25,1}(\gamma)=D_{25,1,1}(\gamma)+D_{25,1,2}(\gamma),$$

for

$$D_{25,1,1}(\gamma):=\int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_{15}(\gamma,\beta)\,\mathrm{d}\beta,$$
$$D_{25,1,2}(\gamma):=\int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{16}(\gamma,\beta)\,\mathrm{d}\beta.$$

Where the kernels are given by

$$\mathbf{S}_{15}(\gamma,\beta)=\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^3Q_2(\gamma,\beta)},$$
$$\mathbf{S}_{16}(\gamma,\beta)=\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^3Q_2(\gamma,\beta)}.$$

To deal with $D_{25,1,1}$, we expand $\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta$, then we have

$$D_{25,1,1}(\gamma)=(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{\mathbb{R}}\mathbf{S}_{15}(\gamma,\beta)\,\mathrm{d}\beta-\int_{\mathbb{R}}(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma-\beta)(\gamma,\beta)\mathbf{S}_{15}(\gamma,\beta)\,\mathrm{d}\beta. \tag{3.6.82}$$

In order to obtain an $L^2(\partial S_r)$ bound, we require the $L^\infty(\partial S_r)$ estimate of the first integral in (3.6.82). For the *in* part, we have

$$\begin{aligned}&\int_{|\beta|<1}\mathbf{S}_{15}(\gamma,\beta)\,\mathrm{d}\beta\\&=\int_{|\beta|<1}\left(\frac{\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\right)\frac{\beta\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\&+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta^2\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\&+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2}{Q_2(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta\\&+\frac{\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\partial_\alpha^2 d_1^2(\gamma)}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta.\end{aligned} \tag{3.6.83}$$

We note that the last integral in (3.6.83) can be bounded by using the estimates for J_6, as provided in the estimates for the decomposition (3.4.77). Thus

$$\left|\int_{|\beta|<1}\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta\right|_*\le c_R.$$

Then, from inequalities (3.6.77), (A5), (A6), (A4), (A2), (A4) and Lemma 32 we deduce that

$$\left|\int_{|\beta|<1}\mathbf{S}_{15}(\gamma,\beta)\,\mathrm{d}\beta\right|_*\le c_R.$$

Regarding the *out* part, once again by using the same inequalities, we can deduce the following bound

$$\left|\mathbf{S}_{15}(\gamma,\beta)\right|_*\le c_R|\beta|^{-2} \tag{3.6.84}$$

and hence

$$\left|\int_{|\beta|>1}\mathbf{S}_{15}(\gamma,\beta)\,\mathrm{d}\beta\right|_*\le c_R.$$

Then, by taking the $L^2(\partial S_r)$ norm, we get

$$\left\|(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma)\int_{\mathbb{R}}\mathbf{S}_{15}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

To estimate the second integral in (3.6.82), we use a similar decomposition. For the *in* part we have

$$\begin{aligned}&\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma-\beta)\mathbf{S}_{15}(\gamma,\beta)\,\mathrm{d}\beta\\&=\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma-\beta)\left(\frac{\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{\beta}-\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\right)\frac{\beta\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\&+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta^2\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\&+\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma-\beta)\left(\frac{\beta^2}{Q_2(\gamma,\beta)}-\frac{1}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta\\&+\frac{\partial_\alpha(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\partial_\alpha^2 d_1^2(\gamma)}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma-\beta)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta.\end{aligned} \tag{3.6.85}$$

The bound for the last integral in the previous decomposition (3.6.85) follows from the estimates for J_6, see (3.4.77). Thus, by taking the $L^2(\partial S_r)$ norm and making use of the Minkowski's integral inequality, together with the inequalities (3.6.77), (A5), (A6), (A4), (A2), (A4) and Lemma 32, we deduce

$$\left\|\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma-\beta)\mathbf{S}_{15}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

The bound for the *out* part, follows from estimate (3.6.84). We obtain that

$$\|D_{25,1,1}\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}. \tag{3.6.86}$$

For $D_{25,1,2}$, we expand $\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)$, inside the kernel $\mathbf{S}_{16}(\gamma,\beta)$, then

$$D_{25,1,2}(\gamma):=D_{25,1,2,1}(\gamma)+D_{25,1,2,2}(\gamma),$$

for

$$\begin{aligned}D_{25,1,2,1}(\gamma)&:=\int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_1(\gamma,\beta)\mathbf{S}_{16,1}(\gamma,\beta)\,\mathrm{d}\beta,\\ D_{25,1,2,2}(\gamma)&:=\int_{\mathbb{R}}\Delta(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma,\beta)\mathbf{S}_{16,2}(\gamma,\beta)\,\mathrm{d}\beta\end{aligned}$$

where

$$\begin{aligned}\mathbf{S}_{16,1}(\gamma,\beta)&=\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2\Delta(\mathbf{d}^1+\mathbf{d}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^3Q_2(\gamma,\beta)},\\ \mathbf{S}_{16,2}(\gamma,\beta)&=2\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2\beta(2\gamma-\beta)}{Q_1(\gamma,\beta)^3Q_2(\gamma,\beta)}\end{aligned}$$

The estimate for $D_{25,1,2,1}$. follows from the estimation for $D_{25,1,1}$. We obtain

$$\|D_{25,1,2,1}\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}. \tag{3.6.87}$$

To estimate $D_{25,1,2,2}$, we expand $\Delta(\mathbf{d}^1-\mathbf{d}^2)_2(\gamma,\beta)$. We have

$$D_{25,1,2,2}(\gamma)=(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma)\int_{\mathbb{R}}\mathbf{S}_{16,2}(\gamma,\beta)\,\mathrm{d}\beta-\int_{\mathbb{R}}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\mathbf{S}_{16,2}(\gamma,\beta)\,\mathrm{d}\beta. \tag{3.6.88}$$

We decompose the *in* part as follows

$$\begin{aligned}\frac{1}{2}\int_{|\beta|<1}\mathbf{S}_{16,2}(\gamma,\beta)d\beta&=\int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\ &\quad+\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}\left(\frac{\beta^2(2\gamma-\beta)}{Q_2(\gamma,\beta)}-\frac{2\gamma}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta\\ &\quad+\frac{2\gamma\partial_\alpha^2 d_1^2(\gamma)}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\int_{|\beta|<1}\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta.\end{aligned}$$

Using Corollary 2 and combining the estimates for (3.6.81) and (3.6.83), we deduce

$$\left|\int_{|\beta|<1}\mathbf{S}_{16,2}(\gamma,\beta)\,\mathrm{d}\beta\right|_*\le c_R.$$

The *out* part, can be bounded by considering the following estimate

$$\left|\mathbf{S}_{16,2}(\gamma,\beta)\right|_*\le c_R|\beta|^{-2}. \tag{3.6.89}$$

Then, by taking the $L^2(\partial S_r)$ norm, we deduce

$$\left\|(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma)\int_{\mathbb{R}}\mathbf{S}_{16,2}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}. \tag{3.6.90}$$

Next, we estimate the second integral in (3.6.88). We have the following decomposition

$$\begin{aligned}
&\frac{1}{2}\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\mathbf{S}_{16,2}(\gamma,\beta)\,\mathrm{d}\beta\\
&=\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\left(\frac{\partial_\alpha\Delta d_1^2(\gamma,\beta)}{\beta}-\partial_\alpha^2 d_1^2(\gamma)\right)\frac{\beta^2(2\gamma-\beta)\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3Q_2(\gamma,\beta)}\,\mathrm{d}\beta\\
&+\partial_\alpha^2 d_1^2(\gamma)\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\left(\frac{\beta^2(2\gamma-\beta)}{Q_2(\gamma,\beta)}-\frac{2\gamma}{|\partial_\alpha\mathbf{z}^2(\gamma)|^2}\right)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta\\
&+\frac{2\gamma\partial_\alpha^2 d_1^2(\gamma)}{|\partial_\alpha\mathbf{z}^1(\gamma)|^2}\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\frac{\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta.
\end{aligned}$$

By taking the $L^2(\partial S_r)$ norm and using the Minkowski's integral inequality, we derive

$$\left\|\int_{|\beta|<1}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\mathbf{S}_{16,2}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

Finally, by considering the bound (3.6.89) we can obtain an estimate for the *out* part. This yields to

$$\left\|\int_{\mathbb{R}}(\mathbf{d}^2-\mathbf{d}^1)_2(\gamma-\beta)\mathbf{S}_{16,2}(\gamma,\beta)\,\mathrm{d}\beta\right\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}. \tag{3.6.91}$$

Putting together the estimates (3.6.90) and (3.6.91) we arrive to the following inequality

$$\|D_{25,1,2,2}\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}.$$

and therefore from the previous inequality and (3.6.87), we obtain that

$$\|D_{25,1,2}\|_{L^2(\partial S_r)}\le c_R\|(\mathbf{d}^1-\mathbf{d}^2)_2\|_{L^2(\partial S_r)}. \tag{3.6.92}$$

Combining the previous bound (3.6.92) and the estimation (3.6.86), we deduce

$$\|D_{25,1}\|_{L^2(\partial S_r)}\le c_R\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}.$$

The estimates for $D_{25,2}$ and $D_{25,3}$ follows the same argument. Then we can infer the following bound

$$\|D_{25}\|_{L^2(\partial S_r)}\le c_R\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}. \tag{D27}$$

We combine the $L^2(\partial S_r)$ estimates (D24), (D25), (D26) and the previous (D27), then we conclude that

$$\|\mathrm{J}_6(\mathbf{d}^1)-\mathrm{J}_6(\mathbf{d}^2)\|_{L^2(\partial S_r)}\le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]. \tag{L11}$$

From estimates (L7), (L8), (L9), (L10) and (L11), finally we arrive to

$$\begin{aligned}
\|\mathrm{E}_1(\mathbf{d}^1)-\mathrm{E}_1(\mathbf{d}^2)\|_{L^2(\partial S_r)}&\le\sum_{i=2}^{6}\|\mathrm{J}_i(\mathbf{d}^1)-\mathrm{J}_i(\mathbf{d}^2)\|_{L^2(\partial S_r)}\\
&\le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)}+\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}\\
&\qquad+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}+\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big].
\end{aligned}$$

■

Combining the last inequality (L6) with the estimate (L5), we obtain the following

$$\begin{aligned}
&\|\partial_\alpha^2F_1(\mathbf{d}^1)-\partial_\alpha^2F_1(\mathbf{d}^2)\|_{L^2(\partial S_r)}\\
&\qquad\le c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)}+\|\mathbf{d}^1-\mathbf{d}^2\|_{X_{r,2}}+\|\partial_\alpha^3(\mathbf{d}^1-\mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].
\end{aligned} \tag{L12}$$

3.6.2 Lipschitz Condition of $\partial_\alpha^2 F_2$

Now we deal with the second coordinate, we have the following lemma.

Lemma 15. *Given two deviations* $\mathbf{d}^1, \mathbf{d}^2 \in O_R$, *the following inequality holds*

$$\begin{aligned} &\|\partial_\alpha^2 F_2(\mathbf{d}^1) - \partial_\alpha^2 F_2(\mathbf{d}^2)\|_{L^2(\partial S_r)} \\ &\qquad \le c_R \Big[\|\mathbf{d}^1 - \mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)} + \|\mathbf{d}^1 - \mathbf{d}^2\|_{X_{r,2}} + \|\partial_\alpha^3 (\mathbf{d}^1 - \mathbf{d}^2)_2\|_{L^2(\partial S_r)} \Big]. \end{aligned} \tag{L13}$$

Proof. For the second coordinate, we decompose

$$\partial_\alpha^2 F_2(\mathbf{d})(\gamma) = \mathcal{F}_1(\mathbf{d})(\gamma) + \mathcal{F}_2(\mathbf{d})(\gamma),$$

for

$$\begin{aligned} \mathcal{F}_1(\mathbf{d})(\gamma) &= \partial_\alpha^2 \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)} \partial_\alpha \Delta d_2(\gamma,\beta)\,\mathrm{d}\beta, \\ \mathcal{F}_2(\mathbf{d})(\gamma) &= 2\partial_\alpha^2 \int_{\mathbb{R}} \frac{\Delta z_1(\gamma,\beta)\beta}{Q(\gamma,\beta)}\,\mathrm{d}\beta. \end{aligned}$$

To estimate the difference $\mathcal{F}_1(\mathbf{d}^1) - \mathcal{F}_2(\mathbf{d}^2)$, we make use of the decomposition (3.4.113). By replacing $\partial_\alpha \Delta d_1^i(\gamma,\beta)$ by $\partial_\alpha \Delta d_2^i(\gamma,\beta)$. Hence we deduce

$$\|\mathcal{F}_1(\mathbf{d}^1) - \mathcal{F}_1(\mathbf{d}^2)\|_{L^2(\partial S_r)} \le c_R \Big[\|\mathbf{d}^1 - \mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)} + \|\mathbf{d}^1 - \mathbf{d}^2\|_{X_{r,2}} + \|\partial_\alpha^3 (\mathbf{d}^1 - \mathbf{d}^2)_2\|_{L^2(\partial S_r)} \Big]. \tag{L14}$$

We have the following expression for $\mathcal{F}_2(\mathbf{d})(\gamma)$. We decompose

$$\frac{1}{2}\mathcal{F}_2(\mathbf{d})(\gamma) = \mathrm{J}_7(\mathbf{d})(\gamma) + \mathrm{J}_8(\mathbf{d})(\gamma) + \mathrm{J}_9(\mathbf{d})(\gamma) + \mathrm{J}_{10}(\mathbf{d})(\gamma),$$

for

$$\begin{aligned} \mathrm{J}_7(\mathbf{d})(\gamma) &:= \int_{\mathbb{R}} \beta \frac{\partial_\alpha^2 \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)}\,\mathrm{d}\beta, \\ \mathrm{J}_8(\mathbf{d})(\gamma) &:= -2\int_{\mathbb{R}} \beta \frac{\partial_\alpha \Delta d_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \partial_\alpha Q(\gamma,\beta)\,\mathrm{d}\beta, \\ \mathrm{J}_9(\mathbf{d})(\gamma) &:= -\int_{\mathbb{R}} \beta \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^2} \partial_\alpha^2 Q(\gamma,\beta)\,\mathrm{d}\beta, \\ \mathrm{J}_{10}(\mathbf{d})(\gamma) &:= 2\int_{\mathbb{R}} \beta \frac{\Delta z_1(\gamma,\beta)}{Q(\gamma,\beta)^3} \partial_\alpha Q(\gamma,\beta)^2\,\mathrm{d}\beta. \end{aligned}$$

For the first difference with J_7, we have the following decomposition

$$\mathrm{J}_7(\mathbf{d}^1)(\gamma) - \mathrm{J}_7(\mathbf{d}^2)(\gamma) = \mathcal{D}_5(\gamma) + \mathcal{D}_6(\gamma),$$

for

$$\begin{aligned} \mathcal{D}_5(\gamma) &:= \int_{\mathbb{R}} \beta \frac{\partial_\alpha^2 \Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)}\,\mathrm{d}\beta, \\ \mathcal{D}_6(\gamma) &:= \int_{\mathbb{R}} \beta \partial_\alpha^2 \Delta d_1^2(\gamma,\beta) \left[\frac{1}{Q_1(\gamma,\beta)} - \frac{1}{Q_2(\gamma,\beta)} \right] \mathrm{d}\beta. \end{aligned}$$

The $L^2(\partial S_r)$ bound for $\mathcal{D}_5(\gamma)$, follows from the estimates for F_1. Then we deduce

$$\|\mathcal{D}_5\|_{L^2(\partial S_r)} \leq c_R \|\partial_\alpha^2(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

To estimate $\mathcal{D}_6$, we make use of the equation (3.5.3).

$$\mathcal{D}_6(\gamma) := \mathcal{D}_{6,1}(\gamma) + \mathcal{D}_{6,1}(\gamma),$$

for

$$\mathcal{D}_{6,1}(\gamma) = \int_{\mathbb{R}} \Delta(\mathbf{d}^2 - \mathbf{d}^1)_1(\gamma,\beta)\partial_\alpha^2 \Delta d_1^2(\gamma,\beta)\left[\frac{\beta\Delta(\mathbf{z}^1 + \mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\right] \mathrm{d}\beta,$$
$$\mathcal{D}_{6,2}(\gamma) = \int_{\mathbb{R}} \Delta(\mathbf{d}^2 - \mathbf{d}^1)_2(\gamma,\beta)\partial_\alpha^2 \Delta d_1^2(\gamma,\beta)\left[\frac{\beta\Delta(\mathbf{z}^1 + \mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)Q_2(\gamma,\beta)}\right] \mathrm{d}\beta.$$

