Topics on noncommutative Fourier multipliers

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Contexto y Resultados Principales

Objetivo de la tesis

El objetivo de esta tesis es estudiar las álgebras de von Neumann usando métodos que provienen del análisis real, como los multiplicadores de Fourier, la teoría de interpolación o los operadores maximales, entre algunos otros. Siguiendo el hilo conductor de varias preguntas abiertas en la comunidad científica, iremos adaptando técnicas clásicas al ámbito no conmutativo y arrojando así algo de luz sobre las cuestiones expuestas.

Las álgebras de von Neumann son álgebras de operadores sobre espacios de Hilbert cerradas en la topología débil*. En el presente trabajo, las trataremos como una generalización de las funciones acotadas $L_\infty(\Omega)$ en un espacio de medida Ω, y por tanto como una clase adecuada de objetos donde tratar de replicar resultados del análisis armónico.

Mientras que las álgebras de von Neumann conmutativas son isomorfas precisamente a los espacios de la forma $L_\infty(\Omega)$, las no conmutativas pueden presentarse de formas y tipos muy diversos. De hecho, su clasificación sigue siendo un desafío para la comunidad matemática. Nosotros nos centraremos en una clase de álgebras de von Neumann que, siendo lo suficientemente amplia, permanece más cerca del caso L_∞: las álgebras traciales.

Una traza en una álgebra de von Neumann es un funcional positivo con ciertas propiedades adicionales, que enunciaremos con detalle más adelante. Baste por ahora decir que en el caso conmutativo $L_\infty(\Omega)$, las trazas son exactamente los funcionales que provienen de la integración respecto a una medida. En otras palabras, las trazas son *una noción generalizada de integral*, de tal modo que pueden utilizarse para definir espacios L_p sobre álgebras de von Neumann. A estos espacios se los denomina precisamente *espacios L_p no conmutativos*.

El desarrollo de una teoría de análisis armónico en estos espacios L_p no conmutativos ha sido una línea de investigación muy activa y fructífera en las últimas décadas. Algunos avances recientes en esta dirección incluyen el desarrollo de una teoría de interpolación [JX07, Dir15, CR23], generalizaciones de las transformadas de Hilbert y de Riesz [MR17, PdlST23, JMP18], y el descubrimiento de varios resultados para multiplicadores suaves que emulan el teorema clásico de Hörmander-Mikhlin [CAGPPT23, MRX22, JMP14].

Esta tesis tratará precisamente de continuar esas líneas de investigación. El cuerpo del desarrollo se halla dividido en tres bloques bien diferenciados. En el primero, presentaremos resultados que abordan una pregunta abierta de [CAGPP20] sobre martingalas en el ámbito no conmutativo, la cual requiere además estudiar ciertos teoremas de interpolación de operadores maximales en espacios de Orlicz. El segundo bloque se orienta a extender las técnicas de [GPPX22] para estudiar nuevas transformadas de Hilbert en retículos del grupo de Lie $\mathrm{SL}_2(\mathbb{C})$. Finalmente, en el tercer y último bloque estudiaremos una versión en Schur de la transformada de Riesz, y dearrollaremos varias aplicaciones en forma de teoremas de acotación en S_p para varias clases de multiplicadores de Schur.

Martingalas, Operadores Maximales e Interpolación

El teorema de diferenciación de Lebesgue establece que al promediar una función en $L_1(\mathbb{R}^2)$ sobre bolas cada vez más pequeñas, se recupera el valor de la función original en casi todo punto. Jessen, Marcinkiewicz y Zygmund [JMZ35] demostraron que, al reemplazar bolas por rectángulos con lados paralelos a los ejes, uno es capaz de obtener un resultado análogo al de Lebesgue. Sin embargo, también probaron esta versión del teorema sólo se cumple para funciones en $L\log L(\mathbb{R}^2)$, siendo esta la clase óptima de funciones donde se puede obtener dicho teorema.

La clave de su demostración consitió en encontrar una desigualdad débil para el operador maximal fuerte, definido de manera análoga al operador maximal de Hardy-Littlewood usando rectángulos con ejes paralelos a los ejes en lugar de bolas. El resto de la prueba se basa en un argumento general de Stein que relaciona convergencia y desigualdades maximales. Lance [Lan76] demostró que este argumento también funciona en espacios no conmutativos y Conde, González-Pérez y Parcet [CAGPP20] estudiaron la convergencia de sucesiones de martingalas de dos parámetros $(\mathbb{E}_n \otimes \mathbb{E}_m)_{n,m}$ en álgebras de von Neumann utilizando el mismo método: probar desigualdades débiles de operadores maximales.

En el caso no conmutativo, obtuvieron las desigualdades deseadas para funciones en $L\log^{2+\varepsilon} L(\mathcal{M}\otimes\mathcal{M})$ para todo $\varepsilon > 0$. Sin embargo, no pudieron determinar si esta es la clase óptima de funciones para la que el resultado se cumple.

En mi trabajo con González-Pérez y Parcet [PPG24], resolvimos parcialmente este problema demostrando que la clase óptima no puede ser $L\log^{\alpha} L(\mathcal{M}\otimes\mathcal{M})$ para $\alpha < 2$ (el caso $\alpha = 2$ permanece abierto). La idea fue mostrar que, de ser así, la norma L_p del operador maximal crecería mucho más lentamente cuando $p \to 1$ de lo que realmente crece. Para ello desarrollamos un teorema de interpolación que proporciona un control adecuado de la norma L_p del operador interpolado. Nuestra demostración parte de un refinamiento de algunas técnicas de Cadilhac y Ricard [CR23], que simplifican la prueba original de Junge y Xu [JX07].

Nuevos Multiplicadores de Fourier en Retículos

Dado un grupo abeliano G, su grupo dual es el conjunto de homomorfismos de G al círculo unitario $\mathbb{T}$. La transformada de Fourier relaciona funciones en G con funciones en su grupo dual, generalizando la relación de Fourier clásica entre $\mathbb{T}$ y $\mathbb{Z}$. Si G no es abeliano, esta relación se pierde. En ese caso, en lugar del grupo dual, se trabaja con el álgebra de von Neumann $\mathcal{L}G$, generada por la representación regular izquierda $\lambda\colon G \to B(L_2(G))$.

Los elementos $f \in \mathcal{L}G$ tienen una descomposición de Fourier $f = \int_G \widehat{f}(g)\lambda_g\, dg$ para cierta función $\widehat{f} \in L_1(G)$. Un multiplicador de Fourier T_m en $\mathcal{L}G$ es un operador definido formalmente por $T_m(\lambda_g) = m(g)\lambda_g$, donde m es una función en G. Si T_m puede extenderse a un operador acotado $T_m\colon L_p(\mathcal{L}G) \to L_p(\mathcal{L}G)$, diremos que T_m es acotado en L_p o L_p-acotado.

Los multiplicadores de Fourier en $L_\infty(\mathcal{L}G)$ pueden emplearse para demostrar ciertas propiedades de álgebras de operadores que resultarían inaccesibles de otro modo. Concretamente, Cowling y Haagerup [CH89] probaron que el álgebra de von Neumann de $\mathrm{SL}_2(\mathbb{C})$ tiene la llamada propiedad w*-CBAP usando la existencia de multiplicadores en L_p con un decaimiento adecuado, pero los ejemplos de tales multiplicadores son realmente muy escasos. Parece razonable esperar que existan familias mucho más grandes de multiplicadores acotados en $L_p(\mathcal{L}\mathrm{SL}_2(\mathbb{C}))$ para $1 < p < \infty$, como sucede en el caso euclídeo.

Una obstrucción a este programa fue descubierta recientemente por Parcet, de la Salle y Tablate [PdlST23], quienes demostraron que sólo existe un multiplicador de Fourier (localmente) L_p-

acotado sobre el álgebra de von Neumann de $\mathrm{PSL}_2(\mathbb{C})$ que sea idempotente. En cambio, González-Pérez, Parcet y Xia [GPPX22] mostraron que este hecho no se cumple para los retículos de $\mathrm{PSL}_2(\mathbb{C})$. De hecho, ellos encontraron un multiplicador idempotente que es acotado en L_p sobre el álgebra de von Neumann de $\mathrm{PSL}_2(\mathbb{Z}[i])$, pero no sobre $\mathrm{PSL}_2(\mathbb{C})$.

En mi trabajo [Gar24], extiendo este resultado a la familia de retículos conocida como grupos de Bianchi, provando así la existencia de multiplicadores idempotentes en una familia mucho más rica de subgrupos de $\mathrm{SL}_2(\mathbb{C})$. La prueba utiliza argumentos de [GPPX22], combinados con resultados de [Sta17] sobre la acción de los grupos de Bianchi en el semiplano superior. Este resultado prepara un futuro estudio de posibles extensiones de estos multiplicadores al grupo $\mathrm{SL}_2(\mathbb{C})$, invitando a realizar nuevas investigaciones sobre multiplicadores L_p-acotados en este espacio.

Multiplicadores de Riesz en Versión Schur

Un multiplicador de Schur S_M es una aplicación que envía una matriz $(A_{i,j})_{i,j\in I}$ a $(M(i,j)A_{i,j})_{i,j\in I}$, donde M es una función definida en $I\times I$. Si S_M define un operador acotado en $L_p(B(\ell_2(I)),\mathrm{tr})$, se dice que es un multiplicador de Schur acotado en S_p. Los multiplicadores de Schur y de Fourier se relacionan a través del teorema de transferencia [NR11], el cual sugiere que los multiplicadores de Schur son una clase de operadores cercana a los multiplicadores de Fourier, pero con una rigidez menor.

En mi trabajo con González-Pérez, Parcet y Ricard [GPPGR24] proponemos un punto de vista que permite unificar las demostraciones de varios resultados clave en esta área y establecemos al mismo tiempo algunos resultados nuevos de acotación en S_p de multiplicadores de Schur. Para ello, estudiamos primero una versión en Schur de las transformadas de Riesz clásicas, R_u. Dada una sucesión de vectores $u=(u_i)_{i\in I}$ en un espacio de Hilbert, definimos R_u como el multiplicador de Schur de valores en un espacio de Hilbert dado por

$$R_u\big((A_{i,j})_{i,j\in I}\big) \;=\; \Big(\frac{u_i-u_j}{\|u_i-u_j\|}A_{i,j}\Big)_{i,j\in I}.$$

En nuestro trabajo probamos que, al igual que en el caso clásico, la norma L_p de la función cuadrado asociada a $R_u(A)$ es equivalente a la norma L_p de A. A partir de este resultado establecemos la acotación en S_p del multiplicador S_M en los siguientes casos:

1. *Factorización a través de un espacio de Hilbert:* hay acotación S_p cuando M puede factorizarse a través de un espacio de Hilbert de la forma

$$M(i,j) \;=\; \Big\langle \frac{u_i-u_j}{\|u_i-u_j\|}, \frac{v_i-v_j}{\|v_i-v_j\|}\Big\rangle.$$

   (Esta es una extensión del criterio de Grothendieck-Haagerup, que caracteriza los multiplicadores acotados en L_∞.)

2. *Diferencias divididas α-Hölder:* hay acotación S_p cuando $M(x,y)=\frac{|f(x)-f(y)|^\beta}{|x-y|^\alpha}$, donde $0<\alpha<\beta$ y f es una función (α/β)-Hölder y no decreciente. (Esto es una versión fuerte de la conjetura de Arazy, demostrada originalmente por Potapov y Sukochev [PS11].)

Además, también recuperamos el teorema de Hörmander-Mikhlin de [CAGPPT23] y el teorema de Marcinkiewicz de [CLM24], unificando así una parte importante de la teoría actual en S_p sobre multiplicadores de Schur.

Palabras clave: análisis armónico no conmutativo, multiplicadores de Fourier, multiplicadores de Schur, operadores maximales, interpolación.

Overview and Main Results

The goal of this thesis is to study an specific kind of algebras, known as von Neumann algebras, with methods coming from real analysis. These methods include the theory of Fourier multipliers, interpolation and maximal operators, among others. Each of these topics is well-stablished in mathematical analysis, not only for their intrinsic value but also for their wide-ranging applications across various fields in mathematics. Here, we aim to investigate how these techniques can be applied within the framework of von Neumann algebras.

Von Neumann algebras are weak*-closed algebras of operators acting on Hilbert spaces. They can be regarded as a generalization of the bounded functions $L_\infty(\Omega)$ on a measure space Ω. More concretely, any commutative von Neumann algebra is isomorphic (in the proper category) to an algebra $L_\infty(\Omega)$. On the other hand, non-commutative von Neumann algebras may appear in a variety of kinds and forms, and their classification constitutes still today a challenge for mathematicians. We will focus on a class of von Neumann algebras that, being still broad enough, remain closer to the L_∞ case: the tracial ones.

A trace on a von Neumann algebra is a positive functional with some extra properties, that we will enunciate in detail later ... Let's just say for now that in the commutative case $L_\infty(\Omega)$, traces are exactly the functionals coming from integration against measures. In other words, traces are *generalized* integrals. They can be used to define L_p spaces on von Neumann algebras, which are sometimes called *non-commutative* L_p-spaces.

The development of a theory of harmonic analysis in non-commutative L_p-spaces has been a very active and fruitful line of research over the past decades. Recent advances in this direction include a theory of interpolation [JX07, Dir15, CR23], generalizations of the Hilbert and Riesz transforms [MR17, PdlST23, JMP18] and a deeper understanding on smooth multipliers through results like the Hörmander-Mikhlin theorem [CAGPPT23, MRX22, JMP14].

The present work aims to continue the research of the aforementioned authors, and can be divided in three blocks. In the first block, we will present results addressing an open question of [CAGPP20], which requires the study of interpolation with applications to martingales in the non-commutative setting. The second block is aimed to extending the techniques in [GPPX22] in order to study new Hilbert transforms on lattices of the Lie group $\mathrm{SL}_2(\mathbb{C})$. Finally, in the last block we will prove some new results on Schur multipliers, together with several applications ranging from simplified proofs of the Hörmander-Mikhlin theorem in [CAGPPT23] and the Marcinkiewicz criteria in [CLM24], to new generalizations of the Grothendieck-Haagerup's S_∞-criteria and the Arazy's divided differences conjecture.

Martingales, Maximal Operators, and Interpolation

The Lebesgue differentiation theorem states that averaging a function in $L_1(\mathbb{R}^2)$ over balls shrinking to a point is a process that recovers the value of the function almost everywhere. This is a natural generalization of the fundamental theorem of calculus, except for the fact that it requires the choice of balls as the basic kind of region to average over. Jessen, Marcinkiewicz and Zygmund [JMZ35] proved that replacing balls with rectangles with edges parallel to the axes still leads to

a result analogous to the Lebesgue differentiation theorem. This version of the theorem applies only to functions in $L\log L(\mathbb{R}^2)$. Furthermore, the authors showed that this is the optimal class of functions for which the result can hold.

The key step involved finding a weak inequality for the strong maximal operator, defined as the Hardy-Littlewood maximal operator but using rectangles instead of balls. The rest of the proof relies on a general argument of Stein relating convergence and maximal inequalities. Lance [Lan76] showed that this argument also works for non-commutative spaces, and Conde, González-Pérez and Parcet [CAGPP20] studied convergence of two-parameter martingale sequences $(\mathbb{E}_n \otimes \mathbb{E}_m)_{n,m}$ on von Neumann algebras using the same method: proving weak maximal inequalities.

In the non-commutative case, they obtained the desired inequalities for functions in $L\log^{2+\varepsilon} L(\mathcal{M}\otimes \mathcal{M})$ for any $\varepsilon > 0$. However, they were unable to determine whether this is the optimal class of functions for which the result holds.

In my work with González-Pérez and Parcet [PPG24], we partially solved this problem by proving that the optimal class cannot be $L\log^{\alpha} L(\mathcal{M}\otimes\mathcal{M})$ for $\alpha < 2$ (the case $\alpha = 2$ remains open). The idea was to show that in that case, the L_p-norm of the maximal operator would increase much more slowly as p tends to 1 than it is known to do. Therefore, the key point was to prove an interpolation theorem which gives a suitable control on the L_p-norm of the associated maximal operator. We proved this result by refining some techniques from Cadilhac and Ricard [CR23], which simplify the original proof provided by Junge and Xu [JX07].

New Fourier Multipliers on Lattices

For an Abelian group G, its dual group is the group of homomorphisms from G to the unit circle $\mathbb{T}$. The Fourier transform relates functions in G and its dual group, generalizing the classical Fourier relation between $\mathbb{T}$ and $\mathbb{Z}$. If G is not Abelian, this relation is lost. In that case, instead of the dual group, we work with the von Neumann algebra $\mathcal{L}G$ generated by the left regular representation $\lambda\colon G \to B(L_2(G))$.

Elements $f \in \mathcal{L}G$ have a Fourier decomposition $f = \int_G \widehat{f}(g)\lambda_g\, dg$ for some $\widehat{f} \in L_1(G)$. A Fourier multiplier T_m on $\mathcal{L}G$ is an operator formally defined as $T_m(\lambda_g) = m(g)\lambda_g$, where m is a function on G. If T_m can be extended to a bounded operator $T_m\colon L_p(\mathcal{L}G) \to L_p(\mathcal{L}G)$, we say that T_m is bounded on L_p or L_p-bounded.

Fourier multipliers on $L_\infty(\mathcal{L}G)$ can be used to prove certain operator-algebraic properties which would be inaccessible otherwise. Concretely, Cowling and Haagerup [CH89] showed that the von-Neumann algebra of $\mathrm{SL}_2(\mathbb{C})$ has the w*-CBAP, but examples of such multipliers are very scarce. It seems reasonable to expect that much larger families of multipliers would be bounded in $L_p(\mathcal{L}\mathrm{SL}_2(\mathbb{C}))$, for $1 < p < \infty$, as happens in the Euclidean case. This suggests the need to develop a general theory of L_p-boundedness for Fourier multipliers in this setting.

An obstruction to this program was recently found by Parcet, de la Salle, and Tablate [PdlST23]. They proved that there is exactly one (locally) L_p-bounded Fourier multiplier on the group von Neumann algebra of $\mathrm{PSL}_2(\mathbb{C})$ which is idempotent. On the other hand, González-Pérez, Parcet, and Xia [GPPX22] showed that this is not the case for lattices of $\mathrm{PSL}_2(\mathbb{C})$. They found another idempotent multiplier which is L_p-bounded on the von Neumann algebra of $\mathrm{PSL}_2(\mathbb{Z}[i])$, though not on $\mathrm{PSL}_2(\mathbb{C})$.

In my work [Gar24], I extended this result to a broad family of lattices called the Bianchi groups, proving the existence of idempotent L_p Fourier multipliers in a much more rich family of subgroups of $\mathrm{SL}_2(\mathbb{C})$. The proof is based on the argument of [GPPX22], together with some results in [Sta17],

where the author developes a detailed theory of the action of Bianchi groups in the upper-half plane. This result paves the way for studying the extensions of these multipliers to the whole group $\mathrm{SL}_2(\mathbb{C})$, inviting further investigation into the properties of L_p-bounded multipliers in this space.

A unified perspective for several results on Schur multipliers

A Schur multiplier S_M is a map that sends a matrix $(A_{i,j})_{i,j\in I}$ to $(M(i,j)A_{i,j})_{i,j\in I}$, where M is a function on $I\times I$. If S_M defines a bounded operator on $L_p(B(\ell_2(I)),\mathrm{tr})$, we call it an S_p-bounded Schur multiplier. Schur and Fourier multipliers are connected through the main result in [NR11], which suggests that Schur multipliers are a less rigid counterpart of Fourier multipliers.

In my work with González-Pérez, Parcet and Ricard [GPPGR24], we unified the proofs of several key results in this area and established new ones by adopting a different approach. Firstly, we studied a Schur version R_u of the classical Riesz transforms. For a sequence of vectors $u=(u_i)_{i\in I}$ in a Hilbert space, we define R_u as the Hilbert-valued Schur multiplier

$$R_u((A_{i,j})_{i,j\in I})=\left(\frac{u_i-u_j}{\|u_i-u_j\|}A_{i,j}\right)_{i,j\in I}.$$

We proved that, as in the classical case, the L_p-norm of the square function related to $R_u(A)$ is equivalent to the L_p-norm of A. Building on this result, we established the S_p-boundedness of S_M in the following cases:

1. *Hilbert factorization:* when M can be factorized through a Hilbert as

$$M(i,j)=\Big\langle \frac{u_i-u_j}{\|u_i-u_j\|},\frac{v_i-v_j}{\|v_i-v_j\|}\Big\rangle.$$

   (This is an extension of the celebrated Grothendick-Haagerup criteria, which characterizes the L_∞-bounded multipliers.)

2. *α-Hölder divided differences:* when $M(x,y)=\frac{|f(x)-f(y)|^\beta}{|x-y|^\alpha}$, where $0<\alpha<\beta$ and f is an (α/β)-Hölder, non-decreasing function. (This strengthens Arazy's conjecture, originally proven by Potapov and Sukochev [PS11].)

In addition, we also recovered the Hörmander-Mikhlin theorem from [CAGPPT23] and the Marcinkiewicz theorem from [CLM24], summarizing a significant amount of the current theory on S_p-bounded Schur multipliers.

Keywords: noncommutative harmonic analysis, Fourier multipliers, Schur multipliers, maximal operators, interpolation.

Chapter 1

Prerequisites

1.1 Basic definitions in von Neumann algebras

This is an introductory chapter to fix some definitions and notation. We are going to use as general references [Tak02, Tak03] for the sections on von Neumann algebras and group von Neumann algebras, [CdlS15, NR11] for Schur multipliers and the transference Fourier-Schur, [ER88] for operator spaces and [PX03] for the theory of noncommutative L_p-spaces.

Topologies and von Neumann Algebras. Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. Then a $*$-subalgebra $\mathcal{M} \subset B(\mathcal{H})$ is a *von Neumann algebra* if it contains the identity and satisfies any (and therefore all) of the following equivalent conditions:

i) $\mathcal{M}$ is closed for weak limits, ie. if $(T_n) \subset \mathcal{M}$ is a sequence and there exist $T \in B(\mathcal{H})$ such that
$$\langle T_n\xi, \eta\rangle \to \langle T\xi, \eta\rangle, \quad \text{for all } \xi, \eta \in \mathcal{H},$$
then $T \in \mathcal{M}$.

ii) $\mathcal{M}$ is closed for strong limits, ie. if $(T_n) \subset \mathcal{M}$ is a sequence and there exist $T \in B(\mathcal{H})$ such that
$$\|T_n\xi\| \to \|T\xi\|, \quad \text{for all } \xi \in \mathcal{H},$$
then $T \in \mathcal{M}$.

iii) $\mathcal{M}$ coincides with its double commutant $\mathcal{M}''$.

Any von Neumann algebra $\mathcal{M}$ has a unique predual $\mathcal{M}_*$, that is, a Banach space $\mathcal{M}_*$ such that $\mathcal{M}$ is isometrically isomorphic to $(\mathcal{M}_*)^*$. The weak topology induced by $\mathcal{M}_*$ on $\mathcal{M}$ is called the *weak*-topology*. Another posible characterization of this topology is the weak topology induced by the functionals
$$\mathcal{M} \ni x \mapsto \mathrm{tr}(xA), \quad \text{for } A \in S_1(\mathcal{H}),$$
where $S_1(\mathcal{H})$ is the set of trace-class operators, that is, the operators $T \in B(\mathcal{H})$ such that $\mathrm{tr}(|T|)$ is finite.

States and Weights. Let $\mathcal{M}, \mathcal{N}$ be two von Neumann algebras, and $f\colon \mathcal{M} \to \mathcal{N}$ a linear map between them. Then we say that

i) f is *normal* if it is continuous in the weak*-topology,

ii) f is *unital* if $f(1) = 1$,

iii) f is *positive* if it sends positive elements to positive elements,

iv) if f is positive, we say that it is *faithful* when $x \geq 0$ and $f(x) = 0$ implies that $x = 0$.

When $f\colon \mathcal{M} \to \mathbb{C}$ is a positive and unital functional, we say that it is a *state*. For positive maps like states, there is another characterization of normality: that for any increasing and bounded net (x_α) of positive operators, it holds that

$$f(\sup_\alpha x_\alpha) = \sup_\alpha f(x_\alpha).$$

Sometimes we will be interested in working with unbounded functionals. Let $\mathcal{M}_+$ denote the positive elements of $\mathcal{M}$. When a map $f\colon \mathcal{M}_+ \to [0, \infty]$ is homogeneous and respect sums, we say it is a *weight*. Notice that any positive linear functional restricts to a weight, whereas a weight can be extended to a bounded functional on $\mathcal{M}$ if and only if $f(1) < \infty$.

Normality and faithfulness can be defined for weights in the same fashion as for (positive) linear maps. We are going to give two extra definitions: for a weight $f\colon \mathcal{M}_+ \to [0, \infty]$, we say that

i) f is *semifinite* if the space $\{x \in \mathcal{M} :\ f(x^*x) < \infty\}$ is weak*-dense in $\mathcal{M}$,

ii) f is *tracial* if $f(x^*x) = f(xx^*)$ for any $x \in \mathcal{M}$.

Remark 1.1.1. When f is a finite weight and therefore extends to a bounded linear map $\tilde{f}$ on $\mathcal{M}$, the traciality of f is equivalent to $\tilde{f}$ satisfying

$$\tilde{f}(ab) = \tilde{f}(ba), \quad \text{for any } a, b \in \mathcal{M},$$

which follows immediately from the polarization identity.

A tracial weight will be usually called a *trace*. Unless otherwise stated, we will consider all traces to be semifinite, normal and faithful. When the trace is finite, we will consider in addition the trace to be normalized (ie. $\mathrm{tr}(1) = 1$). A pair $(\mathcal{M}, \tau)$ where $\mathcal{M}$ is a von Neumann algebra and τ is a trace will be called a *tracial von Neumann algebra*.

1.2 Noncommutative L_p-Spaces

We are going to briefly show how to construct the L_p spaces associated to a tracial von Neumann algebra $(\mathcal{M}, \tau)$, and how this construction coincides with the usual one in the commutative case.

Noncommutative L_p Spaces - a Simple Approach. An element $x \in \mathcal{M}$ is said to be *finitely supported* if there is a projection p of finite trace so that $x = pxp$. The set

$$\mathcal{M}_{\mathrm{fin}} \ = \ \{\, x \in \mathcal{M} : x \text{ is finitely supported} \,\}$$

serves as an essential domain for defining L_p norms. For $x \in \mathcal{M}_{\mathrm{fin}}$ and $1 \leq p \leq \infty$, we may define using continuous calculus the following norm:

$$\|x\|_\infty \ := \ \|x\|_{\mathcal{M}} \quad \text{if } p = \infty, \quad \text{and} \quad \|x\|_p \ := \ \Big(\tau(|x|^p)\Big)^{1/p} \ \text{ otherwise;}$$

where we have used the spectral calculus of x to define the operator $|x|^p$. It is possible to prove that, since x is finitely supported, $\tau(|x|^p)$ is finite. One then obtains a normed space $(\mathcal{M}_{\mathrm{fin}}, \|\cdot\|_p)$.

Definition 1.2.1. The noncommutative L_p-space, denoted by $L_p(\mathcal{M}, \tau)$, is defined as the completion of $\mathcal{M}_{\mathrm{fin}}$ with respect to $\|\cdot\|_p$.

Notice that in the commutative case $\mathcal{M} \cong L_\infty(X)$, $\mathcal{M}_{\text{fin}}$ are precisely the bounded measurable functions that are supported in a set of finite measure. This process generalizes the construction of the usual L_p spaces, but it has as a disadvantage that we don't really know how a general element in $L_p(\mathcal{M})$ looks like. Therefore we are still going to outline a second approach to these L_p spaces.

Noncommutative L_p Spaces - Unbounded Operators The continuous and Borel functional calculus can be extended to the more general context of unbounded operators in a pretty similar (but more technical) way. If $\mathcal{M} \subset B(\mathcal{H})$ is a von Neumann algebra and T is a (densely defined, closed) positive operator on $\mathcal{H}$, we say that it is *affiliated with* $\mathcal{M}$ if its spectral projections $\mathbf{1}_{[0,\lambda)}(T)$ belong to $\mathcal{M}$. This allows us to extend the trace on $\mathcal{M}$ to any $T \geq 0$ affiliated with $\mathcal{M}$ as follows

$$\tau(T) \;=\; \int_0^\infty \lambda \, d\tau\big(\mathbf{1}_{[0,\lambda)}(T)\big).$$

In general, we will say that an operator T is affiliated with $\mathcal{M}$ if its polar decomposition $T = u|T|$ is such that $u \in \mathcal{M}$ and $|T|$ is affiliated with $\mathcal{M}$. For a general affiliated operator T, we define

$$\|T\|_p \;=\; \big(\tau(|T|^p)\big)^{1/p},$$

Then we let $L_p(\mathcal{M}, \tau)$ be the set of all T with finite p-norm. This is actually a Banach space where $\mathcal{M}_{\text{fin}}$ is dense, providing a concrete model for $L_p(\mathcal{M})$ as a set of unbounded (but affiliated) operators on $\mathcal{H}$.

Remark 1.2.1. In fact, many functional-analytic properties from classical L_p-theory carry over to the noncommutative setting, including duality and interpolation theorems.

The Schatten Classes. We close this section with an important example of noncommutative spaces: the *Schatten classes*.

Definition 1.2.2. For a given Hilbert space $\mathcal{H}$, the *Schatten class* $S_p(\mathcal{H})$ is defined as the set of bounded operators $T \in B(\mathcal{H})$ such that $\operatorname{tr}(|T|^p) < \infty$ if $1 \leq p < \infty$, or $B(\mathcal{H})$ when $p = \infty$.

The Schatten classes are the L_p spaces associated to $(B(\mathcal{H}), \operatorname{tr})$. In this case, all elements in the L_p spaces are bounded operators, and indeed they are contained in one another ($S_p(\mathcal{H}) \subset S_q(\mathcal{H})$ if $p \leq q$).

1.3 Group von Neumann algebras and Fourier multipliers

Group von Neumann algebras are a case of much interest when studying von Neumann algebras. Here we are going to explain how, given a certain group G, we can construct its associated von Neumann algebra $\mathcal{L}G$. This von Neumann algebra can be studied from the group using Fourier multipliers, and this point of view generalized indeed the classical Fourier multipliers studied in $\mathbb{R}$ and $\mathbb{T}$.