The estimate of $\mathcal{D}_6$, follows the same lines as the estimates of D_9, see equation (3.6.20), by replacing $\partial_\alpha \Delta d_1^2(\gamma,\beta)$ by β. We deduce the following bound

$$\|\mathcal{D}_6\|_{L^2(\partial S_r)} \leq c_R\Big[\|\mathbf{d}^1 - \mathbf{d}^2\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^2 - \mathbf{d}^1)\|_{L^2(\partial S_r)}\Big].$$

Thus

$$\begin{aligned}&\|\mathrm{J}_7(\mathbf{d}^1) - \mathrm{J}_7(\mathbf{d}^2)\|_{L^2(\partial S_r)}\\ &\quad \leq c_R\Big[\|\mathbf{d}^1 - \mathbf{d}^2\|_{C^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^2 - \mathbf{d}^1)\|_{L^2(\partial S_r)} + \|\partial_\alpha^2(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}\Big].\end{aligned} \tag{L15}$$

Next, we estimate the difference with J_8. We decompose as follows

$$-\frac{1}{2}\Big(\mathrm{J}_8(\mathbf{d}^1)(\gamma) - \mathrm{J}_8(\mathbf{d}^2)(\gamma)\Big) = \mathcal{D}_7(\gamma) + \mathcal{D}_8(\gamma) + \mathcal{D}_9(\gamma),$$

for

$$\mathcal{D}_7(\gamma) := \int_{\mathbb{R}} \beta\frac{\partial_\alpha\Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\partial_\alpha Q_1(\gamma,\beta)\,\mathrm{d}\beta,$$
$$\mathcal{D}_8(\gamma) := \int_{\mathbb{R}} \beta\partial_\alpha\Delta d_1^2(\gamma,\beta)\frac{\partial_\alpha Q_1(\gamma,\beta) - \partial_\alpha Q_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta,$$
$$\mathcal{D}_9(\gamma) := \int_{\mathbb{R}} \beta\partial_\alpha\Delta d_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)\left[\frac{1}{Q_1(\gamma,\beta)^2} - \frac{1}{Q_2(\gamma,\beta)^2}\right]\mathrm{d}\beta.$$

To estimate $\mathcal{D}_7$, we expand $\partial_\alpha\Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma,\beta)$, then

$$\mathcal{D}_7(\gamma) = \partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma)\int_{\mathbb{R}}\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta - \int_{\mathbb{R}}\partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma-\beta)\frac{\beta\partial_\alpha Q_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\,\mathrm{d}\beta.$$

Using the estimate for D_{10}, in equation (3.6.25), we deduce

$$\|\mathcal{D}_7\|_{L^2(\partial S_r)} \leq c_R\|\partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

Next, we estimate $\mathcal{D}_8$. We use the equation (3.6.34), to decompose

$$\mathcal{D}_8(\gamma) = \mathcal{D}_{8,1}(\gamma) + \mathcal{D}_{8,2}(\gamma) + \mathcal{D}_{8,3}(\gamma) + \mathcal{D}_{8,4}(\gamma),$$

for

$$\mathcal{D}_{8,1}(\gamma) := \int_{\mathbb{R}} \Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma,\beta)\left[\frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right] \mathrm{d}\beta,$$
$$\mathcal{D}_{8,2}(\gamma) := \int_{\mathbb{R}} \Delta(\mathbf{d}^1 - \mathbf{d}^2)_2(\gamma,\beta)\left[\frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right] \mathrm{d}\beta,$$
$$\mathcal{D}_{8,3}(\gamma) := \int_{\mathbb{R}} \partial_\alpha\Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma,\beta)\left[\frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right] \mathrm{d}\beta,$$
$$\mathcal{D}_{8,4}(\gamma) := \int_{\mathbb{R}} \partial_\alpha\Delta(\mathbf{d}^1 - \mathbf{d}^2)_2(\gamma,\beta)\left[\frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\Delta(\mathbf{z}^1+\mathbf{z}^2)_2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right] \mathrm{d}\beta.$$

We expand $\Delta(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma,\beta)$, we get

$$\begin{aligned}\mathcal{D}_{8,1}(\gamma) = (\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma) &\int_{\mathbb{R}} \frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2} \mathrm{d}\beta \\ &- \int_{\mathbb{R}} (\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma-\beta)\frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2} \mathrm{d}\beta.\end{aligned} \tag{3.6.93}$$

To estimate the first integral above, we decompose the *in* part as follows

$$\begin{aligned}&\int_{|\beta|<1} \frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2} \mathrm{d}\beta \\ &\qquad = \int_{|\beta|<1}\left(\frac{\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)(\gamma,\beta)}{\beta} - \partial_\alpha^2(\mathbf{d}^1+\mathbf{d}^2)_1(\gamma)\right)\frac{\beta^2\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\mathrm{d}\beta \\ &\qquad + \partial_\alpha^2(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma)\int_{|\beta|<1}\frac{\beta^2\partial_\alpha\Delta d_1^2(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\mathrm{d}\beta.\end{aligned}$$

Using (3.4.16), we control the last integral. Thus we can deduce

$$\left|\int_{|\beta|<1} \frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2} \mathrm{d}\beta\right|_* \leq c_R$$

For the *out* part we consider

$$\left|\frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2}\right|_* \leq c_R|\beta|^{-3}.$$

Then

$$\left|\int_{|\beta|>1} \frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2} \mathrm{d}\beta\right|_* \leq c_R\int_{|\beta|>1}|\beta|^{-3}\,\mathrm{d}\beta < c_R.$$

By taking the $L^2(\partial S_r)$ we deduce

$$\left\|(\mathbf{d}^1 - \mathbf{d}^2)_1(\gamma)\int_{\mathbb{R}} \frac{\beta\partial_\alpha \Delta d_1^2(\gamma,\beta)\partial_\alpha\Delta(\mathbf{z}^1+\mathbf{z}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^2} \mathrm{d}\beta\right\|_{L^2(\partial S_r)} \leq c_R\|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

To deal with the second integral in (3.6.93), we follow a similar decomposition as in the previous term, and then apply same estimates with the Minkowski's integral inequality. We infer

$$\|\mathcal{D}_{8,1}\|_{L^2(\partial S_r)} \leq c_R\|(\mathbf{d}^1 - \mathbf{d}^2)_1\|_{L^2(\partial S_r)}.$$

The estimate of the remaining terms, follows the same lines of the previous term. We deduce

$$\|\mathcal{D}_8\|_{L^2(\partial S_r)} \leq c_R\Big[\|\mathbf{d}^1 - \mathbf{d}^2\|_{L^2(\partial S_r)} + \|\partial_\alpha(\mathbf{d}^1 - \mathbf{d}^2)\|_{L^2(\partial S_r)}\Big].$$

Now, we estimate $\mathcal{D}_9$. By using (3.5.3) we can argue as in D_{13}. Thus we can deduce the following

$$\|\mathcal{D}_9\|_{L^2(\partial S_r)} \leq c_R\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}.$$

We deduce

$$\|\mathrm{J}_8(\mathbf{d}^1)-\mathrm{J}_8(\mathbf{d}^2)\|_{L^2(\partial S_r)} \leq c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]. \tag{L16}$$

Now we move to the difference with J_9, then we have the following decomposition

$$-\big(J_9(\mathbf{d}^1)-J_9(\mathbf{d}^2)\big)(\gamma) := \mathcal{D}_{10}(\gamma)+\mathcal{D}_{11}(\gamma)+\mathcal{D}_{12}(\gamma),$$

for

$$\mathcal{D}_{10}(\gamma) := \int_{\mathbb{R}} \frac{\Delta(\mathbf{z}^1-\mathbf{z}^2)_1(\gamma,\beta)\beta}{Q_1(\gamma,\beta)^2}\partial_\alpha^2 Q_1(\gamma,\beta)\,\mathrm{d}\beta,$$
$$\mathcal{D}_{11}(\gamma) := \int_{\mathbb{R}} \frac{\Delta z_1^2(\gamma,\beta)\beta}{Q_1(\gamma,\beta)^2}\partial_\alpha^2 (Q_1-Q_2)(\gamma,\beta)\,\mathrm{d}\beta,$$
$$\mathcal{D}_{12}(\gamma) := \int_{\mathbb{R}} \Delta z_1^2(\gamma,\beta)\beta\partial_\alpha^2 Q_2(\gamma,\beta)\left[\frac{1}{Q_1(\gamma,\beta)^2}-\frac{1}{Q_2(\gamma,\beta)^2}\right]\mathrm{d}\beta.$$

We notice that the $L^2(\partial S_r)$ bounds for $\mathcal{D}_{10},\mathcal{D}_{11}$ and $\mathcal{D}_{12}$ follows from the estimates for D_{19}, D_{20} and D_{21} respectively, see (3.6.51), (3.6.52) and (3.6.64) decompositions. Hence, we obtain that

$$\|\mathrm{J}_9(\mathbf{d}^1)-\mathrm{J}_9(\mathbf{d}^2)\|_{L^2(\partial S_r)} \leq c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{C^{2,\delta}(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}+\|\partial_\alpha^2(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]. \tag{L17}$$

Finally, the term J_{10} is similar to J_6. We use the next decomposition

$$\frac{1}{2}\Big(\mathrm{J}_{10}(\mathbf{d}^1)(\gamma)-\mathrm{J}_{10}(\mathbf{d}^2)(\gamma)\Big) = \mathcal{D}_{13}(\gamma)+\mathcal{D}_{14}(\gamma)+\mathcal{D}_{15}(\gamma),$$

for

$$\mathcal{D}_{13}(\gamma) := \int_{\mathbb{R}} \beta\frac{\Delta(\mathbf{d}^1-\mathbf{d}^2)_1(\gamma,\beta)}{Q_1(\gamma,\beta)^3}\partial_\alpha Q_1(\gamma,\beta)^2\,\mathrm{d}\beta,$$
$$\mathcal{D}_{14}(\gamma) := \int_{\mathbb{R}} \beta\Delta z_1^2(\gamma,\beta)\frac{\partial_\alpha Q_1(\gamma,\beta)^2-\partial_\alpha Q_2(\gamma,\beta)^2}{Q_1(\gamma,\beta)^3}\,\mathrm{d}\beta,$$
$$\mathcal{D}_{15}(\gamma) := \int_{\mathbb{R}} \beta\Delta z_1^2(\gamma,\beta)\partial_\alpha Q_2(\gamma,\beta)^2\left[\frac{1}{Q_1(\gamma,\beta)^3}-\frac{1}{Q_2(\gamma,\beta)^3}\right]\mathrm{d}\beta.$$

By using the decomposition (3.6.74), we conclude the next $L^2(\partial S_r)$ bound

$$\|\mathrm{J}_{10}(\mathbf{d}^1)-\mathrm{J}_{10}(\mathbf{d}^2)\|_{L^2(\partial S_r)} \leq c_R\Big[\|\mathbf{d}^1-\mathbf{d}^2\|_{L^2(\partial S_r)}+\|\partial_\alpha(\mathbf{d}^1-\mathbf{d}^2)\|_{L^2(\partial S_r)}\Big]. \tag{L18}$$

By joining the inequalities (L14), (L15), (L16), (L17) and (L18), we obtain the desired estimate (L13). ■

We combine estimates (L12) with (L13) to obtain the following

$$\|\partial_\alpha^2\mathbf{F}(\mathbf{d}^1)-\partial_\alpha^2\mathbf{F}(\mathbf{d}^2)\|_{L^2(\partial S_{r'})} \leq c_R\|\mathbf{d}^1-\mathbf{d}^2\|_{X_{r,3}}.$$

And by using the Banach scale property (L4), we infer

$$\|\partial_\alpha^3\mathbf{F}(\mathbf{d}^1)-\partial_\alpha^3\mathbf{F}(\mathbf{d}^2)\|_{L^2(\partial S_{r'})} \leq \frac{c_R}{r-r'}\|\mathbf{d}^1-\mathbf{d}^2\|_{X_{r,3}}. \tag{L19}$$

Finally, putting together (L3) and (L19), we conclude

$$\|\mathbf{F}(\mathbf{d}^1)-\mathbf{F}(\mathbf{d}^2)\|_{X_{r',3}} \leq \frac{c_R}{r-r'}\|\mathbf{d}^1-\mathbf{d}^2\|_{X_{r,3}}$$

which is the Lipschitz condition of the main Lemma 6.

3.7 Turning Singularity

In this section we will prove the existence of a curve solution for the Muskat equation, with initial data $\mathbf{d}^0(\alpha, 0) + (\alpha, \alpha^2)$, with $\mathbf{d}^0 \in X_{r,3}$, such that at time $t = 0$, we are in the stable regime, and for a small time $T > 0$, we transition to the unstable regime. We start by assuming the following conditions on the curve $\mathbf{z} = \mathbf{d} + \mathbf{p}$,

A. $\mathbf{z}(0, 0) = (0, 0)$, passes through origin.

B. $z_1(\alpha, 0)$ is an odd function.

C. $\partial_\alpha z_1(\alpha, 0) > 0$ up to the point $\alpha = 0$ where $\partial_\alpha z_1(0, 0) = 0$ and $\partial_\alpha z_2(0, 0) > 0$. That is the slope at the time zero is vertical at the point $(0, 0)$.

We state the main result of this section.

Theorem 2. *There exists a solution in an interval $t \in [-T, T]$, with T small enough, $\mathbf{z} = \mathbf{d} + \mathbf{p}$, with $\mathbf{d} \in X_{r,3}$, which satisfies the arc-chord condition and such that, $\partial_\alpha z_2(0, t) > 0$. In addition, for $t \in (0, T]$*

1. $\partial_\alpha z_1(\alpha, -t) > 0$,
2. $\partial_\alpha z_1(0, t) < 0$.

Remark 3. *This theorem proves the existence of a turning singularity for a inital data of the form $\mathbf{z}_0 = \mathbf{d}_0 + \alpha^2$, with $\mathbf{d}_0 \in X_{r,3}$.*

The conclusion of this theorem is a direct consequence of Theorem 4 and the following lemma.

Lemma 16. *There exists a initial curve $\mathbf{z}(\alpha) = \mathbf{d}(\alpha) + \mathbf{p}(\alpha) = \big(z_1(\alpha), z_2(\alpha)\big)$, with $\mathbf{d} \in X_{r,3}$, which satisfies the arc-chord condition and the properties A, B and C, such that*

$$\partial_t \partial_\alpha z_1(0, 0) < 0.$$

Proof. We define $v_1(\alpha) = \partial_t z_1(\alpha)$ which is given by the next integral

$$v_1(\alpha) := PV \int_{\mathbb{R}} \frac{z_1(\alpha) - z_1(\alpha - \beta)}{|\mathbf{z}(\alpha) - \mathbf{z}(\alpha - \beta)|^2} (\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha - \beta)) \, \mathrm{d}\beta.$$

Now, we compute the derivative $\partial_\alpha v_1(\alpha)$. We find that

$$\partial_\alpha v_1(\alpha) = \int_{\mathbb{R}} \frac{\partial_\alpha \Delta d_1(\alpha, \beta)^2}{Q(\alpha, \beta)} \, \mathrm{d}\beta + \int_{\mathbb{R}} \frac{\Delta z_1(\alpha, \beta)}{Q(\alpha, \beta)} \partial_\alpha^2 d_1(\alpha, \beta) \, \mathrm{d}\beta - \int_{\mathbb{R}} \frac{\Delta z_1(\alpha, \beta) \partial_\alpha \Delta d_1(\alpha, \beta)}{Q(\alpha, \beta)^2} \partial_\alpha Q(\alpha, \beta) \, \mathrm{d}\beta.$$

The next step will be evaluate at $\alpha = 0$, by considering the assumptions over the curve $\mathbf{z}(\alpha, t)$, then we have that

$$\begin{aligned}
&\Delta z_1(0, \beta) = d_1(\beta) + \beta, \\
&\partial_\alpha \Delta d_1(0, \beta) = -\partial_\alpha d_1(\beta) - 1, \\
&\partial_\alpha^2 \Delta d_1(0, \beta) = \partial_\alpha^2 d_1(\beta), \\
&Q(0, \beta) = (d_1(\beta) + \beta)^2 + (d_2(-\beta) + \beta^2)^2.
\end{aligned} \tag{3.7.1}$$

Additionally, we compute the derivative $\partial_\alpha Q(\alpha, \beta)$ and evaluate at $\alpha = 0$, that is

$$\partial_\alpha Q(0, \beta) = -2(d_1(\beta) + \beta)(\partial_\alpha d_1(\beta) + 1) - 2(d_2(-\beta) + \beta^2)(\partial_\alpha d_2(0) - \partial_\alpha d_2(-\beta) + 2\beta). \tag{3.7.2}$$

We substitute (3.7.1) and (3.7.2) in $\partial_\alpha v_1(0)$, to obtain the following identity

$$\begin{aligned}
&\partial_\alpha v_1(0)\\
&= PV\int_{\mathbb{R}} \frac{(1+\partial_\alpha d_1(\beta))^2}{(d_1(\beta)+\beta)^2+(d_2(-\beta)+\beta^2)^2}\,\mathrm{d}\beta\\
&+ PV\int_{\mathbb{R}} \frac{(d_1(\beta)+\beta)\partial_\alpha^2 d_1(\beta)}{(d_1(\beta)+\beta)^2+(d_2(-\beta)+\beta^2)^2}\,\mathrm{d}\beta\\
&- 2PV\int_{\mathbb{R}} \frac{(d_1(\beta)+\beta)(1+\partial_\alpha d_1(\beta))}{\Big((d_1(\beta)+\beta)^2+(d_2(-\beta)+\beta^2)^2\Big)^2}\Big[(d_1(\beta)+\beta)(1+\partial_\alpha d_1(\beta))\Big]\,\mathrm{d}\beta\\
&- 2PV\int_{\mathbb{R}} \frac{(d_1(\beta)+\beta)(1+\partial_\alpha d_1(\beta))}{\Big((d_1(\beta)+\beta)^2+(d_2(-\beta)+\beta^2)^2\Big)^2}\Big[(d_2(-\beta)+\beta^2)\partial_\alpha d_2(0)\Big]\,\mathrm{d}\beta\\
&- 2PV\int_{\mathbb{R}} \frac{(d_1(\beta)+\beta)(1+\partial_\alpha d_1(\beta))}{\Big((d_1(\beta)+\beta)^2+(d_2(-\beta)+\beta^2)^2\Big)^2}\Big[(d_2(-\beta)+\beta^2)(2\beta-\partial_\alpha d_2(-\beta)\Big]\,\mathrm{d}\beta.
\end{aligned} \tag{3.7.3}$$

In the second integral of the last equality (3.7.3), we use integration by parts with respect to β. We infer

$$\begin{aligned}
&PV\int_{\mathbb{R}} \frac{(d_1(\beta)+\beta)\partial_\alpha^2 d_1(\beta)}{(d_1(\beta)+\beta)^2+(-d_2(-\beta)-\beta^2)^2}\,\mathrm{d}\beta\\
&= -PV\int_{\mathbb{R}} \frac{(\partial_\beta d_1(\beta)+1)^2}{(d_1(\beta)+\beta)^2+(d_2(-\beta)+\beta^2)^2}\,\mathrm{d}\beta\\
&+ 2PV\int_{\mathbb{R}} \frac{(\partial_\beta d_1(\beta)+1)^2(d_1(\beta)+\beta)^2}{\Big((d_1(\beta)+\beta)^2+(d_2(-\beta)+\beta^2)^2\Big)^2}\,\mathrm{d}\beta\\
&+ 2PV\int_{\mathbb{R}} \frac{(\partial_\beta d_1(\beta)+1)(d_1(\beta)+\beta)}{\Big((d_1(\beta)+\beta)^2+(d_2(-\beta)+\beta^2)^2\Big)^2}\Big[(d_2(-\beta)+\beta^2)(2\beta-\partial_\beta d_2(-\beta))\Big]\,\mathrm{d}\beta.
\end{aligned} \tag{3.7.4}$$

Plugging the relation (3.7.4) in (3.7.3), we obtain

$$\partial_\alpha v_1(0) = -2\partial_\alpha d_2(0) PV\int_{\mathbb{R}} \frac{(d_1(\beta)+\beta)(1+\partial_\alpha d_1(\beta))}{\Big((d_1(\beta)+\beta)^2+(d_2(-\beta)+\beta^2)^2\Big)^2}(d_2(-\beta)+\beta^2)\,\mathrm{d}\beta.$$

Finally, the change of variable $\beta=-\alpha$ leads to

$$\partial_\alpha v_1(0) = 2\partial_\alpha d_2(0) PV\int_{\mathbb{R}} \frac{(d_1(\alpha)+\alpha)(1+\partial_\alpha d_1(\alpha))(d_2(\alpha)+\alpha^2)}{((d_1(\alpha)+\alpha)^2+(d_2(\alpha)+\alpha^2)^2)^2}\,\mathrm{d}\alpha. \tag{3.7.5}$$

The last integral (3.7.6) completes the first part of the proof. In the second part, we will prove that $\partial_\alpha v_1(0)<0$. We choose a function $d_1 \in X_{r,3}$, with $r<1/2$, subject to the following condtiontions, it is odd, passing through the origin, is smooth, and satifies $\partial_\beta d_1(0) = -1$. The following functions work

$$d_1(\alpha) = -\frac{\alpha}{1+\alpha^2} \quad \text{and} \quad z_1(\alpha) = d_1(\alpha)+\alpha = \frac{\alpha^3}{1+\alpha^2}.$$

For every $\epsilon>0$ we define the following function

$$d_2^\epsilon(\alpha) = \mathrm{e}^{-\epsilon\alpha^2}(\bar z_2(\alpha)-\alpha^2).$$

Here the function $\bar z_2 \in X_{3,r}$, with small r, is chosen as in Lmma 5.3 of [13], thus the curve $\mathbf{z}^* = (z_1, \bar z_2)$, satisfies

$$\partial_\alpha \bar z_2(0) PV\int_{\mathbb{R}} \frac{z_1(\alpha)\partial_\alpha z_1(\alpha)\bar z_2(\alpha)}{(z_1(\alpha)^2+z_2(\alpha)^2)^2}\,\mathrm{d}\alpha < 0. \tag{3.7.6}$$

Let us comment that the construction in Lemma 5.3 of [13] is made for a curve periodic in the horizontal variable, but similar arguments work in the case of a curve with an asymptotically flat interface.

Now, we define

$$\mathbf{z}^\epsilon(\alpha) = (z_1(\alpha), z_2^\epsilon(\alpha)) \quad \text{with} \quad z_2^\epsilon(\alpha) = d_2^\epsilon(\alpha) + \alpha^2.$$

We can then check that $\mathbf{z}^\epsilon$ satisfies the properties A, B and D. Due to the fact that $d_2^\epsilon(0) = \bar{z}_2(0) = 0$, we infer that

$$\mathbf{z}^\epsilon(0) = (z_1(0), z_2^\epsilon(0)) = (0, 0).$$

It is clear that z_1 is an odd function. Additionally, the function z_2^ϵ satisfies

$$\partial_\alpha z_2^\epsilon(0) = \partial_\alpha \bar{z}_2(0) > 0.$$

and therefore the property C fulfilled. We observe that

$$\lim_{\epsilon\to 0} z_2^\epsilon(\alpha) = \bar{z}_2(\alpha), \text{ for } \alpha \in \mathbb{R}.$$

Therefore, by using the dominated convergence theorem, we obtain that

$$\begin{aligned}
\lim_{\epsilon\to 0} v_1^\epsilon(0) &= 2\partial_\alpha \bar{z}_2(0) \lim_{\epsilon\to 0} PV \int_{\mathbb{R}} \frac{z_1(\alpha)\partial_\alpha z_1(\alpha) z_2^\epsilon(\alpha)}{(z_1(\alpha)^2 + z_2^\epsilon(\alpha)^2)^2}\, d\alpha \\
&= 2\partial_\alpha \bar{z}_2(0) \int_{\mathbb{R}} \lim_{\epsilon\to 0} \frac{z_1(\alpha)\partial_\alpha z_1(\alpha) z_2^\epsilon(\alpha)}{(z_1(\alpha)^2 + z_2^\epsilon(\alpha)^2)^2}\, d\alpha \\
&= 2\partial_\alpha \bar{z}_2(0) \int_{\mathbb{R}} \frac{z_1(\alpha)\partial_\alpha z_1(\alpha) \bar{z}_2(\alpha)}{(z_1(\alpha)^2 + \bar{z}_2(\alpha)^2)^2}\, d\alpha < 0.
\end{aligned}$$

Thus, there exists $\epsilon^* > 0$, such that the curve $\mathbf{z}^{\epsilon^*}(\alpha) = (z_1(\alpha), z_2^{\epsilon^*}(\alpha))$ satisfies the required conditions, which indicates a change in the sign of the Rayleigh-Taylor condition

$$\partial_\alpha v_1^{\epsilon^*}(0) < 0$$

and this completes the proof.

■

Proof of the main Theorem 2. We take an initial data $\mathbf{z}^0(\alpha) = (d_1(\alpha), d_2^{\epsilon^*}(\alpha)) + (\alpha, \alpha^2)$. We observe that $\mathbf{z}^0(\alpha)$ is analytic and satisfies the conditions of Lemma 16. Thus, by using Theorem 4, the Cauchy-Kowaleski's Theorem, which guaranteed the existence of solutions with this initial data. ■

Chapter 4

Necessary Lemmas

4.1 Necessary Lemmas I

This section is devoted to the necessary lemmas used in the energy estimates. More precisely, we study the integrability and decay properties of the kernels K and G defined in (1.4.2). Throughout the section, we will use often the auxiliary globally Lipschitz function $F\colon \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) := \frac{1}{1+x^2}.$$

We start with the following lemma.

Lemma 17. *The truncated Hilbert transform of the rational function*

$$r(x) = \frac{x^m}{(1+x^2)^n},$$

for $m, n \in \mathbb{N}_+$ and $m < 2n$ is bounded. That is

$$|H_{|y|<1} r(x)| < c \quad \textit{and} \quad |H_{|y|>1} r(x)| < c.$$

Proof. Using the definition of the Hilbert transform we have

$$Hr(x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{1}{y} \frac{(x-y)^m}{(1+(x-y)^2)^n} \,\mathrm{d}y.$$

We know that the Hilbert transform of rational function is again a rational function and $|Hr(x)| \leq c$. Firstly, we estimate the *in* part. We decompose the integrand using partial fractions as follows

$$\frac{1}{x-y} \frac{y^m}{(1+y^2)^n} = \frac{b(x)}{x-y} + \sum_{k=1}^{n} \frac{a_k(x) y + c_k(x)}{(1+y^2)^k},$$

where $b(x)$, $a_k(x)$ and $c_k(x)$ are bounded terms. We obtain that

$$\begin{aligned} H_{|y|<1} r(x) = {} & \frac{1}{\pi} b(x) \int_{|x-y|<1} \frac{1}{x-y} \,\mathrm{d}y + \frac{1}{\pi} \sum_{k=1}^{n} a_k(x) \int_{|x-y|<1} \frac{y}{(1+y^2)^k} \,\mathrm{d}y \\ & + \frac{1}{\pi} \sum_{k=1}^{n} c_k(x) \int_{|x-y|<1} \frac{1}{(1+y^2)^k} \,\mathrm{d}y. \end{aligned}$$

We deduce that $|H_{|y|<1}r(x)| < c$. The bound for the *out* part is easy because

$$H_{|y|>1}r(x) = Hr(x) - H_{|y|<1}r(x),$$

thus $|H_{|y|>1}r(x)| < c$ which completes the proof. ∎

Lemma 18. *Let $g \in H^s(\mathbb{R})$ with $s \geq 3$, then*

$$\left\| PV \int_{\mathbb{R}} \frac{1}{\alpha} K(\,\cdot\,, \alpha)\, \mathrm{d}\alpha \right\|_{L^\infty} \leq c\,(1 + \|g\|_{C^2})^3. \tag{4.1.1}$$

Proof. Notice that by definition

$$K(x,\alpha) = F(\Delta_\alpha h).$$

We decompose the integral in the next way

$$PV \int_{\mathbb{R}} \frac{1}{\alpha} K(x,\alpha)\, \mathrm{d}\alpha = \int_{\mathbb{R}} \frac{1}{\alpha} F(\Delta_\alpha f)\, \mathrm{d}\alpha + \int_{\mathbb{R}} \frac{1}{\alpha} \big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]\, \mathrm{d}\alpha \tag{4.1.2}$$

where the first term is the Hilbert transform of F, that is

$$PV \int_{\mathbb{R}} \frac{1}{\alpha} F(\Delta_\alpha f)\, \mathrm{d}\alpha = PV \int_{\mathbb{R}} \frac{1}{\alpha} \frac{1}{1 + (2x-\alpha)^2}\, \mathrm{d}\alpha = \pi H F(2x),$$

this Hilbert transform is a rational function and is bounded. To deal with the second term in (4.1.2) we split it in the *in* and *out* parts. We compute the difference and observe

$$F(\Delta_\alpha h) - F(\Delta_\alpha f) = \Delta_\alpha g\, B(x,\alpha)$$

where

$$B(x,\alpha) = -2\Delta_\alpha f F(\Delta_\alpha f) F(\Delta_\alpha h) - \Delta_\alpha g F(\Delta_\alpha f) F(\Delta_\alpha h)$$

is a bounded term $|B(x,\alpha)| \leq 2$. Adding and subtracting $\partial_x g(x)$ we have the next decomposition

$$\begin{aligned} \int_{|\alpha|<1} \frac{1}{\alpha} \big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]\, \mathrm{d}\alpha &= \int_{|\alpha|<1} \frac{1}{\alpha} (\Delta_\alpha g - \partial_x g(x)) B(x,\alpha)\, \mathrm{d}\alpha \\ &\quad + \partial_x g(x) \int_{|\alpha|<1} \frac{1}{\alpha} B(x,\alpha)\, \mathrm{d}\alpha. \end{aligned} \tag{4.1.3}$$

Now, from the Fundamental Theorem of Calculus we have the next bound

$$|\Delta_\alpha g - \partial_x g(x)| \leq c\, \|\partial_x^2 g\|_{L^\infty} |\alpha|. \tag{4.1.4}$$

Using the bound for $B(x,\alpha)$ and the last inequality we obtain that

$$\left| \int_{|\alpha|<1} \frac{1}{\alpha} (\Delta_\alpha g - \partial_x g(x)) B(x,\alpha)\, \mathrm{d}\alpha \right| \leq c\, \|\partial_x^2 g\|_{L^\infty}.$$

For the second integral in (4.1.3), adding and subtracting the terms $\partial_x g(x)$ and $F(\partial_x h(x))$ we obtain the next decomposition

$$\begin{aligned} \int_{|\alpha|<1} \frac{1}{\alpha} B(x,\alpha)\, \mathrm{d}\alpha = &-2 \int_{|\alpha|<1} \frac{1}{\alpha} \Delta_\alpha f F(\Delta_\alpha f) \big[F(\Delta_\alpha h) - F(\partial_x h(x))\big]\, \mathrm{d}\alpha \\ &- F(\partial_x h(x)) \int_{|\alpha|<1} \frac{\Delta_\alpha f}{\alpha} F(\Delta_\alpha f)\, \mathrm{d}\alpha \\ &- \int_{|\alpha|<1} \frac{1}{\alpha} \big(\Delta_\alpha g - \partial_x g(x)\big) F(\Delta_\alpha f) F(\Delta_\alpha h)\, \mathrm{d}\alpha \\ &- \partial_x g(x) \int_{|\alpha|<1} \frac{1}{\alpha} F(\Delta_\alpha f) \big[F(\Delta_\alpha h) - F(\partial_x h(x))\big]\, \mathrm{d}\alpha \\ &- \partial_x g(x) F(\partial_x h(x)) \int_{|\alpha|<1} \frac{1}{\alpha} F(\Delta_\alpha f)\, \mathrm{d}\alpha. \end{aligned}$$

Using the Lipschitz condition of F and the Fundamental Theorem of Calculus we deduce that

$$|F(\Delta_\alpha h) - F(\partial_x h(x))| \leq c\,|\Delta_\alpha h - \partial_x h(x)| \leq c\left(1 + \|\partial_x^2 g\|_{L^\infty}\right)|\alpha| \tag{4.1.5}$$

and from Lemma 17 we find that

$$\left|\int_{|\alpha|<1} \frac{1}{\alpha} F(\Delta_\alpha f)\,\mathrm{d}\alpha\right|, \left|\int_{|\alpha|<1} \frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)\,\mathrm{d}\alpha\right| < c,$$

in the last integral we recall the definition of f. Therefore we conclude from (4.1.4) and (4.1.5) that

$$\left|\partial_x g(x)\int_{|\alpha|<1}\frac{1}{\alpha}B(x,\alpha)\,\mathrm{d}\alpha\right| \leq c\,(1 + \|g\|_{C^2})^3.$$

The bound for the *out* part in the second term of (4.1.3) can be deduced from the Lipschitz condition of F

$$|F(\Delta_\alpha h) - F(\Delta_\alpha f)| \leq 2\|g\|_{L^\infty}|\alpha|^{-1}. \tag{4.1.6}$$

Then using the fact that $B(x,\alpha)$ is bounded, we conclude that

$$\left|\int_{|\alpha|>1}\frac{1}{\alpha}\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]\,\mathrm{d}\alpha\right| \leq c\,\|g\|_{L^\infty}.$$

and this completes the proof. ∎

The following result presents a similar estimate to the previous lemma, but now for the kernel G.