Group Algebras of Discrete Groups. Let G be a discrete group consider the Hilbert space $\ell_2(G)$. The *left regular representation* of G is given by

$$\lambda_g : \ell_2(G) \to \ell_2(G), \quad \big(\lambda_g \, \xi\big)(h) \;=\; \xi(g^{-1}h),$$

for $\xi \in \ell_2(G)$ and $g, h \in G$.

Definition 1.3.1. The *group von Neumann algebra* of a discrete group G, denoted $\mathcal{L}G$, is defined as the von Neumann algebra generated by the image of λ, that is

$$\mathcal{L}(G) \;=\; \{\lambda_g : g \in G\}'' \;\subset\; B(\ell_2(G)).$$

Concretely, elements of $\mathcal{L}(G)$ can be formally seen as (possibly infinite) linear combinations

$$x \;=\; \sum_{g\in G} \alpha_g\, \lambda_g,$$

where $\{\alpha_g\} \subset \mathbb{C}$ is a suitable family of coefficients called the *Fourier coefficients of* x, and usually denoted by $\widehat{x}(g) = \alpha_g$. Furthermore, $\mathcal{L}(G)$ admits a faithful normal tracial state via

$$\tau(x) \;=\; \langle x\,\delta_e, \delta_e\rangle_{\ell_2(G)},$$

where δ_e is the unit vector supported at the identity $e \in G$.

Remark 1.3.1. Notice that, if we have written x as a linear combination $x = \sum_g \widehat{x}(g)\lambda_g$, then $\tau(x) = \widehat{x}(e)$. This is exactly what happens when we are working with the Fourier expansion of a function on $\mathbb{T}$, and τ is just the functional given by integration. We will explore in more detail this connection very soon.

Group Algebras of Locally Compact Groups. The previous constructions can be generalized to the case when G is a topological group which is unimodular, locally compact and Hausdorff. In that case, we consider the Hilbert space $L_2(G)$, constructed using the left Haar measure μ on G. The left regular representation $\lambda\colon G \to B(L_2(G))$ is again defined as

$$\lambda_g\big(\xi\big)(h) \;=\; \xi(g^{-1}h),$$

In a similar manner, the von Neumann algebra $\mathcal{L}G$ is the algebra generated by $\{\lambda_g\colon g \in G\}$ in $B(L_2(G))$, and an element $x \in \mathcal{L}(G)$ can be (at least formally) expressed as

$$x \;=\; \int_G \widehat{x}(g)\,\lambda_g\; d\mu(g).$$

However, unlike the discrete case, the natural trace τ on $\mathcal{L}(G)$ given by

$$\tau\big(x^*x\big) \;=\; \int_G \big|\widehat{x}(g)\big|^2\, d\mu(g)$$

will define a (possibly unbounded) weight on $\mathcal{L}G$, instead of a tracial state.

Fourier Multipliers on $\mathcal{L}G$ A *Fourier multiplier* on $\mathcal{L}G$ is a map T_m defined on the finite linear span $\{\lambda_g\}_{g\in G}$ by

$$T_m(\lambda_g) \;=\; m(g)\,\lambda_g,$$

for some function $m : G \to \mathbb{C}$. One then asks when this definition extends to a bounded operator

$$T_m : L_p(\mathcal{L}G) \;\longrightarrow\; L_p(\mathcal{L}G).$$

When this extension exists, we say that T_m or even just m is a bounded L_p multiplier on $\mathcal{L}G$. Fourier multipliers play a central role in modern analysis, since they often provide key tools for understanding operator-algebraic properties (e.g. amenability, the completely bounded approximation property, etc.).

Fourier Multipliers on Abelian Groups. When G is abelian and locally compact, its group von Neumann algebra $\mathcal{L}(G)$ is naturally isomorphic to $L_\infty(\widehat{G})$, where $\widehat{G}$ is the dual group of G —that is, the set of homomorphisms $\chi\colon G \to \mathbb{T}$. This connection goes beyond and allows us to identify Fourier multipliers on both contexts. Let's prove it.

Proposition 1.3.2. *Let G be an unimodular abelian group. Then $\mathcal{L}G$ and $L_\infty(\widehat{G})$ are $*$-isomorphic.*

Proof. Observe that $L_\infty(\widehat{G})$ can be embedded into $B\big(L_2(\widehat{G})\big)$ via multiplication operators. On the other hand, let $\mathcal{F}\colon L_2(G) \to L_2(\widehat{G})$ be the Fourier transform on G. We claim that

$$\mathcal{F}^*(\mathcal{L}G)\,\mathcal{F} = L_\infty(\widehat{G}).$$

Indeed, a straightforward calculation shows that $\mathcal{F}^*\lambda_g\,\mathcal{F}$ is the multiplication by the evaluation at g. By the Stone–Weierstrass theorem, the von Neumann algebra generated by these evaluation functions contains $C_0(\widehat{G})$, so it must coincide with $L_\infty(\widehat{G})$. This completes the proof.

□

This proposition goes even beyond its statement, since the proof shows how *Fourier multipliers are precisely the usual multipliers* on $L_2(\widehat{G})$, and concretely on the abelian groups $\mathbb{R}$ and $\mathbb{T}$, whose dual groups are respectively $\mathbb{R}$ and $\mathbb{Z}$.

1.4 Schur multipliers, operators spaces and a transference principle

In this section we introduce Schur multipliers in two different contexts: the *discrete* one, and the *measure-theoretic* one. Schur multipliers are interesting mathematical objects in their own, but also because of their connection with Fourier multipliers. To understand this connection, we have to introduce first the notion of completely bounded operators, which comes from the theory of operator spaces.

Definition of Schur Multipliers. A *(discrete) Schur multiplier* on an index set I is a map S_M acting on matrices $(A_{i,j})_{i,j\in I}$ as

$$S_M\big((A_{i,j})_{i,j\in I}\big) \;=\; \big(M(i,j)\,A_{i,j}\big)_{i,j\in I},$$

where $M(i,j)$ is a complex-valued function on $I \times I$. When S_M defines a bounded operator on the Schatten class $S_p(\ell^2 I)$ (defined as the noncommutative L_p spaces associated to $(B(\ell^2 I), \mathrm{tr})$), we say that it is a Schur multiplier on S_p.

A parallel notion of Schur multipliers can be developed in the setting of functions on measure spaces. Suppose we have a measure space (X,μ), so the operators of the form

$$T_k(f)(x) = \int_X k(x,y) f(y)\,dy,$$

where $k \in L_2(X \times X)$ is a square-integrable function, are exactly all the operators in $S_2(L_2(X,\mu))$. We can formally define Schur multipliers S_M as multipliers of the form

$$S_M(T_k) = T_{M\,\cdot\,k}, \qquad \text{for } T_k \in S_p \cap S_2(L_2(X)).$$

When such a multiplier extends to a bounded multiplier in the whole $S_p(L_2(X))$, we say that it is a S_p-bounded Schur multiplier. Notice that both notions (the discrete and the measure-theoretic one) coincide when μ is the counting measure on X.

Remark 1.4.1. The class of Schur multipliers is not independent of the choice of the base space. That means for example, that even if $L_2(\mathbb{R})$ and $\ell_2(\mathbb{Z})$ are the same as Hilbert space, the class of Schur multipliers are different classes of operators in each case.

Operator Spaces and Completely Bounded Norms. To handle Schur multipliers in full generality, it is important to work in the framework of *operator spaces*. An operator space is, informally, a Banach space E equipped with a family of norms $\|\cdot\|_n$ on matrix amplifications $\mathbb{M}_n(E)$ of every size. These norms must satisfy in addition some compatibility axioms in order to be an operator space. The reader can find the details in [ER88].

Example 1.4.2. Any closed linear subspace $E \subset B(\mathcal{H})$ is an operator space, with the natural norms induced by the identifications $\mathbb{M}_n(E) \subset B(H \otimes \ell_2(\{1, \dots, n\}))$. (And actually, any operator space is of this form.)

The feature of operator spaces that is of our interest is that they allow one to talk about *complete boundedness* of maps.

Definition 1.4.1. A linear map $T : E \to F$ between operator spaces is completely bounded if the induced map

$$\mathrm{Id}_{M_n} \otimes T : M_n(E) \longrightarrow M_n(F)$$

is uniformly bounded for all n, and the *cb-norm* $\|T\|_{\mathrm{cb}}$ is then the supremum of these norms.

The Fourier-Schur Transference. The completely bounded norm of an operator is necessary to understand the connection between Schur and Fourier multipliers. The next theorem has been proven in [NR11] for the discrete case, and a bit later in [CdlS15] for the continuous case.

Theorem 1.4.3. *Let G be a locally compact group and $m\colon \mathrm{G} \to \mathbb{C}$ a bounded function. Consider the map $M(g,h) = m(g^{-1}h)$. If m defines a completely L_p-bounded Fourier multiplier on $L_p(\mathcal{L}G)$, then M defined a completely S_p-bounded Schur multiplier on $S_p(L_2(G))$ and the following inequality holds*

$$\|S_M\|_{cb-S_p} \leq \|T_m\|_{cb-L_p}.$$

For certain groups, including abelian and even the solvable ones, this inequality is known to be indeed an equality of norms. Whether it can be improved to an equality in the general case constitutes a problem that remains open. In any case, this has been a seminal result connecting Schur and Fourier multipliers, and has served as a motivation for people working in harmonic analysis to become interested in the Schur theory.

Chapter 2

Interpolation, Maximal Operators and Martingales

2.1 Introduction and main results

Maximal inequalities in the context of noncommutative von Neumann algebras have a long history that traces back to the independent works of Cuculescu [Cuc71] on martingale sequences and Lance [Lan76], and later Yeadon [Yea77, Yea80], on ergodic theory. More concretely, Cuculescu proved that, if $\mathbf{E}_n : \mathcal{N} \to \mathcal{N}_n$ is a sequence of unital conditional expectations from a semifinite von Neumann algebra $(\mathcal{N}, \tau)$ into a filtration $(\mathcal{N}_n)_n$, then, for every positive element $x \in L_1(\mathcal{N})$ and scalar $\lambda \geq 0$, there exists a projection $e_\lambda \in \mathscr{P}(\mathcal{N})$ such that

i) $e_\lambda \, \mathbf{E}_n(x) \, e_\lambda \leq \lambda \, e_\lambda$, for every $n \geq 0$, and

ii) $\tau(\mathbf{1} - e_\lambda) \lesssim \dfrac{\|x\|_1}{\lambda}$.

It is trivial to verify that, when $\mathcal{N} = L_\infty(\Omega)$ is Abelian, these two conditions are equivalent to the weak type $(1,1)$ of the maximal Doob operator $f \mapsto \sup_n |\mathbf{E}_n[f]|$, see [Doo53]. Like in the classical setting, the above noncommutative weak type $(1,1)$ implies a generalization of almost everywhere convergence, introduced by Lance, called *bilateral almost uniform* convergence. Indeed, a sequence $(x_n)_n$ of τ-measurable operators, see [Ter81] for the precise definition, converges to x bilaterally almost uniformly if, for every $\varepsilon > 0$, there is a projection $e \in \mathscr{P}(\mathcal{N})$ such that $\tau(\mathbf{1} - e) < \varepsilon$ and

$$\big\| e \, (x_n - x) \, e \big\|_\infty \to 0.$$

Asymmetrical versions of the above definition can be defined by placing the projection e only on the left or on the right. Those would be referred to as *almost uniform convergence*. Thus, Cuculescu's inequality implies the bilateral almost uniform convergence of $\mathbf{E}_n[x]$ for every $x \in L_1(\mathcal{N})$.

While the tools for handling noncommutative weak types of maximal operators were set up early on, the strong types only appeared later. The main idea here is that the classical maximal estimates can be interpreted as mixed-norm bounds in $L_p[\ell_\infty]$, ie

$$\Big\| \sup_n |f_n| \Big\|_p = \big\| (f_n)_n \big\|_{L_p[\ell_\infty]}.$$

The noncommutative analogues of Banach space-valued L_p-spaces appeared firstly in the work of Pisier [Pis98], who defined a way of constructing $L_p(\mathcal{N}; E)$, where E needs to have a finer structure than that given by its Banach space norm —called an *operator space structure*— see [Pis03, ER00]. Also, the algebra $\mathcal{N}$ in Pisier's definition needed to be hyperfinite. More general von Neumann

algebras were considered by Junge in [Jun04], where noncommutative operator space-valued L_p-spaces were defined for QWEP von Neumann algebras. In the particular case of $L_p(\mathcal{N};\ell_\infty)$, a definition independent of the approximation properties of $\mathcal{N}$ was reached in the works of Junge and Xu [Jun02, JX03, JX05]. Once the spaces $L_p(\mathcal{M};\ell_\infty)$ were defined, the strong type (p,p) for the maximal operator associated to a family of maps $x \mapsto S_n(x)$ can be expressed just as the boundedness of $S = (S_n)_n : L_p(\mathcal{N}) \to L_p(\mathcal{N};\ell_\infty)$. These mixed-norm spaces satisfy many natural properties. For instance, it was show in a remarkable article of Junge [Jun02] that the maximal function associated to a family of conditional expectations has strong type (p,p) for $1 < p$. Nevertheless, their compatibility with Marcinkiewicz interpolation is very subtle. The noncommutative generalization of the Marcinkiewicz interpolation theorem was obtained by Junge and Xu [JX03] and their proof turned out to be remarkably involved, see [Dir15] as well. In fact, and contrary to the classical case, the operators S_n are required to be positivity-preserving and the operator norms in L_p explode like

$$c(p) := \big\|S = (S_n)_n : L_p(\mathcal{N}) \to L_p(\mathcal{N};\ell_\infty)\big\| \sim \big(p'\big)^2 \quad \text{as} \quad p \to 1^+,$$

while in the classical case the constant $c(p)$ grows merely like p' as $p \to 1^+$, where p' will represent the conjugate exponent of p in the forthcoming discussion. This growth was later shown to be optimal [JX05, Hu09] and not a shortcoming of Junge/Xu's proof. One of the underlying reasons for this technicality has to do with the fact that the projections e_λ in i)–ii) should behave as noncommutative generalization of the measurable sets $\{|\sup_n S_n(x)| \le \lambda\}$ and thus it would be expected that the integral of $e_\lambda^\perp = 1 - e_\lambda$, as a positive measurable operator, should have an L_p-norm comparable to the mixed $L_p[\ell_\infty]$-norm of $(S_n(x))_n$. Nevertheless this is not the case as explained in [JX05] or [CAGPP20, Remark 3.5].

Recently, noncommutative maximal inequalities have received renewed attention. For instance, maximal ergodic theorems for group actions of polynomial-growth groups [HLW21], Fourier multipliers [HWW20] and amenable groups [CW22] have been obtained. Part of our work will be built on a recent article of Cadilhac and Ricard that has clarified many of the technical steps behind the original proof of the noncommutative Marcinkiewicz interpolation for maximal inequalities [CR23].

The strong maximal function

Let $\mathcal{M}\bar{\otimes}\mathcal{N}$ be the spacial tensor product of two finite von Neumann algebras $(\mathcal{M},\tau_\mathcal{M})$ and $(\mathcal{N},\tau_\mathcal{N})$ endowed with normal and faithful tracial states and let $(\mathcal{M}_n)_n$ and $(\mathcal{N}_n)_n$ be ascending sequence of (unital) von Neumann subalgebras $\mathcal{M}_n \subset \mathcal{M}_{n+1} \subset \cdots \subset \mathcal{M}$ and $\mathcal{N}_n \subset \mathcal{N}_{n+1} \subset \cdots \subset \mathcal{N}$. We will usually refer to these ascending chains of subalgebras as *filtrations*. We will furthermore assume that the increasing union of the subalgebras is weak-$*$ dense inside $\mathcal{M}$ and $\mathcal{N}$ respectively. We will denote by $\mathbf{E}_n \otimes \mathbf{E}_m : \mathcal{M}\bar{\otimes}\mathcal{N} \to \mathcal{M}_n\bar{\otimes}\mathcal{N}_m$ the tensor product of the conditional expectations onto $\mathcal{M}_n$ and $\mathcal{N}_m$ respectively. Observe that, if both $\mathcal{N}$ and $\mathcal{M}$ are isomorphic to $L_\infty([0,1])$ and the subalgebras are given by $\mathcal{N}_n = \mathcal{M}_n = L_\infty([0,1];\Sigma_n)$, where Σ_n is taken to be the σ-algebra generated by the dyadic sets of length 2^{-n}, then the maximal operator associated to the family of conditional expectations $(\mathbf{E}_n \otimes \mathbf{E}_m)_{n,m}$ is a dyadic model for the classical *strong maximal operator*. This operator was studied early on. Indeed, Jessen, Marcinkiewicz and Zygmund [JMZ35] proved, using our previous notation, that

$$f \in L\,\log L([0,1]^2) \quad \Longrightarrow \quad \big(\mathbf{E}_n \otimes \mathbf{E}_m\big)[f] \to f \ \text{ almost everywhere.}$$

This was shown using an argument that relies on the lim sup of a family of operators and the fact that the one-dimensional maximal function maps $L\log L$ into L_1. They also proved that $L\log L$ is the largest Orlicz class for which there is convergence almost everywhere to the original data. The optimality was proved by showing that almost everywhere convergence for an Orlicz

class L_Φ implies a *restricted weak Orlicz type* estimate. The full weak Orlicz type (Φ, Φ), where $\Phi(s) = t\,(1 + \log_+ s)$ was later proved using geometric arguments by Córdoba and Feffermann [CF75]. An alternative proof of the weak Orlicz type (Φ, Φ) using a Fubini-type argument to combine two different operators with weak type $(1, 1)$ was given in [Guz72].

It is natural to ask if these results hold for general finite von Neumann algebras. Let $\Phi : [0, \infty) \to [0, \infty]$ be a locally finite, non-decreasing and convex function with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. Recall the following definition of (maximal) weak Orlicz type (Φ, Φ).

Definition 2.1.1. Let $(\mathcal{N}, \tau_\mathcal{N})$ and $(\mathcal{M}, \tau_\mathcal{M})$ be semifinite von Neumann algebras. A sequence $(S_n)_{n\geq 0}$ of operators $S_n : L_1(\mathcal{N}) \cap \mathcal{N} \to L_0(\mathcal{M})$ is said to be of (maximal) weak type (Φ, Φ) if there exists a $C \geq 0$ such that for every $x \geq 0$ any $\lambda > 0$ there is a projection $e_\lambda \in \mathscr{P}(\mathcal{M})$ satisfying:

i) $\big\|e_\lambda\, S_n(x)\, e_\lambda\big\|_\infty \leq \lambda$ for all $n \geq 0$, and

ii) $\tau_\mathcal{M}(1 - e_\lambda) \leq \tau_\mathcal{N} \circ \Phi\Big(C\,\dfrac{|x|}{\lambda}\Big)$.

We will denote the optimal constant $C \geq 0$ in Definition 2.1.1 by $[S]_{(\Phi,\Phi)}$. Clearly, when $\mathcal{N} = L_\infty(\Omega)$ is Abelian, the definition recovers the classical one. The following problem, whose solution would generalize the results of [JMZ35, Guz72, CF75], was stated in [CAGPP20].

Problem 2.1.1 (from [CAGPP20, Conjecture B3]). Let $(\mathcal{M}, \tau_\mathcal{M})$ and $(\mathcal{N}, \tau_\mathcal{N})$ be two finite von Neumann algebras and $\mathbf{E}_n \otimes \mathbf{E}_m : \mathcal{M}\bar{\otimes}\mathcal{N} \to \mathcal{N}_n\bar{\otimes}\mathcal{M}_m$ be the tensor product of two conditional expectations associated to filtrations $(\mathcal{M}_n)_n$ and $(\mathcal{N}_n)_n$.

i) Is the optimal weak Orlicz type of the maximal operator associated to $(\mathbf{E}_n \otimes \mathbf{E}_m)_{n,m}$ given by (Φ, Φ), where $\Phi(s) = s\,(1 + \log_+ s)^2$?

ii) Is the largest Orlicz class $L_\Phi(\mathcal{M}\bar{\otimes}\mathcal{N})$ for which $(\mathbf{E}_n\otimes\mathbf{E}_m)(x)$ converges bilaterally almost uniformly for every $x \in L_\Phi$ as $n, m \to \infty$ given by $\Phi(s) = s\,(1 + \log_+ s)^2$?

Although the problem above makes perfect sense in the case of semifinite von Neumann algebras, the crux of the problems lies in the local integrability, thus we will restrict our discussion to the finite case. Observe that, contrary to the classical case, the conjectured optimal here is $L\log^2 L$ instead of $L\log L$. This is the reasonable thing to ask for since, by [Hu09], the noncommutative Doob maximal maps $L\log^2 L$ into $L_1[\ell_\infty]$ and that bound is known to be optimal. Partial answers to Problem 2.1.1ii) have been obtained. For instance in [CAGPP20, Theorem A] it was shown, using an argument that mimics the lim sup argument from [JMZ35], that there is bilateral almost uniform convergence for every $x \in L\log^2 L$. This result can be obtained as well from earlier results in [HS18]. Whether there are larger Orlicz classes than $L\log^2 L$ for which almost uniform convergence holds is an open problem, although it doesn't seem likely.

Problem 2.1.1i) can be decoupled into two halves. The first is whether the conjectured weak Orlicz type (Φ, Φ) holds. The second is whether the Orlicz type cannot be lowered below $\Phi(s) = s\,(1 + \log_+ s)^2$. That is, if Ψ is another increasing and convex function with $\Psi \in o(\Phi)$, then, $\mathscr{E} = (\mathbf{E}_n \otimes \mathbf{E}_m)_{n,m}$ cannot be of weak type (Ψ, Ψ). Regarding the first problem, it was also shown in [CAGPP20, Theorem B3] that the noncommutative strong maximal operator has weak Orlicz type (Θ, Θ), where $\Theta(s) = L\log^{2+\varepsilon} L$, for every $\varepsilon > 0$. Our main result in this article solves, with an argument relying on interpolation, the second part of the problem. Thus, the optimal weak Orlicz type for the noncommutative strong maximal cannot be lowered below $L\log^2 L$. To sharpen the weak Orlicz type from $L\log^{2+\varepsilon} L$ to $L\log^2 L$ remains as a challenging open problem.

Our results

As said before, this article gives a solution to one half of Problem 2.1.1i). The key is a Marcinkiewicz-type interpolation theorem asserting that, if $\mathscr{E} = (\mathbf{E}_n \otimes \mathbf{E}_m)_{n,m}$ were of weak Orlicz type (Ψ, Ψ), with $\Psi(s) \in o\big(s\,(1+\log_+ s)^2\big)$, then the operator norm in L_p would satisfy that

$$c(p) := \big\|(\mathbf{E}_n \otimes \mathbf{E}_m)_{n,m} : L_p(\mathcal{N}\bar{\otimes}\mathcal{M}) \to L_p(\mathcal{N}\bar{\otimes}\mathcal{M};\ell_\infty)\big\| \in o\big((p')^4\big) \quad \text{as} \ \ p \to 1^+,$$

but this would contradict the known optimal bound for the norm for the noncommutative strong maximal on L_p, which is, by a Fubini-type argument, the square of the optimal norm of the noncommutative Doob maximal. Thus, the optimal weak Orlicz type for the noncommutative strong maximal can't be lowered below $L\log^2 L$.

Our main tool will be the following Marcinkiewicz-like interpolation theorem for noncommutative maximal functions. Even though we are interesting in applications to finite von Neumann algebras, we will formulate our result in the semifinite context since —we suspect— it can be a useful tool with applications beyond Problem 2.1.1. Here q_Φ and p_Φ denote the upper and lower Matuszewska-Orlicz indices [Mal85] of a Young function Φ, which coincide with the usual Boyd indices [Boy69] of the respective Orlicz spaces. We will also assume in all our statements that $q_{\Phi_1} < p_{\Phi_2}$.

Theorem 2.1.2. *Let $(\mathcal{N}, \tau_\mathcal{N})$ and $(\mathcal{M}, \tau_\mathcal{M})$ be von Neumann algebras endowed with n.s.f. traces and let Φ_1, Φ_2 be locally finite Young functions. Given $S = (S_n)_{n\geq 1}$, a family of positive linear maps $S_n : L_1(\mathcal{N}) \cap L_\infty(\mathcal{N}) \to L_0(\mathcal{M})$ such that S is of maximal weak type (Φ_1, Φ_1) and (Φ_2, Φ_2) simultaneously, we have that there exists a function $p \mapsto F(p; \Phi_1, \Phi_2) : (q_{\Phi_1}, p_{\Phi_2}) \to (0, \infty)$ such that:*

$$\big\|S : L_p(\mathcal{N}) \to L_p(\mathcal{M};\ell_\infty)\big\| \lesssim \max\Big\{[S]_{(\Phi_1,\Phi_1)},\, [S]_{(\Phi_2,\Phi_2)}\Big\}\, F(p;\Phi_1,\Phi_2),$$

for every $q_{\Phi_1} < p < q_{\Phi_2}$, where the term $F(p;\Phi_1,\Phi_2)$ is given by

$$F(p;\Phi_1,\Phi_2) := \inf_{k_0}\left\{\Big[\sum_{k>k_0}\Big(2^{-k}\big(\Phi_1(2^k)+\Phi_2(2^{k_0})\big)^{\frac{1}{p}}\Big)^{\frac{1}{2}} + \sum_{k\leq k_0}\Big(2^{-k}\,\Phi_2(2^k)^{\frac{1}{p}}\Big)^{\frac{1}{2}}\Big]^2\right\}. \qquad (\mathrm{Ct}_p)$$

The proof of this theorem follows closely the steps of [CR23]. In fact, in their Remark [CR23, Remark 3.6] it is already stated that a qualitative result like our Theorem 2.1.2 would hold for rearrangement invariant spaces $E(\mathcal{M},\tau)$ with q_E and p_E being their Boyd indices. Our addition is to turn the qualitative statement into a quantitative one by bounding the norm in terms of p. It is this explicit bound in terms which allows us to solve one half of Problem 2.1.1ii).

The expression defining $F(p;\Phi_1,\Phi_2)$ can be better understood in the following manner. Let $(a_k)_{k\in\mathbf{Z}}$ and $(b_n)_{k\in\mathbf{Z}}$ be non-decreasing sequences. Their concatenation at k_0 can be defined as the non-decreasing sequence $a_k \#_{k_0} b_k$ given by a_k when $k \leq k_0$ and $a_{k_0} + b_k$ when $k > k_0$. Then, we can rewrite the expression as

$$F(p;\Phi_1,\Phi_2) = \inf_{k_0\in\mathbf{Z}}\Big\|\Big(2^{-k}\big(\Phi_2(2^k)\#_{k_0}\Phi_1(2^k)\big)^{\frac{1}{p}}\Big)_{k\in\mathbf{Z}}\Big\|_{\ell_{\frac{1}{2}}(\mathbf{Z})}.$$

In Remark 2.3.5 we will see that the $\frac{1}{2}$-quasi-norm can be improved to a $\frac{p}{p+1}$-quasi-norm when the operator S is evaluated over projections. Nevertheless, both give the same growth order when $p \to 1^+$. Also, the maximum of $[S]_{(\Phi_1,\Phi_1)}$ and $[S]_{(\Phi_1,\Phi_1)}$ in Theorem 2.1.2 above can be improved to an expression that recovers the geometric mean of both quantities when the weak types Orlicz types are just weak types (p_1,p_1) and (p_2,p_2) respectively, see Remark 2.3.4. While the quantity $F(p;\Phi_1,\Phi_2)$ can be hard to compute in explicit terms for general Orlicz spaces, it can

be asymptotically estimated for many reasonable Orlicz classes like $L\log^{\alpha} L$ spaces, see Corollary 2.4.1. Before stating our result for the strong maximal, the following monotonicity theorem is needed.

Lemma 2.1.3. *Let $F(p;\Phi)$ be the constant defined by (Ct_p) when the second Orlicz type just coincides with the strong type (∞,∞). Assume that $F(p;\Phi)\to\infty$ as $p\to q_\Phi^+$ and take $\Psi\in o(\Phi)$ another Young function with $q_\Phi = q_\Psi$, then $F(p;\Psi)\in o\big(F(p;\Phi)\big)$ as $p\to q_\Phi^+$.*

As a corollary of the preceding monotonicity lemma and the interpolation result we get

Corollary 2.1.4. *Let $\mathbf{E}_n\otimes\mathbf{E}_m : \mathcal{M}\bar{\otimes}\mathcal{N}\to\mathcal{M}_n\bar{\otimes}\mathcal{N}_m$ be as before. If the noncommutative strong maximal associated to $(\mathbf{E}_n\otimes\mathbf{E}_m)_{n,m}$ has weak type (Ψ,Ψ), then $\Psi\notin o\big(s\,(1+\log_+ s)^2\big)$*

2.2 Some topics related to noncommutative function spaces

Noncommutative Orlicz spaces

Given a tracial von Neumann algebra $(\mathcal{M},\tau)$, it is possible to define the space of τ-measurable operators $L_0(\mathcal{M},\tau)$, as the space of unbounded operators affiliated to $\mathcal{M}$ satisfying that

$$x\in L_0(\mathcal{M},\tau) \iff \exists\lambda>0,\ \tau\big(\mathbf{1}_{(\lambda,\infty)}(|x|)\big)<\infty,$$

where $|x| = (x^*x)^{\frac{1}{2}}$ is and $\mathbf{1}_{(\lambda,\infty)}$ is applied to the unbounded operator via functional calculus. The space $L_0(\mathcal{M};\tau)$ is a complete metric vector space with the topology of convergence in measure, see [Ter81]. The noncommutative L_p-spaces associated to a pair $(\mathcal{M},\tau)$ are simply given by

$$L_p(\mathcal{M},\tau) = \big\{x\in L_0(\mathcal{M},\tau) : \tau\big(|x|^p\big)<\infty\big\}$$

while their associated norm is given by $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}$. As it is common, we will use the convention that $L_\infty(\mathcal{M},\tau)=\mathcal{M}$ and drop the dependency on τ whenever there is no ambiguity.

A similar definition of noncommutative Orlicz spaces is available. Let $\Phi:[0,\infty)\to[0,\infty]$ be a function. We will say that it is a *Young function* if

- Φ is non-decreasing,
- Φ is convex,
- $\Phi(0)=0$ and $\lim\limits_{t\to\infty}\Phi(t)=\infty$.