Lemma 19. *Let $g \in H^s(\mathbb{R})$ with $s \geq 3$, then*

$$\left\|PV\int_{\mathbb{R}}\frac{1}{\alpha}G(\,\cdot\,,\alpha)\,\mathrm{d}\alpha\right\|_{L^\infty} \leq c\,(1 + \|g\|_{C^2})^2. \tag{4.1.7}$$

Proof. Using the function F we rewrite the integral as

$$\begin{aligned} PV\int_{\mathbb{R}}\frac{1}{\alpha}G(x,\alpha)\,\mathrm{d}\alpha &= -4\int_{\mathbb{R}}\frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)F(\Delta_\alpha h)\,\mathrm{d}\alpha \\ &\quad - 2\int_{\mathbb{R}}\frac{1}{\alpha}\Delta_\alpha g F(\Delta_\alpha f)F(\Delta_\alpha h)\,\mathrm{d}\alpha := -4G_1 - 2G_2. \end{aligned}$$

We start with the bound for the *in* part in G_1. Notice that adding and subtracting $F(\partial_x h(x))$, we obtain the next decomposition

$$G_1^{in} = \int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)\big[F(\Delta_\alpha h) - F(\partial_x h(x))\big]\,\mathrm{d}\alpha + F(\partial_x h(x))\int_{|\alpha|<1}\frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)\,\mathrm{d}\alpha.$$

Then, in a similar way to the Lemma 18, we use the Lipschitz condition of F to obtain that

$$|G_1^{in}| \leq c\,(1 + \|g\|_{C^2}).$$

Now for the *out* part we add and subtract $F(\Delta_\alpha f)$. We find that

$$G_1^{out} = \int_{|\alpha|>1}\frac{1}{\alpha}\Delta_\alpha f F(\Delta_\alpha f)\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]\,\mathrm{d}\alpha + \int_{|\alpha|>1}\frac{1}{\alpha}\Delta_\alpha f[F(\Delta_\alpha f)]^2\,\mathrm{d}\alpha.$$

Using the Lipschitz condition (4.1.6) we obtain that

$$|F(\Delta_\alpha h) - F(\Delta_\alpha f)| \leq 2\|g\|_{L^\infty}|\alpha|^{-1}$$

and from Lemma 17 we have

$$\left| \int_{|\alpha|>1} \frac{1}{\alpha} \Delta_\alpha f [F(\Delta_\alpha f)]^2 \, \mathrm{d}\alpha \right| < c, \tag{4.1.8}$$

hence

$$|G_1^{out}| \leq c\,(1 + \|g\|_{L^\infty}).$$

In order to estimate G_2, for the *in* part we add and subtract the terms $\partial_x g(x)$ and $F(\partial_x h(x))$ to obtain the next decomposition

$$\begin{aligned} G_2^{in} &= \int_{|\alpha|<1} \frac{1}{\alpha} (\Delta_\alpha g - \partial_x g(x)) F(\Delta_\alpha f) F(\Delta_\alpha h) \, \mathrm{d}\alpha \\ &\quad + \partial_x g(x) \int_{|\alpha|<1} \frac{1}{\alpha} F(\Delta_\alpha f) \big[F(\Delta_\alpha h) - F(\partial_x h(x))\big] \, \mathrm{d}\alpha \\ &\quad + \partial_x g(x) F(\partial_x h(x)) \int_{|\alpha|<1} \frac{1}{\alpha} F(\Delta_\alpha f) \, \mathrm{d}\alpha, \end{aligned}$$

which are the terms appearing in G_1^{in} and (4.1.3). Hence

$$|G_2^{in}| \leq c\,(1 + \|g\|_{C^2})^2.$$

Finally, for the *out* part G_2^{out}, we observe

$$|G_2^{out}| \leq \int_{|\alpha|>1} \frac{|g(x) - g(x-\alpha)|}{\alpha^2} |F(\Delta_\alpha f) F(\Delta_\alpha h)| \, \mathrm{d}\alpha \leq 2\|g\|_{L^\infty} \int_{|\alpha|>1} |\alpha|^{-2} \, \mathrm{d}\alpha$$

and this completes the proof. ∎

In the next lemma we prove similar estimates now for the derivative in x of the kernel $K(x, \alpha)$.

Lemma 20. *Let $g \in H^s(\mathbb{R})$ with $s \geq 3$, then*

$$\left\| PV \int_{\mathbb{R}} \frac{1}{\alpha} \partial_x K(\,\cdot\,, \alpha) \, \mathrm{d}\alpha \right\|_{L^\infty} \leq c\,(1 + \|g\|_{C^{2,\delta}})^2 \qquad \textit{for} \quad \delta \in (0, s - 5/2]. \tag{4.1.9}$$

Proof. First we note that

$$\partial_x K(x, \alpha) = F'(\Delta_\alpha h) \partial_x \Delta_\alpha h.$$

For the *in* part we add and subtract $\partial_x^2 h(x)$ and $F'(\partial_x h(x))$ and decompose in the following way

$$\begin{aligned} PV \int_{|\alpha|<1} \frac{1}{\alpha} \partial_x K(x, \alpha) \, \mathrm{d}\alpha &= \int_{|\alpha|<1} \frac{F'(\Delta_\alpha h)}{\alpha} \big[\partial_x \Delta_\alpha h - \partial_x^2 h(x)\big] \, \mathrm{d}\alpha \\ &\quad + \partial_x^2 h(x) \int_{|\alpha|<1} \frac{1}{\alpha} \big[F'(\Delta_\alpha h) - F'(\partial_x h(x))\big] \, \mathrm{d}\alpha. \end{aligned}$$

Using the inequalities

$$\begin{aligned} &|\partial_x \Delta_\alpha h - \partial_x^2 h(x)| \leq c |\partial_x^2 g|_{C^\delta} \cdot |\alpha|^\delta, \\ &|F'(\Delta_\alpha h) - F'(\partial_x h(x))| \leq c\,|\Delta_\alpha h - \partial_x h(x)|, \end{aligned}$$

it follows the next bound

$$\left| \int_{|\alpha|<1} \frac{1}{\alpha} \partial_x K(x, \alpha) \, \mathrm{d}\alpha \right| \leq c\,(1 + \|g\|_{C^{2,\delta}})^2. \tag{4.1.10}$$

For the *out* part, by adding and subtracting the term $F'(\Delta_\alpha f)$ we obtain that

$$PV\int_{|\alpha|>1}\frac{1}{\alpha}\partial_x K(x,\alpha)\,\mathrm{d}\alpha=\int_{|\alpha|>1}\frac{1}{\alpha}\big[F'(\Delta_\alpha h)-F'(\Delta_\alpha f)\big]\partial_x\Delta_\alpha h\,\mathrm{d}\alpha \\ +\int_{|\alpha|>1}\frac{1}{\alpha}F'(\Delta_\alpha f)(\partial_x\Delta_\alpha h)\,\mathrm{d}\alpha. \tag{4.1.11}$$

Notice that

$$|F'(\Delta_\alpha h)-F'(\Delta_\alpha f)|\le c\,|\Delta_\alpha g|,$$
$$|\partial_x\Delta_\alpha h|\le c\,(1+\|\partial_x^2 g\|_{L^\infty}).$$

Hence the following bound is automatic

$$\left|\int_{|\alpha|>1}\frac{1}{\alpha}\big[F'(\Delta_\alpha h)-F'(\Delta_\alpha f)\big]\partial_x\Delta_\alpha h\,\mathrm{d}\alpha\right|\le c\,(1+\|\partial_x^2 g\|_{L^\infty})\int_{|\alpha|>1}\frac{|g(x)-g(x-\alpha)|}{\alpha^2}\,\mathrm{d}\alpha \\ \le c\,(1+\|\partial_x^2 g\|_{L^\infty})\|g\|_{L^\infty}.$$

For the second integral in the right hand side of (4.1.11) we expand

$$\partial_x\Delta_\alpha h=2+\partial_x\Delta_\alpha g,$$

and decompose

$$\int_{|\alpha|>1}\frac{1}{\alpha}F'(\Delta_\alpha f)(\partial_x\Delta_\alpha h)\,\mathrm{d}\alpha=2\int_{|\alpha|>1}\frac{1}{\alpha}F'(\Delta_\alpha f)\,\mathrm{d}\alpha+\int_{|\alpha|>1}\frac{\partial_x g(x)-\partial_x g(x-\alpha)}{\alpha^2}F'(\Delta_\alpha f)\,\mathrm{d}\alpha.$$

Notice that

$$F'(\Delta_\alpha f)=-2\Delta_\alpha f F(\Delta_\alpha f)^2\quad\text{and}\quad |F'(\Delta_\alpha f)|<2.$$

Using the estimate (4.1.8) we obtain that

$$\left|\int_{|\alpha|>1}\frac{1}{\alpha}\partial_x K(x,\alpha)\,\mathrm{d}\alpha\right|\le c\,(1+\|g\|_{C^1}).$$

The last bound together with the estimate (4.1.10) completes the proof. ■

In the following lemma we obtain a bound for $\Lambda K(x,0)$ where $K(x,0)$ is the kernel at zero.

Lemma 21. *Let $g\in H^s(\mathbb{R})$ with $s\ge 3$, then we have the next bound*

$$\|\Lambda K(x,0)\|_{L^\infty}\le c\,(1+\|g\|_{C^{2,\delta}})\quad \textit{for}\quad \delta\in(0,s-5/2]. \tag{4.1.12}$$

Proof. By definition of the operator Λ we have

$$\Lambda K(x,0)=\frac{1}{\pi}PV\int_{\mathbb{R}}\frac{K(x,0)-K(y,0)}{(x-y)^2}\,\mathrm{d}y,$$

where

$$K(x,0)=\frac{1}{1+(\partial_x h(x))^2}.$$

We denote $K(x,0):=K(x)$ and split in the *in* and *out* parts. We change variables $y=x-y$ to obtain that

$$\Lambda K(x)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{K(x)-K(x-y)}{y^2}\,\mathrm{d}y$$

and write as

$$\int_{|y|<1} \frac{K(x)-K(x-y)}{y^2}\,\mathrm{d}y = \frac{1}{2}\int_{|y|<1} \frac{K(x)-K(x-y)}{y^2}\,\mathrm{d}y + \frac{1}{2}\int_{|y|<1} \frac{K(x)-K(x+y)}{y^2}\,\mathrm{d}y,$$

hence

$$\begin{aligned}\Lambda K(x,0) &= \frac{1}{2\pi}\int_{|y|<1} \frac{2K(x)-K(x+y)-K(x-y)}{y^2}\,\mathrm{d}y + \frac{1}{\pi}\int_{|x-y|>1} \frac{K(x)-K(y)}{(x-y)^2}\,\mathrm{d}y \\ &= I^{in} + I^{out}.\end{aligned}$$

Using the Fundamental Theorem of Calculus we obtain the formulas

$$K(x)-K(x-y) = \int_0^1 K'(x+(1-s)y)\partial_x^2 h(x+(1-s)y)\,\mathrm{d}s \cdot y$$

and

$$K(x)-K(x+y) = -\int_0^1 K'(x+sy)\partial_x^2 h(x+sy)\,\mathrm{d}s \cdot y.$$

Let us recall that $\partial_x^2 h(x) = 2 + \partial_x^2 g(x)$. Thus, we have the next estimate

$$\begin{aligned}|2K(x)-K(x-y)-K(x+y)| &\le \|\partial_x K\|_{L^\infty}\int_0^1 |\partial_x^2 g(x+(1-s)y) - \partial_x^2 g(x+sy)|\,\mathrm{d}s\cdot|y| \\ &\le c\,|y|\sup_{x\neq y}|\partial_x^2 g(x+(1-s)y) - \partial_x^2 g(x+sy)| \\ &\le c|y|^{1+\delta}|\partial_x^2 g|_{C^\delta},\end{aligned}$$

where $\delta = 1/2$. Hence the *in* part on ΛK is bounded by

$$|I^{in}| \le c\,\|g\|_{C^{2,\delta}}\int_{|y|<1} |y|^{\delta-1}\,\mathrm{d}y.$$

Now, for the *out* part is enough to see that $0 < K(x) \le 1$, for all $x \in \mathbb{R}$. Hence

$$\begin{aligned}|I^{out}| &\le \frac{1}{\pi}\int_{|x-y|>1} \frac{|K(x)-K(y)|}{(x-y)^2}\,\mathrm{d}y \\ &\le \frac{2}{\pi}\int_{|x-y|>1} \frac{1}{|x-y|^2}\,\mathrm{d}y < \infty,\end{aligned}$$

and this completes the proof. ∎

In the following lemma we recall that $\Phi(x,\alpha)$ is the derivative of the difference

$$\Phi(x,\alpha) := \partial_\alpha\left[\frac{K(x,\alpha)-K(x,0)}{\alpha}\right].$$

Lemma 22. *Let $g \in H^s(\mathbb{R})$ with $s \ge 3$, then for every $\delta \in (0, s-5/2]$, we have*

$$\big|\Phi(x,\alpha)\big| \le c\,(1+\|g\|_{C^{2,\delta}})^2|\alpha|^{\delta-1}, \quad \textit{for} \quad x\in\mathbb{R}. \tag{4.1.13}$$

Proof. Recall that $F(\Delta_\alpha h) = K(x,\alpha)$ and write

$$F(\alpha) = K(x,\alpha) \;\text{ and }\; F(0) = K(x,0).$$

Then we integrate in the next way

$$\begin{aligned}\Phi(x,\alpha) &= -\frac{F(\alpha)-F(0)}{\alpha^2} + \frac{1}{\alpha}F'(\alpha)\\ &= -\frac{1}{\alpha^2}\int_0^\alpha F'(z)\,\mathrm{d}z + \frac{F'(\alpha)}{\alpha}\\ &= \frac{1}{\alpha^2}\int_0^\alpha \int_z^\alpha F''(w)\,\mathrm{d}w\,\mathrm{d}z.\end{aligned}$$

Hence

$$|\Phi(x,\alpha)| \leq c\,|\partial_\alpha^2 K(x,\alpha)|$$

A direct computation yields to

$$\partial_\alpha^2 K(x,\alpha) = F''(\Delta_\alpha h)[\partial_\alpha \Delta_\alpha h]^2 + F'(\Delta_\alpha h)\partial_\alpha^2 \Delta_\alpha g. \tag{4.1.14}$$

Using the Fundamental Theorem of Calculus we obtain the next indentities

$$\partial_\alpha^2 \Delta_\alpha g(x) = \frac{1}{\alpha}\int_0^1\int_0^1 \Big[\partial_x^2 g(x+(rs-1)\alpha) - \partial_x^2 g(x-\alpha)\Big](2s)\,\mathrm{d}r\,\mathrm{d}s,$$

$$\partial_\alpha \Delta_\alpha h(x) = \int_0^1 (s-1)\partial_x^2 h(x+(s-1)\alpha)\,\mathrm{d}s,$$

where the integrands are bounded by

$$\begin{aligned}\left|\partial_x^2 g(x+(rs-1)\alpha) - \partial_x^2 g(x-\alpha)\right| &\leq c\,\|g\|_{C^{2,\delta}}|\alpha|^\delta,\\ |\partial_x^2 h(x+(s-1)\alpha)| &\leq c\,(1+\|\partial_x^2 g\|_{L^\infty}).\end{aligned}$$

It follows from equation (4.1.14) that

$$|\Phi(x,\alpha)| \leq c\,|\partial_\alpha^2 K(x,\alpha)| \leq c\,(1+\|g\|_{C^{2,\delta}})^2|\alpha|^{\delta-1}, \tag{4.1.15}$$

which completes the proof. ∎

Lemma 23. *Let $g\in H^s(\mathbb{R})$ with $s\geq 3$, the kernel $K(x,\alpha)$ belongs to $L_x^2(\mathbb{R})$, that is*

$$\int_{\mathbb{R}} K(x,\alpha)^2\,\mathrm{d}x \leq c\,(1+\|\partial_x g\|_{L^\infty}). \tag{4.1.16}$$

Proof. Notice that

$$K(x,\alpha)^2 < K(x,\alpha) < 1$$

and the lower bound

$$\Delta_\alpha h \geq 2x - \alpha - \|\partial_x g\|_{L^\infty}.$$

Using the last lower bound, we have

$$K(x,\alpha) \leq \frac{1}{1+(2x-\alpha)^2} \quad \text{if} \quad x \geq \|\partial_x g\|_{L^\infty}$$

and $K(x,\alpha) < 1$ for any $x \in \mathbb{R}$. Then we split

$$\int_0^\infty K(x,\alpha)\,\mathrm{d}x \le \int_0^{\|\partial_x g\|_{L^\infty}} \mathrm{d}x + \int_{\|\partial_x g\|_{L^\infty}}^\infty \frac{1}{1+(2x-\alpha)^2}\,\mathrm{d}x.$$

The first integral is bounded by $\|\partial_x g\|_{L^\infty}$, while for the second one, the change of variable $z = 2x-\alpha$ implies that

$$\int_0^\infty \frac{1}{1+(2x-\alpha)^2}\,\mathrm{d}x \le \frac{1}{2}\int_{\mathbb{R}} \frac{\mathrm{d}z}{1+z^2} < \infty.$$

which completes the proof. ∎

Lemma 24. *Let $g \in H^s(\mathbb{R})$ with $s \ge 3$, the kernel $G(x,\alpha)$ belongs to $L^2_x(\mathbb{R})$, that is*

$$\int_{\mathbb{R}} G(x,\alpha)^2\,\mathrm{d}x \le c\,(1+\|\partial_x g\|_{L^\infty})^3. \tag{4.1.17}$$

Proof. From the definition (1.4.2) we have that

$$G(x,\alpha) = -\frac{2\Delta_\alpha f + \Delta_\alpha g}{(1+(\Delta_\alpha f)^2)(1+(\Delta_\alpha h)^2)} = -(2\Delta_\alpha f + \Delta_\alpha g)K(x,\alpha)F(\Delta_\alpha f).$$

We decompose the sum and observe

$$|G(x,\alpha)| \le 2|\Delta_\alpha f|F(\Delta_\alpha f)K(x,\alpha) + \|\partial_x g\|_{L^\infty}K(x,\alpha) \le (2+\|\partial_x g\|_{L^\infty})K(x,\alpha).$$

Then

$$G(x,\alpha)^2 \le (2+\|\partial_x g\|_{L^\infty})^2 K(x,\alpha)^2.$$

Now we integrate

$$\int_{\mathbb{R}} G(x,\alpha)^2\,\mathrm{d}x \le (2+\|\partial_x g\|_{L^\infty})^2 \int_{\mathbb{R}} K(x,\alpha)^2\,\mathrm{d}x$$

then the proof follows from Lemma 23. ∎

Lemma 25. *Let $g \in H^s(\mathbb{R})$ with $s \ge 3$. The partial derivate with respect to α of the kernel $K(x,\alpha)$ belongs to $L^2_x(\mathbb{R})$, that is*

$$\int_{\mathbb{R}} \partial_\alpha K(x,\alpha)^2\,\mathrm{d}x \le c\,(1+\|\partial_x g\|_{L^\infty})^3. \tag{4.1.18}$$

Proof. Recall that $K(x,\alpha) = F(\Delta_\alpha h)$, then the derivative with respect to α is given by

$$\partial_\alpha K(x,\alpha) = F'(\Delta_\alpha h)\partial_\alpha \Delta_\alpha h.$$

Now we observe

$$F'(\Delta_\alpha h) \le 2K(x,\alpha)$$

and from the Fundamental Theorem of Calculus we have

$$|\partial_\alpha \Delta_\alpha h| \le 2 + \|\partial_x^2 g\|_{L^\infty},$$

which implies that

$$|\partial_\alpha K(x,\alpha)|^2 \le c\,(1+\|\partial_x^2 g\|_{L^\infty})^2 K(x,\alpha).$$

then the estimate follows from Lemma 23. ∎

Lemma 26. *Let $g \in H^s(\mathbb{R})$ with $s \ge 3$, we have*

$$\left\| PV\int_{|\alpha|>1} \frac{1}{\alpha}\partial_x^2 K(\,\cdot\,,\alpha)\,\mathrm{d}\alpha \right\|_{L^\infty} \le c\,(1+\|g\|_{C^{2,\delta}})^2 \quad \textit{for} \quad \delta \in (0,1). \tag{4.1.19}$$

Proof. Using the indentity (2.1.21) we have

$$\partial_x^2 K(x,\alpha) = (\partial_x^2\Delta_\alpha g)B_1(x,\alpha) + (\partial_x\Delta_\alpha h)^2 B_2(x,\alpha)$$

where $B_1(x,\alpha)$ and $B_2(x,\alpha)$ are bounded terms. We decompose the integral in the next way

$$\begin{aligned}\int_{|\alpha|>1} \frac{1}{\alpha}\partial_x^2 K(x,\alpha)\,\mathrm{d}\alpha = &\int_{|\alpha|>1} \frac{1}{\alpha^2}\big(\partial_x^2 g(x) - \partial_x^2 g(x-\alpha)\big)B_1(x,\alpha)\,\mathrm{d}\alpha\\ &+ \int_{|\alpha|>1} \frac{1}{\alpha}(2+\partial_x\Delta_\alpha g)^2 B_2(x,\alpha)\,\mathrm{d}\alpha.\end{aligned}$$

For the first integral in the right hand side, we note that $|B_1(x,\alpha)| = |F'(\Delta_\alpha h)| \leq 2$ and

$$|\partial_x^2 g(x) - \partial_x^2 g(x-\alpha)| \leq |\partial_x^2 g|_{C^\delta}\cdot|\alpha|^\delta, \quad \text{for} \quad \delta\in(0,1).$$

Therefore

$$\left|\int_{|\alpha|>1} \frac{1}{\alpha^2}\big(\partial_x^2 g(x) - \partial_x^2 g(x-\alpha)\big)B_1(x,\alpha)\,\mathrm{d}\alpha\right| \leq c\,\|g\|_{C^{2,\delta}} \int_{|\alpha|>1} |\alpha|^{2-\delta}\,\mathrm{d}\alpha,$$

which is integrable. To get the bound for the second integral we observe

$$B_2(x,\alpha) = -2F(\Delta_\alpha h)^2 + 8(\Delta_\alpha h)^2 F(\Delta_\alpha h)^3,$$

then we proceed as in Lemma 20 to obtain that

$$\left|\int_{|\alpha|>1} \frac{1}{\alpha}(\partial_x\Delta_\alpha h)^2 B_2(x,\alpha)\,\mathrm{d}\alpha\right| \leq c\,(1+\|\partial_x g\|_{L^\infty})^2,$$

and this completes the proof. ■

Lemma 27. *Let $g\in H^s(\mathbb{R})$ with $s\geq 3$, then*

$$\left|PV\int_{|\alpha|>1} \frac{1}{\alpha}K(x,\alpha)^3\,\mathrm{d}\alpha\right| \leq c\,(1+\|g\|_{L^\infty}), \tag{4.1.20}$$

Proof. Using $K(x,\alpha) = F(\Delta_\alpha h)$ and adding a subtracting $F(\Delta_\alpha f)$ we have the next decomposition

$$\begin{aligned}K(x,\alpha)^3 = &F(\Delta_\alpha h)^2\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big] + F(\Delta_\alpha h)\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]F(\Delta_\alpha f)\\ &+ \big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]F(\Delta_\alpha f)^2 + F(\Delta_\alpha f)^3 := \Xi(x,\alpha) + F(\Delta_\alpha f)^3\end{aligned}$$

Using the Lipschitz condition (4.1.6), we see that $|\Xi(x,\alpha)|/\alpha$ is integrable for $|\alpha|>1$. Finally, by using Lemma 17, we infer

$$\left|\int_{|\alpha|>1} \frac{1}{\alpha}K(x,\alpha)^3\,\mathrm{d}\alpha\right| \leq c\,\|g\|_{L^\infty} + \left|\int_{|\alpha|>1} \frac{1}{\alpha}F(\Delta_\alpha f)^3\,\mathrm{d}\alpha\right| \leq c\,(1+\|g\|_{L^\infty}),$$

and this completes the proof. ■

In the next lemma we recall the definition (2.1.29)

$$\gamma(x,\alpha) = 24(\Delta_\alpha h)K(x,\alpha)^3 - 48(\Delta_\alpha h)^3 K(x,\alpha)^4.$$

Lemma 28. *Let $g \in H^s(\mathbb{R})$ with $s \geq 3$, we have the next bound*

$$\left| PV \int_{|\alpha|>1} \frac{1}{\alpha} \gamma(x,\alpha)\,\mathrm{d}\alpha \right| \leq c\,(1+\|g\|_{L^\infty})^3. \tag{4.1.21}$$

Proof. Recall that $K(x,\alpha) = F(\Delta_\alpha h)$. Using the definition (2.1.29) we expand $\Delta_\alpha h$ and $(\Delta_\alpha h)^3$ to obtain that

$$\begin{aligned}\gamma(x,\alpha) &= 24\Delta_\alpha f K(x,\alpha)^3 + 24\Delta_\alpha g K(x,\alpha)^3 - 48(\Delta_\alpha f)^3 K(x,\alpha)^4 \\ &\quad - 48\cdot 3(\Delta_\alpha f)^2 \Delta_\alpha g K(x,\alpha)^4 - 48\cdot 3(\Delta_\alpha f)(\Delta_\alpha g)^2 K(x,\alpha)^4 \\ &\quad - 48(\Delta_\alpha g)^3 K(x,\alpha)^4.\end{aligned} \tag{4.1.22}$$

The second and last terms in (4.1.22) are easily bounded by

$$|24\Delta_\alpha g K(x,\alpha)^3 - 48\Delta_\alpha g K(x,\alpha)^4| \leq c\,\frac{\|g\|_{L^\infty}}{|\alpha|} + c\frac{\|g\|^3_{L^\infty}}{|\alpha|^3}.$$

For the fourth term, by adding and subtracting $\Delta_\alpha g$, we obtain the next decomposition

$$(\Delta_\alpha f)^2 \Delta_\alpha g K(x,\alpha)^4 = (\Delta_\alpha h)^2 \Delta_\alpha g K(x,\alpha)^4 - 2\Delta_\alpha h(\Delta_\alpha g)^2 K(x,\alpha)^4 + (\Delta_\alpha g)^3 K(x,\alpha)^4. \tag{4.1.23}$$

Hence the fourth term in (4.1.22) is bounded by

$$|(\Delta_\alpha f)^2 \Delta_\alpha g K(x,\alpha)^4| \leq c\,\frac{\|g\|_{L^\infty}}{|\alpha|} + c\,\frac{\|g\|^2_{L^\infty}}{|\alpha|^2} + c\,\frac{\|g\|^3_{L^\infty}}{|\alpha|^3}.$$

In a similar way the fifth term is bounded by

$$|\Delta_\alpha f(\Delta_\alpha g)^2 K(x,\alpha)^4| \leq c\,\frac{\|g\|^2_{L^\infty}}{|\alpha|^2} + c\,\frac{\|g\|^3_{L^\infty}}{|\alpha|^3}.$$

For the first term adding and subtracting $F(\Delta_\alpha f)$, we have the next decomposition

$$\begin{aligned}\Delta_\alpha f K(x,\alpha)^3 &= \Delta_\alpha f F(\Delta_\alpha h)^2 \big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big] \\ &\quad + \Delta_\alpha f F(\Delta_\alpha h)\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]F(\Delta_\alpha f) \\ &\quad + \Delta_\alpha f\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]F(\Delta_\alpha f)^2 + \Delta_\alpha f F(\Delta_\alpha f)^3.\end{aligned} \tag{4.1.24}$$

Using the Lipschitz condition (4.1.6) and estimates from Lemma 17 we obtain that

$$\begin{aligned}\left| \int_{|\alpha|} \frac{1}{\alpha} \Delta_\alpha f K(x,\alpha)^3\,\mathrm{d}\alpha \right| &\leq c\,\|g\|_{L^\infty} + \left| \int_{|\alpha|>1} \frac{1}{\alpha} \Delta_\alpha f F(\Delta_\alpha f)^3\,\mathrm{d}\alpha \right| \\ &\leq c\,(1+\|g\|_{L^\infty}).\end{aligned}$$

Similarly we find that

$$\left| \int_{|\alpha|} \frac{1}{\alpha} (\Delta_\alpha f)^3 K(x,\alpha)^4\,\mathrm{d}\alpha \right| \leq c\,(1+\|g\|_{L^\infty}).$$

We conclude the proof by using the decay at infinity for the remaining terms. ■

Lemma 29. *Let $g \in H^s(\mathbb{R})$ with $s \geq 3$, we have the next bound*

$$\left| PV \int_{|\alpha|>1} \frac{1}{\alpha} B_4(x,\alpha)\,\mathrm{d}\alpha \right| \leq c\,(1+\|g\|_{C^1})^2. \tag{4.1.25}$$

Proof. Using the definition (2.1.25)

$$B_4(x,\alpha) = 3\big[-2K(x,\alpha)^3 + 8(\Delta_\alpha h)^2 K(x,\alpha)^4\big]\partial_x \Delta_\alpha h.$$

We expand the terms $\partial_x \Delta_\alpha h$ and $(\Delta_\alpha h)^2$ in $B_4(x,\alpha)$ to obtain the following decomposition

$$B_4(x,\alpha) = \Psi(x,\alpha) - 12K(x,\alpha)^3 + 48(\Delta_\alpha f)^2 K(x,\alpha)^4$$

where

$$\Psi(x,\alpha) := 96\Delta_\alpha f \Delta_\alpha g K(x,\alpha)^4 - 6\partial_x \Delta_\alpha g K(x,\alpha)^3 + 24\partial_x \Delta_\alpha g (\Delta_\alpha h)^2 K(x,\alpha)^4.$$

We note that

$$|\Psi(x,\alpha)| \leq c\left(\|g\|_{C^1} + \|g\|_{C^1}^2\right)|\alpha|^{-1},$$

hence $|\Psi(x,\alpha)|/\alpha$ is integrable for $|\alpha| > 1$. For the remaining terms in the decomposition, we follow the proofs of Lemma 27 and Lemma 28. ∎

Lemma 30. *Let $g \in H^s(\mathbb{R})$ with $s \geq 3$, then*

$$\left|PV \int_{|\alpha|>1} \frac{1}{\alpha}\Gamma(x,\alpha)\,\mathrm{d}\alpha\right| \leq c\,(1 + \|g\|_{L^\infty}). \tag{4.1.26}$$

Proof. Using the identity (2.1.41), we decompose the integral in two terms

$$\int_{|\alpha|>1} \frac{1}{\alpha}\Gamma(x,\alpha)\,\mathrm{d}\alpha = \int_{|\alpha|>1} \frac{1}{\alpha}\Gamma_1(x,\alpha)\,\mathrm{d}\alpha + \int_{|\alpha|>1} \frac{1}{\alpha}\Gamma_2(x,\alpha)\,\mathrm{d}\alpha, \tag{4.1.27}$$

for

$$\begin{aligned}\Gamma_1(x,\alpha) &:= -2(\Delta_\alpha f)^2\big[F(\Delta_\alpha h)^3 F(\Delta_\alpha f) + F(\Delta_\alpha h)^2 F(\Delta_\alpha f)^2 + F(\Delta_\alpha h)F(\Delta_\alpha f)^3\big],\\ \Gamma_2(x,\alpha) &:= -\Delta_\alpha g \Delta_\alpha f\big[F(\Delta_\alpha h)^3 F(\Delta_\alpha f) + F(\Delta_\alpha h)^2 F(\Delta_\alpha f)^2 + F(\Delta_\alpha h)F(\Delta_\alpha f)^3\big].\end{aligned}$$

Notice

$$|\Gamma_2(x,\alpha)| \leq 2\|g\|_{L^\infty}|\alpha|^{-1},$$

then the second integral in (4.1.27) is bounded. While for the first one, we proceed in a similar way to (4.1.24) by adding and subtracting $F(\Delta_\alpha f)$. Then we have

$$\begin{aligned}(\Delta_\alpha f)^2 F(\Delta_\alpha h)^3 F(\Delta_\alpha f) = {}& (\Delta_\alpha f)^2 F(\Delta_\alpha f)^2\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]F(\Delta_\alpha f)\\ &+ (\Delta_\alpha f)^2 F(\Delta_\alpha f)\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]F(\Delta_\alpha f)^2\\ &+ (\Delta_\alpha f)^2\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]F(\Delta_\alpha f)^3 + (\Delta_\alpha f)^2 F(\Delta_\alpha f)^4.\end{aligned}$$