Observe that Young functions, with the conventions that will be used throughout this text, can take the value ∞ at finite arguments. This may seem strange but all the conditions, including convexity, make sense for functions with values in $[0,\infty]$ with the convention that $t\cdot\infty=\infty$ if $t\neq 0$.

Young functions are used to define Orlicz spaces $L_\Phi(\mathcal{M},\tau)$ in the following way. Let $x\in L_0(\mathcal{M},\tau)$ be a τ-measurable operator affiliated to $\mathcal{M}$. We have that $x\in L_\Phi(\mathcal{M},\tau)$ if and only if, for some scalar $\lambda>0$

$$\tau\left\{\Phi\Big(\frac{|x|}{\lambda}\Big)\right\}<\infty.$$

Moreover, L_Φ is a Banach space with norm given by

$$\|x\|_\Phi = \inf\left\{\lambda>0 :\ \tau\circ\Phi\Big(\frac{|x|}{\lambda}\Big)\le 1\right\}.$$

When $(\mathcal{M},\tau)$ is a measure space we have that $L_\Phi(\mathcal{M},\tau)$ coincides with the classical Orlicz space $L_\Phi(\Omega)$, see [RF61, RR02b, RR02a]. Notice as well that Orlicz spaces generalize L_p-spaces by taking $\Phi(t)=t^p$. This includes $L_\infty(\mathcal{M})$ if we take the Young function $\Phi(t)=\infty\cdot\mathbf{1}_{[1,\infty)}(t)$.

Given two Young functions Φ_1 and Φ_2 we say that Φ_2 grows as fast as Φ_1 if $\Phi_1(t)\lesssim\Phi_2(t)$ for t larger that some quantity. Observe that, although $\lesssim$ is clearly an order relation, it is not a total order. Nevertheless, in many situations it is possible to compare a Young function $\Phi(t)$ with powers t^α in an optimal way, for that, we need to introduce the following indices.

Definition 2.2.1. Let Φ be a finite positive Young function, and let M_Φ be the function:

$$M_\Phi(t)=\sup_{s>0}\frac{\Phi(st)}{\Phi(s)}.$$

The (lower and upper, resp.) Matuszewska-Orlicz indices are defined as:

$$p_\Phi=\lim_{t\to0}\frac{\log M_\Phi(t)}{\log t},\qquad q_\Phi=\lim_{t\to\infty}\frac{\log M_\Phi(t)}{\log t}.$$

A few observations are in order. First, notice that the existence of the limits that define q_Φ and p_Φ is justified by the fact that M_Φ is sub-multiplicative, ie $M_\Phi(t_1\,t_2)\le M_\Phi(t_1)\,M_\Phi(t_2)$, together with Fekete's subadditivity lemma [Fek23]. Next, the definition of M_Φ immediately implies that $M_\Phi(t)$ is the best constant such that

$$\Phi(st)\le M_\Phi(t)\Phi(s),\qquad\text{for all } s,t>0. \tag{2.2.1}$$

The Matuszewska-Orlicz indices encode which are the best polynomial approximations of $M_\Phi(t)$ near zero and infinity, that is

$$\begin{aligned} M_\Phi(t) &= \exp\big(q_\Phi\log t+o(\log t)\big) \quad \text{as} \quad t\to\infty\\ M_\Phi(t) &= \exp\big(p_\Phi\log t+o(\log t)\big) \quad \text{as} \quad t\to0^+ \end{aligned}$$

From these identities it is easily obtained that when $M_\Phi(t)$ goes to infinity, it grows faster than $t^{q_\Phi-\varepsilon}$ but slower than $t^{q_\Phi+\varepsilon}$ for all $\varepsilon>0$, and similarly near zero. Notice also that

$$\frac{1}{M_\Phi(t^{-1})}\lesssim\Phi(t)\lesssim M_\Phi(t)\qquad\text{for all } t>0,$$

and so one can deduce also bounds on the growth and decay of Φ via M_Φ. Indeed, it holds that

$$\min\left\{t^{p_\Phi-\varepsilon},t^{q_\Phi+\varepsilon}\right\}\lesssim_{(\varepsilon)}\Phi(t)\lesssim_{(\varepsilon)}\max\left\{t^{p_\Phi-\varepsilon},t^{q_\Phi+\varepsilon}\right\}\quad\text{for every }\ t>0, \tag{2.2.2}$$

for every $\varepsilon>0$, see [Mal85]. Furthermore, p_Φ and q_Φ are the largest and smallest indices for which this holds. Further details about Matuszewska-Orlicz indices and the function M_Φ associated to a Young function Φ can be found in [Mal85].

Distribution functions and singular numbers

Let $x\in L_0(\mathcal{M},\tau)$ be a τ-measurable operator, we define the following non-negative, non-increasing functions on $s>0$ given by

$$\begin{aligned} \lambda_s(x) &= \tau\big(\mathbf{1}_{(s,\infty)}(|x|)\big)\\ \mu_s(x) &= \inf\left\{t>0:\lambda_t(x)\le s\right\} \end{aligned}$$

The first one is referred to as the (noncommutative) distribution function, and the second one as the generalized singular numbers. The first can be seen as a generalization of the classical distribution function $\lambda_s(f) = \mu\big\{|f| > s\big\}$ while the second, which is just the pseudo-inverse of the first, generalizes the decreasing re-arrangement of f. Both functions $\mu_s(x), \lambda_s(x) : [0,\infty) \to [0,\infty]$ are non-increasing and lower semicontinuous.

The map that associates an element $x \in L_0(\mathcal{M},\tau)$ with its generalized singular numbers $\mu_s(x)$ has many useful properties that generalize those of the decreasing rearrangement. For instance it holds that for every positive element $x \in L_0(\mathcal{N},\tau_{\mathcal{N}})$

$$\tau_{\mathcal{N}}(x) \;=\; \int_0^\infty \mu_s(x)\, ds.$$

Furthermore, it holds in general that $\|x\|_p = \|\mu(x)\|_{L_p(0,\infty)}$, for every $1 \le p \le \infty$. The map $x \mapsto \mu(x)$ is also order preserving in the sense that, if $x \le y$ then $\mu_s(x) \le \mu_s(y)$ for every $s > 0$. Despite these nice properties, the map $x \mapsto \mu_s(x)$ shares certain pathologies with the classical decreasing rearrangement. In particular it fails to be sub-linear, meaning that it is not possible to bound the decreasing rearrangement of a sum by the sum of the decreasing rearrangements. This pathology can be fixed by the introduction of the following order relation. Let $f, g : (0,\infty) \to [0,\infty]$ be two non-increasing functions, we have the majorization relation given by

$$f \preceq g \quad \text{ if and only if } \quad \int_0^s f(t)\, dt \le \int_0^s g(t)\, dt,\ \forall s > 0$$

This relation is quite useful, see [CR23]. In particular it holds that

$$\text{if } \quad x = \sum_k x_k \quad \text{ then } \quad \mu_s(x) \preceq \sum_k \mu_s(x_k). \tag{2.2.3}$$

We will use that if $\mu_s(x) \preceq \mu_s(y)$ then $\|x\|_p \le \|y\|_p$ for every $p \in [1,\infty]$. Both properties are easily obtained from the fact that

$$\int_0^s \mu_t(x)\, dt = \|x\|_{L_1(\mathcal{M})+sL_\infty(\mathcal{M})}.$$

We will also use the notation D_η for the dilation $(D_\eta f)(t) = f(\eta^{-1}t)$. In many situation $D_\eta\big(\mu_s(x)\big)$ would implicitly be used to mean dilation in the real variable s.

We will need the following lemma involving generalized singular numbers from [CR23].

Lemma 2.2.1 ([CR23, Lemma 2.1]). *Let $p \in (0,\infty)$ and $x \in L_p^+(\mathcal{N})$. There exist a sequence $(r_n)_{n\in\mathbf{Z}}$ of finite projections in $\mathcal{N}$ so that*

i) $x = \displaystyle\sum_{n\in\mathbb{Z}} 2^{-n} r_n \in L_p(\mathcal{N}),$

ii) $\mu(x^\alpha) \;\preceq\; \displaystyle\sum_{n\in\mathbf{Z}} 2^{-n\alpha}\mu(r_n) \;\le\; \frac{1}{1-2^{-\alpha}}\mu(x^\alpha),\ \forall \alpha > 0.$

The above lemma is highly constructive. Indeed, take a number $t \in \mathcal{R}_+$ and consider its binary expansion. We can fix $d_n(t)$ as the characteristic function of the t with a 1 on its $-n^{\text{th}}$ binary digit. Identity i) follows by spectral calculus. The first inequality in ii) follows from (2.2.3) while the second requires an explicit computation.

Mixed $L_p(\ell_\infty)$-spaces

These mixed spaces have already been covered extensively in the literature, see [JX03, JP10]. We briefly recall that if $(x_n)_{n\geq 0}$ is a bounded sequence in $L_p(\mathcal{N})$ we have that $(x_n)_n \in L_p(\mathcal{N};\ell_\infty)$ if and only if there is a factorization $x_n = \alpha\, v_n\, \beta$ where $\alpha, \beta \in L_{2p}(\mathcal{N})$ and $(v_n)_n \in \ell_\infty \bar{\otimes} \mathcal{N}$. The norm of the space will be given by

$$\big\|(x_n)_n\big\|_{L_p(\mathcal{N};\ell_\infty)} := \inf\Big\{ \|\alpha\|_{2p} \Big(\sup_n \|v_n\| \Big) \|\beta\|_{2p} : x_n = \alpha\, v_n\, \beta \Big\}.$$

Observe that, without loss of generality, we can choose $\|\alpha\|_{2p}^2 = \|\beta\|_{2p}^2 = \|(x_n)_n\|_{L_p[\ell_\infty]}$ and v_n to be a sequence of contractions. The norm above has simpler forms when the sequence $x = (x_n)_n$ is either self-adjoint or positive. In the first case we have that it is possible to take v_n self-adjoint and $\beta = \alpha^*$. In the case of positive sequences of operators $x_n \geq 0$, it holds that

$$\big\|(x_n)_n\big\|_{L_p(\mathcal{N};\ell_\infty)} = \inf\Big\{ \|\alpha\|_p : \ \ 0 \leq x_n \leq \alpha, \ \ \forall n \geq 0 \Big\}. \tag{2.2.4}$$

We will use this formula repeatedly. Observe that, contrary to many other natural noncommutative function spaces, the mixed $L_p[\ell_\infty]$-norm of $(x_n)_n$ is not equal to that of $(|x_n|)_n$ or $(|x_n^*|)_n$. In fact, the quantities associated to the $L_p[\ell_\infty]$-norm of $|x_n|$ and $|x_n^*|$ give rise to asymmetric mixed spaces as studied in [JP10, HJP16].

2.3 Proof of the main theorem

We will say that two (orthogonal) projections p, q are disjoint if $pq = 0$. In the forthcoming discussion we will need the following elementary lemma whose proof we omit.

Lemma 2.3.1. *If S is a positive operator on a Hilbert space H, and e is an orthogonal projection such that $e^\perp S e^\perp = 0$, then $S = eSe$.*

Similarly, the proof of our main theorem requires the following diagonal bound lemma.

Lemma 2.3.2 ([CR23, Lemma 3.1.]). *Let $(q_k)_{0\leq k\leq n}$ be a sequence of disjoint projections in $\mathcal{N}$. The for any sequence $(d_k)_{0\leq k\leq N}$ of positive integers and any $x \in L_0^+(\mathcal{N})$ it holds that*

$$\Big(\sum_{k=0}^n q_k \Big)\, x \, \Big(\sum_{k=0}^n q_k \Big) \leq \Big(\sum_{k=0}^n \frac{1}{d_k} \Big) \sum_{k=0}^n d_k\, q_k\, x\, q_k$$

and the same follows for infinite sums.

We will prove Theorem 2.1.2 first when x is a finite projection. Then, we will decompose a general x into a sum of projections following Lemma 2.3.3. Observe that, since projections are positive, we can use formula (2.2.4).

Indeed, let $r \in \mathcal{N}$ be a finite projection. We want to manufacture a uniform bound for $S_n(r)$ from the projection in Definition 2.1.1i). First, we will normalize the operator $S = (S_n)_{n\geq 0}$ by dividing it by the maximum of $[S]_{(\Phi_1,\Phi_1)}$ and $[S]_{(\Phi_2,\Phi_2)}$ so that, without loss of generality, we can assume that $[S]_{(\Phi_i,\Phi_i)} \leq 1$, for $i \in \{1,2\}$. Now, fix a monotone sequence of positive scalars $\lambda_k = \eta\, 2^{-k}$, where $\eta > 0$ is a parameter that we will optimize in Remark 2.3.4 and that can be assumed to be one in Proposition 2.3.3. By definition we have that, for $i \in \{1,2\}$ and any given scalar $\lambda_k > 0$,

there is a projection $e^i_{\lambda_k}$ satisfying that

$$\begin{cases} \big\| e^i_{\lambda_k}\, S_n(r)\, e^i_{\lambda_k} \big\|_\infty \le \lambda_k \quad \text{for all } \ n \ge 0, \\ \tau\big(\mathbf{1} - e^i_{\lambda_k}\big) \le \tau \circ \Phi_i\Big(\dfrac{r}{\lambda_k}\Big) \end{cases}$$

Now, fix an index $k_0 \in \mathbf{Z}$ such that $\Phi_2(1/\lambda_{k_0}) < \infty$. We can define a sequence of projections $(e_k)_{k\in\mathbf{Z}}$ as

$$e_k \ = \ e^1_{\lambda_k}\, \mathbf{1}_{(k_0,\,\infty)}(k) + e^2_{\lambda_k}\, \mathbf{1}_{(-\infty,\,k_0]}(k).$$

Although $(e_k)_k$ may not be decreasing, it can be used to produce a decresing sequence in a natural way as follows

$$\widetilde{e}_k = \bigwedge_{j\le k} e_j. \tag{2.3.1}$$

Since $\widetilde{e}_k$ is decreasing $q_k = \widetilde{e}_k - \widetilde{e}_{k+1}$ is a projection that should be understood as the region over which the maximal lays in $(\lambda_{k+1}, \lambda_k)$. We define z_r as

$$z_r := \bigg(\sum_{k\in\mathbf{Z}} \frac{1}{d_k}\bigg) \sum_{k\in\mathbf{Z}} d_k\, \lambda_k\, q_k \tag{2.3.2}$$

for some $(d_k)_{k\in\mathbf{Z}}$ to be determined later. We have the following

Proposition 2.3.3. *Let $(\mathcal{N}, \tau_{\mathcal{N}})$ and $(\mathcal{M}, \tau_{\mathcal{M}})$ be two semifinite von Neumann algebras and $S = (S_n)_{n\ge 0}$ be a family of operators of (maximal) weak type (Φ_1, Φ_1) and (Φ_2, Φ_2) simultaneously as described in Theorem 2.1.2. Given a finite projection $r \in \mathcal{N}$, z_r, defined as in* (2.3.2)*, satisfies that*

i) The projections in (2.3.1) *satisfy that $\big\|\widetilde{e}_k\, S_n(r)\, \widetilde{e}_k\big\|_\infty \le \lambda_k$ and that*

$$\tau(\mathbf{1} - \widetilde{e}_k) \ \le \ 2 \begin{cases} (\tau\circ\Phi_2)\Big(\dfrac{r}{\lambda_k}\Big) & \text{when } k \le k_0 \\ (\tau\circ\Phi_2)\Big(\dfrac{r}{\lambda_{k_0}}\Big) + (\tau\circ\Phi_1)\Big(\dfrac{r}{\lambda_k}\Big) & \text{when } k > k_0 \end{cases} \tag{2.3.3}$$

ii) $S_n(r) \le z_r$, for every $n \ge 0$.

iii) For some $k_0 \in \mathbf{Z}$ as above and a choice of d_k independent of r it holds that

$$\begin{aligned} \mu(z_r) \preceq \bigg(\sum_{k\in\mathbf{Z}} \frac{1}{d_k}\bigg) \Bigg[\sum_{k\ge k_0} d_k\, \lambda_k\, D_{2\,\Phi_2(\lambda_{k_0}^{-1}) + 2\,\Phi_1(\lambda_{k+1}^{-1})}(\mu(r)) \\ + \sum_{k<k_0} d_k\, \lambda_k\, D_{2\,\Phi_2(\lambda_{k+1}^{-1})}(\mu(r)) \Bigg], \end{aligned} \tag{2.3.4}$$

iv) For every $q_{\Phi_1} < p < p_{\Phi_2}$, it is possible to optimize $(d_k)_k$ to obtain that

$$\|z_r\|_p \lesssim \max\big\{[S]_{(\Phi_1,\Phi_1)},\, [S]_{(\Phi_2,\Phi_2)}\big\}\, F(p;\Phi_1,\Phi_2)\, \|r\|_p,$$

where $F(p,\Phi_1,\Phi_2)$ is the factor defined in (Ct_p).

Proof. For i), start by noticing that, for any finite projection $r \in \mathcal{N}$ and Young function Φ, it holds by spectral calculus that

$$\tau \circ \Phi\Big(\frac{r}{\lambda}\Big) = \tau(r)\, \Phi\Big(\frac{1}{\lambda}\Big).$$

First, let $k \leq k_0$. The fact that $\big\|\widetilde{e}_k\, S_n(r)\, \widetilde{e}_k\big\|_\infty \leq \lambda_k$ is trivially verified in any case. For the second, we have that

$$\begin{aligned} \tau\big(\mathbf{1} - \widetilde{e}_k\big) \;\leq\; \sum_{j\leq k} \tau\big(\mathbf{1} - e_j\big) \;&\leq\; \sum_{j\leq k} \Phi_2\Big(\frac{1}{\lambda_j}\Big)\, \tau(r) \\ &\leq\; \Big(\sum_{j\leq k} M_{\Phi_2}\Big(\frac{\lambda_k}{\lambda_j}\Big)\Big)\, \Phi_2\Big(\frac{1}{\lambda_k}\Big)\, \tau(r), \end{aligned}$$

where we have multiplied and divided by λ_k inside the argument of Φ_2 and used identity (2.2.1). By convexity, we have that $M_\Phi(s) \leq s$ for every $0 \leq s \leq 1$, thus, the constant is bounded by

$$\sum_{j\leq k} \frac{\lambda_k}{\lambda_j} = \sum_{j\leq 0} 2^j = 2,$$

which is independent of k. Observe that, in the case of a non-finite Φ_2, we still have that $\Phi_2(s\,t) \leq s\,\Phi_2(t)$ for $0 \leq s \leq 1$ and the computation above is still valid. Now we need to tackle the case of $k_0 < k$. In that case, we have that

$$\begin{aligned} \tau\big(\mathbf{1} - \widetilde{e}_k\big) \;&=\; \sum_{j\leq k} \tau\big(\mathbf{1} - e_j\big) \\ &=\; \sum_{j\leq k_0} \Phi_2\Big(\frac{1}{\lambda_j}\Big)\, \tau(r) \;+\; \sum_{k_0<j\leq k} \Phi_1\Big(\frac{1}{\lambda_j}\Big)\, \tau(r) \\ &\leq\; \sum_{j\leq k_0} M_{\Phi_2}\Big(\frac{\lambda_{k_0}}{\lambda_j}\Big)\, \Phi_2\Big(\frac{1}{\lambda_{k_0}}\Big)\, \tau(r) \;+\; \sum_{k_0<j\leq k} M_{\Phi_1}\Big(\frac{\lambda_k}{\lambda_j}\Big) \Phi_1\Big(\frac{1}{\lambda_k}\Big)\, \tau(r) \\ &\leq\; 2\left[\Phi_2\Big(\frac{1}{\lambda_{k_0}}\Big) + \Phi_1\Big(\frac{1}{\lambda_k}\Big)\right] \tau(r). \end{aligned}$$

For ii), first notice that the limit projection $\widetilde{e}_\infty = \inf_k \widetilde{e}_k$ verifies that $\widetilde{e}_\infty\, S_n(r)\, \widetilde{e}_\infty = 0$ for all n. Since $S_n(r)$ is positive, by Lemma 2.3.1 it holds that $\widetilde{e}_\infty^{\perp}\, S_n(r)\, \widetilde{e}_\infty^{\perp} = S_n(r)$. On the other hand, since $\lim\limits_{t\to 0} \Phi_2(t) = 0$, we have that $\widetilde{e}_k \to \mathbf{1}$ weakly as $k \to -\infty$. Thus, the projections

$$g_N = \sum_{-N\leq k\leq N} q_k$$

satisfy that $g_N \to \mathbf{1} - \widetilde{e}_\infty$. Therefore $g_N\, S_n(r)\, g_N \to S_n(r)$. Applying Lemma 2.3.2 as well as i) above gives

$$\begin{aligned} g_N\, S_n(r)\, g_N &\leq \Big(\sum_{|k|\leq N} \frac{1}{d_k}\Big) \sum_{|k|\leq N} d_k\, q_k\, S_n(r)\, q_k \\ &\leq \Big(\sum_{|k|\leq N} \frac{1}{d_k}\Big) \sum_{|k|\leq N} d_k\, \lambda_k\, q_k \end{aligned}$$

and taking limits as $N \to \infty$ yields the bound ii).

It holds that $q_k \leq \mathbf{1} - \widetilde{e}_{k+1}$, thus point iii) follows from an application of the L_p-convergence of the sum defining z_r in addition to the bound:

$$\begin{aligned}\mu(q_{k-1}) = \mathbf{1}_{[0,\tau(q_{k-1})]} &\leq \mathbf{1}_{[0,\tau(\mathbf{1}-\tilde{e}_k)]} \\ &\leq D_{2\,\Phi_2(\lambda_{k_0}^{-1})+2\,\Phi_1(\lambda_k^{-1})}(\mu(r))\,\mathbf{1}_{(k_0,\infty)}(k) + D_{2\,\Phi_2(\lambda_k^{-1})}(\mu(r))\,\mathbf{1}_{(-\infty,k_0]}(k).\end{aligned}$$

Using that the relation $\prec$ is sub-additive, ie (2.2.3), we get the desired identity.

For iv), we use the triangular inequality together with the fact that $\tau(q_k) \leq \tau(\mathbf{1} - \tilde{e}_{k+1})$ and the bounds in (2.3.3) to obtain that

$$\begin{aligned}\|z_r\|_p &= \left\| c_d \sum_{k\in\mathbf{Z}} d_k\,\lambda_k\,q_k \right\|_p \\ &\leq c_d \sum_{k\in\mathbf{Z}} d_k\,\lambda_k\,\tau(\mathbf{1}-\widetilde{e}_{k+1})^{\frac{1}{p}}\end{aligned} \tag{2.3.5}$$

Observe that, making the change of variable $k \mapsto k-1$ and renaming the d_{k-1} by d_k, we have that

$$\begin{aligned}\|z_r\|_p &\leq 2\,c_d \sum_{k\in\mathbf{Z}} d_k\,\lambda_k\,\tau(\mathbf{1}-\widetilde{e}_k)^{\frac{1}{p}} \\ &\leq 4\left(\sum_{k\in\mathbf{Z}}\frac{1}{d_k}\right)\left[\sum_{k>k_0} d_k\,\lambda_k\left(\Phi_2\Big(\frac{1}{\lambda_k}\Big)+\Phi_1\Big(\frac{1}{\lambda_k}\Big)\right)^{\frac{1}{p}} + \sum_{k\leq k_0} d_k\,\lambda_k\,\Phi_2\Big(\frac{1}{\lambda_k}\Big)^{\frac{1}{p}}\right]\end{aligned} \tag{2.3.6}$$

Now we need to choose a sequence $(d_k)_k$ that is close to a minimizer for the functional above. Since one of the multiplicative terms increases with $(d_k)_k$ and the other decreases, we can try making the two terms similar, this yield the following choice of coefficients

$$d_k = \lambda_k^{-\frac{1}{2}}\left(\Phi_2\Big(\frac{1}{\lambda_k}\Big)+\Phi_1\Big(\frac{1}{\lambda_k}\Big)\right)^{-\frac{1}{2p}}\mathbf{1}_{(k_0,\infty)}(k) + \lambda_k^{-\frac{1}{2}}\,\Phi_2\Big(\frac{1}{\lambda_k}\Big)^{-\frac{1}{2p}}\mathbf{1}_{(-\infty,k_0]}(k).$$

Substituting in the equation gives that

$$\|z_r\|_p \leq 4\left[\sum_{k>k_0}\lambda_k^{\frac{1}{2}}\left(\Phi_1\Big(\frac{1}{\lambda_k}\Big)+\Phi_2\Big(\frac{1}{\lambda_{k_0}}\Big)\right)^{\frac{1}{2p}} + \sum_{k\leq k_0}\lambda_k^{\frac{1}{2}}\,\Phi_2\Big(\frac{1}{\lambda_k}\Big)^{\frac{1}{2p}}\right]^2\|r\|_p. \tag{2.3.7}$$

We have chosen k_0 such that $\Phi_2(\lambda_{k_0}^{-1}) < \infty$ and therefore, we can take infimum on the right hand side over k_0 satisfying that condition. If Φ_2 is finite everywhere, we already have the required statement. Otherwise, the k_0's for which $\Phi_2(\lambda_{k_0}^{-1}) = \infty$ give a right hand side that is infinite, and thus the infimum is unaffected. Since we have re-scaled our operator S, a simple scaling argument gives the factor $\max\{[S]_{(\Phi_1,\Phi_1)}, [S]_{(\Phi_2,\Phi_2)}\}$. The only thing that remains to be proven is that $F(p;\Phi_1,\Phi_2)$ is finite when $q_{\Phi_1} < p < p_{\Phi_2}$. For simplicity, we can both multiply and divide the term $\Phi_2(\lambda_{k_0}^{-1})$ by $\Phi_1(\lambda_k^{-1})$, then using that $\Phi_1(\lambda_{k_0}^{-1}) \leq \Phi_1(\lambda_k^{-1})$, we have that

$$F(p;\Phi_1,\Phi_2) \leq \left(1+\frac{\Phi_2(\lambda_{k_0}^{-1})}{\Phi_1(\lambda_{k_0}^{-1})}\right)^{\frac{1}{p}}\left[\sum_{k>k_0}\lambda_k^{\frac{1}{2}}\,\Phi_1\Big(\frac{1}{\lambda_k}\Big)^{\frac{1}{2p}} + \sum_{k\leq k_0}\lambda_k^{\frac{1}{2}}\,\Phi_2\Big(\frac{1}{\lambda_k}\Big)^{\frac{1}{2p}}\right]^2$$

Since $q_{\Phi_1} < p < p_{\Phi_2}$, there is an $\varepsilon > 0$ such that $q_{\Phi_1} + \varepsilon < p < p_{\Phi_2} - \varepsilon$. An application of (2.2.2) shows that $\Phi_1(\lambda_k^{-1}) \leq 2^{k(q_{\Phi_1}+\varepsilon)}$ for k positive and big enough, as well as $\Phi_2(\lambda_k^{-1}) \leq 2^{k(p_{\Phi_2}-\varepsilon)}$ when k is negative and big. All this together gives

$$F(p;\Phi_1,\Phi_2) \lesssim_{(\varepsilon,\Phi_1,\Phi_2,k_0)} \left[\sum_{k>k_0} 2^{\frac{k}{2}\left(\frac{q_{\Phi_1}+\varepsilon}{p}-1\right)} + \sum_{k\leq k_0} 2^{\frac{k}{2}\left(\frac{p_{\Phi_2}-\varepsilon}{p}-1\right)}\right]^2 < \infty$$

and that concludes iv). □

Proof of Theorem 2.1.2. We only have to prove the result for $x \geq 0$ in L_p. Using Lemma 2.2.1 we find a decomposition $x = \sum 2^{-m} r_m$, with r_m a finite spectral projection of x. Pick z_m such that $S_n(r_m) \leq z_m$ as in Proposition 2.3.3. Define $z = \sum 2^{-m} z_m$, so $S_n(x) \leq z$. To see that z is well defined, we just have to show that the sum $\sum 2^{-m} z_m$ converges absolutely in L_p-norm. Actually, we'll have to prove the convergence of a coarser sum a few steps ahead, so we omit this step.

We control first the decreasing rearrangement of z using (2.2.3), Proposition 2.3.3iii) and Lemma 2.2.1ii) for $\alpha = 1$:

$$\begin{aligned}
\mu(z) &\preceq \sum_{m\in\mathbb{Z}} 2^{-m}\mu(z_m) \\
&\preceq c_d \left(\sum_{k>k_0} \sum_{m\in\mathbb{Z}} 2^{-m} d_k \lambda_k D_{2\,\Phi_2(\lambda_{k_0}^{-1})+2\,\Phi_1(\lambda_k^{-1})}(\mu(r_m)) + \sum_{k\leq k_0} \sum_{m\in\mathbb{Z}} 2^{-m}\, d_k\, \lambda_k\, D_{2\,\Phi_2(\lambda_k^{-1})}(\mu(r_m)) \right) \\
&= c_d \left(\sum_{k>k_0} d_k\, \lambda_k\, D_{2\,\Phi_2(\lambda_{k_0}^{-1})+2\,\Phi_1(\lambda_k^{-1})} \Big(\sum_m 2^{-m}\mu(r_m) \Big) + \sum_{k\leq k_0} d_k\, \lambda_k\, D_{2\,\Phi_2(\lambda_k^{-1})} \Big(\sum_m 2^{-m}\mu(r_m) \Big) \right) \\
&\leq 2\, c_d \left(\sum_{k>k_0} d_k\, \lambda_k\, D_{2\,\Phi_2(\lambda_{k_0}^{-1})+2\,\Phi_1(\lambda_k^{-1})}(\mu(x)) + \sum_{k\leq k_0} d_k\, \lambda_k\, D_{2\,\Phi_2(\lambda_k^{-1})}(\mu(x)) \right)
\end{aligned}$$

Now we compare the L_p-norms of z and x:

$$\begin{aligned}
\|z\|_p &= \|\mu(z)\|_p \\
&\leq 2\, c_d \left(\sum_{k>k_0} d_k\, \lambda_k \left\| D_{2\,\Phi_2(\lambda_{k_0}^{-1})+2\,\Phi_1(\lambda_k^{-1})}(\mu(x)) \right\|_p + \sum_{k\leq k_0} d_k\, \lambda_k \left\| D_{2\,\Phi_2(\lambda_k^{-1})}(\mu(x)) \right\|_p \right) \\
&\leq 4\, c_d \left(\sum_{k>k_0} d_k\, \lambda_k \left(\Phi_1\Big(\frac{1}{\lambda_k}\Big) + \Phi_2\Big(\frac{1}{\lambda_{k_0}}\Big) \right)^{\frac{1}{p}} + \sum_{k\leq k_0} d_k\, \lambda_k\, \Phi_2\Big(\frac{1}{\lambda_k}\Big)^{\frac{1}{p}} \right) \|x\|_p \\
&\lesssim F(p;\Phi_1,\Phi_2)\, \|x\|_p,
\end{aligned}$$

where $F(p;\Phi_1,\Phi_2)$ is the same function that appears in equation (2.3.7). □

There are several things to clarify in the preceding proofs. Let us start saying that, although in Proposition 2.3.3 we use $\lambda_k = 2^{-k}$ all the computations work verbatim with $\lambda_k = \eta\, 2^{-k}$ for some $\eta > 0$, the advantage of this extra degree of freedom is that now we can optimize the constant in terms of $[S]_{(\Phi_1,\Phi_1)}$ and $[S]_{(\Phi_2,\Phi_2)}$.

Remark 2.3.4. Both in the classical Marcinkiewicz theorem and in the noncommutative one, when the Young functions are given by $\Phi_i(t) = t^{p_i}$, we have that the dependency of the operator L_p-norm of $S = (S_n)_{n\geq 0}$ on the weak type quasi norms $[S]_{(\Phi_i,\Phi_i)}$ is not a supremum but a geometric mean, ie:

$$\big\| S : L_p(\mathcal{N}) \to L_p(\mathcal{M};\ell_\infty) \big\| \lesssim_{(p,\Phi_1,\Phi_2)} [S]^{\theta}_{(p_1,p_1)}\, [S]^{1-\theta}_{(p_2,p_2)} \quad \text{for} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

A similar dependency would be desirable when interpolating between two weak Orlicz types, changing of course the exponents p_1 and p_2 by the adequate Boyd indices. For the sake of brevity we will denote the weak Orlicz type quasi-norms of S with respect to Φ_1 and Φ_2 by C_1 and C_2 respectively.

Assume that Φ_2 is finite everywhere. Choosing $\lambda_k = \eta\, 2^{-k}$ and repeating the computations on Proposition 2.3.3 gives that

$$\tau(\mathbf{1} - \widetilde{e}_k) \;\leq\; 2 \begin{cases} \Phi_2\Big(C_2 \dfrac{2^k}{\eta}\Big)\, \tau(r) & \text{if } k \leq k_0, \\ \left[\Phi_2\Big(C_2 \dfrac{2^{k_0}}{\eta}\Big) + \Phi_1\Big(C_1 \dfrac{2^k}{\eta}\Big)\right] \tau(r) & \text{if } k > k_0. \end{cases}$$

Repeating the computations in the bound of z_r yields that

$$\begin{aligned} \|z_r\|_p &\lesssim c_d \left[\sum_{k\leq k_0} d_k\, \eta\, 2^{-k}\, \Phi_2\Big(C_2 \frac{2^k}{\eta}\Big)^{\frac{1}{p}} + \sum_{k>k_0} d_k\, \eta\, 2^{-k} \left(\Phi_2\Big(C_2 \frac{2^{k_0}}{\eta}\Big) + \Phi_1\Big(C_1 \frac{2^k}{\eta}\Big)\right)^{\frac{1}{p}} \right] \|r\|_p \\ &\lesssim c_d \left[\eta\, M_{\Phi_1}\Big(\frac{C_1}{\eta}\Big)^{\frac{1}{p}} + \eta\, M_{\Phi_2}\Big(\frac{C_2}{\eta}\Big)^{\frac{1}{p}}\right] \\ &\qquad \left[\sum_{k\leq k_0} d_k\, 2^{-k}\, \Phi_2(2^k)^{\frac{1}{p}} + \sum_{k>k_0} d_k\, 2^{-k} \left(\Phi_2(2^{k_0}) + \Phi_1(2^k)\right)^{\frac{1}{p}} \right] \|r\|_p \end{aligned}$$

Minimizing the expression above in d_k and following closely the computations on the proof of Theorem 2.1.2 yields that, for every $q_{\Phi_1} < p < p_{\Phi_2}$

$$\big\|S : L_p(\mathcal{N}) \to L_p(\mathcal{M};\ell_\infty)\big\| \lesssim_{(\Phi_1,\Phi_2)} \inf_{\eta>0} \left[\eta\, M_{\Phi_1}\Big(\frac{C_1}{\eta}\Big)^{\frac{1}{p}} + \eta\, M_{\Phi_2}\Big(\frac{C_2}{\eta}\Big)^{\frac{1}{p}}\right] F(p;\Phi_1,\Phi_2).$$

When $M_{\Phi_1}(t) = t^{p_1}$ and $M_{\Phi_2}(t) = t^{p_2}$, the infimum above can be computed in terms of $\eta > 0$ and it yields precisely the geometric mean of C_1 and C_2 appearing in the classical case. This is useful in many cases. Similar computations can be repeated in the case of an operator of weak type (Φ_1, Φ_1) and strong type (∞, ∞), in that case, we have that, taking $k_0 = 0$ for simplicity, the inequality (2.3.3) yields 0 in the first line as long as $\eta > C_2$. Therefore, the dependency on C_1 and C_2 becomes

$$\big\|S : L_p(\mathcal{N}) \to L_p(\mathcal{M};\ell_\infty)\big\| \lesssim_{(\Phi_1)} \inf_{\eta>C_2} \left\{\eta\, M_{\Phi_1}\Big(\frac{C_1}{\eta}\Big)^{\frac{1}{p}}\right\} \left[\sum_{k\geq 0} 2^{-\frac{k}{2}}\, \Phi_1(2^k)^{\frac{1}{2p}}\right]^2$$

and again the infimum on η yields the desired geometric mean when $M_{\Phi_1}(t) = t^{p_1}$.

Remark 2.3.5. An important observation is that the computations in Proposition 2.3.3 are not optimal. Indeed, it is possible to improve the asymptotic behaviour of the constant in terms of p when we evaluate S over projections. Let z_r be the element defined by (2.3.2). Since it is given by scalar combinations of orthogonal projections we can actually estimate its L_p-norm explicitly

$$\begin{aligned} \|z_r\|_p &= c_d \left[\sum_{k\in\mathbf{Z}} d_k^p\, \lambda_k^p\, \tau(\mathbf{1} - \widetilde{e}_{k+1})\right]^{\frac{1}{p}} \\ &\lesssim \left(\sum_{k\in\mathbf{Z}} \frac{1}{d_k}\right) \left[\sum_{k>k_0} d_k^p\, \lambda_k^p \left(\Phi_2\Big(\frac{1}{\lambda_{k_0}}\Big) + \Phi_1\Big(\frac{1}{\lambda_k}\Big)\right) + \sum_{k\leq k_0} d_k^p\, \lambda_k^p\, \Phi_2\Big(\frac{1}{\lambda_k}\Big)\right]^{\frac{1}{p}} \|r\|_p \end{aligned} \tag{2.3.8}$$

Then optimizing d_k with the following choice of d_k

$$d_k \;=\; \lambda_k^{-\frac{p}{p+1}} \left(\Phi_1\Big(\frac{1}{\lambda_k}\Big) + \Phi_2\Big(\frac{1}{\lambda_{k_0}}\Big)\right)^{\frac{-1}{p+1}} \mathbf{1}_{(k_0,\infty)}(k) + \lambda_k^{-\frac{p}{p+1}}\, \Phi_2\Big(\frac{1}{\lambda_k}\Big)^{\frac{-1}{p+1}} \mathbf{1}_{(-\infty,k_0]}(k) \tag{2.3.9}$$

gives that

$$\|z_r\|_p \leq \left[\sum_{k>k_0} \lambda_k^{\frac{p}{p+1}} \left(\Phi_2\Big(\frac{1}{\lambda_{k_0}}\Big) + \Phi_1\Big(\frac{1}{\lambda_k}\Big)\right)^{\frac{1}{p+1}} + \sum_{k\leq k_0} \lambda_k^{\frac{p}{p+1}}\, \Phi_2\Big(\frac{1}{\lambda_k}\Big)^{\frac{1}{p+1}} \right]^{\frac{p+1}{p}} \|r\|_p \qquad (2.3.10)$$

This allows us to refine the statement of Proposition 2.3.3iv) by changing $F(p;\Phi_1,\Phi_2)$ by a function $G(p;\Phi_1,\Phi_2)$ given by

$$\begin{aligned} G(p;\Phi_1,\Phi_2) &= \inf_{k_0} \left\{ \left[\sum_{k>k_0} \left(2^{-k}\left(\Phi_2(2^{k_0}) + \Phi_1(2^k)\right)^{\frac{1}{p}}\right)^{\frac{p}{p+1}} + \sum_{k\leq k_0} \left(2^{-k}\,\Phi_2(2^k)^{\frac{1}{p}}\right)^{\frac{p}{p+1}} \right]^{\frac{p+1}{p}} \right\}. \\ &= \inf_{k_0} \left\{ \left\| \left(2^{-k}\big(\Phi_2(2^k)\#_{k_0}\Phi_1(2^k)\big)^{\frac{1}{p}}\right)_{k\in\mathbf{Z}} \right\|_{\frac{p}{p+1}} \right\}, \qquad (2.3.11) \end{aligned}$$

where $\Phi_1(2^k)\#_{k_0}\Phi_2(2^k)$ is the concatenation of the sequences $(\Phi_1(2^k))_{k\leq k_0}$ and $(\Phi_1(2^k)+\Phi_2(2^k))_{k>k_0}$ as defined in the Introduction. Observe that, when $\Phi_1(t) = t^{p_1}$ and $\Phi_2(t) = t^{p_2}$, this expression becomes

$$G(p;\Phi_1,\Phi_2) \in O\bigg(\Big(\frac{1}{p-p_1} + \frac{1}{p_2-p}\Big)^{\frac{p+1}{p}}\bigg).$$

It is unclear to us whether this behaviour holds for the whole norm of $S : L_p(\mathcal{M}) \to L_p(\mathcal{N};\ell_\infty)$ and not just its restriction to projections, see [CR23, Remark 3.10] for more on the optimal growth of the constant.

2.4 Applications

In this section we prove that the weak Orlicz type of the maximal operator associated to $(\mathbf{E}_n \otimes \mathbf{E}_m)_{n,m}$ can not be improved to $\Phi_\alpha(s) = s\,(1+\log_+ s)^\alpha$ for $\alpha < 2$, nor to any other Orlicz space L_Ψ with $\Psi \in o(\Phi_2)$. This result is contained in Corollary 2.4.1 below, which we include since it gives a quantitative version of Theorem 2.1.2 just in terms of the logarithmic exponent. In the proof we crucially use the Lemma 2.1.3 from the Introduction, which we now prove.

Proof of Lemma 2.1.3. Recall that the space L_∞ coincides with the Orlicz space L_{χ_∞} generated by $\chi_\infty = \infty\cdot 1_{(1,\infty)}$. The function $F(p;\Phi,\chi_\infty)$ from Theorem 2.1.2 has the form

$$\begin{aligned} F(p;\Phi,\chi_\infty) &= \inf\left\{ \left[\sum_{k>k_0} 2^{-\frac{k}{2}}\,\Phi(2^k)^{\frac{1}{2p}} \right]^2 :\ k_0 \ \text{ s.t. } \chi_\infty(2^{k_0}) < \infty \right\}, \\ &= \left[\sum_{k\geq 0} 2^{-\frac{k}{2}}\,\Phi(2^k)^{\frac{1}{2p}} \right]^2, \end{aligned}$$

and the same formula holds for $F(p;\Psi,\chi_\infty)$.

We need to prove that

$$\lim_{p\to q_\Phi^+} \frac{F(p;\Psi,\chi_\infty)}{F(p;\Phi,\chi_\infty)} = 0.$$

First let's define for every $r\in(0,1)$ the sets

$$\begin{aligned} A_n(r) &= \{k\geq 0 \colon \psi(2^k) \leq r^n \varphi(2^k)\} \quad \text{for } n\geq 1, \\ E_n(r) &= \begin{cases} A_1(r)^c & \text{if } n=0, \\ A_n(r)\smallsetminus A_{n+1}(r) & \text{if } n\geq 1. \end{cases} \end{aligned}$$

The family of sets $(E_n(r))_{n\geq 0}$ is a numerable partition of the positive integers into finite subsets. It also holds that

$$\begin{aligned} F(p;\Psi,\chi_\infty)^{1/2} &\leq \sum_{k\in E_0(r)} 2^{-k/2}\Psi(2^k)^{1/2p} + \sum_{n\geq 1} r^{n/2p} \sum_{k\in E_n(r)} 2^{-k/2}\Phi(2^k)^{1/2p} \\ &\leq \sum_{k\in E_0(r)} 2^{-k/2}\Psi(2^k)^{1/2p} + \left(\frac{r^{1/2p}}{1-r^{1/2p}}\right) F(p;\Phi,\chi_\infty)^{1/2}. \end{aligned}$$

Since the left term is just a finite sum, it is bounded for all possible p. This proves that for all $r\in(0,1)$,

$$\limsup_{p\to q_\Phi^+} \frac{F(p;\Psi,\chi_\infty)}{F(p;\Phi,\chi_\infty)} \leq \left(\frac{r^{1/2q_\Phi}}{1-r^{1/2q_\Phi}}\right)^2,$$

which means that the limit must be 0. □

Corollary 2.4.1. *Let $S=(S_n)_{n\geq 0}$ be family of positivity-preserving operators like those of Theorem 2.1.2. Assume they are of maximal weak type $(\Phi_\alpha,\Phi_\alpha)$ and strong type (∞,∞), where $\Phi_\alpha(s)=s\,(1+\log_+ s)^\alpha$ and $\alpha>0$. Then*

i) $S=(S_n)_{n\geq 0}$ is of (maximal) strong type (p,p), and the function

$$c(p) := \big\|S : L_p(\mathcal{N}) \to L_p(\mathcal{M};\ell_\infty)\big\| \in O\big((p')^{2+\alpha}\big) \qquad as \quad p\to 1^+$$

ii) As a consequence, the strong maximal operator on Problem 2.1.1 is not of weak Orlicz type $(\Phi_\alpha,\Phi_\alpha)$ for any $\alpha<2$.

Proof. So from Theorem 2.1.2 we know that the function

$$F(p;\Phi_\alpha,\chi_\infty) = \left(\sum_{k\geq 0} 2^{-k(\frac{1}{2}-\frac{1}{2p})}(1+k)^{\frac{\alpha}{2p}}\right)^2$$

verifies $\big\|S : L_p(\mathcal{N}) \to L_p(\mathcal{M};\ell_\infty)\big\| \lesssim F(p;\Phi_\alpha,\chi_\infty)$ for all $p\in(1,\infty)$, with a constant just depending on the Orlicz quasi-norms.

The first part of the statement follows from the fact that $F(p;\Phi_\alpha,\chi_\infty)\in O((p')^{2+\alpha})$. To prove this, define $r(p)=2^{\frac{1}{2}-\frac{1}{2p}}$ and $\alpha(p)=\frac{\alpha}{2p}$. The function $F(p;\Phi_\alpha,\chi_\infty)$ has the same order as

$$\left(\sum_{k\geq 0} r(p)^{-k}k^{\alpha(p)}\right)^2 \in O\Big(\frac{1}{(1-r(p))^{2(\alpha(p)+1)}}\Big).$$

And a routine check shows that this term is $O\big((1-\frac{1}{p})^{2+\alpha}\big)$, proving the claim.

The second part of the statement is just an application of the fact that function assigning each p to the p-norm of the aforesaid maximal operator is known not to be in $o((p')^4)$, see [Hu09]. □

Chapter 3

Cotlar identities for Hilbert transforms on lattices of $\mathrm{SL}_2(\mathbb{C})$

3.1 Introduction and main results

The boundedness problem for Fourier multipliers on L_p-spaces has always played a central role in harmonic analysis. One of the most studied examples is the Hilbert transform, defined as $\widehat{Hf}(\xi) = i\,\mathrm{sign}(\xi)\widehat{f}(\xi)$ for $f \in L_2(\mathbb{R})$. Although H was already known to be bounded in $L_p(\mathbb{R})$ for $1 < p < \infty$, in 1955 Cotlar [Cot55] gave a very simple proof of this fact using the following identity:

$$|Hf|^2 = 2H(f \cdot Hf) - H(H(|f|^2)). \tag{Classical Cotlar}$$

This is known nowadays as the Cotlar identity. His proof uses that H is bounded in $L_2(\mathbb{R})$ and that, by a recursive use of (Classical Cotlar), it also must be bounded in every $p = 2^k$ for $k \geq 1$. Interpolation and the fact that H is self-adjoint complete the proof.

Mei and Ricard [MR17] introduced the Cotlar identity in the noncommutative setting in order to study Hilbert transforms over free groups and amalgamated free products of von Neumann algebras. In the recent work of González-Pérez, Parcet and Xia [GPPX22] the authors developed a systematic approach to study Cotlar identities for Fourier multipliers in non-abelian groups. Let G be an unimodular group, $\mathcal{L}G$ the von Neumann algebra of G and $G_0 \subset G$ an open subgroup. Consider $m\colon G \to \mathbb{C}$ a symbol on G and T_m the corresponding Fourier multiplier on $\mathcal{L}G$. Then the formula:

$$(m(g^{-1}) - m(h))(m(gh) - m(g)) = 0, \quad \text{for all } g \in G \setminus G_0,\, h \in G, \tag{Cotlar}$$

is a translation of (Classical Cotlar) for T_m in terms of its symbol. The main result in [GPPX22] states that any m which is bounded, left G_0-invariant and verifies (Cotlar) defines a bounded multiplier in $L_p(\mathcal{L}\,G)$ for all $1 < p < \infty$.

The subgroup G_0 represents a set in which the Cotlar identity may fail. In the argument, this failure is balanced by the invariance of m with respect to G_0. Therefore this formulation of the theorem allows more flexibility in terms of the multiplier than the original one. However, the hypothesis of invariance can be relaxed even further. If $\chi\colon G_0 \to \mathbb{T}^1$ is a character, it is enough for the result to hold that m verifies:

$$m(gh) = \chi(g)m(h) \quad \text{for all } g \in G_0,\, h \in G.$$

We say in this case that m is *left* (G_0, χ)*-equivariant,* and of course the G_0-invariance is recovered when χ is the trivial character.

Hilbert transform in $\mathrm{PSL}_2(\mathbb{C})$

Recall that $\mathrm{PSL}_2(\mathbb{C})$, which is the quotient of the 2×2 complex matrices with determinant 1 by its center, can be identified with the group of orientation-preserving isometries of the three dimensional hyperbolic space $\mathbb{H}^3$. This identification can be made explicit in various ways. Here we give one using the upper-space model of $\mathbb{H}^3$ and quaternions. Let i, j, k denote the usual three quaternionic units, and let's define:

$$\mathbb{H}^3 = \{x + yi + rj \colon x, y, r \in \mathbb{R},\, r > 0\}.$$

Doing so, $\mathbb{H}^3$ is exactly the subspace $\mathbb{C} + \mathbb{R}_{>0} j$ of the quaternions. Now, for a given $\omega \in \mathbb{H}^3$ we set:

$$g \cdot \omega = (a\omega + b)(c\omega + d)^{-1}, \quad \text{for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{C}).$$

It is possible to compute the inverse of a quaternion using its conjugate and modulus. This leads to a more explicit formula for the action of $g \in \mathrm{PSL}_2(\mathbb{C})$ on the element $\omega = z + rj \in \mathbb{C} + \mathbb{R}_{>0} j$, namely:

$$g \cdot \omega = \frac{a\bar{c}|z + rj|^2 + b\bar{d} + a\bar{d}z + b\bar{c}\bar{z} + rj}{|c(z + rj) + d|^2}$$

This is a well-defined action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{H}^3$. Indeed, $\mathrm{PSL}_2(\mathbb{C})$ acts by orientation-preserving isometries on $\mathbb{H}^3$ when equipped with the usual Riemannian metric:

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2},$$

and it is the full group of such isometries (see [Ahl85] for more details).

On the other hand, a group G acting on a set X induces a multiplier on G as follows: first choose a point $x_0 \in X$ and two disjoint subsets $X^+, X^- \subset X$. Let m be the map $m \colon G \to \mathbb{C}$ defined for each $g \in G$ as:

$$m(g) = \begin{cases} 1 & \text{if } g \cdot x_0 \in X^+, \\ -1 & \text{if } g \cdot x_0 \in X^-, \\ 0 & \text{otherwise.} \end{cases}$$

Even if the final multiplier depends on x_0 and also on the partition given by X^+ and X^-, the boundedness of the multiplier is preserved by changing x_0 for any other point in the same G-orbit or using the sets $\{g \cdot X^+, g \cdot X^-\}$, with $g \in G$, instead of $\{X^+, X^-\}$. Back to the action of $\mathrm{PSL}_2(\mathbb{C})$ on the hyperbolic space, we are choosing the base point in $\mathbb{H}^3$ given by j in our quaternionic parametrization, and the following partition:

$$\Sigma^+ = \{\omega \in \mathbb{H}^3 \colon \mathrm{Re}(\omega) > 0\}, \quad \text{and} \quad \Sigma^- = \{\omega \in \mathbb{H}^3 \colon \mathrm{Re}(\omega) < 0\}.$$

This procedure induces a symbol m in $\mathrm{PSL}_2(\mathbb{C})$ that is explicitly given by:

$$m(g) = \mathrm{sign}(\mathrm{Re}(a\bar{c} + b\bar{d})), \quad \text{with } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{C}). \tag{1}$$

The dividing frontier $\Sigma = \mathbb{H}^3 \setminus (\Sigma^+ \cup \Sigma^-) = \{\omega \in \mathbb{H}^3 \colon \mathrm{Re}(\omega) = 0\}$ is a hyperbolic plane, which determines the symbol m up to a sign. Since the action of $\mathrm{PSL}_2(\mathbb{C})$ is transitive both on points and hyperbolic planes in $\mathbb{H}^3$, the boundedness of the multiplier defined by m on $L_p(\mathcal{L}\,\mathrm{PSL}_2(\mathbb{C}))$ will remain the same under any other choice of the kind. Also, it is worth noticing that m is easily shown invariant under the action of two groups:

i) The right action of the group $\mathrm{PSU}(2)$, which is the image of the unitary group $\mathrm{U}(2)$ under the projection $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$.

ii) The left action of the group $G_0 \leq \mathrm{PSL}_2(\mathbb{C})$ defined by:

$$G_0 = \left\{ \begin{bmatrix} x & iy \\ iz & w \end{bmatrix} : x, y, z, w \in \mathbb{R},\ xw + yz = 1 \right\}. \tag{2}$$

In [GPPX22] the authors proved that, when restricted to the lattices $\mathrm{PSL}_2(\mathbb{Z})$ and $\mathrm{PSL}_2(\mathbb{Z}[i])$, this function defines an L_p-bounded Fourier multiplier for every $1 < p < \infty$. They posed three related questions, namely:

i) Is this multiplier bounded in $L_p(\mathcal{L}\,\mathrm{PSL}_2(\mathbb{C}))$?

ii) Is its restriction bounded in $L_p(\mathcal{L}\,\mathrm{PSL}_2(\mathbb{R}))$?

iii) Are there more lattices $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ for which the restriction of m still defines a multiplier bounded in $L_p(\mathcal{L}\,\Gamma)$?

The two first questions are negatively answered by the work of Parcet, de la Salle and Tablate. Concretely, by [PdlST23, Corollary B2] and the fact that the Lie algebra of $\mathrm{PSL}_2(\mathbb{C})$ is simple (as a real Lie algebra) solves the problem.

In the present work we tackle the third question. Our main result concerns the family of groups $\Gamma_n = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-n}])$, and it can be stated follows:

Theorem 3.1.1. *For any integer $n > 0$, the symbol m restricted to the group Γ_n defines a bounded Fourier multiplier in $L_p(\mathcal{L}\,\Gamma_n)$ for all $1 < p < \infty$, whose norm satisfies:*

$$\|T_m \colon L_p(\mathcal{L}\,\Gamma_n) \to L_p(\mathcal{L}\,\Gamma_n)\| \lesssim \left(\frac{p^2}{p-1}\right)^{\beta}, \quad \textit{where } \beta = 1 + \log_2(1+\sqrt{2}).$$

The proof consist in identifying a subgroup $K_n \leq \Gamma_n$ and a suitable character $\chi \colon K_n \to \mathbb{T}^1$ for which m is left (K_n, χ)-equivariant, and then proving by hand that (Cotlar) holds. Using results and ideas from [Sta17], we refined the argument in [GPPX22] for the case $n = 1$, where the authors defined an auxiliary symbol $\widetilde{m}$ that is indeed K_1-invariant, and carried out the analogous computations that we present here in more generality.

Bianchi groups are another natural family of lattices in $\mathrm{PSL}_2(\mathbb{C})$ to consider, which were introduced by Bianchi in [Bia92] as a generalization of the group $\mathrm{PSL}_2(\mathbb{Z})$. For every square-free positive integer $n > 0$, we define the n-th Bianchi group as $\Gamma'_n = \mathrm{PSL}_2(\mathcal{O}_{-n})$, where $\mathcal{O}_{-n}$ denotes the ring of integers of the quadratic extension $\mathbb{Q}(\sqrt{-n})$. The explicit definition of Γ'_n depends on the class of n modulus 4, since:

$$\mathcal{O}_{-n} = \begin{cases} \mathbb{Z}[\sqrt{-n}] & \text{if } n \not\equiv -1 \,(\mathrm{mod}\, 4), \\ \mathbb{Z}\left[\frac{1+\sqrt{-n}}{2}\right] & \text{otherwise.} \end{cases}$$

Therefore this family extends the one featuring in Theorem 3.1.1 when $n \equiv -1 \,(\mathrm{mod}\, 4)$. In this case, the problem is that the set where (Cotlar) fails is bigger in Γ'_n than in Γ_n. This set cannot be contained in a subgroup with respect to which m has some kind of invariance, and this is why the Cotlar identity cannot hold in every Bianchi group Γ'_n with $n \equiv -1 \,(\mathrm{mod}\, 4)$.

The question of whether m defines a bounded multiplier on $L_p(\mathcal{L}\,\Gamma'_n)$ is left open in this case, but we are still able to prove that, choosing a different hyperbolic plane Σ to induce our multiplier, one can still get symbols that satisfy Cotlar identity in most of Bianchi groups (all but $n = 3$).

Moreover, this approach also allows us to prove the Cotlar identity for the original m in $\mathrm{PSL}_2(\mathbb{Z}[i])$ in a much simpler way than any of the previous proofs.

3.2 Previous results on the Cotlar identity and the action of Bianchi groups

The Cotlar identity in group von Neumann algebras

The study of general conditions for a symbol m that ensure the L_p-boundedness of T_m has been an active area of research both in the classical and the noncommutative case. As discussed in the Introduction, the key result we are going to use concerns the following version of Cotlar identity for noncommutative Fourier multipliers:

Theorem 3.2.1. *[GPPX22, Theorem A] Let G be a discrete group, $G_0 \leq G$ a subgroup and $\chi\colon G_0 \to \mathbb{T}^1$ some character. Let T_m be a Fourier multiplier whose symbol $m\colon G \to \mathbb{C}$ is bounded and left (G_0, χ)-equivariant. If m satisfies the identity:*

$$(m(g^{-1}) - m(h))(m(gh) - m(g)) = 0, \text{ for all } g \in G \setminus G_0 \text{ and } h \in G$$

the T_m is bounded in L_p for all $1 < p < \infty$. Moreover, its norm satisfies:

$$\|T_m\colon L_p(\mathcal{L}\,G) \to L_p(\mathcal{L}\,G)\| \lesssim \left(\frac{p^2}{p-1}\right)^{\beta}, \quad \text{with } \beta = \log_2(1+\sqrt{2}).$$

The subgroup G_0 gives a range of flexibility to this result with respect to the original one of Cotlar: taking a big subgroup G_0 increases the chances for the formula to hold, but makes it harder for m to satisfy the invariance hypothesis.

Hyperbolic planes, their boundaries and Möebius transformations

As we said, the group $\mathrm{PSL}_2(\mathbb{C})$ acts transitively on the set of pairs (p, Σ) where Σ is an hyperbolic plane embedded in $\mathbb{H}^3$ and p is a point in Σ. When working with the upper-half space model $\mathbb{H}^3 = \mathbb{C} + \mathbb{R}_{>0} j$, hyperbolic planes can be identified with half-planes and semispheres perpendicular (in the Euclidean sense) to $\mathbb{C}$. This induces a bijection between the set of hyperbolic planes in $\mathbb{H}^3$ and the set of generalized circles in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, that is, the set of lines and circles in $\mathbb{C}$. We will denote by $\partial\Sigma$ the generalized circle associated to Σ by this correspondence, and we will call it the boundary of Σ.

Notice also that, for any given hyperplane Σ and any $g \in \mathrm{PSL}_2(\mathbb{C})$, it holds that $\partial(g \cdot \Sigma) = g \cdot \partial\Sigma$, where g is acting by Möbius transformation on the right-hand side. The action of Bianchi groups by Möbius transformations have been extensively studied in [Sta17]. We will introduce now several results and concepts in that article, that we will make use of.

Let $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the extended real line. The Bianchi group Γ'_n acts on $\widehat{\mathbb{R}}$ in a controlled way:

Proposition 3.2.2. *[Sta17, Proposition 4.4] If $n \neq 3$ and $g \in \Gamma'_n$, then $\widehat{\mathbb{R}}$ and $g \cdot \widehat{\mathbb{R}}$ may only intersect tangentially.*

On the other hand, for a given $g \in \mathrm{PGL}_2(\mathbb{C})$ with $|\det(g)| = 1$, the quantities:

$$\alpha = i(a\overline{d} - b\overline{c}), \quad \beta = -2\mathrm{Im}(c\overline{d}), \quad \beta' = -2\mathrm{Im}(a\overline{b}), \quad \text{where } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{3.2.1}$$

describe the image of $\widehat{\mathbb{R}}$ by g in the following way:

$$g \cdot \widehat{\mathbb{R}} = \left\{ X/Y \in \widehat{\mathbb{C}} \colon \beta X\overline{X} - \alpha Y\overline{X} - \overline{\alpha} X\overline{Y} + \beta' Y\overline{Y} = 0 \right\}.$$

Proposition 3.2.3. *[Sta17, Propositions 3.5 and 3.7] The coefficients α, β and β' defined as above verify that:*

i) $\beta\beta' = |\alpha|^2 - 1$,

ii) the generalized circle $g \cdot \widehat{\mathbb{R}}$ goes through 0 if and only if $b' = 0$,

iii) the generalized circle $g \cdot \widehat{\mathbb{R}}$ is indeed a line if and only if $b = 0$,

iv) if $b = 0$, then α is a unit vector perpendicular to $g \cdot \widehat{\mathbb{R}}$,

v) if $b \neq 0$, then $g \cdot \widehat{\mathbb{R}}$ is a circle of center α/β and radius $1/|\beta|$.