Using the estimate (4.1.6) and Lemma 17 we obtain that

$$\left|\int_{|\alpha|>1} \frac{1}{\alpha}(\Delta_\alpha f)^2 F(\Delta_\alpha h)^3 F(\Delta_\alpha f)\,\mathrm{d}\alpha\right| \leq c\,\|g\|_{L^\infty} + \left|\int_{|\alpha|>1} \frac{1}{\alpha}(\Delta_\alpha f)^2 F(\Delta_\alpha f)^4\,\mathrm{d}\alpha\right| \leq c\,(1 + \|g\|_{L^\infty}).$$

The remaining terms in Γ_1 are bounded similarly and this finishes the proof. ∎

Lemma 31. *Let $g \in H^s(\mathbb{R})$ with $s \geq 3$, then*

$$\left|PV \int_{|\alpha|>1} \frac{1}{\alpha}\Theta(x,\alpha)\,\mathrm{d}\alpha\right| \leq c\,(1 + \|g\|_{L^\infty})^2. \tag{4.1.28}$$

Proof. Using the identity (2.1.44) we decompose in the next way

$$\int_{|\alpha|>1} \frac{1}{\alpha}\Theta(x,\alpha)\,\mathrm{d}\alpha := \int_{|\alpha|>1} \frac{1}{\alpha}\Theta_1(x,\alpha)\,\mathrm{d}\alpha + \int_{|\alpha|>1} \frac{1}{\alpha}\Theta_2(x,\alpha)\,\mathrm{d}\alpha, \tag{4.1.29}$$

for

$$\begin{aligned}\Theta_1(x,\alpha) &:= -2(\Delta_\alpha f)^4\big[F(\Delta_\alpha h)^3F(\Delta_\alpha f) + F(\Delta_\alpha h)^3 F(\Delta_\alpha f)^2 \\ &\qquad + F(\Delta_\alpha h)^2F(\Delta_\alpha f)^3 + F(\Delta_\alpha h)F(\Delta_\alpha f)^4\big], \\ \Theta_2(x,\alpha) &:= -\Delta_\alpha g(\Delta_\alpha f)^3\big[F(\Delta_\alpha h)^3F(\Delta_\alpha f) + F(\Delta_\alpha h)^3 F(\Delta_\alpha f)^2 \\ &\qquad + F(\Delta_\alpha h)^2F(\Delta_\alpha f)^3 + F(\Delta_\alpha h)F(\Delta_\alpha f)^4\big].\end{aligned}$$

Notice

$$|\Theta_2(x,\alpha)| \le c\,(1+\|\partial_x g\|_{L^\infty})\|g\|_{L^\infty}|\alpha|^{-1},$$

then the second integral in (4.1.29) is bounded. While for Θ_1 we proceed in a similar way to Γ_1 in the previous lemma. By adding and subtracting $F(\Delta_\alpha f)$, we find that

$$\Theta_1(x,\alpha) = -2(\Delta_\alpha f)^4F(\Delta_\alpha h)^4\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]F(\Delta_\alpha f) + \tilde{\Theta}(x,\alpha) + c\,(\Delta_\alpha f)^4F(\Delta_\alpha f)^5,$$

where

$$|\tilde{\Theta}(x,\alpha)| \le c\|g\|_{L^\infty}|\alpha|^{-1}.$$

We compute directly

$$F(\Delta_\alpha h) - F(\Delta_\alpha f) = -\Delta_\alpha g(2\Delta_\alpha f + \Delta_\alpha g)F(\Delta_\alpha h)F(\Delta_\alpha f).$$

Then expanding the sum we obtain that

$$\begin{aligned}\Big| -2(\Delta_\alpha f)^4F(\Delta_\alpha h)^4\big[F(\Delta_\alpha h) - F(\Delta_\alpha f)\big]F(\Delta_\alpha f)\Big| &\le \big|2(\Delta_\alpha f)^5\Delta_\alpha g F(\Delta_\alpha h)^3F(\Delta_\alpha f)^2\big| \\ &\quad + \big|2(\Delta_\alpha f)^4(\Delta_\alpha g)^2F(\Delta_\alpha h)^3F(\Delta_\alpha f)^2\big| \\ &\le c\,\|g\|_{L^\infty}|\alpha|^{-1} + c\,\|g\|_{L^\infty}^2|\alpha|^{-2}\end{aligned}$$

and therefore

$$\left|\int_{|\alpha|>1}\frac{1}{\alpha}\Theta_1(x,\alpha)\,\mathrm{d}\alpha\right| \le c\,(1+\|g\|_{L^\infty})^2,$$

which completes the proof. ■

4.2 Necessary Lemmas II

This section is devoted to the necessary lemmas used in Cauchy-Kowalevski's Theorem. More precisely, we will present some properties of the behavior of the kernel near to the origin and far from the origin. Additionally, we will show the integrability properties of the kernel $Q^{\mathrm{p}}(\gamma,\beta)$. Unlike the previous section, along this section $\Delta f(\alpha,\beta)$ denotes,

$$\Delta f(\alpha,\beta) = f(\alpha) - f(\alpha-\beta).$$

Recall that $c_R > 0$ is a constant that depends only on R and this constant may change from one line to another. We start with the following lemma.

Lemma 32. *Given a deviation* $\mathbf{d} \in O_R$, *we have the next inequality*

$$\left|\frac{\beta^2}{Q(\gamma',\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right|_* \le c_R|\beta|. \tag{4.2.1}$$

Proof. We compute directly the difference, we find that

$$\frac{\beta^2}{Q(\gamma',\beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2} = \frac{\beta^2|\partial_\alpha \mathbf{z}(\gamma')|^2 - Q(\gamma',\beta)}{Q(\gamma',\beta)|\partial_\alpha \mathbf{z}(\gamma')|^2}.$$

In the numerator, we observe that

$$\begin{aligned}\beta^2|\partial_\alpha \mathbf{z}(\gamma')|^2 - Q(\gamma',\beta) &= (\beta\partial_\alpha z_1(\gamma') - \Delta z_1(\gamma',\beta))(\beta\partial_\alpha z_1(\gamma') + \Delta z_1(\gamma',\beta)) \\ &\quad + (\beta\partial_\alpha z_2(\gamma') - \Delta z_2(\gamma',\beta))(\beta\partial_\alpha z_2(\gamma') + \Delta z_2(\gamma',\beta)).\end{aligned} \tag{4.2.2}$$

By using the Fundamental Theorem of Calculus we find the following formula

$$\beta\partial_\alpha z_1(\gamma') - \Delta z_1(\gamma',\beta) = \int_0^1 \big(\partial_\alpha z_1(\gamma') - \partial_\alpha z_1(\gamma' + (s-1)\beta)\big)\,\mathrm{d}s\,\beta.$$

By definition, we notice that $\partial z_1(\gamma') = \partial_\alpha d_1(\gamma) + 1$, therefore we obtain the next bound

$$|\beta\partial_\alpha z_1(\gamma') - \Delta z_1(\gamma',\beta)|_* \leq \|\partial_\alpha^2 d_1\|_{L^\infty(\partial S_{r'})}|\beta|^2.$$

and

$$|\beta\partial_\alpha z_1(\gamma') + \Delta z_1(\gamma',\beta)|_* \leq 2(1 + \|\partial_\alpha d_1\|_{L^\infty(\partial S_{r'})})|\beta|.$$

Then using the previous estimates together with the arc-chord condition (A2), we find that

$$\begin{aligned}&\left|\frac{(\beta\partial_\alpha z_1(\gamma') - \Delta z_1(\gamma',\beta))(\beta\partial_\alpha z_1(\gamma') + \Delta z_1(\gamma',\beta))}{Q(\gamma',\beta)|\partial_\alpha \mathbf{z}(\gamma')|^2}\right|_* \\ &\qquad = \left|\frac{\beta^2}{Q(\gamma',\beta)}\frac{(\beta\partial_\alpha z_1(\gamma') - \Delta z_1(\gamma',\beta))}{\beta^2}\frac{(\beta\partial_\alpha z_1(\gamma') + \Delta z_1(\gamma',\beta))}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right|_* \\ &\qquad \leq 2R^4\|\partial_\alpha^2 d_1\|_{L^\infty(\partial S_{r'})}(1 + \|\partial_\alpha d_1\|_{L^\infty(\partial S_{r'})})\cdot|\beta| \\ &\qquad \leq 2(1+R)^6|\beta| \\ &\qquad \leq c_R|\beta|.\end{aligned}$$

For the second term in (4.2.2), we see that

$$\begin{aligned}&\frac{(\beta\partial_\alpha z_2(\gamma') - \Delta z_2(\gamma',\beta))}{Q(\gamma',\beta)}\frac{(\beta\partial_\alpha z_2(\gamma') + \Delta z_2(\gamma',\beta))}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \\ &\qquad = \frac{\beta^2}{Q(\gamma',\beta)}\frac{\beta\partial_\alpha z_2(\gamma') - \Delta z_2(\gamma',\beta)}{\beta^2}\frac{\beta\partial_\alpha z_2(\gamma')}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \\ &\qquad + \frac{\Delta z_2(\gamma',\beta)}{Q(\gamma,\beta)}\frac{\beta\partial_\alpha z_2(\gamma') - \Delta z_2(\gamma',\beta)}{|\partial_\alpha \mathbf{z}(\gamma')|}.\end{aligned}$$

By definition, we notice that $\partial_\alpha z_2(\gamma') = \partial_\alpha d_2(\gamma') + 2\gamma'$, then using again the Fundamental Theorem of Calculus we have the following bound

$$|\beta\partial_\alpha z_2(\gamma') - \Delta z_2(\gamma',\beta)|_* \leq (2 + \|\partial_\alpha^2 d_2\|_{L^\infty(\partial S_{r'})})|\beta|^2.$$

Then we deduce

$$\begin{aligned}\left|\frac{\beta^2}{Q(\gamma',\beta)}\frac{\beta\partial_\alpha z_2(\gamma') - \Delta z_2(\gamma',\beta)}{\beta^2}\frac{\beta\partial_\alpha d_2(\gamma')}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right|_* &\leq R^4\|\partial_\alpha d_2\|_{L^\infty(\partial S_{r'})}(2 + \|\partial_\alpha^2 d_1\|_{L^\infty(\partial S_{r'})})|\beta| \\ &\leq (1+R)^6|\beta| \\ &\leq c_R|\beta|.\end{aligned}$$

In order to obtain the bound for the second term, first we see

$$\begin{aligned}
&\left|\frac{2\gamma'}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right|_* \\
&\le \left|\frac{2\gamma'}{|\partial_\alpha \mathbf{p}(\gamma')|^2}\right|_* + \left|2\gamma'\left[\frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2} - \frac{1}{|\partial_\alpha \mathbf{p}(\gamma')|^2}\right]\right|_* \\
&\le \left|\frac{2\gamma'}{1+(2\gamma')^2}\right|_* + \left|2\gamma'\left[\frac{-\partial_\alpha d_1(\gamma')(2+\partial_\alpha d_1(\gamma')) - \partial_\alpha d_2(\gamma')(4\gamma' + \partial_\alpha d_2(\gamma'))}{(1+(2\gamma')^2)(\partial_\alpha z_1(\gamma')^2 + \partial_\alpha z_2(\gamma')^2)}\right]\right|_* \\
&\le \left|\frac{2\gamma'}{1+(2\gamma')^2}\right|_* + \left|\frac{2\gamma'}{1+(2\gamma')^2}\right|_* \left|2\frac{\partial_\alpha d_1(\gamma')}{\partial_\alpha z_1(\gamma')^2 + \partial_\alpha z_2(\gamma')^2}\right|_* \\
&+ \left|\frac{2\gamma'}{1+(2\gamma')^2}\right|_* \left|\frac{\partial_\alpha d_1(\gamma')^2}{\partial_\alpha z_1(\gamma')^2 + \partial_\alpha z_2(\gamma')^2}\right|_* \\
&+ \left|\frac{(2\gamma')^2}{1+(2\gamma')^2}\right|_* \left|2\frac{\partial_\alpha d_2(\gamma')}{\partial_\alpha z_1(\gamma')^2 + \partial_\alpha z_2(\gamma')^2}\right|_* \\
&+ \left|\frac{2\gamma'}{1+(2\gamma')^2}\right|_* \left|\frac{\partial_\alpha d_2(\gamma')^2}{\partial_\alpha z_1(\gamma')^2 + \partial_\alpha z_2(\gamma')^2}\right|_* < c_R.
\end{aligned} \tag{A8}$$

Hence we have

$$\left|\frac{\beta^2}{Q(\gamma',\beta)}\frac{\beta\partial_\alpha z_2(\gamma') - \Delta z_2(\gamma',\beta)}{\beta^2}\frac{2\gamma'\beta}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right| \le 2^4(1+R)^7|\beta| < c_R|\beta|.$$

For the last part, we expand $\Delta z_2(\gamma',\beta) = \Delta d_2(\gamma',\beta) + \beta(2\gamma'-\beta)$, we obtain that

$$\begin{aligned}
\left|\frac{\Delta d_2(\gamma',\beta)}{Q(\gamma',\beta)}\frac{\beta\partial_\alpha z_2(\gamma') - \Delta z_2(\gamma',\beta)}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right|_* &\le \|\partial_\alpha d_2\|_{L^\infty(\partial S_{r'})}(2 + \|\partial_\alpha^2 d_2\|_{L^\infty(\partial S_{r'})})R^4|\beta| \\
&\le 2^6(1+R)^6|\beta| \\
&\le c_R|\beta|.
\end{aligned}$$

We notice that

$$\begin{aligned}
&\left|\frac{\beta^2(2\gamma'-\beta)}{Q(\gamma',\beta)}\right|_* \\
&\le \left|\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathbf{p}}(\gamma',\beta)}\right|_* + \left|\beta^2(2\gamma'-\beta)\left[\frac{1}{Q(\gamma',\beta)} - \frac{1}{Q^{\mathbf{p}}(\gamma',\beta)}\right]\right|_* \\
&\le \left|\frac{(2\gamma'-\beta)}{1+(2\gamma'-\beta)^2}\right|_* + \left|\beta^2(2\gamma'-\beta)\left[\frac{-\Delta d_2(\gamma',\beta)(2\beta(\gamma'-\beta) + \Delta d_2(\gamma,\beta))}{(\beta^2+\beta^2(2\gamma'-\beta)^2)Q(\gamma',\beta)}\right]\right|_* \\
&\le \left|\frac{(2\gamma'-\beta)}{1+(2\gamma'-\beta)^2}\right|_* + \left|\frac{(2\gamma'-\beta)}{1+(2\gamma'-\beta)^2}\right|_* \left|2\frac{\Delta d_1(\gamma',\beta)}{\beta}\frac{\beta^2}{Q(\gamma,\beta)}\right|_* \\
&+ \left|\frac{(2\gamma'-\beta)}{1+(2\gamma'-\beta)^2}\right|_* \left|\frac{\Delta d_1(\gamma',\beta)^2}{\beta^2}\frac{\beta^2}{Q(\gamma,\beta)}\right|_* \\
&+ \left|\frac{(2\gamma'-\beta)^2}{1+(2\gamma'-\beta)^2}\right|_* \left|2\frac{\Delta d_2(\gamma',\beta)}{\beta}\frac{\beta^2}{Q(\gamma,\beta)}\right|_* \\
&+ \left|\frac{(2\gamma'-\beta)}{1+(2\gamma'-\beta)^2}\right|_* \left|\frac{\Delta d_2(\gamma',\beta)^2}{\beta^2}\frac{\beta^2}{Q(\gamma,\beta)}\right|_* \le c_R.
\end{aligned} \tag{A9}$$

Finally, we conclude that

$$\left|\frac{\beta^2(2\gamma'-\beta)}{Q(\gamma',\beta)}\frac{\beta\partial_\alpha z_2(\gamma') - \Delta z_2(\gamma',\beta)}{\beta}\frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2}\right|_* \le c_R|\beta|$$

which completes the proof. ■

Corollary 2. *Given a deviation* $\mathbf{d} \in O_R$*, we have the next inequality*

$$\left| \frac{\beta \Delta p_2(\gamma', \beta)}{Q(\gamma', \beta)} - \frac{2\gamma'}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \right|_* \leq c_R |\beta|. \tag{4.2.3}$$