3.3 Description of the set where Cotlar identity fails

Throughout the rest of the paper, we will denote by τ and τ' the following matrices:

$$\tau = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \tau' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let m be the function defined in (1) and set $\Gamma_n = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{-n}])$. As we shall prove later, our function $m\big|_{\Gamma_n}$ is invariant (through a suitable character) with respect to:

$$K_n = \{ g \in \Gamma_n \colon m(g) = 0 \}, \tag{3.3.1}$$

which turns out to be a subgroup of Γ_n. The goal of this section is to give an explicit description of this set. Along our proof, we will also give a description of the analogous set

$$K'_n = \{ g \in \Gamma'_n \colon m(g) = 0 \} \tag{3.3.2}$$

for Γ'_n the Bianchi group of discriminant $-n$. These subsets K'_n are defined only for square-free integers, and moreover $K'_n = K_n$ whenever $n \not\equiv -1 \pmod 4$.

The main theorem of this section (namely, Theorem 3.3.2) allows us to decompose K_n and K'_n as a combination of the four following disjoint sets:

$$\begin{aligned}
K_n^+ &= \left\{ \begin{bmatrix} x & y\sqrt{-n} \\ z\sqrt{-n} & w \end{bmatrix} : x,y,z,w \in \mathbb{Z},\ xw + nyz = 1 \right\}, \\
K_n^- &= \left\{ \begin{bmatrix} x\sqrt{-n} & y \\ z & w\sqrt{-n} \end{bmatrix} : x,y,z,w \in \mathbb{Z},\ nxw + yz = -1 \right\}, \\
L_n^+ &= \left\{ \begin{bmatrix} a & -\overline{a} \\ c & \overline{c} \end{bmatrix} : a,c \in \mathcal{O}_{-n},\ \mathrm{Re}(a\overline{c}) = \frac{1}{2} \right\}, \quad \text{and} \\
L_n^- &= \left\{ \begin{bmatrix} a & \overline{a} \\ c & -\overline{c} \end{bmatrix} : a,c \in \mathcal{O}_{-n},\ \mathrm{Re}(a\overline{c}) = -\frac{1}{2} \right\}.
\end{aligned} \tag{3.3.3}$$

Lemma 3.3.1. *Let $g \in \mathrm{PSL}_2(\mathbb{C})$, $G_0 \leq \mathrm{PSL}_2(\mathbb{C})$ be the group defined in (2) and:*

$$L = \left\{ \begin{bmatrix} a & -\overline{a} \\ c & \overline{c} \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{C}) \colon a,c \in \mathbb{C},\ 2\mathrm{Re}(a\overline{c}) = 1 \right\}.$$

Then it holds that:

i) $g \cdot i\widehat{\mathbb{R}} = i\widehat{\mathbb{R}}$ *if and only if* $g \in G_0 \cup \tau' G_0$,

ii) $g \cdot \mathbb{S}^1 = i\widehat{\mathbb{R}}$ *if and only if* $g \in L \cup \tau' L$.

Proof. Notice that $g \cdot i\widehat{\mathbb{R}} = i\widehat{\mathbb{R}}$ if and only if $\sigma(g) \cdot \widehat{\mathbb{R}} = \widehat{\mathbb{R}}$. It is well-known that:

$$\{g \in \mathrm{PSL}_2(\mathbb{C})\colon g \cdot \widehat{\mathbb{R}} = \widehat{\mathbb{R}}\} = \mathrm{PSL}_2(\mathbb{R}) \cup \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \mathrm{PSL}_2(\mathbb{R}).$$

The first point of the statement follows immediately.

We claim now that any $g \in L$ verifies $g \cdot \mathbb{S}^1 = i\widehat{\mathbb{R}}$. This is because $g^{-1} \cdot 0$, $g^{-1} \cdot \infty$ and $g^{-1} \cdot i$ can be very easily checked to be all complex numbers in $\mathbb{S}^1$. On the other hand, given any $g_0 \in \mathrm{PSL}_2(\mathbb{C})$ such that $g_0 \cdot \mathbb{S}^1 = i\widehat{\mathbb{R}}$, the set of $g \in \mathrm{PSL}_2(\mathbb{C})$ satisfying $g \cdot \mathbb{S}^1 = i\widehat{\mathbb{R}}$ decomposes as $G_0 g_0 \cup \tau' G_0 g_0$. Since $G_0 L \subset L$, the second point of the statement follows. □

Theorem 3.3.2. *Let* $n \geq 1$ *be an integer with* $n \neq 3$*, and* K_n*,* K_n' *the sets defined in* (3.3.1) *and* (3.3.2)*, respectively. Let also* $\Sigma = \{\omega \in \mathbb{H}^3\colon \mathrm{Re}(\omega) = 0\}$*. Then, it holds that:*

i) K_n *is the stabilizer of* Σ *in* Γ_n*.*

ii) If $n \equiv -1 \pmod 4$ *is a square free integer, then* K_n' *is the union of the stabilizer of* Σ *in* Γ_n'*, and the elements* $g \in \Gamma_n'$ *such that* $g \cdot \mathbb{S}^1 = i\widehat{\mathbb{R}}$*.*

Equivalently, $K_n = K_n^+ \cup K_n^-$ *and* $K_n' = K_n^+ \cup K_n^- \cup L_n^+ \cup L_n^-$*.*

Proof. Let g be the matrix

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{C}).$$

Denote by σ and σ' the automorphisms of $\mathrm{PSL}_2(\mathbb{C})$ given by conjugation by τ and τ', respectively. Notice that $\sigma'(g^t) = g^{-1}$, so it follows that $g \cdot \Sigma = \Sigma$ if and only if $\sigma(g^t) \cdot \widehat{\mathbb{R}} = \widehat{\mathbb{R}}$, and $g \cdot \mathbb{S}^1 = i\widehat{\mathbb{R}}$ if and only if $\sigma(g^t) \cdot \widehat{\mathbb{R}} = \mathbb{S}^1$. On the other hand, the quantities α, β and β' defined in (3.2.1) for $\sigma(g^t)$ are the following:

$$\alpha = i(a\overline{d} + c\overline{b}), \quad \beta = 2\mathrm{Re}(b\overline{d}), \quad \beta' = 2\mathrm{Re}(a\overline{c}).$$

It holds that $m(g) = 0$ if and only if $\beta = \beta'$. Also, this implies that $|\alpha|^2 + |\beta|^2 = 1$. We consider now two cases:

i) If $g \in \Gamma_n$, then $\beta, \beta' \in 2\mathbb{Z}$, so we conclude that $\beta = \beta' = 0$ and $|\alpha| = 1$. Therefore, $\sigma(g^t) \cdot \widehat{\mathbb{R}}$ is a line that goes through 0 and has as orthogonal vector α (see Proposition 3.2.3). Notice also that $\alpha \in i\mathbb{Z}[\sqrt{-n}]$. If $n > 1$, then $\alpha \in \{-i, i\}$, so we get $\sigma(g^t) \cdot \widehat{\mathbb{R}} = \widehat{\mathbb{R}}$. If $n = 1$, $\sigma(g^t) \in \Gamma_1$. By Proposition 3.2.2 , $\sigma(g^t) \cdot \widehat{\mathbb{R}}$ is tangent to $\widehat{\mathbb{R}}$, so they must be the same line.

ii) If $g \in \Gamma_n'$ with $n \equiv -1 \pmod 4$ and $n \neq 3$, then $\beta, \beta' \in \mathbb{Z}$ and $\alpha \in i\mathcal{O}_{-n}$. If $\beta = 0$, then $|\alpha| = 1$ and therefore $\alpha \in \{i, -i\}$. This leads to $\sigma(g^t) \cdot \widehat{\mathbb{R}} = \widehat{\mathbb{R}}$ in the same fashion as before. On the other hand, if $|\beta| = 1$ and $|\alpha| = 0$, then $\sigma(g^t) \cdot \widehat{\mathbb{R}} = \mathbb{S}^1$ by Proposition 3.2.3.

The rest of the statement follows from Lemma 3.3.1. □

Remark 3.3.3. Whereas K_n is always a subgroup of Γ_n, $K_n' \subset \Gamma_n'$ is not, since it is not closed under products neither taking inverses.

Remark 3.3.4. The theorem does not apply for K_3'. Notice that $\mathcal{O}_{-3} = \mathbb{Z}[\xi_3]$ where ξ_3 denotes a primitive 3-root of the unit. A matrix as simple as:

$$u = \begin{bmatrix} \xi_3 & 0 \\ 0 & \overline{\xi_3} \end{bmatrix}$$

will be in K_3' but not in $K_3 \cup L_3$. Also, since m is right PSU(2)-equivariant, if we pick any $g \in K_3$ then $gu \in K_3'$, but this product will not be in $K_3 \cup L_3$ in general.

3.4 Proof of the Cotlar Identity

The sets K_n^+ and K_n^- defined in (3.3.3) verify certain relations related to the invariance of m: $\tau' K_n^+ = K_n^+ \tau' = K_n^-$. These identities, together with the fact that K_n^+ is a subgroup of Γ_n, implies easily that:

$$K_n^+ K_n^-,\, K_n^- K_n^+ \subset K_n^- \quad \text{and} \quad K_n^- K_n^- \subset K_n^+.$$

We claim now that, because of these inclusions, the function $\chi \colon K_n \to \mathbb{T}^1$ defined as:

$$\chi(g) = \begin{cases} 1 & \text{if } g \in K_n^+, \\ -1 & \text{if } g \in K_n^-, \end{cases}$$

is a character. The following three lemmas prove that $m\big|_{\Gamma_n}$ is left (K_n, χ)-equivariant.

Lemma 3.4.1. *Let $g \in \mathrm{PSL}_2(\mathbb{C})$ and let $r_1(g)$ and $r_2(g)$ denote the first and second rows of g, respectively. There exist an unitary matrix $u \in \mathrm{PSU}(2)$ such that:*

$$g = \begin{bmatrix} s^{-1} & s^{-1}t \\ 0 & s \end{bmatrix} u,$$

with $s = |r_2(g)|$ and $t = \langle r_1(g), r_2(g) \rangle$, where the bracket represents the scalar product in $\mathbb{C}^2$.

Proof. This is just an explicit statement of the ANK decomposition for $\mathrm{PSL}_2(\mathbb{C})$. It can be proven directly as follows. Let u be the (only) unitary matrix such that $r_2(g)u^* = (0, s)$ with $s > 0$. Thus, $s = |r_2(g)|$. On the other hand, using that $\det(gu^*) = 1$, we get that $r_1(g)u^* = (s^{-1}, \omega)$ for some $\omega \in \mathbb{C}$. This ω can be computed using that $\omega = s^{-1}\langle r_1(gu^*), r_2(gu^*) \rangle = s^{-1}\langle r_1(g), r_2(g) \rangle$, which is the definition of $s^{-1}t$. □

Lemma 3.4.2. *For any $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{C})$, it holds that:*

$$\mathrm{Im}(b\overline{c} - a\overline{d})^2 - 4\mathrm{Re}(a\overline{c})\mathrm{Re}(b\overline{d}) \leq 1.$$

Moreover, if $g \in \Gamma_n$, then the right-hand side of the above inequality can be improved to 0.

Proof. Same computations as in the proof of [GPPX22, Lemma 5.3] shows that the left-hand side of the above expression can be written as $p(X) = -4X(1 + X)$, where $X = na_2d_2 + b_1c_1$. This proves the statement for $g \in \mathrm{PSL}_2(\mathbb{C})$. If $g \in \Gamma_n$, then X is an integer and therefore $p(X) \in 4\mathbb{Z}$, which proves the second part of the statement. □

Lemma 3.4.3. *The symbol $m\big|_{\Gamma_n}$ is right K_n-invariant and left (K_n, χ)-equivariant.*

Proof. It is immediate to check that $m(\omega g) = -m(g)$. On the other hand, K_n^+ and K_n^- are contained respectively in G_0 and ωG_0, where G_0 is the group defined in the Introduction by (2). Since m is invariant by the left action of G_0, it follows that $m\big|_{\Gamma_n}$ is left (K_n, χ)-equivariant.

For the right invariance, let's take $g \in \Gamma_n$ and $h \in K_n$. If $g \in K_n$, it is immediate that $m(gh) = m(g) = 0$, so we rule out this case. Let's write g and h as

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} s^{-1} & s^{-1}t \\ 0 & s \end{bmatrix} u,$$

where we used Lemma 3.4.1 to decompose h in a product of two matrices, such that $u \in \mathrm{PSU}(2)$, $s > 0$ and $t = \langle r_1(h), r_2(h) \rangle$. Recall that r_1 and r_2 represent the first and second rows of our matrices, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{C}^2$. Since $h \in K_n$, t is purely imaginary, which allows us to write:

$$\begin{aligned} \mathrm{Re}\langle r_1(gh), r_2(gh) \rangle &= \mathrm{Re}(a\bar{c})(1 + (\mathrm{Im}t)^2)s^{-2} + \mathrm{Re}(b\bar{d})s^2 + \mathrm{Im}(b\bar{c} - a\bar{d})\mathrm{Im}t \\ &= \begin{bmatrix} s^{-1}\mathrm{Im}t & s \end{bmatrix} \begin{bmatrix} 2\mathrm{Re}(a\bar{c}) & \mathrm{Im}(b\bar{c} - a\bar{d}) \\ \mathrm{Im}(b\bar{c} - a\bar{d}) & 2\mathrm{Re}(b\bar{d}) \end{bmatrix} \begin{bmatrix} s^{-1}\mathrm{Im}t \\ s \end{bmatrix} + s^{-2}\mathrm{Re}(a\bar{c}) \end{aligned} \tag{3.4.1}$$

The Lemma 3.4.2 says that the determinant of the matrix in (3.4.1) is always non-negative. Therefore, this matrix will be semidefinite positive if $\mathrm{Re}(a\bar{c}) \geq 0$ and $\mathrm{Re}(b\bar{d}) \geq 0$ and semidefinte negative otherwise. In both cases, it implies that $m(gh)m(g) \geq 0$. Since $gh \notin K_n$, it follows that $m(gh) = m(g)$, proving the statement. □

In the proof of the Theorem 3.1.1, we will make use of two inequalities that we introduce now as two independent lemmas.

Lemma 3.4.4. *Let* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_n$. *Then*

$$m(g)m(g^t)\mathrm{Re}(a\bar{d} + b\bar{c}) \geq 0,$$

where g^t *denotes the transpose of* g.

Proof. If g or g^t are in K_n, the result is immediate. If they are not, we know $m(g)$ has the same sign as $\mathrm{Re}(a\bar{c})$ or $\mathrm{Re}(b\bar{d})$, depending on which one is non-zero. We'll suppose that both $\mathrm{Re}(a\bar{c})$ and $\mathrm{Re}(a\bar{b})$ are non-zero, since the rest of the cases comes from applying this one to $\tau' g$, $g\tau'$ or $\tau' g \tau'$.

Under this hypothesis, the statement is equivalent to

$$\mathrm{Re}(a\bar{c})\mathrm{Re}(a\bar{d})\mathrm{Re}(a\bar{d} + b\bar{c}) \geq 0.$$

From the proof of [GPPX22, Proposition 5.8] we know that the left-hand side of the inequality equals $p(X) = (AX + B)(2X + 1)$ with $A = n(a_1^2 + a_2^2)$, $B = na_2^2$ and $X = b_1c_1 + na_2d_2$. Since X is an integer and the roots of the polynomial p have modulus lesser or equal than 1, we conclude the statement. □

Lemma 3.4.5. *For any* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{C})$, *it holds that* $\mathrm{Re}(a\bar{c})\mathrm{Re}(b\bar{d}) \geq -\frac{1}{4}$. *Moreover, if* $g \in \Gamma_n$, *the right-hand side of the inequality can be improved to* 0.

Proof. Suppose that $\mathrm{Re}(a\bar{c})\mathrm{Re}(b\bar{d}) < 0$. Then, by multiplying the equation $ad - bc = 1$ by $\overline{cd}$ and taking real part, we get that:

$$|\mathrm{Re}(b\bar{d})||c|^2 + |\mathrm{Re}(a\bar{c})||d|^2 = |\mathrm{Re}(c\bar{d})| \leq |c||d|.$$

Now we claim that any positive numbers $x, y, \alpha, \beta > 0$ satisfying

$$\alpha x^2 + \beta y^2 \leq xy \tag{3.4.2}$$

must verify $\alpha\beta \leq \frac{1}{4}$. To prove the claim, just notice that (3.4.2) is equivalent to $\alpha u^2 - u + \beta \leq 0$ with $u = x/y$, and this can only happen if the discriminant $1 - 4\alpha\beta$ is greater than or equal to 0.

If $g \in \Gamma_n$, then both $\mathrm{Re}(a\overline{c})$ and $\mathrm{Re}(b\overline{d})$ are integers, so $\mathrm{Re}(a\overline{c})\mathrm{Re}(b\overline{d})$ must be indeed non-negative. □

Proof of Theorem 3.1.1. We are going to prove that the symbol $m\big|_{\Gamma_n}$ satisfies (Cotlar) relative to K_n, that is:

$$(m(g^{-1}) - m(h))(m(gh) - m(g)) = 0, \text{ for all } g \in \Gamma_n \setminus K_n \text{ and } h \in \Gamma_n.$$

If $h \in K_n$, the equality follows from the right K_n-invariance of m proven in Lemma 3.4.3. Now, suppose that $h \notin K_n$ and $m(g^{-1}) \neq m(h)$. We have to prove that $m(gh) = m(g)$. Since the hypothesis $m(g^{-1}) \neq m(h)$ implies that $gh \notin K_n$, it suffies to prove that $m(gh)m(g) \geq 0$. We write:

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} s^{-1} & s^{-1}t \\ 0 & s \end{bmatrix} u,$$

using Lemma 3.4.1 to decompose h into an upper-triangular matrix and an unitary one. Now, a computation shows that:

$$\begin{aligned} m(gh)m(g) = \mathrm{sign}\Big(&\mathrm{Re}(a\overline{c} + b\overline{d})\mathrm{Re}(a\overline{c})s^{-2}(1 + (\mathrm{Re}t)^2) \\ &+ \mathrm{Re}(a\overline{c} + b\overline{d})\mathrm{Re}(a\overline{d} + b\overline{c})\mathrm{Re}t \\ &+ \mathrm{Re}(a\overline{c} + b\overline{d})\big[\mathrm{Re}(a\overline{c})s^{-2}(\mathrm{Im}t)^2 + \mathrm{Re}(b\overline{d})s^2 + \mathrm{Im}(b\overline{c} - a\overline{d})\mathrm{Im}t\big]\Big) \\ = \mathrm{sign}\Big(&(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})\Big) \end{aligned}$$

Now notice that (I) is non-negative because of Lemma 3.4.5 and the fact that $s > 0$. Also, (II) is non-negative because $\mathrm{Re}t$ has the same sign as $m(h) = -m(g^{-1}) = m(g^t)$, so we can apply Lemma 3.4.4. Finally, (III) is non-negative because of Lemma 3.4.2, which implies that each factor of the product has the same sign as $m(g)$ or is zero. □

Remark 3.4.6. We still don't know if the Fourier multiplier given by $m\big|_{\Gamma'_n}$ is bounded or not in $L_p(\mathcal{L}\,\Gamma'_n)$, but what can be proven is that this symbol do not verify a Cotlar identity as in Theorem 3.2.1 with respect to any possible subgroup of Γ'_n. To see this, suppose that $m\big|_{\Gamma'_n}$ is (G_0, χ)-equivariant for some subgroup $G_0 \leq \Gamma'_n$ and some character χ on G_0. We claim that $G_0 \cap L_n^+ = \varnothing$. Firstly, notice that for any $l \in L_n^+$ and $g \in \mathrm{PSL}_2(\mathbb{C})$, it holds that:

$$m(lg) = \mathrm{sign}\,(|r_1(g)| - |r_2(g)|)$$

where $r_1(g)$ and $r_2(g)$ denotes the first and second rows of g as complex vectors in $\mathbb{C}^2$. Let $a\colon \Gamma'_n \to \Gamma'_n$ be the map that permutes the two rows of a matrix and multiplies the first column by -1. If $G_0 \cap L_n^+ \neq \varnothing$, by the formula above it would hold that for any $h \in \Gamma'_n$ and any $l \in G_0 \cap L_n^+$:

$$m(lh) = \chi(l)m(h) = \chi(l)m(a(h)) = m(la(h)) = -m(lh),$$

which is of course impossible. On the other hand, fix an $l \in L_n^+$ whose inverse is not in K'_n. Then in order to Cotlar identity to hold, one needs that:

$$m(l^{-1}) = m(h), \quad \text{for any } h \in \Gamma'_n \text{ such that } m(lh) \neq 0.$$

Pick $h \in \Gamma'_n$ any element which verifies this equation. Let h' be given by $h' = \tau' h \tau'$, where τ' is the matrix defined at the beginning of Section 3.3. Notice that $m(lh') = -m(lh) \neq 0$, but $m(h') = -m(h)$. Therefore Cotlar identity must fail when applied to l and h'.

3.5 Addenda: another L_p-bounded multiplier on Bianchi groups

Our initial choice of the hyperbolic plane

$$\Sigma = \{\omega \in \mathbb{H}^3 \colon \mathrm{Re}(\omega) = 0\}$$

was motivated by the result in [GPPX22], where the authors proved that the corresponding multiplier m satisfies the Cotlar identity in $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}[i])$. However, we have seen that this identity fails for m when restricted to a general Bianchi group Γ'_n. This failure is connected to the geometry of circles in the orbit of $\partial\Sigma$ under the action of Γ'_n, so it is natural to ask for multipliers induced by planes with a better-behaved boundary.

Concretely, we now consider the multiplier $\widetilde{m}$ induced by the hyperplane

$$\widetilde{\Sigma} = \{\omega \in \mathbb{H}^3 \colon \mathrm{Im}(\omega) = 0\},$$

and the partition

$$\widetilde{\Sigma}^+ = \{\omega \in \mathbb{H}^3 \colon \mathrm{Im}(\omega) > 0\}, \quad \widetilde{\Sigma}^- = \{\omega \in \mathbb{H}^3 \colon \mathrm{Im}(\omega) < 0\}.$$

We claim that m satisfies the Cotlar identity on $\mathrm{SL}_2(\mathbb{Z}[i])$ if and only if $\widetilde{m}$ does so. Let σ be the automorphism of $\mathrm{PSL}_2(\mathbb{C})$ given by conjugation by τ, where τ is the matrix defined at the beginning of Section 3.3. A simple computation shows that:

$$\begin{aligned} \widetilde{m}(\sigma(g)) &= \mathrm{sign}\,\mathrm{Im}\left[(aj+bi)\overline{(-cij+d)}\right] \\ &= \mathrm{sign}\,\mathrm{Re}\left(a\overline{c}+b\overline{d}\right) = m(g). \end{aligned}$$

The automorphism σ leaves $\mathrm{SL}_2(\mathbb{Z}[i])$ invariant, so σ restricts to an automorphism of this group, proving our claim. Therefore, this point of view also generalizes the results of [GPPX22] in a different way than we did before.

Lemma 3.5.1. *For any $n \geq 1$ with $n \neq 3$, the set $\widetilde{K}_n = \{g \in \Gamma'_n \colon \widetilde{m}(g) = 0\}$ coincides with the stabilizer of $\widetilde{\Sigma}$ under the action of Γ'_n. Indeed, $\widetilde{m}(g)$ can be written as:*

$$\widetilde{m}(g) = \begin{cases} 1 & \text{if } g \cdot \widehat{\mathbb{R}} \text{ lies on the upper half-plane,} \\ -1 & \text{if } g \cdot \widehat{\mathbb{R}} \text{ lies on the lower half-plane,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Notice first that two hyperbolic planes Σ_1 and Σ_2 verify $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ if and only if $\partial\Sigma_1$ and Σ_2 intersect in at least two points. If $g \in \Gamma'_n$ and $\widetilde{m}(g) = 0$, it is because $g \cdot \widetilde{\Sigma} \cap \widetilde{\Sigma} \neq \emptyset$. Since $\partial\widetilde{\Sigma} = \widehat{\mathbb{R}}$, this implies that $g \cdot \widehat{\mathbb{R}}$ and $\widehat{\mathbb{R}}$ intersect in at least two points, but because of Proposition 3.2.2 it means that they are the same generalized circle. Therefore $g \cdot \widehat{\mathbb{R}} = \widehat{\mathbb{R}}$ and $g \cdot \widetilde{\Sigma} = \widetilde{\Sigma}$. The rest of the statement follows similarly. □

Lemma 3.5.2. *The multiplier $\widetilde{m}$ restricted to Γ'_n is right $\widetilde{K}_n$-invariant and left $(\widetilde{K}_n, \chi)$-equivariant, for some character χ on $\widetilde{K}_n$.*

Proof. Both the invariance and the equivariance follow easily from Lemma 3.5.1. just taking the character χ defined as:

$$\chi(g) = \begin{cases} 1 & \text{if } g \text{ send the upper half-plane to itself,} \\ -1 & \text{otherwise.} \end{cases}$$

□

Theorem 3.5.3. *The symbol $\widetilde{m}$ satisfies the Cotlar identity on every Bianchi group Γ'_n with $n \neq 3$.*

Proof. Because of Lemma 3.5.2, it is enough to prove that

$$(\widetilde{m}(g^{-1}) - \widetilde{m}(h))(\widetilde{m}(gh) - \widetilde{m}(g)) = 0, \text{ for all } g, h \in \Gamma'_n \setminus \widetilde{K}_n.$$

Let's suppose that $\widetilde{m}(g^{-1}) \neq \widetilde{m}(h)$. Then, without lost of generality, we can suppose that there is a generalized circle C_g in the upper half-plane and a generalized circle C_h in the lower half-plane such that $g \cdot C_g = \widehat{\mathbb{R}}$ and $h \cdot \widehat{\mathbb{R}} = C_h$. Since $\widehat{\mathbb{R}}$ and C_h lie both on the exterior of C_g, they are mapped into the same half-plane by g. That is, $g \cdot C_h = gh \cdot \widehat{\mathbb{R}}$ and $g \cdot \widehat{\mathbb{R}}$ lie on the same side of $\widehat{\mathbb{C}} \setminus \widehat{\mathbb{R}}$, and therefore $\widetilde{m}(gh) = \widetilde{m}(g)$. □

Chapter 4

Riesz-Schur transforms

4.1 Introduction and main results

Riesz transforms are the archetypes of singular integrals, as they are multidimensional generalizations of the Hilbert transform. Given a square-integrable function $f : \mathbb{R}^n \to \mathcal{C}$, the Euclidean Riesz transforms R_j acting on f are defined as follows:

$$R_j f(x) = \partial_j (-\Delta)^{-\frac{1}{2}} f(x) = \int_{\mathbb{R}^n} \Big\langle \frac{\xi}{|\xi|}, e_j \Big\rangle \widehat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} \, d\xi.$$

The study of Euclidean Riesz tranforms has a long history. Dimension-free estimates for the full Riesz transform $\mathcal{R} f = \nabla (-\Delta)^{-\frac{1}{2}} f$ go back to the work of Gundy-Varopoulos and Stein [GV79, Ste83]. Shortly after, a probabilistic interpretation by Meyer pioneered significant generalizations [Bak85, Bak87, Gun86, Mey84, Pis88] for diffusion processes.

The remarkable work of Lust-Piquard incorporated Abelian groups and the first noncommutative models to the picture [LP98, LP99, LP04]. More recently, Riesz transforms in the context of von Neumann algebras have been systematically investigated in [JMP18, JM10]. Given an index set Γ and a Hilbert space $\mathcal{H}$, we will consider Riesz-Schur transforms $\mathcal{R}_u$ defined for matrices $A \in B(\ell_2(\Gamma))$ as follows:

$$\mathcal{R}_u(A) = \Big(\frac{u_j - u_k}{\|u_j - u_k\|} A_{jk} \Big)_{j,k \in \Gamma} \quad \text{for some} \quad u = (u_j)_{j \in \Gamma} \subset \mathcal{H}.$$

The main theorem in this chapter is a version of a classical result for Euclidean Riesz transforms. Although the best constants are known to be dimension-free, we sketch a proof where we show that, dropping that assumption, the argument would be rather simple:

Theorem 4.1.1 (Square-function inequality for Euclidean Riesz transforms). *Let R_j denote the Riesz transforms in the different directions of $\mathbb{R}^n$, for $j = 1, \dots, n$. Then there exist (finite) constants A_p and B_p such that for any function $f \in L_p(\mathbb{R}^n)$ and any $1 < p < \infty$ it holds that:*

$$A_p \|f\|_p \leq \|(\sum |R_j f|^2)^{1/2}\|_p \leq B_p \|f\|_p.$$

Sketch of the proof. We present a short argument from [Ste70]. The constant B_p can be easily found since it is known that the Riesz transforms are bounded operators from $L_p(\mathbb{R}^n)$ to itself. On the other hand, it is easy to see that:

$$\langle f, g \rangle = \sum_{i,j=1,\dots,n} \langle R_i f, R_j g \rangle,$$

for f, g sufficiently regular. Then the constant A_p can be found using a triangular inequality, since:

$$\|f\|_p = \sup_{\|g\|_q \leq 1} \langle f, g \rangle \leq nB_p \| \sum_i |R_i f| \|_p \leq A_p \| (\sum_i |R_i f|^2)^{1/2} \|_p.$$

□

The main result of this chapter generalizes this theorem to a much more general setting, recovering dimension-free estimates for a non-diagonal version of the Riesz transform:

Theorem 4.1.2 (Square-function inequality for Riesz-Schur transforms). *Let Γ be any index set and fix any family $(u_j)_{j\in\Gamma}$ of vectors in some Hilbert space $\mathcal{H}$. Consider a semifinite von Neumann algebra $(\mathcal{M}, \tau)$. Then the following inequalities hold for $x = \sum_{j,k} x_{jk} \otimes e_{jk} \in L_p(\mathcal{M}; S_p(\Gamma))$ and any $1 < p < \infty$ up to universal constants*

$$\Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \otimes \frac{u_j - u_k}{\|u_j - u_k\|} \Big\|_{RC_p} \lesssim c_p \Big\| \sum_{j,k\in\Gamma} x_{j,k} \otimes e_{j,k} \Big\|_p, \tag{RS$_1$}$$

$$\Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \Big\|_p \lesssim \Big\| \sum_{\substack{j,k\in\Gamma \\ u_j = u_k}} x_{jk} \otimes e_{jk} \Big\|_p + c_{p'} \Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \otimes \frac{u_j - u_k}{\|u_j - u_k\|} \Big\|_{RC_p}. \tag{RS$_2$}$$

Here $c_p = \max\big\{p, (\frac{p}{p-1})^{\frac{3}{2}}\big\}$ and we adopt the convention that $\frac{0}{0} = 0$ when $u_j = u_k$.