Proof. We add and subtract

$$\frac{2\gamma' \beta^2}{Q(\gamma', \beta)}$$

Then

$$\frac{\beta \Delta p_2(\gamma', \beta)}{Q(\gamma', \beta)} - \frac{2\gamma'}{|\partial_\alpha \mathbf{z}(\gamma')|^2} = \frac{\beta^2}{Q(\gamma', \beta)} \Big[\Delta p_2(\gamma', \beta) - 2\gamma' \Big] + 2\gamma' \left[\frac{\beta^2}{Q(\gamma', \beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \right].$$

Using the definition $\Delta p_2(\gamma', \beta) = \beta(2\gamma' - \beta)$, we get

$$\left| \frac{\beta \Delta p_2(\gamma', \beta)}{Q(\gamma', \beta)} - \frac{2\gamma'}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \right|_* \leq \left| \frac{\beta^2}{Q(\gamma', \beta)}(-\beta) \right|_* + \left| 2\gamma' \left[\frac{\beta^2}{Q(\gamma', \beta)} - \frac{1}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \right] \right|_*.$$

Then by plugging $2\gamma'$, in the estimates of previous Lemma 32, and using estimates (A8), we infer

$$\left| \frac{\beta \Delta p_2(\gamma', \beta)}{Q(\gamma', \beta)} - \frac{2\gamma'}{|\partial_\alpha \mathbf{z}(\gamma')|^2} \right|_* \leq c_R |\beta|.$$

■

Corollary 3. *Given a deviation* $\mathbf{d} \in O_R$*, we have the next inequality*

$$\left| \frac{\beta^2 \Delta p_2(\gamma', \beta)^2}{Q(\gamma', \beta)^2} - \frac{(2\gamma')^2}{|\partial_\alpha \mathbf{z}(\gamma')|^4} \right|_* \leq c_R |\beta|. \tag{4.2.4}$$

Lemma 33. *Given a deviation* $\mathbf{d} \in O_R$*, we have the next inequality*

$$\left| \frac{\beta^2}{Q(\gamma', \beta)} - \frac{\beta^2}{Q^{\mathbf{P}}(\gamma', \beta)} \right|_* \leq c_R |\beta|^{-1}.$$

Proof. We compute the difference

$$\beta^2 \left[\frac{1}{Q(\gamma', \beta)} - \frac{1}{Q^{\mathbf{P}}(\gamma', \beta)} \right] = - \frac{\beta^2}{Q(\gamma, \beta)} \frac{\beta^2}{Q^{\mathbf{P}}(\gamma, \beta)} \frac{\Delta d_1(\gamma', \beta)(2\beta + \Delta d_1(\gamma', \beta))}{\beta^2} - \frac{\Delta d_2(\gamma', \beta)(2\beta(2\gamma' - \beta) + \Delta d_2(\gamma', \beta))}{Q(\gamma', \beta) Q^{\mathbf{P}}(\gamma', \beta)} \beta^2.$$

We use the following two inequalities

$$|\Delta d_1(\gamma', \beta)|_* \leq 2 \|d_1\|_{L^\infty(\partial S_{r'})}$$

and

$$\left| \frac{2\beta + \Delta d_1(\gamma', \beta)}{\beta^2} \right|_* \leq 2(1 + \|d_1\|_{L^\infty(\partial S_{r'})}) |\beta|^{-1} < c_R |\beta|^{-1},$$

to conclude that

$$\left| - \frac{\beta^2}{Q(\gamma', \beta)} \frac{\beta^2}{Q^{\mathbf{P}}(\gamma', \beta)} \frac{\Delta d_1(\gamma', \beta)(2\beta + \Delta d_1(\gamma', \beta))}{\beta^2} \right|_* \leq 4(1 + R)^4 |\beta|^{-1} < c_R |\beta|^{-1}.$$

For the second part we find that

$$\left| -\frac{\beta^2}{Q(\gamma',\beta)}\frac{\beta^2}{Q^{\mathrm{p}}(\gamma',\beta)}\frac{\Delta d_2(\gamma',\beta)^2}{\beta^2}\right|_* \leq 4R^2\|d_2\|^2_{L^\infty(\partial S_{r'})}|\beta|^{-2} < c_R|\beta|^{-1}.$$

To deal with the final part, we see

$$-2\frac{\beta^2\Delta d_2(\gamma',\beta)\beta(2\gamma'-\beta)}{Q(\gamma',\beta)Q^{\mathrm{p}}(\gamma',\beta)} = -2\frac{\beta^2}{Q(\gamma',\beta)}\frac{\beta^2(2\gamma'-\beta)}{Q^{\mathrm{p}}(\gamma',\beta)}\frac{\Delta d_2(\gamma',\beta)}{\beta}.$$

Then, it follows that

$$\left|\frac{\Delta d_2(\gamma',\beta)(2\beta(2\gamma'-\beta)+\Delta d_2(\gamma',\beta))}{Q(\gamma',\beta)Q^{\mathrm{p}}(\gamma',\beta)}\beta^2\right|_* \leq c\,(1+R)^4|\beta|^{-1} < c_R|\beta|^{-1}.$$

and this completes the proof. ■

Lemma 34. *The following integral is bounded*

$$\left|PV\int_{|\beta|>1}\frac{1}{\beta}\frac{1}{1+(\zeta-\beta)^2}\,\mathrm{d}\beta\right|_* < c, \quad \textit{for} \quad \zeta\in\mathbb{C}.$$

Proof. We take $\zeta=\alpha+ir$, then we rewrite the integral, as follows

$$\begin{aligned}\int_{|\beta|>1}\frac{1}{\beta}\frac{1}{1+(\zeta-\beta)^2}\,\mathrm{d}\beta &= \int_{|\beta-\alpha|>1}\frac{1}{(\alpha-\beta)}\frac{1}{1+(\beta+ir)^2}\,\mathrm{d}\beta\\ &= \int_{|\beta-\alpha|>1}\frac{1}{(\alpha-\beta)}\frac{1}{(\beta+ir-i)}\frac{1}{(\beta+ir+i)}\,\mathrm{d}\beta.\end{aligned}$$

We decompose in the next way

$$\begin{aligned}&\int_{|\beta-\alpha|>1}\frac{1}{(\alpha-\beta)}\frac{1}{(\beta+ir-i)}\frac{1}{(\beta+ir+i)}\,\mathrm{d}\beta\\ &\qquad = A_1\int_{|\beta-\alpha|>1}\frac{1}{\alpha-\beta}d\beta + A_2\int_{|\beta-\alpha|>1}\frac{1}{(\beta+ir-i)}\,\mathrm{d}\beta\\ &\qquad + A_3\int_{|\beta-\alpha|>1}\frac{1}{(\beta+ir+i)}\,\mathrm{d}\beta\end{aligned}$$

where $A_1, A_2, A_3\in\mathbb{C}$ are bounded terms. Recall that the integral is taken in the sense of the princial value. Hence

$$A_1\int_{|\beta-\alpha|>1}\frac{1}{\alpha-\beta}\,\mathrm{d}\beta = 0.$$

For the remaining terms, we have

$$\begin{aligned}\int_{|\beta-\alpha|>1}\frac{1}{\beta+ir-i}\,\mathrm{d}\beta &= \log(\beta+ir-i)\Big|_{|\beta-\alpha|>1},\\ \int_{|\beta-\alpha|>1}\frac{1}{\beta+ir+i}\,\mathrm{d}\beta &= \log(\beta+ir+i)\Big|_{|\beta-\alpha|>1}.\end{aligned}$$

We find that

$$\begin{aligned}&\int_{|\beta-\alpha|>1}\frac{1}{\beta+ir-i}\,\mathrm{d}\beta\\ &= \lim_{R\to\infty}\Big[\log(\alpha+R-ir-i)-\log(\alpha+1+ir-i)+\log(\alpha-1+ir-i)-\log(\alpha-R+ir-i)\Big]\\ &= \lim_{R\to\infty}\log\left(\frac{\alpha+R-ir-i}{\alpha-R-ir-i}\right)+\log\left(\frac{\alpha-1-ir-i}{\alpha+1-ir-i}\right),\end{aligned}$$

we observe that the limit and the remaining term are bounded and this completes the proof. ■

Lemma 35. *The next integral is bounded*

$$\left| \int_{|\beta|<1} \frac{1}{\beta} \frac{\zeta - \beta}{1 + (\zeta - \beta)^2} \, \mathrm{d}\beta \right|_* \leq c, \quad \textit{for} \quad \zeta \in \mathbb{C}.$$

Proof. As in the previous lemma, we take $\zeta = \alpha + ir$, then

$$\int_{|\beta|<1} \frac{1}{\beta} \frac{\zeta - \beta}{1 + (\zeta - \beta)^2} \, \mathrm{d}\beta = \int_{|\beta-\alpha|>1} \frac{1}{\alpha - \beta} \frac{\beta + ir}{1 + (\beta + ir)^2} \, \mathrm{d}\beta$$

Now we decompose the integrand as follows

$$\begin{aligned} &\int_{|\beta-\alpha|<1} \frac{1}{\alpha - \beta} \frac{\beta + ir}{1 + (\beta + ir)^2} \, \mathrm{d}\beta \\ &\qquad = A_1 \int_{|\beta-\alpha|<1} \frac{\mathrm{d}\beta}{\alpha - \beta} + A_2 \int_{|\beta-\alpha|<1} \frac{\mathrm{d}\beta}{\beta + ir - i} + A_3 \int_{|\beta-\alpha|<1} \frac{\mathrm{d}\beta}{\beta + ir + i}, \end{aligned}$$

where $A_1, A_2, A_3 \in \mathbb{C}$ are bounded terms. We conclude the proof by observing that the integrals are finite. ■

Lemma 36. *The following integral is finite*

$$\left| \int_{|\beta|>1} \frac{1}{\beta} \frac{1}{(1 + (\zeta - \beta)^2)^2} \, \mathrm{d}\beta \right|_* \leq c, \quad \textit{for} \quad \zeta \in \mathbb{C}.$$

Proof. Taking $\zeta = \alpha + ir$, we change the variables to rewrite

$$\int_{|\beta|>1} \frac{1}{\beta} \frac{1}{(1 + (\zeta - \beta)^2)^2} \, \mathrm{d}\beta = \int_{|\beta-\alpha|>1} \frac{1}{\alpha - \beta} \frac{1}{(1 + (\beta + ir)^2)^2} \, \mathrm{d}\beta.$$

Now we decompose

$$\begin{aligned} &\int_{|\beta-\alpha|>1} \frac{1}{\alpha - \beta} \frac{1}{(1 + (\beta + ir)^2)^2} \, \mathrm{d}\beta \\ &\qquad = A_1 \int_{|\beta-\alpha|>1} \frac{\mathrm{d}\beta}{\alpha - \beta} + A_2 \int_{|\beta-\alpha|>1} \frac{\mathrm{d}\beta}{\beta + ir - i} + A_3 \int_{|\beta-\alpha|>1} \frac{\mathrm{d}\beta}{\beta + ir + i} \\ &\qquad + A_4 \int_{|\beta-\alpha|>1} \frac{\mathrm{d}\beta}{(\beta + ir - i)^2} + A_5 \int_{|\beta-\alpha|>1} \frac{\mathrm{d}\beta}{(\beta + ir + i)^2}, \end{aligned}$$

where each $A_i \in \mathbb{C}$ is a bounded term. Following the argument used in Lemma 34 we complete the proof. ■

Lemma 37. *The next integral is bounded*

$$\left| \int_{|\beta|>1} \frac{1}{\beta} \frac{\zeta - \beta}{(1 + (\zeta - \beta)^2)^2} \, \mathrm{d}\beta \right|_* \leq c, \quad \textit{for} \quad \zeta \in \mathbb{C}.$$

Proof. This result follows from the previous lemmas, by decomposing the integral into several finite terms. ■

Lemma 38. *The next integral is bounded*

$$\left| \int_{\mathbb{R}} \frac{1}{(1 + (\alpha + ir)^2)^2} \, \mathrm{d}\alpha \right|_* \leq c.$$

Proof. We split in the *in* and *out* parts,

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{(1+(\alpha+ir)^2)^2}\,\mathrm{d}\alpha &= \int_{|\alpha|<1} \frac{1}{(1+(\alpha+ir)^2)^2} d\alpha + \int_{|\alpha|>1} \frac{1}{(1+(\alpha+ir)^2)^2}\,\mathrm{d}\alpha \\ &< c_R + \int_{|\alpha|>1} \frac{1}{(1+(\alpha+ir)^2)^2}\,\mathrm{d}\alpha.\end{aligned}$$

For the second integral we decompose in the following way

$$\begin{aligned}\int_{|\alpha|>1} \frac{1}{(1+(\alpha+ir)^2)^2}\,\mathrm{d}\alpha &= \int_{|\beta|>1} \frac{1}{(\alpha+ir-i)^2}\frac{1}{(\alpha+ir+i)^2}\,\mathrm{d}\beta \\ &= A_1 \int_{|\beta|>1} \frac{1}{(\alpha+ir-i)}\,\mathrm{d}\beta + A_2 \int_{|\beta|>1} \frac{1}{(\alpha+ir-i)^2}\,\mathrm{d}\beta \\ &+ A_3 \int_{|\beta|>1} \frac{1}{(\alpha+ir+i)}\,\mathrm{d}\beta + A_4 \int_{|\beta|>1} \frac{1}{(\alpha+ir+i)^2}\,\mathrm{d}\beta.\end{aligned}$$

The coefficients $A_i \in \mathbb{C}$ are bounded. Then, by repeating the arguments used in Lemma 34 we conclude that the last integral is bounded. This completes the proof. ■

Chapter 5

Steady-state solutions for the Muskat problem with surface tension

In this chapter we deal with the following equations

$$\gamma \frac{z_1' z_2'' - z_1'' z_2'}{(z_1'^2 + z_2'^2)^{3/2}} + \mathrm{g}\,(\rho_+ - \rho_-) z_2 = const$$

where $(z_1, z_2)\colon \mathbb{R} \to \mathbb{R}^2$ is a curve and z_2, the second component of the curve is an odd function. We set

$$\lambda = \mathrm{g}(\rho^+ - \rho^-)/\gamma.$$

In [33], Ehrnstrom, Escher and Matioc proved that there exists 2π-periodic solutions provided the parameter $\lambda > \lambda_*$, where λ_* is finite. They consider the case when the interface is locally the graph of a function $(x, f(x))$. Instead of that we consider a curve $(g(y), y)$ and we are able to show that there exists 2π-periodic solutions for the case $\lambda^* < \lambda < \lambda_*$, moreover we show that if $\lambda < \lambda^*$ the solutions there are no longer periodic. The proof in the main theorem is obtained by analyzing a explicit formula (5.1.3) for the period. Moreover, we describe some numerical examples that indicates $\lambda^* \sim \lambda_*/7$. The results in this chapter has been published in [58].

5.1 Steady-State solutions

In this section we study the steady-state solution with the conditions (1.2.4). We will impose that the curve is 2π-periodic in the horizontal variable. In order to do it we will parameterize the curve as $\mathbf{z}(y) = (h(y), y)$ for $y \in (\pi/2, \pi/2)$ in such a waty that h satisfies

$$-\frac{h''}{(1 + h'^2)^{3/2}} + \lambda y = 0, \quad h(0) = 0, \quad h'(0) = -\alpha < 0. \tag{5.1.1}$$

After solving (5.1.1) in $y \in (-\pi/2, \pi/2)$ we will use reflections with respect to $x = \pi/2$ and $x = 3\pi/2$ to construct a 2π-periodic in the horizontal variable solution $\mathbf{z}$ (see figures 5.1, 5.2, 5.3 and 5.4).

To prove the main theorem we have three previous lemmas, the first about the existence of 2π-periodic solutions, the second related to the function $\lambda \mapsto \alpha$ and the third on the intersection of the solution curves. We leave at the end of the section the proof of the main theorem.