Remark 4.1.3. Notice that the right-hand side of (RS$_2$) is finite whenever the left hand side is so, since the map

$$\sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \mapsto \sum_{\substack{j,k\in\Gamma \\ u_j = u_k}} x_{jk} \otimes e_{jk}$$

is a conditional expectation into the subalgebra of matrices supported on $\{(i,j) \in \Gamma \times \Gamma : \ u_i = u_j\}$.

The RC_p norm is a generalization on Hilbert spaces of the L_p norm of the associated square-function. We will define in detail what this norm means in Section 4.2. By now, we will just make some remarks comparing this result to some others in the literature:

i) This result is a nonToeplitz and basis-free extension of the main result in [JMP18]. Indeed, it recovers the dimension-free estimates from [JMP18, LP04, Ste83] by Fourier-Schur transference results [CdlS15, NR11, PRS22].

ii) The Euclidean case follows by taking $\mathcal{H} = \mathbb{R}^n$ and $u_x = x$, followed by Theorem 1.19 of [LdlS11] and Fourier-Schur transference.

iii) The constants in the result match the best-known so far in this context. Arhancet and Kriegler also investigated it in [AK22, Theorem 3.3]. They followed [JMP18] estimating L_p-norms of the Gaussian gradient δ_ψ —equivalently the Dirac operator— and their statement is equivalent to ours via a Khintchine type inequality.

The main contribution in Theorem 4.1.2 is a cleaner statement and prominently a much simpler proof, which has facilitated to uncover interesting applications below. None of the usual harmonic analysis or probabilistic methods are needed. On the contrary, the argument of the proof is modeled on Grothendieck's inequality [Pis01, Chapter 5]. This becomes quite enlightening and it is not accidental that Theorem 4.1.2 leads to a sufficient condition for Schatten p-boundedness of Schur multipliers which resembles the Grothendieck's S_∞ criterion. Let's recall first what this criterion says:

Theorem 4.1.4 (Grothendieck's S_∞ criterion). *A Schur multiplier S_M is bounded on $S_\infty(\ell_2\Gamma)$, and indeed completely bounded, if and only if there exist a Hilbert space $\mathcal{H}$ and two bounded functions $\nu, \eta\colon \Gamma \to \mathcal{H}$ such that $M(j,k) = \langle \nu_j, \eta_k \rangle$.*

See [Gro56, Pis01, Pis12] for a reference on the proof of this result. Our work leads as well to a new version of the Arazy's conjecture (proven by Sukochev and Potapov in [PS11]). The Arazy's conjecture says that the multiplier S_M with symbol

$$M(x,y) = \frac{f(x) - f(y)}{x - y}$$

satisfies a bound of the kind

$$\|S_M\colon S_p(\mathbb{R}^n) \to S_p(\mathbb{R}^n)\| \leq c_p \|f\|_{\mathrm{Lip}}$$

for $1 < p < \infty$, where $\|\cdot\|_{\mathrm{Lip}}$ is the Lipschitz norm [Ara82, Sch38].

Theorem 4.1.5. *Consider arbitrary vectors $\{u_j, u_j', w_j, w_j' : j \in \Gamma\}$ in some Hilbert space $\mathcal{H}$ and let $\Lambda : \mathcal{H} \to \mathcal{H}$ be a bounded linear map. Then, the following inequalities hold for any $1 < p < \infty$ up to absolute constants*:

i) S_p-factorization condition. *The symbols*

$$M(j,k) = \Big\langle \frac{u_j + u_k'}{\|u_j + u_k'\|}, \Lambda\Big(\frac{w_j + w_k'}{\|w_j + w_k'\|}\Big) \Big\rangle$$

define completely bounded Schur multipliers with $\|S_M\|_{\mathrm{cb}(S_p(\Gamma))} \lesssim \max\big\{p, \frac{p}{p-1}\big\}^{\frac{5}{2}}$.

ii) Square root of Arazy's divided differences. *Given a nondecreasing Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, the symbol*

$$M_f(x,y) = \sqrt{\frac{f(x) - f(y)}{x - y}}$$

defines a completely bounded multiplier with $\|S_{M_f}\|_{\mathrm{cb}(S_p(\mathcal{R}))} \lesssim \max\big\{p, \frac{p}{p-1}\big\}^{\frac{5}{2}} \|f\|_{\mathrm{lip}}^{\frac{1}{2}}$.

Our last result provides a very simple proof of two recent criteria which have received a lot of attention.

Theorem 4.1.6. *Given $1 < p < \infty$*:

i) Marcinkiewicz-Schur multipliers. *If $M : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$*

$$\big\|S_M\big\|_{\mathrm{cb}(S_p(\mathbb{Z}))} \lesssim \sup_{j,k \in \mathbb{Z}} \mathbf{Var}_{|\cdot| \sim 2^k} \Big\{ \big|M(j + \cdot, j)\big| + \big|M(j, j + \cdot)\big| \Big\},$$

where $\mathbf{Var}_{|\cdot| \sim 2^k} f = \sum_{2^k \leq |n| < 2^{k+1}} |f(n+1) - f(n)|$ for any function f on the integers.

ii) Hörmander-Mikhlin-Schur multipliers. *If $M \in \mathcal{C}^{[\frac{n}{2}]+1}(\mathbb{R}^{2n} \setminus \{0\})$*

$$\big\|S_M\big\|_{\mathrm{cb}(S_p(\mathbb{R}^n))} \lesssim \sum_{|\gamma| \leq [\frac{n}{2}]+1} \Big\| |x - y|^{|\gamma|} \Big\{ \big|\partial_x^\gamma M(x,y)\big| + \big|\partial_y^\gamma M(x,y)\big| \Big\} \Big\|_\infty.$$

In conclusion, Riesz-Schur transforms recover and refine well-known estimates for Hilbert transforms —Theorem 4.1.2 for $\dim \mathcal{H} = 1$— and Riesz transforms, Arazy's divided differences, Marcinkiewicz-Schur and Hörmander-Mikhlin-Schur multipliers [CLM24, CAGPPT23, Gro56, JMP18, LP04, Sch38, Ste83]. To the best of our knowledge, these references give all known sufficient conditions on symbols for S_p-bounded Schur multipliers and $p \geq 1$.

4.2 Hilbert-valued L_p-spaces

Remark 4.2.1. Throughout this chapter, and for convenience in the notation, the scalar product in complex Hilbert spaces will be antilinear on the left and linear on the right.

The Schatten p-class $S_p(\Gamma)$ is nothing but $L_p(B(\ell_2(\Gamma)), \mathrm{tr})$ and $L_p(\mathcal{M}\bar{\otimes}B(\ell_2(\Gamma)), \tau \otimes \mathrm{tr})$ will be identified with a space of matrices $(x_{jk})_{j,k\in\Gamma}$ with coefficients in $L_p(\mathcal{M})$ or with sums of the form $\sum_{j,k} x_{jk} \otimes e_{jk}$ where e_{jk} are the standard matrix units. Any such matrix belongs to $L_p(\mathcal{M}\bar{\otimes}B(\ell_2(\Gamma)), \tau \otimes \mathrm{tr})$ when

$$\sup_{\substack{\Sigma\subset\Gamma \\ \Sigma \text{ finite}}} \Big\| \sum_{j,k\in\Sigma} x_{jk} \otimes e_{jk} \Big\|_p < \infty.$$

The space $L_p(\mathcal{M}, \mathcal{H}_{rc})$

We begin by recalling some basic facts on p-row and p-column operator spaces, drawing primarily on [Pis03] as our reference.

Row and column Hilbert spaces. Let $\mathcal{H}$ be a complex Hilbert space, and choose a fixed unit vector $\xi \in \mathcal{H}$. For $u \in \mathcal{H}$, define the rank-one operators

$$\Lambda^c_u(\nu) \;=\; \langle \xi, \nu\rangle\, u \quad \text{and} \quad \Lambda^r_u(\nu) \;=\; \langle u, \nu\rangle\, \xi.$$

Notice that Λ^r_u is the adjoint of Λ^c_u. The map

$$\Lambda^c \colon u \mapsto \Lambda^c_u \qquad \text{and} \qquad \Lambda^r \colon \bar{u} \mapsto \Lambda^r_u$$

can be understood respectively as a linear maps from $\mathcal{H}$ and $\bar{\mathcal{H}}$ to $B(\mathcal{H})$. Indeed, they are isometric embeddings in the spaces $S^2(B(\mathcal{H}))$ and $\bar{S}^2(B(\mathcal{H}))$ of matrices in $B(\mathcal{H})$ with finite norm

$$\|x\|_{S^2} = \big(\mathrm{tr}(\mathrm{x}^*\mathrm{x})\big)^{1/2} \qquad \text{and} \qquad \|x\|_{\bar{S}^2} = \big(\mathrm{tr}(\mathrm{x}\mathrm{x}^*)\big)^{1/2} \qquad \text{(resp.)}.$$

Definition 4.2.1. The images of the maps Λ^c and Λ^r, denoted by $\mathcal{H}_c$ and $\mathcal{H}_r$, are called the *column* and *row Hilbert spaces*, respectively.

We often identify elements of $\mathcal{H}_c$ and $\mathcal{H}_r$ with their preimages in $\mathcal{H}$ or the conjugate space $\bar{\mathcal{H}}$ whenever no confusion arises. Notice that, choosing a basis (δ_j) of $\mathcal{H}$ and fixing $\xi = \delta_0$, then one can identify Λ^r and Λ^c with the linear or antilinear embeddings of $\mathcal{H}$ given by

$$\Lambda^c(\delta_j) = e_{j0} \qquad \text{and} \qquad \Lambda^r(\delta_j) = e_{0j},$$

justifying the usual choice of column/row notation.

Column/row-valued noncommutative $\mathbf{L_p}$-spaces. Let $\mathcal{M}$ be a von Neumann algebra with trace τ. For a Hilbert space $\mathcal{H}$, the *column-valued* space $L_p(\mathcal{M}; \mathcal{H}_c)$, sometimes denoted C_p or $C_p(\mathcal{M})$, is defined as follows:

Definition 4.2.2. The *column L_p-space* $L_p(\mathcal{M}; \mathcal{H}_c)$ is defined as the completion of the image of

$$\mathcal{M} \otimes \mathcal{H} \ni x \otimes u \;\longmapsto\; x \otimes \Lambda^c_u \in \mathcal{M}\,\bar{\otimes}\,B(\mathcal{H})$$

in the L_p-norm, whenever $1 \le p < 1$, and in the weak* topology when $p = \infty$.

The *row-valued* space $L_p(\mathcal{M};\mathcal{H}_r)$, also denoted R_p or $R_p(\mathcal{M})$), is defined analogously by replacing Λ^c with Λ^r.

Using the fact that $\Lambda_u^r\Lambda_u^c$ and $\Lambda_u^c\Lambda_u^r$ are rank-one projections onto the lines $\mathbb{C}\xi$ and $\mathbb{C}u$ (resp.), it is straightforward to check the following norm identities for finite sums:

$$\Big\|\sum_{k=1}^n x_k\otimes u_k\Big\|_{C_p} \;=\; \Big\|\Big(\sum_{j,k=1}^n \langle u_j,u_k\rangle\, x_j^*\, x_k\Big)^{\frac12}\Big\|_p, \qquad \Big\|\sum_{k=1}^n x_k\otimes \overline{u_k}\Big\|_{R_p} \;=\; \Big\|\Big(\sum_{j,k=1}^n \langle u_j,u_k\rangle\, x_j\, x_k^*\Big)^{\frac12}\Big\|_p.$$

Proposition 4.2.2 (Homogeneity of C_p and R_p). *If $\Lambda:\mathcal{H}\to\mathcal{H}$ is a bounded linear (resp. antilinear) map, then* $\mathrm{id}\otimes\Lambda$ *extends to a completely bounded map on $L_p(\mathcal{M};\mathcal{H}_c)$ (resp. $L_p(\mathcal{M};\mathcal{H}_r)$) with the same norm if $1\le p<\infty$, or to a weak-$*$ continuous map if $p=\infty$.*

Proposition 4.2.3 (Duality of row and column spaces). *For $1\le p<\infty$ with conjugate exponent $p'=\frac{p}{p-1}$, we have*

$$C_p^* \;=\; R_{p'}, \qquad R_p^* \;=\; C_{p'},$$

via the duality bracket

$$\big\langle\, x\otimes u,\; y\otimes\overline{w}\,\big\rangle_{C_p,\,R_{p'}} \;=\; \tau(x\,y)\,\langle w,\,u\rangle_{\mathcal{H}}.$$

The column structure is also *associative* in the sense that we have the following (complete) isometric embedding

$$L_p\big(\mathcal{M};\;(\mathcal{H}\otimes\mathcal{K})_c\big)\hookrightarrow L_p\big((\mathcal{M}\bar{\otimes}B(\mathcal{H}));\;\mathcal{K}_c\big),$$

defined just by using the map Λ^c from $\mathcal{H}_c$ to $B(\mathcal{H})$. A completely analogous result holds for row spaces.

The row-column space $\mathbf{RC_p}$. To view (C_p,R_p) as a compatible pair of Banach spaces, we identify a real subspace $\mathcal{H}_{\mathbb{R}}\subset\mathcal{H}$ such that $\mathcal{H}\cong\mathbb{C}\otimes_2\mathcal{H}_{\mathbb{R}}$ as real Hilbert spaces. This induces an isometric antilinear map

$$J:\mathcal{H}\;\longrightarrow\;\mathcal{H}$$

which acts by complex conjugation on the $\mathbb{C}$-factor. In particular, J extends to all of $\mathcal{H}$. With this, we have continuous linear maps $C_p,R_p\to L_p(\mathcal{M})^\Gamma$ respectively extending

$$x\otimes u\mapsto\big(\langle e_j,u\rangle x\big)_{j\in\Gamma}\quad\text{and}\quad x\otimes\bar u\mapsto\big(\langle e_j,J(u)\rangle x\big)_{j\in\Gamma}.$$

Another way to say it is that $x\otimes u\in C_p$ corresponds to $x\otimes\overline{J(u)}\in R_p$.

We define $RC_p=R_p+C_p$ when $p\le2$ and $RC_p=R_p\cap C_p$ when $p\ge2$. Concretely, for an elementary tensor,

$$\Big\|\sum_{k=1}^n x_k\otimes u_k\Big\|_{RC_p}=\begin{cases}\inf\Big\{\Big\|\sum_{k=1}^m y_k\otimes v_k\Big\|_{C_p}\;+\;\Big\|\sum_{k=1}^r z_k\otimes\overline{J(w_k)}\Big\|_{R_p}\Big\} & \text{for } 1\le p\le2,\\ \max\Big\{\Big\|\sum_{k=1}^n x_k\otimes u_k\Big\|_{C_p},\;\Big\|\sum_{k=1}^n x_k\otimes\overline{J(u_k)}\Big\|_{R_p}\Big\} & \text{for } 2\le p\le\infty.\end{cases}$$

Here the infimum is taken over all ways of writing $\sum_{k=1}^n x_k\otimes u_k$ as $\sum_{k=1}^m y_k\otimes v_k\;+\;\sum_{k=1}^r z_k\otimes w_k$ in $C_p\oplus R_p$. We then let RC_p be the completion of simple tensors in $L_p(\mathcal{M})\otimes\mathcal{H}$ under these norms.

As with R_p and C_p, the space RC_p is homogeneous: if $\Lambda:\mathcal{H}\to\mathcal{H}$ is bounded, then $\mathrm{id}\otimes\Lambda$ extends to a completely bounded map on RC_p with $\|\mathrm{id}\otimes\Lambda\|_{\mathrm{cb}}=\|\Lambda\|$. Finally, duality also extends naturally from row and column spaces, yielding

$$RC_p^*\;\simeq\;RC_{p'}\quad\text{for}\quad1<p<\infty.$$

For simple tensors, the duality bracket is given by

$$\big\langle\, x\otimes u,\ y\otimes w\big\rangle_{RC_p,\,RC_{p'}} \;=\; \tau\big(x\,y\big)\,\langle\, J(w),\,u\rangle_{\mathcal{H}}. \tag{4.2.1}$$

Noncommutative Khintchine inequalities

Gaussian Hilbert spaces and classical Khintchine inequality. Let $(\Omega,\mathcal{B},\mu)$ be a probability space. A *Gaussian Hilbert space* on Ω is a closed subspace $G(\Omega)\subset L_2(\Omega)$ where every function is a (real-valued) centered Gaussian. Any subset of Gaussian variables $S\subset G(\Omega)$ are jointly Gaussian, and therefore they will be mutually independent if and only if they are an orthogonal family. Therefore, Gaussian spaces are always generated (as Hilbert spaces) by a set of mutually independent Gaussian variables, and the converse is also true.

The *classical Khintchine inequality* asserts that for any sequence of numbers $(x_n)_n$ and any sequence $(\gamma_n)_n$ of normalized, independent Gaussians on Ω, the following equivalence of norm holds for $1\le p<\infty$:

$$\Big\|\sum_n x_n\gamma_n\Big\|_p \approx_p \|(x_n)_n\|_{\ell_2}.$$

In particular, any Gaussian Hilbert space $G(\Omega)$ is a closed subspace of L_p for any $1\le p<\infty$. On the other hand, for each $1\le p\le\infty$ the exponentials $\big\{e^{i\gamma}:\gamma\in G(\Omega)\big\}$ belong also to $L_p(\Omega)$. Moreover, if the σ-algebra generated by $G(\Omega)$ is the full σ-algebra $\mathcal{B}$, then such exponentials span a norm-dense subspace of $L_p(\Omega)$ for $1\le p<\infty$, and weak*-dense if $p=\infty$.

Gaussian functor and non-commutative Khintchine inequality. Given a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, one may construct a measure space $(\Omega,\mathcal{B},\mu)$ and an isometry $W\colon\mathcal{H}_{\mathbb{R}}\to G(\Omega)\subset L_2(\Omega)$ such that the random variables in $G(\Omega)$ generate the σ-algebra $\mathcal{B}$. This construction is unique up to isomorphism (see [Nel73]), motivating the following terminology.

Definition 4.2.3 (Gaussian functor of a Hilbert space). For a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, its *Gaussian functor* (or *boson functor*) $\Gamma(\mathcal{H})$ is by definition the commutative von Neumann algebra $(L_\infty(\Omega),\mu)$, where $(\Omega,\mathcal{B},\mu)$ and W are as above.

Let $\mathcal{H}$ be the complexification of $\mathcal{H}_{\mathbb{R}}$. If $h\in\mathcal{H}$, then we can write $h=a+i\,b$ with $a,b\in\mathcal{H}_{\mathbb{R}}$, and define $W(h)=W(a)+i\,W(b)$. The next result follows by a straightforward change of basis from [LP86] and from [LPP91, Remark III.7].

Theorem 4.2.4 (Noncommutative Khintchine inequality). *There exist universal constants $A,B>0$ such that for any tracial von Neumann algebra $(\mathcal{M},\tau)$, any real Hilbert space $\mathcal{H}_{\mathbb{R}}$, any $1<p<\infty$, any integer $n\ge1$, and elements $x_k\in L_p(\mathcal{M})$, $u_k\in\mathcal{H}$ for $k\le n$, we have*

$$A^{-1}\Big\|\sum_{k=1}^n x_k\otimes u_k\Big\|_{RC_p} \;\le\; \Big\|\sum_{k=1}^n x_k\otimes W(u_k)\Big\|_p \;\le\; B\,\sqrt{p}\,\Big\|\sum_{k=1}^n x_k\otimes u_k\Big\|_{RC_p}.$$

Let $\mathrm{X}_p=\overline{\mathrm{span}}\{\,x\otimes W(u):\,x\in L_p(\mathcal{M}),\,u\in\mathcal{H}\}\subset L_p(\mathcal{M}\,\bar\otimes\,L_\infty(\Omega))$. By the Khintchine inequalities, we obtain explicit isomorphisms

$$\sigma_p:\mathrm{X}_p\;\longrightarrow\;RC_p$$

compatible with the usual duality bracket. Concretely, for simple tensors $(f,g)\in L_p\times L_{p'}$,

$$\big\langle\sigma_p(f),\,\sigma_{p'}(g)\big\rangle_{RC_p,\,RC_{p'}} \;=\; \int_\Omega \tau\big(f(w)\,g(w)\big)\,d\mu(w)\quad\text{for}\quad 1<p<\infty.$$

In addition, letting

$$\mathbb{E} : L_p(\mathcal{M} \bar{\otimes} L_\infty(\Omega)) \ \ni \ f \ \longmapsto \ \int_\Omega f(w)\, d\mu(w) \ \in \ L_p(\mathcal{M})$$

denote the natural conditional expectation, we can compare X_p with the row and column spaces R_p and C_p (for $p > 2$) via the relations

$$\|x\|_{R_p} \ = \ \Big\| \mathbb{E}\big(\sigma_p^{-1}(x)\, \sigma_p^{-1}(x)^*\big) \Big\|_{p/2}^{1/2} \quad \text{and} \quad \|x\|_{C_p} \ = \ \Big\| \mathbb{E}\big(\sigma_p^{-1}(x)^*\, \sigma_p^{-1}(x)\big) \Big\|_{p/2}^{1/2}.$$

The Gaussian projection. Let Q be the *Gaussian projection*, i.e. the orthogonal projection of $L_2(\Omega)$ onto the closed subspace $\{\, W(h) : \, h \in \mathcal{H}\}$. For any von Neumann algebra $(\mathcal{M}, \tau)$, the operator

$$\mathrm{id} \otimes Q \ : \ L_2(\mathcal{M} \bar{\otimes} L_\infty(\Omega)) \ \longrightarrow \ L_2(\mathcal{M} \bar{\otimes} L_\infty(\Omega))$$

extends to an L_p-projection for $1 < p < \infty$, and we have the following bound on the norm:

Lemma 4.2.5. *The map* $\mathrm{id} \otimes Q$ *extends to a bounded projection on* $L_p(\mathcal{M} \bar{\otimes} L_\infty(\Omega))$ *for all* $1 < p < \infty$*, with norm bounded by* $c\sqrt{\max\{p, p/(p-1)\}}$ *for some constant* $c > 0$*.*

Proof. Let $p > 2$ and $x \in L_p \cap L_2(\mathcal{M} \bar{\otimes} L_\infty(\Omega))$ a finite tensor. If we denote by $Q^\perp$ the complementary projection of Q, then for any $f, g \in L_p \cap L_2(\Omega)$ the following equality holds

$$\int_\Omega fg = \int_\Omega Q(f)Q(g) \ + \ \int_\Omega Q^\perp(f)Q^\perp(g),$$

and in particular, it follows that

$$\mathbb{E}\big(x^*x - (\mathrm{id} \otimes Q)(x)^*(\mathrm{id} \otimes Q)(x)\big) = \mathbb{E}\big((\mathrm{id} \otimes Q^\perp)(x)^*(\mathrm{id} \otimes Q^\perp)(x)\big) \geq 0.$$

Using the previously defined isometry $\sigma_p \colon X_p \to RC_p$, we get

$$\big\|\sigma_p\big((\mathrm{id} \otimes Q)(x)\big)\big\|_{C_p} \ = \ \Big\|\mathbb{E}\big((\mathrm{id} \otimes Q)(x)^*\, (\mathrm{id} \otimes Q)(x)\big)\Big\|_{p/2}^{\frac{1}{2}} \ \leq \ \Big\|\mathbb{E}(x^*x)\Big\|_{p/2}^{\frac{1}{2}} \ \leq \ \|x\|_p.$$

By similar arguments for R_p the Khintchine inequalities in Theorem 4.2.4, together with density arguments, the result follows for $p > 2$. The case $1 < p < 2$ is obtained by duality. □

Remark 4.2.6. Recall that for any pair of random variables in a Gaussian Hilbert space, it is the same to be independent or orthogonal as elements in $L_2(\Omega)$. Therefore, if γ is a centered Gaussian and f is some measurable function on $\mathbb{R}$, the Gaussian projection $Q\big(f(\gamma)\big)$ can be computed as:

$$Q\big(f(\gamma)\big) = \left(\int_{\mathbb{R}} f(x)\, d\gamma(x)\right) \frac{\gamma}{\|\gamma\|_{L_2(\Omega)}^2}.$$

ORC_p-bounded families of operators

In this section, we adapt the classical notion of *R-boundedness* [HvNVW17, Chapter 8] to the noncommutative setting of $L_p(\mathcal{M})$-spaces. Recall that, in the commutative framework, a family

of bounded operators $\{T_i\}_i$ on a Banach space X is R-bounded if there is a constant C such that for all finite choices $(T_1, \ldots, T_n)$ in the family and all $(x_1, \ldots, x_n) \in X^n$, we have

$$\left\| \sum_{k=1}^{n} r_k \, T_k x_k \right\|_{L^2(\Omega;X)} \;\leq\; C \left\| \sum_{k=1}^{n} r_k \, x_k \right\|_{L^2(\Omega;X)},$$

where $(r_k)_{k=1}^{\infty}$ is a sequence of independent Rademacher functions. Such a condition is stronger than simple uniform boundedness and is often crucial in harmonic analysis and operator theory.

Noncommutative setting: ORC_p-boundedness. In order to define analogs of R-boundedness in the noncommutative case, we replace the classical Rademacher functions by suitable row/column structures in $L_p(\mathcal{M})$.

Definition 4.2.4 (OC_p-bounded families of operators). Given a sequence of operators $(\phi_n)_{n\in\mathbb{Z}}$ where each $\phi_n : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ is bounded, we say that $(\phi_n)_{n\in\mathbb{Z}}$ is *OC_p-bounded* if there exists a bounded operator

$$\phi : L_p(\mathcal{M}; \ell_2(\mathbb{Z})_c) \;\longrightarrow\; L_p(\mathcal{M}; \ell_2(\mathbb{Z})_c)$$

such that

$$\phi\big(x \otimes \delta_n\big) \;=\; \phi_n(x) \otimes \delta_n \quad \text{for all } x \in L_p(\mathcal{M}) \text{ and all integers } n.$$

Similarly, a family Φ of operators (each $L_p(\mathcal{M}) \to L_p(\mathcal{M})$) is then OC_p-bounded if every sequence $(\phi_n)_{n\in\mathbb{Z}} \subset \Phi$ satisfies the above condition with a uniform bound on $\|\phi\|$.

By replacing the column space $L_p(\mathcal{M}; \ell_2(\mathbb{Z})_c)$ with the row space $L_p(\mathcal{M}; \ell_2(\mathbb{Z})_r)$ or their row-column version, one obtains analogous notions of OR_p- and ORC_p-boundedness. As we see, row and column vector-valued elements of $L_p(\mathcal{M})$ play the same role that Rademacher functions do in the classical setting. In the next proposition we list some basic properties of this kind of operators. The statements are for OC_p-bounded families, but are also valid in for families that are OR_p- or ORC_p-bounded.

Proposition 4.2.7. *Let Φ and Φ' be two families of operators in $L_p(\mathcal{M})$ that are OC_p bounded with constants C_Φ and $C_{\Phi'}$ respectively. Then:*

i) the family of operators

$$\Phi + \Phi' = \{S + S' : \; S \in \Phi, \; S' \in \Phi'\}$$

is OC_p-bounded with constant less or equal than $C_\Phi + C_{\Phi'}$,

ii) for any constant $C > 0$, the family

$$C\Phi = \{CS : \; S \in \Phi\}$$

is OC_p-bounded with constant CC_Φ.

We will illustrate these concepts with concrete examples in Section 4.4, coming from Riesz–Schur transforms in the noncommutative setting. For general background, see [JLMX06]. In particular, the following classical lemma (Lemma 4.2 in [JLMX06]) connects these notions to important properties in operator-valued harmonic analysis.

Lemma 4.2.8. *If a set Φ is OC_p-bounded for $1 \leq p < \infty$, then the closure of its absolute convex hull for the point-weak topology is also OC_p-bounded with the same constant. The same holds for OR_p-bounded or ORC_p-bounded families of operators.*

Recall that if X and Y are Banach spaces, the point-weak topology in $B(X,Y)$ is the topology induced by functionals Y^*. If $X = Y$ are Hilbert spaces, then it coincides with the weak operator topology introduced in Chapter 1. Notice also that when Y is separable and reflexive, the unit ball in $B(X,Y)$ is compact in the point-weak topology, as happens in the Hilbert case. A concrete application of this fact is illustrated by the following lemma:

Lemma 4.2.9. *Let $K \subset B(S_p(\mathbb{Z}))$ be a bounded set of Schur multipliers, with $1 < p < \infty$. Then an operator S is in the point-weak closure of K if and only if*

i) S is a Schur multiplier with symbol M, and

ii) there is a net (S_{M_α}) in K such that (M_α) converges pointwise to M.

Proof. First, suppose that there is a net (S_{M_α}) converging to S in the point-weak topology. Then $\langle S_{M_\alpha}(e_{jk}), e_{j'k'}\rangle$ converges to $\langle S(e_{jk}), e_{j'k'}\rangle$ for every matrix units e_{jk} and $e_{j'k'}$. It follows easily that S is a Schur multiplier with symbol $M(j,k) = \lim_\alpha M_\alpha(j,k)$.

On the other hand, say that (S_{M_α}) is a net in K such that $M = \lim_\alpha M_\alpha$ pointwise. Since K is bounded, (S_{M_α}) has a subnet (S_{M_β}) that converges in the point-weak topology to some $S_M \in B(S_p(\mathbb{Z}))$. Obviosly $\lim_\beta M_\beta = \lim_\alpha M_\alpha = M$, proving the claim. □

These two lemmas give a useful way to work with OC_p-bounded families of Schur multipliers. Lemma 4.2.8 shows that taking the absolute convex hull and passing to the point-weak closure preserves the OC_p-boundness, whereas Lemma 4.2.9 shows that this closure can be described explicitly by pointwise convergence of the symbols.