The idea is integrate the equation (5.1.1) directly over the interval $[0, y]$ to determine conditions in the parameters λ and α. After the integration we have the next equation

$$\frac{h'(y)}{(1 + h'(y)^2)^{1/2}} = \frac{\lambda}{2} y^2 - \frac{\alpha}{(1 + \alpha^2)^{1/2}}.$$

In order to simplify the notations we put $\beta := \dfrac{\alpha}{(1+\alpha^2)^{1/2}}$. Taking squares and since h' and $(\lambda/2)y^2 - \beta$ has the same sign, we have

$$h(y) = \int_0^y \frac{\left(\frac{\lambda}{2}s^2 - \beta\right)}{\sqrt{1 - \left(\frac{\lambda}{2}s^2 - \beta\right)^2}}\,\mathrm{d}s. \tag{5.1.2}$$

We observe from equation (5.1.2) that there exists a zero for h' and a positive value where h' explodes, that is

$$h'\left(\sqrt{\frac{2}{\lambda}\beta}\right) = 0 \quad \text{and} \quad h'\left(\sqrt{\frac{2}{\lambda}(1+\beta)}\right) = \infty,$$

also at this point the curve is no longer the graph of the function h. We define the period of the solution curve as the integral

$$T(\lambda,\alpha) := \int_0^{\sqrt{2\lambda^{-1}(1+\beta)}} \frac{\left(\frac{\lambda}{2}s^2 - \beta\right)}{\sqrt{1 - \left(\frac{\lambda}{2}s^2 - \beta\right)^2}}\,\mathrm{d}s. \tag{5.1.3}$$

Lemma 39. *If $\lambda \in (0, \lambda_*]$, the period (5.1.3) of the solution curve satisfies*

$$T(\lambda,\infty) < \pi/2 \leq T(\lambda, 0).$$

Proof. Taking the change of variable $\tau = s/\sqrt{2\lambda^{-1}(1+\beta)}$, we get the next expression for the period

$$T(\lambda,\alpha) = \sqrt{\frac{2}{\lambda}} \int_0^1 \frac{g_\tau(\alpha)}{\sqrt{1+g_\tau(\alpha)}} \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}}, \tag{5.1.4}$$

where $g_\tau(\alpha) = (1+\beta)\tau^2 - \beta$. The derivatives respect to β and α of g_τ, and β with respect to α, are

$$\frac{\partial}{\partial\beta} g_\tau(\alpha) = \tau^2 - 1 < 0 \quad \text{and} \quad \frac{\partial\beta}{\partial\alpha} = \frac{1}{(1+\alpha^2)^{3/2}} > 0.$$

Then from the chain rule the derivative respect to α is

$$g_\tau'(\alpha) = \frac{\partial}{\partial\beta} g_\tau(\alpha) \frac{\partial\beta}{\partial\alpha} = \frac{\tau^2 - 1}{(1+\alpha^2)^{3/2}} < 0$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{g_\tau(\alpha)}{\sqrt{1+g_t(\alpha)}}\right) = g_\tau'(\alpha)\frac{2+g_\tau(\alpha)}{(1+g_\tau(\alpha))^{3/2}} < 0 \tag{5.1.5}$$

and therefore the period $T(\lambda,\alpha)$ is decreasing with respect to α. Now, if we want to determine the pair (λ,α) such that $T(\lambda,\alpha) = \pi/2$, we will determine conditions on λ observing its behavior as α goes to zero and infinity. We will see explicitly that the following inequalities are satisfied

$$\lim_{\alpha\to\infty} T(\lambda,\alpha) < \pi/2 < \lim_{\alpha\to 0} T(\lambda,\alpha).$$

For the first inequality, we compute $T(\lambda, 0)$ from (5.1.4), that is

$$\lim_{\alpha\to 0} T(\lambda,\alpha) = \int_0^1 \frac{g_\tau(0)}{\sqrt{1+g_\tau(0)}} \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}}.$$

When $\alpha = 0$, $\beta = 0$ and we have

$$T(\lambda, 0) = \frac{1}{2}\frac{1}{\sqrt{2\lambda}} \int_0^1 \frac{\tau^{-1/4}}{(1-\tau)^{1/2}}\,\mathrm{d}\tau = \frac{1}{2}\frac{1}{\sqrt{2\lambda}} \int_0^1 \tau^{3/4-1}(1-\tau)^{1/2-1}\,\mathrm{d}\tau. \tag{5.1.6}$$

Therefore, if we take
$$\lambda < \frac{1}{2\pi^2} B^2\left(\frac{3}{4}, \frac{1}{2}\right) = \lambda_*,$$
we get
$$T(\lambda, 0) = \frac{1}{2}\frac{1}{\sqrt{2\lambda}} B\left(\frac{3}{4}, \frac{1}{2}\right) > \frac{\pi}{2}.$$

The next step is compute the limit when $\alpha \to \infty$ in equation (5.1.3). We observe for a fixed $\lambda < \lambda_*$ the limit in the integral satisfies

$$\begin{aligned}
\lim_{\alpha\to\infty} T(\lambda, \alpha) &= \lim_{\alpha\to\infty} \sqrt{\frac{2}{\lambda}} \int_0^1 \frac{g_\tau(\alpha)}{\sqrt{1+g_\tau(\alpha)}} \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}} \\
&= \sqrt{\frac{2}{\lambda}} \left(\int_0^{1/\sqrt{2}} + \int_{1/\sqrt{2}}^1 \right) \frac{g_\tau(\infty)}{\sqrt{1+g_\tau(\infty)}} \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}} \\
&= \sqrt{\frac{2}{\lambda}} \int_0^{1/\sqrt{2}} \frac{g_\tau(\infty)}{\sqrt{1+g_\tau(\infty)}} \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}} + \frac{1}{\sqrt{\lambda}} \left(\sqrt{2} + \log\left(\frac{\sqrt{2}}{2+\sqrt{2}} \right) \right).
\end{aligned} \tag{5.1.7}$$

We can see in the last integral, that for a fixed $\lambda < \lambda_*$
$$\lim_{\alpha\to\infty} T(\lambda, \alpha) < 0$$
moreover $T(\lambda, \alpha) \to -\infty$, because the first integral goes to $-\infty$. ■

The value λ_* is the critical value found by Ehrnstrom, Escher and Matioc. When the parameter $\lambda \in (\lambda_*, 1]$ the steady solution correspond to a curve parameterized by the graph $(x, f(x))$ with $f(0) = 0$ and $f'(0) = \alpha$ (see [33]). The next lemma provides a relation between λ and α.

Lemma 40. *Let $\lambda \in (0, \lambda_*]$, then there exists a unique $\alpha(\lambda) \in [0, \infty)$ such that $T(\lambda, \alpha(\lambda)) = \pi/2$ and the mapping $\alpha\colon (0, \lambda_*] \to [0, \infty)$ is smooth, bijective and decreasing.*

Proof. From lemma 39 we have $T(\lambda, \infty) < \pi/2 \le T(\lambda, 0)$ for $\lambda \in (0, \lambda_*]$. Also we known that $T(\lambda, \alpha)$ is smooth respect to λ and α, hence for $\lambda \in (0, \lambda_*]$ there exists a unique $\alpha(\lambda) \in (0, \infty)$ such that $T(\lambda, \alpha(\lambda)) = \pi/2$. Choose the pair $(\lambda, \alpha(\lambda)) \in (0, \lambda_*] \times [0, \infty)$ such that $T(\lambda, \alpha(\lambda)) = \pi/2$, then
$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda} T(\lambda, \alpha(\lambda)) = \partial_\lambda T(\lambda, \alpha(\lambda)) + \partial_\alpha T(\lambda, \alpha(\lambda))\, \alpha'(\lambda),$$
because $\partial_\alpha T, \partial_\lambda T < 0$ we have $\alpha'(\lambda) < 0$. From (5.1.6) we get $\lim_{\lambda\to\lambda_*} \alpha(\lambda) = 0$, and from (5.1.7), $\lim_{\lambda\to 0} \alpha(\lambda) = \infty$. This completes the proof. ■

To complete the proof of the main theorem we need to know if the curves solutions has intersections. We know from (5.1.3) that the function h has a minimum at the point $y = \sqrt{2\lambda^{-1}\beta}$, define
$$I(\lambda, \alpha) := \int_0^{\sqrt{2\lambda^{-1}\beta}} \frac{\left(\frac{\lambda}{2}y^2 - \beta\right)}{\sqrt{1 - \left(\frac{\lambda}{2}y^2 - \beta\right)^2}}\, \mathrm{d}y$$
which is the value of h at $\sqrt{2\lambda^{-1}\beta}$. Now we want to determine if the fact that $\lambda \in (0, \lambda_*]$ is enough to have
$$I(\lambda, \alpha(\lambda)) > -\frac{\pi}{2},$$

this property is important because we want that the solution curve has not intersections. Taking the change of variable $\tau = s/\sqrt{2\lambda^{-1}\beta}$ we can rewrite the integral as

$$I(\lambda,\alpha) = \sqrt{\frac{2}{\lambda}} \int_0^1 \frac{\beta(\tau^2-1)}{\sqrt{1-\beta^2(1-\tau^2)^2}} \beta^{1/2}\, \mathrm{d}\tau,$$

we have the next lemma.

Lemma 41. *There exists $\lambda^* < \lambda_*$ positive such that if $\lambda \in (\lambda^*, \lambda_*]$ then $I(\lambda, \alpha(\lambda)) > -\pi/2$ and $I(\lambda^*, \alpha(\lambda^*)) = -\pi/2$.*

Proof. We observe

$$\partial_\lambda I(\lambda,\alpha) = \int_0^1 \frac{\beta(\tau^2-1)}{\sqrt{1-\beta^2(1-\tau^2)^2}} \beta^{1/2}\, \mathrm{d}\tau \left(-\frac{1}{\sqrt{2}}\lambda^{-3/2}\right) > 0$$

because $0 < \tau < 1$, then $I(\lambda, \alpha)$ is increasing with respect to λ. Take the pair $(\lambda, \alpha(\lambda)) \in (0, \lambda_*] \times [0, \infty)$, such that $T(\lambda, \alpha(\lambda)) = \pi/2$. Now we have

$$\frac{\pi}{2} = I(\lambda, \alpha(\lambda)) + \sqrt{\frac{2}{\lambda}} \int_{\sqrt{\frac{\beta}{\beta+1}}}^1 \frac{g_\tau(\alpha)}{\sqrt{1+g_\tau(\alpha)}} \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}},$$

then we have

$$\begin{aligned}
\lim_{\lambda\to\lambda_*} I(\lambda) &= \frac{\pi}{2} - \lim_{\lambda\to\lambda_*} \sqrt{\frac{2}{\lambda}} \int_{\sqrt{\frac{\beta}{\beta+1}}}^1 \frac{g_\tau(\alpha)}{\sqrt{1+g_\tau(\alpha)}} \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}} \\
&= \frac{\pi}{2} - \sqrt{\frac{2}{\lambda_*}} \int_0^1 \frac{g_\tau(0)}{\sqrt{1+g_\tau(0)}} \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}} \\
&= \frac{\pi}{2} - \frac{\pi}{2} = 0,
\end{aligned}$$

the last inequality is due to $\lim_{\lambda\to\lambda_*} \alpha(\lambda) = 0$. Hence from $\lim_{\lambda\to 0} \alpha(\lambda) = \infty$, we get

$$\begin{aligned}
\lim_{\lambda\to 0} I(\lambda) &= \frac{\pi}{2} - \lim_{\lambda\to 0} \sqrt{\frac{2}{\lambda}} \int_{\sqrt{\frac{\beta}{\beta+1}}}^1 \frac{g_\tau(\alpha)}{\sqrt{1+g_\tau(\alpha)}} \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}} \\
&= \frac{\pi}{2} - \left(\lim_{\lambda\to 0} \sqrt{\frac{2}{\lambda}}\right) \left(\lim_{\alpha\to\infty} \int_{\sqrt{\frac{\beta}{\beta+1}}}^1 \frac{g_\tau(\alpha)}{\sqrt{1+g_\tau(\alpha)}} \frac{\mathrm{d}\tau}{\sqrt{1-\tau^2}}\right) \\
&= \frac{\pi}{2} - \left(\lim_{\lambda\to 0} \sqrt{\frac{2}{\lambda}}\right) \left(\sqrt{2} + \log\left(\frac{\sqrt{2}}{2+\sqrt{2}}\right)\right) = -\infty
\end{aligned}$$

therefore with respect to λ we have

$$I(0) < -\frac{\pi}{2} < I(\lambda_*).$$

The continuity of I over $(0, \lambda_*]$ implies that there exists $\lambda^* > 0$ such that $I(\lambda^*) = -\pi/2$. We take $\lambda \in (\lambda^*, \lambda_*]$ then by lemma 39, there exists $\alpha(\lambda) \in [0, \infty)$ such that $T(\lambda, \alpha(\lambda)) = \pi/2$. Since $I(\lambda)$ is increasing with respect to λ and $I(\lambda^*) = -\pi/2$ we have

$$-I(\lambda^*) < I(\lambda) < I(\lambda_*),$$

which means that

$$I(\lambda) > -\frac{\pi}{2}.$$

■

Now we can proceed to the proof of the main result theorem 3.

Proof the main theorem 3. Take $\lambda \in (0, \lambda_*]$, then from lemmas 39 and 40 we find $\alpha(\lambda)$ such that $T(\lambda) = \pi/2$. Now if $\lambda > \lambda^*$ by lemma 41 we have value $I(\lambda) > -\pi/2$. Hence by reflections we can construct a curve 2π-periodic which does not have self intersections and is it solution of the steady equation (5.1.1). ■

5.2 Numerical examples

In this section we show some numerical examples of steady-state solutions for different values of λ. Here we present numericals solutions, see figures 5.1, 5.2, 5.3 and 5.4, they explain the behavior of the solution curves when λ approaches to the value λ^*. These examples were produced using Mathematica. The limit curve z_{λ^*} (figure 5.3) remains 2π-periodic but has self-intersections, also for $\lambda_*/16$ (figure 5.4) we observe that the curve solution has self-intersections, then we can infer that $\lambda_*/16 < \lambda^*$.

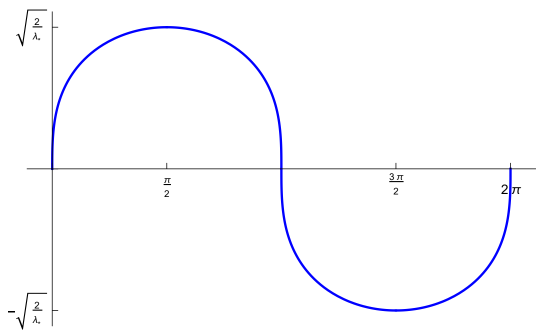


Figure 5.1: Curve solution for λ_* and $\beta = 0$.

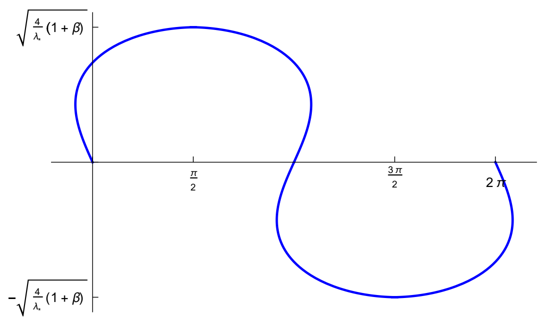


Figure 5.2: Curve solution for $\lambda_*/2$ and $\beta \sim 0.225$.

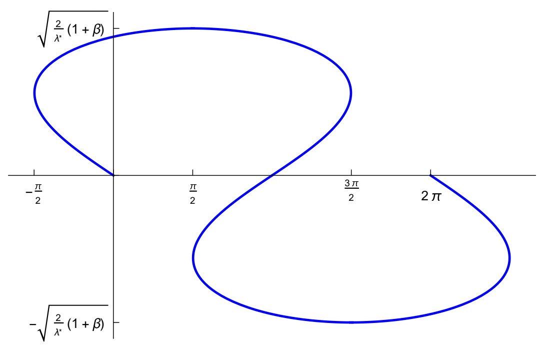


Figure 5.3: Curve solution for $\lambda^* \sim \lambda_*/7$ and $\beta \sim 0.46$

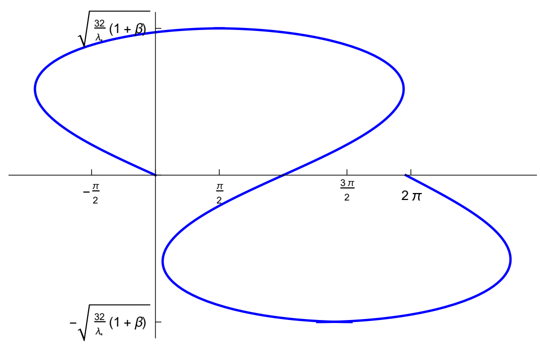


Figure 5.4: Curve solution for $\lambda_*/16$ and $\beta \sim 0.525$

Remark 4. *It remains to have a analytical proof of the value λ^* that lead us to an explicit value for λ^*.*

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