Convergence for truncated sums of ORC_p-bounded sequences. Let $(\Delta_n)_{n\in\mathbb{Z}}$ be a partition of $\Gamma \times \Gamma$ in disjoint sets, and $(S_{\Delta_n})_{n\in\mathbb{Z}}$ be the corresponding Schur multipliers on $S_p(\Gamma)$, each S_{Δ_n} having symbol $1_{(j,k)\in\Delta_n}$. We further assume that the sequence $(S_{\Delta_n})_{n\in\mathbb{Z}}$ is *completely unconditional on $S_p(\Gamma)$ for $1 < p < \infty$*, meaning that finite sums $\sum_n \epsilon_n S_{\Delta_n}$ and their matrix amplifications are uniformly bounded under almost any choice of signs (ϵ_n).

Using the noncommutative Khintchine inequalities, we obtain for $x = (x_{jk})_{j,k}$ in $L_p(\mathcal{M} \bar{\otimes} B(\ell_2(\Gamma)))$:

$$D_p^{-1} \Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \Big\|_p \lesssim \Big\| \sum_{n\in\mathbb{Z}} \sum_{j,k\in\Gamma} 1_{(j,k)\in\Delta_n} x_{jk} \otimes e_{jk} \otimes \delta_n \Big\|_{RC_p} \lesssim D_p \Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \Big\|_p . \tag{4.2.2}$$

In particular, any sequence of completely unconditional sets Δ_n verifies the following lemma:

Lemma 4.2.10. *Let Φ be an ORC_p-bounded family of operators with constant K_p. Consider any sequence $(S_{M_n})_{n\in\mathbb{Z}}$ of Schur multipliers in Φ and any sequence $(\Delta_n)_{n\in\mathbb{Z}}$ satisfying* (4.2.2)*. Then the symbol*

$$M(j,k) \;=\; \sum_{n\in\mathbb{Z}} 1_{(j,k)\in\Delta_n}\, M_n(j,k) \quad \textit{for} \quad j,k \in \Gamma$$

defines a Schur multiplier on $S_p(\Gamma)$ with complete norm bounded by $K_p\, D_p^2$.

Proof. We are going to prove the statement for $p \geq 2$, since the other case follows by duality.

$$\begin{aligned} \Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \Big\|_p &\gtrsim_{D_p^{-1}} \Big\| \sum_{n\in\mathbb{Z}} \sum_{j,k\in\Gamma} 1_{(j,k)\in\Delta_n} x_{jk} \otimes e_{jk} \otimes \delta_n \Big\|_{RC_p} \\ &\gtrsim_{{K_p}^{-1}} \Big\| \sum_{n\in\mathbb{Z}} \sum_{j,k\in\Gamma} 1_{(j,k)\in\Delta_n} M_n(j,k) x_{jk} \otimes e_{jk} \otimes \delta_n \Big\|_{RC_p} \end{aligned}$$

$$\gtrsim_{D_p^{-1}} \Big\| \sum_{n\in\mathbb{Z}} \sum_{j,k\in\Gamma} 1_{(j,k)\in\Delta_n} M_n(j,k) x_{jk} \otimes e_{jk} \Big\|_p .$$

The two first inequalities follows from the definition of ORC_p and the hypothesis on Δ_n, whereas the last inequality uses the that the RC_p norm is greater or equal than the C_p (or the R_p) norm. □

An illustrative example of this corollary is provided by the diagonal dyadic partitions

$$\Delta_n = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, 2^n \leq |x-y|_\infty < 2^{n+1}\}. \tag{4.2.3}$$

The proof that these partitions satisfy the inequality (4.2.2) closely parallels the classical Littlewood-Paley theorem in $\mathbb{R}^n$ (see, for instance, [Duo01]). We now sketch the argument, with an emphasis on keeping track of the constants that arise in the computations.

Theorem 4.2.11. *Let $p \geq 2$ and (Δ_n) denote the dyadic partition defined in Equation (4.2.3), and let $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a smooth function supported on $[1/2, 4]$ and equal to 1 on $[1,2]$. Define the dilated functions $\varphi_n(t) = \varphi(2^n t)$, and let $\tilde{\varphi}_n(x) = \varphi_n(|x|_\infty)$ be their extensions to $\mathbb{R}^n$.*

i) For every $A \in S_p(\mathbb{R}^n)$, it holds that

$$\big\| \sum_{n\in\mathbb{Z}} S_{\tilde{\varphi}_n}(A) \otimes \delta_n \big\|_{RC_p} \lesssim \left(\frac{p^2}{p-1}\right) \big\|A\big\|_{S_p},$$

where $S_{\tilde{\varphi}_n}$ denotes the Toeplitz Schur multiplier associated with $\tilde{\varphi}_n$.

ii) For any $\sum_n A_n \otimes \delta_n \in L_p\big(B(L_2(\mathbb{R}^n)); (\ell_2\mathbb{Z})_{rc}\big)$, it holds that

$$\big\| \sum_{n\in\mathbb{Z}} S_{\Delta_n}(A_n) \otimes \delta_n \big\|_{RC_p} \lesssim \left(\frac{p^2}{p-1}\right)^n \big\| \sum_{n\in\mathbb{Z}} A_n \otimes \delta_n \big\|_{RC_p}.$$

Proof. The first point follows from the Khintchine inequality and the classical Hörmander-Mikhlin theorem. Let $T_{\tilde{\varphi}_n}$ denote the Fourier multiplier with symbol $\tilde{\varphi}_n$. Then, by transference,

$$\begin{aligned}\big\| \sum_{n\in\mathbb{Z}} S_{\tilde{\varphi}_n}(A) \otimes \delta_n \big\|_{RC_p} &\lesssim \big\| \sum_{n\in\mathbb{Z}} \varepsilon_n S_{\tilde{\varphi}_n}(A) \big\|_{L_p(\Omega;\, S_p(\mathbb{R}^n))} \\ &\lesssim \Big(\int_\Omega \big\| \sum_{n\in\mathbb{Z}} \varepsilon_n(\omega) T_{\tilde{\varphi}_n} \big\|_{cb(L_p(\mathbb{R}^n))}^p \, d\omega \Big)^{1/p} \big\|A\big\|_{S_p(\mathbb{R}^n)}.\end{aligned}$$

Note that $(\sum_n \varepsilon_n(\omega) T_{\tilde{\varphi}_n})_\omega$ is a family of Hörmander-Mikhlin operators with norm uniformly bounded by $p^2/(p-1)$, which proves the first point.

For the second point, observe that the map $A_n \otimes e_{n1} \mapsto S_{\Delta_n}(A_n) \otimes e_{n1}$ extends to a Toeplitz Schur multiplier on $L_p(B(L_2(\mathbb{R}^n)) \bar{\otimes} B(\ell_2(\mathbb{Z})))$, with Toeplitz symbol $m(x,n) = \mathbf{1}_{2^n \leq |x|_\infty < 2^{n+1}}$. The corresponding Fourier multiplier acts on sequences (f_n) in $L_p(\mathbb{R}^n)$ by

$$T_m \,:\, (f_n) \mapsto (T_{m_n} f_n),$$

where $m_n(x) = m(x,n)$. Using a standard trick (see [Duo01], Corollary 8.2), one can express each T_{m_n} as a product of n differences of Hilbert transforms. In particular, this implies

$$\big\|T_m\big\|_{L_p} \lesssim \left(\frac{p^2}{p-1}\right)^n,$$

concluding the proof of the second point. □

We claim now that that this result implies that

$$D_p^{-1} \left\|A\right\|_p \lesssim \Big\| \sum_{j,k,n\in\mathbb{Z}} \sum_{j,k\in\Gamma} S_{\Delta_n}(A) \otimes \delta_n \Big\|_{RC_p} \lesssim D_p \left\|A\right\|_p,$$

for $D_p \approx \left(\frac{p^2}{p-1}\right)^{n+1}$. The right-hand side inequality follows directly from the theorem, whereas the other inequality requires noticing that, for $A, B \in S_p(\mathbb{R}^n)$ with compact support it holds that

$$\begin{aligned} \big|\mathrm{tr}(AB)\big| &= \big| \sum_{n\in\mathbb{Z}} \mathrm{tr}(S_{\Delta_n}(A)\, S_{\Delta_n}(B))\big| \\ &= \big|\big\langle \sum_{n\in\mathbb{Z}} S_{\Delta_n}(A) \otimes \delta_n, \sum_{n\in\mathbb{Z}} S_{\Delta_n}(B) \otimes \delta_n \big\rangle_{RC_p, RC_{p'}}\big| \\ &\lesssim D_{p'} \big\| \sum_{n\in\mathbb{Z}} S_{\Delta_n}(A) \otimes \delta_n \big\|_{RC_p} \left\|B\right\|_{S_{p'}}. \end{aligned}$$

Then the claim follows from the appreciation that $D_p \approx D_{p'} \approx \max(p,p')^{n+1}$.

Corollary 4.2.12. *The partition (Δ_n) in Theorem 4.2.11 satisfies* (4.2.2) *with constant*

$$D_p \approx \left(\frac{p^2}{p-1}\right)^{n+1} \qquad \text{for } \ p \geq 2.$$

Remark 4.2.13. Analogous results can be obtained for discrete dyadic partitions in $B(\ell_2(\mathbb{Z}^n))$ using the very same argument and transference.

4.3 Square function inequalities

In this section we prove the dimension-free estimates for Riesz-Schur transforms in Theorem 4.1.2. As explained in the Introduction to this chapter, the proof does not follow any of the standard approaches from harmonic analysis or probability theory. After the proof, we shall show how to recover the trigonometric Riesz transform estimates from [JMP18] in the amenable case.

Riesz-Schur transforms

FRom now on, we will use the sign function sgn defined in 0 as $\mathrm{sgn}(0) = 1$, so that $\mathrm{sgn}(x)^2 = 1$ for every $x \in \mathbb{R}$. Fix also the constant $C_p = \max\{p, p'\}$.

Lemma 4.3.1. *Let (Ω,μ) be a probability measure space. Then, the following norm equivalence holds in $L_p(\mathcal{M}\bar{\otimes}B(\ell_2(\Gamma))\bar{\otimes}L_\infty(\Omega))$ for any $1 < p < \infty$ and any family $(f_j)_{j\in\Gamma}$ of real-valued measurable functions on Ω*

$$\Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \otimes \mathrm{sgn}(f_j - f_k) \Big\|_p \approx_{C_p} \Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \Big\|_p.$$

Proof. It suffices to assume (x_{jk}) is finitely supported. Since

$$\Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \otimes \mathrm{sgn}(f_j - f_k) \Big\|_p^p = \int_\Omega \Big\| \sum_{j,k\in\Gamma} \mathrm{sgn}(f_j(\omega) - f_k(\omega)) x_{jk} \otimes e_{jk} \Big\|_p^p d\mu(\omega),$$

it suffices to check the pointwise equivalence for all $\omega \in \Omega$.

Without loss of generality, we assume that Γ is finite. For a fixed point $\omega \in \Omega$, we define S_M as the Schur multiplier on $B(\ell_2\Gamma)$ associated with the matrix

$$M(i,j) = \operatorname{sgn}\big(f_i(\omega) - f_j(\omega)\big).$$

Choose a total order $\leq$ on Γ such that $i \leq j$ implies $f_i(\omega) \leq f_j(\omega)$. Then define an order-preserving map $\pi\colon \Gamma \to \mathbb{Z}$, and consider the corresponding unitary operator $U\colon \ell_2(\Gamma) \to \ell_2(\mathbb{Z})$ defined by $U(\delta_i) = \delta_{\pi(i)}$.

Let $\mathbb{E}\colon B(\ell_2\mathbb{Z}) \to B(\ell_2\mathbb{Z})$ be the diagonal conditional expectation given by:

$$\mathbb{E}(e_{n,m}) = \begin{cases} e_{n,m}, & \text{if } n = m \text{ or } (n,m) = (\pi(i),\pi(j)) \text{ for some } i,j \in \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following identity holds:

$$US_MU^* = \mathbb{E} + H \circ (\mathrm{id} - \mathbb{E}),$$

where H denotes the usual Hilbert transform, understood as a Schur multiplier.

This shows that

$$\|S_M : S_p(\Gamma) \to S_p(\Gamma)\|_{cb} \lesssim \|H : S_p(\mathbb{Z}) \to S_p(\mathbb{Z})\|_{cb},$$

proving one side of the inequality. The other side follows from the identity $S_M \circ S_M = \mathrm{id}$.

□

Proof of Theorem 4.1.2. By approximation we can assume that Γ is finite, say $\Gamma = \{1, 2, \ldots, n\}$ and let $M_n = B(\ell_2(\Gamma))$. Assume for the moment that $u_j \in \mathcal{H}_{\mathbb{R}}$ and consider its gaussian functor. Using the Remark 4.2.6, and since $W(u_j - u_k)$ is a centered gaussian variable with L_2-norm $\|u_j - u_k\|$ we have (assuming $u_j \neq u_k$) that

$$Q\big(\operatorname{sgn}\big(W(u_j - u_k)\big)\big) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \operatorname{sgn}(s) s e^{-s^2/2} ds \, \frac{W(u_j - u_k)}{\|u_j - u_k\|} = \sqrt{\frac{2}{\pi}} \frac{W(u_j - u_k)}{\|u_k - u_k\|}.$$

Let $x = \sum_{j,k} x_{jk} \otimes e_{jk}$ and consider

$$\begin{aligned} \mathcal{R}(x) &= \sum_{j,k \in \Gamma} x_{jk} \otimes e_{jk} \otimes \frac{u_j - u_k}{\|u_j - u_k\|}, \\ \mathcal{H}(x) &= \sum_{j,k \in \Gamma} x_{jk} \otimes e_{jk} \otimes \operatorname{sgn}\big(W(u_j) - W(u_k)\big). \end{aligned}$$

Then we use the identity

$$\sigma_p^{-1}(\mathcal{R}(x)) = \sqrt{\frac{\pi}{2}} (\mathrm{id} \otimes Q)(\mathcal{H}(x))$$

and Lemma 4.3.1 for $f_j = W(u_j)$ to deduce (RS_1). Indeed, we have

$$\begin{aligned} \big\|\mathcal{R}(x)\big\|_{R_p} &= \big\|\mathbb{E}\big(\sigma_p^{-1}(\mathcal{R}(x))\sigma_p^{-1}(\mathcal{R}(x))^*\big)\big\|_{p/2}^{1/2} \\ &= \sqrt{\frac{\pi}{2}} \big\|\mathbb{E}\big((\mathrm{id} \otimes Q)(\mathcal{H}(x))(\mathrm{id} \otimes Q)(\mathcal{H}(x))^*\big)\big\|_{p/2}^{1/2} \lesssim \|\mathcal{H}(x)\|_p \lesssim C_p \|x\|_p. \end{aligned}$$

The inequalities here are proven in the same way as in Lemma 4.2.5. The same holds for the C_p-norm and (RS_1) follows for $p \geq 2$. When $1 < p < 2$, we use the noncommutative Khintchine inequality in Theorem 4.2.4, followed by Lemmas 4.2.5 and 4.3.1.

$$\big\|\mathcal{R}x\big\|_{RC_p} \lesssim \Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \otimes Q\big(\operatorname{sign} W(u_j - u_k)\big)\Big\|_p \lesssim \frac{1}{\sqrt{1-p}} C_p \|x\|_p$$

The use of Lemma 4.2.5 (which wasn't directly used for $p \geq 2$) produces the asymmetry in the constants in the statement of Theorem 4.1.2.

The proof of (RS_1) is complete for $u_j \in \mathcal{H}_{\mathbb{R}}$ and for the general case, let $\mathcal{K}_{\mathbb{R}}$ be $\mathcal{H}$ as a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{K}_{\mathbb{R}}} = \operatorname{Re} \langle \cdot, \cdot \rangle_{\mathcal{H}}$. Let $\mathcal{K} = \mathcal{K}_{\mathbb{R}} + \mathrm{i}\mathcal{K}_{\mathbb{R}}$ be the trivial complexification of $\mathcal{K}_{\mathbb{R}}$. Its norm is made so that for $u, w \in \mathcal{K}_{\mathbb{R}}$ we get $\|u + \mathrm{i}w\|_{\mathcal{K}}^2 = \|u\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2$. The map $\Lambda : \mathcal{K} \to \mathcal{H}$ given by $\Lambda(u + \mathrm{i}w) = u + \mathrm{i}w$ for $u, w \in \mathcal{K}_{\mathbb{R}} = \mathcal{H}$ is a well-defined linear map between complex Hilbert spaces, and its norm is bounded by $\sqrt{2}$. This gives (RS_1) for $u_j \in \mathcal{H}$ (loosing a factor $\sqrt{2}$ in the constant) since it holds for $u_j \in \mathcal{K}_{\mathbb{R}}$ and RC_p is homogeneous.

Let us now consider the lower estimate (RS_2). If the matrix (x_{jk}) is supported by $\{(j,k) \in \Gamma \times \Gamma : u_j \neq u_k\}$, then the estimate follows by duality. Indeed, we know from (RS_1) that the maps defined by

$$\mathcal{R}_p : L_p(\mathcal{M} \otimes M_n) \ni x \otimes e_{jk} \mapsto x \otimes e_{jk} \otimes \frac{u_j - u_k}{\|u_j - u_k\|} \in RC_p(\mathcal{M} \otimes M_n),$$

$$\overline{\mathcal{R}}_{p'} : L_{p'}(\mathcal{M} \otimes M_n) \ni x \otimes e_{jk} \mapsto x \otimes e_{jk} \otimes \frac{J(u_j - u_k)}{\|u_j - u_k\|} \in RC_p(\mathcal{M} \otimes M_n),$$

have norms $\lesssim c_p, c_{p'}$ respectively. Given a pair $(x, y) \in L_p(\mathcal{M} \otimes M_n) \times L_{p'}(\mathcal{M} \otimes M_n)$ and using the duality bracket (4.2.1) for RC_p and $RC_{p'}$, we deduce the following identity

$$\big\langle \mathcal{R}_p(x), \overline{\mathcal{R}}_{p'}(y) \big\rangle_{RC_p, RC_{p'}} = \sum_{j,k\in\Gamma} \tau(x_{jk} y_{kj}) \frac{\langle u_j - u_k, u_k - u_j \rangle}{\|u_j - u_k\|^2} = -(\mathrm{tr} \otimes \tau)(xy),$$

with the support assumption. Taking the sup over all y in the unit p'-ball gives $\|x\|_p \lesssim c_{p'} \|\mathcal{R}(x)\|_{RC_p}$. The general case follows by using the triangular inequality. □

Trigonometric Riesz transforms

We start with a simple remark which will allow us to connect our estimates for Riesz-Schur transforms in Theorem 4.1.2 to the dimension-free estimates for trigonometric Riesz transforms in [JMP18].

Lemma 4.3.2. *Let $p \geq 2$ and consider a family of contractions $\Lambda_j : \mathcal{H} \to \mathcal{H}$. Then the following inequality holds for $(x_{jk})_{j,k\in\Gamma} \in L_p(\mathcal{M} \otimes B(\ell_2(\Gamma)))$ and any family of vectors $(\xi_{jk})_{j,k\in\Gamma}$ in $\mathcal{H}$:*

$$\Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \otimes \Lambda_j(\xi_{jk}) \Big\|_{C_p} \leq \Big\| \sum_{j,k\in\Gamma} x_{jk} \otimes e_{jk} \otimes \xi_{jk} \Big\|_{C_p}.$$

The same estimate holds in R_p with contractions Λ_k varying over columns $k \in \Gamma$.

Proof. First, observe that rewriting e_{jk} as $e_{1k} \otimes e_{j1}$ does not alter the C_p-norm of the tensors. Further more, consider the tensors

$$\sum_{j,k} x_{jk} \otimes e_{jk} \otimes u_{jk} \ \in \ L_p\big(\mathcal{M} \bar{\otimes} B(\ell_2\Gamma);\, \mathcal{H}_c\big)$$

and

$$\sum_{j,k} x_{jk} \otimes e_{1k} \otimes e_{j1} \otimes u_{jk} \ \in \ L_p\big(\mathcal{M} \bar{\otimes} B(\ell_2\Gamma);\, \big(S^2(\Gamma) \otimes H\big)_c\big),$$

These two tensors have the same C_p-norm. Hence, the only remaining step is to check the boundedness of the diagonal map

$$e_{j1} \otimes u \ \longmapsto \ e_{j1} \otimes \Lambda_j(u),$$

which follows directly from the homogeneity of the C_p-norm. This completes the proof. □

We are now in position to recover some results from [JMP18]. When $\Gamma = \mathrm{G}$ is a locally compact group, consider an orthogonal cocycle $\beta : \mathrm{G} \to \mathcal{H}_{\mathbb{R}}$ associated to an orthogonal action α. That is, $\beta(gh) = \beta(g) + \alpha_g(\beta(h))$ and we find the

$$\beta(g^{-1}) - \beta(h^{-1}) = -\alpha_{g^{-1}}(\beta(gh^{-1})) = \alpha_{h^{-1}}(\beta(hg^{-1})) \quad \text{for} \quad g, h \in \mathrm{G}. \tag{4.3.1}$$

Let $x = \sum_{g,h} x_{g,h} \otimes e_{g,h}$ be a simple tensor in $L_p(\mathcal{M}\bar{\otimes}B(\ell_2(\mathrm{G})))$. For convenience in the statement and to match the formulation in [JMP18], we assume that $x_{g,h} = 0$ whenever $\beta(gh^{-1}) = 0$, which is equivalent to $\beta(hg^{-1}) = 0$. Then, we deduce the following estimate for $2 \le p < \infty$ directly from Theorem 4.1.2 with $u_g = \beta(g^{-1})$ together with identities (4.3.1) and Lemma 4.3.2

$$\begin{aligned}\Big\| \sum_{g,h\in\mathrm{G}} x_{g,h} \otimes e_{g,h} \Big\|_p &\approx_{c_p} \Big\| \sum_{g,h\in\mathrm{G}} x_{g,h} \otimes e_{g,h} \otimes \frac{\alpha_{g^{-1}}(\beta(gh^{-1}))}{\|\beta(gh^{-1})\|} \Big\|_{RC_p} \\ &\approx \Big\| \sum_{g,h\in\mathrm{G}} x_{g,h} \otimes e_{g,h} \otimes \frac{\beta(hg^{-1})}{\|\beta(hg^{-1})\|} \Big\|_{R_p} + \Big\| \sum_{g,h\in\mathrm{G}} x_{g,h} \otimes e_{g,h} \otimes \frac{\beta(gh^{-1})}{\|\beta(gh^{-1})\|} \Big\|_{C_p}.\end{aligned}$$

When G is amenable and using [LdlS11, Theorem 1.19] and Fourier-Schur transference, one recovers that for $x = \int_G x_g \otimes \lambda(g)\, dg \in L_p(\mathcal{M}\bar{\otimes}\mathcal{L}(\mathrm{G}))$ with $x_g = 0$ when $\beta(g) = 0$ and any $2 < p < \infty$

$$\|x\|_p \ \approx_{c_p} \ \Big\| \int_{\mathrm{G}} x_g \otimes \lambda(g) \otimes \frac{\beta(g)}{\|\beta(g)\|}\, dg \Big\|_{C_p} + \Big\| \int_{\mathrm{G}} x_g \otimes \lambda(g) \otimes \frac{\beta(g^{-1})}{\|\beta(g^{-1})\|}\, dg \Big\|_{R_p}. \tag{4.3.2}$$

This is the main result in [JMP18]—the case $1 < p < 2$ follows by duality. This includes nonunimodular groups, not considered in [JMP18]. In addition, using local transference from [PRS22] we get local forms of (4.3.2) for nonamenable groups.

4.4 Schur multipliers in Schatten p-classes

Given an index set Γ, the Schur multiplier with symbol $M : \Gamma \times \Gamma \to \mathcal{C}$ is denoted by S_M. The cb-norm on S_p is denoted by $\|S_M\|_{p,\mathrm{cb}}$. Given $1 \le p \le \infty$, we recall the following well known facts

$$\|S_M\|_{p,\mathrm{cb}} = \|S_M\|_{p',\mathrm{cb}} = \|S_{\bar{M}}\|_{p,\mathrm{cb}} = \|S_{M_{\mathrm{op}}}\|_{p,\mathrm{cb}}, \tag{4.4.1}$$

where $M_{\mathrm{op}}(j,k) = M(k,j)$. Indeed, the equality $\|S_M\|_{p,\mathrm{cb}} = \|S_{M_{\mathrm{op}}}\|_{p',\mathrm{cb}}$ follows by duality. Considering $(\mathcal{M}_{\mathrm{op}}, \tau)$ instead of $(\mathcal{M}, \tau)$ gives $\|S_M\|_{p,\mathrm{cb}} = \|S_{M_{\mathrm{op}}}\|_{p,\mathrm{cb}}$ and the last identity comes from a conjugation with the involution.

New criteria for S_p-multipliers

First criteria: S_p-factorization condition. We start with the proof of the first criteria in Theorem 4.1.5, the S_p-factorization condition.

Proof of Theorem 4.1.5, Part 1. Assume first that $u'_k = -u_k$ and $w'_k = -w_k$. We are using Theorem 4.1.2 by fixing some $\mathcal{H}_{\mathbb{R}} \subset \mathcal{H}$ with associated involution J. Let us consider the following maps

$$\mathcal{R}_p : S_p(\Gamma) \ni e_{jk} \mapsto e_{jk} \otimes \frac{u_j - u_k}{\|u_j - u_k\|} \in RC_p(B(\ell_2(\Gamma))),$$

$$\overline{\mathcal{R}}_{p'} : S_{p'}(\Gamma) \ni e_{jk} \mapsto e_{jk} \otimes \frac{J(w_j - w_k)}{\|w_j - w_k\|} \in RC_p(B(\ell_2(\Gamma))),$$

Let $\mathbf{\Lambda} = \mathrm{id} \otimes \Lambda$ be the complete contraction given by homogeneity of RC_p. We have for $x \in S_p$ and $y \in S_q$

$$\mathrm{tr}\big(S_M(x)y\big) = \sum_{j,k\in\Gamma} x_{jk}y_{kj}\Big\langle \frac{u_j - u_k}{\|u_j - u_k\|}, \Lambda\Big(\frac{w_j - w_k}{\|w_j - w_k\|}\Big)\Big\rangle = -\big\langle \mathbf{\Lambda}^*\mathcal{R}_p(x), \overline{\mathcal{R}}_{p'}(y)\big\rangle.$$

Thus $S_M = -\overline{\mathcal{R}}^*_{p'}\mathbf{\Lambda}^*\mathcal{R}_p$ and the estimate follows from Theorem 4.1.2. The general case follows by a standard 2×2 trick. Take as index set two disjoints copies $\Gamma_1 \cup \Gamma_2$ of Γ and set

$$\widehat{M}(j,k) = \Big\langle \frac{h_j - h_k}{\|h_j - h_k\|}, \Lambda\Big(\frac{g_j - g_k}{\|g_j - g_k\|}\Big)\Big\rangle \quad \text{with} \quad (h_j, g_j) = \begin{cases} +(u_j, w_j) & \text{if } j \in \Gamma_1, \\ -(u'_j, w'_j) & \text{if } j \in \Gamma_2. \end{cases}$$

Then, the $\Gamma_1 \times \Gamma_2$ corner of $\widehat{M}$ equals $(M(j,k))_{j,k\in\Gamma}$ and the result follows. □

S_p-boundedness for angles of vectors The argument above can be modified, at least for real Hilbert spaces and $\Delta = \mathrm{id}$, to prove the boundedness of a very similar multiplier. One just has to reproduce the above duality argument using instead of $\mathcal{R}_p$ and $\overline{\mathcal{R}}_{p'}$ the gaussian tensors $\mathcal{T}_p$ and $\mathcal{T}_{p'}$ given by

$$\mathcal{T}_p : S_p(\Gamma) \ni e_{jk} \mapsto e_{jk} \otimes \frac{W(u_j) - W(u_k)}{\|u_j - u_k\|} \in L_p\big(B(\ell_2(\Gamma))\bar{\otimes}L_\infty(\Omega)\big),$$

where W is a gaussian field on $\mathcal{H}$. Them Lemma 4.3.1, together with Pisier's formula in [Pis01, Lemma 5.5]

$$\int_\Omega \mathrm{sgn}(W(u))\mathrm{sgn}(W(w))d\mu = \frac{2}{\pi}\arcsin\langle u, w\rangle$$

gives us the boundedness of the Schur multiplier with symbol

$$\arcsin\Big\langle \frac{u_j - u_k}{\|u_j - u_k\|}, \frac{w_j - w_k}{\|w_j - w_k\|}\Big\rangle,$$

with a norm bounded by $\max\{p, p/p-1\}^2$.

Can we improve the constants in the S_p-factorization condition? As we have pointed in the Introduction, classical Grothendieck symbols in S_∞ arise from Theorem 4.1.5 i) by fixing $w'_j = 0 = u_k$. Of course, the constants remain uniformly bounded in that case. In the following result, we show that we can actually improve the constants when only one term vanishes in Theorem 4.1.5 i).

Corollary 4.4.1. *The symbols on* $\Gamma \times \Gamma$

$$M_1(j,k) = \Big\langle \frac{u_j - u'_k}{\|u_j - u'_k\|}, \frac{w_j}{\|w_j\|}\Big\rangle \quad \textit{and} \quad M_2(j,k) = \Big\langle \frac{u_j - u_k}{\|u_j - u'_k\|}, \frac{w_k}{\|w_k\|}\Big\rangle$$

satisfy the estimate $\|S_{M_1}\|_{p,\mathrm{cb}}, \|S_{M_2}\|_{p,\mathrm{cb}} \lesssim \max\big\{p, \frac{p}{p-1}\big\}$ *for every* $1 < p < \infty$.

Proof. By symmetry, it suffices to prove the statement for M_1 and $p > 2$ (see Equation (4.4.1)). Let's assume that $u_i' = u_i$, since the general case follows from a 2×2 trick as happened in the general S_p-factorization condition.

Let $(\delta_j)_{j\in\Gamma}$ denote the canonical basis of $\ell_2(\Gamma)$. It is clear that the map

$$\rho : S_p(\Gamma) \ni e_{j,k} \mapsto e_{j,k} \otimes \delta_j \in L_p(B(\ell_2(\Gamma)); \ell_2(\Gamma)_c)$$

is a complete isometry for all $p \geq 1$. Composing it with Theorem 4.1.5 and using the associativity of column Hilbert spaces, we get that the map

$$V : e_{jk} \mapsto e_{jk} \otimes \Big(\delta_j \otimes \frac{u_j - u_k}{\|u_j - u_k\|}\Big)$$

decomposes as

$$V\colon L_p(\ell_2\Gamma) \xrightarrow{R_u} L_p\big(\ell_2\Gamma;\ \mathcal{H}_c\big) \hookrightarrow L_p\big(\ell_2\Gamma \bar{\otimes} B(\mathcal{H})\big) \xrightarrow{\rho\otimes\mathrm{id}} L_p\big(\ell_2\Gamma \bar{\otimes} B(\mathcal{H});\ (\ell_2\Gamma)_c\big),$$

and restricting its image we get a cb-map $S_p(\Gamma) \to L_p(B(\ell_2(\Gamma)); (\ell_2(\Gamma) \otimes_2 \mathcal{H})_c)$. In addition, the map

$$U : \ell_2(\Gamma) \otimes_2 \mathcal{H} \to \ell_2(\Gamma), \qquad U(\delta_j \otimes h) = \Big\langle \frac{J(w_j)}{\|w_j\|}, h \Big\rangle\, \delta_j$$

is a contraction. By the homogeneity of C_p, we get that $(\mathrm{id} \otimes U)V$ is completely bounded, as J is isometric

$$(\mathrm{id} \otimes U)V(e_{jk}) = \Big\langle \frac{u_j - u_k}{\|u_j - e_k\|}, \frac{w_j}{\|w_j\|} \Big\rangle e_{jk} \otimes \delta_j = \rho \circ S_{M_1}(e_{jk}).$$

Since ρ is a complete isometry, we conclude $\|S_{M_1}\|_{p,\mathrm{cb}} \leq \|(\mathrm{id} \otimes U)V\|_{p,\mathrm{cb}} \lesssim p$. □

Generalized triangular truncations in $\ell_2(\mathbb{Z})$. The Schur multiplier with symbol the characteristic function $M(i,j) = \mathbf{1}_{i\leq j}$ is usually called (upper) triangular truncation. Their boundedness is equivalent to the boundedness of the usual Hilbert transform in $\mathbb{Z}$. We have already seen some generalization of this kind of multipliers, when we considered the non-diagonal Hilber transform $M_\varphi(i,j) = \mathrm{sgn}(\varphi(i) - \varphi(j))$ in the proof of Lemma 4.3.1. Now we introduce another kind of generalized triangular truncations on $\ell_2(\mathbb{Z})$, namely, the family of Schur multipliers whose symbols are given by

$$\big\{M_a(j,k) = \mathrm{sgn}(a_j - k) : a : \mathbb{Z} \to \mathbb{Z}\big\} \tag{4.4.2}$$

We start with a version of Corollary 4.4.1, given by the next lemma:

Proposition 4.4.2. *The family of Schur multipliers*

$$M_n(j,k) = \Big\langle \frac{u_j^{(n)} - u_k'}{\|u_j^{(n)} - u_k'\|}, \frac{w_j^{(n)}}{\|w_j^{(n)}\|} \Big\rangle$$

is OC_p-bounded for any sequences $u_j^{(n)}, u_k', w_j^{(n)} \in \mathcal{H}$ and $1 < p < \infty$.

Proof. We view the space $L_p(\mathcal{M} \bar{\otimes} B(\ell_2\Gamma);\ (\ell_2\mathbb{Z})_c)$ as consisting of elements of the form

$$x_{jk} \otimes e_{jk} \otimes e_{n0} \in \mathcal{M} \bar{\otimes} \ell_2\Gamma \bar{\otimes} B(\ell_2\mathbb{Z}).$$

Oberserve that the element $e_{jk} \otimes e_{n0} \in B(\ell_2\Gamma)\bar{\otimes}B(\ell_2\mathbb{Z})$ naturally corresponds to the matrix $e_{(j,n),(k,0)} \in B(\ell_2(\Gamma \times \mathbb{Z}))$. Therefore, the OC_p-boundedness of the operator

$$M(x_{jk} \otimes e_{jk} \otimes \delta_n) = M_n(x_{jk} \otimes e_{jk}) \otimes \delta_n$$

in $L_p(\mathcal{M}\bar{\otimes}B(\ell_2\Gamma);\, (\ell_2\mathbb{Z})_c)$ corresponds to the boundedness of the Schur multiplier $\tilde{M}$ in $\mathcal{M} \otimes B(\ell_2(\Gamma \times \mathbb{Z}))$ given by

$$\tilde{M}((j,n),(k,m)) = \Big\langle \frac{u_j^{(n)} - u_k'}{\|u_j^{(n)} - u_k'\|}, \frac{w_j^{(n)}}{\|w_j^{(n)}\|} \Big\rangle$$

which is bounded by Corollary 4.4.1. □

Taking $\mathcal{H}_{\mathbb{R}} = \mathbb{R}$ and $(u_j, u_k', w_j) = (a_j, -k, 1)$ the symbols M_a (see Equation (4.4.2)) are represented by

$$M_a(j,k) = \Big\langle \frac{u_j + u_k'}{\|u_j + u_k'\|}, \frac{w_j}{\|w_j\|} \Big\rangle = \operatorname{sgn}(a_j - k),$$

so they fit in the hypothesis of Proposition 4.4.2.

Corollary 4.4.3. *The family of generalized triangular truncations defined in Equation* (4.4.2) *is* OC_p*-bounded with constant* $\max\{p, p/p-1\}$.

Remark 4.4.4. Of course, there are analogous results for OR_p-boundedness of a very similar family of multipliers, and for the respective generalized triangular truncations taking $(u_j, u_k', w_j) = (j, -a_k, 1)$.

Multipliers on $\mathbb{Z}$ of bounded variation on columns. Now we are going to work with a slightly more general family of generalized triangular truncations. Taking a random choice of signs $\varepsilon_j \in \{-1, 0, 1\}$ and defining $(u_j, u_k', w_j) = (a_j, -k, \varepsilon_j)$, Proposition 4.4.2 says that Schur multipliers with symbols in the family Φ defined below constitute a OC_p-bounded family of operators for $1 < p < \infty$:

$$\Phi = \{M_{a,\epsilon}(j,k) = \epsilon_j \operatorname{sgn}(a_j - k)\}. \tag{4.4.3}$$

Lemma 4.4.5. *The family of Schur multipliers on $S_p(\mathbb{Z})$ with symbols in*

$$\big\{M(j,k) = \Delta(j) :\ \sup_{j\in\mathbb{Z}} |\Delta(j)| \le C\big\}$$

is an OC_p-bounded family of multipliers with constant C for any $1 \le p < \infty$.

Proof. It follows from the fact that the map

$$x_{jk}^{(n)} \otimes e_{jk} \otimes e_{n1} \mapsto \Delta^{(n)}(j) x_{jk}^{(n)} \otimes e_{jk} \otimes e_{n1}$$

extends to a bounded operator in $L_p\big(\mathcal{M}\bar{\otimes}B(\ell_2\mathbb{Z} \otimes \mathcal{H})\big)$ with norm $\sup_{j,n} |\Delta^{(n)}(j)|$. □

Corollary 4.4.6. *The set of bounded functions M on $\mathbb{Z}$ such that*

$$\sum_{k\in\mathbb{Z}} |M(j,k) - M(j,k-1)| \le 1 \qquad \textit{and} \qquad \|M\|_\infty \le 1$$

defines a OC_p-bounded family of Schur multipliers.

Proof. Let $\tilde{\Phi} = \frac{1}{2}\Phi$, where Φ is the family of multipliers defined in Equation 4.4.3. A symbol $\tilde{M}_{a,\epsilon} \in \tilde{\Phi}$ is a function that has in each column j exactly one jump of height 1 if $\epsilon_j \neq 0$, and this jump may be positive if $\epsilon_j = 1$ or negative if $\epsilon_j = -1$.

Given M as in the statement of the corollary, define $dM(j,k) = M(j,k) - M(j,k-1)$ the jump in the position k of the row j. For each row j and every integer $m \geq 0$, we are going to associate a (possibly empty) sequence of non-decreasing integers

$$I^{(j,m)} = (k_1^{(j,m)}, \ldots, k_N^{(j,m)})$$

where $k \in \mathbb{Z}$ appears exactly n_k times if $n_k/2^m \leq |dM(j,k)| < (n_k+1)/2^m$.

The column-variation condition of M implies that each $I^{(j,m)}$ has at most 2^m entries. Thus let's take a sequence of at most 2^m functions $a_r^{(m)} \colon \mathbb{Z} \to \mathbb{Z}$ that are defined (when possible) as

$$a_r^{(m)}(j) = \text{possition } r \text{ in the vector } I^{(j,m)}.$$

Now take signs $\epsilon_r^{(j,m)} \colon \mathbb{Z} \to \{-1, 0, 1\}$ matching the jump (if any) of the symbol M in the position $(j, a_r^{(m)}(j))$, and define

$$\tilde{M}^{(m)}(j,k) = \frac{1}{2^m} \sum_r \tilde{M}_{a_r^{(m)}, \epsilon_r^{(m)}}.$$

This $\tilde{M}^{(m)}$ is a symbol that lies in the absolute convex hull of $\tilde{\Phi}$ and verifies that

$$\sup_{j,k \in \mathbb{Z}} |dM(j,k) - d\tilde{M}^{(m)}(j,k)| < \frac{1}{2^m}.$$

We claim that $\tilde{M}^{(m)}$ tends pointwise to M up to some bounded function only depending on rows. To see this, let's consider the functions Δ and $\tilde{\Delta}^{(m)}$ defined as

$$\Delta(j) = \lim_{k \to -\infty} M(j,k), \qquad \text{and} \qquad \tilde{\Delta}^{(m)} \lim_{k \to -\infty} \tilde{M}^{(m)}(j,k).$$

Notice that $M(j,k) = \sum_{k' \leq k} dM(j,k') + \Delta(j)$, and similarly for $\tilde{M}$. Therefore, if we define $M^{(m)}$ as

$$M^{(m)}(j,k) = \tilde{M}^{(m)}(j,k) \; - \; \tilde{\Delta}^{(m)}(j) + \Delta(j),$$

the following inequality holds for every $(j,k) \in \mathbb{Z} \times \mathbb{Z}$:

$$|M(j,k) - M^{(m)}(j,k)| \leq \sum_{k \in \mathbb{Z}} |dM(j,k) - d\tilde{M}^{(m)}(j,k)|.$$

By the dominated convergence theorem, the sum on the right-hand side goes to 0 when m tends to infinity. This implies that M is in the pointwise-closure of the absolute convex hull of some family of OC_p-bounded Schur symbols. Such family is built up from Φ' and the multipliers considered in Lemma 4.4.5 using operations in Proposition 4.2.7, which implies that the family of the statement is indeed an OC_p-bounded family of operators. □

Second criteria: square roots of Arazy's divided differences Recall that our second criteria in Theorem 4.1.5 says that symbols of the form

$$M(x,y) = \sqrt{\frac{f(x) - f(y)}{x - y}},$$

are bounded when f is a Lipschitz, increasing function. The proof follows from a concrete case of the S_p-factorization condition, as we shall see now.

Proof of Theorem 4.1.5, Part 2. We claim that the symbols

$$\tilde{M}(j,k) = \frac{\|\Lambda(w_j) - \Lambda(w_k)\|}{\|w_j - w_k\|}$$

define completely bounded multipliers with $\|S_{\tilde{M}}\|_{\mathrm{cb}(S_p(\Gamma))} \lesssim \max\left\{p, \frac{p}{p-1}\right\}^{\frac{5}{2}}$. Indeed, we just apply Theorem 4.1.5 i) with $(u_j, u_k', w_j, w_k') = (\Lambda(w_j), -\Lambda(w_k), w_j, -w_k)$.

Next we take $\Gamma = \mathbb{R}$. We claim that, by approximation by finite matrices (see [LdlS11, Theorem 1.19]), the multiplier $M(x,y)$ corresponding to the square root of Arazy's divided differences is bounded on $\mathcal{H} = L_2(\mathbb{R})$ with the same norm as $\tilde{M}$. Let it be the vector $w_x = 1_{[0,x]}$ for $x \in \mathbb{R}$, and define $\Lambda : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ as the pointwise product by $\sqrt{f'}$, which exists a.e and is bounded by $\sqrt{\|f\|_{\mathrm{lip}}}$. Then

$$\frac{\|\Lambda(w_x) - \Lambda(w_y)\|}{\|w_x - w_y\|} = \sqrt{\frac{f(x) - f(y)}{x - y}}.$$

□

Remark 4.4.7. Theorem 4.1.5 ii) can actually be improved as follows. Given any $0 < \beta < 1$ and a nondecreasing Lipschitz function $f : \mathbb{R} \to \mathbb{R}$, we consider the symbol

$$M_{f\beta}(x,y) = \Big(\frac{f(x) - f(y)}{x - y}\Big)^{\beta}.$$

It can be shown that for $1 < p < \infty$, the completely bounded norm satisfies

$$\Big\|S_{M_{f\beta}} : S_p(\mathbb{R}) \to S_p(\mathbb{R})\Big\|_{\mathrm{cb}} \lesssim \max\left\{p, \frac{p}{p-1}\right\}^{2-\beta} \|f\|_{\mathrm{Lip}}^{\beta}. \tag{4.4.4}$$

The proof follows from the three-lines lemma for operator-valued analytic functions in L_p, combined with Theorem 4.1.6 ii). Specifically, consider the family of symbols M_{fz} for $0 \le \mathrm{Re}(z) \le 1$ and the associated analytic function $S_{M_{fz}}(A)$ for $A \in S_p(\mathbb{R})$.

- **Case** $\mathrm{Re}(z) = 0$**:** We estimate the Schur multipliers $|f(x) - f(y)|^{\mathrm{is}}$ and $|x - y|^{\mathrm{is}}$. Standard results (see [CAGPPT22, Lemma 1.1]) show that the former is controlled by the latter. From Theorem 4.1.6 ii), the Schur multiplier with symbol $|x - y|^{\pm \mathrm{is}}$ has norm bound
$$(1 + |s|^2) \max\{p, p/(p-1)\}^2.$$
- **Case** $\mathrm{Re}(z) = 1$**:** Applying Theorem 4.1.6 ii), we obtain the bound
$$(1 + |s|) \max\{p, p/(p-1)\} \|f\|_{\mathrm{Lip}}.$$
Since the dependence on s is subexponential, we conclude (4.4.4) via interpolation.

A similar argument applies to α-divided differences of α-Hölder functions as in [CAGPPT23, Corollary 3.4]. Namely, if $0 < \alpha < 1$ and $f : \mathbb{R} \to \mathbb{R}$ is a nondecreasing α-Hölder function, the symbols

$$M_{f\alpha\beta}(x,y) = \frac{|f(x) - f(y)|^{\beta}}{|x - y|^{\alpha\beta}}$$

define completely S_p-bounded multipliers for

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \min\{\alpha, 1/2\}.$$

Remark 4.4.8. Although (4.4.4) improves Theorem 4.1.5 ii), it illustrates how Theorem 4.1.5 i) or (4.4) may be understood as a generalized Hilbert-valued divided difference, and other generalizations besides those in Remark 4.4.7 are conceivable.

Hörmander-Mikhlin-Schur multipliers

In this section we give a simple proof of the main results in [CLM24, CAGPPT23] using mainly Theorem 4.1.2. The approach towards HMS multipliers from [CAGPPT23] is based an a new proof of the Hörmander-Mikhlin theorem from [JMP18, PRS22] which relies on Riesz transforms for fractional laplacians.

A Sobolev-like norm for the HMS theorem Our results will make use of a Sobolev space slightly different from the usual one, as the next definition shows.

Definition 4.4.1. Given $0 < \varepsilon < 1$, the Hilbert space $W^2_{n,\varepsilon}(\mathbb{R}^n)$ is defined as the completion of $\mathcal{C}^\infty_c(\mathbb{R}^n \setminus \{0\})$ for the norm

$$\|f\|_{W^2_{n,\varepsilon}} = \big\| \, |\cdot|^{\frac{n}{2}+\varepsilon} \widehat{|\cdot|^\varepsilon f} \, \big\|_{L_2(\mathbb{R}^n)}.$$

Essentially, we added a regularization term $|\cdot|^\varepsilon$ to the usual definition of the Sobolev norm. This adds some features that are definitely good for our purpose, because it makes the norm dilation invariant, and also because the space $W^2_{n,\varepsilon}$ consists just of bounded continuous functions. For a proof of these two facts, we refer the reader to [PRS22]. Moreover, in that work the authors give a nice characterization of functions in $W^2_{n,\epsilon}$ in terms of cocycles and Riesz transforms, which we transcribe now in the following lemma.

Lemma 4.4.9. *There exist a complex Hilbert space $\mathcal{H}$ with a strongly continuous unitary action $\alpha : \mathbb{R}^n \curvearrowright \mathcal{H}$ and an α-cocycle $\beta : \mathbb{R}^n \to \mathcal{H}$ such that for any $f \in W^2_{n,\varepsilon}(\mathbb{R}^n)$ there exists $u_f \in \mathcal{H}$ such that*

$$f(x) = \Big\langle \frac{\beta(x)}{\|\beta(x)\|}, u_f \Big\rangle \quad \textit{and} \quad \|f\|_\infty \lesssim_{n,\varepsilon} \|u_f\|_{\mathcal{H}} \approx_{n,\varepsilon} \|f\|_{W^2_{n,\varepsilon}}.$$

In particular, we find $f(x-y) = \big\langle \frac{\beta(x)-\beta(y)}{\|\beta(x)-\beta(y)\|}, \alpha_{-y}(u_f) \big\rangle = \big\langle \frac{\beta(x)-\beta(y)}{\|\beta(x)-\beta(y)\|}, -\alpha_{-x}(u_f) \big\rangle$.

Remark 4.4.10. This lemma, together with Proposition 4.4.2, implies that any bounded set of $W^2_{n,\varepsilon}(\mathbb{R}^n)$ defines an OC_p-bounded family Toeplitz Schur multipliers on $\mathbb{R}^n$ for $1 < p < \infty$. Furthermore, let $(f_x)_{x\in\mathbb{R}^n}$ be a family of functions in the unit ball of $W^2_{n,\varepsilon}(\mathbb{R}^n)$ for some $0 < \varepsilon < 1$. Then the Schur multiplier on $\mathbb{R}^n$ with symbol

$$M(x,y) = f_x(x-y)$$

for $x, y \in \mathbb{R}^n$ satisfies the bound $\|S_M\|_{p,\mathrm{cb}} \lesssim_{n,\varepsilon} C_p$ for $1 < p < \infty$, and the family of such Schur multipliers is OC_p-bounded with the same constant.

Remark 4.4.11. According to [PRS22, Lemma 2.2] we know that, for functions with support in a compact set not containing 0, the $W^2_{n,\varepsilon}$-norm is smaller (up to a constant that depends on the support) than the Sobolev norm

$$\|f\|_{H_{\frac{n}{2}+\varepsilon}} = \big\| \, (1+|\cdot|^2)^{\frac{n}{4}+\frac{\varepsilon}{2}} \widehat{f} \, \big\|_{L_2(\mathbb{R}^n)}.$$

Therefore, for compactly supported functions in $H_{\frac{n}{2}+\varepsilon}$ define Toeplitz Schur multipliers. This is obvious though, since the Fourier transform of elements in this Sobolev space belong to $L_1(\mathbb{R}^n)$. In fact, if (f_x) are functions in the unit ball of $H_{\frac{d}{2}+\varepsilon}$ we get a OC_∞-bounded family without any assumption on the support because

$$f_x(x-y) = \int_{\mathbb{R}^n} (1+|s|^2)^{\frac{d}{4}+\frac{\varepsilon}{2}} \widehat{f_x}(s) e^{2\mathrm{i}\pi\langle s,x\rangle} \frac{1}{(1+|s|^2)^{\frac{d}{4}+\frac{\varepsilon}{2}}} e^{-2\mathrm{i}\pi\langle s,y\rangle} ds = \big\langle \xi_x, \eta_y \big\rangle_{L_2},$$

where $\sup_x \|\xi_x\|_2 \lesssim_{d,\varepsilon} 1$ and $\sup_y \|\eta_y\|_2 \lesssim_{d,\varepsilon} 1$. Thus the OC_∞-boundedness comes from the S_p-factorization condition (as η_y does not depend of f_x) and we also get OC_p-boundedness with a constant independent of p for every $p \geq 1$. This illustrates that the spaces $W^2_{n,\varepsilon}(\mathbb{R}^n)$ are definitely larger, allowing more singularities on the symbols considered. Therefore a Hörmander-Mikhlin theorem in terms of these spaces —like the one given below— is formally stronger.

Proofs of HMS and Marcinkiewicz-Schur theorems In what follows and again by approximation, we shall work with $\Gamma = \mathbb{R}^n$. Take $\Delta_k = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : 2^k \leq |x-y|_\infty < 2^{k+1}\}$. This decomposition was already considered in Theorem 4.2.11, and it satisfies (4.2.2) with $D_p \lesssim \max\{p, p/p-1\}^{n+1}$. Other smoother choices of Littlewood-Paley decomposition yield better constants, but we shall not consider them for the sake of simplicity on our presentation. In the following result we fix $0 < \varepsilon < 1$ and $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n \setminus \{0\})$ even and identically 1 on Δ_0.

Theorem 4.4.12. *Let $M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a continuous function such that*

$$\sup_{x\in\mathbb{R}^n} \sup_{k\in\mathbb{Z}} \big\|\varphi(2^k\cdot)M(x,x-\cdot)\big\|_{W^2_{n,\varepsilon}} + \sup_{y\in\mathbb{R}^n} \sup_{k\in\mathbb{Z}} \big\|\varphi(2^k\cdot)M(y-\cdot,y)\big\|_{W^2_{n,\varepsilon}} \leq 1.$$

Then S_M defines a completely bounded Schur multiplier on S_p for every $1 < p < \infty$.

Proof. Let $M_k(x,y) = \varphi(2^{-k}(x-y))M(x,y)$ for $x, y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. It is clear that

$$M(x,y) = \sum_{k\in\mathbb{Z}} 1_{(x,y)\in\Delta_k} M_k(x,y), \quad \text{for } x \neq y$$

because $\varphi(2^{-k}(x-y)) = 1$ if $(x,y) \in \Delta_k$.

We claim that the family S_{M_k} is ORC_p-bounded, and therefore M defines a bounded Schur multiplier in S_p for $1 < p < \infty$ (see Lemma 4.2.10). To prove the claim, consider first the functions

$$f_{x,k}(y) = \varphi(2^{-k}y)M(x,x-y) \quad \text{for} \quad (x,k) \in \mathbb{R}^n \times \mathbb{Z}.$$

We have $M_k(x,y) = f_{x,k}(x-y)$ and our hypothesis means that $\|f_{x,k}\|_{W^2_{n,\varepsilon}} \leq 1$. In particular, the family $\{S_{M_k} : k \in \mathbb{Z}\}$ is OC_p-bounded by Remark 4.4.10. Similarly let $g_{k,y}(x) = \varphi(2^{-k}x)M(y-x,y)$. Then $M_k(x,y) = g_{k,y}(y-x)$ and the same argument shows that indeed the family $\{S_{M_k} : k \in \mathbb{Z}\}$ is OR_p-bounded, and thus ORC_p-bounded. □

Proof of Theorem 4.1.6, Part 1. The argument is essentially the same as in Theorem 4.4.12. Let M be a multiplier as in the statement of the theorem, and consider the decomposition

$$M(j,k) = \begin{cases} \sum_{n\in\mathbb{Z}} \mathbf{1}_{(j,k)\in\Delta_n} M_n(j,k) & \text{if } i \neq j \\ M(j,j) & \text{otherwise,} \end{cases}$$

where $\Delta_n = \{(j,k) \in \mathbb{Z}\times\mathbb{Z} : 2^n \leq |j-k| < 2^{n+1}\}$ and $M_n(j,k)$ is just $M_n = \mathbf{1}_{\Delta_n} \cdot M$. Denoting by $\mathbb{E}$ the diagonal conditional expectation on $S_p(\mathbb{Z})$, the operator S_M can be decomposed as

$$S_M = \sum_{n\in\mathbb{Z}} S_{\Delta_n} S_{M_n} + S_M \circ \mathbb{E}.$$

Notice that $S_M \circ \mathbb{E}$ is bounded in every S_p (even for $p = 1, \infty$); on the other hand, the multipliers S_{M_n} have (uniformly) bounded variation both on rows and columns, so they are an ORC_p-bounded family for every $1 < p < \infty$ by Corollary 4.4.6. The sequence (Δ_n) verifies (4.2.2) since it is a dyadic diagonal partition, so we conclude that S_M is a bounded S_p-multiplier. □

Proof of Theorem 4.1.6, Part 2. According to [PRS22, Lemma 2.2], Theorem 4.4.12 is strictly stronger than the main result of [CAGPPT23], both in its Hörmander-Mikhlin and Sobolev formulations. This already justifies the validity of Theorem 4.1.6 (ii). □

Remark 4.4.13. Our argument for the Hörmander-Mikhlin theorem yields to dimension-depending constants because we used a non-smooth partition of identity. A more refined approach would employ smooth functions φ_n supported on dyadic coronas, as introduced in Theorem 4.2.11. If, in addition, the functions φ_n are chosen to satisfy the condition

$$\sum_{n\in\mathbb{Z}} \varphi_n(x)^2 = 1, \qquad \text{for all } x \in \mathbb{R}^n,$$

then the kernel $M(x,y)$ can be decomposed as

$$M(x,y) = \sum_{n\in\mathbb{Z}} \varphi_n(x-y)^2 M(x,y) = \sum_{n\in\mathbb{Z}} \varphi_n(x-y) M_n(x,y),$$

with $M_n(x,y) = \varphi_n(x-y)M(x,y)$.

Now the condition

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}} \big\|\varphi(2^k\cdot)M(x,x-\cdot)\big\|_{W^2_{n,\varepsilon}} + \sup_{y\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}} \big\|\varphi(2^k\cdot)M(y-\cdot,y)\big\|_{W^2_{n,\varepsilon}} \le 1.$$

for this particular φ, this implies that M can be written as a weighted sum of an ORC_p-bounded sequence of multipliers, weighted by the φ_n. Using the notation from Lemma 4.2.10, together with Theorem 4.2.11 (i) and Corollary 4.4.3, we find that the constants K_p and D_p behave asymptotically like $c_p = \frac{p^2}{1-p}$. As a result, the norm of S_M as a Schur multiplier on S_p is asymptotically bounded by c_p^3. Using this argument, it seems the closest one could get to the optimal bound, which is just $\|S_M : S_p \to S_p\| \lesssim c_p$ (see [CAGPPT23]).

Conclusiones

En esta tesis hemos estudiado álgebras de von Neumann desde la perspectiva del análisis armónico no conmutativo, abordando varios problemas fundamentales relacionados con multiplicadores de Fourier, desigualdades maximales y teoría de interpolación. La tesis está organizada en torno a tres temas principales.

En primer lugar, investigamos operadores maximales y desigualdades martingálicas en contextos no conmutativos. Proporcionamos una solución parcial a un problema abierto sobre la clase óptima de espacios de Orlicz para desigualdades maximales fuertes no conmutativas, mostrando que esta clase óptima no puede ser menor que $L\log^2 L$. Nuestra demostración se basó en un refinamiento de técnicas de interpolación originalmente desarrolladas por Cadilhac y Ricard, lo que permitió un control preciso de las normas de operadores maximales en espacios de Orlicz.

En segundo lugar, exploramos multiplicadores de Fourier en retículos del grupo de Lie $\mathrm{SL}_2(\mathbb{C})$. Extendimos métodos anteriores (identidades de Cotlar) para demostrar la existencia de nuevos multiplicadores idempotentes de Fourier acotados en espacios L^p asociados a ciertos retículos específicos.

Finalmente, analizamos multiplicadores de Schur en clases de Schatten mediante la introducción de un marco unificado basado en transformadas de Riesz en versión Schur. Establecimos nuevas desigualdades tipo "square-function" y obtuvimos nuevos resultados de acotación para multiplicadores de Schur en espacios de Schatten. Además, proporcionamos demostraciones simplificadas y unificadas de resultados clave como el teorema de Hörmander-Mikhlin y el criterio de Marcinkiewicz para multiplicadores de Schur.

En resumen, esta tesis supone un avance significativo en la resolución de cuestiones abiertas en el análisis armónico no conmutativo e introduce técnicas innovadoras. Los métodos desarrollados sugieren direcciones prometedoras para futuras investigaciones, especialmente en el estudio de multiplicadores de Fourier en grupos más generales y en nuevas aplicaciones de las transformadas de Riesz en la teoría de multiplicadores de Schur.

Conclusions

In this thesis, we have studied von Neumann algebras through the lens of noncommutative harmonic analysis, addressing several fundamental problems related to Fourier multipliers, maximal inequalities, and interpolation theory. The thesis is structured around three main topics.

First, we investigated maximal operators and martingale inequalities in noncommutative settings. We provided a partial solution to an open problem concerning the optimal Orlicz space class for noncommutative strong maximal inequalities, showing that the optimal space cannot be smaller than $L \log^2 L$. Our proof relied on a refinement of interpolation techniques originally developed by Cadilhac and Ricard, enabling precise control of maximal operator norms in Orlicz spaces.

Second, we explored Fourier multipliers on lattices in the Lie group $\mathrm{SL}_2(\mathbb{C})$. We extended previous methods —Cotlar identities— to demonstrate the existence of new idempotent Fourier multipliers bounded on L^p spaces associated with particular lattices.

Finally, we analyzed Schur multipliers in Schatten classes by introducing a unified framework based on Schur analogs of Riesz transforms. We established novel square-function inequalities and obtained new boundedness results for Schur multipliers in Schatten spaces. Additionally, we provided streamlined and unified proofs of key results such as the Hörmander-Mikhlin theorem and the Marcinkiewicz criterion for Schur multipliers.

Overall, this thesis significantly advances the resolution of open questions in noncommutative harmonic analysis and introduces innovative techniques. The methods developed here suggest promising directions for future research, particularly in the study of Fourier multipliers on groups and further applications of Riesz transforms in the theory of Schur multipliers.

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